Navigation of a Quadratic Potential with Ellipsoidal Obstacles

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***Abstract*— Successful navigation of a convex quadratic potential in a space with ellipsoidal obstacles can be attained with Rimon-Koditschek artificial potentials in spaces where the ellipsoids are not too eccentric (flat). This paper proposes a modification to gradient dynamics that allows successful navigation of an environment with a quadratic cost and ellipsoidal obstacles regardless of their eccentricity. This is accomplished by altering gradient dynamics with the addition of a second order curvature correction that is intended to imitate worlds with spherical obstacles in which Rimon-Koditschek potentials are known to work. Convergence to the goal is proven for all environments with a single obstacle. In worlds with multiple obstacles convergence is guaranteed in cases when the obstacles are not tightly packed around the agent's target. Results are numerically verified with a discretized version of the proposed flow dynamics.**

I. Introduction

Motion planning considers the problem of defining a path from the robots current location to its goal such that it is able to avoid obstacles that might be present in the environment [1]. Motion control and trajectory planning are applied to solve a variety of problems including surveillance [2], [3] and search and rescue [4], [5]. A specific solution for this problem is given by artificial potentials. These combine an attractive potential whose minimum is the agent's goal with repulsive potentials at the boundaries of the obstacles. When these potentials are combined in such a way that they do not posit local minima other than the desired destination, they are termed navigation functions. The defining properties of the navigation functions ensure that a gradient field of its negative gradient repels a point robot away from the obstacles and attracts the point robot towards its objective from almost every initial condition [6]. Using navigation functions, the agent can follow a smooth trajectory to converge asymptotically to its target in complex worlds [7]. Navigation functions are simple yet effective. They have been adapted so that they work in finite time [8], [9] and extended to solve problems with multiple agents [10] and non stationary obstacles [11].

An important advantage of navigation functions is that they can operate with local information as long as the gradients of the navigation function can be measured locally. Without going into technical details, this is possible if we can measure the distances to nearby obstacles and locally estimate their curvature, as well as measure the gradient of the natural potential we intend to minimize [12]. A drawback of navigation functions is that they only work under some restrictions on the geometry of the environment. Notably, they always work when the potential and obstacles are spherical [13]–[16]. This motivates the use of diffeomorphism to spherical worlds [17], [18] but these require knowledge of the entire environment. Of more practical significance is the fitting of spheres around the obstacles to reduce the problem to the spherical environment case [19].

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To expand on this thread, we consider the case where the point agent has local information about ellipsoidal obstacles (Sec. II). Ellipsoidal worlds are practical and important to consider. They are practical because of results that allow curvature estimation through ellipsoidal fitting [20]. They are important because an ellipsoid fit is, in general, tighter than a sphere fit. Having tighter fits reduces the likelihood of the obstacle fits intersecting each other; a situation that would violate a crucial assumption for navigation functions to work [6]. An important limitation of using ellipsoidal fits is that navigation in ellipsoidal words does not work on all geometric configurations. For navigation functions to work, obstacles cannot be too wide in comparison to the level sets of the objective function [12], [21]. This condition becomes limiting as the number of obstacles increases (see Figure 4).

Our main contribution is to correct gradient dynamics with second order information so that navigation to the goal is possible regardless of the curvature of the obstacles. Our correction is reminiscent of Newton's method [22, Ch. 9] using the Hessian as a local change of coordinates that renders the world spherical (Sec. III). However, the method is not equivalent to a change of coordinates since different curvature corrections are applied to the objective and the obstacle. The resulting controller ceases to be the gradient of a navigation function but it is one that we can nonetheless show converges to the goal regardless of the eccentricity of the obstacles and the natural potential (Sec. III-A). This holds for a space with a single obstacle that we consider first to illustrate ideas. In the case of a configuration with multiple obstacles (Sec. IV) we can propose a dynamical system that uses switching to adapt the correction to the curvature of the nearest obstacle (Sec. IV-A). This method can also be proven to converge irrespectively of the eccentricity of the obstacles but it may fail if a sufficient condition on their spatial arrangement does not hold (Sec. IV-B). The condition is technical and unlikely to be violated in practice. For this to happen obstacles have to be arranged in a tight configuration that encircles the goal. Numerical results showcase success in spaces where navigation functions fail (Sec. V)

II. Potential, Obstacles, and Navigation

We consider the problem of a point agent navigating a quadratic potential in a space with ellipsoidal punctures. Formally, let $\mathcal{X} \subset \mathbb{R}^n$ be a non empty compact convex set that we call the workspace and let $f_0 : \mathcal{X} \to \mathbb{R}_+$ be a convex strictly quadratic function that we call the potential. A point agent is interested in reaching the target destination $x^* \in \mathcal{X}$ which is defined as the minimizer of the potential. Thus, for some positive definite matrix Q we can write

$$x^* = \operatorname*{argmin}_{x\in\mathcal{X}} f_0(x) = \operatorname*{argmin}_{x\in\mathcal{X}} (x-x^*)Q(x-x^*). \qquad (1)$$

In most navigation problems the potential is just the Euclidean distance to the goal [16], [23] but arbitrary quadratic functions are of interest in some situations [21]. For future reference denote the minimum and maximum eigenvalues of Q as $0 < \lambda_{\text{min}} \leq \lambda_{\text{max}}$.

The workspace $\mathcal{X}$ is populated by m ellipsoidal obstacles $\mathcal{O}_i \subset \mathcal{X}$ for $i = 1, \ldots, m$ which are closed and have a non empty interior. Each of these obstacles is represented as the sublevel set of a proper convex quadratic function $\beta_i : \mathbb{R}^n \to \mathbb{R}$. We introduce the obstacle shape matrices A_i and denote their minimum and maximum eigenvalues as $0 < \mu^i_{\text{min}} \leq \mu^i_{\text{max}}$. Further, we introduce ellipsoid centers x^i and ellipsoid radiuses r_i to define

$$\beta_i(x) = \frac{1}{2}(x - x^i)^\top A_i (x - x^i) - \frac{1}{2} r_i^2. \tag{2}$$

The obstacle $\mathcal{O}_i$ is now defined as the zero sublevel set of the quadratic function $\beta_i(x)$,

$$\mathcal{O}_i = \left\{ x \in \mathcal{X} \,\middle|\, \beta_i(x) \leq 0 \right\}. \tag{3}$$

From (2) and (3) it follows that $\mathcal{O}_i$ is an ellipsoid centered at x^i with axes given by the eigenvectors of A_i. The length of the axis along the kth eigenvector is $\mu_i^k r_i$. In particular, the length of the minor axis is $r_i \mu^i_{\text{min}}$ and the length of the major axis is $r_i \mu^i_{\text{max}}$.

We further introduce a concave quadratic function $\beta_0 : \mathbb{R}^n \to \mathbb{R}$ so that we write the workspace as a superlevel set,

$$\mathcal{X} = \left\{ x \in \mathbb{R}^n \,\middle|\, \beta_0(x) \geq 0 \right\}. \tag{4}$$

The navigation problem we want to solve is one in which the agent stays in the interior of the workspace at all times, does not collide with any obstacle and approaches the goal at least asymptotically. For a formal specification we define the free space as the complement of the obstacle set relative to the workspace,

$$\mathcal{F} := \mathcal{X} \setminus \Big(\bigcup_{i=1}^{m} \mathcal{O}_i \Big), \tag{5}$$

so that we can specify the agent's goal as that of finding a trajectory $x(t)$ such that

$$x(t) \in \mathcal{F} \text{ for all } t \geq 0, \text{ and } \lim_{t \to \infty} x(t) = x^*. \tag{6}$$

A common approach to solve (6) is to construct a navigation function [6]. We explain this in the following section after introducing two assumptions that will be used in forthcoming sections.

Assumption 1 Target and initial point in free space. *The target x^* defined in* (1) *lies in the free space defined in* (4)*, $x^* \in \mathcal{F}$. Furthermore, the initial position at time $t = 0$ is bounded away from the obstacles. That is, there exists $\varepsilon > 0$ such that*

$$\min_{x \in \cup_{i=1}^m \mathcal{O}_i} \left\| x(0) - x \right\| \geq \varepsilon. \tag{7}$$

Assumption 2 Obstacles do not intersect. *The intersection of any two obstacles is empty, namely, $\mathcal{O}_i \cap \mathcal{O}_j = \emptyset$ for all $i \neq j$. Furthermore, there exists strictly positive $\rho > 0$ such that for all i and j, $\beta_i(x) \leq \rho \implies \beta_j(x) \geq \rho$ where $\beta_i(x)$ and $\beta_j(x)$ are the ellipsoidal functions defining obstacles $\mathcal{O}_i$ and $\mathcal{O}_j$ [cf.* (2) *and* (3)*].*

Both of these assumptions are minimal restrictions. Assumption 1 requires the target and the initial goal to be in free space. We further bound how close the initial condition can be to the border of an obstacle to preclude arbitrarily difficult starting conditions. Assumption 2 states that obstacles are disjunct. This is needed because if obstacles intersect they effectively become a single non-convex obstacle. Similar to the case of the initial condition we bound the separation between obstacles to preclude arbitrarily difficult environments. Both are usual assumptions for this problem [14], [17], [21].

A. Navigation Functions

A navigation function is a twice continuously differentiable function defined on free space that satisfies three properties: (i) It has a unique minimum which belongs to the interior of free space. (ii) All of the its critical points in free space are nondegenerate. (iii) Its maximum is attained at the boundary of free space. These three properties guarantee that if an agent follows the negative gradient of the navigation function it will converge to the minimum of the navigation function without running into the boundary of free space for almost every initial condition [6]. Thus, it is possible to recast (6) as the problem of finding a navigation function whose minimum is at the goal destination x^*. A family of navigation functions that can be constructed to navigate a convex potential such as f_0 in a space of nonintersecting convex obstacles such as the $\mathcal{O}_i$ we consider here is the Rimon-Koditschek potential [14], [17], [21].

