Unit IV
Graph Theory

- Graph Terminology and Special Types of Graphs
- Representing Graphs
- Graph Isomorphism Connectivity
- Euler and Hamilton Paths
- The handshaking lemma
- Single source shortest path-Dijkstra's Algorithm
- Planar Graphs, Graph Colouring

Graph Terminology

- Mathematical structure which is used to show a particular function with the help of connecting a set of points.
- To create a pairwise relationship between objects.
- **Applications of Graph Theory**
- Mathematics and Computer science
- Computer graphics and networks, biology
- GPS (Global positioning system)

Graph Terminology

Definition of Graph Theory

- a study of points and lines
- type of subfield that is used to deal with the study of a graph.
- graph theory is the study of the relationship between edges and vertices.
- a graph can be represented with the help of pair G(V, E).

    V -the finite set vertices

    E - the finite set edges.
- Hence, the graph basically contains the non-empty set of edges E and set of vertices V.
- Ex: a graph G = (V, E), where
- V = {a, b, c, d}, and E = {(a, b), (a, c), (b, c), (c, d)}.

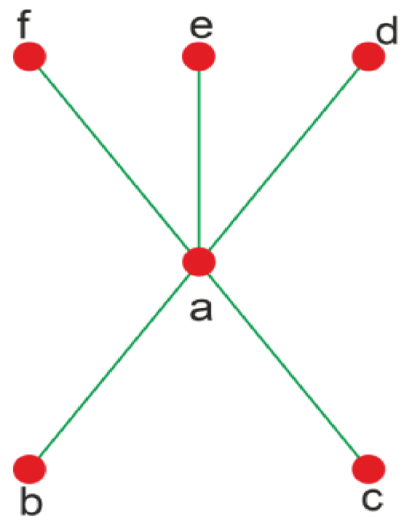
Graph Terminology(Examples)

G1 G2 G3

Graph Terminology(Degree of a vertex)

- The number of edges connecting (Incident) to/on that vertex v with self-loops counted twice

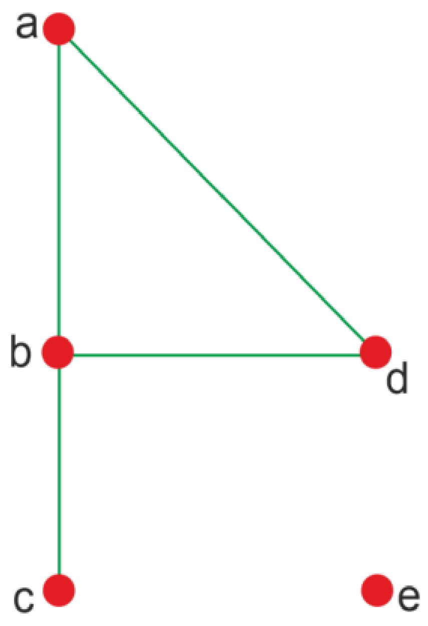
- deg(v)



The vertex 'a' has degree 5, and all the other vertices have a degree 1.
If any vertex has degree 1, then that type of vertex will be known as the 'end vertex'.

Graph Terminology(Degree of a vertex)

- **Degree of a vertex in an Undirected graph(**no directed edge)



Deg(a) = 2 Deg(b) = 3 Deg(d) = 2

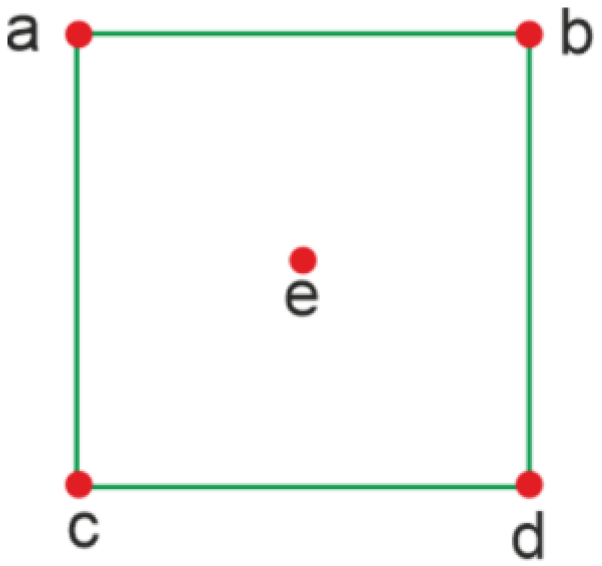
Deg(c) = 1. The vertex c is also known as the **pendent** vertex

Deg(a) = 0 The vertex e can also be called the isolated vertex.

Internal vertex

Graph Terminology(Degree of a vertex)

- **Degree of a vertex in an Undirected graph(**no directed edge)

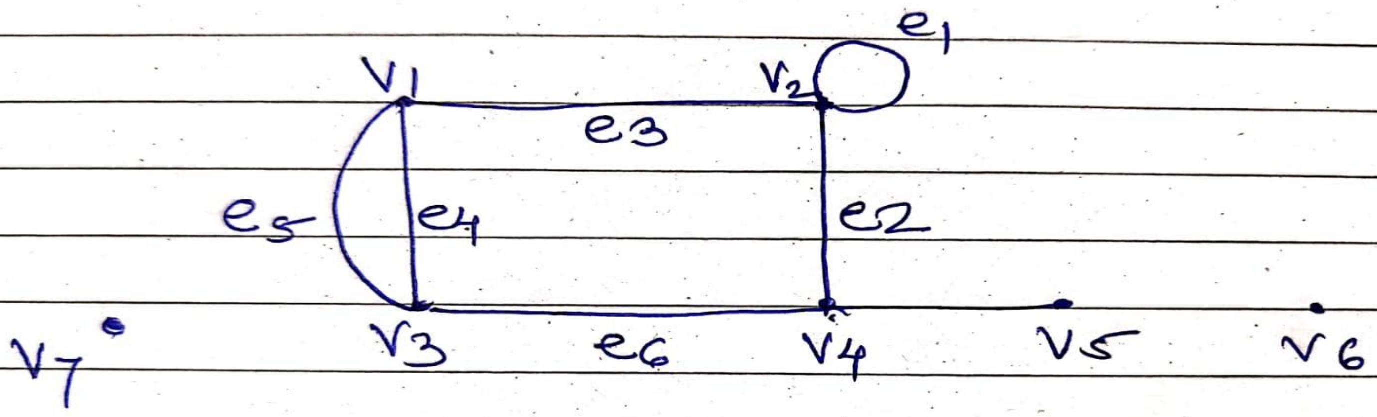


Degree of vertex a = deg(a) = 2
Degree of vertex b = deg(b) = 2
Degree of vertex c = deg(c) = 2
Degree of vertex d = deg(d) = 2
Degree of vertex e = deg(e) = 0(isolated vertex)

Graph Terminology(Degree of a vertex)

Examples

Q:- Find the degree of all vertex (undirected)



$deg(v_1) = 3$

$deg(v_2) = 4$

$deg(v_3) = 3$

$deg(v_4) = 3$

$deg(v_5) = 1$] Pendent

$deg(v_6) = 0$

$deg(v_7) = 0$ } Isolated.

Graph Terminology(Degree of a vertex)

- **Degree of a vertex in an directed graph**
- In-degree :

-Number of edges coming to the vertex

- To count the number of edges that ends at the vertex.

- out-degree:

-Number of edges coming out from the vertex.

-To count the number of edges that begins from the vertex.

Degree of a vertex

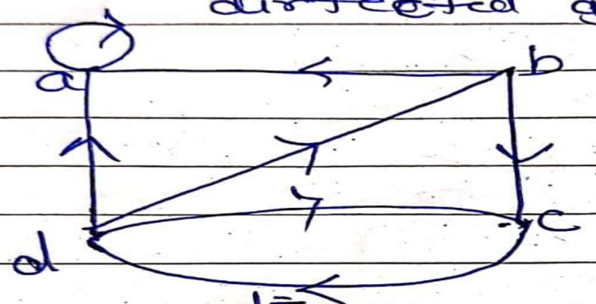
Deg(v) = deg-(v) + deg+(v)

Degree of a vertex in an Directed graph

$deg(v_3)=3$

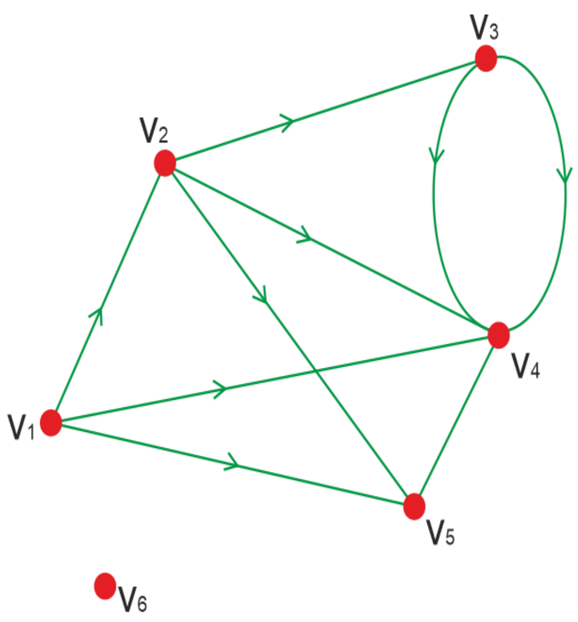
$deg(v_4)=3$

Q: Find the degrees of vertex of directed graph.



Vertex	Indegree $d^-(v)$	Outdegree $d^+(v)$	Total degree
a	3	1	4
b	1	2	3
c	2	1	3
d	1	3	4

Degree of a vertex in an Directed graph



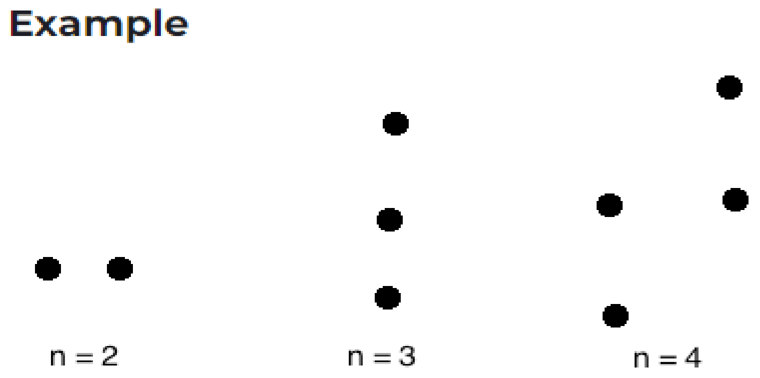
In-degree:
In-degree of a vertex v1 = deg(v1) = 1
In-degree of a vertex v2 = deg(v2) = 1
In-degree of a vertex v3 = deg(v3) = 1
In-degree of a vertex v4 = deg(v4) = 5
In-degree of a vertex v5 = deg(v5) = 1
In-degree of a vertex v6 = deg(v6) = 0

Out-degree:
Out-degree of a vertex v1 = deg(v1) = 2
Out-degree of a vertex v2 = deg(v2) = 3
Out-degree of a vertex v3 = deg(v3) = 2
Out-degree of a vertex v4 = deg(v4) = 0
Out-degree of a vertex v5 = deg(v5) = 2
Out-degree of a vertex v6 = deg(v6) = 0

Degree of vertex
Degree of a vertex v1 = deg(v1) = 1+2 = 3
Degree of a vertex v2 = deg(v2) = 1+3 = 4
Degree of a vertex v3 = deg(v3) = 1+2 = 3
Degree of a vertex v4 = deg(v4) = 5+0 = 5
Degree of a vertex v5 = deg(v5) = 1+2 = 3
Degree of a vertex v6 = deg(v6) = 0+0 = 0

Special Types of Graphs

- Different types of graphs depending upon the number of vertices, number of edges, interconnectivity, and their overall structure
- Null Graph(Empty Graph):No edges between its vertices



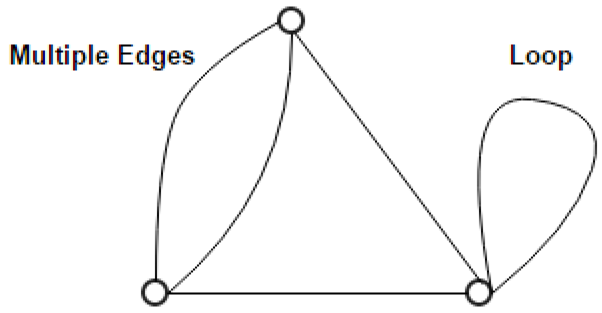
- Trivial Graph

A **trivial graph** is the graph which has only one vertex.

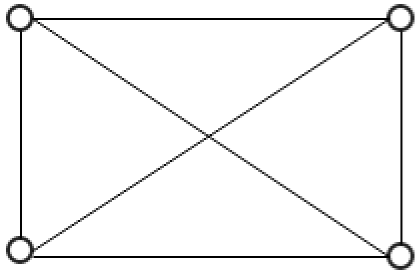
Special Types of Graphs

- Simple Graph
- A **simple graph** is the undirected graph with **no parallel edges** and **no loops**.
- n vertices, the degree of every vertex is at most n -1.

Example



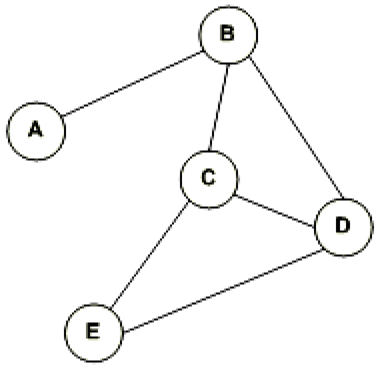
Not a Simple Graph



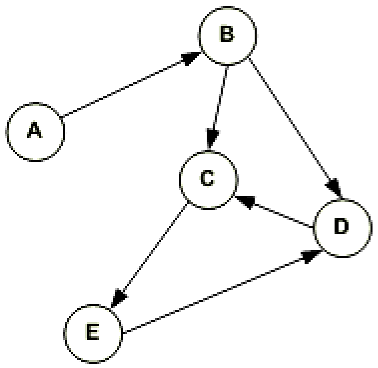
Simple Graph

Special Types of Graphs

- Undirected Graph: edges are **not directed**.

