COL703: Logic for Computer Science (Aug-Nov 2022)

Lectures 13 & 14 (Predicate Logic)

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September 22nd and 29th, 2022

The need for a richer language

- the logical aspects of natural and artificial languages are much richer
  - than what propositional logic can capture
  - limited to sentence components like not, and, or, if ... then
- consider the following declarative sentence

  Every student is younger than some instructor.
- a propositional atom denoting this fails to capture the finer logical structure of this sentence.

The need for a richer language

Every student is younger than some instructor.

this is about being a student, being an instructor, and being younger than somebody else

we would like a mechanism to express these with their logical relationships

this is what we will use predicates for

Predicates

- we can use predicates S, I, and Y

- $S(\text{John})$ – John is a student
- $I(\text{Paul})$ – Paul is an instructor
- $Y(\text{John, Paul})$ – John is younger than Paul

- the meaning of these symbols must be specified exactly

Predicates

- can we now express "Every student is younger than some instructor."

Predicates

- can we now express "Every student is younger than some instructor."
- we don't want to write down every instance of S(.)
- use variables as place holders for concrete values
- $S(x)$ – x is a student
- $I(x)$ – x is an instructor
- $Y(x, y)$ – x is younger than y

Quantifiers

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- Every student is younger than some instructor.
- $\forall$ (for all) and $\exists$ (there exists)

Quantifiers

- can we now express "Every student is younger than some instructor."
- Every student is younger than some instructor.
- $\forall$ (for all) and $\exists$ (there exists)
- the quantifiers always come attached to a variable
- $\forall x$ (for all x) and $\exists z$ (there exists z)

Example

- can we now express "Every student is younger than some instructor."

Example

- can we now express "Every student is younger than some instructor."
- $\forall x\ (S(x) \rightarrow (\exists y\ (I(y) \wedge Y(x, y))))$
- for every x, if x is a student, then there is some y such that y is an instructor and x is younger than y
- predicates can have any (finite) number of arguments (arity)

Another example

- Not all birds can fly.
- $B(x)$ – x is a bird
- $F(x)$ – x can fly

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- $B(x)$ – x is a bird
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- $\neg(\forall x\ (B(x) \rightarrow F(x)))$

Another example

- Not all birds can fly.
- $B(x)$ – x is a bird
- $F(x)$ – x can fly
- $\neg(\forall x \ (B(x) \to F(x)))$
- alternatively, $\exists x \ (B(x) \land \neg F(x))$

Another example

- Not all birds can fly.
- $B(x)$ – x is a bird
- $F(x)$ – x can fly
- $\neg(\forall x \ (B(x) \rightarrow F(x)))$
- alternatively, $\exists x \ (B(x) \land \neg F(x))$
- does this formula evaluate to `true` in the world we currently live in?

Another example

Every child is younger than its mother.

Another example

Every child is younger than its mother.

$$\forall x \; \forall y \; ((C(x) \land M(x,y)) \to Y(x,y))$$

Another example

Andy and Paul have the same maternal grandmother.

Another example

Andy and Paul have the same maternal grandmother.

$$\forall x\ \forall y\ \forall u\ \forall v\ ((M(x, y) \wedge M(y, Andy) \wedge M(u, v) \wedge M(v, Paul)) \rightarrow x = u)$$

Another example

Andy and Paul have the same maternal grandmother.

$$\forall x \; \forall y \; \forall u \; \forall v \; ((M(x, y) \wedge M(y, Andy) \wedge M(u, v) \wedge M(v, Paul)) \rightarrow x = u)$$

- function symbols can help us avoid the inelegent encoding
- equality has been used as a special predicate

As a formal language

there are two sorts of things involved in a predicate logic formula:

objects that we are talking about – constants, variables, $m(a)$, $g(x, y)$

expressions denoting objects are called terms

the other sort of things are formulas

Terms

- Any variable is a term.
- If $c \in \mathcal{F}$ is a nullary function, then c is a term.
- If $t_1, t_2, \ldots, t_n$ are terms and $f \in \mathcal{F}$ has arity $n > 0$, then $f(t_1, t_2, \ldots, t_n)$ is a term.
- Nothing else is a term.

