1. A bitstring $x \in \{0,1\}^5$ is stored in vulnerable memory, subject to corruption – for example, on a spacecraft. An α-ray strikes the memory and resets one bit to a random value (both the new value and which bit is affected are chosen uniformly at random). A second α-ray strikes the memory and resets one bit (again chosen uniformly at random). What is the probability that the resulting bitstring is identical to x?

2. Two teams, A and B, play a best-of-five series of football games. Team A wins each game with probability 60%. Compute the probability that Team A wins the series.

3. Let A and B be arbitrary events in a finite sample space. Prove that if $\texttt{Pr}[B] = 0$, then A and B are independent.

4. **[4 marks]** Let A and B be arbitrary events in a finite sample space. Prove that A and B are independent if and only if A and $\overline{B}$ are independent.

5. Suppose A and B are mutually exclusive events – i.e., $A \cap B = \emptyset$. Prove or disprove the following claim: A and B cannot be independent.

6. Let A and B be two events such that $\texttt{Pr}[A \mid B] = \texttt{Pr}[B \mid A]$. Which of the following is true? Justify your answer briefly.

   (a) A and B must be independent.

   (b) A and B must not be independent.

   (c) A and B may or may not be independent (there's not enough information to tell).

7. Alice wishes to send a 3-bit message `011` to Bob, over a noisy channel that corrupts (flips) each transmitted bit independently with same probability. To combat the possibility of her transmitted message differing from the received message, she adds a parity bit to the end of her message (so that the transmitted message is `0110`). Bob checks that he receives a message with an even number of ones, and if so interprets the first three received bits as the message.

   (a) Assume that each bit is flipped with probability 1%. Conditioned on receiving a message with an even number of ones, what is the probability that the message Bob received is the message that Alice sent?

   (b) What if the probability of error is 10% per bit?

8. Suppose you have two coins – one fair and one p-biased. You choose one of them uniformly at random and flip it. What is $\texttt{Pr}$[you chose the biased coin | observed result sequence], in each of the following cases.

   (a) $p = 2/3$, observed result sequence = H

   (b) $p = 3/4$, observed result sequence = HHHT

   (c) $p = 3/4$, observed result sequence = HTTTHT

9. We have shown linearity of expectation – the expectation of a sum equals the sum of the expectations – even when the random variables in question aren't independent. It turns out that the expectation of a product equals the product of the expectations when the random variables are independent, but not in general when they are dependent.

   (a) Let X and Y be independent random variables. Prove that $\texttt{E}[X \cdot Y] = \texttt{E}[X] \cdot \texttt{E}[Y]$.

(b) When X and Y are dependent random variables, prove that $\mathtt{E}[X \cdot Y]$ is not necessarily equal to $\mathtt{E}[X] \cdot \mathtt{E}[Y]$.

(c) When X and Y are dependent random variables, prove that $\mathtt{E}[X \cdot Y]$ is also not necessarily unequal to $\mathtt{E}[X] \cdot \mathtt{E}[Y]$.