Unit III

Counting Principles

- The Basics of Counting, rule of Sum and Product,
- Permutations and Combinations
- Binomial Coefficients and Identities
- Generalized Permutations and Combinations
- Algorithms for generating Permutations and Combinations.

Basic Principle of Counting

- many a times one needs to find out the number of all possible outcomes for a series of events.

Example

- In how many ways can a panel of judges comprising of 6 men and 4 women be chosen from among 50 men and 38 women?
- How many different 10 lettered PAN numbers can be generated such that the first five letters are capital alphabets, the next four are digits and the last is again a capital letter

Basic Principle of Counting

Mathematical Theory – Counting

Counting mainly encompasses

- Fundamental counting rule(SUM & PRODUCT)
- The permutation rule
- The combination rule.

Counting applications

- ❑ Computer science
- ❑ Mathematics

for example:

- determining complexity of algorithms
- computing discrete probability of some events

Examples of applications of counting problems

Counting can be applied to computing (for example):

- How many passwords of a given length and specified constraints on the used symbols are potentially possible
- How many different IP addresses are possible in a given protocol (e.g.
- IPv4 or IPv6)
- How many different telephone numbers in a given country or network
- are possible
- How many different diving licence plates in a given country are possible

Examples of applications of counting problems

- Counting can be applied to computing (for example):
- the number of operations to be executed by a loop (or a nested loop) in a computer program
- How many different chemical molecules with a given properties are possible
- How many different ways of finding a path in a given graph are possible, etc.

The sum rule:

- If a task can be done in n_1 ways and a second task in n_2 ways, and if these two tasks cannot be done at the same time, then there are $n_1 + n_2$ ways to do either task.
- **Example 1:**
- The department will award a free computer to either a CS student or a CS professor. How many different choices are there, if there are 530 students and 15 professors?
- There are 530 + 15 = 545 choices.

The sum rule:

- **Example 2:**
  - A student can choose a computer project from one of three lists. The three lists contain23, 15 and 19 possible projects respectively. How many possible projects are there to choose from?
  - Solution : 23+15+19=57 Projects
  - **Example 3:** Suppose that either a member of the mathematics faculty or a student who is a mathematics major is chosen as a representative to a university committee. How many different choices are there for this representative if there are 37 members of the mathematics faculty and 83 mathematics majors and no one is both a faculty member and a student?
  - **Solution : 37+83=120**

The sum rule:

- **Generalized sum rule:**
- If we have tasks $T_1, T_2, \ldots, T_m$ that can be done in $n_1, n_2, \ldots, n_m$ ways, respectively, and no two of these tasks can be done at the same time, then there are $n_1 + n_2 + \ldots + n_m$ ways to do one of these tasks.

- If we consider two tasks A and B which are disjoint

(i.e. $A \cap B = \emptyset$), then mathematically $|A \cup B| = |A| + |B|$

The product rule:

- Suppose that a procedure can be broken down into two successive tasks. If there are n_1 ways to do the first task and n_2 ways to do the second task after the first task has been done, then there are $n_1 n_2$ ways to do the procedure

- n1 × n2ways

The product rule:

- **Example 1:**

- In class, there are 4 boys and 10 girls if a boy and a girl have to be chosen for the class monitor, 4 x 10 = 40 ways.
- **Solution:**

  The students can choose class monitor in
- 4 x 10 = 40 ways.

The product rule:

- **Example 1:**
- How many different license plates are there that containing exactly three English letters ?
- **Solution:**
- There are 26 possibilities to pick the first letter, then 26 possibilities for the second one, and 26 for the last one.
- So there are $26 \cdot 26 \cdot 26 = 17576$ different license plates.

The product rule:

Example 3 The chair of an auditorium (The great Hall) is to be labeled with a letter and a positive integer not exceeding 100. What is the largest number of chairs that can be labeled differently?

