ACOL 202 Lecture 29

Probability

We are interested in a random process that chooses a particular outcome from a set of possibilities.

Sample space

Let S be a sample space (the set of possible outcomes).

A probability function $Pr : S \to \mathbb{R}$ describes, for each outcome $s \in S$, the fraction of the time s occurs.

It must satisfy the following conditions:

also sometimes called a probability distribution over S

$$\sum_{s \in S} Pr[s] = 1$$

$$Pr[s] \geq 0 \qquad \forall s \in S$$

<u>Example</u> Draw a card from a perfectly shuffled deck of cards (52 cards)

$$\Pr[c] = \frac{1}{52}$$

$$\sum_{c \in C} \Pr[c] = 1$$

__Example__ From a single sample space, one may choose an outcome in different ways.

__Process 1__

$$S = \{0, 1, 2, \ldots, 7\}$$

3 fair __coins__

HHH	7
HHT	6
HTH	
HTT	
⋮	
TTT	0

$$\Pr[4] = \frac{1}{8}$$

$$\Pr[7] = \frac{1}{8}$$

Process 2

Toss 7 fair coins and pick the number that is same as the number of heads that appear

128

$\boxed{H\ H\ H\ H\ H\ H\ H}$

H H H H H H T

H H H H H T H

H H H H T H H

$Pr[7] = \frac{1}{128}$

$Pr[4] = \frac{\binom{7}{4}}{128}$

Event Let S be a sample space with a prob. function Pr. An event is a subset of S.

$$Pr[E] = \sum_{s \in E} Pr[s]$$

Example Two coins are flipped. What is the prob. of getting at least one head.

$$\overbrace{HH, HT, TH}^{3/4}, TT$$

$$\frac{1}{4} \quad \frac{1}{4} \quad \frac{1}{4} \quad \frac{1}{4}$$

Let S be a sample space and let $A \subseteq S$ and $B \subseteq S$ be two events.

