ACOL 202

Lecture 24
(April 25th)

The number of ways to rearrange a sequence containing $k$ elements $\{x_1, \dots, x_k\}$ where $x_i$ appears $n_i$ times is

$$\frac{(n_1 + n_2 + \dots + n_k)!}{n_1!\, n_2! \dots n_k!}$$

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## Example

How many different ways can we write 10, 800 as a product of primes ?

$$10800 = 2^4 3^3 5^2$$

## Pigeonhole principle

When there are more things than the kind of things then there is more than one of a kind.

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If there are $n$ pigeonholes and there are $n+1$ pigeons then there is at least one pigeonhole with more than one pigeon.

Formally

Let $A$ and $B$ be sets such that $|A| > |B|$.

Let $f: A \to B$ be any function.

Then there exist distinct elements $a$ and $a' \in A$ such that $f(a) = f(a')$

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Consider 17 propositional logic formulas over atoms $p$ and $q$.

| $p$ | $q$ | Formula |
|---|---|---|
| T | T | F |
| T | F | T |
| F | T | F |
| F | F | T |

← 16 different possibilities

Each of these 4 values could be True or False. So, total of $2^4 = 16$ possibilities.

Therefore, if there are 17 formulas, two of them must be logically equivalent.

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## Example

Show that in any group of 5 people, there are two who have an identical number of friends within the group.

At the same time it cannot be the case that someone has 0 friends and someone else has 4 friends.

$0_f$ $\quad 1_f \quad 2_f \quad 3_f \quad 4_f$

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Show that any set of 12 integers contains two integers whose difference is divisible by 11.

When divided by 11, at least two of these 12 integers will have the same remainder.

$$i_1 = 11 a_1 + r$$

$$i_2 = 11 a_2 + r$$

$$i_1 - i_2 = \underline{11 (a_1 - a_2)}$$

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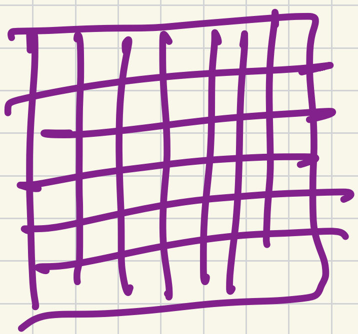
**Exercise** Suppose there are $n^2+1$ points in a 1-by-1 square. Show that there must be at least one point within a distance of $\frac{\sqrt{2}}{n}$ of another point.

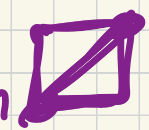


$\frac{1}{n} \times \frac{1}{n}$

$n^2$

$1/n$

$1/n$



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# Combinations and Permutations

**Example** Suppose you are running a printing shop and you have 17 jobs in the queue. You wish to pick 4 jobs to run.

How many different ways can you select these 4 out of 17 jobs?

Ans. It depends.

(On whether the order matters or not.)

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Ordered 4-tuple

$17 \times 16 \times 15 \times 14$

An unordered subset of size 4.

$$\frac{17 \times 16 \times 15 \times 14}{4!}$$

We can also look at this differently.

17! different permutations.

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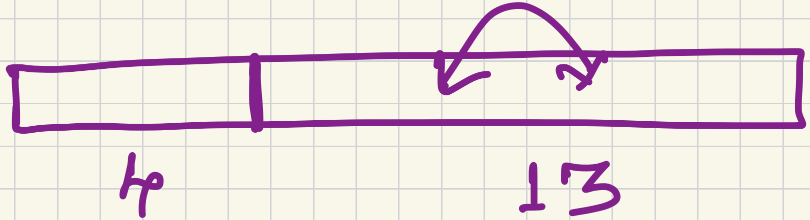


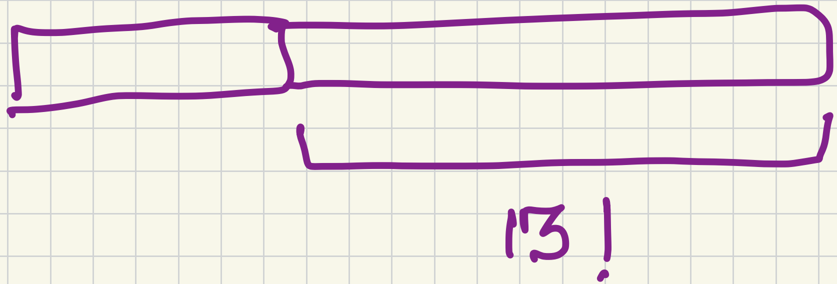


$$\frac{17!}{4!\,13!} = \frac{17\times16\times15\times14\times\cancel{13!}}{\cancel{13!}}$$

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Permutation - where the order of the chosen elements matters

Combinations - " " " " " " " " " " does not matter

Consider non-negative integers $n$ and $k$ with $k \le n$.