In Backus Naur form we may write

$$t ::= x \mid c \mid f(t, \ldots, t)$$

where x ranges over a set of variables var, c over nullary function symbols in $\mathcal{F}$, and f over those elements of $\mathcal{F}$ with arity $n > 0$.

Formulas

- If $P \in \mathcal{P}$ is a predicate symbol of arity $n \geq 1$, and if $t_1, t_2, \ldots, t_n$ are terms over $\mathcal{F}$, then $P(t_1, t_2, \ldots, t_n)$ is a formula.
- If ϕ is a formula, then so is $(\neg\phi)$.
- If ϕ and ψ are formulas, then so are $(\phi \wedge \psi)$, $(\phi \vee \psi)$ and $(\phi \rightarrow \psi)$.
- If ϕ is a formula and x is a variable, then $(\forall x\, \phi)$ and $(\exists x\, \phi)$ are formulas.
- Nothing else is a formula.

Note how the arguments given to predicates are always terms. This can also be seen in the Backus Naur form (BNF) for predicate logic:

$$\phi ::= P(t_1, t_2, \ldots, t_n) \mid (\neg\phi) \mid (\phi \wedge \phi) \mid (\phi \vee \phi) \mid (\phi \rightarrow \phi) \mid (\forall x\, \phi) \mid (\exists x\, \phi)$$

Binding properties

- $\neg$, $\forall y$, $\exists y$ bind most tightly
- then $\vee$ and $\wedge$
- then $\rightarrow$, which is right-associative

Another example

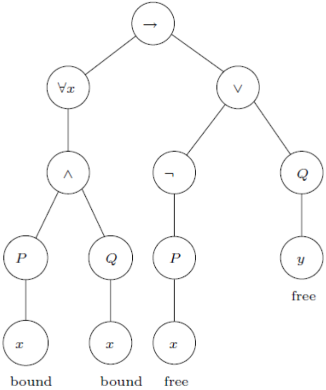
Every son of my father is my brother.

Parse trees

$\forall x \; ((P(x) \to Q(x)) \wedge S(x, y))$

Free and bound variables

$$(\forall x\ (P(x) \land Q(x))) \to (\neg P(x) \lor Q(y))$$



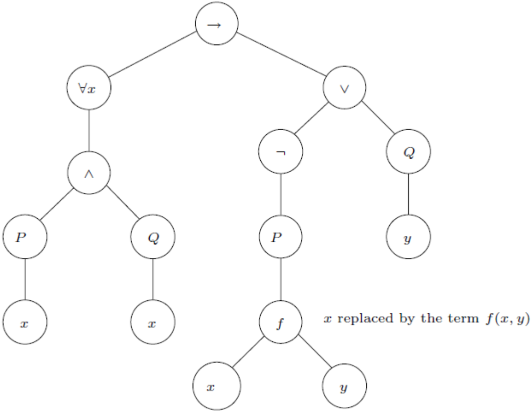
Substitution

given a variable x, a term t, and a formula ϕ

we define $\phi[t/x]$ to be the formula obtained by replacing each free occurrence of variable x in ϕ with t

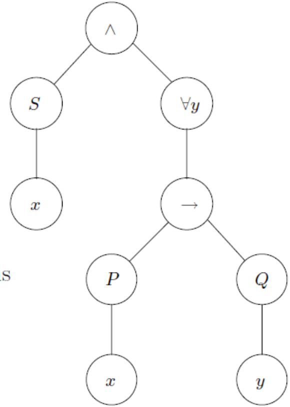
Substitution: example

example: $((\forall x\ (P(x) \wedge Q(x))) \rightarrow (\neg P(x) \vee Q(y)))[f(x,y)/x]$



Undesired side-effects of substitution

substitution must be avoided if t is not free for x in ϕ



the term $f(y, y)$ is not free for x in this formula

Proof theory (natural deduction rules)

(from the book by Huth and Ryan, pages 107-117)

Next week

- Quantifier equivalences
- Semantics of predicate logic
- Undecidability result

Thank you!