CS771A

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Assgn – 0

Ans 5(a)

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$$

$$\det(A-\lambda I) = \begin{vmatrix} 1-\lambda & 2 & 3 \\ 4 & 5-\lambda & 6 \\ 7 & 8 & 9-\lambda \end{vmatrix} = (1-\lambda)\left[(5-\lambda)(9-\lambda) - 48\right] - 2\left[(9-\lambda)4 - 42\right] + 3\left[32 - 7(5-\lambda)\right]$$

$$= (1-\lambda)\left[45+\lambda^2-14\lambda-48\right] - 2\left[36-4\lambda-42\right] + 3\left[32-35+7\lambda\right]$$

$$= (1-\lambda)\left[\lambda^2-14\lambda-3\right] - 2\left[-4\lambda-6\right] + 3\left[-3+7\lambda\right]$$

$$= \lambda^2-14\lambda-3-\lambda^3+14\lambda^2+3\lambda+8\lambda+12-9+21\lambda$$

$$= -\lambda^3+15\lambda^2+18\lambda$$

Every matrix satisfies its characteristic eqn :

$$\Rightarrow -A^3+15A^2+18A = 0$$

$$\Rightarrow A^3-15A^2-17A = A$$

Ans 5(b)

$$A = \begin{bmatrix} 3 & 2 & 4 \\ 2 & 0 & 2 \\ 4 & 2 & 3 \end{bmatrix}$$

To find the Eigen Values :

$$\det(A-\lambda I) = 0$$

$$\det(A-\lambda I)=0 \Rightarrow \begin{vmatrix} 3-\lambda & 2 & 4 \\ 2 & -\lambda & 2 \\ 4 & 2 & 3-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (3-\lambda)(-\lambda(3-\lambda)-4) - 2(2(3-\lambda)-8) + 4(4+4\lambda) = 0$$

$$\Rightarrow (3-\lambda)[-3\lambda+\lambda^2-4] - 2[-2\lambda-2] + 16 + 16\lambda = 0$$

$$\Rightarrow -9\lambda + 3\lambda^2 - 12 + 3\lambda^2 - \lambda^3 + 4\lambda + 4\lambda + 4 + 16 + 16\lambda = 0$$

$$\Rightarrow -\lambda^3 + 6\lambda^2 + 15\lambda + 8 = 0$$

$$\Rightarrow \lambda^3 - 6\lambda^2 - 15\lambda - 8 = 0$$

$$\Rightarrow (\lambda+1)[\lambda^2 - 7\lambda - 8] = 0$$

$$\Rightarrow (\lambda+1)[\lambda^2 - 8\lambda + \lambda - 8] = 0$$

$$\Rightarrow (\lambda+1)[\lambda(\lambda-8) + 1(\lambda-8)] = 0$$

$$\Rightarrow (\lambda+1)^2(\lambda-8) = 0$$

$$\Rightarrow \lambda = -1, -1, 8$$

One root $= -1$

$$\begin{array}{r} \lambda+1 \,)\, \lambda^3 - 6\lambda^2 - 15\lambda - 8 \,(\, \lambda^2 - 7\lambda - 8 \\ \lambda^3 + \lambda^2 \\ \hline -7\lambda^2 - 15\lambda - 8 \\ -7\lambda^2 - 7\lambda \\ \hline -8\lambda - 8 \\ -8\lambda - 8 \end{array}$$

To find the eigen vectors:

① $(A+I)X = 0 \Rightarrow \begin{bmatrix} 4 & 2 & 4 \\ 2 & 1 & 2 \\ 4 & 2 & 4 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \left[\begin{array}{ccc|c} 4 & 2 & 4 & 0 \\ 2 & 1 & 2 & 0 \\ 4 & 2 & 4 & 0 \end{array}\right] \Rightarrow \left[\begin{array}{ccc|c} 4 & 2 & 4 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$

y, z are free variables

Set $y = \alpha$, $z = \beta$

$$4x + 2y + 4z = 0$$
$$4x = -2y - 4z$$
$$x = -\frac{1}{2}\alpha - \beta$$

$$\text{Eigenvector} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -\frac{1}{2}\alpha - \beta \\ \alpha \\ \beta \end{bmatrix}$$

$$= \alpha \begin{bmatrix} -\frac{1}{2} \\ 1 \\ 0 \end{bmatrix} + \beta \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

2 Eigen Bases

(2) $(A-8I)X=0 \Rightarrow \begin{bmatrix} -5 & 2 & 4 \\ 2 & -8 & 2 \\ 4 & 2 & -5 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \left[\begin{array}{ccc|c} -5 & 2 & 4 & 0 \\ 2 & -8 & 2 & 0 \\ 4 & 2 & -5 & 0 \end{array}\right]$

$\Downarrow R_3 \to R_3 + R_1$

$\left[\begin{array}{ccc|c} -5 & 2 & 4 & 0 \\ 2 & -8 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \xleftarrow{R_3 \to R_3 + \frac{1}{2}R_2} \left[\begin{array}{ccc|c} -5 & 2 & 4 & 0 \\ 2 & -8 & 2 & 0 \\ -1 & 4 & -1 & 0 \end{array}\right]$

z is the free variable.