Sol: $\underbrace{26}_{\text{letter}} \times \underbrace{100}_{\boxed{\begin{array}{l}1 \le x \le 100 \\ x \in N\end{array}}} = 2600$ ways to label chairs.

The product rule:

In class, there are 4 boys and 10 girls if a boy and a girl have to be chosen for the class monitor, the students can choose class monitor in $4 \times 10 = 40$ ways.

Generalized product rule:

- If we have a procedure consisting of sequential tasks $T_1, T_2, \ldots, T_m$ that can be done in $n_1, n_2, \ldots, n_m$ ways, respectively, then there are $n_1 \cdot n_2 \cdot \ldots \cdot n_m$ ways to carry out the procedure.
- Mathematically, if a task B arrives after a task A, then

$$|A \times B| = |A| \times |B|$$

Permutations

- A **permutation** is an arrangement of some elements in which order matters. In other words a Permutation is an ordered Combination of elements.
- Examples

  - From a set S ={x, y, z} by taking two at a time, all permutations are –

    $xy, yx, xz, zx, yz, zy.$
  - We have to form a permutation of three digit numbers from a set of numbers $S = \{1, 2, 3\}$. Different three digit numbers will be formed when we arrange the digits. The permutation will be = 123, 132, 213, 231, 312, 321

Permutations

The number of permutations of 'n' different things taken 'r' at a time is denoted by n_{P_r}

$$n_{P_r} = \frac{n!}{(n-r)!}$$

where $n! = 1.2.3.\dots(n-1).n$

Permutations

There are 3 Types of Permutation :

<u>Type I</u> : Let $0 \leq r \leq n$. The number of ways to have an ordered sequence of n distinct elements taken 'r' at time is called as r permutations of 'n' elements and is denoted by $P(n,r)$ or ${}^nP_r$

Permutations(Type-I)

The first place is filled by n ways, second is filled by (n-1) ways, in this way r^{th} place is filled by n-r+1 ways.

$\therefore$ ${}^nP_r$ is given as

$${}^nP_r \text{ or } P(n,r) = n(n-1)(n-2)\cdots\cdots(n-r+1)$$

$$\boxed{P(n,r) = \frac{n!}{(n-r)!}} \quad \text{where } 0 \le r \le n$$

Permutations

Solved Ex: (Type - I)

① Find permutation of the set $A = \{1, 2, 3, 4\}$ taking two elements at time.

Soln

Here $n = 4$ as $4 = \{1, 2, 3, 4\}$ and $r = 2$

$$\therefore {}^{4}P_{2} = \frac{4!}{4-2!} = \frac{4!}{2!} = 3 \times 4$$

$$= \boxed{12 \text{ Permutation}}$$

Permutations

Type II

The number of ways in which of the n elements can be arranged, where r_1 element of one kind, r_2 elements are of another kind and so on till r_k elements are of another kind is given as

$$\boxed{\frac{P(n,r)}{r_1!\,r_2!\cdots r_k!}}$$

Where $r = r_1 + r_2 + \cdots + r_k$

and $0 \leq r \leq n$

Permutations

Solved Example: (Type II)

① In how many ways the letters of the word "MISSISSIPPI" be arranged ?

Soln Here $n = 11$

$r_1 = 4$ (Letter I, 4 times repeated)

$r_2 = 4$ (Letter S, 4 times repeated)

$r_3 = 2$ (Letter P, 2 times repeated)

$\therefore$ Total arrangements $= \dfrac{P(n,r)}{4!\,4!\,2!} = \dfrac{11!}{4!\,4!\,2!}$

$= \boxed{34650}$

Permutations

(2) How many words of three distinct letters can be formed from the letters of the word MAST ?

<u>Solⁿ</u>

There are four distinct letters in the word "MAST"

1- Letter	2- Letter	3- Letter
↑	↑	↑
Filled by 4 ways	Filled by 3 ways	Filled by 2 ways

That is $4 \times 3 \times 2 = \boxed{24 \text{ Words}}$

Permutations

Problem 1 – From a bunch of 6 different cards, how many ways we can permute it?

