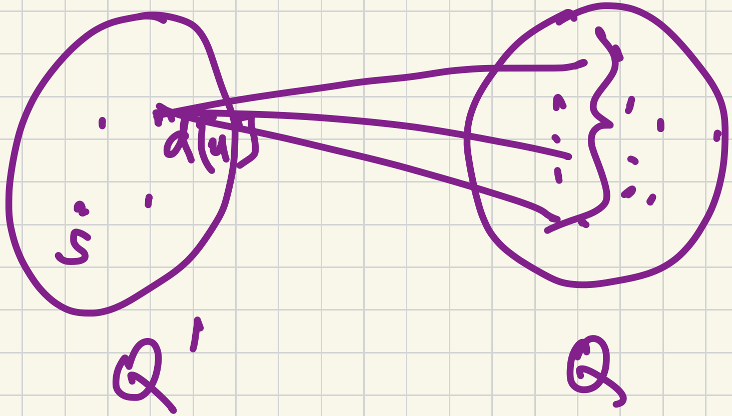
ACOL 202

Lecture 28



d[v]

Correctness

C1. For every v in Q', $d[v]$ is the length of the shortest path between s and v.

C2. For every v in Q, $d[v]$ is the length of the shortest path between s and v that only uses vertices in Q' (except the last vertex)

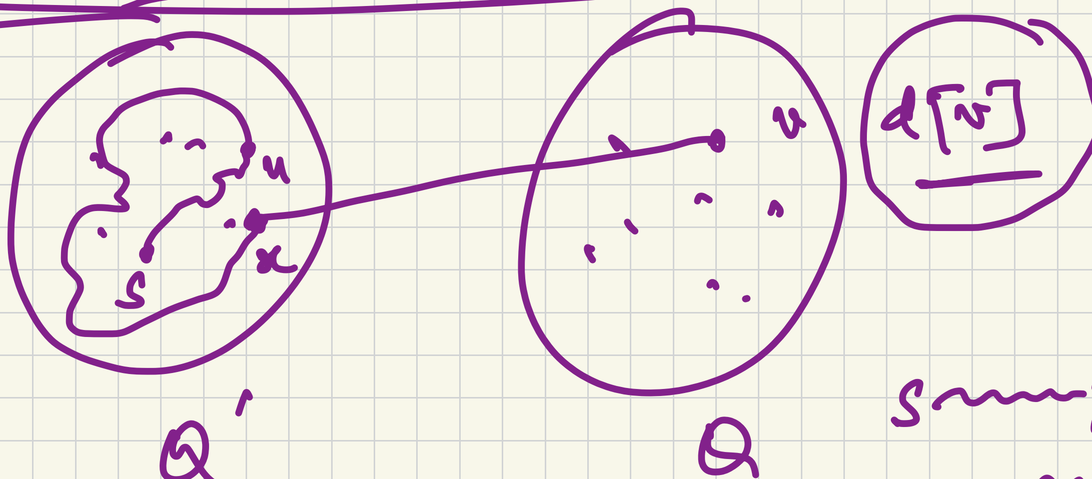
Step

Both these claims are true in the beginning.

Suppose u has the smallest d-value in Q.