$\binom{n}{k}$ $\quad {}^{n}C_{k}$ $\quad C(n,k)$

is defined as $\binom{n}{k} = \dfrac{n!}{k!\,(n-k)!}$

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How many 8-bit strings are there with exactly two 1's?

$$\binom{8}{2} \qquad \frac{8!}{6!\,2!} = \frac{8\times 7}{2} = 28$$

$$\_\ \_\ \_\ \_\ \_\ \_\ \_\ \_$$

$$1_2 \quad 1_2 \quad 1_2$$

$$0+1+2+3+\cdots+7$$

$$28 = \frac{7\times 8}{2}$$

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What about choosing an element more than once ?

( i.e, if repetition is allowed )

- order matters
  - rep allowed — $n^k$
  - rep not allowed — $\dfrac{n!}{(n-k)!}$
- order does not matter
  - rep allowed — $\dbinom{n+k-1}{k}$
  - rep not allowed — $\dbinom{n}{k}$

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Claim The number of ways to select $k$ out of $n$ elements when order does not matter but repetition is allowed is $\binom{n+k-1}{k}$.

$$\boxed{X} = \left\{ x \in \left(\mathbb{Z}^{\geq 0}\right)^n \;\middle|\; \sum_{i=1}^{n} x_i = k \right\}$$

$$\boxed{S} = \left\{ x \in \{0,1\}^{n+k-1} \;\middle|\; x \text{ contains exactly } (n-1) \text{ 1's and } k \text{ zeros} \right\}$$

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$$\Big[\quad \underbrace{000\ldots0}_{x_1}\ \underset{1}{\underset{\big\downarrow}{1}}\ \underbrace{00\ldots0}_{x_2}\ \underset{2}{\underset{\big\downarrow}{1}}\ \ldots\ldots\ \underset{(n-1)}{\underset{\big\downarrow}{1}}\ \underbrace{0000\ldots0}_{x_n}$$

$$\binom{n-1+k}{k} = \binom{n+k-1}{n-1}$$

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Claim

$$\binom{n}{k} = \binom{n}{n-k}$$

$$\frac{n!}{k!\,(n-k)!} \qquad \frac{n!}{(n-k)!\,(n-(n-k))!}$$

$$= \binom{n}{n-k}$$

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## Pascal's identity

For any integer $n \geq 1$ and $k \in \{0, 1, \ldots, n\}$

$$\binom{n-1}{k} + \binom{n-1}{k-1} = \binom{n}{k}$$

Algebraic proof : exercise

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# The Binomial Theorem

$$(x+y)(x+y) = xx + xy + yx + yy$$

$$= x^2 + 2xy + y^2$$

$$= \binom{2}{0} x^2 + \binom{2}{1} xy + \binom{2}{2} y^2$$

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$$\boxed{\text{Claim}}$$

For any $a, b \in \mathbb{R}$
$n \in \mathbb{Z}^{\geq 0}$

$$(a+b)^n = \sum_{i=0}^{n} \binom{n}{i} a^i b^{n-i}$$

Proof idea

$$(x+y)^3 = (x+y)\left[\binom{2}{0}x^2 + \binom{2}{1}xy + \binom{2}{2}y^2\right]$$

$$= \underline{\binom{2}{0}x^3} + \underline{\binom{2}{1}x^2y} + \underline{\binom{2}{2}}xy^2 + \underline{\binom{2}{0}x^2y} + \underline{\binom{2}{1}}xy^2 + \underline{\binom{2}{2}y^3}$$

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$$\binom{2}{0}x^3 + \underbrace{\left(\binom{2}{1}+\binom{2}{0}\right)}_{\binom{3}{1}} x^2y + \underbrace{\left(\binom{2}{2}+\binom{2}{1}\right)}_{\binom{3}{2}} xy^2 + \binom{2}{2}y^3$$

$$\binom{3}{0} \qquad \binom{3}{1} \qquad \binom{3}{2} \qquad \binom{3}{3}$$

**Exercise**

[ You should prove this by induction and use Pascal's identity as above.

# Pascal's Triangle

$$\binom{n}{k} \quad \binom{n}{n-k}$$

$$\binom{0}{0}$$

$$\binom{1}{0} \quad \binom{1}{1}$$

$$\binom{2}{0} \quad \binom{2}{1} \quad \binom{2}{2}$$

$$\binom{3}{0} \quad \binom{3}{1} \quad \binom{3}{2} \quad \binom{3}{3}$$

| | |
|---|---|
| 1 | 1 |
| 2 | 1 1 |
| 4 | 1 2 1 |
| 8 | 1 3 3 1 |
| 16 | 1 4 6 4 1 |
| 32 | |

Claim

$$\sum_{i=0}^{n} \binom{n}{i} = 2^n$$

$$2^n = (1+1)^n$$

$$= \sum_{i=0}^{n} \binom{n}{i} \cancel{1^i \, 1^{n-i}}$$

Analysis of Algorithms

Graph Theory

Probability

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