Problem 2 – In how many ways can the letters of the word 'READER' be arranged?

Problem 3 – In how ways can the letters of the word 'ORANGE' be arranged so that the consonants occupy only the even positions?

Type III

The number of permutations of 'n' elements 'r' at a time when each element may repeated once, twice ..... upto 'r' times in any arrangement is given by n^r

Ex: 1

A dice rolled 3 times. find the no. of faces that can appear on the top.

Sol^n

Every time, die is rolled the face appearing on the top can be any one of the six faces 1 to 6. So when the die is rolled 3 times.

The total no. of faces appear on top $= 6^3 = \underline{216}$

Ex : 2

How many auto license plates can be made if each is identified by 2 letters followed by 4 digits?

Soln

The first two position can be occupied by $26 \times 26 = 26^2$ ways. The next 4 position occupied by $10 \times 10 \times 10 \times 10 = 10^4$

Total no. of auto License plates $= 26^2 \times 10^4$

Cobmination

- A **Combination** is an arrangement of items in which order does not matter.

The number of Combinations of n items chosen r at a time, is given by the formula

$$ {}_nC_r = \frac{n!}{r!(n-r)!} \quad \text{where } 0 \le r \le n . $$

Ex. 1 In how many ways we can select 6 balls out of 10 balls in the box?

Sol^n 6 balls can be selected out of 10 balls by

$$^{10}C_6 \text{ ways} = \frac{10!}{6!4!} = \frac{7 \times 8 \times 9 \times 10}{1 \times 2 \times 3 \times 4} = \underline{\underline{210 \text{ ways}}}$$

Ex. 2

Out of 4 officiers & 10 clerks, a committee of 2 officers and 3 clerks is to be formed. In how many ways committee can be done if :

i) Any officer and any clerk can be included

ii) A particular clerk must be in Committee.

iii) A particular officer cannot be in committee.

Sol

There are 4 officers and 10 clerks out of which a committee of 2 officers and 3 clerk is to be formed.

i) As we know, a committee consist of 2 officers and 3 clerks.

- No. of ways to select 2 officers out of 4

$$^4C_2 = \frac{4!}{2! \times 2!} = \frac{4 \times 3 \times \cancel{2!}}{\cancel{2!} \times 2!} = \frac{12}{2} = \underline{\underline{6 \text{ ways}}}$$

- The number of ways to select 3 clerks out of 10

$$^{10}C_3 = \frac{10!}{3! \times 7!} = \frac{10 \times 9 \times 8 \times \cancel{7!}}{3! \times \cancel{7!}}$$

$$= \frac{10 \times 9 \times 8}{6} = \underline{\underline{120}} \text{ ways}$$

The no. of ways to form a committee consist of 2 officiers and 3 clerk

$$= {}^{4}C_2 \times {}^{10}C_3 = 6 \times 120 = \underline{\underline{720}} \text{ ways}$$

ii) When particular clerk is always included in the committee then we must select remaining 2 clerks out of remaining 9 clerks.

$$= {}^{4}C_2 \times {}^{1}C_1 \times {}^{9}C_2 = 6 \times 1 \times 36 = \underline{\underline{216}} \text{ ways}$$

iii) When a particular officer can not be in the committee then we must select 2 officers from remaining 3 officers.

$$= {}^{3}C_2 \times {}^{10}C_3 = 3 \times 120 = \underline{\underline{360 \text{ ways}}}$$

Binomial Coefficients and Identities

- Used to find the expansion of algebraic identity **$(ax + by)^n$**
- Binomial theorem is used to find the expansion of **two terms** hence it is called the Binomial Theorem
- ***Exponent value*** in the Binomial expansion can also be a negative value or fraction value
- $(a + b)^n$ where a and b are any numbers and n is a non-negative integer. It can be expanded into the sum of terms involving powers of a and b.