$Q' \leftarrow Q' \cup \{u\}$ $\qquad$ $Q \leftarrow Q \setminus \{u\}$

Edge-relaxation

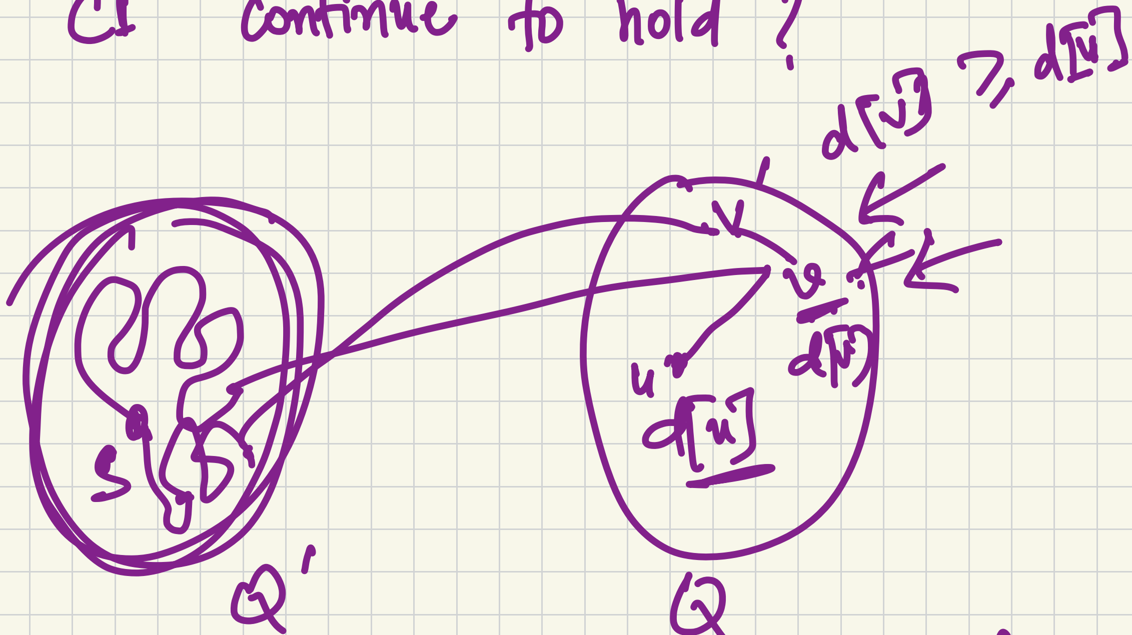


$s \rightsquigarrow u - v$

$s \rightsquigarrow u \rightsquigarrow x - v$

Clearly C2 continues to hold after this step (assuming that C1 was true earlier).

Why does C1 continue to hold?



<u>Exercise</u> You will complete the proof.

Graphs

Edges

Binary Relation

Given sets A and B, a binary relation $R : A \to B$ is a subset of $A \times B$.

$(a, b) \in R$ is denoted by $a R b$
$a \sim_R b$

Every relation $R : A \to B$ can be represented as a bipartite graph.

Matrix representation (seen example $\to$ adjacency matrix)

Inverse relation $\quad b R^{-1} a$ iff aRb

Composition $\quad R : B \to C \mid S : A \to B \mid R \circ S : A \to C$

$a (R \circ S) c$ iff $\exists\, b \in B$ such that aSb and bRc.

Relations on one set $\quad R \subseteq A \times A$

set of cities $\quad aRa'$ if there is a direct flight from a to a'.

$\mathbb{Z}$ $\quad a R a'$ iff $a \equiv a' \pmod 7$

$\mathbb{N}$ $\quad a R a'$ iff $a \leq a'$

Reflexive $\quad \forall x \in A \quad xRx$

Symmetric $\quad \forall x, y \in A \quad xRy \Rightarrow yRx$

Transitive $\quad \forall x, y, z \in A \quad xRy$ and yRz $\Rightarrow xRz$

$\mathbb{R}, \leq$

reflexive ✓

symmetric ✗ (antisymmetric) $\forall x, y$ $\quad xRy$ and $yRx \Rightarrow x = y$.

transitive ✓

$\sim$ on $\mathbb{Z}$

$x \sim y$ if x and y have the same parity (even or odd)

reflexive
symmetric
transitive
$\Big\}$ equivalence relation

$a \sim b$ iff $a \equiv b \pmod 5$

. . . 0, 5, 10, 15, 20, ...
. . . 1, 6, 11, 16, 21,
. . . 2, 7, 12, 17 . .
. . . 3, 8, 13, 18 . .
. . . 4, 9, 14, 19 . .

Partial Order

Irreflexive

$\forall x \in X \quad \underbrace{(x,x) \notin R}_{xRx \text{ is not true}}$

Let A be a set.

A relation on A is a partial order if it is reflexive, antisymmetric and transitive.

strict partial order if it is irreflexive, antisymmetric and transitive.

