COL703: Logic for Computer Science (Aug-Nov 2022)

Lectures 19 & 20 (Undecidability results, Predicate Resolution)

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Compactness

- Compactness of sets of ground formulas – A set of ground quantifier-free formulas has a model iff every finite subset of it has a model.
- Compactness of closed formulas – A set of first-order sentences has a model iff every finite subset of it has a model.
- Löwenheim Skolem Theorem – If a set of closed formulas has a model, then it has a model with a domain (universe) which is at most countable.

Semi-decidability of validity

Validity of first-order formulas is semi-decidable[1].

Proof:

[1] a semi-decision procedure for validity should return "valid" if a valid formula is given as input, but otherwise may compute forever

Semi-decidability of validity

Validity of first-order formulas is semi-decidable[1].

Proof:

Semi-Decision Procedure for Validity
Input: Closed formula F
Output: Either that F is valid or compute forever

Compute a Skolem-form formula G equisatisfiable with $\neg F$
Let $G_1, G_2, \ldots$ be an enumeration of the Herbrand expansion $E(G)$
for $n = 1$ to ∞ **do**
begin
  if $\Box \in \text{Res}^*(G_1 \cup \ldots \cup G_n)$ **then** stop and output "F is valid"
end

[1] a semi-decision procedure for validity should return "valid" if a valid formula is given as input, but otherwise may compute forever

Let us try this on the formula

$$\exists x \forall y \; P(x, y) \rightarrow \forall y \exists x \; P(x, y)$$

Undecidability results

Post's Correspondence Problem (PCP) is undecidable.

Undecidability of validity follows from undecidability of PCP.

Since F is unsatisfiable iff $\neg F$ is valid, satisfiability must also be undecidable.

Satisfiability is not even semi-decidable (because, for any F, either F is valid or $\neg F$ is satisfiable).

Proof

Reference material:

`https://www.cs.ox.ac.uk/people/james.worrell/lecture13-2015.pdf`

Closed formula for a general PCP instance

Given a general instance $\mathbf{P} = \{(x_1, y_1), \ldots, (x_k, y_k)\}$ of PCP we have the formulas

$$
\begin{aligned}
F_1 &= \bigwedge_{i=1}^{k} P(f_{x_i}(e), f_{y_i}(e)) \\
F_2 &= \forall u \, \forall v \, \bigwedge_{i=1}^{k} (P(u, v) \to P(f_{x_i}(u), f_{y_i}(v))) \\
F_3 &= \exists u \, P(u, u) \, .
\end{aligned}
$$

The PCP instance **P** has a solution iff $F_1 \wedge F_2 \to F_3$ is valid.

Next week

- Resolution for Predicate Logic
- Soundness and Completeness

Thank you!