Course Name: Discrete Mathematics

Unit 2 – Relations and Functions

Unit II	Relations and Functions	(07 Hours)
Relations and their Properties, n-ary relations and their applications, Representing relations, Closures of relations, Equivalence relations, Partial orderings, Partitions, Hasse diagram, Lattices, Chains and Anti-Chains, Transitive closure and Warshall's algorithm. **Functions-** Surjective, Injective and Bijective functions, Identity function, Partial function, Invertible function, Constant function, Inverse functions and Compositions of functions, The Pigeonhole Principle.		

Unit 2 – Relations and Functions

Session 1

- Introduction
- Cartesian product – Properties
- Relations – Domain/range of relation

Relations and their properties

- Relationships between elements of sets are represented using the structure called a **relation**
- A subset of Cartesian product of the sets
- Example: a student and his/her ID

Binary relation

- The most direct way to express a relationship between elements of two sets is to use **ordered pairs** made up of two related elements
- **Binary relation**: Let A and B be sets. A binary relation from A to B is a subset of $A \times B$
- A binary relation from A to B is a set R of ordered pairs where the 1st element comes from A and the 2nd element comes from B

n-ary relations

- aRb denotes that $(a,b)\in R$
- When (a,b) belongs to R, a is said to be related to b by R
- Likewise, **n-ary relations** express relationships among n elements
- Let $A_1, A_2, \ldots, A_n$ be sets. An n-ary relation of these sets is a subset of $A_1 \times A_2 \times \ldots \times A_n$. The sets $A_1, A_2, \ldots, A_n$ are called the **domains** of the relation, and n is called its **degree**

Examples

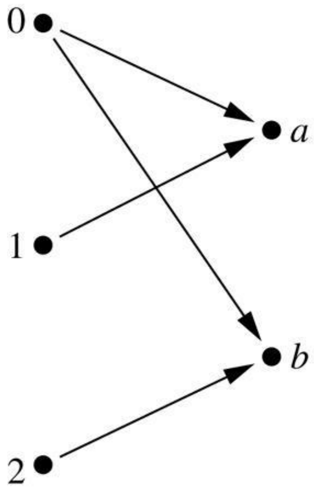
1. Let A be the set of students and B be the set of courses

- Let R be the relation that consists of those pairs (a, b) where $a \in A$ and $b \in B$
- If Jason is enrolled only in CSE20, and John is enrolled in CSE20 and CSE21
- The pairs (Jason, CSE20), (John,CSE20), (John, CSE 21) belong to R
- But (Jason, CSE21) does not belong to R

2. Let A be the set of all cities, and let B be the set of the 50 states in India.

- Define a relation R by specifying (a,b) belongs to R if city a is in state b
- For instance, (Chennai, Tamilnadu), (Indore, Gujrat), (Ahmedabad, Gujrat), (Rajkot, Gujrat), (Pune, Maharashtra), (Nashik, Maharashtra), and (Aurangabad, Maharashtra) are in R

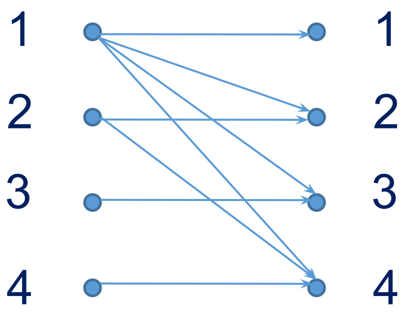
- Let A={0, 1, 2} and B={a, b}. Then {(0, a), (0, b), (1, a), (2, b)} is a relation from A to B
- That is 0Ra but not 1Rb



R	a	b
0	×	×
1	×	
2		×

Relation on a set

- A relation on the set A is a relation from A to A, i.e., a subset of A x A
- Let A be the set {1, 2, 3, 4}. Which ordered pairs are in the relation R={(a,b)|a divides b}?
- R={(1,1),(1,2),(1,3),(1,4),(2,2),(2,4),(3,3),(4,4)}



R	1	2	3	4
1	X	X	X	X
2		X		X
3			X	
4				X

Consider these relations on set of integers

$R_1=\{(a,b)|a\leq b\}$

$R_2=\{(a,b)|a>b\}$

$R_3=\{(a,b)|a=b \text{ or } a=-b\}$

$R_4=\{(a,b)|a=b\}$

$R_5=\{(a,b)|a=b+1\}$

$R_6=\{(a,b)|a+b\leq 3\}$

Which of these relations contain each of the pairs (1,1), (1,2), (2,1), (1, -1) and (2, 2)?

(1,1) is in R_1, R_3, R_4 and R_6; (1,2) is in R_1 and R_6; (2,1) is in R_2, R_5, and R_6; (1,-1) is in R_2, R_3, and R_6; (2,2) is in R_1, R_3, and R_4

- How many relations are there on a set with n elements?
- A relation on a set A is a subset of A x A
- As A x A has n^2 elements, there are 2^{n^2} subsets
- Thus there are 2^{n^2} relations on a set with n elements
- That is, there are $2^{3^2} = 2^9 = 512$ relations on the set {a, b, c}

Cartesian product
If A = {{1, 2}, {3}} and
B = {(a, b),(c, d)},

then A × B = {({1, 2},(a, b)),({1, 2},(c, d)),({3},(a, b)),({3},(c, d))}.

Cartesian Product is not commutative
For the sets A and B one paragraph above,
B × A = {((a, b), {1, 2}),((a, b), {3}),((c, d), {1, 2}),((c, d), {3})}.

This example shows that,
in general, A × B $\neq$ B × A. The underlying reason is that if A and B are non-empty and one set, say A, contains an element x which is not in B, then A × B contains an ordered pair with first component equal to x, but B × A contains no such ordered pair.

The set is A × B is called Universal Relations and $\emptyset$ is the void relation

Operation on relations

Let A, B and C be sets. Then

(a) $A \times (B \cap C) = (A \times B) \cap (A \times C)$;

(b) $A \times (B \cup C) = (A \times B) \cup (A \times C)$;

(c) $(A \cap B) \times C = (A \times C) \cap (B \times C)$;

(d) $(A \cup B) \times C = (A \times C) \cup (B \times C)$.

We prove part (b) and leave the proofs of the remaining parts as an exercise.

We have $(x, y) \in A \times (B \cup C) \Leftrightarrow x \in A$ and $y \in B \cup C \Leftrightarrow (x \in A)$ and $(y \in B$ or $y \in C) \Leftrightarrow [(x \in A)$ and $(y \in B)]$ or $[(x \in A)$ and $(y \in C)]$ (by a distributive law of logic) $\Leftrightarrow [(x, y) \in A \times B]$ or $[(x, y) \in A \times C] \Leftrightarrow (x, y) \in (A \times B) \cup (A \times C)$.

Combining relations

Let A = {1, 2, 3} and B = {1, 2, 3, 4, }. The relations R_1 = {(1,1), (2,2), (3,3)} and
R_2 = {(1,1), (1,2), (1,3), (1,4)} can be combined to obtain:

$R_1 \cup R_2$ = {(1,1), (1,2), (1,3), (1,4), (2,2), (3,3)}
$R_1 \cap R_2$ = {(1,1)}
$R_1 - R_2$ = {(2,2), (3,3)}
$R_2 - R_1$ = {(1,2), (1,3), (1,4)}

Domain and Range

The set which contains all the first elements of all the ordered pairs of relation R is known as the **domain** of the relation.

$D(R) = \{ a \in A \mid \text{for some } b \in B , (a, b)\ b \in R \}$

The set which contains all the second elements, on the other hand, is known as the **range** of the relation.

$Rn(R) = \{ b \in B \mid \text{for some } a \in A , (a, b)\ b \in R \}$

Let us define a set N = {9,10,11,12,13}. Now, let us define a relation A from N to N such that in the ordered pair (x,y) in A , y is two more than x.

This can be represented in different ways:

Set-builder method : $A = \{ (x, y) : y = x + 2, (x , y) \in N \}$

Roster method: A = { (9,11), (10,12), (11,13) }

Domain of A is not N. It is equal to {9,10,11}. And Range of A is {11,12,13}

If A = {2, 4, 6, 8) B = {5, 7, 1, 9}.

Let R be the relation 'is less than' from A to B. Find Domain (R) and Range (R).

Under this relation (R), we have R = {(4, 5); (4, 7); (4, 9); (6, 7); (6, 9), (8, 9) (2, 5) (2, 7) (2, 9)}

Therefore, Domain (R) = {2, 4, 6, 8} and Range (R) = {1, 5, 7, 9}

End of Session 1

Unit 2 – Relations and Functions

Session 2

- Representation –
  - Matrix,
  - Directed graph,
  - Tabular form,
  - Arrow diagram
  - Examples

Representation –

matrix, directed graph, tabular form, arrow diagram & examples

① Matrix

For a relation R, a matrix is formed called as relation matrix and is denoted by M_R

let, $A = \{a_1, a_2, a_3, \cdots a_n\}$
$B = \{b_1, b_2, b_3, \cdots b_n\}$
and $R \subseteq A \times B$ then,
relation matrix $M_R = [M_{ij}]$ can be defined as

$$M_R = [m_{ij}] = \begin{cases} 0 \text{ if } (a_i, b_j) \notin R \\ 1 \text{ if } (a_i, b_j) \in R \end{cases}$$

Ex:

let $A = \{1, 2, 3, 4\}$, $B = \{a, b, c, d\}$

and relation $R = \{(1, a), (1, d), (2, b), (3, c), (4, d)\}$

then,

Relation Matrix:

$$M_R = [m_{ij}]_{4\times 4} = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{cccc} a & b & c & d \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \end{array}$$

A represents rows

B —"— Columns

(2) Directed Graph:

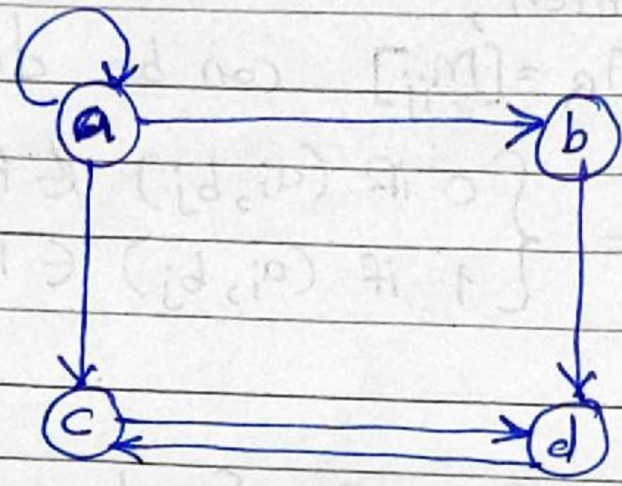
This representation is used only when a finite set relates with itself

Ex:

$A = \{a, b, c, d\}$

R is relation from A to A i.e. $R: A \to A$

$R = \{(a,a), (a,c), (b,d), (c,d), (d,a), (d,c), (a,b)\}$



③ Tabular form:

Ex: Let, A and B are finite sets & R is the relation from A to B.

$A = \{1, 2, 3, 4\}$, $B = \{a, b, c, d\}$

$R = \{(1,c), (2,b), (3,d), (4,b), (4,d)\}$

	a	b	c	d
1			✓	
2		✓		
3				✓
4		✓		✓

row = elements in A

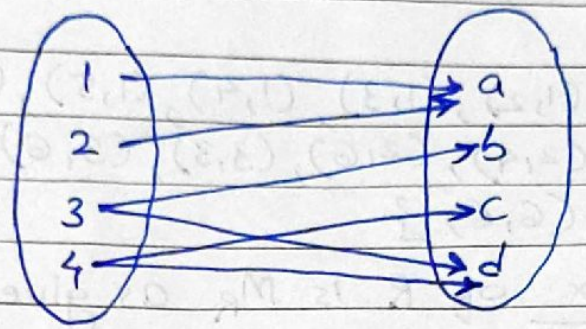
Column = elements in B

Date / /

④ Arrow Diagram Representation

Ex: $A = \{1, 2, 3, 4\}$, $B = \{a, b, c, d\}$

$R = \{(1, a), (2, a), (3, b), (3, d), (4, c), (4, d)\}$



Two ellipses $\underline{\underline{A}}$ $\underline{\underline{B}}$

Ex. Assignments

① $A = \{1, 2, 3, 4, 5, 6\}$ and
$R = \{(a,b) \mid a \text{ divides } b\}$
List all the ordered pairs and find relation matrix of R.

Soln

$R = \{(1,1), (1,2), (1,3), (1,4), (1,5), (1,6),$
$(2,2), (2,4), (2,6), (3,3), (3,6), (4,4),$
$(5,5), (6,6)\}$

Relation Matrix of R is M_R as given below

$$
\begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{array}
\begin{array}{c}
\begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\
\begin{bmatrix}
1 & 1 & 1 & 1 & 1 & 1 \\
0 & 1 & 0 & 1 & 0 & 1 \\
0 & 0 & 1 & 0 & 0 & 1 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix}
\end{array}
$$

Ex. 2 If $A = \{a, b, c\}$, $B = \{b, c\}$ then find $(A \times B) \cap (B \times A)$

Sol^n

$A \times B = \{(a,b), (a,c), (b,b), (b,c), (c,b), (c,c)\}$

$B \times A = \{(b,a), (b,b), (b,c), (c,a), (c,b), (c,c)\}$

$(A \times B) \cap (B \times A) = \{(b,b), (b,c), (c,b), (c,c)\}$

Ex. If $A = \{2, 3, 4, 6, 8\}$

$B = \{2, 3, 4, 6, 8\}$

xRy iff $x+y \leq 10$. find it's relation matrix

Sol^n

$A \times B = \{(2,2), (2,3), (2,4), (2,6), (2,8), (3,2),$
$(3,3), (3,4), (3,6), (3,8), (4,2), (4,3), (4,6)$
$(4,8), (6,2), (6,3), (6,4), (6,6), (6,8), (8,2),$
$(8,3), (8,4), (8,6), (8,8)\}$

$R = \{(2,2), (2,3), (2,4), (2,6), (2,8), (3,2), (3,3),$
$(3,4), (3,6), (4,2), (4,3), (4,6), (6,2), (6,3)$
$(6,4), (8,2)\}$

$$m_R = \begin{array}{c} \\ 2 \\ 3 \\ 4 \\ 6 \\ 8 \end{array} \begin{array}{c} \begin{array}{ccccc} 2 & 3 & 4 & 6 & 8 \end{array} \\ \begin{bmatrix} 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

Additional Reference Slides

Matrix review

- We will only be dealing with zero-one matrices
  - Each element in the matrix is either a 0 or a 1

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 \end{bmatrix}$$

- These matrices will be used for Boolean operations
  - 1 is true, 0 is false

Matrix transposition

- Given a matrix **M**, the transposition of **M**, denoted $\mathbf{M}^t$, is the matrix obtained by switching the columns and rows of **M**

$$\mathbf{M} = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$$

$$\mathbf{M}^t = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}$$

$$\mathbf{M} = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \\ 13 & 14 & 15 & 16 \end{bmatrix}$$

$$\mathbf{M}^t = \begin{bmatrix} 1 & 5 & 9 & 13 \\ 2 & 6 & 10 & 14 \\ 3 & 7 & 11 & 15 \\ 4 & 8 & 12 & 16 \end{bmatrix}$$