Set $z = \alpha$

$$\begin{aligned} -5x + 2y + 4z &= 0 \\ 2x - 8y + 2z &= 0 \end{aligned} \Rightarrow \begin{aligned} -5x + 2y &= -4\alpha \\ 2x - 8y &= -2\alpha \end{aligned}$$

$\Downarrow$

$$\begin{aligned} -20x + 8y &= -16\alpha \\ 2x - 8y &= -2\alpha \\ \hline -18x &= -18\alpha \\ x &= 1\alpha \end{aligned}$$

$\Rightarrow y = \dfrac{-4\alpha + 5x}{2}$

$= \dfrac{-4\alpha + 5\alpha}{2} = \dfrac{\alpha}{2}$

$X = \begin{bmatrix} \alpha \\ \alpha/2 \\ \alpha \end{bmatrix} = \alpha \begin{bmatrix} 1 \\ 1/2 \\ 1 \end{bmatrix}$

3rd Eigen basis

P = Matrix of Eigen ~~Mat~~ Vectors

$= \begin{bmatrix} 1 & -1 & 2 \\ 2 & 0 & 1 \\ 0 & 1 & 2 \end{bmatrix}$

Calculating P^{-1}

$$\left[\begin{array}{ccc|ccc} -1 & -1 & 2 & 1 & 0 & 0 \\ 2 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 2 & 0 & 0 & 1 \end{array}\right] \Rightarrow \left[\begin{array}{ccc|ccc} 1 & 1 & -2 & -1 & 0 & 0 \\ 2 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 2 & 0 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{R_2 \to R_2 - 2R_1} \left[\begin{array}{ccc|ccc} 1 & 1 & -2 & -1 & 0 & 0 \\ 0 & -2 & 5 & 2 & 1 & 0 \\ 0 & 1 & 2 & 0 & 0 & 1 \end{array}\right] \xrightarrow{R_2 \leftrightarrow R_3} \left[\begin{array}{ccc|ccc} 1 & 1 & -2 & -1 & 0 & 0 \\ 0 & 1 & 2 & 0 & 0 & 1 \\ 0 & -2 & 5 & 2 & 1 & 0 \end{array}\right]$$

$$\xrightarrow[R_3 \to R_3 + 2R_2]{R_1 \leftarrow R_1 - R_2} \left[\begin{array}{ccc|ccc} 1 & 0 & -4 & -1 & 0 & -1 \\ 0 & 1 & 2 & 0 & 0 & 1 \\ 0 & 0 & 9 & 2 & 1 & 2 \end{array}\right] \xrightarrow{R_3 \to \frac{1}{9}R_3} \left[\begin{array}{ccc|ccc} 1 & 0 & -4 & -1 & 0 & -1 \\ 0 & 1 & 2 & 0 & 0 & 1 \\ 0 & 0 & 1 & 2/9 & 1/9 & 2/9 \end{array}\right]$$

$$\xrightarrow[R_1 \leftarrow R_1 + 4R_3]{R_2 \leftarrow R_2 - 2R_3} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -1/9 & 4/9 & -1/9 \\ 0 & 1 & 0 & -4/9 & -2/9 & 5/9 \\ 0 & 0 & 1 & 2/9 & 1/9 & 2/9 \end{array}\right]$$

$$P^{-1} = \begin{bmatrix} -1/9 & 4/9 & -1/9 \\ -4/9 & -2/9 & 5/9 \\ 2/9 & 1/9 & 2/9 \end{bmatrix}$$

$\Rightarrow A = PDP^{-1}$ where $D = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 8 \end{bmatrix}$

$A^5 = PD^5P^{-1}$ Now $\boxed{A^k = PD^kP^{-1}}$

$$A^5 = \begin{bmatrix} -1 & -1 & 2 \\ 2 & 0 & 1 \\ 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} (-1)^5 & 0 & 0 \\ 0 & (-1)^5 & 0 \\ 0 & 0 & 8^5 \end{bmatrix} \begin{bmatrix} -1/9 & 4/9 & -1/9 \\ -4/9 & -2/9 & 5/9 \\ 2/9 & 1/9 & 2/9 \end{bmatrix}$$

$$= \frac{1}{9} \cdot \begin{bmatrix} -1 & -1 & 2 \\ 2 & 0 & 1 \\ 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 32768 \end{bmatrix} \begin{bmatrix} -1 & 4 & -1 \\ -4 & -2 & 5 \\ 2 & 1 & 2 \end{bmatrix}$$

$$= \frac{1}{9} \begin{bmatrix} +1 & 1 & 65536 \\ -2 & 0 & 32768 \\ 0 & -1 & 65536 \end{bmatrix} \begin{bmatrix} -1 & 4 & -1 \\ -4 & -2 & 5 \\ 2 & 1 & 2 \end{bmatrix} = \frac{1}{9} \begin{bmatrix} 131067 & 65538 & 131076 \\ 65538 & 32760 & 65538 \\ 131076 & 65538 & 131067 \end{bmatrix}$$

$$= \begin{bmatrix} 14563 & 7282 & 14564 \\ 7282 & 3640 & 7282 \\ 14564 & 7282 & 14563 \end{bmatrix} //$$

Ans 6

$P(A_1) = \frac{1}{2}$ $P(A_2) = \frac{1}{3}$ $P(A_3) = \frac{1}{4}$

Given: Mutually Independent Events.

$$P[(A_1 \cap A_2) \cup A_3^c] = P(A_1 \cap A_2) + P(A_3^c) - P(A_1 \cap A_2 \cap A_3^c)$$

$$= P(A_1)P(A_2) + (1 - P(A_3)) - P(A_1)P(A_2)(1 - P(A_3))$$

$$= P(A_1)P(A_2)[1 - (1 - P(A_3))] + (1 - P(A_3))$$

$$= P(A_1)P(A_2)[P(A_3)] + (1 - P(A_3))$$

$$= \frac{1}{2} \cdot \frac{1}{3} \cdot \frac{1}{4} + \frac{3}{4}$$

$$= \frac{1}{24} + \frac{3}{4} = \frac{18+1}{24} = \frac{19}{24}$$

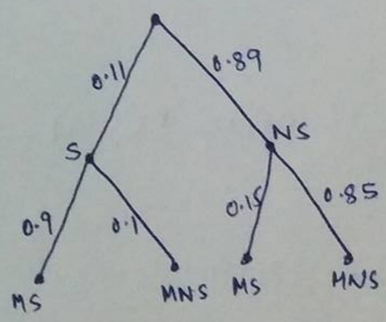
Ans 7

S - Actual spam mail

NS - Actual Non Spam Mail

MS - Marked Spam mail

MNS - Marked Non-Spam mail



a) $P(MS) = P(MS|S) + P(MS|NS)$

$= (0.11)(0.9) + (0.89)(0.15)$

$= 0.099 + 0.1335$

$= 0.2325$

P.T.O

$\Rightarrow$

(b) $P(S|MS) = \dfrac{P(MS|S)\,P(S)}{P(MS|S)\,P(S) + P(MS|NS)\,P(NS)}$

$= \dfrac{(0.9)(0.11)}{(0.9)(0.11) + (0.15)(0.89)} = \dfrac{0.099}{0.2325} = \dfrac{990}{2325} = \dfrac{66}{155}$