To explain the construction of these potentials we use the definition of the obstacles and the workspace provided in (3) and (4) we can write an analytical expression for free space. To that end, define the function $\beta : \mathbb{R}^n \to \mathbb{R}$ to be the product of all the obstacle equations,

$$\beta(x) = \prod_{i=1}^{m} \beta_i(x). \tag{8}$$

By Assumption 2, only the function β_i can be negative inside of obstacle $\mathcal{O}_i$. It follows that $\beta(x)$ is negative if and only if the argument x is inside of some obstacle. We can therefore define the free space as the set of points where the product of $\beta(x)$ and $\beta_0(x)$ is positive,

$$\mathcal{F} = \left\{ x \in \mathbb{R}^n \,\middle|\, \beta_0(x)\beta(x) > 0 \right\}. \tag{9}$$

We can now think of the product $\beta_0(x)\beta(x)$ as a barrier function that we must guarantee stays positive during navigation. Then Rimon-Koditschek potential does so by introducing a parameter $k > 0$ and defining the function $\varphi_k : \mathcal{F} \to \mathbb{R}_+$ as

$$\varphi_k(x) = \frac{f_0(x)}{\Big({f_0}^k(x) + \beta_0(x)\beta(x) \Big)^{1/k}}. \tag{10}$$

The intuition supporting this definition is that k controls the importance of the barrier $\beta_0(x)\beta(x)$ relative to the potential ${f_0}^k(x)$, thereby repelling the agent and preventing it from crossing into the obstacle space – see also (12). Irrespectively of intuition, it is known that $\varphi_k(x)$ is a navigation function when k is sufficiently large under some restrictions on the shape of the obstacles, the potential function, and position of the goal [14], [21].

The operative phrase in the previous sentence is "under some restrictions." The potential $\varphi_k(x)$ in (10) is not always a valid navigation function because for some geometries it can have several local minima as critical points. For the case of a quadratic potential and ellipsoidal obstacles that we consider here $\varphi_k(x)$ is known to be a valid potential when [21, Theorem 3]

$$\frac{\lambda_{\max}}{\lambda_{\min}} \times \frac{\mu^i_{\max}}{\mu^i_{\min}} < 1 + \frac{d_i}{r_i \mu^i_{\max}}, \tag{11}$$

where $d_i = \|x_i - x^*\|$ is the distance from the center of the ellipsoid to the goal and, we recall, $0 < \lambda_{\min} \leq \lambda_{\max}$ are potential eigenvalues, $0 < \mu^i_{\min} \leq \mu^i_{\max}$ are obstacle eigenvalues, and r_i is the ellipsoid radius. When (11) may fail, φ_k fails to be a navigation function because it presents a local minimum on the side of the obstacle opposite the target with a non negligible region of attraction. To satisfy the inequality we need the curvature of the obstacles to be thin relative to the curvature of the potential. This is what happens on the example in the left in Figure 1 where the level sets of the potential and the obstacle are circular. The inequality is violated when the curvature of the obstacle is too wide with respect to the objective function. This is what happens on the example in the right of Figure 1 where the ellipsoidal obstacle is wide and the level sets of the objective are narrow. This geometry generates a local minima in $\varphi_k(x)$ irrespectively of the value of k.

The important consequence that follows from (11) is that the navigation function $\varphi_k(x)$ in (10) may fail to solve the navigation problem specified in (6). Indeed, it will fail whenever the obstacles are wide with respect to the potential level sets. The main contribution of this paper is to introduce an alternative dynamical systems that will solve (6) in *all* environments with quadratic potentials and ellipsoidal obstacles.

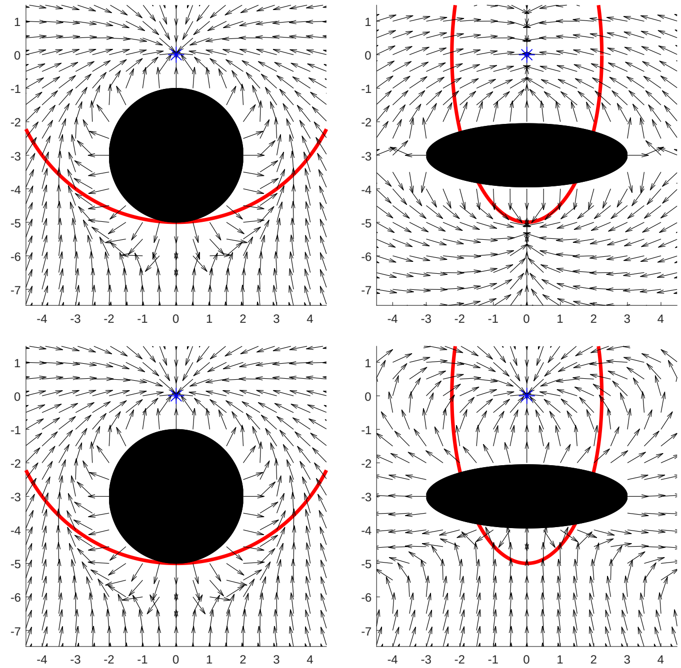


Fig. 1. Navigation Function Flows. The left plots show a sphere world that can be navigated with a navigation function $\varphi_k(x)$ of the form specified in (10). The obstacle adds a critical point that can be made to be a saddle by choosing sufficiently large k (left top). The right plots show a space that can't be navigated with a function of this form. No matter how large k we always have an attractive minima (right top). This happens because the obstacle is too wide relative to the level sets of the potential function. The bottom two figures show the proposed corrected flow which eliminates the local minima. A level set of the objective function is shown in red.

III. Curvature Corrected Navigation Flows

The idea of the navigation potential in (10) is that its negative gradient defines navigation dynamics. It is instructive to write the dynamical system $\dot{x} = -\nabla\varphi_k(x)$ by explicitly evaluating the navigation function gradient.

$$\dot{x} \;=\; -\nabla\varphi_k(x) \;=\; -\beta(x)\nabla f_0(x) + \frac{f_0(x)}{k}\nabla\beta(x). \tag{12}$$

Observe that in (12) we omit $\beta_0(x)$ for simplicity. This is a minor modification as we explain in Remark 1. The first term, $-\beta(x)\nabla f_0(x)$, in this dynamical system is an attractive potential to the goal and the second term, $(f_0(x)/k)\nabla\beta(x)$, is a repulsive field pushing the agent away from the obstacles. When the agent is close to the obstacle $\mathcal{O}_i$, the product function $\beta(x)$ takes a value close to zero thereby eliminating the first summand in (12) and prompting the agent's velocity to be almost collinear with the vector $\nabla\beta(x)$. In turn, this makes the time derivative $\dot{\beta(x)}$ positive thus preventing $\beta(x)$ from becoming negative. This guarantees that the agent remains in free space [cf., (9)]. When the agent is away from the obstacles, the term that dominates is the negative gradient of $f_0(x)$ which pushes the agent towards the goal x^*. The parameter k balances the relative strengths of these two potentials.

At points where the attractive and repulsive potentials cancel we find critical points. These points can be made saddles when the condition in (11) holds. An important observation here is that the condition is always satisfied when the potential and the obstacles are spherical because in that case the left hand side is $(\lambda_{\max}/\lambda_{\min}) \times (\mu^i_{\max}/\mu^i_{\min}) = 1$. This motivates an approach in which we implement a change of coordinates to render the geometry spherical. The challenge is that the change of coordinates that would render obstacles spherical is not the same change of coordinates that would render the potential spherical. Still, this is a good idea that motivates the curvature corrected dynamics that we present in this section.

To simplify presentation consider first the case of a single obstacle and postpone the general case to Section IV. In this case the function in (8) reduces to $\beta(x) = \beta_1(x)$. To correct the navigation function flow, the idea is to pre-multiply each gradient in the original flow (12) by the Hessian inverse of the corresponding function. This yields the dynamical system

$$\dot{x} = -\beta(x)\nabla^2 f_0(x)^{-1}\nabla f_0(x) + \frac{f_0(x)}{k}\nabla^2\beta(x)^{-1}\nabla\beta(x). \tag{13}$$

If we particularize (12) to the case of a single obstacle, we see that (13) differs from the resulting dynamics in that $\nabla f_0(x)$ in the first term is premultiplied by the Hessian inverse $\nabla^2 f_0(x)^{-1}$ and in that $\nabla\beta(x)$ is premultiplied by the Hessian inverse $\nabla^2\beta(x)^{-1}$. The motivation for this modification is that when the objective and obstacles are spherical all critical points of the navigation function other than the goal are saddles. The Hessian inverse premultiplications in (13) are intended to emulate this situation. To explain this better observe that since we are assuming the objective to be a quadratic function the Hessian inverse gradient product in (13) is simply given by the position of the agent relative to the goal

$$\nabla^2 f_0(x)^{-1}\nabla f_0(x) = x - x^*. \tag{14}$$

Likewise, recalling our definition of the ellipsoids in (2), the corresponding Hessian inverse gradient product is the position of the agent relative to the center of the ellipsoid

$$\nabla^2\beta(x)^{-1}\nabla\beta(x) = x - x^1. \tag{15}$$

Substituting (14) and (15) into our proposed dynamical system in (13) yields

$$\dot{x} = -\beta(x)(x - x^*) + \frac{f_0(x)}{k}(x - x^1). \tag{16}$$

These dynamics are equivalent to (12) if (12) is particularized to an environment with an spherical objective function and an spherical obstacle. Since such a spherical environment satisfies the geometric condition in (11), it is reasonable to expect that a point agent following (13), which reduces to (16), converges to the goal x^* in all environments. A convergence analysis is available in the following section.