- In a “square” matrix, the main diagonal stays unchanged

Matrix join

- A *join* of two matrices performs a Boolean OR on each relative entry of the matrices
  - Matrices must be the same size
  - Denoted by the or symbol: $\vee$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 \end{bmatrix} \vee \begin{bmatrix} 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}$$

Matrix meet

- A *meet* of two matrices performs a Boolean AND on each relative entry of the matrices
  - Matrices must be the same size
  - Denoted by the or symbol: $\wedge$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 \end{bmatrix} \wedge \begin{bmatrix} 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

Relations using matrices

- List the elements of sets *A* and *B* in a particular order
  - Order doesn't matter, but we'll generally use ascending order
- Create a matrix

$$\mathbf{M}_R = [m_{ij}]$$

$$m_{ij} = \begin{cases} 1 & \text{if } (a_i, b_j) \in R \\ 0 & \text{if } (a_i, b_j) \notin R \end{cases}$$

Relations using matrices

- Consider the relation of who is enrolled in which class
  - Let A = { Alice, Bob, Claire, Dan }
  - Let B = { CS101, CS201, CS202 }
  - R = { (a,b) | person a is enrolled in course b }

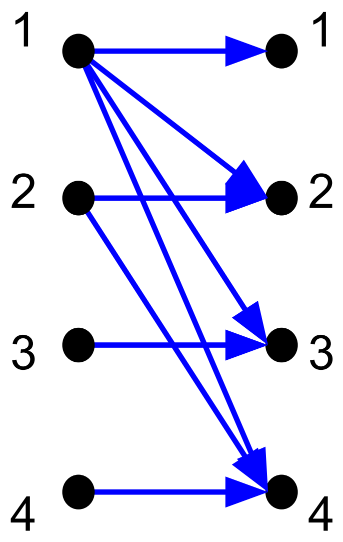
	CS101	CS201	CS202
Alice	X		
Bob		X	X
Claire			
Dan		X	X

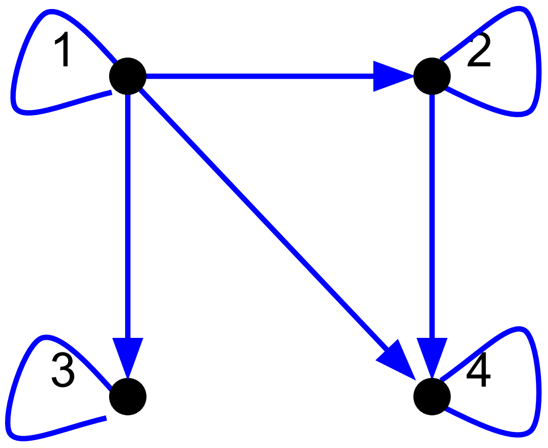
$$\mathbf{M}_R = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

- What is it good for?
  - It is how computers view relations
    - A 2-dimensional array
  - Very easy to view relationship properties
- We will generally consider relations on a single set
  - In other words, the domain and co-domain are the same set
  - And the matrix is square

Representing relations - using directed graphs

- A directed graph consists of:
  - A set *V* of vertices (or nodes)
  - A set *E* of edges (or arcs)
  - If (*a, b*) is in the relation, then there is an arrow from *a* to *b*
- Will generally use relations on a single set
- Consider our relation $R = \{ (a,b) \mid a \text{ divides } b \}$
- Old way:





Unit 2 – Relations and Functions

Session 3

- Fundamental concepts of relation - Inverse, complement, composition & examples
- Properties – Reflexive, irreflexive, symmetric, asymmetric, anti-symmetric, transitive
- Additional References & Examples

Fundamental concepts of relation –

Inverse, complement, composition & examples

① Inverse or Converse $\boxed{R^c}$ $\boxed{R^{-1}}$

Let, R be the relation from A to B i.e $R: A \to B$, the inverse of relation R is denoted by R^{-1} is a relation from B to A.

Ex: $A = \{1, 2, 3, 4\}$; $B = \{a, b, c, d\}$

R is the relation from A to B

$R = \{(1,c), (3,b), (2,d), (4,a), (4,d)\}$

$\therefore$ Converse of relation R is R^c or R^{-1} is

R^C or $R^{-1} = \{ (c,1), (b,3), (d,2), (a,4), (d,4) \}$

I.m.P

1. $(R^c)^C = R$
2. $(R \cup S)^c = R^c \cup S^c$
3. $(R \cap S)^c = R^c \cap S^c$

6. Inverse or converse

Mathematically

R^{-1} or $R^{c} = \{ (y, x) \mid y \in B, x \in A$ and $(x, y) \in R \}$

(2) <u>Complement</u> $\boxed{\overline{R}}$ $\boxed{R'}$

Let, R be the relation from $\underline{A}$ to B. The Complement of R is denoted by $\overline{R}$ or $R' : A \to B$

Mathematically,

$$\overline{R} \text{ or } R' = \{(x,y) \mid (x,y) \notin R \text{ but } x \in A, y \in B\}$$

i.m.p.

$$\boxed{\overline{R} = (A \times B) - R}$$

<u>Ex</u>:

$A = \{1,2,3\}$ $\quad$ $B = \{a,b\}$

$A \times B = \{(1,a), (1,b), (2,a), (2,b), (3,a), (3,b)\}$

$R = \{(2,a), (3,b)\}$

$\overline{R}$ or $R' = (A \times B) - R = \{(1,a), (1,b), (2,b), (3,a)\}$

③ Composition of Relation

Composition of relation formed from already available sequence of relations.

Mathematically,

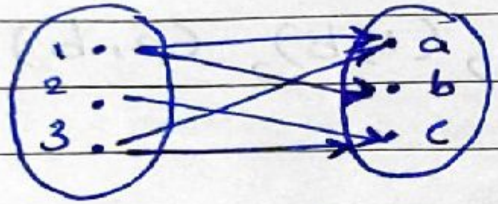
$R \circ S = \{ (a,c) \mid aRb,\ bSc$ i.e. $(a,b) \in R$ and $(b,c) \in S$ for $a \in A,\ c \in C \}$

Ex: $A = \{1,2,3\}$, $B = \{a,b,c\}$, $C = \{p,q,r\}$

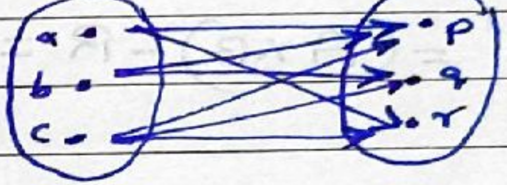
Let, $R : A \to B$, $S : B \to C$

$R : \{(1,a), (1,b), (2,c), (3,a), (3,c)\}$

$S : \{(a,p), (a,r), (b,p), (b,q), (c,q), (c,p), (c,r)\}$

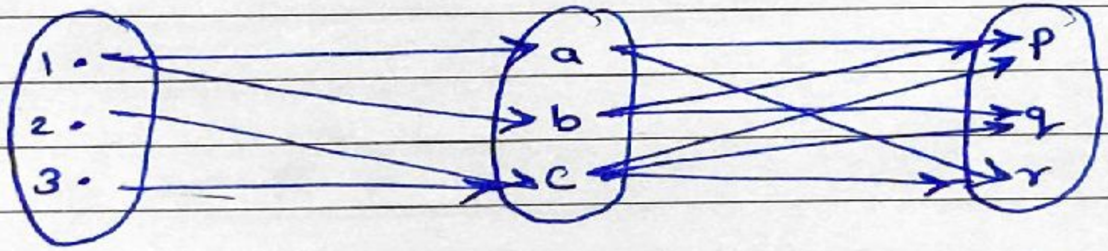


$R : A \to B$



$S : B \to C$

Now Composition of RoS



$RoS = \{(1,p), (1,r), (1,q), (2,p), (2,q), (2,r), (3,p), (3,q), (3,r)\}$

Ex. on Types of Relation

Let $A = \{1,2,3\}$, $B = \{a,b\}$

$R = \{(1,a), (2,b), (3,a)\}$

$S = \{(2,a), (3,b)\}$

Find $\overline{R}$, $\overline{S}$, $\overline{R} \cap \overline{S}$, $\overline{R} \cup \overline{S}$

Sol^n: From set A and B

$A \times B = \{(1,a), (1,b), (2,a), (2,b), (3,a), (3,b)\}$

$R = \{(1,a), (2,b), (3,a)\}$

$S = \{(2,a), (3,b)\}$

i) $\overline{R} = (A \times B) - R = \{(1,b), (2,a), (3,b)\}$

complement of R

ii) $\overline{S} = (A \times B) - S = \{(1,a), (1,b), (2,b), (3,a)\}$

complement of S

iii) $\overline{R} \cap \overline{S} = \{(1,b)\}$

iv) $\overline{R} \cup \overline{S} = \{(1,a), (1,b), (2,a), (2,b), (3,a), (3,b)\}$

Ex: Let $A = \{3, 4, 5, 6, 8\}$

R and S be the relations on A such that

$R = \{(x, y) \mid x = y + 1 \text{ or } y = 2x\}$

$S = \{(x, y) \mid x \text{ divides } y\}$ find $(R \cap S)^c$

Sol^n $R = \{(4,3), (5,4), (6,5), (3,6), (4,8)\}$

$S = \{(3,3), (3,6), (4,4), (4,8), (5,5), (6,6), (8,8)\}$.

$R \cap S = \{(3,6)\}$

$\therefore$ Converse of $R \cap S$ is $(R \cap S)^c = \{6, 3\}$.

Ex. Let $A = \{1, 2, 3\}$, $B = \{a, b, c\}$

$R = \{(1, a), (2, a), (3, c)\}$ Find i) R^c ii) $D(R^c)$ iii) $R(R^c)$

Sol^n $R = \{(1,a), (2,a), (3,c)\}$

i) The converse of R is $R^c = \{(a,1), (a,2), (c,3)\}$

ii) Domain $D(R^c) = \{a, c\}$

iii) Range $R(R^c) = \{1, 2, 3\}$

Properties of Relation

Properties of Relation

① **Reflexive :**

Let, R be a relation on set A.
R is said to be a reflexive relation if $(a,a) \in R$ for every a in A.

R is not reflexive relation, if for some element $a \in A$, $a \not R a$ or $(a,a) \notin R$

Ex. $A = \{1,2,3,4\}$

i) $R_1 = \{(1,1),(2,2),(3,3),(4,4),(1,3),(2,3)\}$

R_1 is reflexive. (Every ele. in A is related to itself)

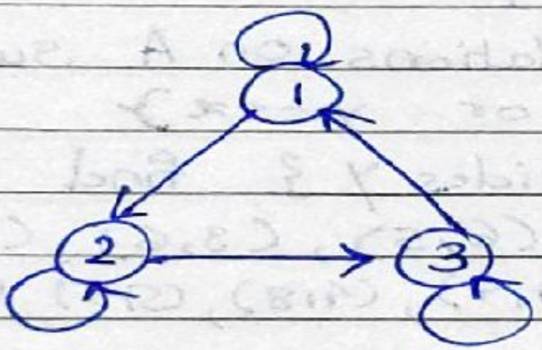
ii) $R2 = \{(1,2),(1,1),(3,4),(2,2),(3,3)\}$

R2 is not reflexive ($4 \in A$ but $(4,4) \notin R$)

Representation:

diagonal = 1

$$\begin{array}{c|ccc} & 1 & 2 & 3 \\ \hline 1 & 1 & 1 & 0 \\ 2 & 0 & 1 & 1 \\ 3 & 0 & 1 & 1 \end{array}$$



each vertex = loop

Ex.

Are the following relations on set $A = \{1,2,3,4\}$ reflexive?

$R_1 = \{(1,1), (2,2), (2,3), (3,3), (4,4)\}$ Yes

$R_2 = \{(1,1), (1,2), (2,3), (3,3), (4,4)\}$ NO

since $(2,2) \notin R_2$

$R_3 = \{(x,y) \in R^2 : x \le y\}$ Yes

since $x \le x$ for any $x \in R$

Irreflexive Relation:

A relation on a set A is called irreflexive if $(a,a) \notin R$ for every element $a \in A$

Ex: Are the following relations on Set $A = \{1,2,3\}$ Irreflexive?

$R_1 = \{(1,2), (1,3), (2,1), (2,3)\}$ Yes

$R_2 = \{(1,2), (1,3), (2,1), (3,3)\}$ No

$R_3 = \{(x,y) \in R^2 : x < y\}$ Yes

Non-reflexive Relation :

A relation R on a set A is called non-reflexive if $(a,a) \in R$ is true for some element $a \in A$ and false for others.

OR

If R is neither reflexive nor irreflexive.

Ex: Determine whether the following relation on set $A = \{1,2,3\}$ are Reflexive, Irreflexive or Non reflexive?

$R_1 = \{(1,1), (1,2), (2,2), (2,3), (3,3)\}$ Reflexive

$R_2 = \{(1,2), (2,2), (2,3), (3,1)\}$ Non-reflexive

$R_3 = \{(1,2), (1,3), (2,3), (3,1)$ Irreflexive

Since $(2,2) \in R_2$ but $(1,1) \notin R_2$ and $(3,3) \notin R_2$

Symmetric Relation

Def. A relation R on set A is called symmetric if $(b,a) \in R$ whenever $(a,b) \in R$
i.e. if $aRb \Rightarrow bRa$ for all $a, b \in A$

Ex. Are the following relations on set $A = \{1,2,3\}$ Symmetric?

Yes: $R_1 = \{(1,1), (1,2), (1,3), (2,2), (2,1), (3,1)$

No: $R_2 = \{(1,2), (1,3), (2,1), (2,3), (3,3)\}$

Ex: Is the following relation R on R Symmetric?

$R_3 = \{(x,y) \in R^2 : x^2 + y^2 = 1\}$ Yes

Since if $x^2 + y^2 = 1$ then $y^2 + x^2 = 1$ too

i.e. if $(x,y) \in R_2$ then $(y,x) \in R_2$.

Asymmetric Relation

Def. A relation R on set A is called asymmetric if $(b,a) \notin R$ whenever $(a,b) \in R$ i.e,
if $aRb \Rightarrow b \not R a$ for $a \neq b$

Ex: Are the following relations on set $A = \{1,2,3\}$ asymmetric?

$R_1 = \{(1,1), (2,1), (2,3), (3,1)\}$ Yes

$R_2 = \{(2,2), (3,3)\}$ No

Antisymmetric Relation

Def. A relation R on a set A is called antisymmetric if aRb and $bRa \Rightarrow a=b$ for all $a,b \in A$

Ex. Is the following relation on set $A=\{1,2,3\}$ antisymmetric?

$R_1 = \{(1,2),(2,2),(2,3)\}$ Yes

Ex: Is the following relation on $\mathbb{R}$ antisymmetric?

$R_3 = \{(x,y) \in \mathbb{R}^2 : x \le y\}$ Yes

Since $x \le y$ and $y \le x$ implies $x=y$ then $(x,y) \in R$ and $(y,x) \in R \Rightarrow x=y$

Ex. Is the following relation on $\mathbb{N}$ antisymmetric?

$R_3 = \{(x,y) \in \mathbb{N} : x \text{ is a divisor of } y\}$ Yes

* NOTE *

Antisymmetric is not the same as not symmetric. A relation may be symmetric as well as antisymmetric at the same time.

