1. We are given an undirected graph G. An articulation point (or cut vertex) is defined as a vertex which, when removed along with associated edges, makes the graph disconnected (or more precisely, increases the number of connected components in G).

   Suppose we run DFS on G, and consider the DFS tree that we get. Prove that a non-root, non-leaf vertex u in the DFS tree is an articulation point if and only if there exists a subtree rooted at a child of u that has no back edges to a proper ancestor of u.

2. **[2 marks]** Suppose we are given a graph G where edge-weights can be positive or negative, however there are no negative cycles in the graph. Our task is to find the shortest path between vertices s and t in G. We know that Dijkstra's single-source shortest path algorithm requires that the edge-weights are non-negative. In the presence of negative edge-weights, let us construct a graph G' as follows. Start with G' same as G, and then take the biggest negative weight in G' ($-w$, say), and increase the weight of every edge in G' by w. G' has no negative weights now, and we can use Dijkstra's algorithm on G' to find shortest path (p', say) between s and t. The original graph must have a path p corresponding to p' because we did not add or remove any edges. We claim that p gives us shortest path between s and t.

   Either argue that the claim is true, or demonstrate (with the help of an example) that it can be falsified.

3. For arbitrary relations R and S, prove that $(R \circ S)^{-1} = (S^{-1} \circ R^{-1})$.

4. Let R be any relation on $A \times B$. Prove or disprove: $\langle x, x \rangle \in R \circ R^{-1}$ for every $x \in A$.

5. What set is represented by the relation $\leq \circ \geq$, where $\leq$ and $\geq$ are relations on $\mathbb{R}$?

6. Let $R \subseteq A \times A$ and $T \subseteq A \times A$ be relations. Prove or disprove the following.

   (a) R is reflexive if and only if R^{-1} is reflexive.

   (b) R is irreflexive if and only if R^{-1} is irreflexive.

   (c) If $R \circ T$ is reflexive, then R and T are both reflexive.

   (d) If R and T are both irreflexive, then $R \circ T$ is irreflexive.

7. **[2 marks]** Prove that R is symmetric if and only if $R \cap R^{-1} = R = R^{-1}$.

8. A relation R on A is asymmetric if, for every $a, a' \in A$, $\langle a, a' \rangle \in R$ implies that $\langle a', a \rangle \notin R$. Prove that, if R is irreflexive and transitive, then R is asymmetric.

9. Prove that $\preceq$ is a partial order if and only if $\preceq^{-1}$ is a partial order.

10. A *cycle* in a relation R is a sequence of k distinct elements $a_0, a_1, \ldots, a_{k-1} \in A$ where $\langle a_i, a_{(i+1) \bmod k} \rangle \in R$ for every $i \in \{0, 1, \ldots, k-1\}$. A cycle is *nontrivial* if $k \geq 2$. Prove that there are no nontrivial cycles in any transitive, antisymmetric relation R.

11. Let A be a finite set with a partial order $\preceq$. Then there is a total order $\preceq_{total}$ on A that is consistent with $\preceq$.