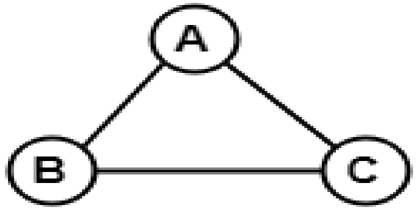


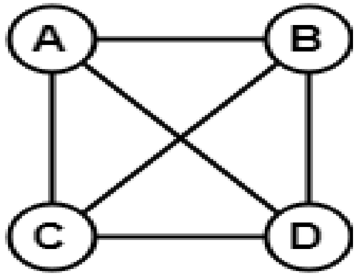
- Directed Graph(**digraphs)edges are directed** by arrows.



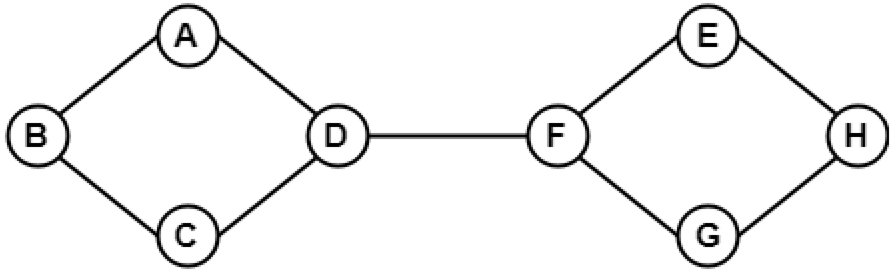
Special Types of Graphs

- Complete Graph
- A graph in which every pair of vertices is joined by exactly one edge is called **complete graph**. It contains all possible edges.



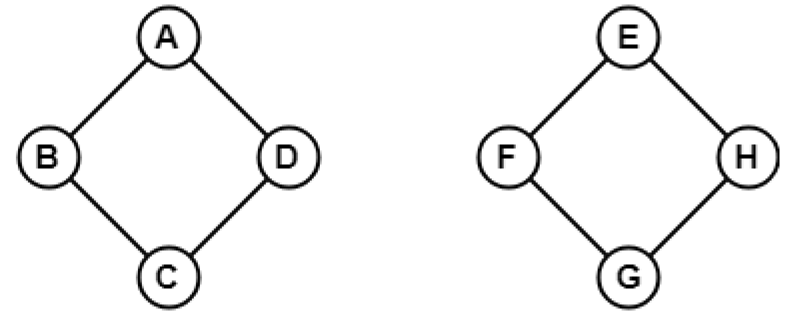


- Connected Graph
- visit from any one vertex to any other vertex. at least one edge or path exists between every pair of vertices.

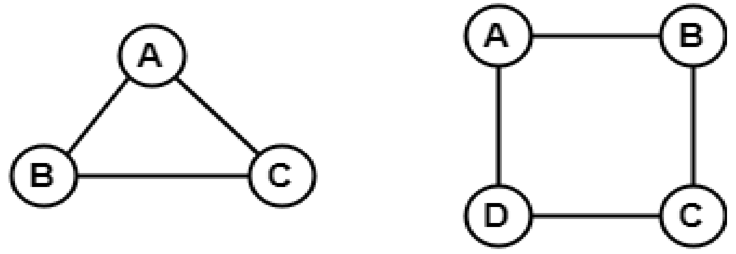


Special Types of Graphs

- Disconnected Graph
- any path does not exist between every pair of vertices.

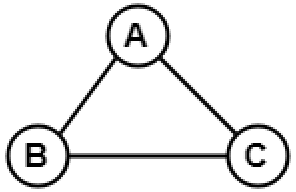


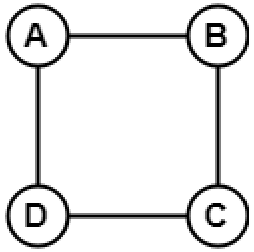
- Regular Graph: degree of all the vertices is same.

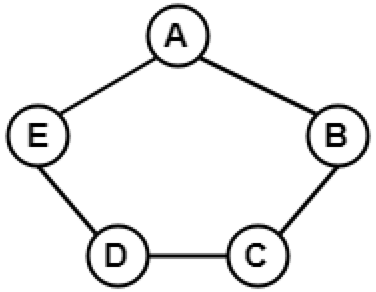


Special Types of Graphs

- Cyclic Graph
- A graph with 'n' vertices (where, n>=3) and 'n' edges forming a cycle of 'n' with all its edges is known as **cycle graph**.
- A graph containing at least one cycle in it is known as a **cyclic graph**.
- n the cycle graph, degree of each vertex is 2.
- The cycle graph which has n vertices is denoted by Cn.

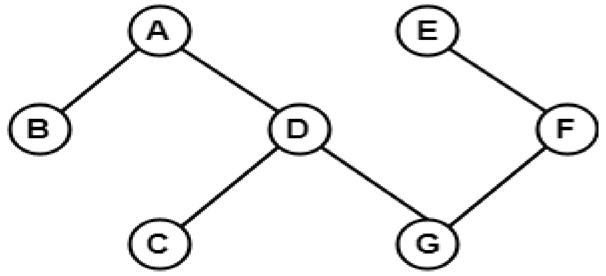






Special Types of Graphs

- Acyclic Graph
- A graph which does not contain any cycle in it is called as an **acyclic graph**.

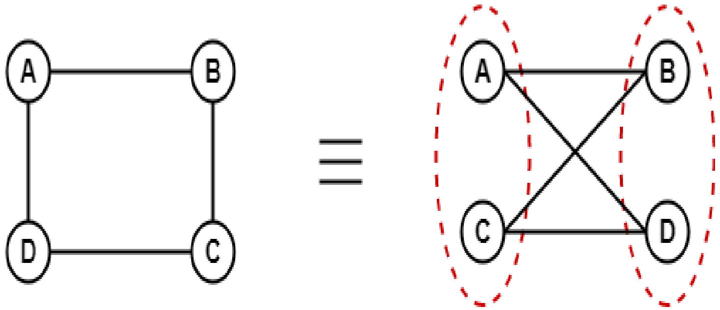


- Bipartite Graph
- A **bipartite graph** is a graph in which the vertex set can be partitioned into two sets such that edges only go between sets, not within them.
- This graph always has two sets, X and Y, with the vertices.
- In this graph, the vertices of set X can only have a connection with the set Y.
- We cannot join the vertices within the same set.

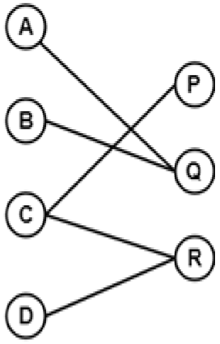
Special Types of Graphs

- Bipartite Graph

Example 1



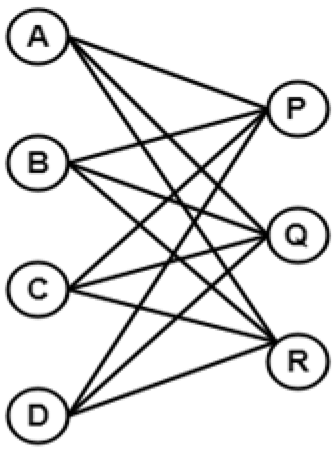
Example 2



Special Types of Graphs

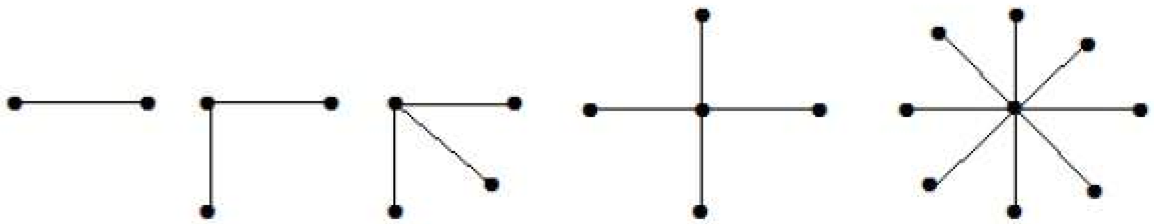
- Complete Bipartite Graph
- bipartite graph in which each vertex in the first set is joined to each vertex in the second set by exactly one edge.
- A complete bipartite graph is a bipartite graph which is complete.

Complete Bipartite graph = Bipartite graph + Complete grap
h



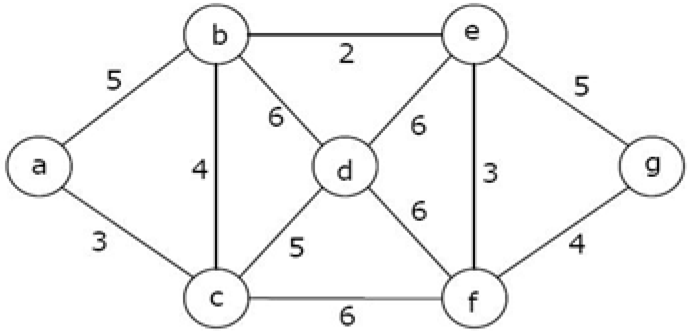
Special Types of Graphs

- Star Graph
- A star graph is a complete bipartite graph in which n-1 vertices have degree 1 and a single vertex have degree (n -1). This exactly looks like a star where (n - 1) vertices are connected to a single central vertex.
- A star graph with n vertices is denoted by S_n.



Special Types of Graphs

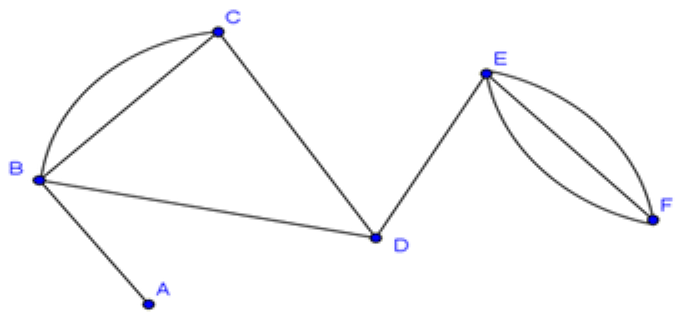
- Weighted Graph
- A weighted graph is a graph whose edges have been labeled with some weights or numbers.
- The length of a path in a weighted graph is the sum of the weights of all the edges in the path.



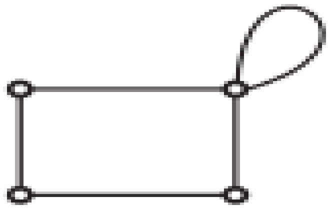
In the above graph, if path is a -> b -> c -> d -> e -> g then the length of the path is 5 + 4 + 5 + 6 + 5 = 25.

Special Types of Graphs

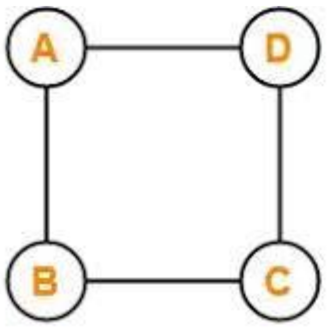
- Multi-graph
- A graph in which there are multiple edges between any pair of vertices or there are edges from a vertex to itself (loop) is called a **multi - graph**.



- Pseudograph:A graph in which loops and parallel edges/multiple edges are allowed



Special Types of Graphs

- Finite graph
- A graph G={V,E} in which both V and E are finite set is called a finite graph
- 

Example of Finite Graph

V={A,B,C,D}

E={(A,B),(A,D),(B,C),(C,D)}

Questions

Define following terms with example.

i) Complete graph

ii) Regular graph

iii) Bipartite graph

iv) Complete bipartitie graph

v) Paths and circuits

Define the following terms with suitable example.

i) Factor of graph

ii) Weighted Graph

iii) Bipartite graph

Explain with example:

i) Bipartite Graph

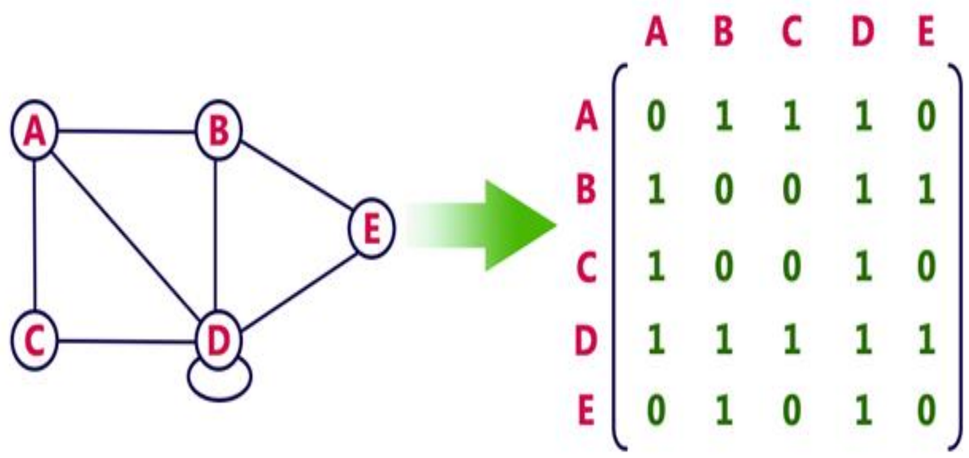
ii) Connected Graphs

Representation of Graphs

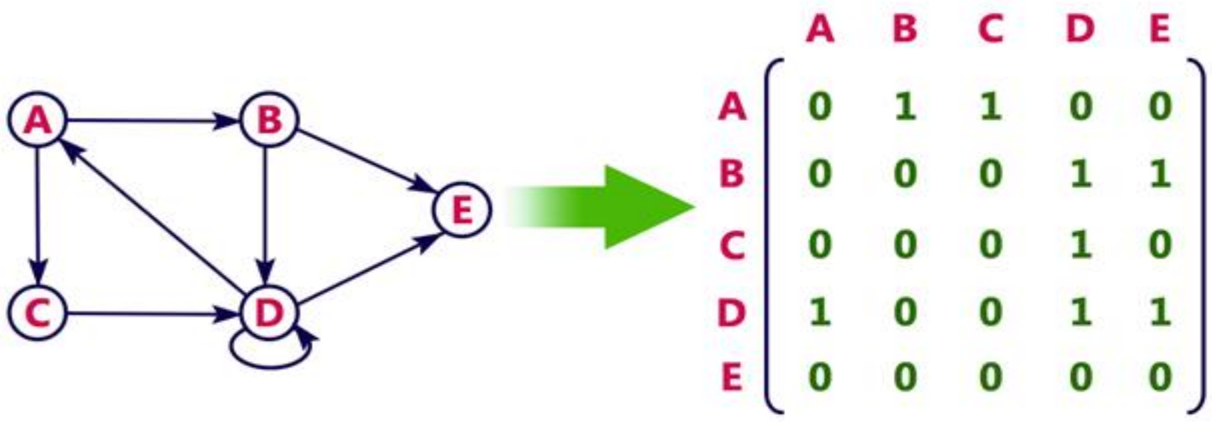
- Adjacency Matrix- **matrix representation**
- Incidence Matrix- **matrix representation**
- Adjacency List -**linked representation**

Adjacency Matrix-

Undirected graph representation



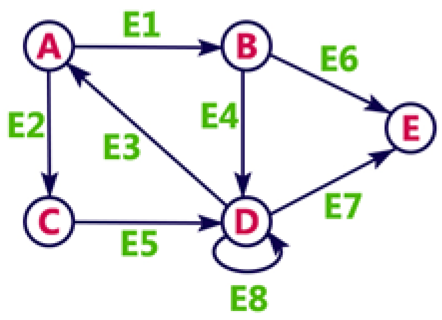
Directed graph represenation



Representation of Graphs

Incidence Matrix- **matrix representation**

- This matrix is filled with either **0 or 1** or -1. Where,
- 0 is used to represent row edge which is not connected to column vertex.
- 1 is used to represent row edge which is connected as outgoing edge to column vertex.
- -1 is used to represent row edge which is connected as incoming edge to column vertex.