Ex. Are the following relations on set $A = \{1, 2, 3, 4\}$ symmetric, antisymmetric or asymmetric?

$R_1 = \{(1,1), (1,2), (2,1), (3,3), (4,4)\}$	Symmetric
$R_2 = \{(1,1), (3,3)\}$	Sym. and antisym.
$R_3 = \{(1,3), (3,2), (2,1)\}$	asymmetric
$R_4 = \{(4,4), (3,3), (1,4)\}$	antisym. and asym.

Transitive Relation

Def. A relation R on a set A is called transitive if whenever $(a,b) \in R$ and $(b,c) \in R$, then $(a,c) \in R$ for $a,b,c \in A$

Ex. Are the following relations on set $A = \{1,2,3,4\}$ transitive?

$R_1 = \{(1,1), (1,2), (2,2), (2,1), (3,3)\}$ Yes

$R_2 = \{(1,3), (3,2), (2,1)\}$ NO

Since $(1,3) \in R_2$ and $(3,2) \in R$
but $(1,2) \notin R$

$R_3 = \{(2,4), (4,3), (2,3), (4,1)\}$ NO

Since $(2,4) \in R_3$ and $(4,1) \in R_3$
but $(2,1) \notin R_3$

Additional References & Examples

Complement of Relation

Compliment of a relation will contain all the pairs where pair do not belong to relation but belongs to Cartesian product.

R = A * B – X

A = { 1, 2} B = { 3, 4} R = { (1, 3) (2, 4) }

Then the complement of R

Rc = { (1, 4) (2, 3) }

Let A = { 1, 2, 3, 4} and B = { a, b} then

AXB = { (1, a), (2, a), (3, a), (4, a), (1, b), (2, b), (3, b), (4, b) }

Let R = { (1, a), (2, a), (3, a), (4, a) }

And S = {(1, a), (2, b), (3, b), (4, b) }

Find Rc , Sc, Rc ∪ Sc and Rc ∩ Sc

Rc = {(1, b), (2, b), (3, b), (4, b) }

Sc = { (1, b), (2, a), (3, a), (4, a) }

Rc ∪ Sc = { (1, a), (2, a), (3, a), (4, a) , (2, b), (3, b), (4, b) }

Rc ∩ Sc = {(1, b)}

Verify Demorgans therom

Properties of relations : Reflexive

- In some relations an element is always related to itself
- Let R be the relation on the set of all people consisting of pairs (x,y) where x and y have the same mother and the same father. Then x R x for every person x
- A relation R on a set A is called **reflexive** if $(a,a) \in R$ for every element $a \in A$
- The relation R on the set A is reflexive if $\forall a((a,a) \in R)$

Example

- Consider these relations on {1, 2, 3, 4}

  R_1={(1,1),(1,2),(2,1),(2,2),(3,4),(4,1),(4,4)}

  R_2={(1,1),(1,2),(2,1)}

  R_3={(1,1),(1,2),(1,4),(2,1),(2,2),(3,3),(4,1),(4,4)}

  R_4={(2,1),(3,1),(3,2),(4,1),(4,2),(4,3)}

  R_5={(1,1),(1,2),(1,3),(1,4),(2,2),(2,3),(2,4),(3,3),(3,4),(4,4)}

  R_6={(3,4)}

  Which of these relations are reflexive?
- R_3 and R_5 are reflexive as both contain all pairs of the (a,a)
- Is the "divides" relation on the set of positive integers reflexive?

Symmetric

- In some relations an element is related to a second element <u>if and only if</u> the 2nd element is also related to the 1st element
- A relation R on a set A is called **symmetric** if $(b,a) \in R$ whenever $(a,b) \in R$ for all $a, b \in A$
- The relation R on the set A is symmetric if $\forall a \forall b ((a,b) \in R \rightarrow (b,a) \in R)$
- A relation is symmetric if and only if <u>a is related to b</u> implies that <u>b is related to a</u>

Antisymmetric

- A relation R on a set A such that for all a, b ∈ A, if (a, b)∈R and (b, a) ∈ R, then a=b is called **antisymmetric**
- Similarly, the relation R is antisymmetric if $\forall a \forall b(((a,b) \in R \land (b,a) \in R) \rightarrow (a=b))$
- A relation is **antisymmetric** if and only if there are no pairs of distinct elements a and b with <u>a related to b</u> and <u>b related to a</u>
- That is, the only way to have <u>a related to b</u> and <u>b related to a</u> is for a and b to be <u>the same</u> element

Symmetric and antisymmetric

- The terms symmetric and antisymmetric are not opposites as a relation can have both of these properties or may lack both of them
- A relation cannot be both symmetric and antisymmetric if it contains some pair of the form (a, b) where $a \neq b$

Example

- Consider these relations on {1, 2, 3, 4}

  $R_1=\{(1,1),(1,2),(2,1),(2,2),(3,4),(4,1),(4,4)\}$

  $R_2=\{(1,1),(1,2),(2,1)\}$

  $R_3=\{(1,1),(1,2),(1,4),(2,1),(2,2),(3,3),(4,1),(4,4)\}$

  $R_4=\{(2,1),(3,1),(3,2),(4,1),(4,2),(4,3)\}$

  $R_5=\{(1,1),(1,2),(1,3),(1,4),(2,2),(2,3),(2,4),(3,3),(3,4),(4,4)\}$

  $R_6=\{(3,4)\}$

  Which of these relations are symmetric or antisymmetric?
- R_2 and R_3 are symmetric: each (a,b) □ (b,a) in the relation
- R_4, R_5, and R_6 are all antisymmetric: no pair of elements a and b with $a \neq b$ s.t. (a, b) and (b, a) are both in the relation

Example

- Which are symmetric and antisymmetric

  $R_1=\{(a,b)|a\leq b\}$

  $R_2=\{(a,b)|a>b\}$

  $R_3=\{(a,b)|a=b \text{ or } a=-b\}$

  $R_4=\{(a,b)|a=b\}$

  $R_5=\{(a,b)|a=b+1\}$

  $R_6=\{(a,b)|a+b\leq 3\}$
- Symmetric: R_3, R_4, R_6. R_3 is symmetric, if a=b (or a=-b), then b=a (b=-a), R_4 is symmetric as a=b implies b=a, R_6 is symmetric as $a+b\leq 3$ implies $b+a\leq 3$
- Antisymmetric: R_1, R_2, R_4, R_5 . R_1 is antisymmetric as $a\leq b$ and $b\leq a$ imply a=b. R_2 is antisymmetric as it is impossible to have a>b and b>a, R_4 is antisymmteric as two elements are related w.r.t. R_4 if and only if they are equal. R_5 is antisymmetric as it is impossible to have a=b+1 and b=a+1

Transitive

- A relation R on a set A is called **transitive** if whenever $(a,b) \in R$ and $(b,c) \in R$ then $(a,c) \in R$ for all $a, b, c \in A$
- Using quantifiers, we see that a relation R is **transitive** if we have

  $\forall a \forall b \forall c \, (((a,b) \in R \land (b,c) \in R) \rightarrow (a,c) \in R)$

Example

- Which one is transitive?

  $R_1=\{(1,1),(1,2),(2,1),(2,2),(3,4),(4,1),(4,4)\}$

  $R_2=\{(1,1),(1,2),(2,1)\}$

  $R_3=\{(1,1),(1,2),(1,4),(2,1),(2,2),(3,3),(4,1),(4,4)\}$

  $R_4=\{(2,1),(3,1),(3,2),(4,1),(4,2),(4,3)\}$

  $R_5=\{(1,1),(1,2),(1,3),(1,4),(2,2),(2,3),(2,4),(3,3),(3,4),(4,4)\}$

  $R_6=\{(3,4)\}$
- R_4 R_5 R_6 are transitive
- R_1 is not transitive as (3,1) is not in R_1
- R_2 is not transitive as (2,2) is not in R_2
- R_3 is not transitive as (4,2) is not in R_3

Example

- Which are symmetric and antisymmetric

  $R_1=\{(a,b)|a\leq b\}$

  $R_2=\{(a,b)|a>b\}$

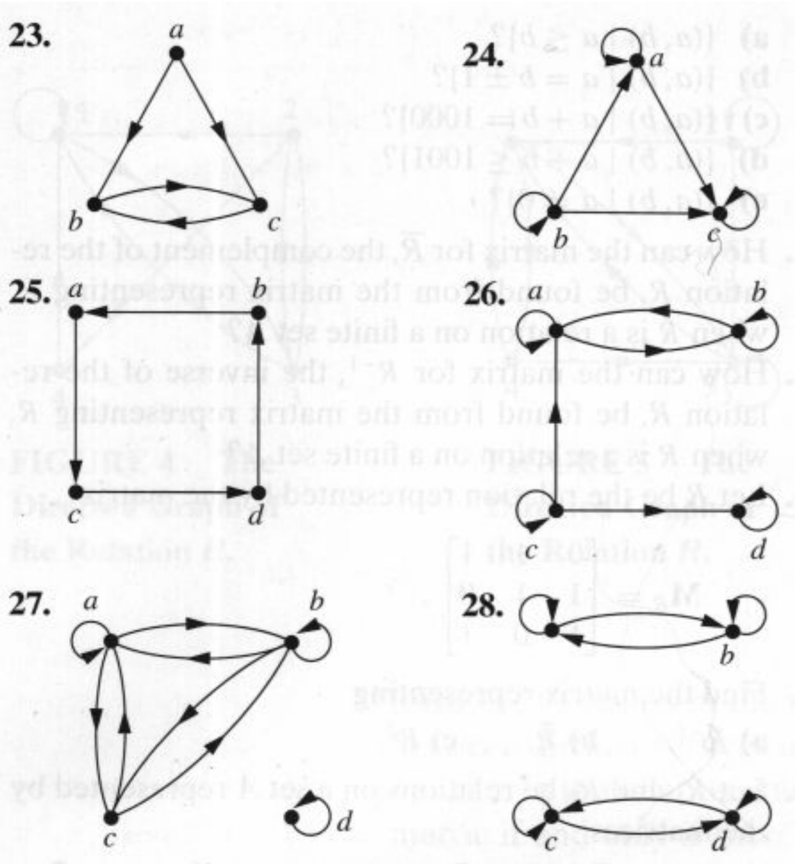
  $R_3=\{(a,b)|a=b \text{ or } a=-b\}$

  $R_4=\{(a,b)|a=b\}$

  $R_5=\{(a,b)|a=b+1\}$

  $R_6=\{(a,b)|a+b\leq 3\}$
- R_1 is transitive as $a\leq b$ and $b\leq c$ implies $a\leq c$. R_2 is transitive
- R_3 , R_4 are transitive
- R_5 is not transitive (e.g., (2,1), (1,0)). R_6 is not transitive (e.g. (2,1),(1,2))

Sample questions



Which of the graphs are reflexive, irreflexive, symmetric, asymmetric, antisymmetric, or transitive

	23	24	25	26	27	28
Reflexive		Y		Y		Y
Irreflexive	Y		Y			
Symmetric					Y	Y
Asymmetric			Y			
Anti-symmetric		Y	Y			
Transitive						Y

End of Session 3

Unit 2 – Relations and Functions

Session 4

- Equivalence Relation
- Equivalences Classes
- Partitions

Equivalence relation

Definition: A relation R on a set A is called an **equivalence relation** if it is **reflexive, symmetric and transitive.**

Equivalence relation

Definition: A relation R on a set A is called an **equivalence relation** if it is **reflexive, symmetric and transitive**.

Example: Let A = {0,1,2,3,4,5,6} and

- R= $\{(a,b) \mid a,b \in A, a \equiv b \bmod 3\}$ (a is congruent to b modulo 3)

Congruencies:

- 0 mod 3 = 0   1 mod 3 = 1   2 mod 3 = 2   3 mod 3 = 0
- 4 mod 3 = 1   5 mod 3 = 2   6 mod 3 = 0

Relation R has the following pairs:

- (0,0)   (0,3), (3,0), (0,6), (6,0)
- (3,3), (3,6) (6,3), (6,6)   (1,1),(1,4), (4,1), (4,4)
- (2,2), (2,5), (5,2), (5,5)

Equivalence relation

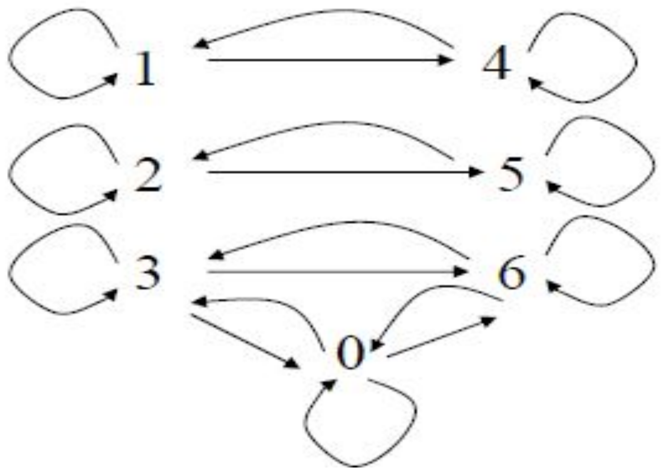
- **Relation R on A={0,1,2,3,4,5,6} has the following pairs:**

  (0,0) (0,3), (3,0), (0,6), (6,0)

  (3,3), (3,6) (6,3), (6,6) (1,1),(1,4), (4,1), (4,4)

  (2,2), (2,5), (5,2), (5,5)

- Is R reflexive? **Yes.**
- Is R symmetric? **Yes.**
- Is R transitive. **Yes.**



Then

- **R is an equivalence relation**.

Which of these relations on {0, 1, 2, 3} are equivalence relations? Determine the properties of an equivalence relation that the others lack

a) { (0,0), (1,1), (2,2), (3,3) }
- Has all the properties, thus, is an equivalence relation

b) { (0,0), (0,2), (2,0), (2,2), (2,3), (3,2), (3,3) }
- Not reflexive: (1,1) is missing
- Not transitive: (0,2) and (2,3) are in the relation, but not (0,3)

c) { (0,0), (1,1), (1,2), (2,1), (2,2), (3,3) }
- Has all the properties, thus, is an equivalence relation

d) { (0,0), (1,1), (1,3), (2,2), (2,3), (3,1), (3,2) (3,3) }
- Not transitive: (1,3) and (3,2) are in the relation, but not (1,2)

e) { (0,0), (0,1) (0,2), (1,0), (1,1), (1,2), (2,0), (2,2), (3,3) }
- Not symmetric: (1,2) is present, but not (2,1)
- Not transitive: (2,0) and (0,1) are in the relation, but not (2,1)

Equivalence class

Definition: Let R be an equivalence relation on a set A. The set $\{ x \in A \mid a\ R\ x\}$ is called **the equivalence class of a,** denoted by $[a]_R$ or simply $[a]$. If $b \in [a]$ then b is called **a representative of this equivalence class**.