A. Convergence Analysis

Before we get to our first result, we present a lemma that will be used in the argument. Let us define for any $\delta > 0$ the following neighborhood of the target

$$\mathcal{B}_{x^*}(\delta) := \left\{ x \in \mathcal{F} | (x - x^*)^\top \nabla^2\beta(x)(x - x^*) \leq \delta \right\}. \tag{17}$$

From Assumption 1, we know that x^* lies in the interior of the free space. Therefore, there exists a $\delta' > 0$ such that $\mathcal{B}_{x^*}(\delta')$ does not intersect with the obstacle or the boundary. We define the following neighborhood of the obstacle $\mathcal{B}_k$

$$\mathcal{B}_k := \left\{ x \in \mathcal{B}^c_{x^*}(\delta') | \beta(x) < \frac{f_0(x)(x - x^*)^\top \nabla\beta(x)}{k\|\nabla^2\beta(x)^{\frac{1}{2}}(x - x^*)\|^2} \right\}. \tag{18}$$

Since the free space is compact and both f_0 and β are continuously differentiable, it follows that the numerator on the right hand since of (18) is bounded. Likewise, since for any $x \in \mathcal{B}^c_{x^*}(\delta')$ we have that $(x - x^*)^\top \nabla^2\beta(x)(x - x^*) > \delta'$ it follows that by increasing k, the area of $\mathcal{B}_k \cap \mathcal{B}^c_{x^*}(\delta')$ decreases. Therefore, for every point $x_0 \in \mathcal{B}^c_{x^*}(\delta')$, there exists a $K^*(x_0)$ such that for all $k > K^*(x_0)$, $x_0 \notin \mathcal{B}_k$. The set $\mathcal{B}_k$ can be thought of as the repulsion zone of the obstacle. By increasing k, we decrease the area of the repulsion zone. If the agent starts outside the repulsion zone, it will not enter the zone. We formalize this with the following lemma.

Lemma 1 *Let k be large enough such that the arbitrary starting point x_0 is not in the set $\mathcal{B}_k$. Then for all $t \in [0, \infty)$, $x(t) \notin \mathcal{B}_k$.*

Proof: See Appendix-A. ■

Lemma 1 confirms that by following the flow prescribed in (13), the agent will never collide with the obstacle. We are now in condition to present the first result of this work, in which we claim that the dynamics (13) converge to x^* for large enough k.

Theorem 1 *Let $f_0(x)$ satisfy Assumption 2 and let $\beta(x)$ be an ellipsoid as in (2). Further let x be the solution of the dynamical system (13) with initial condition x_0. Then, for every point $x_0 \in \mathcal{F}$, there exists a $K^*(x_0)$ such that when $k > K^*(x_0)$, $x(t) \in \mathcal{F}$ for all $t \geq 0$ and $\lim_{t\to\infty} x(t) = x^*$.*

Proof: See Appendix-B. ■

To prove Theorem 1, we show that x^* is asymptotically stable on $\mathcal{F}\backslash\mathcal{B}_k$ in the sense of Lyapunov, where we define the Lyapunov function as

$$V(x) = \frac{1}{2}(x - x^*)^\top \nabla^2\beta(x)(x - x^*). \tag{19}$$

Theorem 1 states that for any starting point $x \in \mathcal{F}$, there exists a k large enough such that the dynamics will cause the point agent to asymptotically converge to the target. As the starting point of the agent approaches the obstacle, the k required approaches infinity. By the latter part of Assumption 1, Theorem 1 holds for a fininte K. This result is different than in the navigation function literature since in those cases the navigation is achieved for a given value of k for all initial positions.

The Lyapunov function (45) evaluated at some point $x \in \mathcal{F}$ is proportional to the volume of an ellipsoid centered at the agent's target with the same eccentricity and orientation of the obstacle where x lies on its border. This result holds for any bounded free space, and therefore we have shown that even when the obstacle is almost flat, the dynamics will allow the agent to maneuver around the obstacle. A limitation of the previous result is that it holds for one obstacle only. As such, in the next sections we generalize the result to multiple obstacles. Before doing so, we present a pertinent remark regarding the fact that the function β_0 is not included in the new dynamics (13).

Remark 1 Since the Lyapunov function is proportional to the volume of an ellipsoid centered at the target with the same eccentricity and orientation of the obstacle, the agent will always be advancing toward the objective where the norm is defined by the Hessian of the obstacle. For large enough k this means that the gradient of the agent at any point on the boundary of the free space will be directed toward the target which is inside the boundary by assumption. This makes it possible to omit β_0 in our flow equation.

IV. EXTENSION TO MULTIPLE OBSTACLES

In the previous section, we showed that an agent following the dynamics (12) will converge asymptotically to its target for a world with one arbitrarily flat ellipsoidal obstacle (Theorem 1). In this section we extend the previous result to the case where the workspace is populated with a finite number of non-intersecting obstacles. To do so, we propose to modify the correction term of the gradients of the obstacle functions β_i in (13). Let us start by observing that the gradient of the obstacle function $\nabla\beta$ becomes a linear combination of the gradients of the m obstacles, where the weights are given by the distances to the other obstacles. To see why this is the case, use the product rule of derivation to write

$$\nabla\beta(x) = \sum_{i=1}^{m} \big(\prod_{j\neq i} \beta_j(x)\big)\nabla\beta_i(x). \tag{20}$$

For ease of notation, we define the omitted product function $\bar{\beta}_i(x) = \prod_{j\neq i}\beta_j(x)$, so (20) becomes

$$\nabla\beta(x) = \sum_{i=1}^{m} \bar{\beta}_i(x)\nabla\beta_i(x). \tag{21}$$

Because $\beta_i(x)$ approaches zero as the agent approaches the border of the obstacle $\mathcal{O}_i$, $\bar{\beta}_l(x)$ approaches zero for all $l \neq i$. This means that $\bar{\beta}_i(x)$ acts as a soft switch for each obstacle

$i = 1, \dots, m$ which makes the gradient of the closest obstacle the most relevant. As such, when we are close to an obstacle $\mathcal{O}_i$, we want to pre-multiply the gradient by the Hessian inverse of said obstacle. We therefore modify the correction term for the obstacle gradient to be the inverse of a linear combination of the Hessians of the m obstacles. Let $\alpha_i : \mathcal{F} \to [0, 1]$ be a soft switch which makes $\alpha_i(x)$ close to 1 when x is close to $\partial\mathcal{O}_i$ and close to 0 when x is away from $\partial\mathcal{O}_i$. Then we define the obstacle gradient correction term $B : \mathcal{X} \to \mathbb{R}^{n\times n}$ as

$$B(x) = \sum_{i=1}^{m} \alpha_i(x)\nabla^2\beta_i(x). \tag{22}$$

With the previous definition, the resulting dynamics for the multiple obstacle case becomes

$$\dot{x} = -\beta(x)\nabla^2 f_0(x)^{-1}\nabla f_0(x) + \frac{f_0(x)}{k}B(x)^{-1}\nabla\beta(x). \tag{23}$$

In the next section we discuss the desired properties of the soft switch to achieve convergence and we suggest an example that can be used in practice.

A. Soft Switch for Local Curvature Correction

As previously argued our interest is to choose $B(x)$ such that $B(x) \approx \nabla^2\beta_i(x)$ when the agent is close to $\mathcal{O}_i$, and thus we let $B(x)$ be equal some linear combination of the obstacle Hessians. The contribution of each obstacle Hessian is controlled by a soft switch $\alpha_i(x)$ for all $i = 1, \dots, m$ which we require $\alpha_i(x)$ to be close to one when in the neighborhood of obstacle $\mathcal{O}_i$ and to be close to zero away of it. We formalize this with the following property and lemma.

Property 1 *For every $\delta > 0$, there exists $\varepsilon(\delta)$ such that $\beta_i(x) < \varepsilon(\delta)$ implies $1-\delta < \alpha_i(x)$ and $\alpha_j(x) < \delta$ for all $j \neq i$. Further, $\beta_i(x) < 1/k^p$ for some $p > 0$ implies $\alpha_i(x) > 1 - O(1/k^p)$ and $\alpha_j(x) < O(1/k^p)$* for all*$j \neq i$ where the notation $O(h)$ implies that there exists a constant L such that $\lim_{h\to 0} = O(h)/h = L$.*

Lemma 2 *Let Assumption 2 hold and let $\alpha_i(x)$ be a soft switch satisfying property 1 for all $i = 1, \dots, m$. Then for all $\delta > 0$, there exists $\epsilon(\delta) > 0$ such that $\beta_i(x) < \epsilon(\delta)$ implies that $B(x)^{-1} = (\alpha_i(x)\nabla^2\beta_i(x))^{-1} + D(x)$ where $\|D(x)\| < \delta$. Further, $\beta_i(x) < 1/k^p$ for $p > 0$ implies $B(x)^{-1} = (\alpha_i(x)\nabla^2\beta_i(x))^{-1} + D(x)$ where $\|D(x)\| < O(1/k^p)$.*

Proof: See Appendix-C ■

For the correction term $B(x)$ with soft switch $\alpha_i(x)$ satisfying property 1, the contribution of the Hessian of the closest obstacle is approximately one while the contributions of the other obstacles are negligible. Similarly, we can show that when $\beta_i(x) < 1/k^p$, for large k

$$\nabla\beta(x) \approx \bar{\beta}_i(x)\nabla\beta_i(x). \tag{24}$$

This comes from the fact that for sufficiently large k, $\bar{\beta}_i(x) > \rho^{m-1}$ (c.f. Assumption 2). As such, $\bar{\beta}_j(x) < O(1/k^p)$ since it includes $\beta_i(x)$ in its product. Thus, when the agent is very close to an obstacle $\mathcal{O}_i$, Lemma 2 and (24) imply the flow (23) is almost exactly equal to (13). We show that the convergence results hold for *any* switch which satisfies property 1. The question remains whether a switch satisfying the aforementioned assumptions exists or not. In the next proposition, we show that property 1 is satisfied by the soft switch defined for $i = 1 \dots m$ by

$$\alpha_i(x) = \frac{\bar{\beta}_i(x)}{\sum_{i=1}^{m}\bar{\beta}_i(x)}. \tag{25}$$

Proposition 1 *Suppose $\alpha_i(x)$ is defined as in* (25) *for all $i = 1, \dots m$. Then $\beta_i(x) < 1/k^p$ for any $p > 0$ implies $\alpha_i(x) > 1 - O(1/k^p)$ and $\alpha_j(x) < O(1/k^p)$ for all $j \neq i$.*

Proof: See Appendix-D ■

At the boundary of obstacle i, all $\bar{\beta}_j$ are zero and $\bar{\beta}_i$ is strictly positive, so $\alpha_i(x)$ is one and all $\alpha_j(x) = 0$. The previous proposition confirms that it is possible to construct switches that satisfy property 1.

