ACOL 202

Extra Lecture
(12th May)

Every positive integer > 1 can be factored into a product of primes.

Let C be the set of all integers > 1 that cannot be factored into primes.

If C is empty we are done. Otherwise, it will have a smallest element.

If the smallest element (n) is prime then we are done.

Otherwise $\because$ n is composite

$$n = a \times b \quad \text{for } 1 < a, b < n$$

$$\therefore \quad a = \underline{p_1 p_2 p_3 \cdots p_k}$$

$$b = \underbrace{q_1 q_2 q_3 \cdots q_m}_{\text{primes}}$$

$$\therefore \quad n = \underbrace{p_1 p_2 \cdots p_k q_1 q_2 \cdots q_m}_{\text{primes}}.$$

We can also use the well-ordering principle to prove that some property $P(n)$ holds for every non-negative integer n.

Example

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$$

for all $n \geq 1$.

$$\boxed{\sum_{i=1}^{k-1} i = \frac{(k-1)k}{2}}$$

$+k$ $\qquad$ $+k$

Infinite Sets

For any pair of finite sets A and B

there is a surjection from A to B
iff $|A| \geq |B|$

injection from A to B
iff $|A| \leq |B|$

bijection from A to B iff
$|A| = |B|$

A is strictly bigger than B

(there is a surjection from A to B but there is no bijection from B to A)

iff $|A| > |B|$

For finite and infinite sets A, B, and C

A surj B, B surj C then A surj C

A bij B and B bij C then A bij C

Schroder Bernstein Theorem

For any pair of sets A and B,
if A surj B and B surj A
then A bij B.

A is finite $b \notin A$

$$|A \cup \{b\}| = |A| + 1$$

Made with Goodnotes

Let A be a set and $b \notin A$.
Then A is infinite iff A bij $A \cup \{b\}$.

A bij $A \cup \{b\}$ $\Rightarrow$ A is infinite

A is finite $\Rightarrow$ no bij between A and $A \cup \{b\}$

A is infinite $\Rightarrow$ A bij $A \cup \{b\}$

$a_0 \quad a_1 \quad a_2 \quad \ldots$

$f(b) = a_0$
$f(a_i) = a_{i+1}$

Made with Goodnotes

Countable sets

A set C is countably infinite iff $\mathbb{N}$ bij C.

A set is countable iff it is either finite or countably infinite.

If A and B are countable, $A \cup B$ is also countable.

$$f : \mathbb{N} \to A$$

$$g : \mathbb{N} \to B$$

$$f(1) \quad f(2) \quad f(3) \quad \dots$$

$$g(1) \quad g(2) \quad g(3) \quad \dots$$

$$h(n) = f\left(\frac{n+1}{2}\right) \quad \text{if } n \text{ is odd}$$

$$= g(n) \quad \text{if } n \text{ is even}$$

The cross product of two countable sets is countable.

$$(a_i, b_j) = c_k$$

$$k = \max(i,j)^2 + i + \max(i-j, 0)$$

	b_0	b_1	b_2	b_3	b_4	$\ldots$
a_0	c_0	c_1	c_4	c_9	c_{16}	
a_1	c_3	c_2	c_5	c_{10}	c_{17}	
a_2	c_8	c_7	c_6	c_{11}	$\vdots$	
a_3	c_{15}	c_{14}	c_{13}	c_{12}	$\vdots$	
a_4	$\ldots$		$\ldots$		$\ldots$	
$\vdots$						

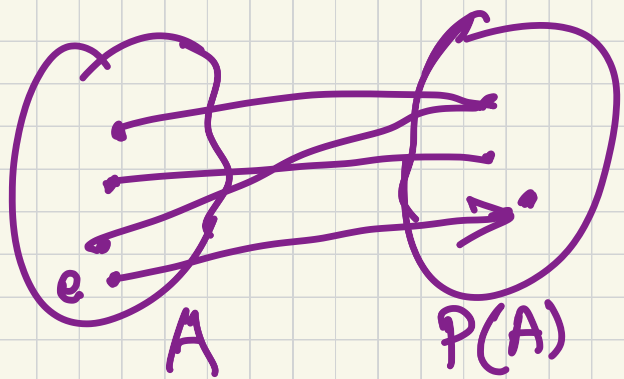
Corollary The set of rational numbers is countable.

Surjection from $\mathbb{Z} \times \mathbb{Z} \to \mathbb{Q}$

$$f(a,b) = \begin{cases} \frac{a}{b} & \text{if } b \neq 0 \\ 0 & \text{otherwise} \end{cases}$$

Theorem For any set A, the power set $P(A)$ is strictly bigger than A.

$g : A \to P(A)$ is not a surjection.



$B = \{a \in A : a \notin g(a)\}$

There is no element e in A such that $f(e) = B$.

Suppose there is such an element e in A such that $f(e) = B$.

Now, there are two cases:

i) $e \notin B$ then, by definition e must be in B because $B = \{a \in A : a \notin f(a)\}$

ii) $e \in B$ then, again, by definition e should not be in B.

Therefore, there cannot be such an e.

Exercise: Convince yourself that this contradiction is correct.

Theorem $\mathbb{R}$ is uncountable.

$$[0,1]$$

$$f : \mathbb{N} \to \underline{[0,1]}$$

$f(n)$ $\to 0.45269 \ldots$

n	$f(n)$
1	$0.3579142634\ldots$
2	$0.14357669 47$
3	0.4516738172
4	$0.66757912\ldots$
5	$0.17328973\ldots$
6	

Even larger infinities

$$\mathbb{N}, \quad \mathcal{P}(\mathbb{N}), \quad \mathcal{P}(\mathcal{P}(\mathbb{N})), \quad \mathcal{P}(\mathcal{P}(\mathcal{P}(\mathbb{N}))) \quad \ldots$$

Cantor's Continuum Hypothesis

There is no set A such that $\mathcal{P}(\mathbb{N})$ is strictly bigger than A and A is strictly bigger than $\mathbb{N}$.