ACOL 202

Lecture 13

Mathematical Induction

To give a proof by mathematical induction of

$\forall n \in \mathbb{Z}^{\geq 0} : P(n)$,

we prove the following:

i) the base case, $P(0)$

ii) the inductive step, $\boxed{P(n) \Rightarrow P(n+1) \text{ for every } n.}$

$\boxed{\forall x\,(P(x) \to Q(x))}$, $\boxed{\forall x\ P(x)}$

$\forall x\,(Q(x))$

x_0

$P(x_0) \to Q(x_0)$

$P(x_0)$

$Q(x_0)$

$\cancel{\forall x}$ $\forall x\, Q(x)$

Claim For any non-negative integer n,

$$\sum_{i=0}^{n} 2^i = 2^{n+1} - 1$$

Proof by induction

Base case $\sum_{i=0}^{0} 2^i = 2^0$

$= 1$

$= 2^{0+1} - 1$

$= R.H.S$

Inductive step.

$\boxed{\cancel{P(k)} \Rightarrow \cancel{P(k+1)}}$

$S(k) \Rightarrow S(k+1)$

$S(n) \Rightarrow S(n+1)$

Assume that $\sum_{i=0}^{n} 2^i = 2^{n+1} - 1$

(Inductive hypothesis)

Consider $\sum_{i=0}^{n+1} 2^i = \sum_{i=0}^{n} 2^i + 2^{n+1}$

$= 2^{n+1} - 1 + 2^{n+1}$

$= 2 \cdot 2^{n+1} - 1 = 2^{n+2} - 1$

$= 2^{(n+1)+1} - 1 =$ RHS.

$$S = 2^0 + 2^1 + \ldots + 2^n$$

$$2S = 2^1 + 2^2$$

Claim $$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$$

Assume $$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$$

$$\sum_{i=1}^{n+1} = \sum_{i=1}^{n} i + (n+1)$$

$$= \frac{n(n+1)}{2} + (n+1)$$

$$= (n+1)\left(\frac{n}{2} + 1\right)$$

$$= \frac{(n+1)(n+2)}{2}$$

Claim Prove by mathematical induction

for all $n \geq 5$, $4n < 2^n$.

Base Case :-

$n = 5$

$\cancel{4n < 2}$ $4n < 2^n$ (Assume)

$4(n+1) = 4n + 4 \ldots\ldots$ (i)

$4 < 2^k$, $k > 2$ $\therefore 4 < 2^n$ as $n \geq 5 \ldots$ (ii)

$\boxed{\begin{aligned} &\text{From (i) \& (ii),} \\ &4n + 4 < 2^n + 2^n = 2 \cdot 2^n \\ &\phantom{4n + 4 < 2^n + 2^n} = 2^{(n+1)} \end{aligned}}$

Claim For any integer $k \geq 0$ we have

$$k+1 \geq H_{2^k} \geq \frac{k}{2} + 1$$

where H_n is the n^{th} harmonic number given by

$$H_n = \sum_{i=1}^{n} \frac{1}{i}$$

Claim $\quad H_{2^k} \le k+1$

Base case

$$H_{2^0} = H_1 = \sum_{i=1}^{1} \frac{1}{i}$$

$$= 1 \le 1$$

Inductive step

$$H_{2^k} \le k+1 \quad (\text{IH})$$

$$H_{2^{k+1}} = \boxed{\frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{2^k}} + \frac{1}{2^k+1} + \frac{1}{2^k+2} + \dots + \frac{1}{2^k+2^k}$$

$$\leq (k+1) + \boxed{\frac{1}{2^k+1} + \frac{1}{2^k+2} + \dots + \frac{1}{2^k+2^k}}$$

$$\leq (k+1) + \frac{1}{2^k+1} + \frac{1}{2^k+1} + \frac{1}{2^{k+1}} + \dots + \frac{1}{2^k+1}$$

$$\leq k+1 \quad \frac{2^k}{2^k+1} \leq k+2$$

Prove that $H_{2^k} \geqslant \frac{k}{2} + 1$

where $H_n = \sum_{i=1}^{n} \frac{1}{i}$

Prove that

$\forall n \geq 2$, $n^3 - n$ is divisible by 6.