The goal of introducing this switch is to emulate the dynamics in the one obstacle case (Section III) in the neighborhood of each obstacle. That is, to eliminate the effect of obstacles that are far away. Hence, it is to expect that such an approach will succeed in avoiding the obstacles as well. In the next section, we introduce the notion of a graph in a configuration. This concept will be required to complete the argument that the agent converges to the target.

B. Defining Graphs on Configurations

Before defining the graph on the configuration we require the following lemma, stating that the space can be divided in hyperplanes which the agent can only traverse in one direction.

Lemma 3 *Let $f_0(x)$ satisfy Assumption 2 and $\mathcal{H}$ be a hyperplane that does not intersect any obstacle and separates the space $\mathcal{X}$ into two halfspaces $\mathcal{H}^1$ and $\mathcal{H}^2$. Without loss of generality assume $x^* \in \mathcal{H}^1$. Let $n \in \mathbb{R}^n$ be a unit vector normal to the hyperplane pointing toward $\mathcal{H}^1$. Then there exists K_1 such that for all $k > K_1$, $n^\top\dot{x} < 0$*

Proof: See Appendix-E ■

The previous lemma is interpreted as follows. Once the agent transitions from $\mathcal{H}_2$ to $\mathcal{H}_1$, it will never be able to go back into $\mathcal{H}_2$. As such, we consider constructing a graph on the configuration which would describe the order in which an agent would visit each obstacle.

In order to show that once the agent visits an obstacle it will never return to the same obstacle, consider the following construction of a graph on an arbitrary configuration. First, to formalize notation, consider a hyperplane $\mathcal{H}_{i,j}$ between obstacle $\mathcal{O}_i$ and $\mathcal{O}_j$. Because the obstacles are strictly convex, we know this strictly separating hyperplane exists [22]. Let $\mathcal{H}^i$ denote the halfspace containing $\mathcal{O}_i$ and $\mathcal{H}^j$ denote the halfspace containing $\mathcal{O}_j$. The agent's target x^* is either in $\mathcal{H}^i$ or $\mathcal{H}^j$. Without loss of generality, let $x^* \in \mathcal{H}^j$. Now we can define a graph $G = (\mathcal{N}, \mathcal{E})$, where the obstacles are nodes $\mathcal{N} = \{1, \dots, m\}$, and the edges in $\mathcal{E}$ represent the ability to move from one obstacle to the other. To place an edge between any pair of obstacles $\mathcal{O}_i, \mathcal{O}_j$, we check two conditions independently of each other:

Condition 1 *There is a hyperplane $\mathcal{H}_{i,j}$ such that $x^* \in \mathcal{H}^j$.*

Condition 2 *There is a hyperplane $\mathcal{H}_{j,i}$ such that $x^* \in \mathcal{H}^i$.*

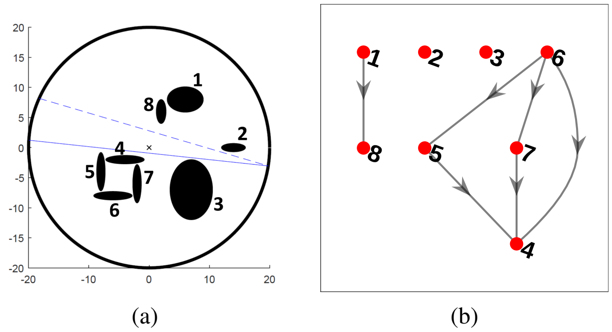


Fig. 2. We consider placing an edge between obstacles $\mathcal{O}_2$ and $\mathcal{O}_3$. The dashed line shows that there exists a hyperplane $\mathcal{H}_{2,3}$ with $x^* \in \mathcal{H}^3$ and the solid line shows there exists a hyperplane $\mathcal{H}_{3,2}$ with $x^* \in \mathcal{H}^2$. As such, there will be no edge placed between nodes 2 and 3 of the corresponding graph, shown in (b).

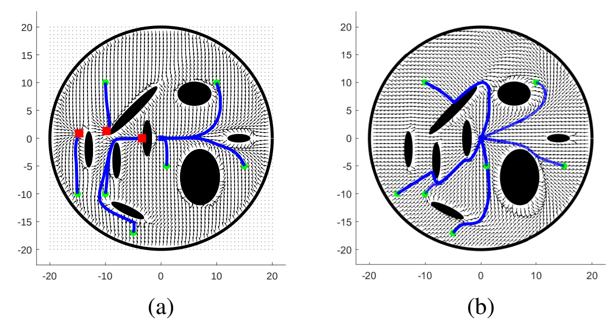


Fig. 3. (a) Trajectories generated by following the negative gradient of the RK Potential which is not a navigation function as condition (11) is violated. Trajectories which converge to a local minimum of $\varphi(x)$ end in a red square. (b) Trajectories generated by following our proposed dynamics. All trajectories converge to the target of the agent located at $(0, 0)$. In both examples, we have $k = 15$

Because there is always a hyperplane between two convex sets, at least one condition will be satisfied for every pair of obstacles. Therefore we define the graph by

$$(i, j) \in \mathcal{E} \text{ if Condition 1 and not Condition 2.} \tag{26}$$

We will illustrate this with an example. Consider the environment in Fig 2 (a). Let us consider whether or not to place an edge between nodes two and three of the corresponding graph. We first consider the hyperplane $\mathcal{H}_{2,3}$ where $x^* \in \mathcal{H}^3$. In the figure, this is the dashed hyperplane. Because we are also able to draw a hyperplane $\mathcal{H}_{3,2}$ between the obstacles such that $x^* \in \mathcal{H}^2$, we do not put an edge between nodes two and three of the graph. By Lemma 3, this means that with large enough k, the trajectory of the agent will never cross the boundaries to hit either $\mathcal{O}_2$ or $\mathcal{O}_3$, where by hit, we mean be ε-close.

The graph corresponding to the configuration is shown in Figure 2 (b). We interpret the graph as follows. Suppose the point agent is placed randomly on the free space. Then following (23), it may eventually hit one of the obstacles represented by the nodes on the graph. If it hits a node that is at the end of the directed acyclic graph (i.e. nodes two, three, four, or eight), it will hit any other obstacle before reaching its target. If the agent hits any other node, then it can either go directly to the target or visit the obstacle which the edge indicates. For example, if the agent is near obstacle one, it will either go to the target, or hit obstacle eight. It is impossible for the agent to visit any other obstacle.

We now have the tools necessary to present the full convergence argument, where we claim that the agent is able to navigate a world with multiple obstacles without requiring condition (11) to hold.

Theorem 2 *Let $f_0(x)$ satisfy Assumption and $\beta_i(x)$ be ellipsoidal obstacles for $i = 1, \ldots, m$ satisfying Assumption 2. If the graph G defined on the configuration does not have any cycles, then for every point $x_0 \in \mathcal{F}$, there exists a K^* such that when $k > K^*$, $x(t) \in \mathcal{F}$ for all $t \geq 0$ and $\lim_{t\to\infty} x(t) = x^*$*

For the full proof, see Appendix-F. A sketch of the proof is as follows. We show that the agent will locally escape any obstacle that it encounters by constructing a Lyapunov function for that obstacle as in (45). This comes from the fact that the soft switch makes the effect of far away obstacles negligible. So if the graph constructed on the configuration is such that there are no cycles, then the agent will eventually visit a final obstacle before converging to the target. Note that Theorem 2 only holds for configurations whose graphs do not contain any cycles. This is a minimal restriction because worlds which contain cycles become increasingly rare as the number of dimensions increases. In the following section, we show numerically that the agent will reach its goal even in the case where the graph G has a cycle. Further, we consider a discretization of the flow (23) and compare with the uncorrected flow to verify Theorem 2.

V. Numerical Results

In this section, we evaluate the performance of our proposed dynamics compared to the performance of the navigation function dynamics. We consider a discrete approximation of the gradient flow (23) using a normalization. We normalize the dynamics to avoid executing speeds that are too large for the system.

$$x_{t+1} = x_t + \eta \frac{g(x_t)}{\|g(x_t)\| + \epsilon} \tag{27}$$

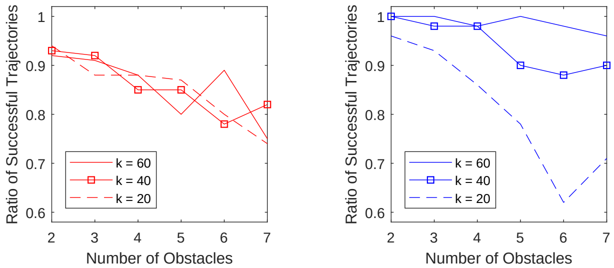
where η is a constant step size. Similarly, we consider a similar discrete approximation for the navigation function dynamics

$$x_{t+1} = x_t + \eta \frac{\nabla\varphi(x_t)}{\|\nabla\varphi(x_t)\| + \epsilon} \tag{28}$$

First, we compare the trajectories from several different initial positions in a world with eight ellipsoidal obstacles, and we show that the corrected dynamics result in a world with only one stable point, the target. We then explore the effect of increasing the number of obstacles of randomly generated ellipsoidal worlds. The obstacles are generated such that condition (11) might fail. Therefore, increasing the number of obstacles results fewer trajectories which successfully reach the target. In contrast, our proposed dynamics perform well even when the number of obstacles is large.

