ACOL 202

Lecture 30 (May 16)

Bayes' Rule

For any two events A and B,

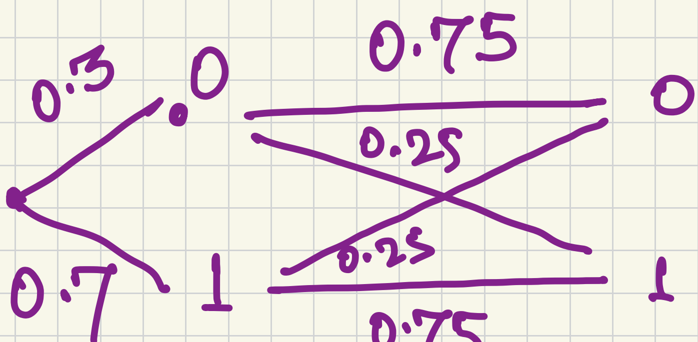
$$\Pr[A|B] = \frac{\Pr[B|A] \cdot \Pr[A]}{\Pr[B]}$$

We know from conditional probability,

$$\Pr[A|B] = \frac{\Pr[A \cap B]}{\Pr[B]}$$

$$\Pr[B|A] = \frac{\Pr[B\cap A]}{\Pr[A]}$$

And then simplify, under the assumption (fact) that $\Pr[A\cap B] = \Pr[B\cap A]$



What is the probability that the receiver receives '1'?

$$\Pr[\text{rcvd } 1] = \Pr[\text{rcvd } 1 \mid \text{send } 1] \cdot \Pr[\text{send } 1]$$

$$+ \Pr[\text{rcvd } 1 \mid \text{send } 0] \cdot \Pr[\text{send } 0]$$

$$= \boxed{\left((0.75 \times 0.7) + (0.25 \times 0.3)\right)}$$

$$= 0.6$$

$$\Pr[\text{sent } 1 \mid \text{rcvd } 1] = \frac{\Pr[\text{rcvd } 1 \mid \text{sent } 1] \cdot \Pr[\text{sent } 1]}{\Pr[\text{rcvd } 1]}$$

$$= \frac{0.75 \times 0.7}{0.6} = 0.875$$

What is the probability that the receiver receiving a 0 was sent as a 0.

$$\Pr[\text{sent } 0 \mid \text{rcvd } 0]$$

$$= \frac{\Pr[\text{rcvd } 0 \mid \text{sent } 0] \cdot \Pr[\text{sent } 0]}{\Pr[\text{rcvd } 0]}$$

$$= \frac{0.75 \times 0.3}{0.4}$$

Two coins in a bag

fair $\qquad$ 0.75 - biased

prior probability

$$\boxed{Pr[fair] = 0.5 = Pr[biased]}$$

evidence

posterior prob.

$$Pr[biased \mid H] = \frac{Pr[H \mid biased] . Pr[biased]}{Pr[H]}$$

$$= \frac{0.75 \times 0.5}{Pr[H \mid biased] . Pr[biased] + P[H \mid fair] . Pr[fair]}$$

$$= \frac{0.75 \times 0.5}{(0.75 \times 0.5) + (0.5 \times 0.5)} = 0.6.$$

Random Variable

a random variable X assigns a numerical value to every outcome in the sample space S.

$$X : \underline{S} \to \underline{\mathbb{R}}$$

Indicator random variable

$X : S \to \{0, 1\}$

flip a fair coin three times (independently)

X = the number of heads

$X(THH) = 2$ $X(HTT) = 1$

Y

$Y(TTT) = 1$

$Y(THT) = 2$

$Y(TTH)$

$Y(HTT) = 0$

Suppose we flip n coins that are fair, independently.

X_i be an indicator random variable that captures whether the ith flip came up heads or not.

$$\underbrace{X_1 + X_2 + X_3 + \cdots + X_n}_{\text{a random variable that gives the total number of heads.}}$$

Independence of random variables.

Two random variable X and Y are independent if every two events of the form $X = x$ and $Y = y$ are independent.

That is,

$$\Pr[X = x \text{ and } Y = y] = \Pr[X = x] \cdot \Pr[Y = y]$$

Expectation

The expectation of a random variable X, denoted $E[X]$, is defined as

$$E[X] = \sum_{x \in S} X(x) \cdot Pr[x]$$

Example Let X denote the no. of heads in 3 independent flips of a fair coin.