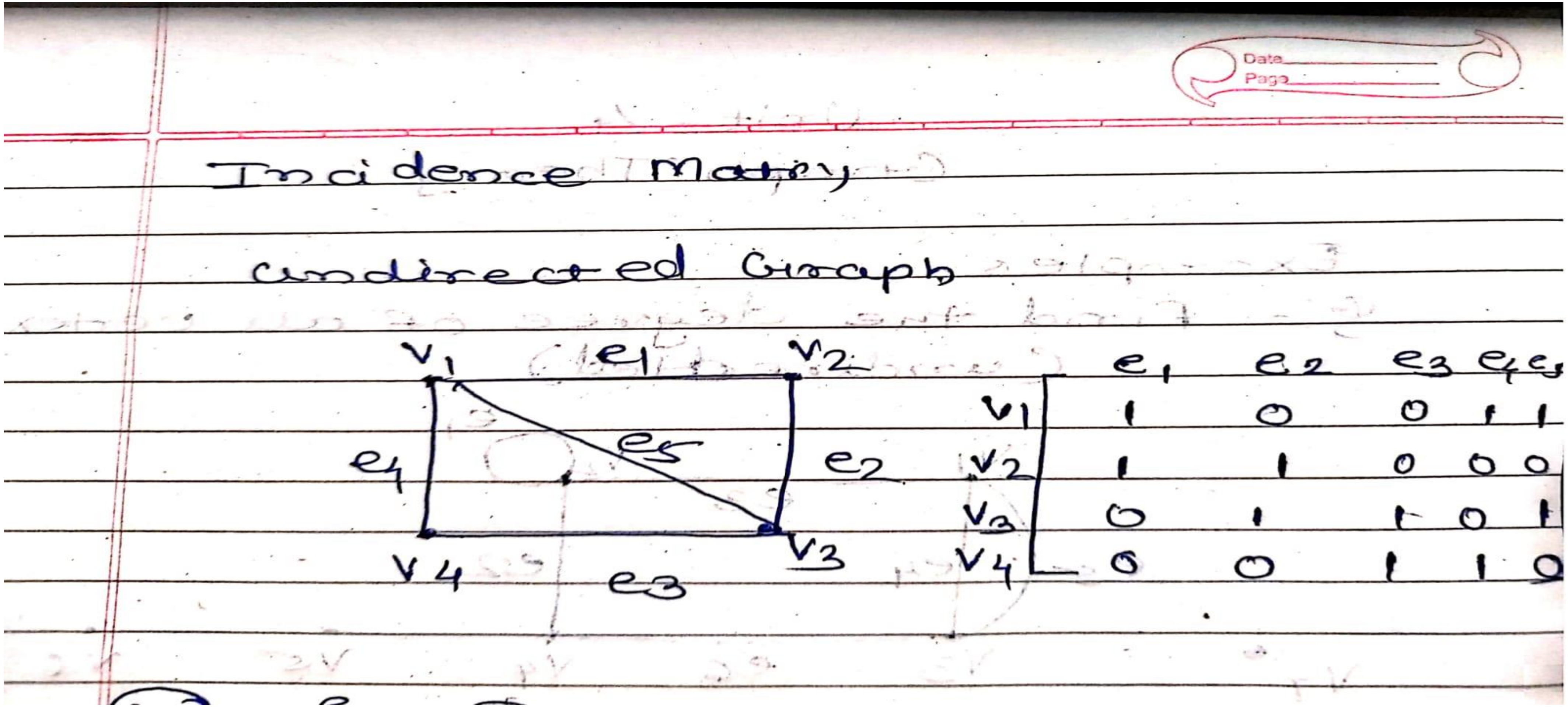


	E1	E2	E3	E4	E5	E6	E7	E8
A	1	1	-1	0	0	0	0	0
B	-1	0	0	1	0	1	0	0
C	0	-1	0	0	1	0	0	0
D	0	0	1	-1	-1	0	1	1
E	0	0	0	0	0	-1	-1	0

directed graph representation.

Representation of Graphs

Incidence Matrix- **matrix representation**



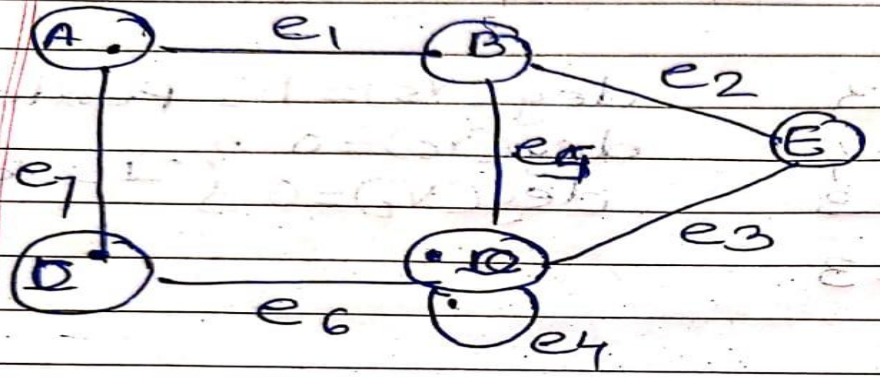
Incidence Matrix

undirected Graph

	e_1	e_2	e_3	e_4	e_5
v_1	1	0	0	1	1
v_2	1	1	0	0	0
v_3	0	1	1	0	1
v_4	0	0	1	1	0

Representation of Graphs

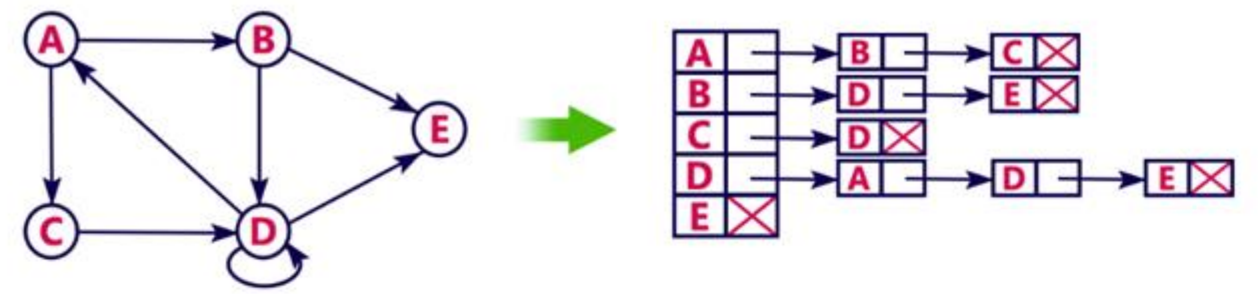
Incidence Matrix- **matrix representation**



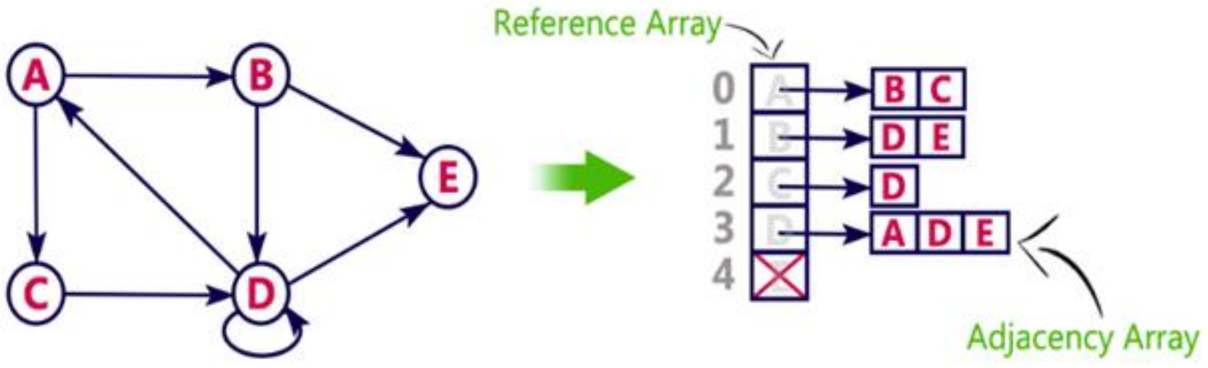
	e1	e2	e3	e4	e5	e6	e7
A	1	0	0	0	0	0	1
B	1	1	0	0	1	0	0
C	0	0	0	0	0	1	1
D	0	0	1	1	1	1	0
E	0	1	1	0	0	0	0

Representation of Graphs

Adjacency List -**linked representation**



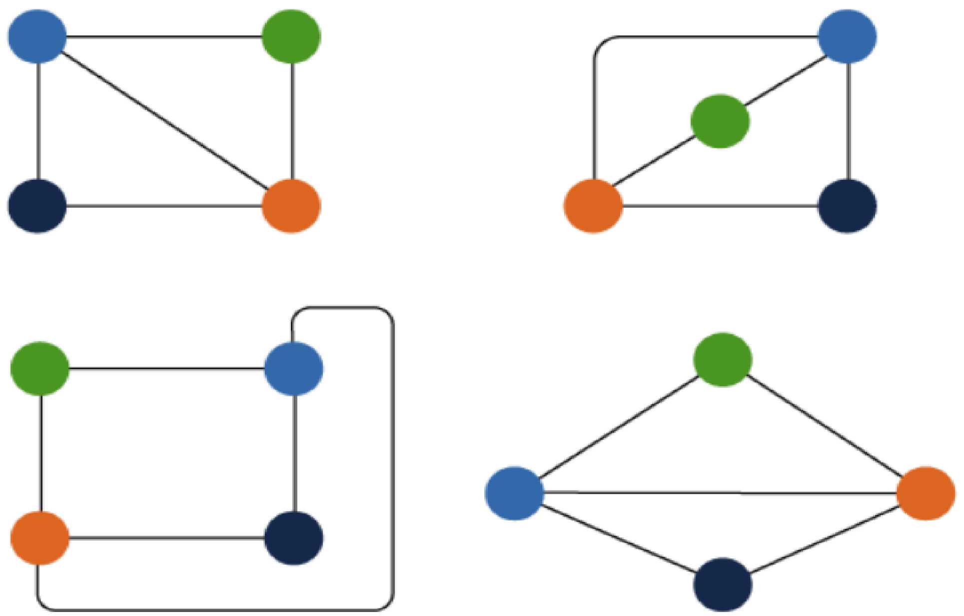
Directed graph representation implemented using
li



Representation using array

Explain the terms adjacency matrix and incidence matrix.

Graph Isomorphism

- Single graph can have more than one form.
- Two different graphs can have the same number of edges, vertices, and same edges connectivity (**isomorphism graphs**)
- 

Two simple graphs G and H are **isomorphic**, denoted $G \cong H$, if there exists a bijection $f : V_G \to V_H$ between vertices of the graph such that if $\{a,b\}$ is an edge in G then $\{f(a), f(b)\}$ is an edge in H.