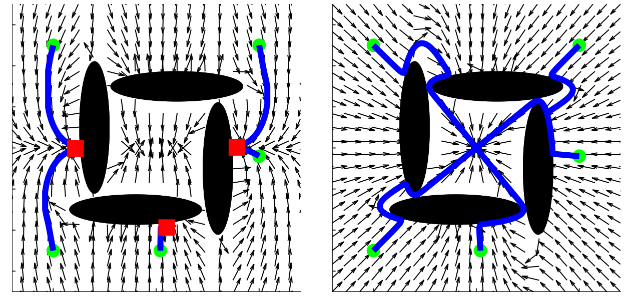
A. Correcting the Field

In this section, we show an ellipsoidal world with eight obstacles and several different initialization points. The objective value of the function is chosen to be $f_0(x) = \|x\|^2$. Figure 3 (a) shows the vector field and some trajectories for the gradient of the navigation function. Indeed, condition (11) is violated. As



(a) (b)

Fig. 4. Ratio of successful simulations given different number of obstacles m and different tuning parameters k. (a) Uncorrected Dynamics: Regardless of the value k, the ratio of successful simulations decreases as the number of obstacles increases. (b) Corrected Dynamics: By increasing k, the ratio of successful simulations remains high regardless of the number of obstacles



(a) (b)

Fig. 5. Trajectories generated in a world whose graph contains a cycle for which Theorem 2 does not hold. (a) All trajectories fail given the uncorrected flow. (b) All trajectories converge to the target despite the presence of a cycle. In this example, we have $k = 10$.

such, four of the trajectories converge to points which are not the target. We selected $k = 15$ because this was the maximum value for k considered in the analysis for worlds which violate the condition [21]. We set $\epsilon = 10^{-4}$ and $\eta = 0.01$ in (28).

We compare the trajectories of the navigation function dynamics where the condition is violated to our proposed dynamics. The Figure 3 (b) shows the agent following (27) with $\epsilon = 10^{-4}$ and $\eta = 0.01$. With the same value of $k = 15$, all of the trajectories converge to the agent's target. Not only does the vector field show that there is only one stable point, it also allows for a visual understanding of Lemma 1. The vector fields around each obstacle show a boundary which the trajectory never enters so long as the agent begins outside. This boundary – of the set $\mathcal{B}_k$ in Lemma 1 – becomes tight as we increase k. To ensure that the agent will reach its target, a k must be chosen so that the sets $\mathcal{B}_k$ for $k = 1 \dots m$ does not intersect with each other.

B. Elliptical Obstacles in $\mathbb{R}^2$

In this section, we explore the effect of increasing the number of obstacles on the percentage of successful trajectories. We define the external shell to be the a spherical with center $(0, 0)$ and radius r_0. The center of each ellipsoid is drawn uniformly from $[-r_0/2, r_0/2]^2$. The maximum semiaxis r_i is drawn uniformly from $[r_0/10, r_0/5]$. The positive definite matrices A_i have minimum eigenvalues 1 and $\mu^i_{\max}$, where $\mu^i_{\max}$ is drawn randomly from $[1, r_0/2]$. The obstacles are then rotated by θ_i where θ_i is drawn randomly from $[-\pi/2, \pi/2]$. The obstacles are redrawn if Assumption 2 is violated. For the objective function, we consider a quadratic cost given by

$$f_0(x) = (x - x^*)^\top Q(x - x^*). \tag{29}$$

where $Q \in \mathcal{M}^{2\times 2}$ is a diagonal matrix with eigenvalues $\text{eig}(Q) = \{1, \lambda\}$ where λ is drawn from $[0, r_0]$. The minimizer of the objective function x^* is drawn uniformly from $[-r_0/2, r_0/2]^2$. The minimizer x^* is redrawn if it violates Assumption 1. Finally, the initial position is drawn uniformly from $[-r_0, r_0]^2$ and is redrawn if it is not in the interior of the free space.

For our experiments, we set $r_0 = 20$ and $\eta = 10^{-4}$. We then vary number of obstacles m from two to seven. For tuning parameters $k = \{20, 40, 60\}$ we run 100 simulations for each $m \in \{2, \dots, 7\}$. Each simulation is terminated successfully when the norm of the difference of x_t and x^* is less than the step size $\eta = 0.01$. A simulation is terminated unsuccessfully if the agent collides with an obstacle - including the outer boundary - or the number of steps reaches 5×10^4. Figure 4 (a) shows the the results of the simulation for the uncorrected dynamics. For all values of k, the ratio of successful trajectories decreases as the number of obstacles increases. In contrast, as k increases for the corrected dynamics, we see that the ratio of successful trials increases. The poor performance of $k = 20$ in the corrected dynamics case is due to the fact that we do not consider the outer obstacle β_0 in the dynamics. This is consistent with Theorem 2, which suggests the choice of a very large k.

C. A Configuration with a Cycle

Theorem 2 holds only for the case when the configuration is such that there are no cycles defined on its corresponding graph (26). An example of this configuration is shown in Figure 5. The cycle runs counterclockwise, meaning the north obstacle leads to the east, the east obstacle leads to the south, and so on. This suggests that Theorem 2 can be further generalized to hold without the restriction to acyclic graphs.

VI. Conclusions

We considered the problem of a point agent navigating to a destination with ellipsoidal obstacles in the way using only local information about its surroundings. In particular, we suggested using second order information to remove the geometric constraint on the eccentricity of the obstacles from the navigation function approach. We constructed the argument of asymptotic convergence to the target for the multiple obstacle case by first considering the single obstacle then extending to multiple obstacles using a soft switch on the correction term. For future work, this approach may be used to extend our results to convex obstacles in general.

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Appendix

A. Proof of Lemma 1

Consider

$$\dot{\beta}(x) = \nabla\beta(x)^\top \dot{x}. \tag{30}$$

We show that $\dot{\beta}(x) > 0$ on the boundary of $\mathcal{B}_k$ which implies that the agent outside $\mathcal{B}_k$ will not enter. Substituting (13) for $\dot{x}$ and using the fact that $\nabla^2 f_0(x)^{-1}\nabla f_0(x) = x - x^*$, we have

$$\dot{\beta}(x) = \nabla\beta(x)^\top \left(-\beta(x)(x-x^*) + \frac{f_0(x)}{k}\nabla^2\beta(x)^{-1}\nabla\beta(x)\right). \tag{31}$$

Distribute and substitute the right hand side of (18) for $\beta(x)$ for points in the boundary of the region $\mathcal{B}_k$

$$\begin{aligned}\dot{\beta}(x) = &-\frac{f_0(x)((x-x^*)^\top\nabla\beta)^2}{k(x-x^*)^\top\nabla^2\beta(x)(x-x^*)} \\ &+\frac{f_0(x)\nabla\beta(x)^\top\nabla^2\beta(x)^{-1}\nabla\beta(x)}{k}.\end{aligned} \tag{32}$$

Because $\nabla^2\beta(x) \succ 0$, we rewrite the quadratic terms $(x-x^*)^\top\nabla^2\beta(x)(x-x^*)$ and $\nabla\beta(x)\nabla^2\beta(x)^{-1}\nabla\beta(x)$ in terms of norms, and we factor out $f_0(x)/k$

$$\begin{aligned}\dot{\beta}(x) = \frac{f_0(x)}{k}\Big(&-\frac{((x-x^*)^\top\nabla\beta(x))^2}{\|\nabla^2\beta(x)^{\frac{1}{2}}(x-x^*)\|^2} \\ &+\|\nabla^2\beta(x)^{-\frac{1}{2}}\nabla\beta(x)\|^2\Big).\end{aligned} \tag{33}$$

We know that $f_0(x)/k > 0$ because $f_0 : \mathcal{X} \to \mathbb{R}_+$ and because $x^* \notin \partial\mathcal{B}_k$ by Assumption 1. Hence what is left to show is that the terms inside the parentheses of (33) is positive. Equivalently,

$$\|\nabla^2\beta(x)^{-\frac{1}{2}}\nabla\beta(x)\|^2 \cdot \|\nabla^2\beta(x)^{\frac{1}{2}}(x-x^*)\|^2 \geq ((x-x^*)^\top\nabla\beta(x))^2 \tag{34}$$

which is true by Cauchy Schwartz. Hence, we have $\dot{\beta}(x)|_{x\in\mathcal{B}_k} \geq 0$ with equality holding when $\nabla^2\beta(x)^{1/2}(x-x^*)$ is aligned with $\nabla^2\beta(x)^{1/2}\nabla\beta(x)$. That is, when

$$x - x^* = a\nabla^2\beta(x)^{-1}\nabla\beta(x) \tag{35}$$

for some $a > 0$. Let us denote the point on the border of $\mathcal{B}_k$ such that $(x-x^*)$ is aligned with $\nabla^2\beta(x)^{-1}\beta(x)$ by x_s. Consider evaluating

$$g_1(x) := \dot{x} \tag{36}$$

at x_s, where the subscript 1 denotes the one obstacle case. Using the fact that $\nabla^2 f_0(x)^{-1}\nabla f_0(x) = (x-x^*)$, (13) becomes

$$g(x_s) = -\beta(x_s)(x_s - x^*) + \frac{f_0(x_s)}{k}\nabla^2\beta(x_s)^{-1}\nabla\beta(x_s). \tag{37}$$