$$Pr[S] = 1$$

$$Pr[\phi] = 0$$

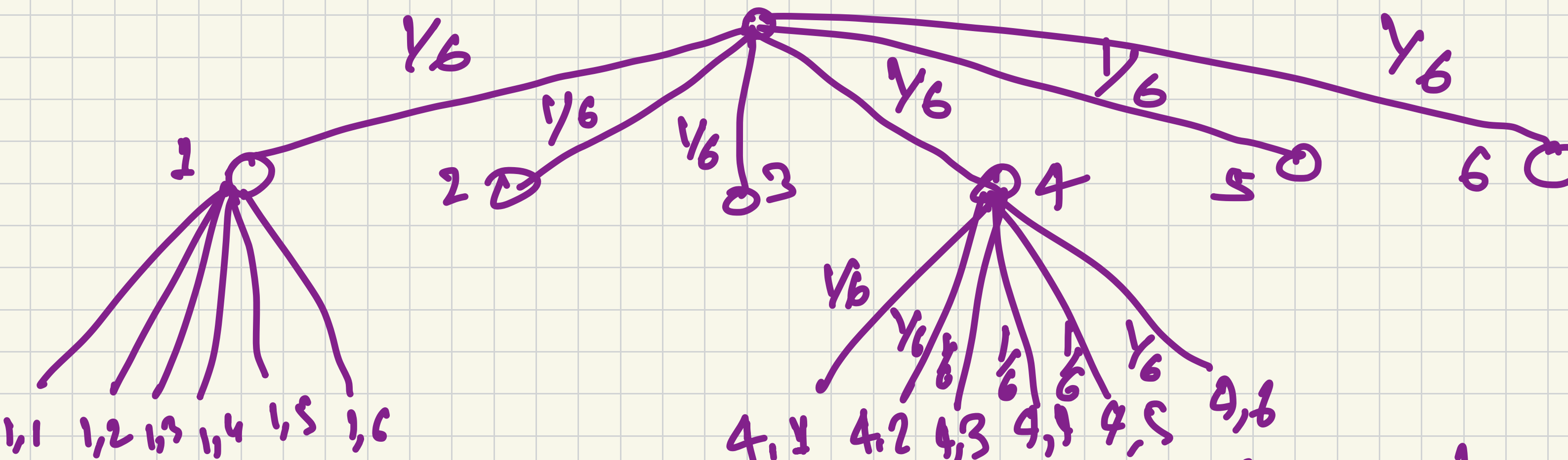
$$Pr[\bar{A}] = 1 - Pr[A]$$

$$Pr[A \cup B] = Pr[A] + Pr[B] - Pr[A \cap B].$$

Tree Diagrams

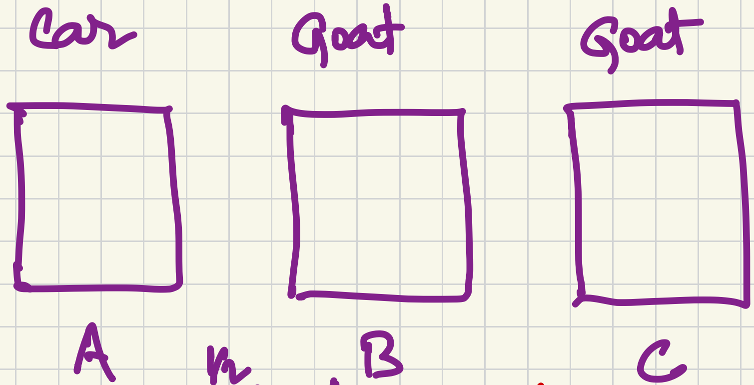
Sequence of random choices rather than a single random choice.

Example Rolling two fair dice, one after another.



$$\Pr[\text{same value twice}] = 6 \cdot \frac{1}{36} = \frac{1}{6}$$

Example



1/3 Car in A
- 1/3 → Pick A
  - 1/2 → open B NS
  - 1/2 → open C NS
- → Pick B 1 → open C S
- → Pick C 1 → open B S

1/3 Car in B
- → Pick A 1 → open C S
- → Pick B
  - → A NS
  - → C NS
- → Pick C → open A S

1/3 Car in C
- → Pick A → open B S
- → Pick B → open A S
- → Pick C
  - → open A NS
  - → open B NS

Pr[switching leads to winning the car]

$= 6/9$

$= 2/3$

Common Probability Distributions

1. Uniform distribution — each outcome is equally likely.

2. Bernoulli distribution (with a parameter p)

Prob. distribution resulting from flipping a p-biased coin

$$Pr[H] = p \qquad Pr[T] = 1-p$$

3. Binomial distribution (with parameter n and p)

distribution over the sample space $\{0, 1, \ldots, n\}$

where $Pr[k]$ denotes the probability of getting exactly k heads when n p-biased coins are flipped.

$$\Pr[k] = \binom{n}{k} p^k (1-p)^{n-k}$$

iv) Geometric distribution (with parameter p)

Consider a p-biased coin which we keep flipping till we get a head.

$\Pr[k]$ denotes the prob. of getting the first head in exactly k flips.

Sample space $\mathbb{Z}^{\geq 1}$

$$\Pr[k] = (1-p)^{k-1} \cdot p$$

Independent and dependent events.

Two events A and B are independent iff $Pr[A \cap B] = Pr[A] \cdot Pr[B]$.

A and B are dependent if they are not independent.

Example Draw a card from a deck of perfectly shuffled cards.

A = drawing an ace B: drawing a heart

$Pr[A] = \frac{1}{13}$ $Pr[B] = \frac{1}{4}$ $Pr[A \cap B] = 1/52$

$\therefore$ Independent.

A : drawing a spade B: drawing a heart

______________________ dependent.

$\Pr[A \cap B] > \Pr[A] \cdot \Pr[B]$

then A & B are said to be positively correlated

$\Pr[A \cap B] < \Pr[A] \cdot \Pr[B]$

. . - - . negatively correlated.

