ACOL 202 Strong Induction

(Weak) Induction $P(n)$ $n \geq 0$

Base case $P(0)$

Inductive step $\boxed{P(n) \Rightarrow P(n+1)}$

$P(0)$ $P(1)$ $P(2)$ . . . . $P(n)$ $P(n+1)$

$\forall\, 0 \leq i \leq n$ $P(i)$ $\quad P(n+1)$

Example

Every integer > 1 is a product of one or more primes.

Base case 2 done.

$\forall \boxed{2 \le k \le n}$

$n+1$

If $(n+1)$ is a prime $(n+1)$

If $(n+1)$ is not a prime number

$$(n+1) = a \times b$$

$$\boxed{2 \leq a, b \leq n}$$

$$n+1 = q_1 q_2 q_3 \ldots q_j \cdot r_1 r_2 r_3 \ldots r_m$$

Exercise

Given $n \in \mathbb{N}$ define a_n recursively as follows:

$a_0 = 1$

$a_1 = 3$

$a_n = 2a_{n-1} - a_{n-2}$

$\boxed{\text{Prove that } a_n = 2n+1 \text{ for all } n \geq 0.}$

Base case

$$a_2 = 2a_1 - a_0$$
$$= 2(3) - 1 = 5$$
$$= 2(2) + 1$$

a_3

a_4

$$a_0 = 2(0) + 1$$
$$= 1$$
$$a_1 = 2(1) + 1$$
$$= 3$$

$$0 \leq k \leq n$$

$$a_{n+1} = 2a_n - a_{n-1}$$

$$\boxed{2(2n+1) - \left(2(n-1)+1\right)}$$

$$4n+2 - (2n-2+1)$$

$$4n+2-2n+2-1$$

$$2n+3$$

$$2(n+1)+1$$

$\cancel{a_2}$

Claim Every amount of postage that is at least 12 cents can be made from 4-cent and 5-cent stamps.

$$\boxed{12 \le a \le k}$$

$$\boxed{k+1 \ge 16}$$

$k+1-4 \ge 12$

$k+1$

$$k+1 \geq 16$$

$$\boxed{k+1-4} \geq 12$$

$$\underline{m(4)} + \underline{p(5)}$$

$$(k+1) \qquad (m+1)\,4, \; p(5)$$

Claim: Every positive integer n can be written as a sum of distinct non-negative integer powers of 2.

$$1 = 2^0$$

$$2 = 2^1$$

$$3 = 2^1 + 2^0$$

Base case

$n = 1$ $\qquad 2^0$

$0 \le k \le n$

$n+1$

Let l be the largest integer such that $2^l \le n+1$

$(n+1) - 2^l \ge 0$

$(n+1) - 2^l \le n$

$$\boxed{(n+1) - 2^{l}} = \dots$$

$$2^{r_1} + 2^{r_2} + 2^{r_3} + \dots + 2^{r_m}$$

$$n+1 = 2^{l} + 2^{r_1} + 2^{r_2} + \dots + 2^{r_m}$$

If l actually equal r_i

then $n+1 = 2^{l} + 2^{r_i} + 2^{r_1} + 2^{r_2} + \dots + 2^{r_{i-1}} + 2^{r_{i+1}} \dots + 2^{r_m}$

Contradiction $= 2^{l+1}$

Exercise

Any convex polygon P with $k \geq 3$ vertices can be decomposed into a set of $k-2$ triangles whose interiors do not overlap.

Structural Induction

Recursively defined structures.

$(\alpha \wedge \beta)$

p

$\underline{\alpha, \beta}$

$\alpha \wedge \beta \quad \alpha \vee \beta \quad \neg\alpha$

$\alpha \Rightarrow \beta$

$\alpha ::= p \mid \neg\alpha \mid (\alpha \wedge \beta) \mid (\alpha \vee \beta)$

Consider the recursively defined set S

$$\begin{cases} a \in S & \text{(base case)} \\ \text{if } x \in S \text{ then } (x) \in S. & \text{(recursive step)} \end{cases}$$

$$\{a,\ (a),\ ((a)),\ (((a)))\}$$

Prove by structural induction that any element in S contains an equal number of left and right paranthesis.