We then substitute the expression for $\beta(x_s)$ given (18) and substitute $(x_s - x^*)/a$ for $\nabla^2\beta(x_s)^{-1}\nabla\beta(x_s)$ (c.f. (35))

$$\begin{aligned}g(x_s) = &-\frac{f_0(x_s)(x_s-x^*)^\top\nabla\beta(x_s)}{k(x_s-x^*)^\top\nabla^2\beta(x_s)(x_s-x^*)}(x_s-x^*) \\ &+\frac{f_0(x_s)}{k}\frac{1}{a}(x_s-x^*).\end{aligned} \tag{38}$$

Again, use (35) and substitute $\nabla^2\beta(x_s)(x_s-x^*)$ with $a\nabla\beta(x_s)$ to obtain

$$g(x_s) = -\frac{f_0(x_s)}{k}\left(\frac{(x_s-x^*)^\top\nabla\beta(x_s)}{a(x_s-x^*)^\top\nabla\beta(x_s)} - \frac{1}{a}\right)(x_s-x^*) \tag{39}$$

Because we are outside $\mathcal{B}_{x^*}(\delta')$, we know that $(x-x^*)^\top\nabla^2\beta_i(x)(x-x^*) > \delta'$. In particular, for x_s, we have $a(x_s-x^*)^\top\nabla\beta(x_s) > \delta'$ (c.f. (35)). Therefore, since $a > 0$, we have that $\dot{x}$ at x_s is equal to zero, indicating that x_s is an equilibrium. Hence, it is impossible to enter the $\mathcal{B}_k$ if the agent starts outside the region. We will now show that this equilibrium is unstable, meaning that the agent will not converge to x_s.

Let $v \in \mathbb{R}^n$ be a unit vector perpendicular to $\nabla^2\beta(x)^{-1}\nabla\beta(x_s)$. Therefore, we want to show that $v^\top J_g(x)v$ evaluated at the point where $x = x_s$ is greater than zero, where

$J_g(x)$ is the Jacobian of the flow. We calculate the Jacobian of (13) to obtain

$$\begin{aligned} J_g(x) = &- \beta(x)\nabla^2 f_0(x)^{-1}\nabla^2 f_0(x) \\ &- \nabla^2 f_0(x)^{-1}\nabla f_0(x)\nabla\beta(x)^\top \\ &+ \frac{1}{k}\nabla^2\beta(x)^{-1}\nabla\beta(x)\nabla f_0(x)^\top \\ &+ \frac{f_0(x)}{k}\nabla^2\beta(x)^{-1}\nabla^2\beta(x). \end{aligned} \tag{40}$$

Evaluate $J_g(x_s)$, and eliminate the second term because $v^\top\nabla^2 f_0(x_s)^{-1}\nabla f_0(x_s) = v^\top(x_s - x^*) = 0$. We also eliminate the third term because $v^\top\nabla^2\beta(x_s)^{-1}\nabla\beta(x_s) = 0$ Hence,

$$v^\top J_g(x_s)v = -\beta(x_s)v^\top v + \frac{f_0(x_s)}{k}v^\top v. \tag{41}$$

Substitute the expression for $\beta(x_s)$ given in (18), and invoke the fact that v is a unit vector,

$$v^\top J_g(x_s)v = -\frac{f_0(x_s)}{k}\left(\frac{(x_s - x^*)^\top\nabla\beta(x_s)}{(x_s - x^*)^\top\nabla^2\beta(x)(x_s - x^*)} - 1\right). \tag{42}$$

Again, using (35), we know $\nabla\beta(x_s) = a\nabla^2\beta(x)(x_s - x^*)$ for some $a > 0$. When $a < 1$, x_s and x^* must be on opposite sides of the obstacle. Use this substitution in (42) to obtain

$$v^\top J_g(x_s)v = \frac{f_0(x_s)}{k}\left(-a\frac{(x_s - x^*)^\top\nabla^2\beta(x_s)(x_s - x^*)}{(x_s - x^*)^\top\nabla^2\beta(x_s)(x_s - x^*)} + 1\right). \tag{43}$$

Since $(x_s - x^*)^\top\nabla^2\beta(x_s)(x_s - x^*) > \delta'$ by our definition of $\mathcal{B}_k$, we simplify

$$v^\top J_g(x_s)v = \frac{f_0(x_s)}{k}\left(-a + 1\right). \tag{44}$$

So x_s is an unstable equilibrium when x^s and x^* are on opposite sides of the obstacle.

B. Proof of Theorem 1

Let

$$V(x) = \frac{1}{2}(x - x^*)^\top\nabla^2\beta(x)(x - x^*), \tag{45}$$

be a Lyapunov function. Because $\nabla^2\beta(x) = A \succ 0$, we know $V(x) > 0$ except for when $x = x^*$ in which case $V(x^*) = 0$. To prove that x^* is asymptotically stable in the sense of Lyapunov, we must show that $\dot{V} < 0$ for all $x \in \mathcal{F}\backslash\mathcal{B}_k$. Differentiate with respect to x

$$\dot{V}(x) = (x - x^*)\nabla^2\beta(x)\dot{x}, \tag{46}$$

and substitute (13) for $\dot{x}$,

$$\begin{aligned} \dot{V}(x) = (x - x^*)^\top\nabla^2\beta(x)\Big(&-\beta(x)\nabla^2 f_0(x)^{-1}\nabla f_0(x) \\ &+ \frac{f_0(x)}{k}\nabla^2\beta(x)^{-1}\nabla\beta(x)\Big). \end{aligned} \tag{47}$$

Because $f_0(x)$ is quadratic, we have $\nabla^2 f_0(x)^{-1}\nabla f_0(x) = (x - x^*)$, so

$$\begin{aligned} \dot{V}(x) = &-\beta(x)(x - x^*)^\top\nabla^2\beta(x)(x - x^*) \\ &+ \frac{f_0(x)}{k}(x - x^*)^\top\nabla\beta(x). \end{aligned} \tag{48}$$

This is less than zero whenever

$$\beta(x) > \frac{f_0(x)(x - x^*)^\top\nabla\beta(x)}{k(x - x^*)^\top\nabla^2\beta(x)(x - x^*)}, \tag{49}$$

which, by definition of $\mathcal{B}_k$, holds for all $x \in \mathcal{F}\backslash(\mathcal{B}_k \cup \mathcal{B}_{x^*}(\delta'))$. We focus now on the region $\mathcal{B}_{x^*}(\delta')$ and notice that we can upper bound (48) using the Cauchy-Schwartz inequality by

$$\begin{aligned} \dot{V}(x) \leq &-\beta(x)(x - x^*)^\top\nabla^2\beta(x)(x - x^*) \\ &+ \frac{f_0(x)}{k}\left\|x - x^*\right\|\left\|\nabla\beta(x)\right\|. \end{aligned} \tag{50}$$

Because $f_0(x)$ is quadratic, the second term is of order $(x - x^*)^3 = \delta'^{3/2}$ while the first term is only of order δ'. Therefore, we choose δ' small enough so that (50) is negative.

By Assumption 1, we know there exists a $K^*(x_0)$ such that $x_0 \notin \mathcal{B}_k$. Further, the agent will never enter $\mathcal{B}_k$ by Lemma 1.

C. Proof of Lemma 2

Notice that it is possible to write $B(x) = A(x) + C(x)$, where $A(x) = \alpha_i(x)\nabla^2\beta_i(x)$, $C(x) = \sum_{j\neq i}\alpha_j(x)\nabla^2\beta_j(x)$. Then using the Kailath Variant matrix identity [24, Ch 4]

$$B(x)^{-1} = A(x)^{-1} + D(x), \tag{51}$$

where

$$D(x) = -A(x)^{-1}C(x)(I + A(x)^{-1}C(x))^{-1}A(x)^{-1}. \tag{52}$$

Use power series expansion on $(I + A(x)^{-1}C(x))^{-1}$,

$$D(x) = -A(x)^{-1}C(x)\left(\sum_{j=0}^{\infty}(-1)^{k+1}\left(A(x)^{-1}C(x)\right)^k\right)A(x)^{-1}. \tag{53}$$

Using the triangle inequality and the Cauchy-Schwartz inequality we bound the norm of $D(x)$ by

$$\|D(x)\| \leq \sum_{k=1}^{\infty}\|A(x)^{-1}\|^{k+1}\|C(x)\|^k \tag{54}$$

Recall that $C(x)$ is the sum of all the Hessians of the obstacles excluding $\mathcal{O}_i$ weighted by $\alpha_j(x)$ for all $j \neq i$. Given that the soft switch α_i for $i = 1, \ldots, m$ satisfies Property 1, we can bound

$$C(x) \prec O\left(\frac{1}{k^p}\right)\sum_{j\neq i}\nabla^2\beta_j(x). \tag{55}$$

Because the obstacles are ellipsoids as defined in (2), $\|\sum_{j\neq i}\nabla^2\beta_j(x)\|$ is bounded by some constant. Therefore, we have the bound on the norm $\|C(x)\|$

$$\|C(x)\| \leq O\left(\frac{1}{k^p}\right). \tag{56}$$

Notice that the right hand side of (54) is a geometric convergent series. Hence, we can further upper bound the right hand side of (54) by

$$\|D(x)\| \leq \frac{\left\|A(x)^{-1}\right\|^2\|C(x)\|}{1 - \|A(x)^{-1}\|\,\|C(x)\|}. \tag{57}$$

To complete the proof, consider taking the following limit of the upper bound of $\|D(x)\|$

$$\lim_{k\to\infty} k^p\frac{\|A(x)^{-1}\|^2\|C(x)\|}{1 - \|A(x)^{-1}\|\,\|C(x)\|} \tag{58}$$

From (56), we know $\lim_{k\to\infty}\|C(x)\| = 0$ and $\lim_{k\to\infty}k^p\|C(x)\| = C_1$. Therefore, since $\|A(x)\|$ is bounded by some a, we have that

$$\lim_{k\to\infty} k^p\frac{\|A(x)^{-1}\|^2\|C(x)\|}{1 - \|A(x)^{-1}\|\,\|C(x)\|} = \frac{a^2C_1}{1}. \tag{59}$$

Therefore, because the limit is bounded by a constant, we have

$$\|D(x)\| \le O\left(\frac{1}{k^p}\right) \tag{60}$$

This completes the proof.