Binomial Coefficients and Identities

- **Binomial Theorem Statement**

$$(a + b)^n = {}^nC_0\, a^n b^0 + {}^nC_1\, a^{n-1} b^1 + {}^nC_2\, a^{n-2} b^2 + \ldots + {}^nC_r\, a^{n-r} b^r + \ldots + {}^nC_n\, a^0 b^n$$

- where $n > 0$ and the ${}^nC_k$ is the binomial coefficient.
- **Binomial Expansion Formula**

$$(a + b)^n = \sum^n{}_r\, {}^nC_r\, a^{n-r}\, b^r$$

n is a positive integer,
a, b are real numbers, and $0 < r \leq n$

$${}^nC_r = n! / [r!\,(n - r)!]$$

Algorithms for generating Permutations and Combinations.

- To find that specific permutation/combination, there has to be a method to generate other arrangements of numbers.
- Generating the permutations of the *n* smallest positive integers and then replacing those integers with any set of *n* elements will create the set of permutations for that set. (Lexicographic ordering)

Definition

The **lexicographic ordering** for a set of permutations {1,2,3,...,n-1,n} has the permutation $a_1a_2...a_n$ precede the permutation $b_1b_2...b_n$ when, for some k, 1 <= *k* <= *n*, $a_1 = b_1$, $a_2 = b_2$, ..., $a_{k-1} = b_{k-1}$, and $a_k < b_k$.

Algorithms for generating Permutations and Combinations.

- Procedure for generating the next permutation in lexicographic order
- for a given $a_1a_2...a_n$.
- If $a_{n-1} < a_n$, swap the two to get the next largest permutation
- If $a_{n-1} > a_n$, then a larger permutation cannot be made from the two integers.
- In that case, look at the final three integers. If $a_{n-2} < a_{n-1}$, then put the smaller of the two integers a_{n-1} and a_n in the a_{n-2} position. Fill the remaining positions in lexicograpic order to complete the permutation (..165 to ...516)
- This procedure can be generalized to produce the next largest permutation for any $a_1a_2...a_n$

Algorithms for generating Permutations and Combinations.

Algorithm

Generating the Next Largest Permutation in Lexicographic Order

(Assuming $\{a_1a_2...a_n\}$ is not $\{n,n-1,...,2,1\}$)

```
NextPermutation(a1a2...an)
{
    j = n - 1;
    while (a[j] > a [j+1])
    {
          j--;
    }
    k = n;
    while (a[j] > a[k])
    {
          k--;
    }
    Swap(a[j],a[k]);
    r = n;
    s = j + 1;
    while (r > s)
    {
         Swap(a[r],a[s]);
         r--;
         s++;
    }
}
```

Algorithms for generating Permutations and Combinations

- **Generating Combination**
- For any *r*-combination, a procedure for creating the next largest combination can be developed.
- A combination $a_1a_2...a_r$ in lexicographic order is given for a set {1,2,3,...,n}.
- In the set {1, 2, 3, 4, 5, 6}, a 4-combination could be {1, 2, 5, 6}. To obtain the next largest combination, find the last a_i in the combination so that a_i != $n - r + i$. (The last element in the combination with a_i != $n - r + i$ is 2.) Replace a_i with $a_i + 1$, and a_j with $a_j + j - i + 1$, for $j = i + 1, i + 2, ..., r$. (This creates the next largest combination, {1, 3, 4, 5}.

Generating Combination Contin..

Algorithm

Generating the Next Largest *r*-Combination in Lexicographic Order

(Assuming $\{a_1, a_2, \dots, a_r\}$ is not $\{n-r+1, \dots, n\}$, and
$a_i < a_j$ when $i < j$)

```
NextCombination(a1a2...ar)
{
    i = r;
    while (a[i] > n - r + i)
    {
          i = i - 1;
    }
    a[i] = a[i] + 1;
    for (j = i + 1; j <= r; j++)
    {
          a[j] = a[i] + j - i;
    }
}
```