1. Prove that $a \bmod b = (a \bmod bc) \bmod b$ for all positive integers a, b, and c.

2. Prove the following properties of divisibility.

   (a) $a \mid b \quad \text{and} \quad b \mid c \quad \Rightarrow \quad a \mid c$

   (b) $a \mid b \quad \text{and} \quad a \mid c \quad \Rightarrow \quad a \mid (b + c)$

   (c) $a \mid b \quad \Rightarrow \quad a \mid bc$

   (d) $ab \mid c \quad \Rightarrow \quad a \mid c \quad \text{and} \quad b \mid c$

3. Prove that the test for divisibility by 3 is correct. First prove that $10^i \bmod 3 = 1$ for any integer $i \geq 0$; then prove the stated claim. Recall that a number is said to be divisible by 3 if the sum of all digits of that number is divisible by 3.

4. **[4 marks]** The divisibility test for 9 is to add up the digits of the given number, and test whether that sum is divisible by 9. State and prove the condition that ensures that this test is correct.

5. Show that, for all $n \geq 3$, we have $f_n \bmod f_{n-1} = f_{n-2}$, where f_i is the i^{th} Fibonacci number.

6. Let p be an arbitrary prime number and let a be an arbitrary nonnegative integer. Prove the following facts.

   (a) If $p \nmid a$, then $gcd(p, a) = 1$.

   (b) For any positive integer k, we have $p \mid a^k$ if and only if $p \mid a$.

   (c) For any integers $n, m \in \{1, \ldots, p-1\}$, we have that $p \nmid nm$.

   (d) For any integer m and any prime number q distinct from p (that is, $p \neq q$), we have $m \equiv_p a$ and $m \equiv_q a$ if and only if $m \equiv_{pq} a$.

   (e) If $0 \leq a < p$, then $a^2 \equiv_p 1$ if and only if $a \in \{1, p-1\}$. (You may use the fact that if $f(x)$ is a polynomial of degree k, and q is a prime, then either $f(a) \bmod q = 0$ for every $a \in \mathbb{Z}_q$, or the equation $f(x) = 0$ has at most k solutions for $x \in \mathbb{Z}_q$. Note that for any integer $n \geq 2$, $\mathbb{Z}_n$ denotes the set $\{0, 1, \ldots, n-1\}$.)