1. Prove that $f(n)$ is $O(g(n))$ if and only if $g(n)$ is $\Omega(f(n))$.

2. Let $f : \mathbb{Z}^{\geq 0} \to \mathbb{Z}^{\geq 0}$ be an arbitrary function. Define the function $g(n) = f(n) + 1$. Prove that $g(n)$ is $O(f(n))$ if and only if $f(n)$ is $\Omega(1)$.

3. The Towers of Hanoi is the following classic puzzle. There are three posts (the "towers"); post A starts with n concentric discs stacked in order of their radius (smallest radius at the top, largest radius at the bottom). We must move all the discs to post B, never placing a disc of larger radius on top of a disc of smaller radius. The easiest way to solve this puzzle is with recursion. You shift the top $(n-1)$ discs first to post C, then move the largest disc to post B, and then once again move all the discs from C to B. Argue that the total number of moves made in this fashion satisfies $T(n) = 2T(n-1) + 1$ and $T(1) = 1$. Prove that $T(n) = 2^n - 1$. (*Hint: Use Induction.*)

4. Prove or disprove the following.

   (a) All 5-node graphs with degrees 1, 1, 1, 1, and 0 are isomorphic.

   (b) All 5-node graphs with degrees 4, 4, 4, 3, and 3 are isomorphic.

   (c) All 5-node graphs with degrees 3, 3, 2, 2, and 2 are isomorphic.

   (d) All n-node, 3-regular graphs are isomorphic.

5. Consider the graph $V = \{1, 2, \ldots, n\}$ and $E = \{\langle i, i-1 \rangle : i \geq 2\}$. For which n is this graph bipartite? Prove that your answer is correct.

6. For which n is the graph $V = \{0, 1, \ldots, n-1\}$ and $E = \{\langle i, i+1 \bmod n \rangle : i \geq 1\}$ bipartite? Prove that your answer is correct.

7. **[4 marks]** Consider a bipartite graph with a set L of nodes in the left column and a set of nodes R on the right column, where $|L| = |R|$. Prove or disprove the following claims.

   (a) The sum of the degrees of the nodes in L must equal the sum of the degrees of the nodes in R.

   (b) The sum of the degrees of the nodes in L must be even.

8. Suppose that G is graph that does not contain a triangle (that is, there is no set of three nodes a, b, and c with the edges $\{a, b\}$ and $\{b, c\}$ and $\{c, a\}$ all appearing in the graph). Prove or disprove: G is bipartite.

9. Prove that any 2-regular graph is planar.