COL703: Logic for Computer Science (Aug-Nov 2022)

Lectures 3 & 4 (Propositional Logic, Soundness and Completeness)

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Recap: Derived rules

- Modus Tollens (MT)
- $\neg\neg$i
- proof by contradiction
- law of excluded middle ($\phi \vee \neg\phi$ is true)

Examples

Use LEM to show the validity of $p \to q \vdash \neg p \vee q$

Provable equivalence

- $\phi \dashv\vdash \psi$
- $p \wedge q \rightarrow r \dashv\vdash p \rightarrow (q \rightarrow r)$
- $p \wedge q \rightarrow p \dashv\vdash r \vee \neg r$

PBC: Classical vs. Intuitionistic Logicians

This week

- Syntax of propositional logic
- Semantics of propositional logic
- Soundness and completeness

Syntax

- formulas are strings over propositional atoms, logical symbols and left- and right-brackets
- but not everything is allowed, of course; e.g. $(\neg)() \vee pq \rightarrow$ does not seem to make any sense
- we would like our formulas to be *well-formed*

Well-formed formulas

- propositional atoms are well-formed formulas
- if ϕ is well-formed, so is $(\neg\phi)$
- if ϕ and ψ are well-formed, so are $(\phi \wedge \psi)$, $(\phi \vee \psi)$, and $(\phi \rightarrow \psi)$
- nothing else is a well-formed formula

Grammar in BNF

$$\phi \ ::= \ p \mid (\neg\phi) \mid (\phi \wedge \psi) \mid (\phi \vee \psi) \mid (\phi \rightarrow \psi)$$

Parse-trees and subformulas

Semantics

- based on the truth value of atomic propositions, and how the logical connectives manipulate the truth values
- $\phi_1, \phi_2, \ldots, \phi_n \vDash \psi$
- truth tables

Example of $\models$ notation

Do the following hold?

- $p \wedge q \models p$
- $p \vee q \models p$
- $\neg q, p \vee q \models p$
- $p \models q \vee \neg q$

Mathematical Induction

Claim. For every well-formed propositional logic formula, the number of left-brackets is equal to the number of right-brackets.

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base case: atomic formulas do not have any brackets

inductive step: argue for all possible logical connectives as root

Soundness of propositional logic

If $\phi_1, \phi_2, \ldots, \phi_n \vdash \psi$ is valid

then it is inconceivable that there there is a valuation in which ψ is false, whereas $\phi_1, \phi_2, \ldots, \phi_n$ are all true.

induction on the length of the (natural deduction) proof

can be tricky though (because of the assumption boxes)

Soundness and Completeness

Soundness If $\phi_1, \phi_2, \ldots, \phi_n \vdash \psi$ is valid, then $\phi_1, \phi_2, \ldots, \phi_n \vDash \psi$ holds.

Completeness If $\phi_1, \phi_2, \ldots, \phi_n \vDash \psi$ holds, then $\phi_1, \phi_2, \ldots, \phi_n \vdash \psi$ is valid.

Next week

- Resolution
- Normal forms
- SAT solving

Thank you!