(c) $P(NS|MNS) = \dfrac{P(MNS|NS)\,P(NS)}{P(MNS|NS)\,P(NS) + P(MNS|S)\,P(S)}$

$= \dfrac{(0.85)(0.89)}{(0.85)(0.89) + (0.1)(0.11)}$

$= \dfrac{1513}{1535}$

(d) $P(\text{Missclassified}) = P(MNS|S) + P(MS|NS)$

$= (0.1)(0.11) + (0.15)(0.89)$

$= 0.1445$

$\boxed{\text{Ans 1}}$ (a)

To show:

$$\sum_{i=1}^{n} P(C_i) - \sum_{1\le i<j\le n} P(C_i \cap C_j) \le P\left(\bigcup_{i=1}^{n} C_i\right) \le \sum P(C_i)$$

(i) $T(n)$: $P\left(\bigcup_{i=1}^{n} C_i\right) \le \sum P(C_i)$

Consider $T(1)$: $P(C_1) \le P(C_1)$

$T(2)$: $P(C_1 \cup C_2) = P(C_1) + P(C_2) - P(C_1 \cap C_2)$

$\le P(C_1) + P(C_2)$

Assume $T(n)$ to be true:

$$P(C_1 \cup C_2 \cup \dots C_n) \leq P(C_1) + P(C_2) + \dots P(C_n) \qquad -①$$

Consider $T(n+1)$:

$$P(C_1 \cup C_2 \cup \dots C_n \cup C_{n+1}) = P(C_1 \cup C_2 \cup \dots C_n) + P(C_{n+1}) - P((C_1 \cup C_2 \dots C_n) \cap C_{n+1}) \qquad -②$$

From ① $P(C_1 \cup C_2 \cup \dots C_n) \leq P(C_1) + \dots P(C_n)$

$$P(C_1 \cup C_2 \cup \dots C_n) + P(C_{n+1}) \leq P(C_1) + \dots + P(C_n) + P(C_{n+1})$$

$$P(C_1 \cup C_2 \cup \dots C_n) + P(C_{n+1}) - P((C_1 \cup C_2 \cup \dots C_n) \cap C_{n+1}) \leq P(C_1) + P(C_2) + \dots + P(C_n) - P((C_1 \cup C_2 \cup \dots C_n) \cap C_{n+1})$$

From ② : $P\left(\bigcup_{i=1}^{n+1} C_i\right) \leq \sum_{i=1}^{n+1} P(C_i)$

(ii) To show:

$$T(n): \sum_{i=1}^{n} P(C_i) - \sum_{1 \leq i < j < n} P(C_i \cap C_j) \leq P\left(\bigcup_{i=1}^{n} C_i\right)$$

Consider $T(2)$:

$$P(C_1) + P(C_2) - P(C_1 \cap C_2) \leq P(C_1 \cup C_2) \quad \text{– Equality holds}$$

Assume $T(n)$ to be true.

$$P(C_1) + P(C_2) + \dots P(C_n) - P(C_1 \cap C_2) - P(C_1 \cap C_2) \dots - P(C_{n-1} \cap C_n) \leq P(C_1 \cup C_2 \cup C_3 \dots C_n) \qquad -①$$

Consider $T(n+1)$:

$$P(C_1 \cup C_2 \cup \dots C_{n+1}) = P(C_1 \cup C_2 \dots C_n) + P(C_{n+1}) - P((C_1 \cup C_2 \cup C_3 \dots C_n) \cap C_{n+1}) \qquad -②$$

From ①: $P(C_1 \cup C_2 \cup \dots C_n) \geq P(C_1) + P(C_2) + \dots P(C_n) - P(C_1 \cap C_2) - P(C_2 \cap C_3) \dots - P(C_{n-1} \cap C_n)$

$$P(C_1 \cup C_2 \cup \dots C_n) + P(C_{n+1}) \geq \sum_{i=1}^{n+1} P(C_i) - P(C_1 \cap C_2) - P(C_2 \cap C_3) - \dots P(C_{n-1} \cap C_n) \qquad -③$$

Consider the term $P((C_1 \cup C_2 \cup C_3 \dots C_n) \cap C_{n+1})$

$$= P((C_1 \cap C_{n+1}) \cup (C_2 \cap C_{n+1}) \dots \cup (C_n \cap C_{n+1})) \leq P(C_1 \cap C_{n+1}) + P(C_2 \cap C_{n+1}) + \dots P(C_n \cap C_{n+1})$$

$$-P((C_1\cup C_2\cup C_3\cdots\cup C_n)\cap C_{n+1}) \geq -\left[P(C_1\cap C_{n+1})+P(C_2\cap C_{n+1})+\cdots+P(C_n\cap C_{n+1})\right]$$

Adding this eqn to ③ gives

$$P(C_1\cup C_2\cup\cdots C_n)+P(C_{n+1})-P((C_1\cup C_2\cup\cdots C_n)\cap C_{n+1}) \geq \sum_{i=1}^{n+1}P(C_i)-\sum_{1\leq i<j<n+1}' P(C_i\cap C_j)$$

From ②

$$\Rightarrow P\left(\bigcup_{i=1}^{n+1}C_i\right) \geq \sum_{i=1}^{n+1}P(C_i)-\sum_{1\leq i<j<n+1}P(C_i\cap C_j)$$

Hence $T(n+1)$ is true.