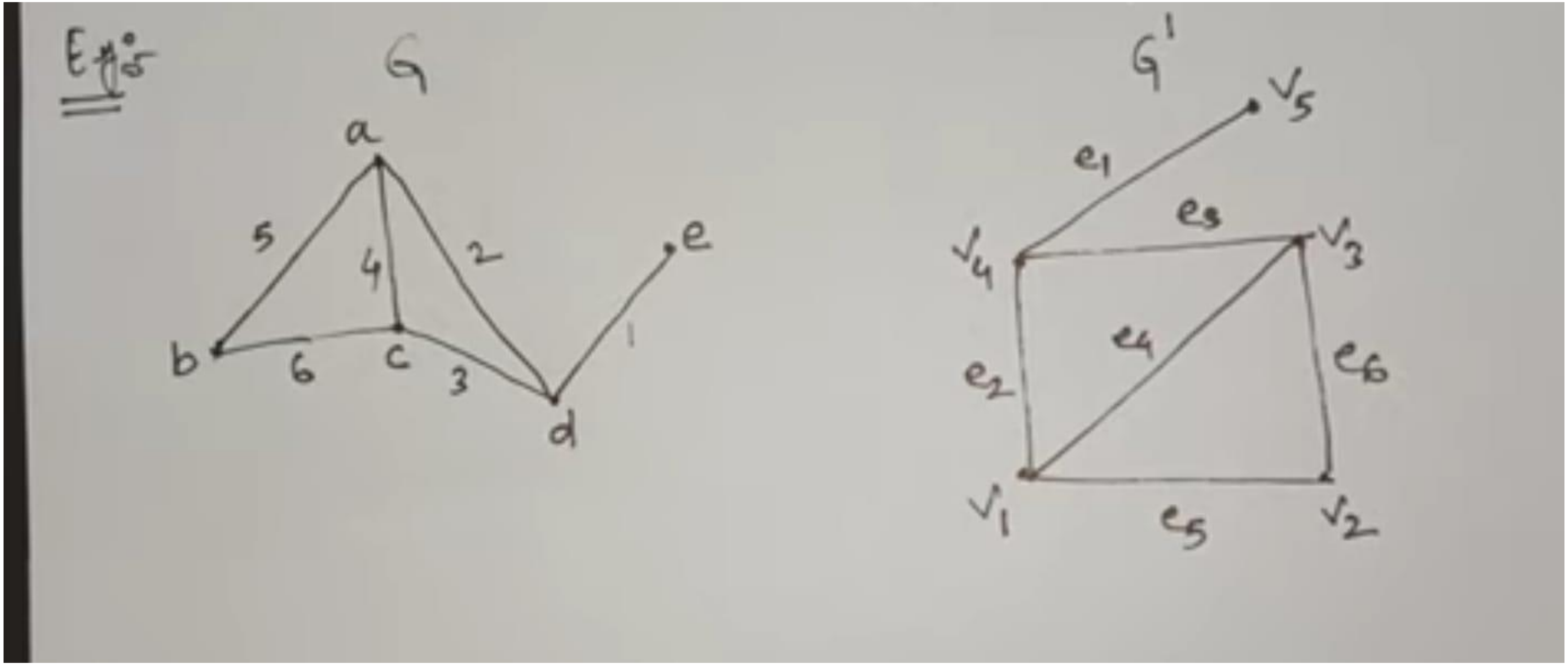
Graph Isomorphism

- Conditions for graph isomorphism

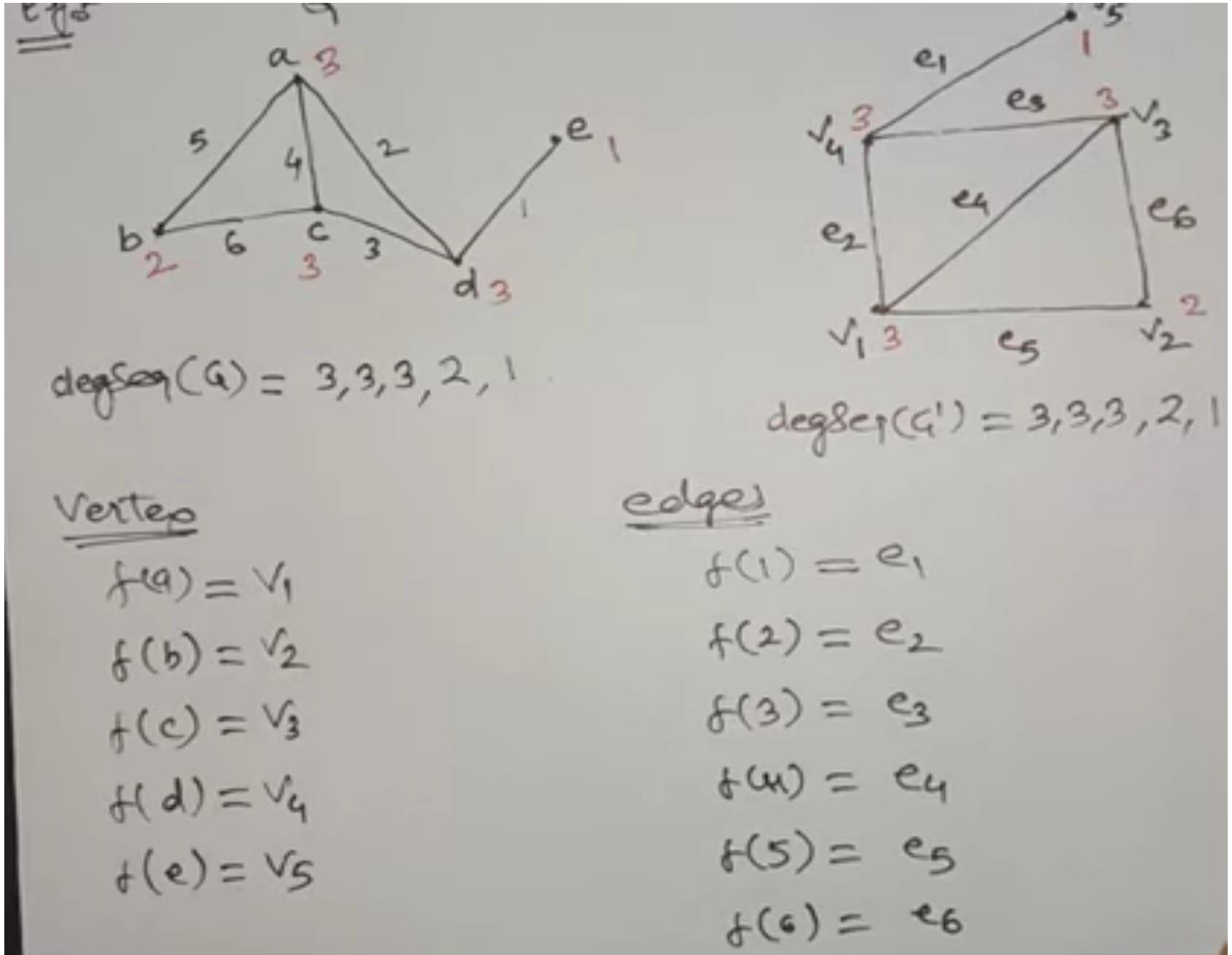
1. There will be an equal number of vertices in the given graphs.
2. There will be an equal number of edges in the given graphs.
3. There will be an equal amount of degree sequence in the given graphs.
4. If the first graph is forming a cycle of length k with the help of vertices {v1, v2, v3, …. vk}, then another graph must also form the same cycle of the same length k with the help of vertices {v1, v2, v3, …. vk}.

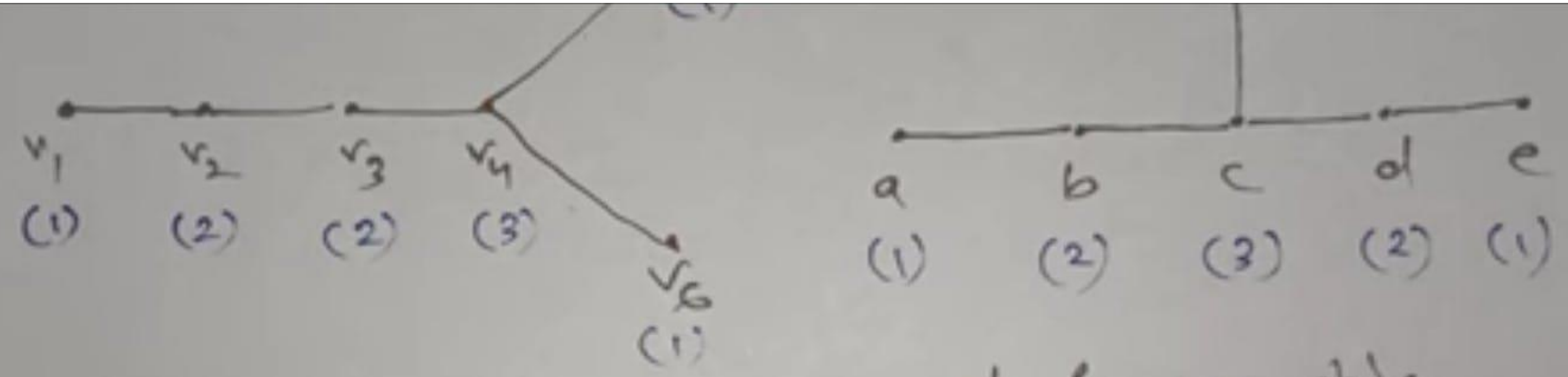
- ***The Degree sequence of a graph can be described as a sequence of degree of all the vertices in ascending order.***

Example



Example


a 3
b 2
c 3
d 3
e 1
5
4
2
6
3
1
e1
e3
e4
e2
e6
e5
v4 3
v3 3
v1 3
v2 2
v5 1
degSeq(G) = 3,3,3,2,1
degSeq(G') = 3,3,3,2,1
Vertex
f(a) = v1
f(b) = v2
f(c) = v3
f(d) = v4
f(e) = v5
edges
f(1) = e1
f(2) = e2
f(3) = e3
f(4) = e4
f(5) = e5
f(6) = e6



These graphs are not isomorphic.

because V_4 is adjacent to two pendent vertex is not preserved.

① Same no. of vertices

② Same no. of edges

③ same degree sequence

$G : 3,2,2,1,1,1$ ✓

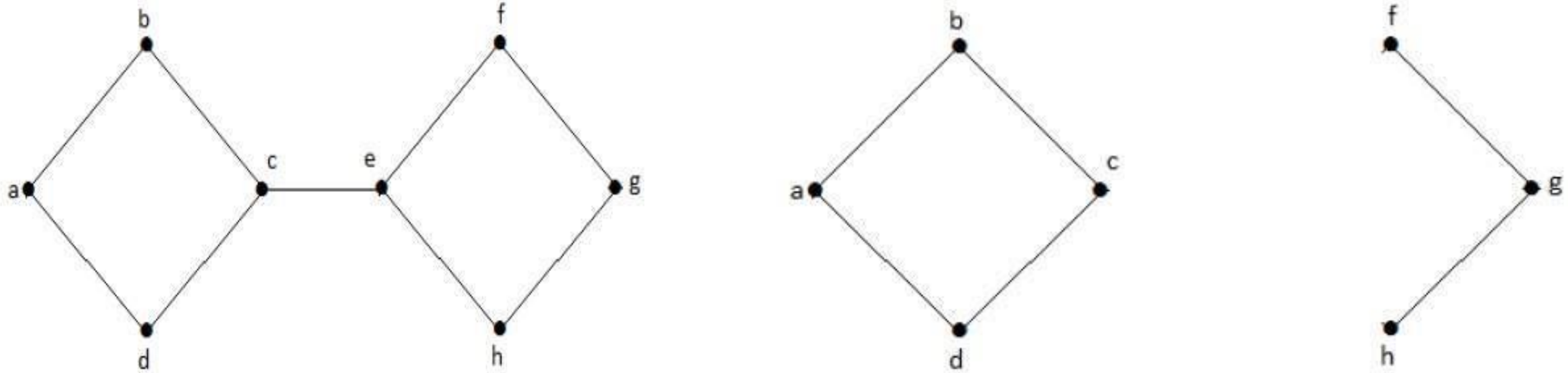
$G' : 3,2,2,1,1,1$ ✓

Connectivity

- From every vertex to any other vertex, there should be some path to traverse
- Connectivity defines whether a graph is connected or disconnected
- Subtopics : Edge connectivity and Vertex connectivity
- **Cut Vertex:**
- Let ‘G’ be a connected graph. A vertex $V \in G$ is called a cut vertex of ‘G’, if ‘G-V’ (Delete ‘V’ from ‘G’) results in a disconnected graph. Removing a cut vertex from a graph breaks it in to two or more graphs.
- A connected graph ‘G’ may have at most (n–2) cut vertices.

Connectivity

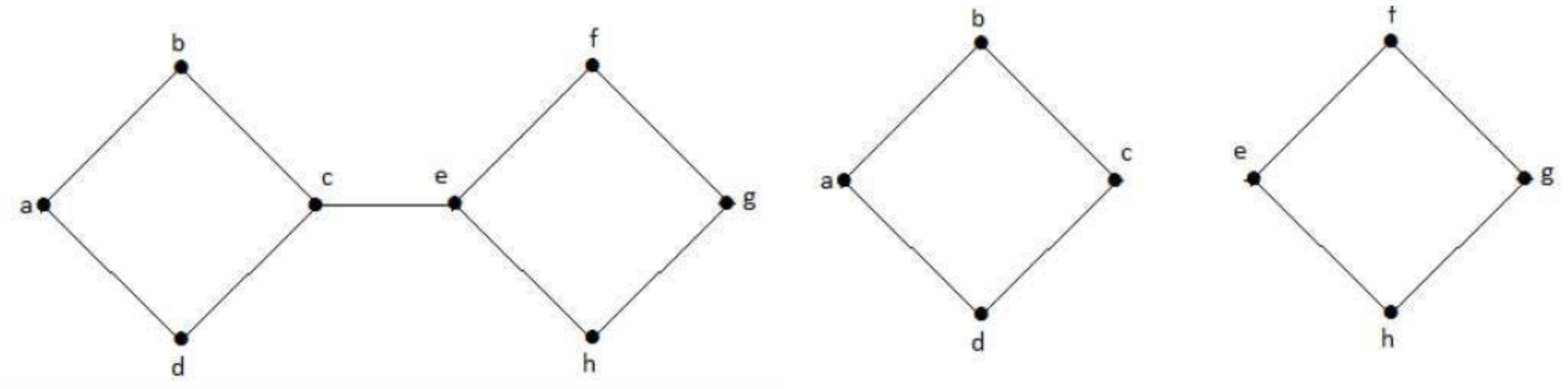
Cut Vertex(Example)



vertices ‘e’ and ‘c’ are the cut vertices.

Connectivity :Cut Edge (Bridge)

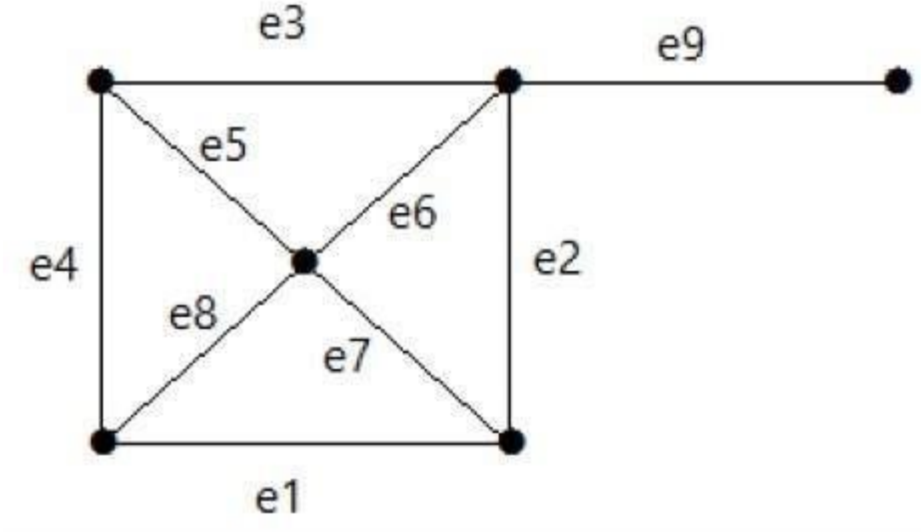
- Let ‘G’ be a connected graph. An edge ‘e’ ∈ G is called a cut edge if ‘G-e’ results in a disconnected graph.
- If removing an edge in a graph results in to two or more graphs, then that edge is called a Cut Edge.