Sample Space $S = \{HHH, HHT, HTH, HTT, THH, THT, TTH, TTT\}$

Expected number of heads in 3 flips

$$E[X] = \frac{1}{8}(3+2+2+1+2+1+1+0) = \frac{12}{8}$$
$$= 1.5$$

The expected no. of heads in 3 flips $= 1.5$.

What is the expected number of aces in a 13-card hand dealt from a 52-card deck?

Let A be a random variable that denotes the no. of aces in a dealt hand.

$$E[A] = \sum_{i=0}^{4} \underline{i} \cdot \underline{Pr[A=i]}$$

$$0 \cdot \frac{\binom{4}{0}\binom{48}{13}}{\binom{52}{13}} + 1 \cdot \frac{\binom{4}{1}\binom{48}{12}}{\binom{52}{13}}$$

$$+ \; \cdots$$

Exercise

$$\underline{1 \text{ Ans.}}$$

Claim Let $X : S \to \mathbb{Z}^{\geq 0}$ be a random variable.

Then $E[X] = \sum_{i=1}^{\infty} \Pr[X \geq i]$

We know that, $E[X] = \sum_{i=0}^{\infty} i \cdot \Pr[X=i]$

$$= \sum_{i=0}^{\infty} \sum_{j=1}^{i} \Pr[X=i]$$

This is summing the rows (see on the left)

$\Pr[X=1]$

$\Pr[X=2] \quad \Pr[X=2]$

$\Pr[X=3] \quad \Pr[X=3] \quad \Pr[X=3]$

$$= \sum_{j=1}^{\infty} \sum_{i=j}^{\infty} \Pr[X=i]$$

This is summing all the columns (see on the left)

$$= \sum_{j=1}^{\infty} \Pr[X \geq j].$$

Exercise Let X be a geometric random variable (with parameter p)

X measures the number of flips of a p-biased coin before we get heads for the first time.

$$E[X] = \sum_{i=1}^{\infty} \underline{\underline{\Pr[X \geq i]}}$$

$$= \sum_{i=1}^{\infty} \Pr[\text{no heads in } (i-1) \text{ flips}]$$

$$= \sum_{i=1}^{\infty} (1-p)^{i-1} = \sum_{i=0}^{\infty} (1-p)^{i} = 1/p.$$

Linearity of Expectation

(the expectation of a sum is the sum of the expectations)

Consider a sample space S. Let $X : S \to \mathbb{R}$ and $Y : S \to \mathbb{R}$ be any two random variables. Then

$$E[X+Y] = E[X] + E[Y]$$

Proof

$$E[X+Y] = \sum_{s \in S} (X+Y)(s) \cdot \Pr[s]$$

$$= \sum_{s \in S} [X(s) + Y(s)] \cdot \Pr[s] = \sum_{s \in S} X(s) \cdot \Pr[s] + \sum_{s \in S} Y(s) \cdot \Pr[s]$$

$$= E[X] + E[Y].$$

Example (revisited) What is the expected number of aces in a 13-card hand dealt from a 52-card deck.

Let us number the cards (in the 13-card hand) from 1 to 13.

Let A_i be an indicator random variable that reports whether the i^{th} card is ace or not.

$E[A_i] = 1 \cdot Pr[ace] = 1 \cdot \frac{1}{13}$

(for all other cards in the sample space A_i will be 0. A_i will be 1 only when the card is an ace.

If A is the number of aces in hand then

$$A = A_1 + A_2 + \ldots + A_{13}$$

$$E[A] = E[A_1] + E[A_2] + \ldots + E[A_3]$$

$$= \frac{1}{13} + \frac{1}{13} + \ldots + \frac{1}{13}$$

$$= 13 \cdot \frac{1}{13} = 1.$$

Tutorial 14 Q4.

Note that the backward direction is same as the forward direction (because $\overline{(\overline{B})} = B$).

Assume A and B are independent. We will prove that A and $\overline{B}$ are independent.

$$A = (A \cap B) \cup (A \cap \overline{B})$$

$$\left.\begin{aligned} Pr[A] &= Pr[A \cap B] + Pr[A \cap \overline{B}] \end{aligned}\right\} \text{$(A \cap B)$ and $(A \cap \overline{B})$ are disjoint}$$

$$= \underbrace{Pr[A] \cdot Pr[B]}_{\text{because A and B are independent}} + Pr[A \cap \overline{B}]$$

$$\therefore Pr[A \cap \overline{B}] = Pr[A] - Pr[A] \cdot Pr[B] = Pr[A](1 - Pr[B])$$

$$= Pr[A] \cdot Pr[\overline{B}].$$