Hence proved by induction.

$\boxed{\text{Ans 1 (b)}}$

To show:

$$T(n) \quad P\left(\bigcap_{i=1}^{n}C_i\right) \geq \sum_{i=1}^{n}P(C_i)-n+1$$

Consider $T(1)$: $P(C_1)\geq P(C_1)$ – Equality holds

$T(2)$: $P(C_1\cap C_2)\geq P(C_1)+P(C_2)-1$

$1\geq P(C_1)+P(C_2)-P(C_1\cap C_2)$

$1\geq P(C_1\cup C_2)$ – True statement.

Assume $T(n)$ to be true:

$$P(C_1\cap C_2\cap C_3\cdots C_n)\geq P(C_1)+P(C_2)+\cdots P(C_n)-n+1 \quad -①$$

Consider $T(n+1)$:

$$P((C_1\cap C_2\cap C_3\cdots C_n)\cap C_{n+1}) = -P((C_1\cap C_2\cap C_3\cdots C_n)\cup C_{n+1}) + P(C_1\cap C_2\cap\cdots C_n)+P(C_{n+1}) \quad -②$$

From ①:

$$P(C_1 \cap C_2 \cap \dots C_n) \geq P(C_1) + P(C_2) + \dots + P(C_n) - n + 1$$

$$P(C_1 \cap C_2 \cap \dots C_n) + P(C_{n+1}) \geq \sum_{i=1}^{n+1} P(C_i) - n + 1 \qquad - ③$$

Note: $P((C_1 \cap C_2 \cap C_3 \dots C_n) \cup C_{n+1}) \leq 1$

$-P((C_1 \cap C_2 \cap \dots C_n) \cup C_{n+1}) \geq -1$

Adding this to eqn 3 gives:

$$P(C_1 \cap C_2 \cap \dots C_n) + P(C_{n+1}) - P[(C_1 \cap C_2 \dots C_n) \cup C_{n+1}] \geq \sum_{i=1}^{n+1} P(C_i) - (n+1) + 1$$

From ②

$$\Rightarrow P\left(\bigcap_{i=1}^{n+1} C_i\right) \geq \sum_{i=1}^{n+1} P(C_i) - (n+1) + 1$$

Hence proved by induction //

$\boxed{\text{Ans 1 (c)}}$

To show:

$$P\left(\bigcap_{i=1}^{n} C_i^c\right) \leq \exp\left[-\sum_{i=1}^{n} P(C_i)\right]$$

We Assume the C_i's are all independent (o/w cannot solve)

$\cancel{LHS = P\left(\bigcap_{i=1}^{n} C_i^c\right) = \prod_{i=1}^{n} P(C_i^c)}$

$=$

P.T.O

$\Longrightarrow$

Since independent,

$$LHS = P\left(\bigcap_{i=1}^{m} C_i^c\right)$$

$$= \prod_{i=1}^{m} P(C_i^c)$$

$$= \prod_{i=1}^{m} (1 - P(C_i))$$

Now consider x such that

$$0 < x < 1$$

then, $1 - x \leq e^{-x}$

$\Rightarrow$

Set $x = P(C_i)$

$$\Rightarrow \quad 1 - P(C_i) \leq e^{-P(C_i)}$$

Taking products

$$\prod_{i=1}^{m} (1 - P(C_i)) \leq e^{-\left(\sum_{i=1}^{m} P(C_i)\right)}$$

$$\Rightarrow \quad P\left(\bigcap_{i=1}^{m} C_i^c\right) \leq e^{-\left(\sum_{i=1}^{m} P(C_i)\right)}$$

//

Hence proved //

$\boxed{\text{Ans 4 a}}$

$$P[X \geq c] \leq \frac{E(X)}{c}$$

Let $A = \{x : X \geq c\}$ and let $f(x)$ = pdf of X

Then:

$$E(X) = \int_{-\infty}^{\infty} x f(x)\,dx = \underbrace{\int_A x f(x)\,dx + \int_{A^c} x f(x)\,dx}_{\text{Since both terms are +ve}}$$

$$\Downarrow$$

$$E(X) \geq \int_A x f(x)\,dx$$

Now, if $x \in A$, then $X \geq c$. Hence we can replace x by c

$$\Rightarrow E(X) \geq c \int_A f(x)\,dx \qquad \left(\because \int_A f(x)\,dx = P(X \in A) = P[X \geq c] \right)$$

$$\Rightarrow E(X) \geq c \cdot P(X \geq c)$$

$$\Rightarrow P(X \geq c) \leq \frac{E(X)}{c} \quad //$$

Hence proved.

Equality holds in the following cases:

(i) $Pr(X = c) = 1$

(ii) $c = +\infty$

(iii) $Pr(X = 0) = 1$

* Yes, Markov's inequality gives a tight bound.

$\boxed{\text{Ans 4 b}}$

In part a substitute $X \to (X-\mu)^2$ & $c = k^2\sigma^2$

$$\Rightarrow P\left[(X-\mu)^2 \geq k^2\sigma^2\right] \leq \frac{E\left[(X-\mu)^2\right]}{k^2\sigma^2} = \frac{\sigma^2}{k^2\sigma^2}$$

$$\Rightarrow P\left[(X-\mu)^2 \geq k^2\sigma^2\right] \leq \frac{1}{k^2}$$

$$\Rightarrow P\left[|X-\mu| \geq k\sigma\right] \leq \frac{1}{k^2} \quad //$$

$\boxed{\text{Ans 4 c}}$

$E(X) = 3 \qquad E(X^2) = 13$

$$\begin{aligned} Var(X) &= E(X^2) - E(X)^2 \\ &= 13 - 9 = 4 \end{aligned}$$

$\sigma^2 = 4$

$\sigma = 2$

Put $k\sigma = \epsilon$, $\epsilon > 0$ in (b)'s ~~en~~ inequality.