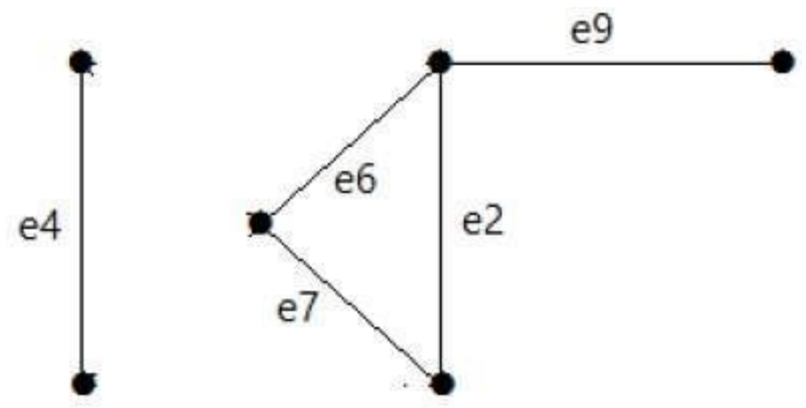


the cut edge is [(c, e)]

Connectivity :Cut Set

- Let ‘G’= (V, E) be a connected graph. A subset E’ of E is called a cut set of G if deletion of all the edges of E’ from G makes G disconnect.
- If deleting a certain number of edges from a graph makes it disconnected, then those deleted edges are called the cut set of the graph.

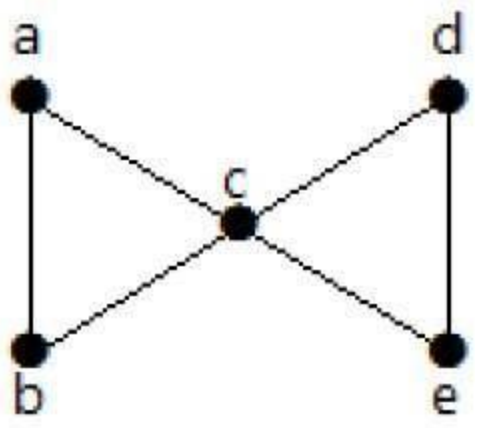


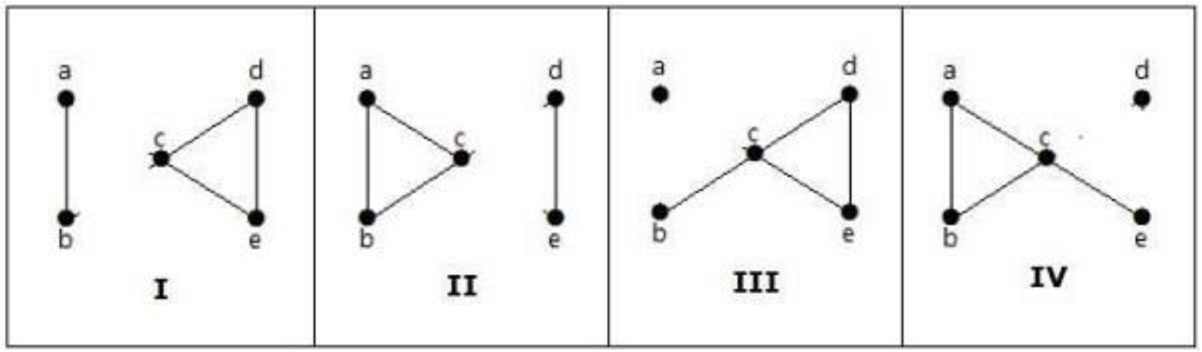


cut set is E1 = {e1, e3, e5, e8}.

Edge Connectivity

- Let ‘G’ be a connected graph. The minimum number of edges whose removal makes ‘G’ disconnected is called edge connectivity of G.
- **Notation** – λ(G)

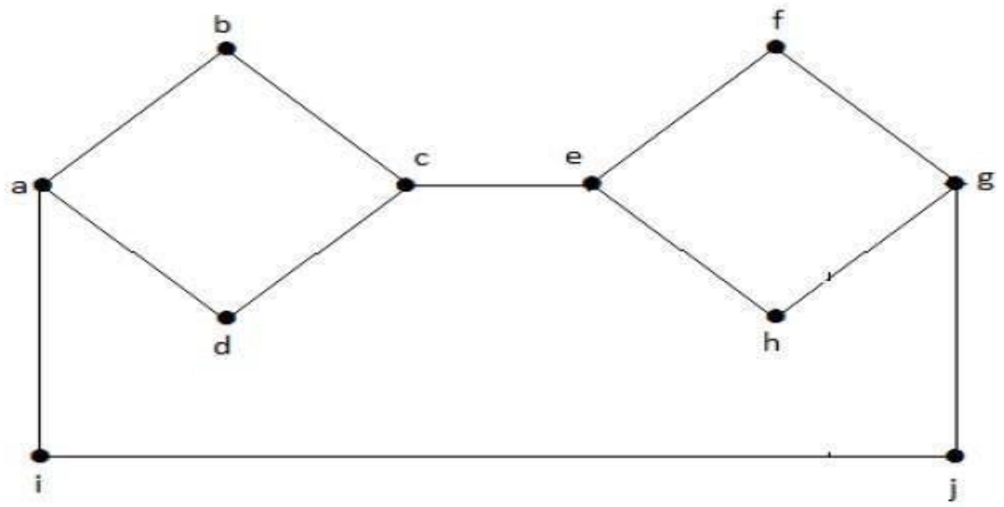




λ(G)) is 2

Vertex Connectivity

- Let 'G' be a connected graph. The minimum number of vertices whose removal makes 'G' either disconnected or reduces 'G' in to a trivial graph is called its vertex connectivity.
- **Notation** − K(G)



If G has a cut vertex, then K(G) = 1.

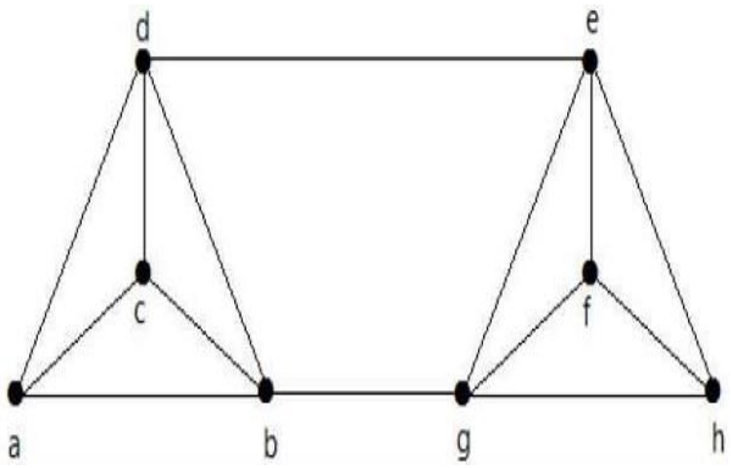
Notation − For any connected graph G,

K(G) ≤ λ(G) ≤ δ(G)

Vertex connectivity (K(G)), edge connectivity (λ(G)), minimum number of degrees of G(δ(G)).

Vertex Connectivity

- Calculate λ(G) and K(G) for the following graph



From the graph,

$\delta(G) = 3$

$K(G) \leq \lambda(G) \leq \delta(G) = 3$ (1)

$K(G) \geq 2$ (2)

Deleting the edges {d, e} and {b, h}, we can disconnect G.

Therefore,

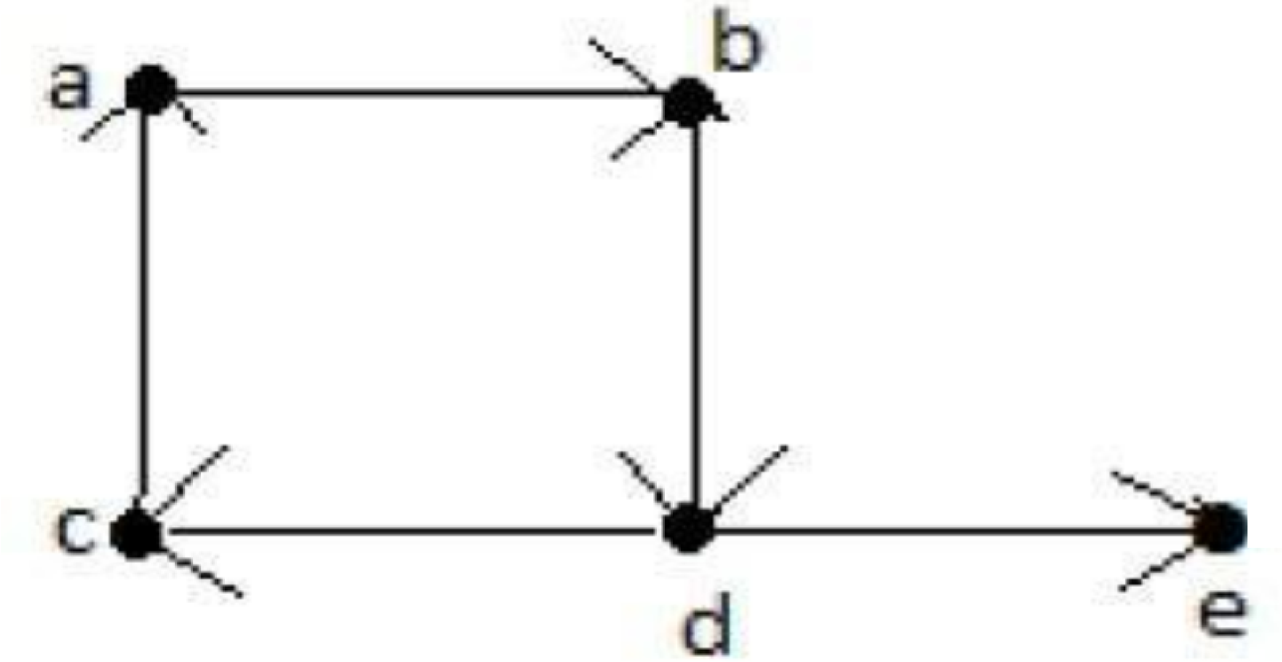
$\lambda(G) = 2$

$2 \leq \lambda(G) \leq \delta(G) = 2$ (3)

From (2) and (3), vertex connectivity $K(G) = 2$

Euler and Hamilton Paths

- Euler’s path contains each edge of ‘G’ exactly once and each vertex of ‘G’ at least once.
- A connected graph G is said to be traversable if it contains an Euler’s path.

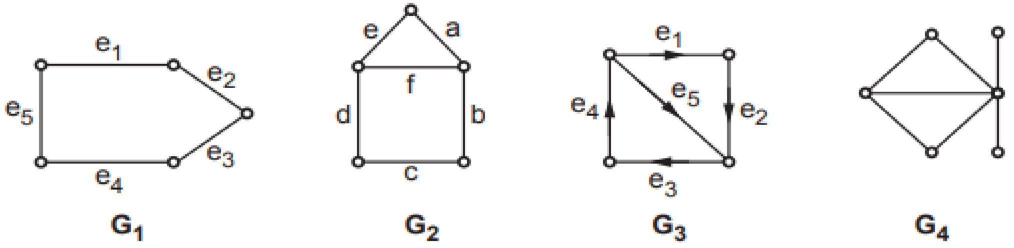


Euler’s Path = d-c-a-b-d-e.

Euler's Circuit

A circuit of a graph which contains every edge of graph exactly once is called the Eulerian circuit.

A graph which has an Eulerian circuit is called as Eulerian graph.



Eulerian circuit $e_2 - e_3 - e_4 - e_5 - e_1$

Eulerian graph.

In graph G_2, Eulerian circuit does not exist.

$\therefore G_2$ is not Eulerian graph.

In graph G_3, Eulerian path is $e_1 \rightarrow e_2 \rightarrow e_3 \rightarrow e_4 \rightarrow e_4 \rightarrow e_5$ but G_3 does not have any Eulerian circuit.

$\therefore G_3$ is not Eulerian graph.

G_4 is also not an Eulerian graph.

Euler’s Paths and Circuit

The existence of Eulerian paths and circuits in a graph depends upon the degree of vertices.

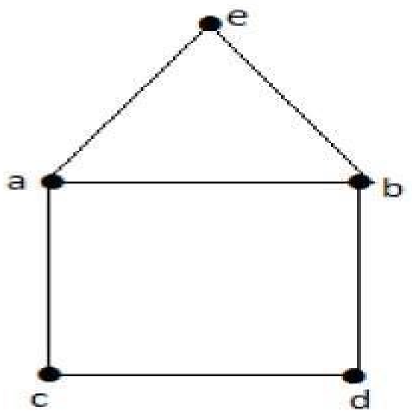
Theorem 1 : An undirected graph possesses an Eulerian path iff it is connected and has either zero or two vertices of odd degree.

Theorem 2 : An undirected graph possesses an Eulerian circuit iff it is connected and its vertices are all of even degree.

Theorem 3 : A directed graph possesses an Eulerian circuit iff it is connected and incoming degree of every vertex is equal to its outgoing degree.

Euler’s Circuit Theorem(Example)

- A connected graph ‘G’ is traversable if and only if the number of vertices with odd degree in G is exactly 2 or 0.
- A connected graph G can contain an Euler’s path, but not an Euler’s circuit, if it has exactly two vertices with an egree.



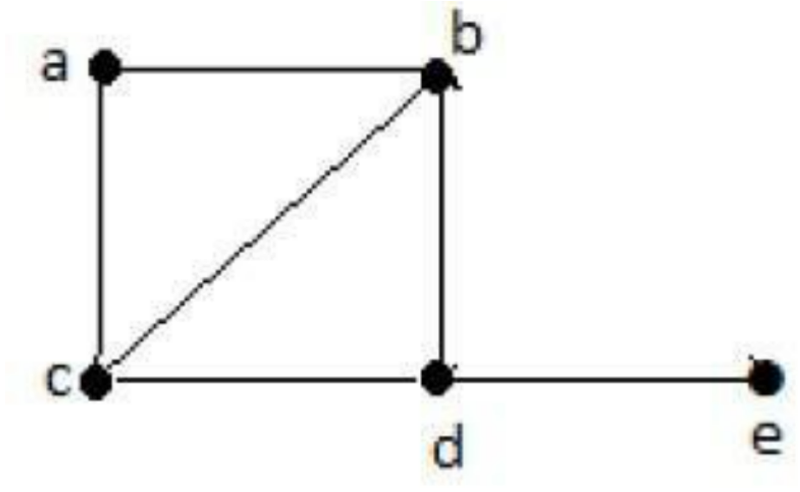
Euler’s Path – b-e-a-b-d-c-a is not an Euler’s circuit, but it is an Euler’s path. Clearly it has exactly 2 odd degree vertices.

Euler’s Paths and Circuit

- **Conditions for Euler Paths and Circuits**
- **Euler Path**: A connected graph has an Euler path if and only if it has exactly zero or two vertices of odd degree.
- **Euler Circuit**: A connected graph has an Euler circuit if and only if every vertex has an even degree.