Example:

- Assume $R=\{(a,b) \mid a \equiv b \bmod 3\}$ for $A=\{0,1,2,3,4,5,6\}$

$R= \{(0,0), (0,3), (3,0), (0,6), (6,0),(3,3), (3,6)\ (6,3), (6,6), (1,1),(1,4), (4,1), (4,4), (2,2), (2,5), (5,2), (5,5)\}$

- **Pick an element** a =0.
- $[0]_R = \{0,3,6\}$
- Element 1: $[1]_R= \{1,4\}$
- Element 2: $[2]_R= \{2,5\}$
- Element 3: $[3]_R= \{0,3,6\} = [0]_R = [6]_R$
- Element 4: $[4]_R= \{1,4\} = [1]_R$ Element 5: $[5]_R= \{2,5\} = [2]_R$

Equivalence class

Example:

- Assume $R=\{(a,b) \mid a \equiv b \bmod 3\}$ for $A=\{0,1,2,3,4,5,6\}$
- $R= \{(0,0), (0,3), (3,0), (0,6), (6,0),(3,3), (3,6) (6,3), (6,6), (1,1),(1,4), (4,1), (4,4), (2,2), (2,5), (5,2), (5,5)\}$

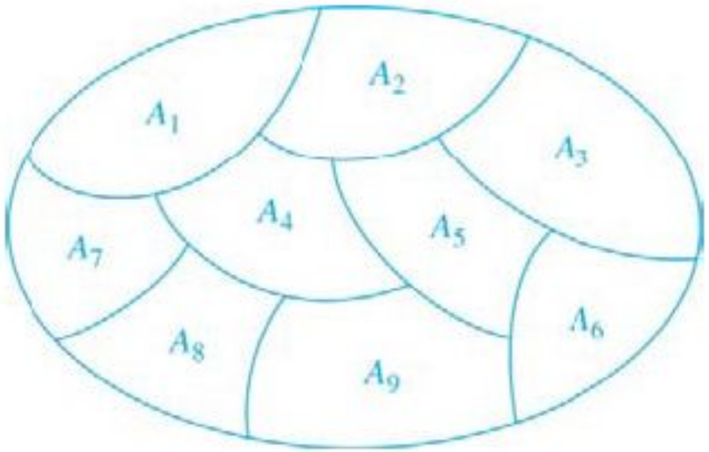
Three different equivalence classes all together:

- $[0]_R = [3]_R =[6]_R = \{0,3,6\}$
- $[1]_R= [4]_R= \{1,4\}$
- $[2]_R= [5]_R= \{2,5\}$

Partition of a set S

Definition: Let S be a set. A collection of nonempty subsets of S $A_1, A_2, \ldots, A_k$ is called **a partition of S** if:

- $A_i \cap A_j = \varnothing,\ i \neq j$ and $S = \bigcup_{i=1}^{k} A_i$



Partition of a set S

Definition: Let S be a set. A collection of nonempty subsets of S $A_1, A_2, \ldots, A_k$ is called **a partition of S** if:

- $A_i \cap A_j = \varnothing,\ i \neq j$ and $S = \bigcup_{i=1}^{k} A_i$

Example: Let S={1,2,3,4,5,6} and

- $A_1 = \{0,3,6\}$ $A_2 = \{1,4\}$ $A_3 = \{2,5\}$
- Is A_1, A_2, A_3 a partition of S ? **Yes.**

- Give a partition of S ?
- {0,2,4,6} {1,3,5}
- {0} {1,2} {3,4,5} {6}

Equivalence classes and partitions

Theorem: Let R be an equivalence relation on a set A. Then the union of all the equivalence classes of R is A:

$$A = \bigcup_{a \in A}^{k} [a]_R$$

Proof: an element a of A is in its own equivalence class $[a]_R$ so union cover A.

Theorem: The equivalence classes form a partition of A.

Proof: The equivalence classes split A into disjoint subsets.

Theorem : Let $\{A_1, A_2, .. A_i ,..\}$ be a partitioning of S. Then there is an equivalence relation R on S, that has the sets A_i as its equivalence classes.

End of Session 4

Unit 2 – Relations and Functions

Session 5

- Closure of relation – reflexive, symmetric, transitive, Warshall's algorithm, examples
- Partial ordered set – HASSE diagram, chains and antichains, elements of POSET
- Lattice – properties, types, principle of duality, examples

Closures of relations

- Let R={(1,1),(1,2),(2,1),(3,2)} on A ={1 2 3}.
- Is this relation reflexive?
- Answer: ?

- Let R={(1,1),(1,2),(2,1),(3,2)} on A ={1 2 3}.
- Is this relation reflexive?
- Answer: **No.** Why?

Closures of relations

- Let R={(1,1),(1,2),(2,1),(3,2)} on A ={1 2 3}.
- Is this relation reflexive?
- Answer: **No.** Why?
- **(2,2) and (3,3) is not in R.**

- The question is what is **the minimal relation** **S $\supseteq$ R** that is reflexive?
- How to make R reflexive with minimum number of additions?
- Answer: ?

Closures of relations

- Let R={(1,1),(1,2),(2,1),(3,2)} on A ={1 2 3}.
- Is this relation reflexive?
- Answer: **No.** Why?
- **(2,2) and (3,3) is not in R.**

- The question is what is **the minimal relation S $\supseteq$** R that is reflexive?
- How to make R reflexive with minimum number of additions?
- **Answer:** Add (2,2) and (3,3)
  - Then S= {(1,1),(1,2),(2,1),(3,2),(2,2),(3,3)}
  - R $\subseteq$ S
  - The minimal set S $\supseteq$ R is called **the reflexive closure of R**

Closures on relations

- **Relations can have different properties:**
    - **reflexive,**
    - **symmetric**
    - **transitive**

- Because of that we define:
    - **symmetric,**
    - **reflexive and**
    - **transitive**

  closures.

Reflexive closure

The set S is called **the reflexive closure of R** if it:

- contains R
- has reflexive property
- is contained in every reflexive relation Q that contains R ($R \subseteq Q$), that is $S \subseteq Q$

Closures

Definition: Let R be a relation on a set A. A relation S on A with property P is called **the closure of R with respect to P** if S is a subset of every relation Q ($S \subseteq Q$) with property P that contains R ($R \subseteq Q$).

OR

Definition : Let A be a set. R be a relation on set A. Closure of relation R is with respect to some property "P " of relation. The P-closure of R is defined as the smallest relation on A containing R & possessing the property P.

Closures

Definition: Let R be a relation on a set A. A relation S on A with property P is called **the closure of R with respect to P** if S is a subset of every relation Q ($S \subseteq Q$) with property P that contains R ($R \subseteq Q$).

Example (symmetric closure):

- Assume R={(1,2),(1,3), (2,2)} on A={1,2,3}.
- What is the symmetric closure S of R?
- S=?

Closures

Definition: Let R be a relation on a set A. A relation S on A with property P is called **the closure of R with respect to P** if S is a subset of every relation Q ($S \subseteq Q$) with property P that contains R ($R \subseteq Q$).

Example (a symmetric closure):

- Assume R={(1,2),(1,3), (2,2)} on A={1,2,3}.
- What is the symmetric closure S of R?
- **S = {(1,2),(1,3), (2,2)} $\cup$ {(2,1), (3,1)}**
  = {(1,2),(1,3), (2,2),(2,1), (3,1)}

Closures

Definition: Let R be a relation on a set A. A relation S on A with property P is called **the closure of R with respect to P** if S is a subset of every relation Q ($S \subseteq Q$) with property P that contains R ($R \subseteq Q$).

Example (transitive closure):

- Assume $R=\{(1,2), (2,2), (2,3)\}$ on $A=\{1,2,3\}$.
- **Is R transitive?**

Closures

Definition: Let R be a relation on a set A. A relation S on A with property P is called **the closure of R with respect to P** if S is a subset of every relation Q ($S \subseteq Q$) with property P that contains R ($R \subseteq Q$).

Example (transitive closure):

- Assume R={(1,2), (2,2), (2,3)} on A={1,2,3}.
- **Is R transitive? No.**
- **How to make it transitive?**
- **S = ?**

Closures

Definition: Let R be a relation on a set A. A relation S on A with property P is called **the closure of R with respect to P** if S is a subset of every relation Q ($S \subseteq Q$) with property P that contains R ($R \subseteq Q$).

Example (transitive closure):

- Assume R={(1,2), (2,2), (2,3)} on A={1,2,3}.
- **Is R transitive? No.**
- **How to make it transitive?**
- **S = {(1,2), (2,2), (2,3)} $\cup$ {(1,3)}**
  = {(1,2), (2,2), (2,3),(1,3)}
- S is the transitive closure of R

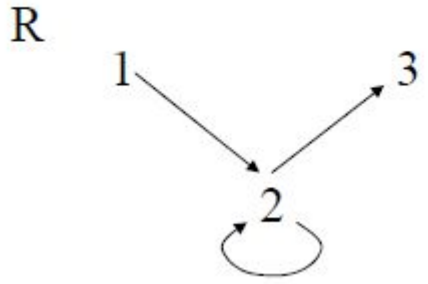
Transitive closure

We can represent the relation on the graph. Finding a transitive closure corresponds to finding all pairs of elements that are connected with a directed path (or digraph).

Example:

Assume R={(1,2), (2,2), (2,3)} on A={1,2,3}.

Transitive closure S = {(1,2), (2,2), (2,3),(1,3)}.



Transitive closure

We can represent the relation on the graph. Finding a transitive closure corresponds to finding all pairs of elements that are connected with a directed path (or digraph).

Example:

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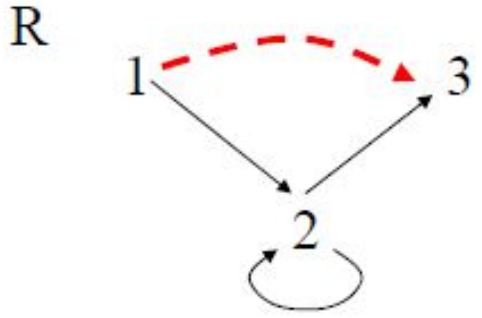

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We can represent the relation on the graph. Finding a transitive closure corresponds to finding all pairs of elements that are connected with a directed path (or digraph).

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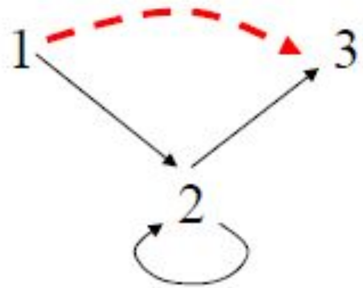
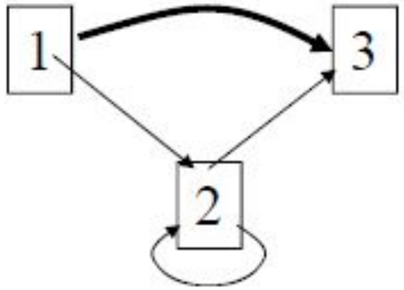
R



S



- Warshall's algorithm is used to determine the **transitive closure(Note: kindly check classroom (Video)**
- Practical and efficient method

Warshall's Algorithm Example:

Find Transitive Closure of the relation R on

A= { 1,2,3,4} defined by
R={(1,2),(1,3),(1,4),(2,1),(2,3),(3,4),(3,2),(4,2),(4,3)}

Let, $A = \{1, 2, 3, 4\}$

$R = \{(1, 2), (1, 3), (1, 4), (2, 1), (2, 3), (3, 4), (3, 2),$

Here, $n(A) = 4$. Therefore we will required W_1, W_2, W_3 and W_4 warshall sets to transitive closure of R.

The relation matrix of R is

$$W_R = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix} \end{array} = W_0$$

▶ **Step I : Find W_1**

To find W_1 from W_0, consider 1^{st} row and 1^{st} column of W_0.

In $\text{column}_1 \Rightarrow 1$ is present at R_2.

In $\text{row}_1 \Rightarrow 1$ is present at C_2, C_3 and C_4.

Thus, make new entries at (R_2, C_2), (R_2, C_3), (R_2, C_4) in W_1.

$$W_1 = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix} \end{array}$$

$$W_1 = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} 0 & \boxed{1} & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix} \end{array}$$

▶ **Step II : Find W_2**

To find W_2 from W_1, consider 2^{nd} column and 2^{nd} row of W_2.

In $\text{column}_2 \Rightarrow 1$ is present at R_1, R_2, R_3 and R_4.

In $\text{row}_2 \Rightarrow 1$ is present at C_1, C_2, C_3 and C_4.

Thus, make new entries at (R_1, C_1), (R_1, C_2), (R_1, C_3), (R_1, C_4), (R_2, C_1), (R_2, C_2), (R_2, C_3), (R_2, C_4), (R_3, C_1), (R_3, C_2), (R_3, C_3), (R_3, C_4), (R_4, C_1), (R_4, C_2), (R_4, C_3), (R_4, C_4).

$$W_2 = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix} \end{array}$$

$$W_2 = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix} \end{array}$$

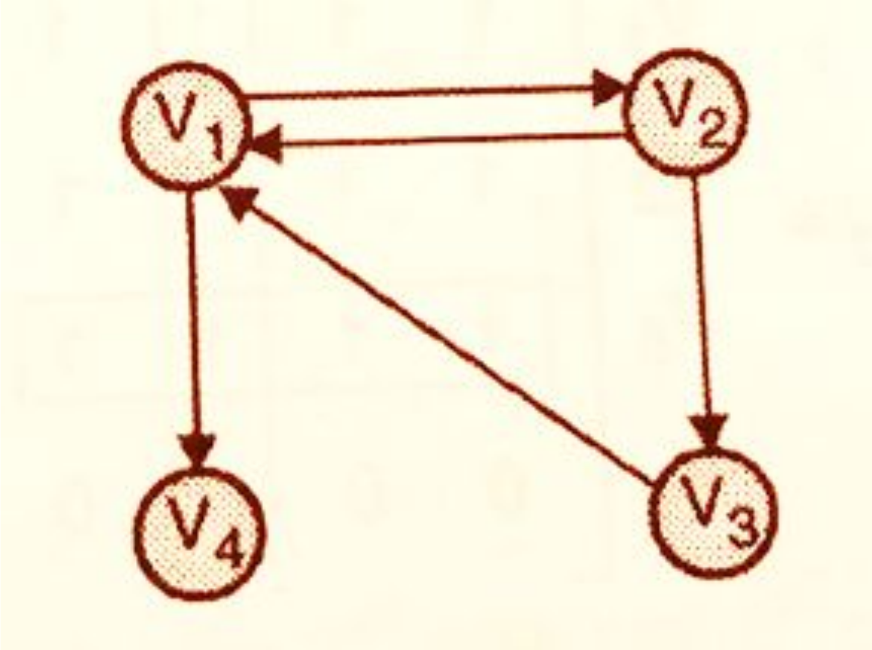
In W_2, every cell contain 1 in it. So no need to find W_3 and W_4.

The relation matrix W_2 is the transitive closure of R.

Therefore $R^* = W_2$

$R^* = \{(1, 1), (1, 2), (1, 3), (1, 4), (2, 1), (2, 2), (2, 3), (2, 4), (3, 1), (3, 2), (3, 3), (3, 4), (4, 1), (4, 2), (4, 3), (4, 4)\}$

Compute the transitive closure of given digraph using Warshall’s algorithm



Partial ordered set

HASSE diagram, chains and antichains, elements of POSET

Partially Ordered Sets

Consider a relation R on a set S satisfying the following properties:

1. R is reflexive, i.e., xRx for every x ∈ S.
2. R is antisymmetric, i.e., if xRy and yRx, then x = y.
3. R is transitive, i.e., xRy and yRz, then xRz.