We get:

$$P(|X-\mu| \geq \epsilon) \leq \frac{\sigma^2}{\epsilon^2} \quad \text{for } \forall\, \epsilon > 0$$

$$\cancel{P(|X-3| \geq 5) \leq \frac{4}{25}}$$

$$P(-2 < X < 8) = P(-2-\mu < X < 8-\mu) = P(-5 < X-\mu < 5) = P(|X-\mu| < 5)$$

$$= 1 - P(|X-\mu| \geq 5)$$

Now $P(|X-\mu| \geq 5) \leq \frac{4}{25} \quad (\epsilon = 5)$

$$-P(|X-\mu| \geq 5) \geq \frac{-4}{25}$$

$$1 - P(|X-\mu| \geq 5) \geq \frac{-4}{25} + 1 = \frac{21}{25}$$

$$P(-2 < X < 8) \geq \frac{21}{25} \quad /$$

$\boxed{\text{Ans 2 c}}$ $E(X)=4$ $E(Y)=-4$ $E(X^2)=E(Y^2)=20$ $cov(X,Y)=-0.5$

(i) $E(2X-Y) = 2E(X)-E(Y)$

$= 2(4)-(-4)$

$= 12$

(ii) $var(X+Y) = V(X)+V(Y)+2\,cov(X,Y)$

$= E(X^2)-[E(X)]^2 + E(Y^2)-[E(Y)]^2 + 2cov(X,Y)$

$= 20-16+20-16+2(-0.5)$

$= 8+(-1) = 7$

(iii) $E((X-Y)^2) = E(X^2+Y^2-2XY)$

$= E(X^2)+E(Y^2)-2E(XY)$

$= E(X^2)+E(Y^2)-2[cov(X,Y)+E(X)E(Y)]$

$= 20+20-2[-0.5+(-16)]$

$= 40+1+32$

$= 73$

$\boxed{\text{Ans 2 (a)}}$ $f(x,y) = \begin{cases} cx^2y & \text{when } 0<x<y<1 \\ 0 & \text{o/w} \end{cases}$

(i) $\int_Y\int_X f(x,y) = 1 \Rightarrow \int_0^1\int_0^y cx^2y\,dx\,dy = 1$

LHS $= \int_0^1 \frac{c\cancel{y}^3}{3}y\,dy = \frac{c}{3}\frac{y^5}{5}\Big|_0^1 = \frac{c}{15}$

$\Rightarrow \frac{c}{15}=1 \Rightarrow c=15$

(ii) $f(x) = \int_0^1 f(x,y)\,dy$

$= \int_0^1 cx^2y\,dy$

$= \left.\dfrac{cx^2y^2}{2}\right|_0^1 = \dfrac{cx^2}{2}$

$f(y) = \int_0^y f(x,y)\,dx$

$= \int_0^y cx^2y\,dx$

$= cy\left.\dfrac{x^3}{3}\right|_0^y = \dfrac{cy^4}{3}$

$f(x) = \dfrac{cx^2}{2}$ $\qquad$ $f(y) = \dfrac{cy^4}{3}$

(iii) $f(x).f(y) = \dfrac{c^2x^2y^4}{6} \neq f(x,y)$

Hence X & Y are not independent.

$f(x|y) = \dfrac{f(x,y)}{f(y)} = \dfrac{cx^2y}{cy^4/3} = 3\,x^2y^{-3} \neq f(x)$

$f(y|x) = \dfrac{f(x,y)}{f(x)} = \dfrac{cx^2y}{\frac{cx^2}{2}} = 2y \neq f(y)$

(iv) Support set:

$P((X+Y)<1) = \iint\limits_{x+y\leq 1} cx^2y\,dx\,dy$

$= \int_0^{1/2}\int_x^{1-x} cx^2y\,dy\,dx = \int_0^{1/2} cx^2 \left.\dfrac{y^2}{2}\right|_x^{1-x} dx = c\int_0^{1/2} x^2\left(\dfrac{(1-x)^2-x^2}{2}\right)dx$

$= c\int_0^{1/2} x^2\left(\dfrac{x^2+1-2x-x^2}{2}\right)dx = \dfrac{c}{2}\int_0^{1/2} x^2(2x-1)\,dx = \dfrac{c}{2}\left[\dfrac{2x^3}{3} - \dfrac{x^3}{3}\right]\Big|_0^{1/2}$

$= 15/192$ //

$\boxed{\text{Ans 2 b}}$

$$f(x) = \begin{cases} \sin(x) & \text{when } 0 < x < \frac{\pi}{2} \\ 0 & \text{o/w} \end{cases}$$

(i) $M_x(t) = E(e^{tX}) = \int_{-\infty}^{\infty} e^{tx} \sin x \, dx$

$$= \int_0^{\frac{\pi}{2}} \underbrace{e^{tx}}_{v} \underbrace{\sin x}_{u} \, dx$$

$$\int e^{tx} \sin x \, dx = \sin x \int e^{tx} dx - \int \underbrace{e^{tx}}_{v} \underbrace{\cos x}_{u} \, dx$$

$$= \sin x \frac{e^{tx}}{t} - \left[\cos x \frac{e^{tx}}{t} + \int e^{tx} \sin x \, dx \right]$$

$$2\int e^{tx} \sin x \, dx = \frac{e^{tx} \sin x}{t} - \frac{e^{tx} \cos x}{t}$$

$$\int e^{tx} \sin x \, dx = \frac{e^{tx} \sin x - e^{tx} \cos x}{2t} + C$$

$$\int_0^{\frac{\pi}{2}} e^{tx} \sin x \, dx = \left. \frac{e^{tx}(\sin x - \cos x)}{2t} \right|_0^{\frac{\pi}{2}}$$