D. Proof of Proposition 1

For any obstacle $\mathcal{O}_i$ with $i = 1 \dots m$ we have that $\beta_i(x) < 1/k^p$. Therefore, there exists a K such that when $k > K$, $\beta_i(x) < 1/k^p < \rho$ which implies $\beta_j(x) > \rho$ for all $j \neq i$ as per Assumption 2. By the definition of $\alpha_i(x)$ as given in (25), the denominator $\sum_i^m \bar{\beta}_i(x)$ is upper bounded by some constant because the free space is enclosed by β_0. Further, the denominator is bounded away from zero because $\beta_j(x) > \rho$ for all $j \neq i$.

Consider the soft switch for an obstacle $\mathcal{O}_j$ where $j \neq i$. Then the numerator $\bar{\beta}_j(x)$ is upper bounded by $O(1/k^p)$ as argued above. As such, we have

$$\alpha_j(x) < O\left(\frac{1}{k^p}\right). \tag{61}$$

Now we must show that $\alpha_i(x) > 1 - O\left(\frac{1}{k^p}\right)$ when $\beta_i(x) < 1/k^p$. Multiply the numerator and denominator of (25) by $1/\bar{\beta}_i(x)$ to obtain the following equivalent expression of $\alpha_i(x)$

$$\alpha_i(x) = \frac{1}{1 + \sum_{j\neq i} \frac{\bar{\beta}_j(x)}{\bar{\beta}_i(x)}}. \tag{62}$$

Apply the definition $\bar{\beta}_i(x) = \prod_{l\neq i} \beta_l(x)$ to the ratio $\bar{\beta}_j(x)/\bar{\beta}_i(x)$ and notice that all terms except for $\beta_i(x)$ and $\beta_j(x)$ cancel. We are then left with

$$\frac{\bar{\beta}_j(x)}{\bar{\beta}_i(x)} = \frac{\beta_i(x)}{\beta_j(x)}. \tag{63}$$

Substitute the right and side of (63) into (62), and factor out the common $\beta_i(x)$

$$\alpha_i(x) = \frac{1}{1 + \beta_i(x)\sum_{j\neq i} \frac{1}{\beta_j(x)}} \tag{64}$$

We expand using a Taylor Series expansion

$$\alpha_i(x) = 1 - \beta_i(x)\sum_{j\neq i} \frac{1}{\beta_j(x)} + \sum_{k=2}^{\infty}(-1)^k \left(\beta_i(x)\sum_{j\neq i} \frac{1}{\beta_j(x)}\right)^k \tag{65}$$

For $k > K$, we have $\beta_j \ge \rho$ for all $j \neq i$ (c.f. Assumption 2). Therefore, we bound $\sum_{j\neq i} 1/\beta_j(x)$ by some positive constant C. As such, since $\beta_i(x) = 1/k^p$, we have the following bound

$$\alpha_i(x) > 1 - O\left(\frac{1}{k^p}\right). \tag{66}$$

This completes the proof.

E. Proof of Lemma 3

Let us start by writing the product $n^\top \dot{x}$ substituting $\dot{x}$ by the dynamics (23)

$$n^\top \dot{x} = n^\top \left(-\beta(x)\nabla^2 f_0(x)^{-1}\nabla f_0(x) + \frac{f_0(x)}{k}B(x)^{-1}\nabla\beta(x)\right). \tag{67}$$

Because $\nabla^2 f_0(x)^{-1}\nabla f_0(x) = (x - x^*)$ by Assumption 2,

$$n^\top \dot{x} = -\beta(x)n^\top(x - x^*) + \frac{f_0(x)n^\top B(x)^{-1}\nabla\beta(x)}{k}. \tag{68}$$

Since $\mathcal{H}$ does not intersect any obstacle, there exists $\varepsilon > 0$ such that for all $x \in \mathcal{H}$ we have that

$$\beta(x) > \varepsilon. \tag{69}$$

Combine (68) with the fact that $n^\top(x - x^*) > 0$, which holds since $x* \in \mathcal{H}^1$, to obtain the bound

$$\beta(x)n^\top(x - x^*) \ge \delta_\varepsilon, \tag{70}$$

for some $\delta_\varepsilon > 0$. Further, because all obstacles are contained in $\{x|\beta_0(x) > 0\}$, n is a unit vector, and the functions f_0, $B(x)$, and $\beta(x)$ are continuous, we can bound the numerator of the second term of (68) by the constant C. That is

$$f_0(x)n^\top B(x)^{-1}\nabla\beta(x) \le C \tag{71}$$

Use (70) and (71) to bound (68)

$$n^\top \dot{x} \le -\delta_\varepsilon + \frac{C}{k} \tag{72}$$

Therefore, when $k > C/\delta_\varepsilon$ we have $n^\top \dot{x} < 0$. This completes the proof.

F. Proof of Theorem 2

In this section, we present the proof of our main result.

Because there is no cycle in the graph, it is only possible to visit a finite number of obstacles. As such, what is left to be shown is that once the agent approaches an obstacle, it will navigate around that obstacle.

Define the region

$$E^i_\varepsilon \triangleq \{x|\beta(x) < \varepsilon, \beta_i(x) < \rho\} \tag{73}$$

Where $\rho > 0$ is as defined in assumption Assumption 2. When $\varepsilon < \rho$, E^i_ε is a connected space.

As such, we have the following lemma.

Lemma 4 *Let $f_0(x)$ be quadratic. Then, there exists a K^i_2 such that for all $k > K^i_2$, the limit set of all $x \in E^i_{\frac{1}{\sqrt{k}}}$ is a subset of $\mathcal{F}\backslash E^i_{\frac{1}{\sqrt{k}}}$*

Proof: We will show this by considering a local Lyapunov function. For simplicity in notation, we assume that $x^* = 0$. For the case of an arbitrary target $x^* \in \mathcal{F}$, simply replace $x - x^*$ for x.

For the obstacle $\mathcal{O}^i$, consider the Lyapunov function

$$V_i(x) = \frac{1}{2}x^\top \nabla^2\beta_i(x)x. \tag{74}$$

Because $\nabla^2\beta_i(x) \succ 0$, we know $V_i > 0$ for all $x \neq x^*$. Taking the derivative with respect to x, we obtain

$$\dot{V}_i(x) = x^\top \nabla^2\beta_i(x)\dot{x}. \tag{75}$$

Substitute (23) for $\dot{x}$

$$\dot{V}_i(x) = x^\top \nabla^2\beta_i(x)(-\beta(x)x + \frac{f_0(x)}{k}B(x)^{-1}\nabla\beta(x)) \tag{76}$$

We want $\dot{V}_i < 0$, therefore after rearranging so that the right hand side of (76) is less than zero, and equivalent condition becomes

$$\beta(x) > \frac{f_0(x)x^\top \nabla^2\beta_i(x)B(x)^{-1}\nabla\beta(x)}{kx^\top \nabla^2\beta_i(x)x} \tag{77}$$

Define the set

$$\mathcal{B}_k^i \triangleq \{x \in \mathcal{B}_{x^*}^c(\delta') | \beta(x) \leq \frac{f_0(x) x^\top \nabla^2 \beta_i(x) B(x)^{-1} \nabla \beta(x)}{k x^\top \nabla^2 \beta_i(x) x}\}, \tag{78}$$

where the set $\mathcal{B}_{x^*}^c(\delta')$ is defined in (17). Then, because $\mathcal{B}_k^i$ shrinks with $1/k$ and $E^i_{\frac{1}{\sqrt{k}}}$ shrinks with $1/\sqrt{k}$, there exists a K_2^i such that for all $k > K_2^i$, $\mathcal{B}_k^i$ is a proper subset of $E^i_{1/\sqrt{k}}$. Hence, $\dot{V}_i < 0$ for all $x \in E^i_{1/\sqrt{k}} \backslash \mathcal{B}_k^i$

What is left to show is that the agent will not enter the region $\mathcal{B}_k^i$ where the Lyapunov function fails. To do this, similar to the case of one obstacle, we check that $\dot{\beta}(x) > 0$ for all points $x \in \mathcal{F}$ where (77) holds with equality.

$$\begin{aligned}\dot{\beta}(x) =& \nabla \beta(x)^\top \dot{x} \\ =& \nabla \beta(x)^\top (-\beta(x) x + \frac{f_0(x)}{k} B(x)^{-1} \nabla \beta(x))\end{aligned} \tag{79}$$

Substitute the right hand side of (77) for $\beta(x)$,

$$\begin{aligned}\dot{\beta}(x) = -\frac{f_0(x)}{k} \Big(& \frac{x^\top \nabla^2 \beta_i(x) B(x)^{-1} \nabla \beta(x) \nabla \beta(x)^\top x}{x^\top \nabla^2 \beta_i(x) x} \\ & - \nabla \beta(x) B(x)^{-1} \nabla \beta(x) \Big) > 0.\end{aligned} \tag{80}$$

Because $f_0(x)/k$ is positive by definition, an equivalent condition for $\dot{\beta} > 0$ is

$$\begin{aligned}& \frac{x^\top \nabla^2 \beta_i(x) B(x)^{-1} \nabla \beta(x) \nabla \beta(x)^\top x}{x^\top \nabla^2 \beta_i(x) x} \\ & - \nabla \beta(x) B(x)^{-1} \nabla \beta(x) < 0.\end{aligned} \tag{81}$$