Then R is called a partial order relation, and the set S together with partial order is called a partially order set or POSET and is denoted by (S, ≤).

Example:

The set N of natural numbers form a poset under the relation '$\leq$' because firstly $x \leq x$, secondly, if $x \leq y$ and $y \leq x$, then we have $x = y$ and lastly if $x \leq y$ and $y \leq z$, it implies $x \leq z$ for all $x, y, z \in N$.

The set N of natural numbers under divisibility i.e., 'x divides y' forms a poset because x/x for every $x \in N$. Also if x/y and y/x, we have $x = y$. Again if x/y, y/z we have x/z, for every $x, y, z \in N$.

Consider a set $S = \{1, 2\}$ and power set of S is P(S). The relation of set inclusion $\subseteq$ is a partial order. Since, for any sets A, B, C in P (S), firstly we have $A \subseteq A$, secondly, if $A \subseteq B$ and $B \subseteq A$, then we have $A = B$. Lastly, if $A \subseteq B$ and $B \subseteq C$, then $A \subseteq C$. Hence, $(P(S), \subseteq)$ is a poset.

Elements of POSET

(1) Maximal element –

If in a poset, an element is not related to any other element, then its called maximal elements.

(2) Minimal element –

If in a poset, no element is related to an element, then its called minimal element.

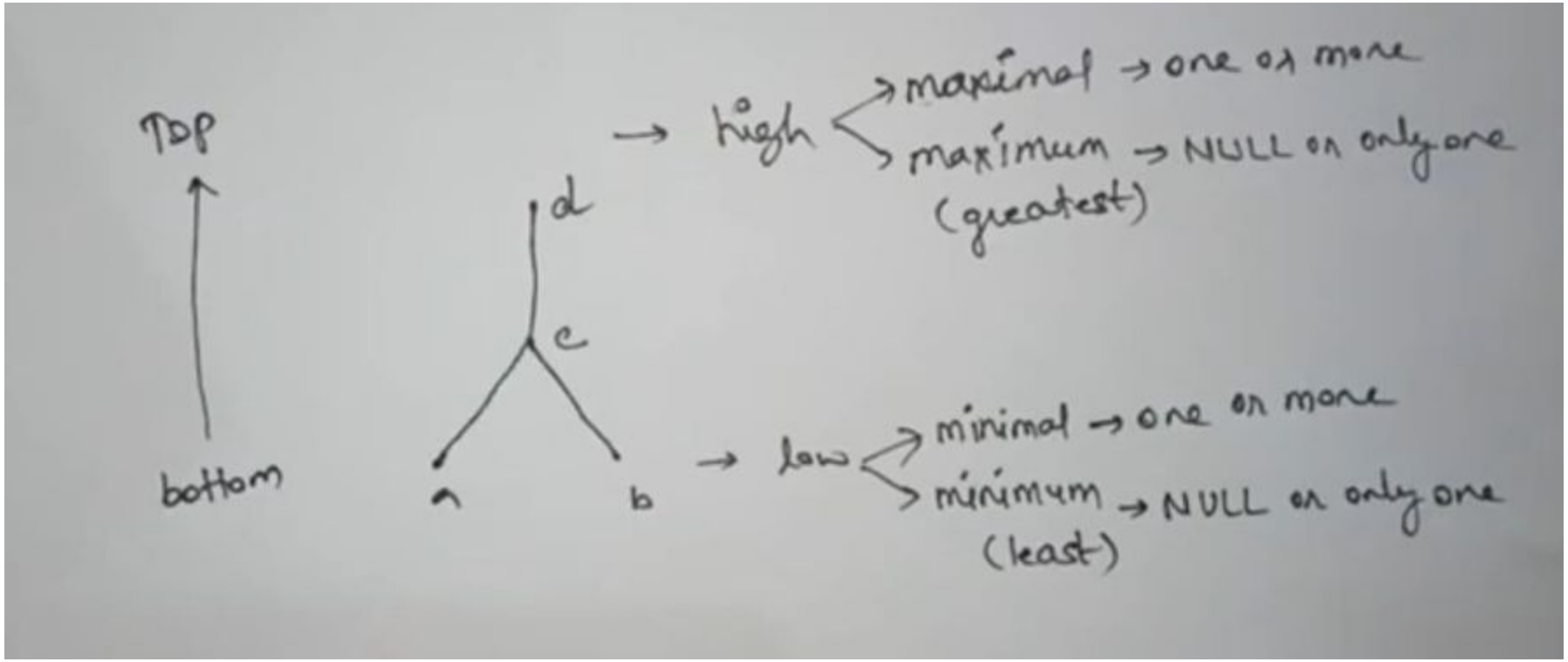
(3) Maximum element –

If it is maximal and every element is related to it.

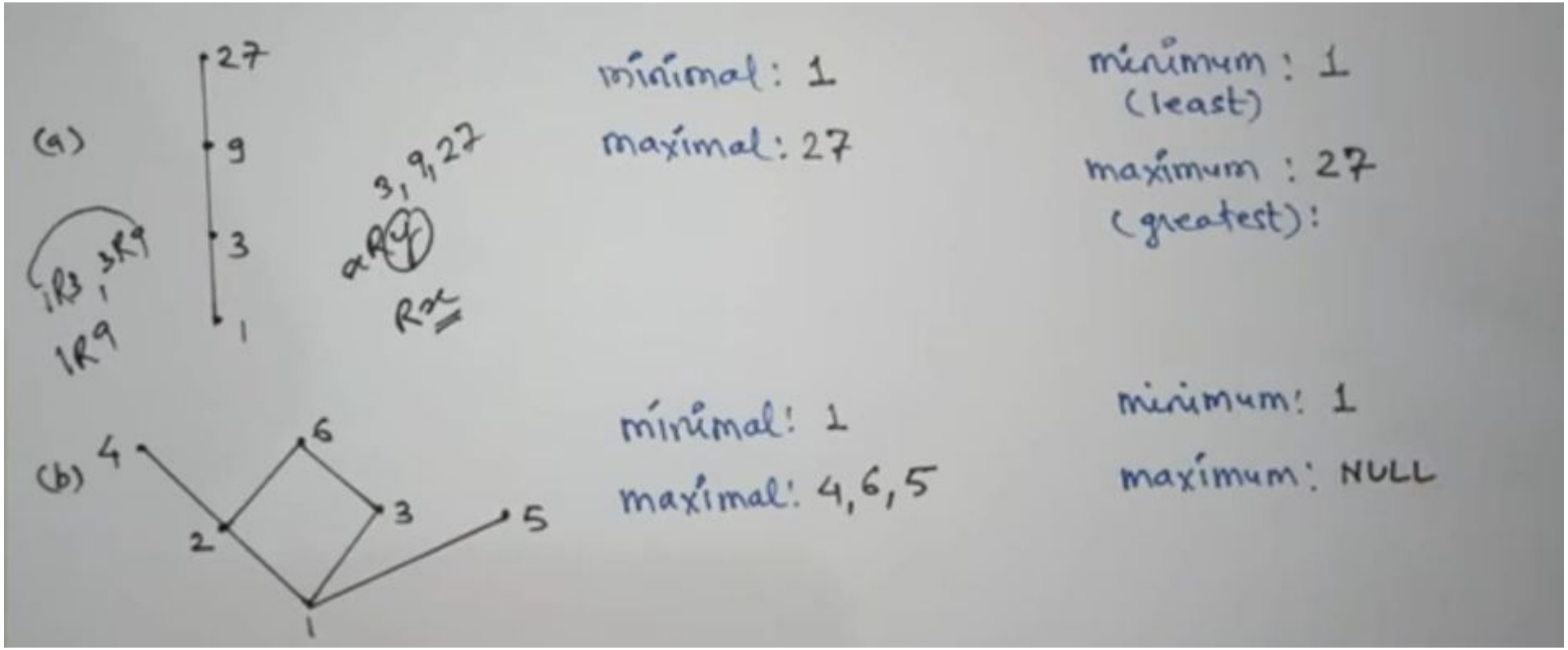
Elements of POSET

(4) Minimum element -

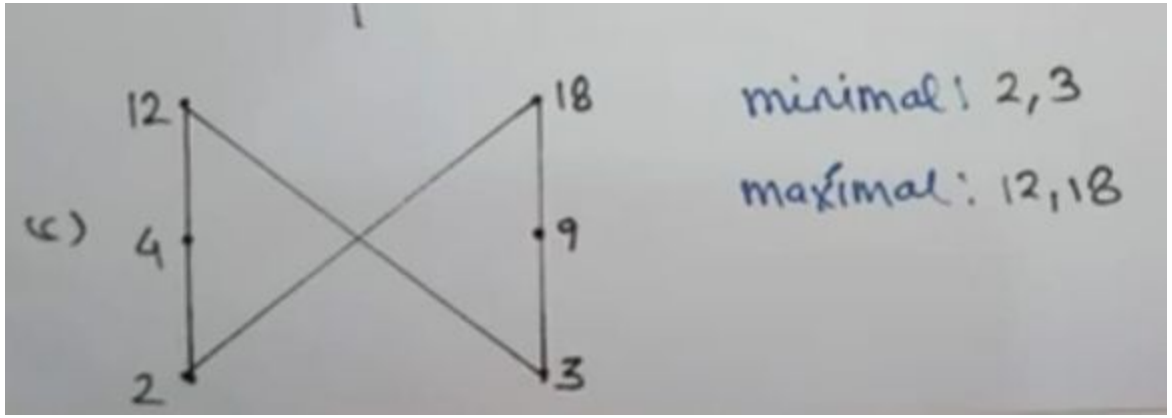
If it is minimal and it is related to every element in Poset.



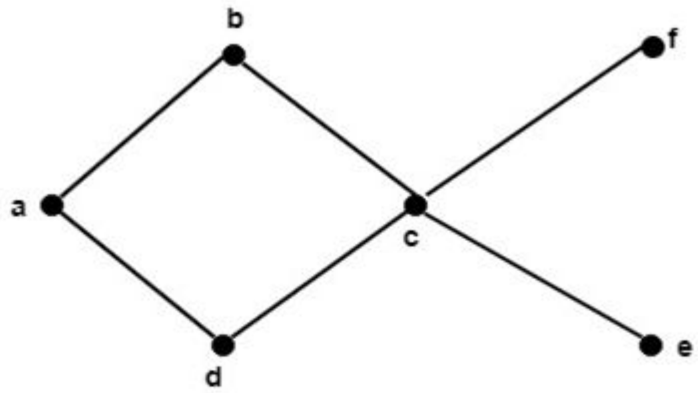
Elements of POSET



Elements of POSET



Example: Determine all the maximal and minimal elements of the poset whose Hasse diagram is shown in fig:



Solution: The maximal elements are b and f.
The minimal elements are d and e.

Comparable Elements:

Consider an ordered set A. Two elements a and b of set A are called comparable if

$$a \underset{R}{\leq} b \qquad \text{or} \qquad b \underset{R}{\leq} a$$

Non-Comparable Elements:

Consider an ordered set A. Two elements a and b of set A are called non-comparable if neither a ≤ b nor b ≤ a.

Example: Consider A = {1, 2, 3, 5, 6, 10, 15, 30} is ordered by divisibility. Determine all the comparable and non-comparable pairs of elements of A.

Solution: The comparable pairs of elements of A are:

{1, 2}, {1, 3}, {1, 5}, {1, 6}, {1, 10}, {1, 15}, {1, 30}
{2, 6}, {2, 10}, {2, 30}
{3, 6}, {3, 15}, {3, 30}
{5, 10}, {5, 15}, {5, 30}
{6, 30}, {10, 30}, {15, 30}

The non-comparable pair of elements of A are:

{2, 3}, {2, 5}, {2, 15}
{3, 5}, {3, 10}, {5, 6}, {6, 10}, {6, 15}, {10, 15}

Linearly Ordered Set:
Consider an ordered set A. The set A is called linearly ordered set or totally ordered set, if every pair of elements in A is comparable.

Example: The set of positive integers I+ with the usual order ≤ is a linearly ordered set.

R = { (a,b) | a<=b } on set S= {1,2,3,4}

A **Hasse diagram** is a *graphical representation* of the relation of elements of a **partially ordered set (poset)** with an implied *upward orientation*. A point is drawn for each element of the partially ordered set (poset) and joined with the line segment according to the following rules:

- If $p<q$ in the poset, then the point corresponding to *p appears lower* in the drawing than the point corresponding to q.
- The two points p and q will be joined by line segment *if p is related to q*.

To draw a Hasse diagram, provided set must be a poset.

A poset or partially ordered set A is a pair, $(B, \leq)$ of a set B whose elements are called the vertices of A and obeys following rules:

1. Reflexivity $\rightarrow p \leq p \ \forall p \in B$
2. Anti-symmetric $\rightarrow p \leq q$ and $q \leq p$ if $p=q$
3. Transitivity $\rightarrow$ if $p \leq q$ and $q \leq r$ then $p \leq r$

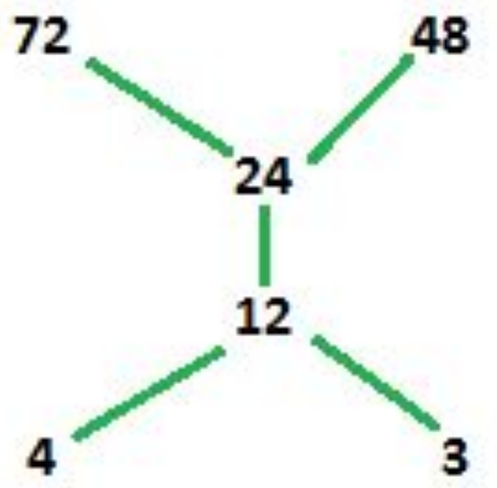
Example-1: Draw Hasse diagram for ({3, 4, 12, 24, 48, 72}, /)

Explanation – According to above given question first, we have to find the poset for the divisibility.

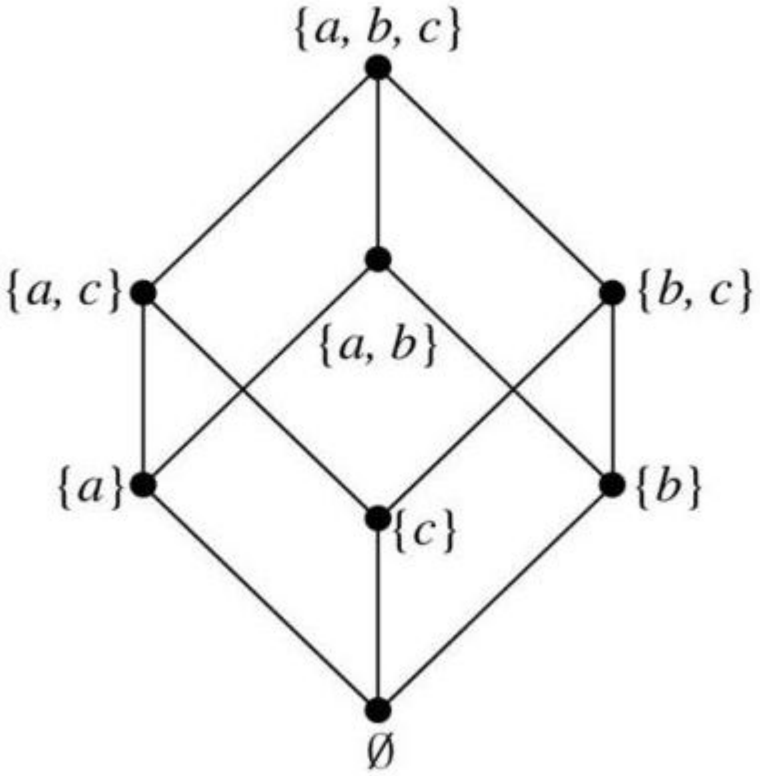
Let the set is A.