Hamiltonian Path

- A connected graph is said to be Hamiltonian if it contains each vertex of G exactly once. Such a path is called a **Hamiltonian path**.



Hamiltonian Path – e-d-b-a-c.

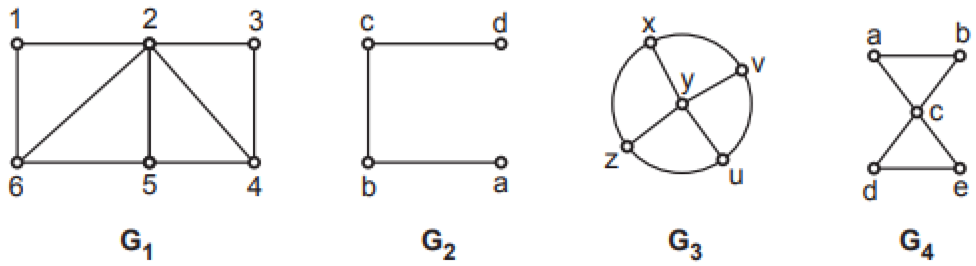
- Euler's circuit contains each edge of the graph exactly once.
- In a Hamiltonian cycle, some edges of the graph can be skipped.

Hamiltonian Path and Circuit

A path in a connected graph G is called a Hamiltonian path if it contains every vertex of G exactly once.

A circuit in a connected graph G is called a Hamiltonian circuit if it contains every vertex of G exactly once.

A graph which has a Hamiltonian circuit is called a Hamiltonian Graph.



In graph G_1, Hamiltonian circuit is 1-2-3-4-5-6-1

$\therefore G_1$ is a Hamiltonian graph.

In graph G_2, Hamiltonian path is a-b-c-d but Hamiltonian circuit does not exist.

$\therefore G_2$ is not Hamiltonian graph.

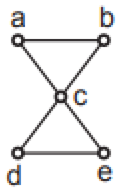
In graph G_3, Hamiltonian circuit is x-y-z-u-v-x

$\therefore G_3$ is Hamiltonian graph but it is not Eulerian.

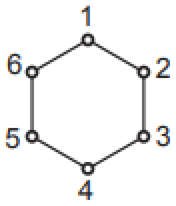
In graph G_4, Hamiltonian path is a-b-c-d-e but Hamiltonian circuit does not exist. $\therefore G_4$ is not Hamiltonian but Eulerian graph.

Hamiltonian Path and Circuit

Theorem 1 : Let G be a simple connected graph on n vertices. If the sum of the degree of each pair of vertices in G is (n – 1) or large then there exists a Hamiltonian path in G.



G_1



G_2

In graph G_1, n = 5, degree sum of every pair of vertices is 4 or more. Hence ∃ a Hamiltonian path in G_1 which is a-b-c-d-e.

In graph G_2 n = 6, degree of every pair of vertices is 4 which is not equal to n – 1 = 5.

Theorem 2 : If G (v, E) is a simple connected graph on n vertices and d (v) = $\frac{n}{2}$; ∀ v ∈ V then G will contain a Hamiltonian circuit.

Theorem 3 : Let G (v, E) be a connected simple graph. If G has a Hamiltonian circuit then for every proper non empty subsets of v, the components in the graph G-S is less than or equal to the number of vertices in S.

Theorem 4 : A Hamiltonian graph contains no cut vertices and hence is 2-connected.

The handshaking lemma

- The Handshaking Theorem(Sum of degree theorem)
- A fundamental principle in graph theory, states that in any undirected graph, the sum of the degrees of all vertices is equal to twice the number of edges in the graph. Symbolically, for a graph with ‘n vertices and ‘E edges:

$$\sum(deg(v)) = 2 \times E \qquad\qquad \sum_{i=1}^{n} d(v_i) = 2 \text{ x } |E|$$

-∑(deg(v)) -represents the sum of the degrees of all vertices in the graph

-deg(v)represents the degree of a vertex ‘v’

-E-represents the edges

Handshaking Theorem

Conclusions in the handshaking theorem

- There must be even numbers for the sum of degree of all the vertices.
- If there are odd degrees for all the vertices, then the sum of degree of these vertices must always remain even.
- If there are some vertices that have an odd degree, then the number of these vertices will be even.

Handshaking Theorem Proof

Step 1: Definitions

Consider an undirected graph with 'n' vertices and 'E' edges. Let's denote the degrees of the vertices as $deg(v_1), deg(v_2), \ldots, deg(v_n)$.

Step 2: Counting Edges

Each edge contributes to the degree of two vertices: one on each end of the edge. Therefore, the total sum of degrees is twice the number of edges:

$$\sum(deg(v)) = 2 \times E$$

Step 3: Calculating Total Degrees

On the left-hand side of the equation, we have the sum of the degrees of all vertices in the graph. Mathematically, this is expressed as:

$$\sum(deg(v)) = deg(v_1) + deg(v_2) + \ldots + deg(v_n)$$

Step 4: Relation to Number of Edges

We've established that the total sum of degrees is twice the number of edges. Therefore, we can substitute $2 \times E$ for $\sum(deg(v))$:

$$2 \times E = deg(v_1) + deg(v_2) + \ldots + deg(v_n)$$

Step 5: Conclusion

This equation demonstrates that the sum of the degrees of all vertices is indeed equal to twice the number of edges:

$$\sum(deg(v)) = 2 \times E$$

Applications of the Handshaking Theorem

- **Computer Networking:**

  -To determine the total number of connections in a network.

  -It helps in understanding the overall structure of the network

  -Used to optimize its performance.

- **Social Networks:**

  -To determine the number of connections between individuals.

  -It helps in understanding the overall structure of the social network

- to identify influencers or key individuals in the network.

Examples

- **Find the number of edges in a graph G with 10 vertices, where each vertex has a degree of 3.**

Solution:

Let G = (V, E) be a graph with 10 vertices, where each vertex has a degree of 3. Using the handshake theorem, we have:

sum of the degrees of all vertices in G = 2 * |E|

$3 * 10 = 2 * |E|$

$|E| = (3 * 10) / 2 = 15$

Therefore, the graph has 15 edges.

Examples

- **Find the number of vertices in a graph G with 12 edges, where each vertex has a degree of 2**

Solution:

Let G = (V, E) be a graph with 12 edges, where each vertex has a degree of 2.
Using the handshake theorem, we have:

sum of the degrees of all vertices in G = 2 * |E|

2 * |V| = 2 * 12

|V| = 12 / 2 = 6

Therefore, the graph has 6 vertices.

Examples

- **Find the sum of the degrees of all vertices in a graph G with 7 edges and 5 vertices.**

Solution:

Let G = (V, E) be a graph with 7 edges and 5 vertices. Using the handshake theorem, we have:

sum of the degrees of all vertices in G = $2 * |E|$

sum of the degrees of all vertices in G = $2 * 7 = 14$

Examples

- **Find the number of edges in a graph G with 6 vertices, where one vertex has a degree of 4, two vertices have a degree of 3, and the rest have a degree of 2.**

Solution:

Let G = (V, E) be a graph with 6 vertices, where one vertex has a degree of 4, two vertices have a degree of 3, and the rest have a degree of 2. Using the handshake theorem, we have:

sum of the degrees of all vertices in G = $2 * |E|$

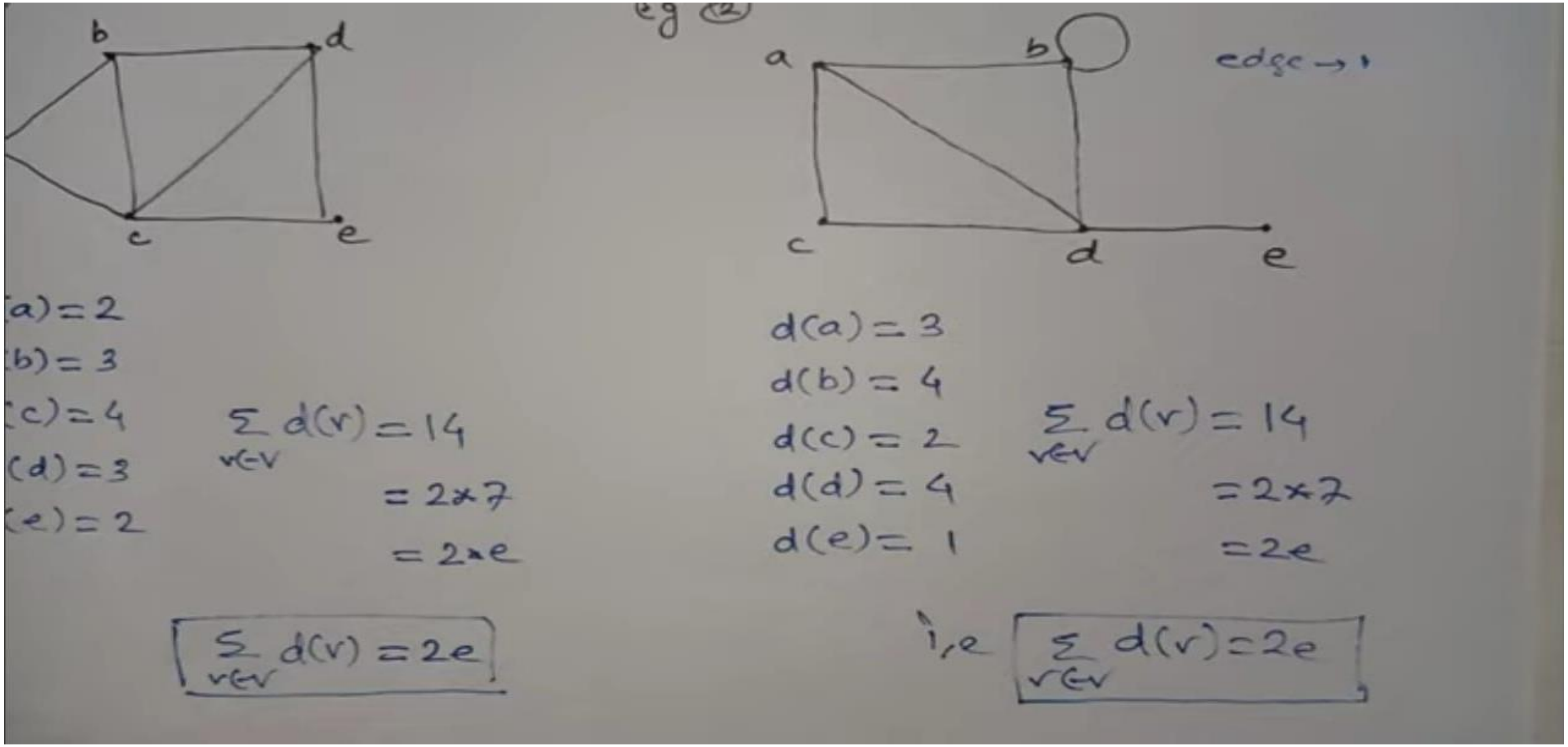
$4 + 3 * 2 + 2 * 3 = 2 * |E|$

$10 = 2 * |E|$

$|E| = 5$

Therefore, the graph has 5 edges.

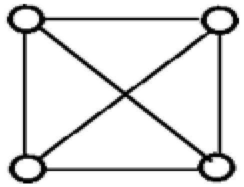
Examples


eg (2)
b
d
c
e
a
edge →
(a) = 2
(b) = 3
(c) = 4
(d) = 3
(e) = 2
Σ d(v) = 14
v∈V
= 2*7
= 2*e
Σ d(v) = 2e
v∈V
d(a) = 3
d(b) = 4
d(c) = 2
d(d) = 4
d(e) = 1
Σ d(v) = 14
v∈V
= 2*7
= 2e
i,e Σ d(v) = 2e
v∈V

Planar Graph

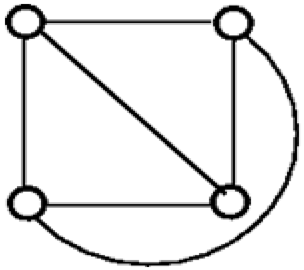
- A **planar graph** is a graph that we can draw in a plane in such a way that no two edges of it cross each other except at a vertex to which they are incident.

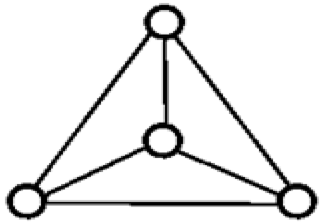
Example

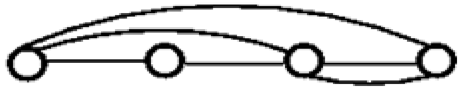


The above graph may not seem to be planar because it has edges crossing each other. But we can redraw the above graph.

The three plane drawings of the above graph are:





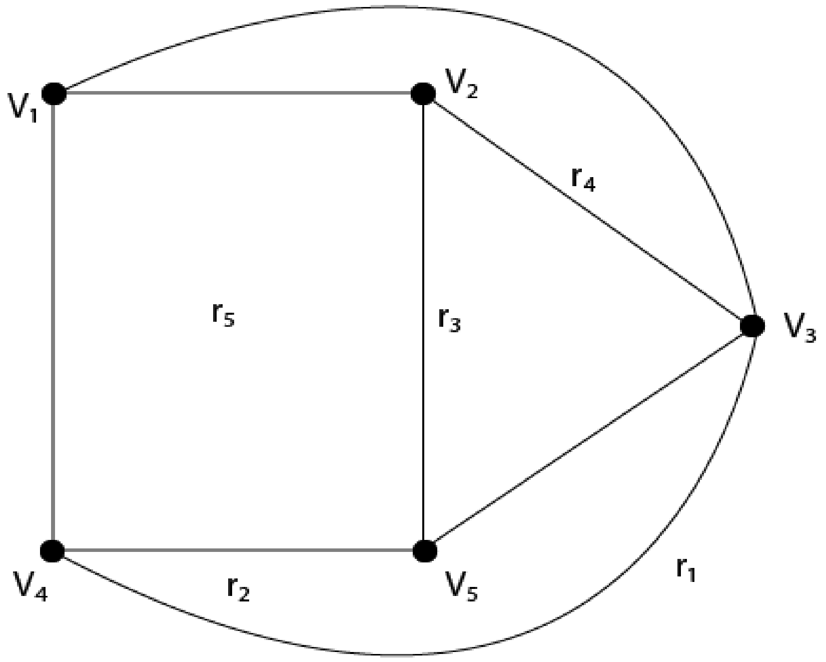


The above three graphs do not consist of two edges crossing each other and therefore, all the above graphs are planar.

