ACOL 202

Lecture 11
(18th Feb)

Claim: Let n be an integer. Then n is even iff n^2 is even.

Counter example

Claim No positive integer is expressible in two different ways as the sum of two perfect squares.

50 $25+25$ $1+49$

65 1^2+8^2 4^2+7^2

Constructive vs. Non-constructive Proofs

Claim: There exist distinct integers $x, y \in \{1901, 1902, 1903, \ldots, 2025\}$ such that the last two digits of x^2 and y^2 are the same.

Claim: There exist irrational numbers a and b such that a^b is rational.

Not constructive

$\sqrt{2}^{\sqrt{2}}$ → rational ✓

→ irrational

$$\left(\sqrt{2}^{\sqrt{2}}\right)^{\sqrt{2}} = 2$$

$$a = \sqrt{2} \qquad b = \boxed{\log_2 9} = p/q$$

$$a^b = 3$$

$$a^b = \sqrt{2}^{\log_2 9} = \cancel{2^{\frac{1}{2}} \cdot 2^{\log_2 9}} = 2^{\frac{1}{2} \cdot \log_2 9}$$

$$= 2^{\log_2 3}$$

$$= 2^{\log_2 3} = 3$$

Why is $\log_2 9$ irrational?

Suppose not. Let $\log_2 9 = p/q$, where p & q are integers and p/q is in its lowest form.

$$q \log_2 9 = p$$

$$\log_2 9^q = p \qquad 9^q = 2^p$$

Claim There are infinitely many prime numbers.

Assume not. Let p be the largest prime.

$\therefore$ $(p!+1)$ must be composite.

Let k be the smallest number > 1 which divides $(p! + 1)$.

Clearly, $k > p$. Why?

(We proved in the last class that $(n! + 1)$ has no factors between 2 and n.)

But k must then be composite.

why? (p was the biggest prime)

There must be a k' that divides k evenly.

k' must also divide $(p!+1)$ evenly.

This contradicts our claim that k was the smallest such number which divides $(p!+1)$.

Claim

If you are caught cheating in the exam then you will get a zero.

You were not caught cheating in the exam.

Therefore, you will not get a zero.

$r \Rightarrow q$

$p \Rightarrow q$

$\neg p$

$\therefore$ Conclude, $\neg q$

Claim

Let $F_n = \{ k \in \mathbb{Z}^{\geq 1}$ such that $k | n \}$ denote the factors of an integer $n \geq 2$.

Then $|F_n|$ is even.

Proof.

$\left.\begin{array}{l} f_{small} \\ f_{big} \end{array}\right\}$ denote the factors that are smaller and bigger than $\sqrt{n}$

Claim $|F_{small}| = |F_{big}|$

$$f_9 = \{1, 3, 9\}$$

Denying the hypothesis

Claim: If you climb Everest you will be famous.

You have not climbed Everest.

$\therefore$ You are not famous.

Read sections 4.3.2, 4.3.3 and ~~the section~~ on common errors in proofs

Confusing correlation with causation } ||

Ad hominem attack

"To the man"

n is even $\Longleftrightarrow$ n^2 is even

$\Rightarrow$ ✓

$\Leftarrow$ n^2 is even $\Rightarrow$ n is even

Contrapositive

n not even $\Rightarrow$ n^2 not even

✓

$n = 2k+1$

$n^2 = (2k+1)^2 = 4k^2 + 4k + 1$

odd