$((x))$

Base case

a has 0 left 0 n

Inductive step

n left n right

n -

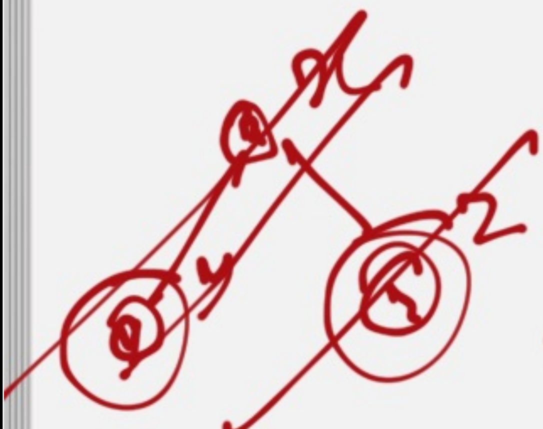
Binary Tree

A binary tree is either

null — an empty tree (null)

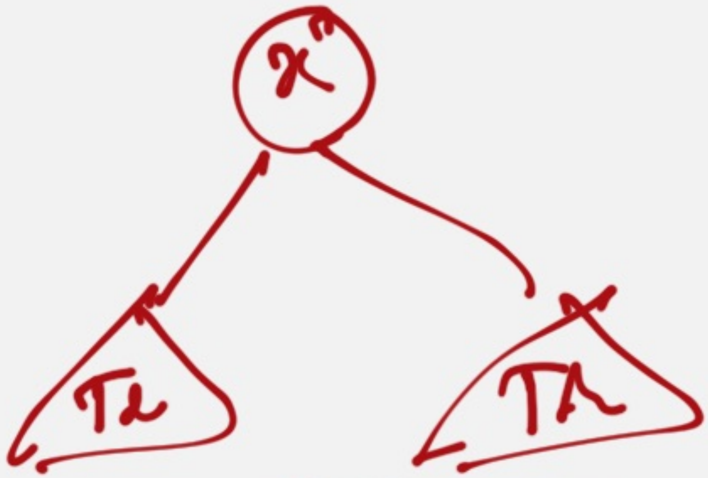
— a root node x,

binary trees { left subtree T_l — right subtree T_r —



Claim: In a binary tree T

$$\text{leaves}(T) \leq \text{internals}(T) + 1$$



$$\text{leaves}(T_l) \leq \text{internals}(T_l) + 1$$

$$\text{leaves}(T_r) \leq \text{internals}(T_r) + 1$$

$$\text{leaves}(T_l) + \text{leaves}(T_r) \leq \text{int}(T_l) + \text{int}(T_r) + 1 + 1$$

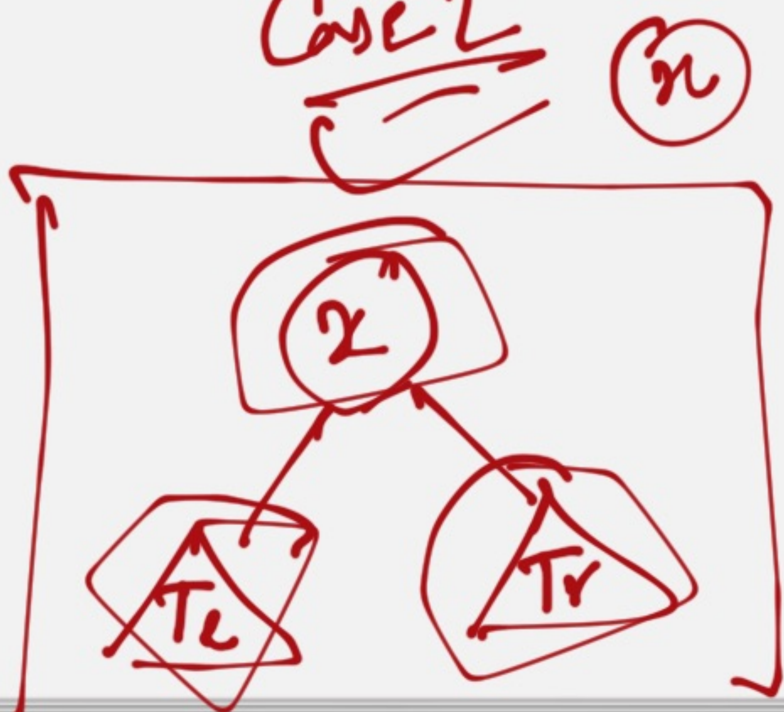
Case 1:

T_l and T_r are empty

Case 2

At least one of T_l, T_r is nonempty

leaves (T) =



Exercise

Consider the set S recursively defined as follows.

(Base case) $3 \in S$

(Recursive step) $x \in S$, and $y \in S$ then $x+y \in S$.

Prove that S is the set of all positive integers that are multiples of 7.

Structural induction is an implicit form of strong induction.