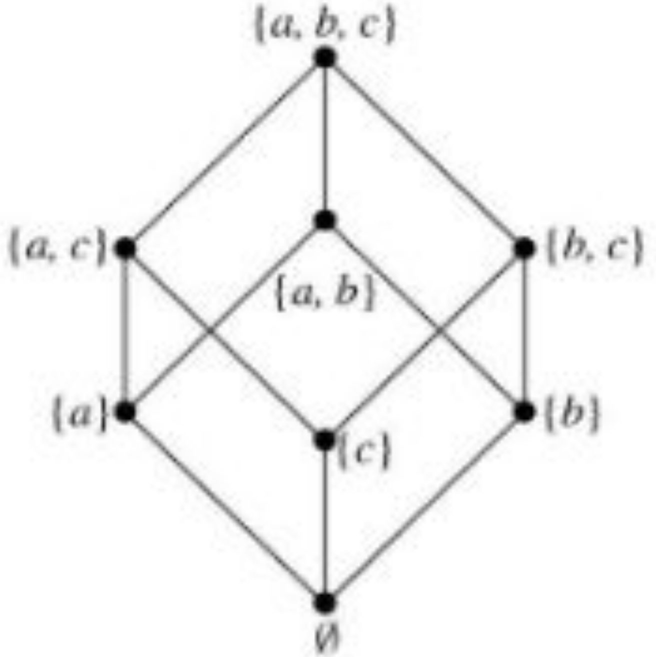
A={(3 $\prec$12), (3 $\prec$24), (3 $\prec$48), (3 $\prec$72), (4 $\prec$12), (4 $\prec$24), (4 $\prec$48), (4 $\prec$72), (12 $\prec$24), (12 $\prec$48), (12 $\prec$72), (24 $\prec$48), (24 $\prec$72)}

So, now the Hasse diagram will be:



- Example 13: Draw the Hasse diagram representing the partial ordering $\{(A, B) \mid A \subseteq B\}$ on the power set $S=\{a ,b, c\}$.

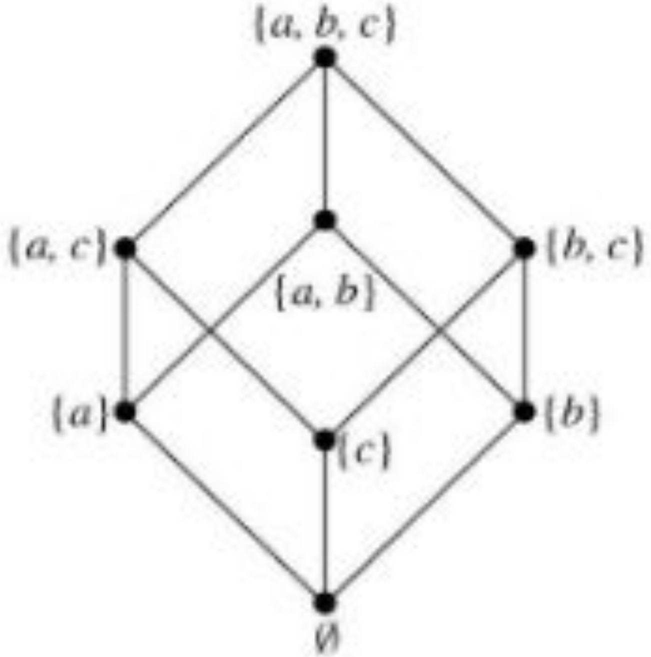




(i) The chains are

$$c_1 = \{ \phi, \{a\}, \{a, b\}, \{a, b, c\} \}$$
$$c_2 = \{ \{a\}, \{a, c\}, \{a, b, c\} \}$$
$$c_3 = \{ \{b, c\}, \{a, b, c\} \}$$

- In the chain c_1, all the elements are related or comparable likewise in chain c_2 all the elements are related.
- The chains c_1, c_2 and c_3 are subsets of POSET $(P(S), \subseteq)$.



The anti-chains are

$$c_1 = \{\{a\}, \{b\}, \{c\}\},$$
$$c_2 = \{\{a, b\}, \{a, c\}, \{b, c\}\},$$

In the anti-chain c_1, all the elements are non-comparable or not related.

A lattice is a poset, a partially ordered set, in which every pair of elements has both a least upper bound and a greatest lower bound.

Upper bound –

An element $u \in A$ is called an upper bound of B if u succeeds every element of B i,e $x \leq u$ for all $x \in B$

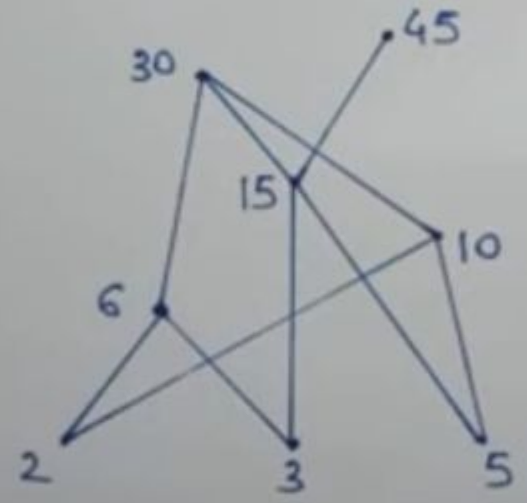
Least upper bound (lub) –

An element $a \in A$ is a least upper bound or supremum for B if 'a' is an upper bound for B and $a \leq b$ where 'b' is any upper bound for B.

Eg: set $A = \{2, 3, 5, 6, 10, 15, 30, 45\}$, subset $B = \{2, 3\}$

$2 \rightarrow \cancel{6}, 10, \cancel{30}$

$3 \rightarrow \cancel{6}, 15, \cancel{30}, 45$



Find

(i) upper bound $\rightarrow$ 6, 30

(ii) Least upper bound $\rightarrow$ 6

Lower bound –

An element $l \in A$ is called a lower bound of B if l preceds every element of B i,e $l \leq x$ for all $x \in B$

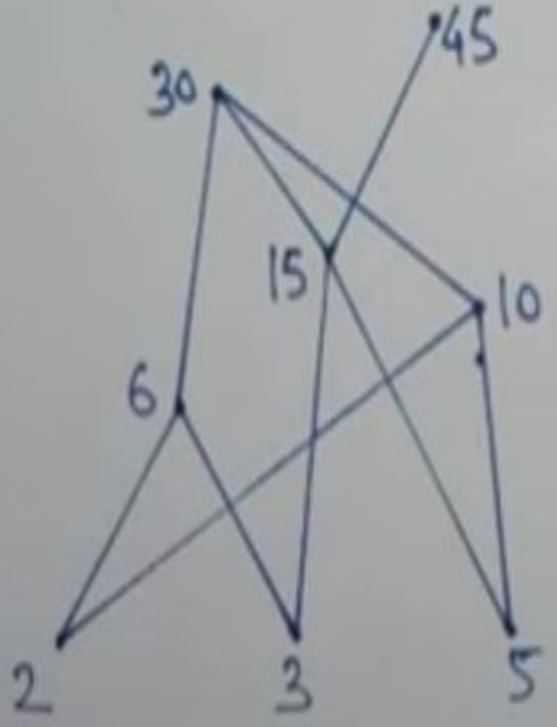
Greatest lower bound (glb) –

An element $a \in A$ is called glb or infimum for B if 'a' is a lower bound for B and $b \leq a$ where b is any lower bound for B

Eg:- set $A = \{2, 3, 5, 6, 10, 15, 30, 45\}$, subset $B = \{30, 45\}$

$30 \to 6, 15, 10, 2, 3, 5$

$45 \to 15, 3, 5$



Find

i) lower bound $\to$ 15, 3, 5

ii) Greatest lower bound $\to$ 15

Lattice

A poset $(P, \leq)$ is called a lattice if every 2-element subset of P has both a least upper bound and a greatest lower bound i.e. if $lub(x, y)$ and $glb(x, y)$ exist for every x and y in P.

In this case we denote

$$x \vee y = lub\{x, y\}$$

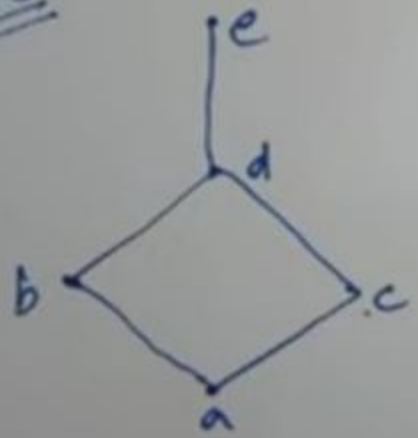
$$x \wedge y = glb\{x, y\}$$

A POSET in which every pair of element has both a least upper bound and greatest lower bound.

A lattic is a mathematical structure with two binary operations (join) and (meet). It is denoted by

$\{L, \vee, \wedge\}$.

Eg ①



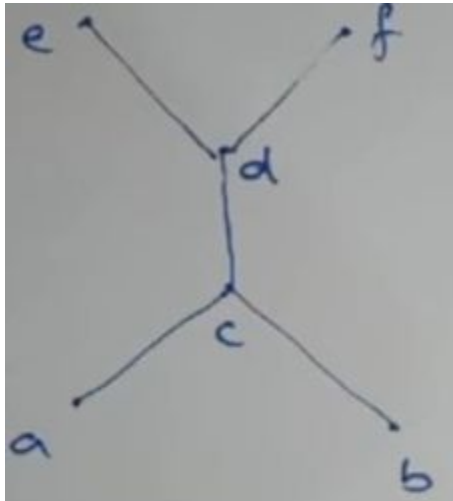
lub table

∨	a	b	c	d	e
a	a	b	c	d	e
b	b	b	d	d	e
c	c	d	c	d	e
d	d	d	d	d	e
e	e	e	e	e	e

glb table

∧	a	b	c	d	e
a	a	a	a	a	a
b	a	b	a	b	b
c	a	a	c	c	c
d	a	b	c	d	d
e	a	b	c	d	e

This poset is a lattice



lub table

$\vee$	a	b	c	d	e	f
a	a	c	c	d	e	f
b	c	b	c	d	e	f
c	c	c	c	d	e	f
d	d	d	d	d	e	f
e	e	e	e	e	e	–
f	f	f	f	f	–	f

glb table

$\wedge$	a	b	c	d	e	f
a	a	–	a	a	a	a
b	–	b	b	b	b	b
c	a	b	c	c	c	c
d	a	b	c	d	d	d
e	a	b	c	d	e	d
f	a	b	c	d	d	f

Lattice:Example

Let A be the set of positive factors of 15 and R be a relation on A s.t. $R = \{xRy \mid x \text{ divides } y, x\ y \in A\}$

Draw Hasse diagram and give and $\wedge$ and $\vee$ *for lattice.*

Solution : We have A = {1, 3, 5, 15}

R = {(1,1) (1,3) (1,5) (1, 15) (3,15) (5,15) (15,15)}

Hasse diagram of R is :

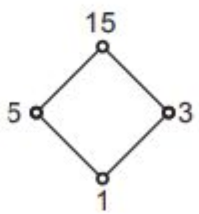


Table for $\wedge$ and $\vee$

$\vee$	1	3	5	15
1	1	3	5	15
3	3	3	15	15
5	5	15	5	15
15	15	15	15	15

$\wedge$	1	3	5	15
1	1	1	1	1
3	1	3	1	3
5	1	1	5	5
15	1	3	5	15

Every pair of elements has lub and glb. It is a lattice.

Lattice:Properties

1) Commutative Law: -

(a) $a \wedge b = b \wedge a$ (b) $a \vee b = b \vee a$

2) Associative Law:-

(a) $(a \wedge b) \wedge c = a \wedge (b \wedge c)$ (b) $(a \vee b) \vee c = a \vee (b \vee c)$