$$= \frac{e^{\frac{\pi}{2}t}(1-0)}{2t} - \frac{e^{\frac{\pi}{2}t}(0-1)}{2t}$$

$$M_x(t) = \frac{e^{\frac{\pi}{2}t}}{t}$$

(ii) $E(x^3) = M_x^{(3)}(0)$

$$= \left. \frac{d^3}{dt^3} M_x(t) \right|_{\text{at } t=0}$$

$$d' = \left. \frac{t e^{\frac{\pi}{2}t} \frac{\pi}{2} - e^{\frac{\pi}{2}t}}{t^2} \right|_{\text{at } t=0} = \left. \frac{e^{\pi/2 t}\left[\frac{\pi}{2}t - 1\right]}{t^2} \right|_{\text{at } t=0}$$

$$d^2 = \frac{t^2\left[e^{\frac{\pi t}{2}}\frac{\pi}{2} + \left(\frac{\pi t}{2}-1\right)\frac{\pi}{2}e^{\frac{\pi}{2}t}\right] - e^{\frac{\pi}{2}t}\left(\frac{\pi t}{2}-1\right)2t}{t^4}$$

$$= \frac{e^{\frac{\pi t}{2}}\left[\frac{\pi t^2}{2} + \frac{\pi t^2}{2}\left(\frac{\pi t}{2}-1\right) - 2t\left(\frac{\pi}{2}t-1\right)\right]}{t^4}$$

$$= \frac{e^{\frac{\pi t}{2}}\left[\frac{\pi t}{2} + \frac{\pi t}{2}\left(\frac{\pi t}{2}-1\right) - 2\left(\frac{\pi t}{2}-1\right)\right]}{t^3}$$

$$= e^{\frac{\pi t}{2}}\left[\frac{\pi t}{2} + \frac{\pi t}{2}\left(\frac{\pi t-2}{2}\right) - \pi t + 2\right] / t^3$$

$$= \frac{e^{\frac{\pi t}{2}}\left[\frac{\pi^2 t^2}{4} - \pi t + 2\right]}{t^3}$$

$$= e^{\frac{\pi t}{2}}\left[\frac{\pi^2 t^2}{4} - \pi t + 2\right] / t^3$$

$$d^3 = \frac{1}{4t^6}\left[t^3\left[\frac{\pi}{2}e^{\frac{\pi t}{2}}(\pi^2t^2 - 4\pi t + 8) + e^{\frac{\pi t}{2}}(2\pi^2 t - 4\pi)\right] - e^{\frac{\pi t}{2}}(\pi^2t^2 - 4\pi t + 8)\,3t^2\right]$$

$$d^3\Big|_{t=0} = \cancel{A} \,//\ \text{too long !!! or something wrong}$$

Ans 8

$$f(\bar{x}) = \log \left| F_0 + x_1F_1 + x_2F_2 + \dots + x_nF_n \right|$$

$$\frac{\partial}{\partial X} \log|X| = (X^{-1})^T = (X^T)^{-1}$$

Put $P = F_0 + x_1F_1 + x_2F_2 + \dots + x_nF_n$

$$\nabla f_i = \frac{\partial f(\bar{x})}{\partial x_i} = \frac{\partial \log|P|}{\partial P} \cdot \frac{\partial P}{\partial x_i}$$

$$= (P^{-1})^T \cdot \frac{\partial P}{\partial x_i}$$

$$= \left((F_0 + x_1F_1 + x_2F_2 + \dots x_nF_n)^{-1}\right)^T \cdot \frac{\partial P}{\partial x_i}$$

$$= \left((F_0 + x_1F_1 + x_2F_2 + \dots + x_nF_n)^{-1}\right)^T \cdot F_i$$

$$\Rightarrow \nabla f = \left((F_0 + x_1F_1 + x_2F_2 + \dots + x_nF_n)^{-1}\right)^T \begin{bmatrix} F_1 & F_2 & F_3 & \dots & F_n \end{bmatrix}$$

Concatenated matrix of all F_i's.

Ans 9 (a)

$\bar{X}$ is $N_3(\mu, \Sigma)$ with

$$\Sigma = \begin{bmatrix} 4 & 1 & 0 \\ 1 & 3 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

No, X_1, X_2 are not independent as is clear from Σ.
$cov(X_1, X_2) \neq 0$

$$f_{X_1,X_2}(x_1,x_2) = N\left(\begin{bmatrix}\mu_1\\ \mu_2\end{bmatrix}, \begin{bmatrix}\Sigma_{11} & \Sigma_{12}\\ \Sigma_{21} & \Sigma_{22}\end{bmatrix}\right)$$

($\because$ Distribution is Normal : Given)

$$f_{X_3} \sim N(\mu_3, \Sigma_{33})$$

From Σ it is clear that $cov(X_1,X_3)=0$ & $cov(X_2,X_3)=0$

$\Rightarrow (X_1,X_2)$ is independent of X_3.