Planar Graph

- Region of a Graph: Consider a planar graph G=(V,E).A region is defined to be an area of the plane that is bounded by edges and cannot be further subdivided.
- A planar graph divides the plans into one or more regions.
- One of these regions will be infinite.
- Finite Region: If the area of the region is finite, then that region is called a finite region.
- **Infinite Region:** If the area of the region is infinite, that region is called a infinite region. A planar graph has only one infinite region.

Planar Graph



Regions : r1,r2,r3,r4,r5
Finite regions :r2,r3,r4,r5
Infinite region, i.e., r1

Properties of Planar Graphs:

1. If a connected planar graph G has e edges and r regions, then $r \leq \frac{2}{3} e$.
2. If a connected planar graph G has e edges, v vertices, and r regions, then $v-e+r=2$.
3. If a connected planar graph G has e edges and v vertices, then $3v-e \geq 6$.
4. A complete graph K_n is a planar if and only if $n<5$.
5. A complete bipartite graph K_{mn} is planar if and only if $m<3$ or $n>3$.

Planar Graph

Example: Prove that complete graph K_4 is planar.

The complete graph K_4 contains 4 vertices and 6 edges.

for a connected planar graph $3v\text{-}e \geq 6$
for K_4, we have $3x4\text{-}6=6$ which satisfies the property (3)
Thus K_4 is a planar graph. Hence Proved.

Euler's Formula

Statement : For any connected planar graph G, with v number of vertices, e number of edges and r number of regions

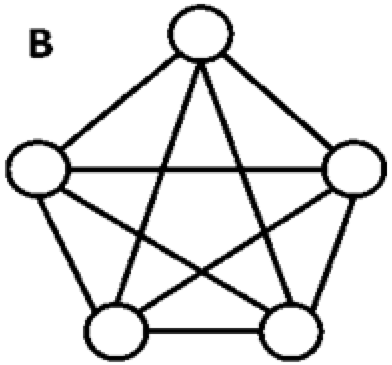
$$\boxed{v - e + r = 2}$$

or $v + r - 2 = e$

Non -Planar Graph

- A graph that is not a planar graph is called a non-planar graph. In other words, a graph that cannot be drawn without at least on pair of its crossing edges is known as non-planar graph.

Example



Properties of Non-Planar Graphs:

Kurathoski's Theorem :A graph is non-planar if and only if it contains a subgraph homeomorphic to K5 or K3,3

Non -Planar Graph

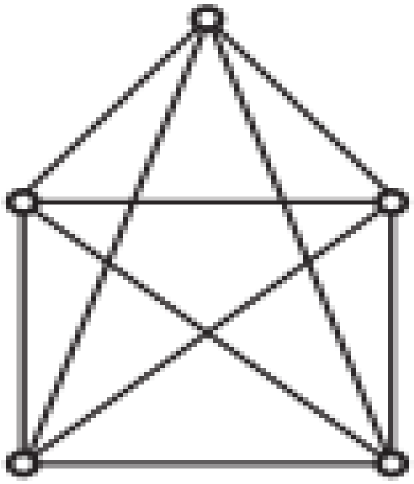
Example1: Show that K_5 is non-planar.

Solution: The complete graph K_5 contains 5 vertices and 10 edges.

for a connected planar graph $3v\text{-}e \geq 6$.

for K_5, we have $3 \times 5\text{-}10=5$ (which does not satisfy property 3 because it must be greater than or equal to 6).

- Thus, K_5 is a non-planar graph.

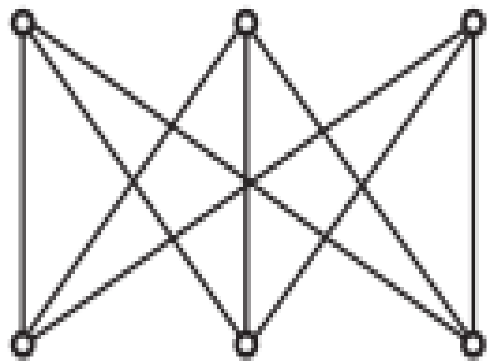


K_5

Non -Planar Graph

Example 2: Show that $K_{3,3}$ is non-planar.

The complete graph $K_{3,3}$ contains 3 vertices and 9 edges.
for a connected planar graph $3v\text{-}e \geq 6$.
for $K_{3,3}$ we have $3 \times 3\text{-}9=0$ (which does not satisfy property 3 because it must be greater than or equal to 6).
Thus, $K_{3,3}$ is a non-planar graph.



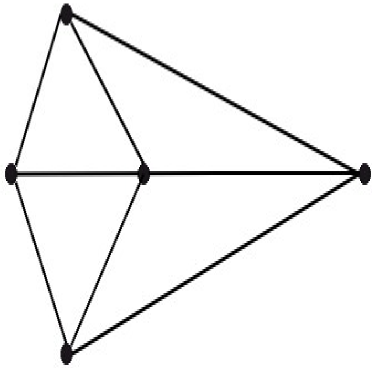
$K_{3,3}$

Graph Coloring

- Suppose that G= (V,E) is a graph with no multiple edges. A vertex coloring of G is an assignment of colors to the vertices of G such that adjacent vertices have different colors. A graph G is M-Colorable if there exists a coloring of G which uses M-Colors.
- Proper Coloring: A coloring is proper if any two adjacent vertices u and v have different colors otherwise it is called improper coloring.

Graph Coloring

- **Example:** Consider the following graph and color C={r, w, b, y}.Color the graph properly using all colors or fewer colors.



The graph shown in fig is a minimum 3-colorable, hence x(G)=3

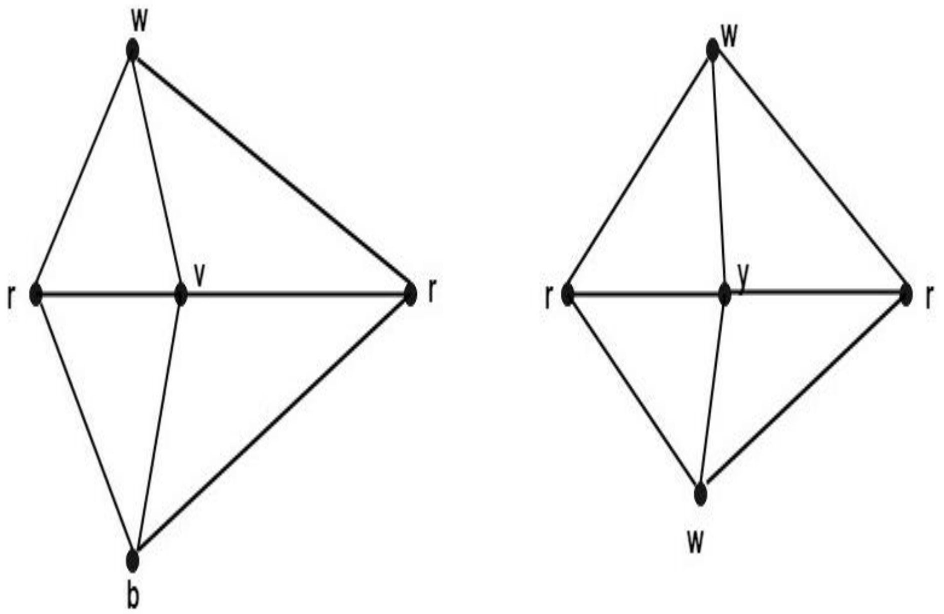


Fig shows the graph properly colored with three colors.

Graph Coloring

- **Chromatic number of G:** The minimum number of colors needed to produce a proper coloring of a graph G is called the chromatic number of G and is denoted by x(G).

A graph G is said to be K-colorable if all vertices of G can be properly colored using at most K different colors. Obviously, a K-colorable graph is K+1 colorable.

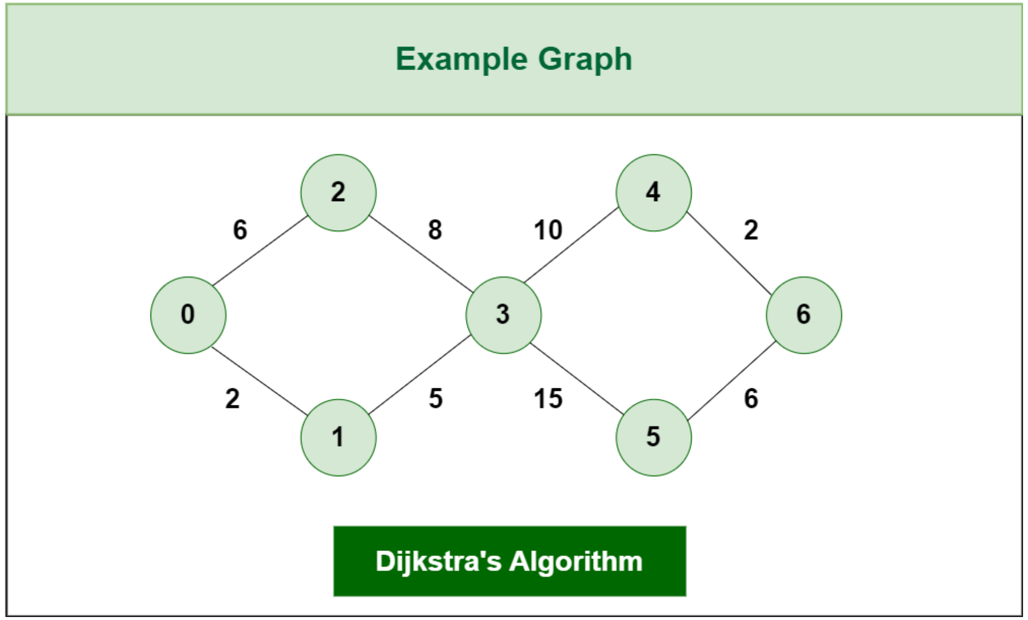
If G is k-colorable then $X(G) \leq K$.

Single source shortest path: **Dijkstra's Algorithm:**

- **Algorithm for Dijkstra's Algorithm:**

1. Mark the source node with a current distance of 0 and the rest with infinity.
2. Set the non-visited node with the smallest current distance as the current node.
3. For each neighbor, N of the current node adds the current distance of the adjacent node with the weight of the edge connecting 0->1. If it is smaller than the current distance of Node, set it as the new current distance of N.
4. Mark the current node 1 as visited.
5. Go to step 2 if there are any nodes are unvisited.

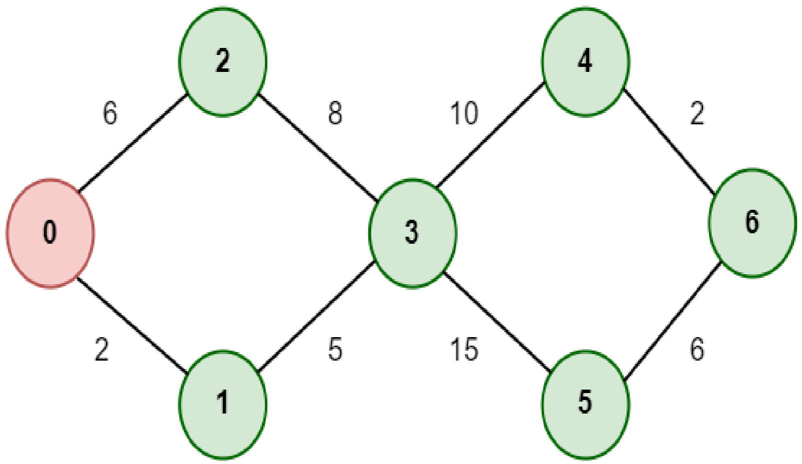
https://youtu.be/Gd92jSu_cZk?si=HWgtcRUSZkCYIiyN



- *The Distance from the source node to itself is 0. In this example the source node is 0.*
- *The distance from the source node to all other node is unknown so we mark all of them as infinity.*

Example: 0 -> 0, 1-> ∞,2-> ∞,3-> ∞,4-> ∞,5-> ∞,6-> ∞.