3) Absorption Law: -

(a) $a \wedge (a \vee b) = a$ (b) $a \vee (a \wedge b) = a$

Function

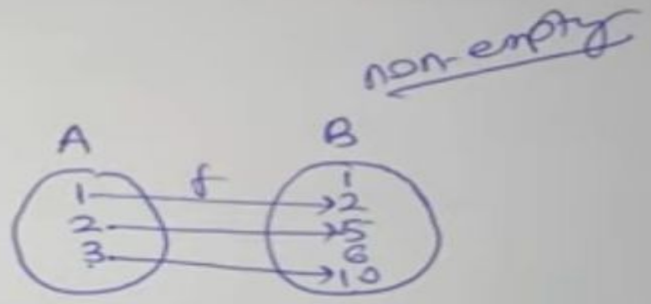
Set → Relation

$f(x) = x^2+1$

$x=1 \quad f(1) = 1^2+1 = 2$

$x=2 \quad f(2) = 5$

$x=3 \quad f(3) = 10$



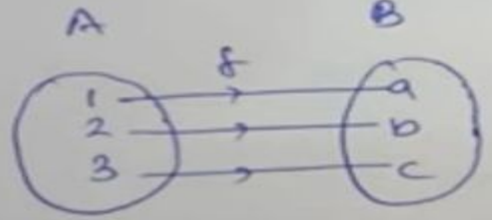


Fig ①

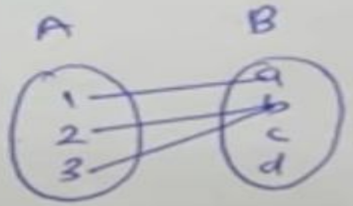


Fig ②

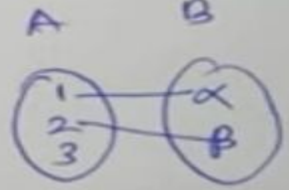


Fig ③

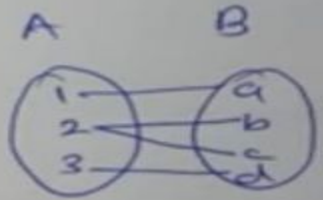
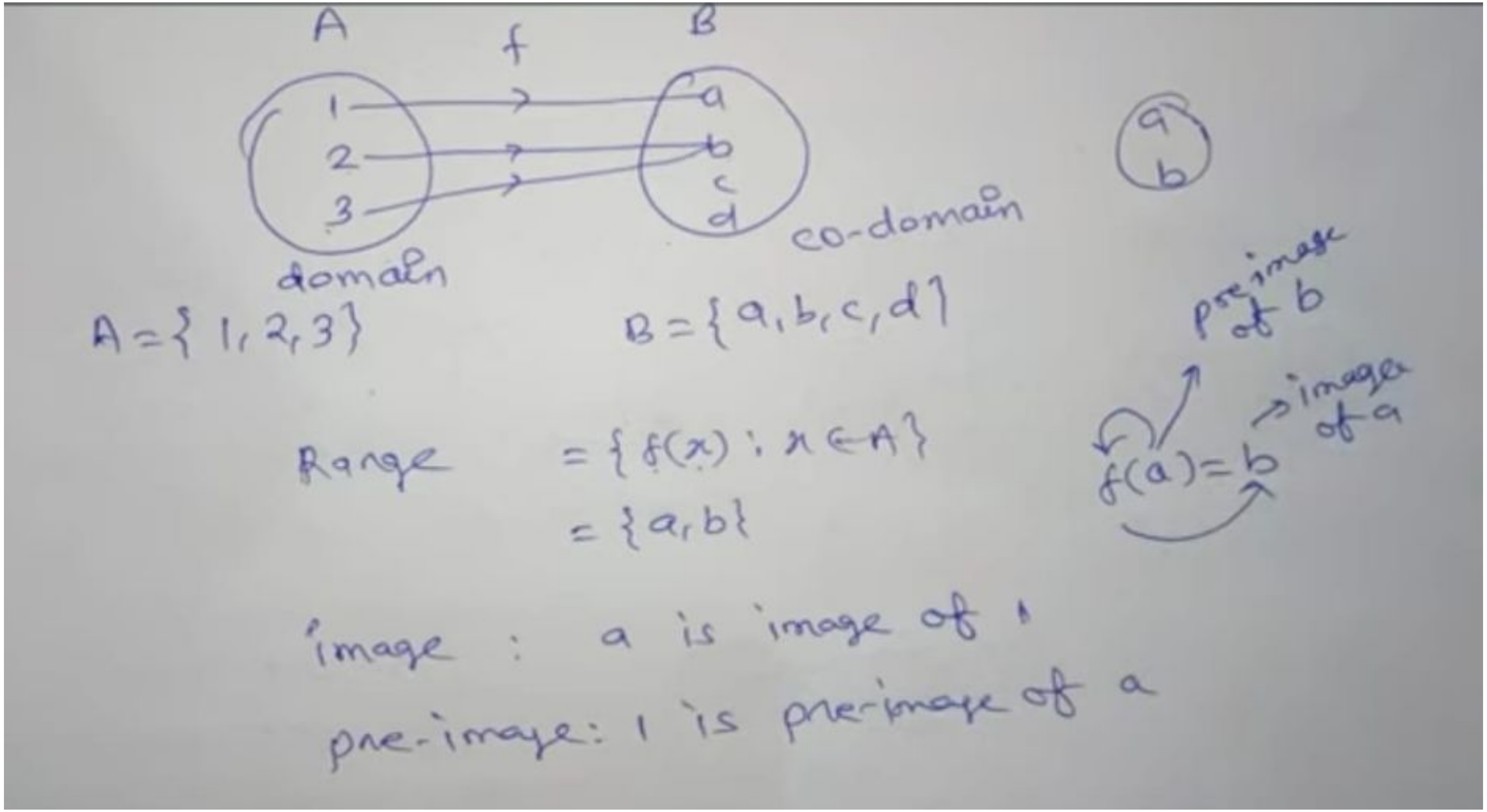


Fig ④

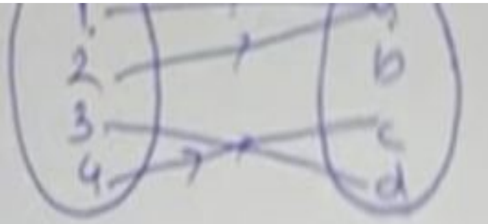
The statement "f is function from A to B" is represented symbollically by

$\boxed{f: A \to B}$ or $\boxed{A \to B}$

Function


A
f
B
1
2
3
a
b
c
d
domain
co-domain
A = {1, 2, 3}
B = {a, b, c, d}
a
b
pre-image of b
image of a
f(a) = b
Range = {f(x) : x ∈ A}
= {a, b}
image : a is image of 1
pre-image: 1 is pre-image of a

Function



If f is a function from A to B. then

(i) set A is called the domain of f denoted by dom$f$

$$\text{dom}.f = \{1,2,3,4\}$$

(ii) set B is called the co-domain of f.

$$\text{co-domain } f = \{a,b,c,d\}$$

(iii) If $(a,b) \in f$, then $f(a)=b$, So b is called the image of a and

Function

(iv) a is called pre-image of b.

(v) The set consisting of all the images of the elements of A under the function f is called the range of f. It is denoted by f(A).

$f(A) = \{ f(x) : \forall\, x \in A \}$ $\quad f(A) = \{a, c, d\}$

(1) **Addition :** Let, f_1 and f_2 be functions from A to B, then $f_1 + f_2$ is defined as

$$\mathbf{f_1 + f_2\,(x) = f_1(x) + f_2\,(x)}$$

(2) **Multiplication :** Let f_1 and f_2 be functions from A to B, then $f_1 \cdot f_2$ are defined as

$$\mathbf{f_1 \cdot f_2(x) = f_1(x) \cdot f_2(x)}$$

Types of function

1) Partial functions :

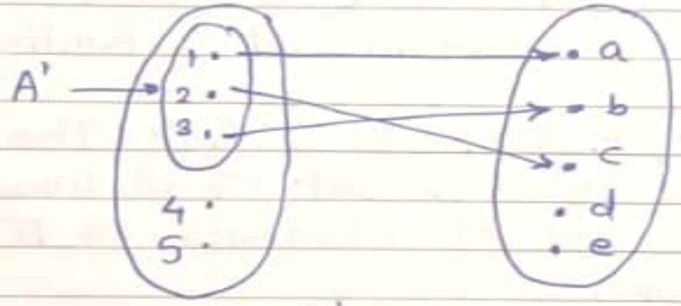
Let , A and B be two non-empty sets. A partial function $f : A' \to B$ for some subset A' of A , where $A' \subset A$ i.e. A' is a proper subset of A .

Ex. $A = \{1, 2, 3, 4, 5\}$, $B = \{a, b, c, d, e\}$

$f : A' \to B$

f is partial function which maps elements of A' (Proper subset of A) to elements of B

$A - A' = \{4, 5\}$ are undefined.

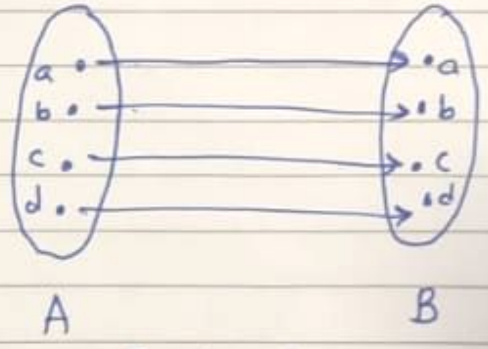


$f : A' \to B$

2) <u>Identity function</u> :

Let, A be a non empty set and f be a function $f: A \to A$ is said to be the identity funⁿ if each element of set A has an image on itself that is $f(a) = a$, $\forall\, a \in A$. It is denoted as I

Ex. Let, $A = \{a, b, c, d\}$, $f: A \to A$



$f: A \to A$

3) Equality of two functions:

Let, f be a function from A to B ($f: A \to B$) and g be a funⁿ from A to B ($g: A \to B$) are equal, when they have same domain, have the same co-domain and map elements of their common domain to the same elements in their common co-domain.

Ex. $f = \{(3,x), (4,y), (5,z)\}$
$g = \{(3,x), (4,y), (5,z)\}$

Here, f and g are equal functions.

4) <u>Constant function:</u>

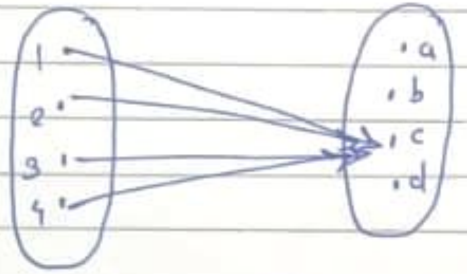
A function is $f: A \to B$ said to be constant funⁿ, if every element of A has the same image in B.

Thus, f be a function $f: A \to B$, $f(x) = c$ for $\forall x \in A$ then $c \in B$ is a constant function

Ex.

$A = \{1, 2, 3, 4\}$, $B = \{a, b, c, d\}$

$f = \{(1,b), (2,b), (3,b), (4,b)\}$



$f: A \to B$

5) Composite function

Let, g be a function from set A to B. Let, f be a function from set B to C. The composition of functions f and g, denoted by $f o g(x)$ or $f(g(x))$ or $f(g(x))$ (read as: "f of g of x").

Ex. $f(x) = 3x + 2$ and $g(x) = x + 5$

composition of f and g is $f o g(x)$ or $f(g(x))$ is

$$f(g(x)) = f(x+5) = 3(x+5) + 2 = 17$$

End of Session 6

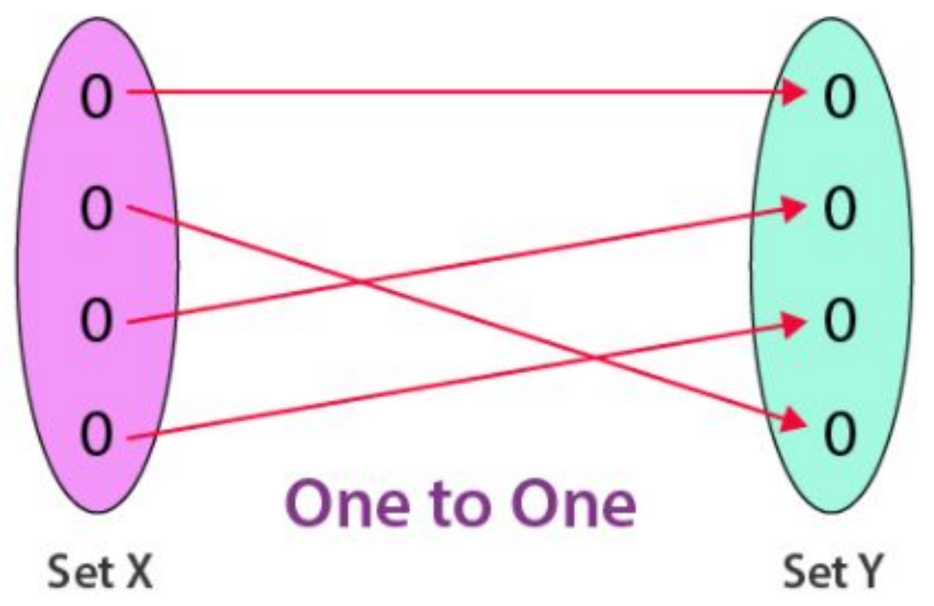
Unit 2 – Relations and Functions

Session 7

- Special types of functions - Surjective, Injective and Bijective functions, Identity function, Partial function, Invertible function, Constant function, Inverse functions and Compositions of functions

One – one function (Injective function)

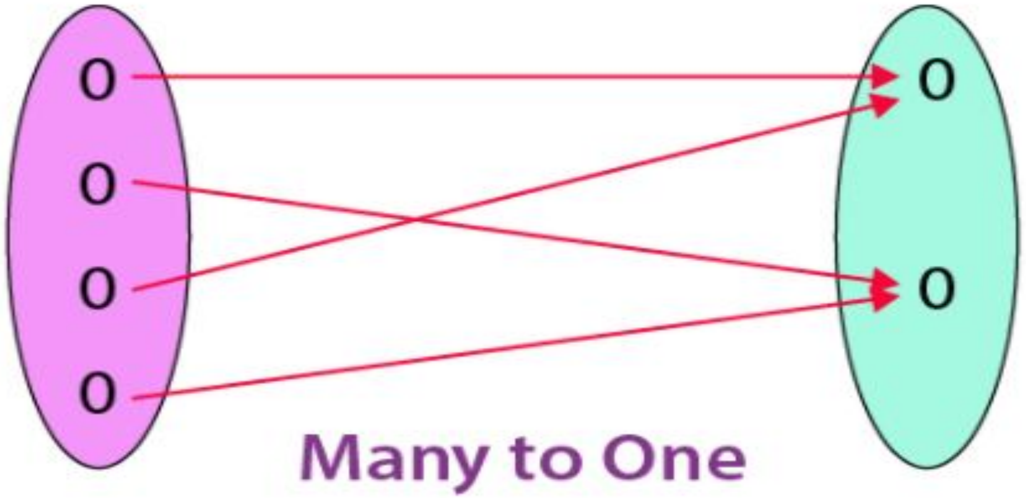
If each element in the domain of a function has a distinct image in the co-domain, the function is said to be one – one function.



For examples f; R R given by f(x) = 3x + 5 is one – one.

Many – one function

On the other hand, if there are at least two elements in the domain whose images are same, the function is known as many to one.



For example f : R R given by f(x) = x^2 + 1 is many one.

Onto Function Definition (Surjective Function)

Onto function could be explained by considering two sets, Set A and Set B, which consist of elements. If for every element of B, there is at least one or more than one element matching with A, then the function is said to be **onto function** or surjective function. The term for the surjective function was introduced by Nicolas Bourbaki.

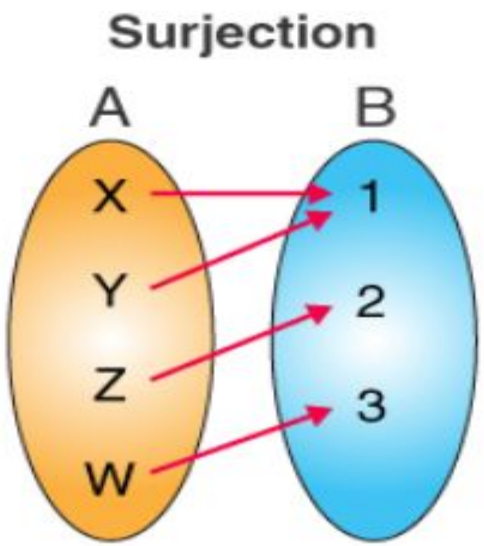


Fig.1

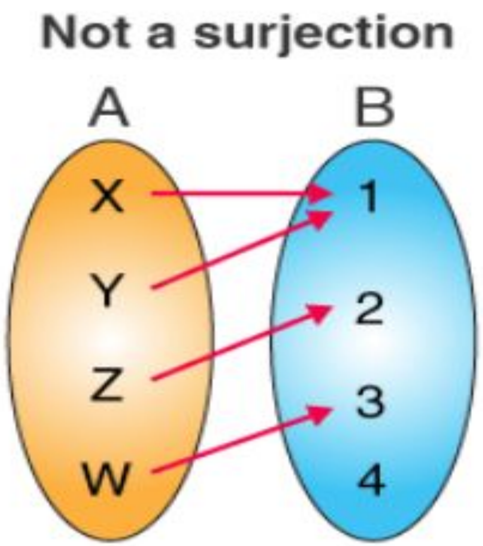
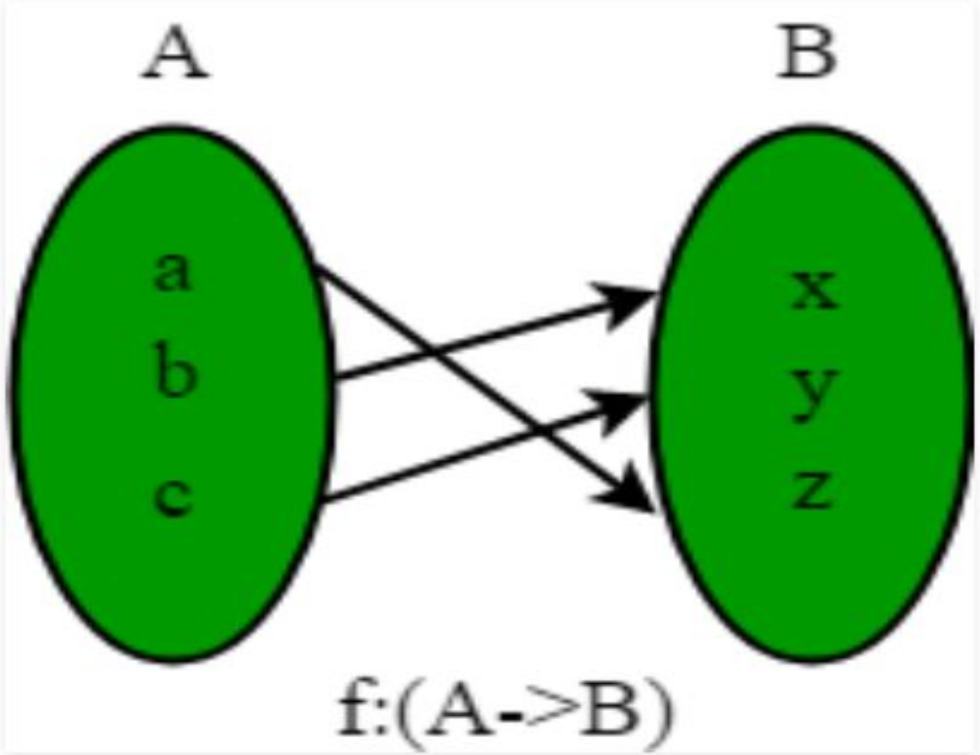