$\boxed{\text{Ans 9 b}}$

Set $P = X_1 + QX_2$ ~~st~~

$Q = -\Sigma_{12}\Sigma_{22}^{-1}$

$$cov(P, X_2) = cov(X_1,X_2) + cov(QX_2, X_2)$$

$$= \Sigma_{12} + Q\,var(X_2)$$

$$= \Sigma_{12} - \Sigma_{12}\Sigma_{22}^{-1}\Sigma_{22}$$

$$= 0$$

$\Rightarrow P$ & X_2 are not-correlated

$\Rightarrow$ They are independent (Because their joint distribution is Normal) — ①

$\Rightarrow E(P) = \mu_1 + Q\mu_2$

$$E(X_1|X_2) = E(P - QX_2|X_2) = E(P|X_2) - E(QX_2|X_2)$$

$$= E(P) - QX_2 \quad \text{— from ①}$$

$$= \mu_1 + Q(\mu_2 - X_2)$$

$$= \mu_1 + \Sigma_{12}\Sigma_{22}^{-1}(X_2 - \mu_2) \,//$$

P.T.O

$\Rightarrow$

$$var(x_1|x_2) = var(p)$$
$$= var(x_1 + Qx_2)$$
$$= var(x_1) + Q(var x_2)Q^T + Q\,cov(x_1, x_2) + cov(x_1, x_2)Q^T$$
$$= \Sigma_{11} + \Sigma_{12}\Sigma_{22}^{-1}\Sigma_{22}\Sigma_{22}^{-1}\Sigma_{21} - 2\Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21}$$
$$= \Sigma_{11} - \Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21} \quad //$$

Hence proved //

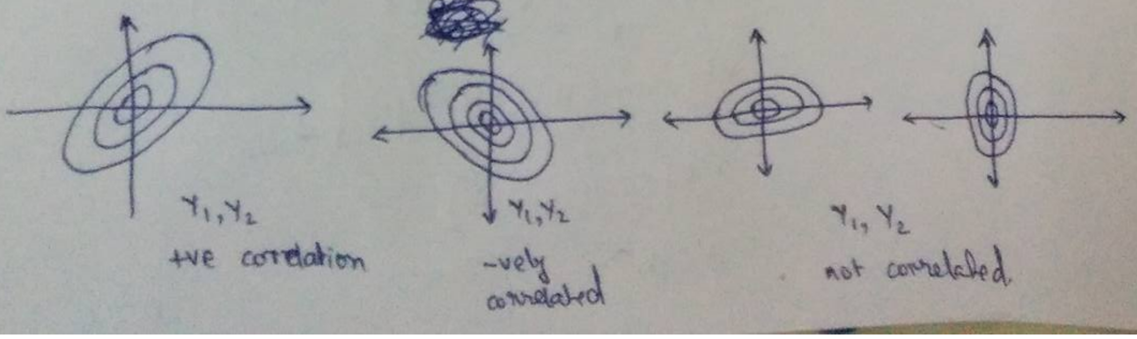
Ans 9 (c)

$$f((y_1, y_2))$$
$$= \frac{1}{(2\pi)^{t/2}} |\Sigma|^{-1/2} \exp\left(\frac{-1}{2}(y-\mu)^T \Sigma^{-1}(y-\mu)\right)$$
$$= \frac{1}{2\pi\, \sigma_{11}\sigma_{22}\sqrt{1-\rho^2}} \exp\left(\frac{-t}{2(1-\rho^2)}\right)$$

Here: $t = \frac{(y_1-\mu_1)^2}{\sigma_{11}^2} - \frac{2\rho(y_1-\mu_1)(y_2-\mu_2)}{\sigma_{11}\sigma_{22}} + \frac{(y_2-\mu_2)^2}{\sigma_{22}^2}$

Correlation b/w y_1 & y_2 = $\rho_{12} = \frac{\sigma_{12}}{\sigma_{11}\sigma_{22}}$

$$\mu = \begin{bmatrix}\mu_1\\ \mu_2\end{bmatrix} \qquad \Sigma = \begin{bmatrix}\sigma_{11} & \sigma_{12}\\ \sigma_{21} & \sigma_{22}\end{bmatrix}$$



Ans 10 (a)

Suppose u, v are two eigen vectors corresponding to distinct eigen values. λ, μ $\Rightarrow Au = \lambda u$

~~Av = μv~~ $Av = \mu v$

$$\lambda \langle u,v \rangle = \langle \lambda u, v \rangle = \langle Au, v \rangle$$

$$= \langle u, A^T v \rangle$$

$$= \langle u, Av \rangle \quad (\because A \text{ is symmetric})$$

$$= \langle u, \mu v \rangle$$

$$= \mu \langle u,v \rangle$$

$$\Rightarrow (\lambda - \mu) \langle u,v \rangle = 0$$

But since $\lambda \neq \mu$ (distinct eigen values)

$$\Rightarrow \langle u,v \rangle = 0$$

$\Rightarrow u,v$ are ~~orthogonal~~ orthonormal (or can be made orthonormal)

But we also know that A is symmetric $\Rightarrow$ Diagonalisable

$\Rightarrow$ We have eigenbases of eigenvectors

$\Rightarrow$ Eigenvectors form orthonormal basis.

Ans 10 (b)

$$\operatorname{cov}(a'x, b'x) = E\left[(a'x - E(a'x))(b'x - E(b'x))'\right]$$

$$= E\left[(a'x - a'\mu)(b'x - b'\mu)'\right]$$

$$= a' E\left[(x-\mu)(x-\mu)'\right] b$$

$$= a' \Sigma b$$

P.T.O

Thus we have

$$Cov(x_a, x_b) = \underline{a}' A \underline{b}$$

$\boxed{\text{Ans 10 (c)}}$ Suppose the values are $D = \{\bar{x}_1, \bar{x}_2, \bar{x}_3, \ldots \bar{x}_N\}$
Let $y = \langle \bar{x}, \bar{w} \rangle$. We need to find $\bar{w}$ such that $var(y)$ is maximised for D. $y_n = \langle x_n, \bar{w} \rangle$ ← we optimise using this

$$\bar{w}^* = \underset{\bar{w}}{\arg\max}\ var(y) = \underset{\bar{w}}{\arg\max}\ E_D(y^2) = \underset{\bar{w}}{\arg\max}\ \frac{1}{N} \sum_n y_n^2$$

