End-term exam (*35 Marks, 20th May, 2 hours*) **ACOL202, IITD-AD, Jan-May 2025**

Name: **Entry No.:**

1. **[6 marks]** Show that whenever 25 girls and 25 boys are seated around a circular table there is always a person both of whose neighbors are boys.

2. **[3 marks]** Is the following statement true? Justify your answer.

   If $\alpha \to \beta$ is satisfiable and α is valid, then β is valid.

3. **[8 marks]** The totient function $\varphi : \mathbb{Z}^{\geq 1} \to \mathbb{Z}^{\geq 0}$, sometimes called Euler's totient function (named after the Swiss mathematician Leonhard Euler), is defined as

   $$\varphi(n) = |\ \{k\ :\ \ 1 \leq k \leq n,\ k \text{ and } n \text{ have no common divisors}\}\ |$$

   For example, $\varphi(6) = 2$, because 1 and 5 have no common divisors with 6, whereas all the others (2, 3, 4, and 6) have a common divisor.

   Fermat-Euler's theorem states that for any a and n that are relatively prime, $a^{\varphi(n)} \equiv_n 1$.

   (a) **[4 marks]** Assuming Fermat-Euler's theorem, prove Fermat's little theorem. Recall the statement of Fermat's little theorem: let p be a prime, and let $a \in \mathbb{Z}_p$ where $a \neq 0$; then $a^{p-1} \equiv_p 1$.

   (b) **[4 marks]** Assuming Fermat-Euler's theorem, prove that a^{-1} in $\mathbb{Z}_n$ is $a^{\varphi(n)-1} \bmod n$, for any $a \in \mathbb{Z}_n$ that is relatively prime to n.

4. **[6 marks]** Alice wishes to send a 3-bit message `011` to Bob, over a noisy channel that corrupts (flips) each transmitted bit independently as follows: the noisy channel flips `0` to `1` with probability p, and it flips `1` to `0` with probability q. To combat the possibility of her transmitted message differing from the received message, she adds a parity bit to the end of her message (so that the transmitted message is `0110`). Bob checks that he receives a message with an even number of ones, and if so interprets the first three received bits as the message that Alice wanted to send. Conditioned on receiving a message with an even number of ones, what is the probability that the message Bob received is the message that Alice sent?

5. **[6 marks]** Suppose the numbers $1, 2, \ldots, 2n$ are written on a whiteboard, where n is an odd integer. Let us say we pick any two of the numbers, i and j, written on the board, write the number $|i - j|$ on the board, and erase i and j. We continue this process until only one integer is written on the board. Prove that this integer must be odd.

6. **[6 marks]** Let us call a logical proposition truth-preserving if the proposition is true under the all-true truth assignment.

   (a) **[4 marks]** Prove the following claim by structural induction on the form of the proposition:

   Any logical proposition that uses only the logical connectives $\vee$ and $\wedge$ is truth-preserving.

   (b) **[2 marks]** Use the claim above to prove that the set $\{\wedge, \vee\}$ is not universal, i.e. there are propositions that cannot be expressed using only $\wedge$ and $\vee$.