We substitute $B(x)^{-1}$ with $A(x)^{-1} + D(x)$ (c.f. (51))

$$\begin{aligned}& \frac{x^\top \nabla^2 \beta_i(x) (\frac{1}{\alpha_i(x)} \nabla^2 \beta_i(x)^{-1} - D(x)) \nabla \beta(x) \nabla \beta(x)^\top x}{x^\top \nabla^2 \beta_i(x) x} \\ & - \nabla \beta(x) (\frac{1}{\alpha_i(x)} \nabla^2 \beta_i(x)^{-1} - D(x)) \nabla \beta(x) < 0.\end{aligned} \tag{82}$$

We distribute and rearrange the terms so that we have

$$\begin{aligned}\frac{1}{\alpha_i(x)} \Big(& \frac{x^\top \nabla \beta(x) \nabla \beta(x)^\top x}{x^\top \nabla^2 \beta_i(x) x} \\ & - \nabla \beta(x) \nabla^2 \beta_i(x)^{-1} \nabla \beta(x) \Big) + E(x) < 0,\end{aligned} \tag{83}$$

where

$$\begin{aligned}E(x) =& \nabla \beta(x) D(x) \nabla \beta(x) \\ & - \frac{x^\top \nabla^2 \beta_i(x) D(x) \nabla \beta(x) \nabla \beta(x)^\top x}{x^\top \nabla^2 \beta_i(x) x}\end{aligned} \tag{84}$$

Because both terms of (84) have $D(x)$, whose norm is bounded as a result of lemma 2, the contribution of $E(x)$ can be made to be arbitrarily small by increasing k.

This, together with the fact that the first two terms of (83) are less than zero by Cauchy Schwartz

$$\|\nabla^2 \beta_i(x)^{-\frac{1}{2}} \nabla \beta(x)\|^2 \cdot \|\nabla^2 \beta_i(x)^{\frac{1}{2}} x\|^2 \geq \|\nabla \beta(x)^\top x\|^2, \tag{85}$$

show that, with large enough k, if the agent begins outside the region $\mathcal{B}_k^i$, it will not enter the region expect for possibly the point where (85) holds with equality. This occurs when $\nabla^2 \beta_i(x)^{\frac{1}{2}} x$ is aligned with $\nabla^2 \beta_i(x)^{-\frac{1}{2}} \nabla \beta(x)$, or x is aligned with $\nabla^2 \beta_i(x)^{-1} \nabla \beta(x)$. Therefore write

$$a \nabla^2 \beta_i(x)^{-1} \nabla \beta(x) = x. \tag{86}$$

for some $a > 0$. To show that $a < 1$, consider the border of $\mathcal{B}_k^i$ with the substitution for $B(x)^{-1}$ in (51)

$$\beta(x) > \frac{f_0(x) x^\top \nabla^2 \beta_i(x) (A(x)^{-1} - D(x)) \nabla \beta(x)}{k x^\top \nabla^2 \beta_i(x) x}. \tag{87}$$

Distribute and use the bound on $D(x)$

$$\beta(x) > \frac{f_0(x) x^\top \nabla^2 \beta_i(x)^{-1} A(x)^{-1} \nabla \beta(x)}{k x^\top \nabla^2 \beta_i(x) x} - \frac{O(\delta_k)}{k}. \tag{88}$$

Since $A(x)^{-1} = \alpha_i(x) \nabla^2 \beta_i(x)^{-1}$, the inequality trivially holds whenever $x^\top \nabla^2 \beta_i(x)^{-1} \nabla \beta(x) < 0$, hence we can say $a < 1$ in (86).

We now take a closer look at the dynamics (23). Using the fact that $\nabla^2 f_0(x)^{-1} \nabla f_0(x) = x$ we write

$$\dot{x} = -\beta(x) x + \frac{f_0(x)}{k} B(x)^{-1} \nabla \beta(x). \tag{89}$$

Now we can use the substitution (51) and the fact that $\nabla \beta(x) = \sum_i^m \bar{\beta}_i(x) \nabla \beta_i(x)$ to obtain

$$\dot{x} = -\beta(x) x + \frac{f_0(x)}{k} (A(x)^{-1} - D(x)) (\sum_i^m \bar{\beta}_i(x) \nabla \beta_i(x)). \tag{90}$$

Distribute to obtain

$$\begin{aligned}\dot{x} =& -\beta(x) x + \frac{f_0(x)}{k} (A(x)^{-1}) \bar{\beta}_i(x) \nabla \beta_i(x) \\ & - \frac{f_0(x)}{k} D(x) (\sum_{j \neq i}^m \bar{\beta}_j(x) \nabla \beta_j(x)) \\ & + \frac{f_0(x)}{k} A(x)^{-1} (\sum_{j \neq i} \bar{\beta}_j(x) \nabla \beta_j(x)) \\ & - \frac{f_0(x)}{k} D(x) \bar{\beta}_i(x) \nabla \beta_i(x).\end{aligned} \tag{91}$$

Since $A(x)^{-1} = \alpha_i(x)^{-1} \nabla^2 \beta_i(x)^{-1}$, we have

$$\begin{aligned}\dot{x} =& -\beta(x) x + \frac{f_0(x) \bar{\beta}_i(x)}{\alpha_i(x) k} \nabla^2 \beta_i(x)^{-1} \nabla \beta_i(x) \\ & - \frac{f_0(x)}{k} D(x) (\sum_{j \neq i}^m \bar{\beta}_j(x) \nabla \beta_j(x)) \\ & + \frac{f_0(x)}{\alpha_i(x) k} \nabla^2 \beta_i(x)^{-1} (\sum_{j \neq i} \bar{\beta}_j(x) \nabla \beta_j(x)) \\ & - \frac{f_0(x)}{k} D(x) \bar{\beta}_i(x) \nabla \beta_i(x).\end{aligned} \tag{92}$$

Substitute the expression for $\beta(x)$

$$\begin{aligned}\dot{x} =& -\frac{f_0(x)x^\top\nabla^2\beta_i(x)\nabla^2\beta_i(x)^{-1}\bar{\beta}_i(x)\nabla\beta_i(x)}{k\alpha_i(x)x^\top\nabla^2\beta_i(x)x}x \\ &+\frac{f_0(x)x^\top\nabla^2\beta_i(x)D(x)\sum_{j\neq 1}\bar{\beta}_j(x)\nabla\beta_j(x)}{kx^\top\nabla^2\beta_i(x)x}x \\ &-\frac{f_0(x)x^\top\nabla^2\beta_i(x)\nabla^2\beta_i(x)^{-1}\sum_{j\neq i}\bar{\beta}_j(x)\nabla\beta_j(x)}{k\alpha_i(x)x^\top\nabla^2\beta_i(x)x}x \\ &+\frac{f_0(x)x^\top\nabla^2\beta_i(x)D(x)\bar{\beta}_i(x)\nabla\beta_i(x)}{kx^\top\nabla^2\beta_i(x)x}x \\ &+\frac{f_0(x)\bar{\beta}_i(x)}{\alpha_i(x)k}\nabla^2\beta_i(x)^{-1}\nabla\beta_i(x) \\ &-\frac{f_0(x)}{k}D(x)(\sum_{j\neq i}^{m}\bar{\beta}_j(x)\nabla\beta_j(x)) \\ &+\frac{f_0(x)}{\alpha_i(x)k}\nabla^2\beta_i(x)^{-1}(\sum_{j\neq i}\bar{\beta}_j(x)\nabla\beta_j(x)) \\ &-\frac{f_0(x)}{k}D(x)\bar{\beta}_i(x)\nabla\beta_i(x).\end{aligned} \tag{93}$$

Which we can group by

$$g(x) := \dot{x} = \frac{1}{k}\times\frac{\bar{\beta}_i(x)}{\alpha_i(x)}(g_1(x)+g_2(x)), \tag{94}$$

where $g_1(x)$ is as defined in (36) for the one obstacle dynamics and

$$\begin{aligned}g_2(x) =& \frac{\alpha_i(x)f_0(x)}{\bar{\beta}_i(x)}\Big(\frac{x^\top\nabla^2\beta_i(x)D(x)\sum_{j\neq 1}\bar{\beta}_j(x)\nabla\beta_j(x)}{x^\top\nabla^2\beta_i(x)x}x \\ &-\frac{x^\top\nabla^2\beta_i(x)\nabla^2\beta_i(x)^{-1}\sum_{j\neq i}\bar{\beta}_j(x)\nabla\beta_j(x)}{k\alpha_i(x)x^\top\nabla^2\beta_i(x)x}x \\ &+\frac{x^\top\nabla^2\beta_i(x)D(x)\bar{\beta}_i(x)\nabla\beta_i(x)}{kx^\top\nabla^2\beta_i(x)x}x \\ &-D(x)(\sum_{j\neq i}^{m}\bar{\beta}_j(x)\nabla\beta_j(x)) \\ &+\frac{1}{\alpha_i(x)}\nabla^2\beta_i(x)^{-1}(\sum_{j\neq i}\bar{\beta}_j(x)\nabla\beta_j(x)) \\ &-D(x)\bar{\beta}_i(x)\nabla\beta_i(x)\Big).\end{aligned} \tag{95}$$

We conclude the proof by showing that as we increase k, the critical point approaches the equivalent saddle point of the one obstacle proof. The norm of the Jacobian of the $g_2(x)$ dies with k, which means there exits a K_2^i such that for all $k > K_2^i$, the saddle point of $g_1(x)$ dominates any local minima created by $g_2(x)$. This concludes our proof. ■

To conclude the proof of Theorem 2, select $K^* = \max\{K_2^1, \ldots, K_2^m\}$.