1. Is the following a tautology? Justify your answer.
   $(\neg q \rightarrow \neg p) \rightarrow ((\neg q \rightarrow p) \rightarrow q)$

   Hint: *Build the truth table for the formula, and check whether the formula evaluates to true for every valuation (every row) or not.*

2. Show using truth table that the following equivalence holds.
   $(p \leftrightarrow q) \equiv (\neg p \leftrightarrow \neg q)$

   Hint: *Build the truth table for the two formulas (on either side of the equivalence relation), and check whether the two formulas evaluate to the same truth value for every valuation (every row) or not.*

3. Let α and β be two formulas in propositional logic. Is it true that $(\alpha \vee \beta)$ is a tautology iff one of them is a tautology?

   Ans: The backward direction holds. If one of α and β is a tautology, then $(\alpha \vee \beta)$ must always be true. The forward direction does not hold. Consider $p \vee \neg p$ which is a tautology, but neither p is a tautology (because p can be false) nor $\neg p$ is a tautology (because p can be true).

4. Show using truth table that the following equivalence holds.
   $(p \rightarrow (q \rightarrow r)) \equiv ((p \wedge q) \rightarrow r)$

   Hint: *Build the truth table for the two formulas (on either side of the equivalence relation), and check whether the two formulas evaluate to the same truth value for every valuation (every row) or not.*

5. Find three formula α, β, and γ such that $(\alpha \wedge \beta \wedge \gamma)$ is unsatisfiable and such that the conjunction of any pair of them is satisfiable.

   Ans: Consider $\alpha = (p \rightarrow q)$, $\beta = (q \rightarrow r)$, and $\gamma = (p \wedge \neg r)$. Clearly, $(\alpha \wedge \beta \wedge \gamma)$ is unsatisfiable. To make γ true, we must make p true and r false. Now, to make α true, q must be true (because p is true). And then to make β true, r must also be true (because q is true). But it is not possible for r to be both false and true at the same time.

   $(\alpha \wedge \beta)$ is satisfied by the valuation $\{p : \text{true}, q : \text{true}, r : \text{true}\}$.
   $(\beta \wedge \gamma)$ is satisfied by the valuation $\{p : \text{true}, q : \text{false}, r : \text{false}\}$.
   $(\gamma \wedge \alpha)$ is satisfied by the valuation $\{p : \text{true}, q : \text{true}, r : \text{false}\}$.

6. Prove that a disjunction of literals $l_1, l_2, \ldots, l_m$, i.e. $(l_1 \vee l_2 \ldots \vee l_m)$, is valid if and only if there are two literals l_i and l_j, $1 \leq i, j \leq m$, such that l_i is $\neg l_j$.

   Ans: If l_i equals $\neg l_j$, then $(l_1 \vee l_2 \ldots \vee l_m)$ evaluates to true for all valuations – either l_j is true, or $\neg l_j$ (i.e., l_i) is true.

For the other direction, we prove the contrapositive. If there is no pair of complementary literals then $(l_1 \vee l_2 \ldots \vee l_m)$ cannot be valid. In particular, the disjunction evaluates to false if we assign false to all the atoms that appear un-negated in it, and true to all the atoms that appear negated in the disjunction. Such an assignment is possible because there is no atom that appears both negated and un-negated.

7. Is the following statement true? Justify your answer.
   If $\alpha \to \beta$ is a tautology and α is a tautology, then β is a tautology.

   Ans: Yes, the statement is true. We will prove it by contradiction.

   Suppose β is not a tautology. Therefore, there must be a valuation (or an assignment) v that makes β false. The same valuation must make α true, because it is a tautology. This means that v makes $\alpha \to \beta$ false (because it makes α true and β false). So, $\alpha \to \beta$ cannot be a tautology. Contradiction!

8. Consider the propositions $(p \to (q \to q))$ and $((p \to q) \to q)$. One of these is a tautology; one of them is not. Which is which? Prove your answer.

   Ans: $(p \to (q \to q))$ is a tautology. This is because $(q \to q)$ always evaluates to true. (An implication is false only when the antecedent is true but the consequent is false. q cannot be made both true and false!)

   $((p \to q) \to q)$ is not a tautology – it evaluates to false when both p and q are false.

9. Either Lily attended the meeting or Lily was not invited. If the boss wanted Lily at the meeting, then she was invited. Lily did not attend the meeting. If the boss did not want Lily at the meeting, and she was not invited, then she is going to be fired. Argue that Lily is going to be fired.

   Ans: Let the work with the following atomic propositions:

   a: Lily attended the meeting.
   i: Lily was invited.
   w: The boss wanted Lily at the meeting.
   f: Lily is going to be fired.

   We know the following (from the problem statement):

   $(a \vee \neg i)$, $(w \to i)$, $\neg a$, $((\neg w \wedge \neg i) \to f)$

   The only valuation that makes these formulas true is $\{a : \text{false}, i : \text{false}, w : \text{false}, f : \text{true}\}$. Thus, if the given statements are true, f must be true, and therefore we know that Lily is going to be fired.

10. Is the following statement true? Justify your answer.
    If $\alpha \to \beta$ is satisfiable and α is satisfiable, then β is satisfiable.

    Ans: No, the statement is not true. The formula $(q \to (p \wedge \neg p))$ is satisfiable. It evaluates to true when q is false (irrespective of the truth value of p). q is also satisfiable – it can be set to true. But $(p \wedge \neg p)$ is not satisfiable.