(Assuming D as 0 mean data)

$$\frac{1}{N} \sum_n y_n^2 = \frac{1}{N} \sum_n (\langle x_n, \bar{w} \rangle)^2 = \frac{1}{N} \sum_n \bar{w}^T x_n x_n^T \bar{w}$$

$$= \bar{w}^T \left(\frac{1}{N} \sum_n x_n x_n^T \right) \bar{w} = \bar{w}^T K \bar{w}$$

where $K = \frac{1}{N} \sum_n \langle x_n, x_n \rangle$ is the cov matrix

Here we consider only $\|w\|^2 = 1 = \bar{w}^T \bar{w}$

Using a Lagrange multiplier λ we enforce this constraint:

$$\bar{w}^* = \underset{\bar{w}}{\arg\max}\ \bar{w}^T K \bar{w} - \lambda(\bar{w}^T \bar{w} - 1)$$

By solving using minimum reconstruction error problem:

we get $\bar{w}^* = \underset{\bar{w}}{\arg\max}\ \bar{w}^T K \bar{w} - \lambda[\bar{w}^T \bar{w} - 1]$

We get k orthogonal directions $\bar{w}_1, \bar{w}_2, \ldots \bar{w}_k$

$\boxed{\text{Ans 3}}\ \boxed{a)}$ $X_1, X_2, X_3 \ldots X_n \sim U(0,1)$

$X_1, X_2, \ldots X_n$ are all independent

a) To find $E[\max(X_1, X_2, \ldots X_n)]$

Let $y = \max(X_1, X_2, \ldots X_n)$

$$P(y \le t) = \prod_{i=1}^{n} P(X_i \le t) \quad \text{for some } t$$

$$E(y) = \underbrace{\int_0^1\int\int \cdots \int_0^1}_{n \text{ times}} \max(x_1, x_2, \ldots x_n)\, dx_1\, dx_2 \ldots dx_n$$

$$P(y \le t) = P(X_1 \le t)^n$$

$$= \left(\frac{t}{\theta}\right)^n \implies f_y = \frac{d}{dt} P(y \le t)$$

$$= \frac{n t^{n-1}}{\theta^n}$$

$$\implies E(y) = \int_0^\theta t\, n \frac{t^{n-1}}{\theta^n}\, dt$$

$$= \int_0^\theta \frac{n}{n+1} \frac{\theta^{n+1}}{\theta^n}\, d\theta = \frac{n}{n+1}\theta$$

Given $\theta = 1$

Hence $E(y) = \dfrac{n}{n+1}$ //

Ans. 3(b)

$$y = \min(x_1, x_2, \dots x_n) = 1 - \max(x_1, x_2 \dots x_n)$$

as $x \in (0,1)$

$$E(y) = 1 - E(\max(x_1, x_2, \dots x_n))$$

$$= 1 - \frac{n}{n+1}$$

$$= \frac{1}{n+1}$$

Ans 3(c)

~~Var~~ $g = \max(x_1, x_2, \dots x_n)$

$$E[g^2] = \int_0^\theta \frac{t^2\, n\, t^{n-1}}{\theta^n} dt =$$

$$= n \int_0^\theta \frac{t^{n+1}}{\theta^n} d\theta$$

$$= \frac{n\theta^2}{n+2}$$

$$V(g) = E(g^2) - E(g)^2$$

$$= \frac{n\theta^2}{n+2} - \left(\frac{n\theta}{n+1}\right)^2 \quad \text{Set } \theta = 1$$

$$= \frac{n}{n+2} - \frac{n^2}{(n+1)^2} = \frac{n(n+1)^2 - n^2(n+2)}{(n+2)(n+1)^2}$$

$$= \frac{n(n^2+2n+1) - n^2(n+2)}{(n+2)(n+1)^2}$$

$$= \frac{n^3 + 2n^2 + n - n^3 - 2n^2}{(n+2)(n+1)^2} = \frac{n}{(n+2)(n+1)^2}$$

$\boxed{\text{Ans 3(d)}}$

$$h = \min(X_1, X_2, \ldots X_n)$$

$$P(h \le t) = 1 - P(\max(X_1, X_2 \ldots X_n) \ge t)$$

$$= 1 - \left(1 - \frac{t}{\theta}\right)^n$$

From above : $f_h = -n \left(1 - \frac{t}{\theta}\right)^{n-1} \left(\frac{-1}{\theta}\right)$

$$= \frac{n}{\theta}\left(1 - \frac{t}{\theta}\right)^{n-1} = n(1-t)^{n-1}$$

Now we consider : $E(h^2) = \int_0^1 t^2 n (1-t)^{n-1}\, dt = n \int_0^1 t^2 (1-t)^{n-1}\, dt$

$$= n \frac{\Gamma(3)\Gamma(n)}{\Gamma(n+3)}$$

$$= \frac{n\, 3!\; n!}{n!\,(n+1)(n+2)(n+3)}$$

$$= \frac{6n}{(n+1)(n+2)(n+3)}$$

$$V(h) = E(h^2) - E(h)^2$$

$$= \frac{6n}{(n+3)(n+2)(n+1)} - \frac{1}{(n+1)^2}$$

$$= \frac{6n(n+1) - (n+2)(n+3)}{(n+3)(n+2)(n+1)^2} = \frac{6n^2 + 6n - [n^2 + 6 + 5n]}{(n+3)(n+2)(n+1)^2}$$

$$= \frac{5n^2 + 1n - 6}{(n+3)(n+2)(n+1)^2} = \cancel{\frac{5n^2 + 5n - 6n - 6}{(n+3)(n+2)(n+1)^2}} = \cancel{\frac{5n(n+1) + 6(n+1)}{(n+3)(n+2)(n+1)^2}}$$

$$\cancel{= \frac{5n+6}{}} = \frac{5n^2 - 5n + 6n - 6}{(n+3)(n+2)(n+1)^2} = \frac{5n(n-1) + 6(n-1)}{(n+3)(n+2)(n+1)^2}$$

$$= \frac{(5n+6)(n-1)}{(n+3)(n+2)(n+1)^2}$$