$\mathbb{Z}$ $\quad a \mid b$ partial order

$a < b$ strict partial order $\quad a \leq b$ partial order

__Comparability__ Let $\leq$ be a partial order on A.

$a, a' \in A$ are said to be comparable if either $a \leq a'$ \| incomparable otherwise

$a' \leq a$

__Total order__ | partial order + every pair of elements comparable

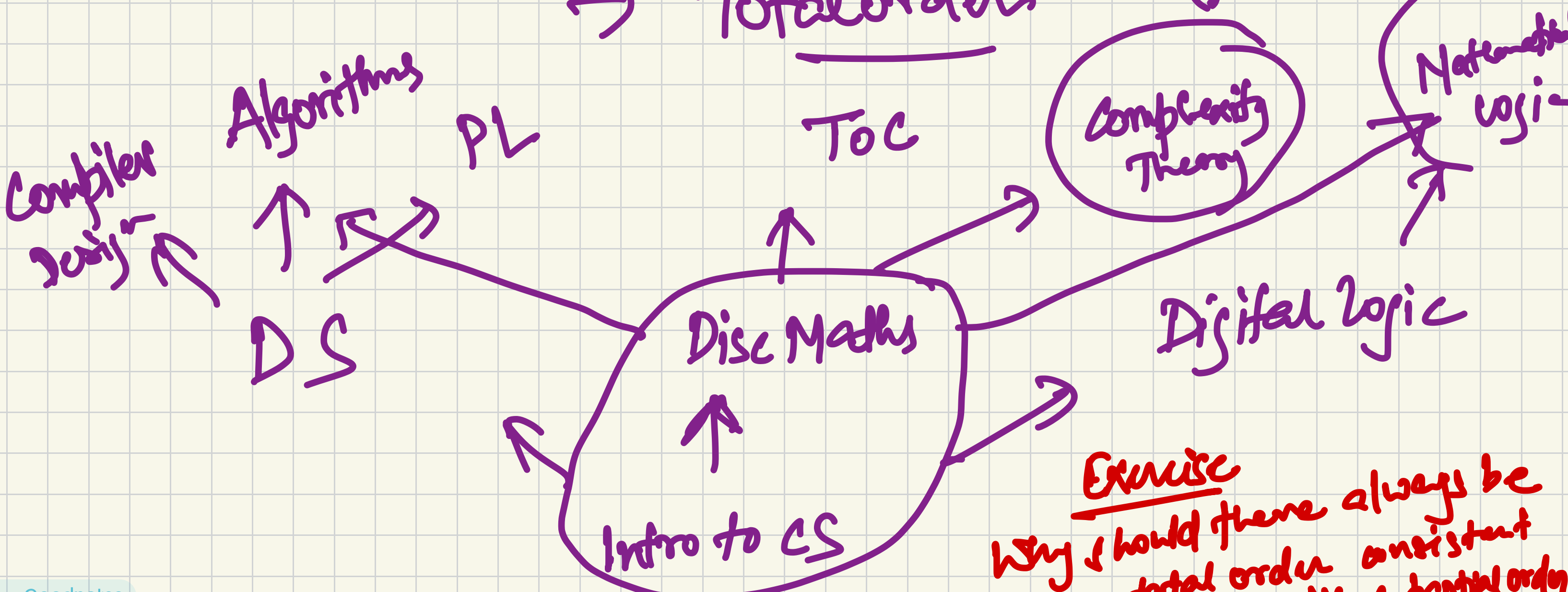
Strict total order | strict partial order + every pair of distinct elements comparable.

$x R y$

if x comes alphabetically no later than y.

→ **Total order**

$\perp$.



Exercise

Why should there always be a total order consistent with a partial order?

The Well-ordering Principle

Every nonempty set of non-negative integers has a smallest element.

We have used it many times without explicitly talking about it.

For example: while proving that $\sqrt{2}$ is irrational, we assumed, for contradiction, that $\sqrt{2} = p/q$ in lowest terms.

How do we know that a fraction can be converted to one in lowest terms?

Suppose not. Let there be positive integers m and n such that m/n cannot be written in lowest terms.

Let C denote the set of numerators of all such fractions.

Clearly, $m \in C$. Thus, C is non-empty.

$\therefore$ It must have a smallest element, say m_0.

By defn of C, there must be a positive integer n_0 such that $\frac{m_0}{n_0}$ cannot be written in lowest terms.

This means that m_0 and n_0 must have a common prime factor $p > 1$.

Consider $\dfrac{m_0/p}{n_0/p}$ which is equal to $\dfrac{m_0}{n_0}$.

If $\dfrac{m_0/p}{n_0/p}$ could be expressed in lowest terms then we could use the same expression for m_0/n_0.

Therefore, it must be the case that $\dfrac{m_0/p}{n_0/p}$ also cannot be expressed in lowest terms. But then m_0/p (the numerator) must have been in C. But $m_0/p < m_0$. contradiction!