Suppose I flip two p-biased coins (independently)

Consider events

A : the first flip comes up heads

B : the second flip comes up heads

C : the two flips match (both heads, or both tails).

Which pairs of these events are independent?

Sample space $\{HH, HT, TH, TT\}$

$A : \{HH, HT\}$

$B : \{HH, TH\}$

$C : \{HH, TT\}$

$A \cap B = B \cap C = A \cap C = \{HH\}$

$\Pr[A] = \Pr[\{HH, HT\}] = p^2 + p \cdot (1-p)$

$= p$

$\Pr[B] = p$

$\Pr[C] = 2p^2 - 2p + 1$

$$\Pr[A \cap B] = \Pr[B \cap C] = \Pr[A \cap C] = p^2$$

$$\Pr[A \cap C] = \Pr[A] \cdot \Pr[C]$$

iff $\quad p^2 = p \cdot (2p^2 - 2p + 1)$

iff $\quad 0 = p \cdot (2p^2 - 2p + 1 - p)$

$$= p \cdot (2p(p-1) - (p-1))$$

$$= \underset{0}{p}\,(\underset{1}{p-1})\,(\underset{1/2}{2p-1}) \leftarrow$$

Conditional Probability

The conditional probability of A given B written $\Pr[A|B]$ is given by

$$\Pr[A|B] = \frac{\Pr[A \cap B]}{\Pr[B]}.$$

(undefined when $\Pr[B] = 0$).

Example I choose a number uniformly at random from $\{1, \ldots, 10\}$.

A = chosen number is odd B : chosen no. is prime

$\Pr[A|B]$ $\Pr[B|A]$

Consider events A and B such that $\Pr[B] \neq 0$.

A and B are independent iff $\Pr[A \mid B] = \Pr[A]$

Conditional independence

Let A, B, and C be events. A and B are said to be conditionally independent given C if $\Pr[A \mid B \cap C] = \Pr[A \mid C]$.

The Chain Rule.

Let A and B be arbitrary events.

Then $\Pr[A \cap B] = \Pr[B] \cdot \Pr[A \mid B]$

More generally, for a collection of events $A_1, A_2, \ldots, A_k$

Convince yourself that this is correct

$$\Pr[A_1 \cap A_2 \cap A_3 \ldots \cap A_k] =$$

$$\Pr[A_1] \cdot \Pr[A_2 \mid A_1] \cdot \Pr[A_3 \mid A_1 \cap A_2] \cdot \ldots$$

$$\ldots \Pr[A_k \mid A_1 \cap A_2 \cap \ldots \cap A_{k-1}]$$

Example

Drawing a heart flush in poker

Let H_i denote that i^{th} card drawn is a heart.

$$\Pr[H_1 \cap H_2 \cap H_3 \cap H_4 \cap H_5]$$

$$= \Pr[H_1] \cdot \Pr[H_2 \mid H_1] \cdot \Pr[H_3 \mid H_{1,2}] \cdot \Pr[H_4 \mid H_{1,2,3}] \cdot \Pr[H_5 \mid H_1 H_2 H_3 H_4]$$

$$= \frac{13}{52} \cdot \frac{12}{51} \cdot \frac{11}{50} \cdot \frac{10}{48} \cdot \frac{9}{48}$$

Theorem (Law of Total Probability)

Let A and B be arbitrary events.

$$\Pr[A] = \Pr[A \mid B] \cdot \Pr[B] + \Pr[A \mid \bar{B}] \cdot \Pr[\bar{B}]$$

$$\Pr[A] = \Pr[(A \cap B) \cup (A \cap \bar{B})]$$

$$= \Pr[A \cap B] + \Pr[A \cap \bar{B}]$$

$$= \underbrace{\Pr[A \mid B] \cdot \Pr[B]}_{\text{chain rule}} + \Pr[A \mid \bar{B}] \cdot \Pr[\bar{B}]$$