Base case ✓

Inductive step

$$(n+1)^3 - (n+1)$$

$$= n^3 + 3n^2 + 3n + 1 - n - 1$$

$$= (n^3 - n) + \underbrace{3n^2 + 3n}_{3n(n+1)}$$

Claim

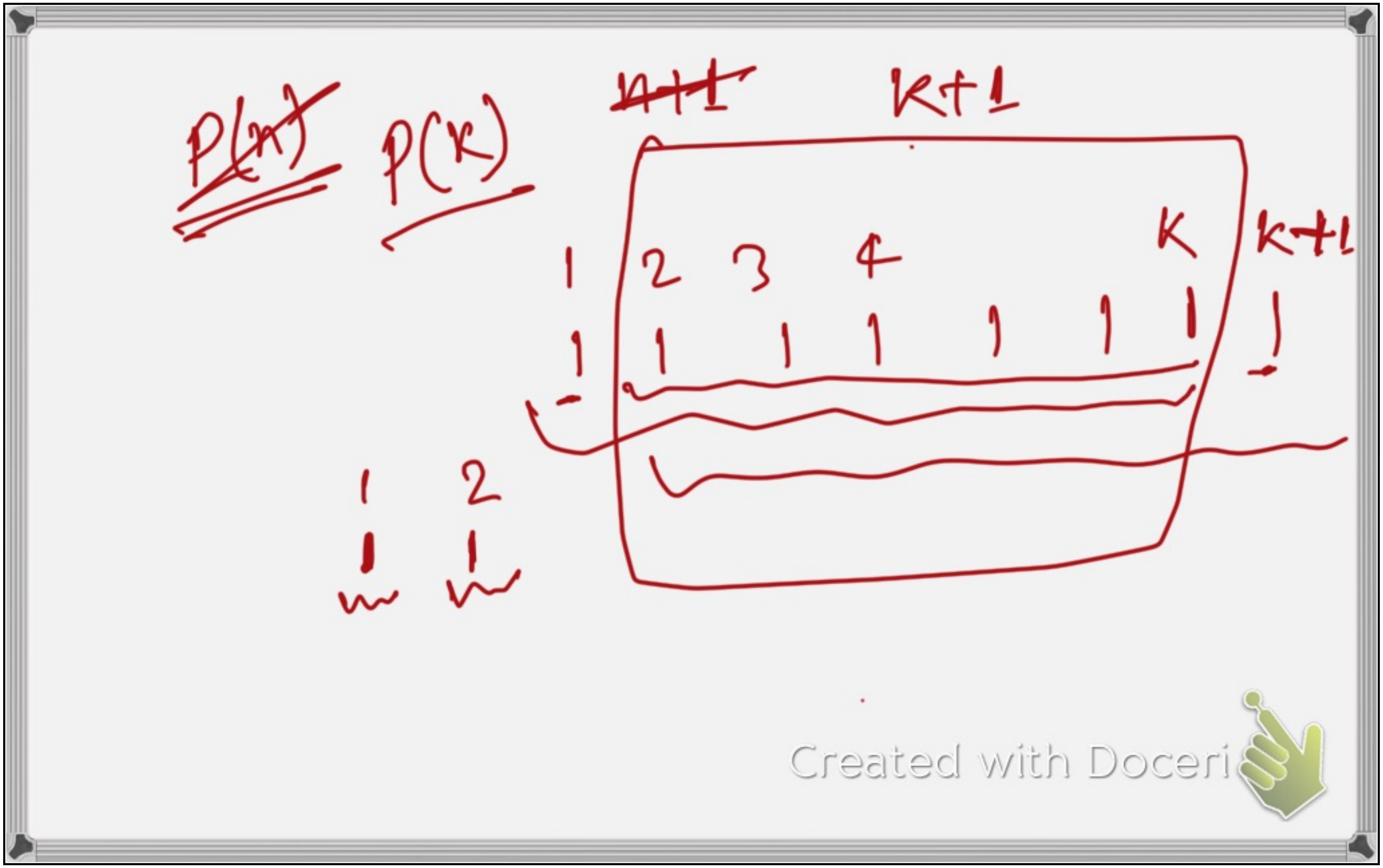
Any set of n horses where $n \geq 1$ have the same color.

$\boxed{1}$

$P(n)$

$P(n+1)$

1	1	1	1	1	1	1
1	2	3	.	..	n	$n+1$


P(n)
P(k)
n+1
k+1
1 2 3 4 k k+1
1 2
Created with Doceri

Exercise

For any positive integer n, a $2^n \times 2^n$ checkerboard with any one square removed can be tiled with L-shaped square tiles covering three squares.