

COL703: Logic for Computer Science (Aug-Nov 2022)

Lectures 17 & 18 (Herbrand's Theorem, Ground Resolution)

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Proofs for recap: Logical equivalence

$$(\forall x \, F \wedge G) \equiv \forall x \, (F \wedge G) \quad (\text{if } x \text{ does not occur free in } G)$$

Proof:

Proofs for recap: Renaming bound variables

Let F denote the formula $Qx \ G$ where Q is a quantifier. Let y be a variable that does not occur in G .

Then $F \equiv Qy \ (G[y/x])$.

Proof:

Proofs for recap: Skolem Form

Let $F = \forall y_1 \forall y_2 \dots \forall y_n \exists z G$ be a rectified formula. Given a function symbol f of arity n that does not appear in F , write

$$F' = \forall y_1 \forall y_2 \dots \forall y_n G[f(y_1, y_2, \dots, y_n)/z].$$

Then F and F' are equisatisfiable.

Proof:

Translation Lemma

If t is a term and F is a formula such that no variable in t occurs bound in F ,
then $\mathcal{A} \models F[t/x]$ iff $\mathcal{A}_{[x \mapsto \mathcal{A}(t)]} \models F$.

Proof: reading exercise

Herbrand structure

universe is the set of ground terms

terms and function symbols being interpreted "as themselves"

built from syntax

Herbrand structure

Definition 1. Let σ be a signature with at least one constant symbol. A σ -structure \mathcal{H} is called a *Herbrand structure* if the following hold:

1. The universe $U_{\mathcal{H}}$ is the set of ground terms over σ .
2. For every constant symbol c in σ we have $c_{\mathcal{H}} = c$.
3. For every k -ary function symbol f in σ and for all ground terms $t_1, t_2, \dots, t_n \in U_{\mathcal{H}}$ we have $f_{\mathcal{H}}(t_1, \dots, t_k) = f(t_1, \dots, t_k)$.

Interpretation of a ground term

Let \mathcal{H} be a Herbrand structure, and t be a ground term.

Then, $\mathcal{H}[[t]] = t$.

Translation Lemma for Herbrand structures

Let \mathcal{H} be a Herbrand structure, F be a formula, and t be a ground term.

Then $\mathcal{H} \models F[t/x]$ if and only if $\mathcal{H}_{[x \mapsto t]} \models F$.

Herbrand's Theorem and Proof

Let $F := \forall x_1 \dots \forall x_n F^*$ be a **closed formula** in Skolem form.

Then F is satisfiable iff it has a Herbrand model.

Proof:

Example

Is the following formula satisfiable?

$$F := \exists x_1 \exists x_2 \exists x_3 (\neg(\neg P(x_1) \rightarrow P(x_2)) \wedge \neg(\neg P(x_1) \rightarrow \neg P(x_3)))$$

Next class

- Ground resolution
- Undecidability result

Thank you!