Fig.2

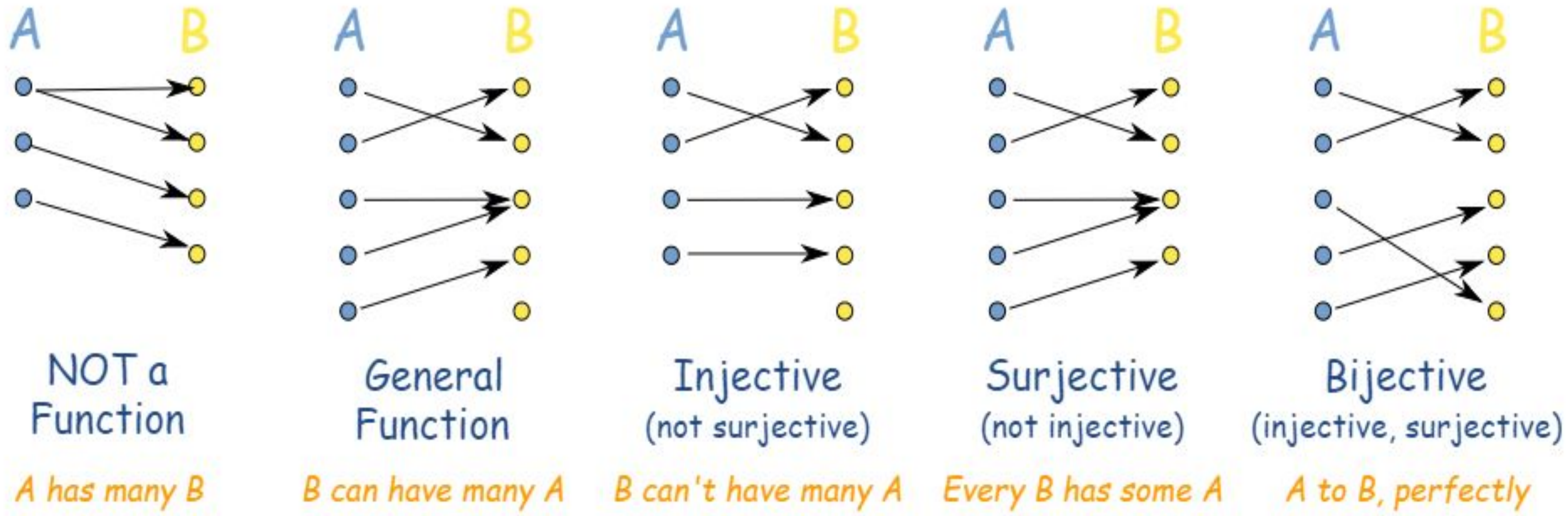
One to one correspondence function(Bijective/Invertible): A function is Bijective function if it is both one to one and onto function.



Inverse Functions: Bijection function are also known as invertible function because they have inverse function property. The inverse of bijection f is denoted as f^{-1}. It is a function which assigns to b, a unique element a such that f(a) = b. hence f^{-1} (b) = a.

Types of Functions

A function is a way of matching the members of a set "A" **to** a set "B":



Composition of Function

Example: $\mathbf{f(x) = 2x+3}$ and $\mathbf{g(x) = x^2}$

$$(g \circ f)(x) = g(f(x))$$

First we apply f, then apply g to that result:

$$(g \circ f)(x) = (2x+3)^2$$

Example: **$f(x) = 2x+3$** and **$g(x) = x^2$**

$$(g \circ f)(x) = g(f(x))$$

First we apply f, then apply g to that result:

$$(g \circ f)(x) = (2x+3)^2$$

we **reverse** the order of f and g?

$$(f \circ g)(x) = f(g(x))$$

First we apply g, then apply f to that result:

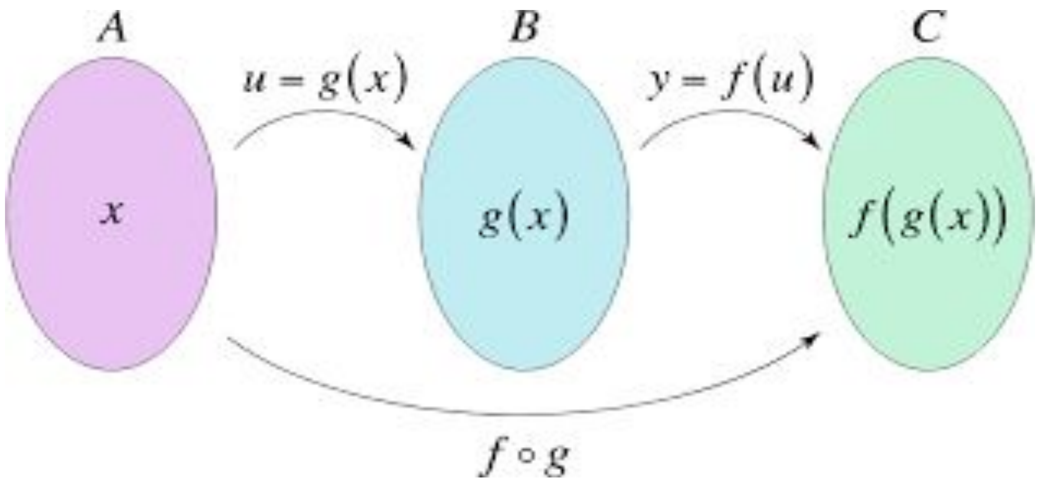
$$(f \circ g)(x) = 2x^2+3$$

Example: **f(x) = 2x+3** and **g(x) = x²**

$$(g \circ f)(x) = g(f(x))$$

First we apply f, then apply g to that result:

$$(g \circ f)(x) = (2x+3)^2$$



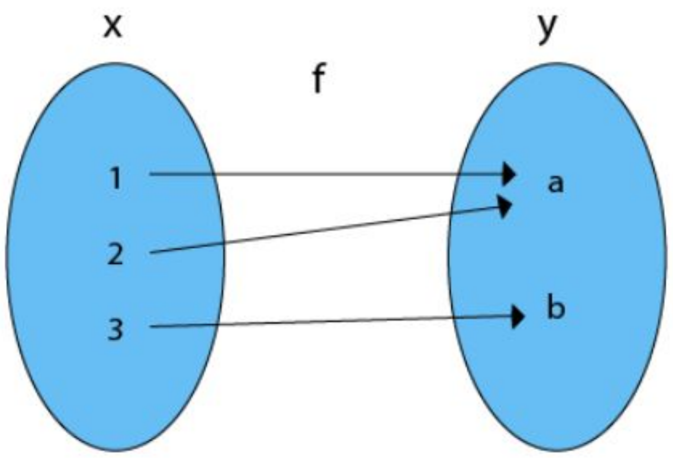
What if we **reverse** the order of f and g?

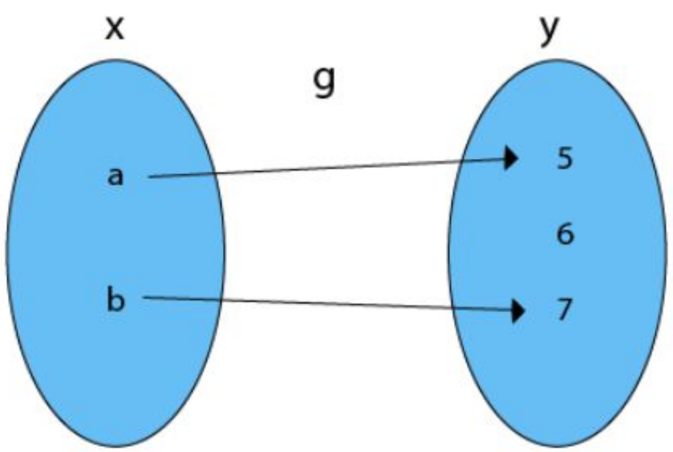
$$(f \circ g)(x) = f(g(x))$$

First we apply g, then apply f to that result:

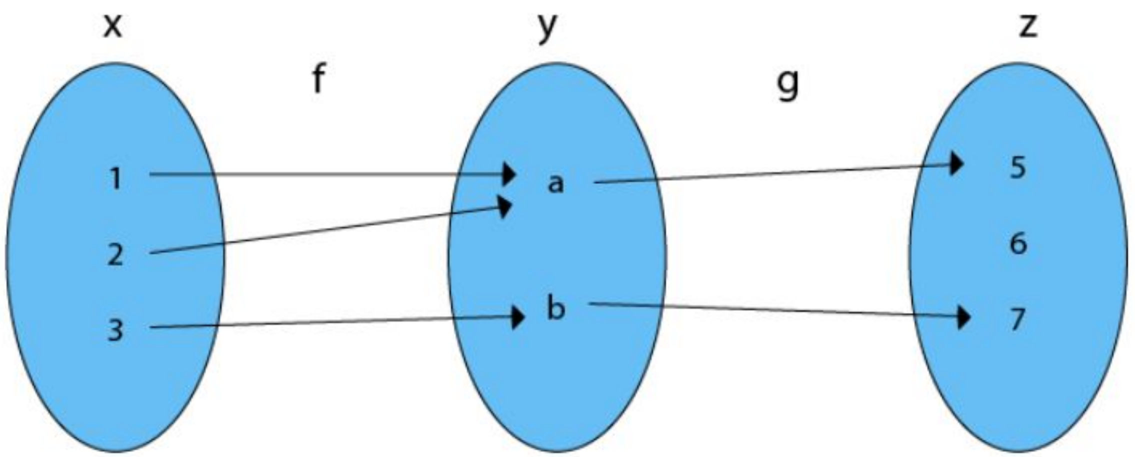
$$(f \circ g)(x) = 2x^2+3$$

Composition of Function





Solution: The composition function gof is shown in fig:

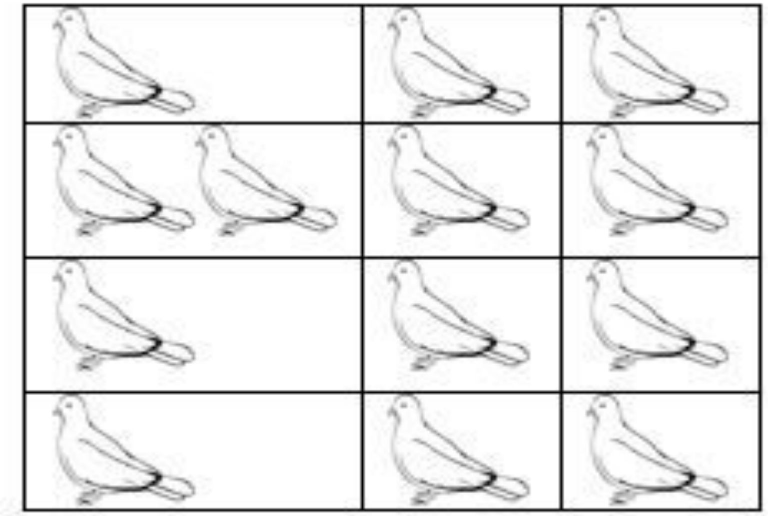



```
(gof) (1) = g [f (1)] = g (a) = 5, (gof) (2) = g [f (2)] = g (a) = 5
(gof) (3) = g [f (3)] = g (b) = 7.
```

The Pigeonhole Principle

- In Discrete Mathematics, it states that if we must put N+1 or more pigeons into N Pigeon Holes, then some pigeonholes must contain two or more.

If n pigeonholes are occupied by n+1 or more pigeons, then at least one pigeonhole is occupied by greater than one pigeon.



Pigeonhole Principle

Generalized pigeonhole principle

- If n pigeonholes are occupied by kn+1 or more pigeons, where k is a positive integer, then at least one pigeonhole is occupied by k+1 or more pigeons.

Pigeonhole Principle

Examples

1. **If (Kn+1) pigeons are kept in n pigeon holes where K is a positive integer, what is the average no. of pigeons per pigeon hole?**

Average number of pigeons per hole = (Kn+1)/n = K + 1/n

Therefore there will be at least one pigeonhole which will contain at least (K+1) pigeons i.e., ceil[K +1/n] and remaining will contain at most K i.e., floor[k+1/n] pigeons. i.e., the minimum number of pigeons required to ensure that at least one pigeon hole contains (K+1) pigeons is (Kn+1).

Pigeonhole Principle

Examples

1. If 11 shoes are selected from 10 pairs of shoes then there must be a pair and matched shoes among the selection.

11 shoes are pigeons 10 pairs are the pigeon holes.

Pigeonhole Principle

Show that if seven numbers from 1 to 12 are chosen then two of them will add up to 13.

A = {1, 2, 3, 4, 5, … 12}

the six different sets each containing 2 numbers that add uptot 13.

A1 = {1, 12}, A2 = {2, 11}, A3 = {3, 10}, A4 = {4, 9}, A5 = {5, 8}, A6 = {6, 7}

Each of the seven numbers chosen must belong to one of these sets. As there are only six sets, by pigeonhole principle two of th $\left\lceil \frac{n-1}{m} \right\rceil + 1$ en numbers must belong to the same set and their sum is 13

Pigeonhole Principle

Show that 7 colours are used to paint 50 bicycles, then at least 8 bicycles will be of same colour.

Solution : By the extended pigeonhole principle, at least $\left\lfloor \frac{n-1}{m} \right\rfloor + 1$ pigeons will occupy one piegeonhole.

Here n = 50, m = 7 and m < n then

$$\left[\frac{50-1}{7}\right] + 1 = 7 + 1 = 8$$

Thus 8 bicycles will be of the same colour.