STEP 1 Start from Node 0 and mark Node 0 as Visited and check for adjacent nodes

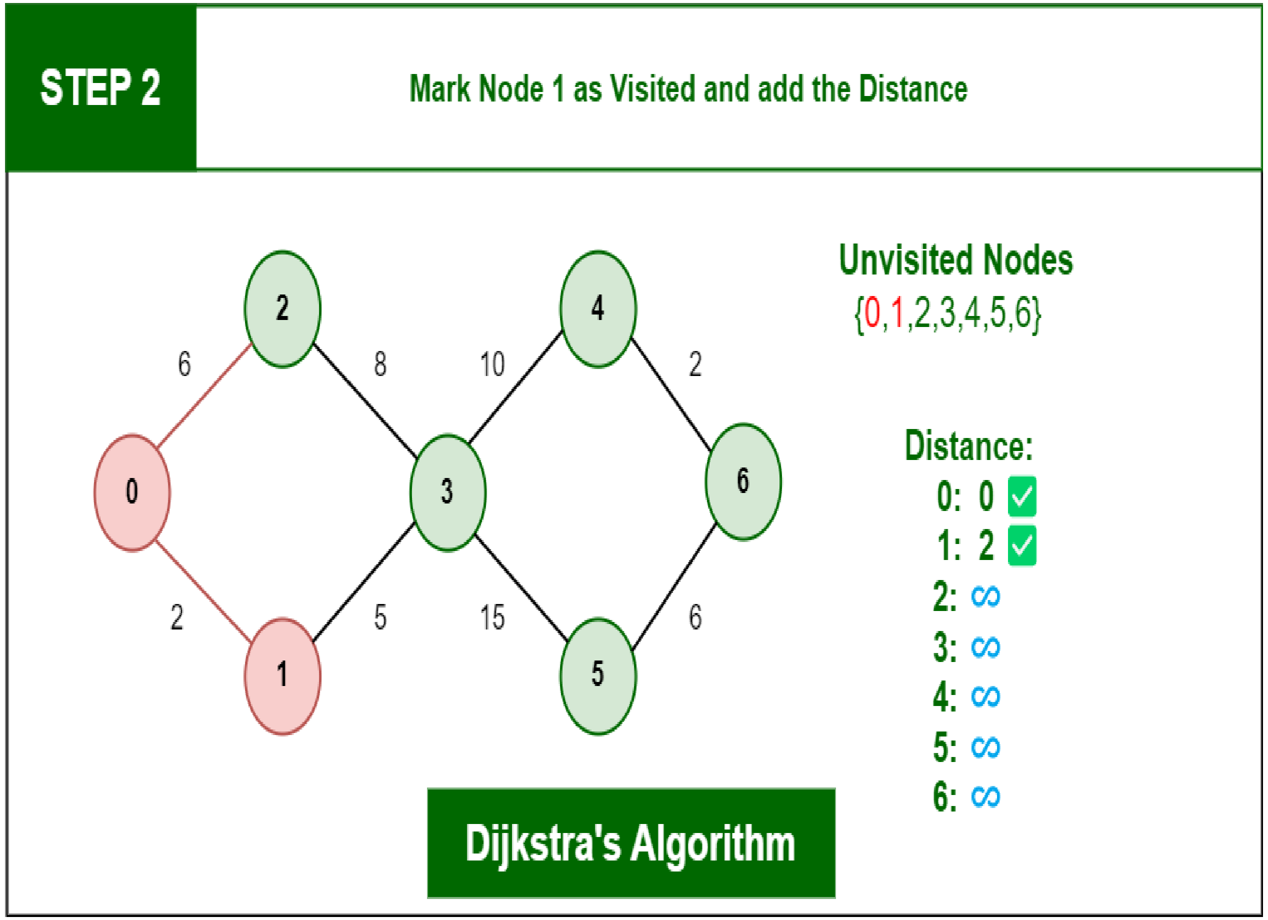


Unvisited Nodes
{0,1,2,3,4,5,6}

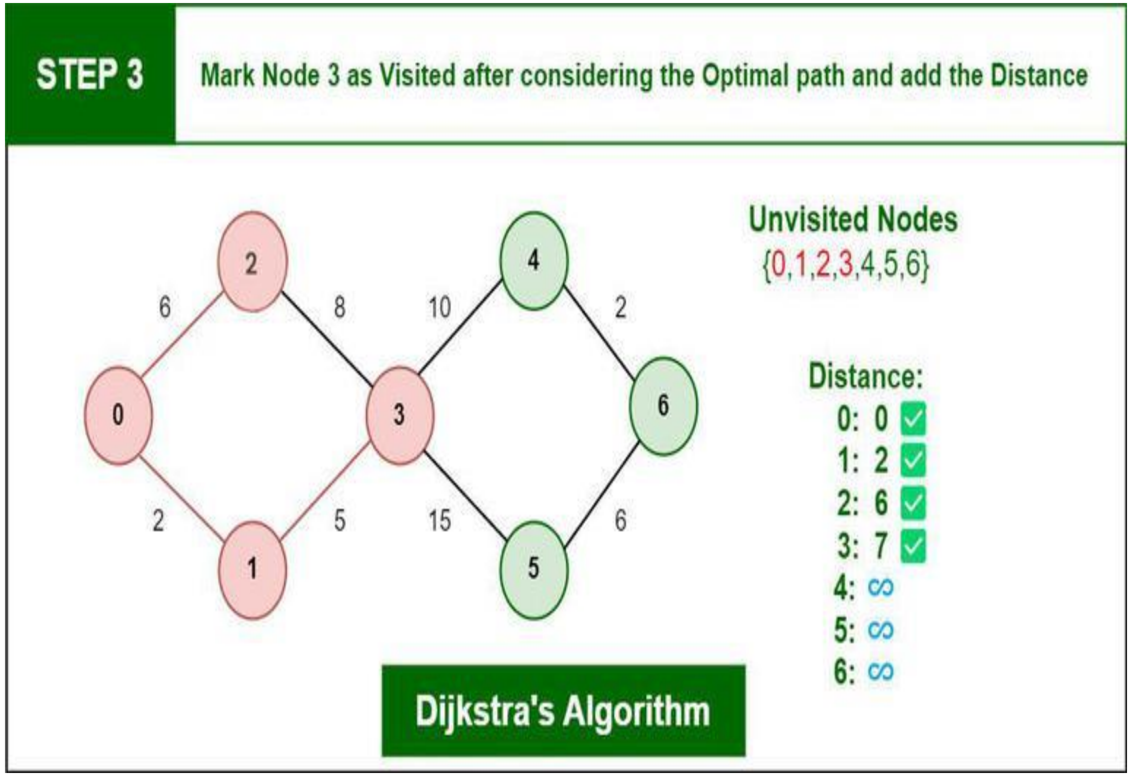
Distance:

0: 0 ✓
1: ∞
2: ∞
3: ∞
4: ∞
5: ∞
6: ∞

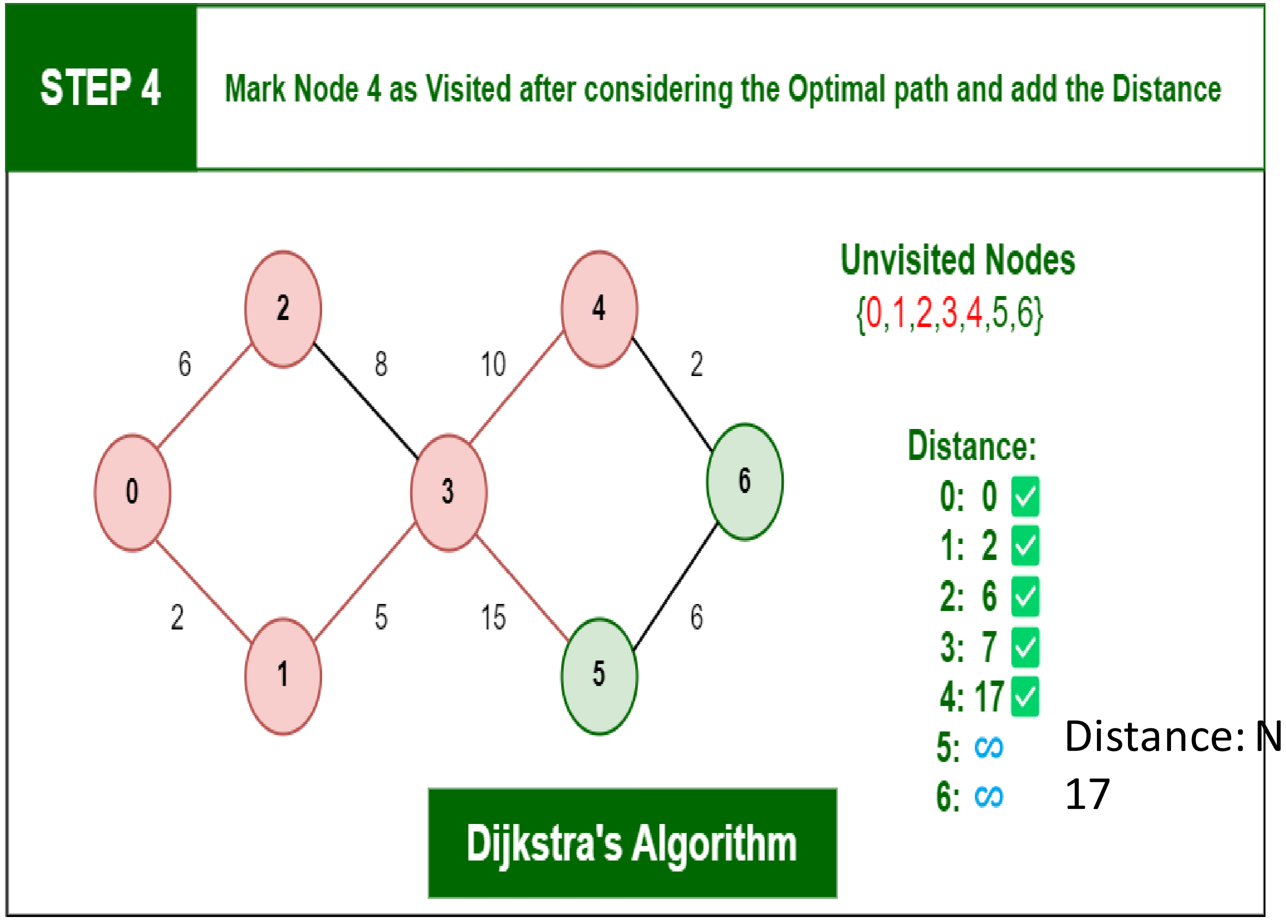
Dijkstra's Algorithm



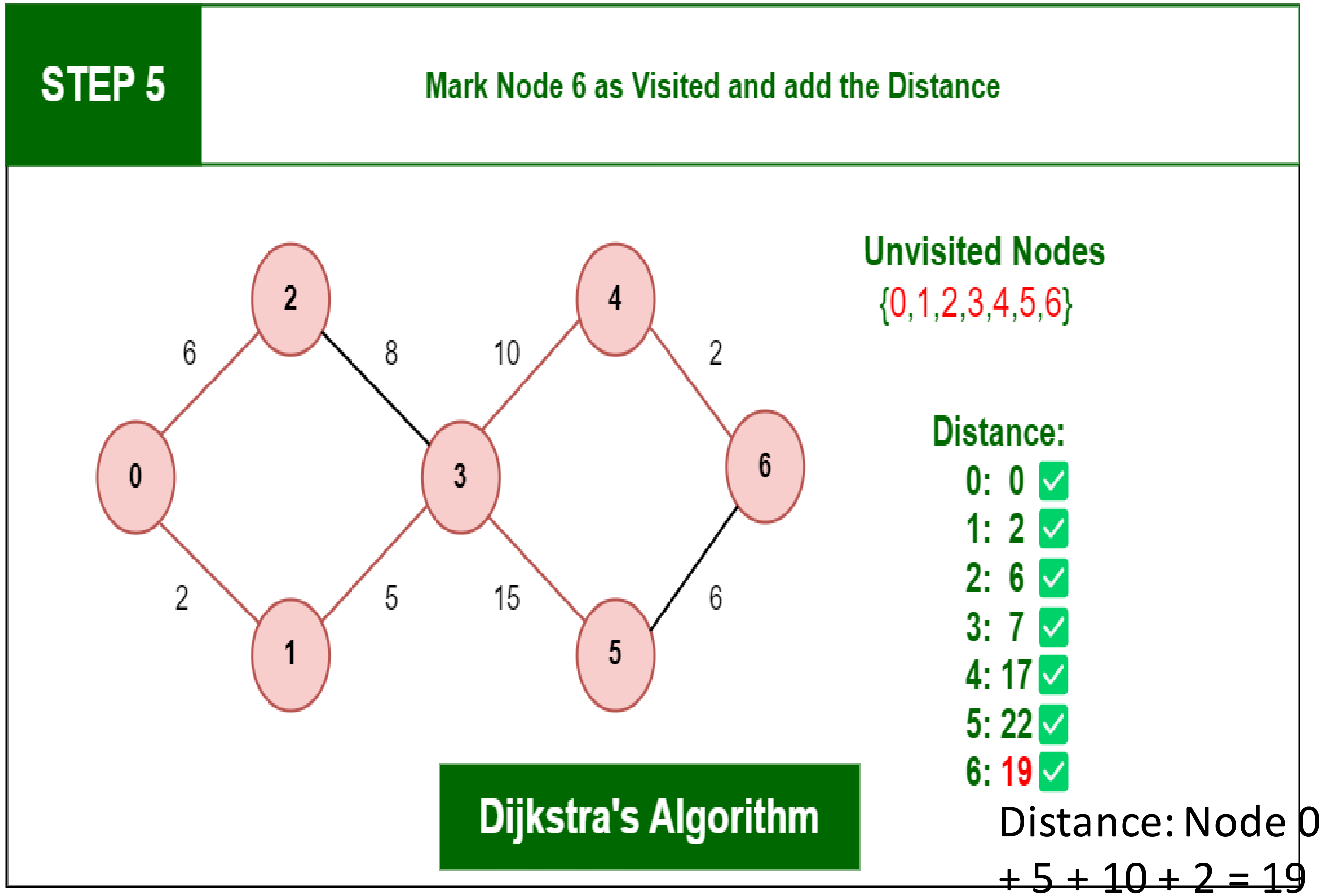
Distance: Node 0 -> Node 1 = 2



Distance: Node 0 -> Node 1 -> Node 3 = 2 + 5 = 7



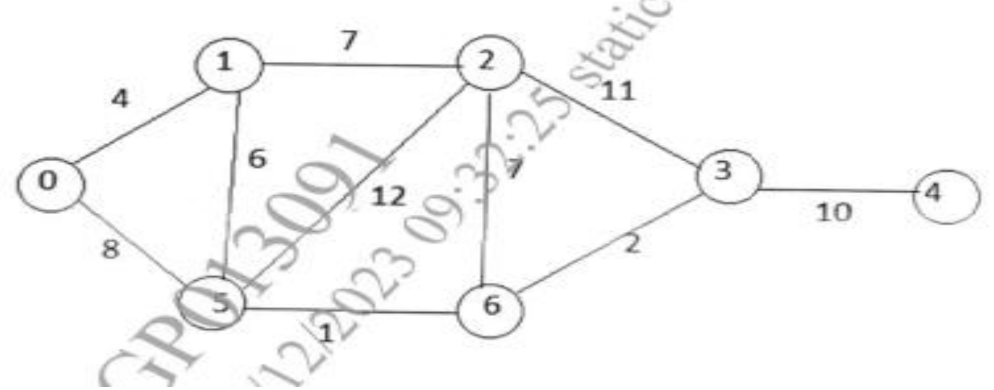
Distance: Node 0 -> Node 1 -> Node 3 -> Node 4 = 2 + 5 + 10 = 17



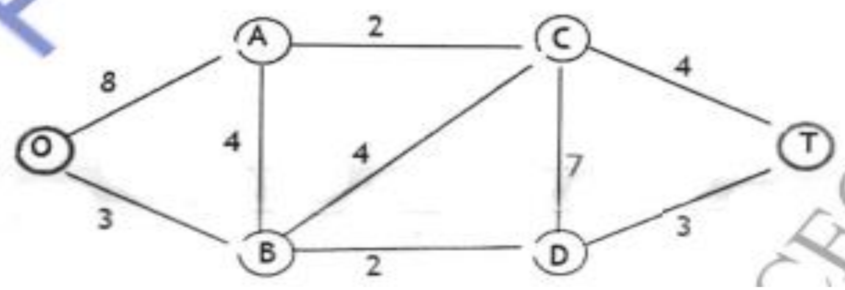
Distance: Node 0 -> Node 1 -> Node 3 -> Node 4 -> Node 6 = 2 + 5 + 10 + 2 = 19

Questions

Find shortest path from vertex ‘0’ to vertex ‘4’ using Dijkstra’s algorithm. [7]



Find shortest path from vertex ‘O’ to Vertex ‘T’ using Dijkstra’s algorithm. [7]



Questions

Find the shortest path between a - z for the given graph using Dijkstra's algorithm **[6]**