1. **[1 mark]** Is the following a tautology? Justify your answer.
   $p \to (q \to p)$

   Ans: Yes; it is a tautology.

p	q	$(q \to p)$	$(p \to (q \to p)$
F	F	T	T
F	T	F	T
T	F	T	T
T	T	T	T

2. **[1 mark]** Are $\neg(p \leftrightarrow q)$ and $\neg p \leftrightarrow q$ logically equivalent? Justify your answer.

   Ans: Yes; they are logically equivalent.

p	$\neg p$	q	$(p \leftrightarrow q)$	$\neg(p \leftrightarrow q)$	$(\neg p \leftrightarrow q)$
F	T	F	T	F	F
F	T	T	F	T	T
T	F	F	F	T	T
T	F	T	T	F	F

3. **[1 mark]** Prove that implication is not associative by giving a truth assignment in which $(p \to (q \to r))$ and $((p \to q) \to r)$ have different truth values.

   Ans: Consider the assignment: $p \mapsto$ False, $q \mapsto$ True, and $r \mapsto$ False.

   Under this assignment, $(p \to (q \to r))$ is True, but $((p \to q) \to r)$ is False.

4. **[1.5 mark]** Consider the following popular puzzle. When asked for the ages of her three children, Mrs. Baker says that Alice is her youngest child if Bill is not her youngest child, and that Alice is not her youngest child if Carl is not her youngest child. Encode these facts, and the necessary background knowledge that only one of the three children can be her youngest child, into propositional logic formulas. Use propositions a, b and c to denote that Mrs. Baker's youngest child is Alice, Bill and Carl, respectively. Find out, with the help of truth tables, who the youngest child is.

   Ans: Bill is the youngest child. There is only one row in the truth table that is consistent with all the given information. And b is true in that row.

a	b	c	$\neg a$	$\neg b$	$\neg c$	$(\neg b \to a)$	$(\neg c \to \neg a)$	Solution?
T	F	F	F	T	T	T	F	Not possible. $(\neg c \to \neg a)$ is given to be True.
F	T	F	T	F	T	T	T	Only possible solution; Bill is the youngest.
F	F	T	T	T	F	F	T	Not possible. $(\neg b \to a)$ is given to be True.
...	...	...	...	...	...	...	...	Other combinations are not possible. Exactly one of a, b, and c must be True.

5. **[1.5 mark]** Let p and q be atomic propositions, and α and β be propositional logic formulas on p and q. Consider the following definitions for α and β:

   - $\alpha = (p \to \neg\beta)$
   - $\beta = (q \to \neg\alpha)$

   Show that there are exactly two pairs of propositional logic formulas (α, β) which satisfy the above definitions. Note that logically equivalent formulas should not be considered as different formulas while solving this problem.

   Ans: From the given definitions, we know that α and β must evaluate to True when, respectively, p and q take the value False. Further, when p is True and q is false, because β evaluates to True, $\neg\beta$ must be False. Thus, α must evaluate to False. Similarly, β must evaluate to False when p is False and q is True.

   When both p and q are True, the only constraint that we have is that α and β are negations of one another. This shows that there are two pairs of formulas satisfying the above definitions (upto logical equivalence).

p	q	α	β	$(p \to \neg\beta)$	$(q \to \neg\alpha)$	Remarks
F	F	F	F	T	T	$\alpha \neq (p \to \neg\beta)$
F	F	F	T	T	T	$\alpha \neq (p \to \neg\beta)$
F	F	T	F	T	T	$\beta \neq (q \to \neg\alpha)$
F	F	T	T	T	T	
F	T	F	F	T	T	$\alpha \neq (p \to \neg\beta)$
F	T	F	T	T	T	$\alpha \neq (p \to \neg\beta)$
F	T	T	F	T	F	
F	T	T	T	T	F	$\beta \neq (q \to \neg\alpha)$
T	F	F	F	T	T	$\alpha \neq (p \to \neg\beta)$
T	F	F	T	F	T	
T	F	T	F	T	T	$\beta \neq (q \to \neg\alpha)$
T	F	T	T	F	T	$\alpha \neq (p \to \neg\beta)$
T	T	F	F	T	T	$\alpha \neq (p \to \neg\beta)$
T	T	F	T	F	T	
T	T	T	F	T	F	
T	T	T	T	F	F	$\alpha \neq (p \to \neg\beta)$