COL703: Logic for Computer Science (Aug-Nov 2022)

Lectures 7 & 8 (Hilbert's Axiomatization, Soundness and Completeness)

Kumar Madhukar

madhukar@cse.iitd.ac.in

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Hilbert's proof system

Axioms and inference rule

Derivations

A derivation of α is a finite sequence of formulas $\beta_1, \beta_2, \ldots, \beta_n$ such that:

- $\beta_n = \alpha$
- each β_i is either an instance of one of the axioms, or modus ponens applied to β_j and β_k such that $j, k < i$.

Example

derivation of $\alpha \to \alpha$

Soundness and Completeness

for all formulas α, $\quad \vdash \alpha \quad$ *iff* $\quad \models \alpha$

Soundness is easy to prove

Completeness

We need to prove that if α is a tautology, then α is derivable.

Suppose we prove that $(\neg\beta \to \neg\alpha) \to (\alpha \to \beta)$ (TODO)
Then, it suffices to prove that if α is not derivable, then α is not a tautology.

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Suppose we prove that $\neg\neg\alpha \to \alpha$. (TODO)

Then, if α is not derivable, $\neg\neg\alpha$ is also not derivable.

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Let us call α consistent if $\nvdash \neg\alpha$.

So, $\neg\alpha$ is consistent. Suppose we prove that if β is consistent then β is satisfiable. (TODO).

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So, $\neg\alpha$ is satisifiable. Therefore, α is not valid (it is not a tautology).

Deduction Theorem

For a set of formulas Γ, and formulas α, β, $\quad \Gamma \cup \{\alpha\} \vdash \beta \quad \text{iff} \quad \Gamma \vdash \alpha \to \beta$.

Our TODO list

- $(\neg\beta \to \neg\alpha) \to (\alpha \to \beta)$
- $\neg\neg\alpha \to \alpha$
- For all formulas β, if β is consistent then β is satisfiable.

Exercises

- Prove: $(\alpha \to \beta) \to (\neg\beta \to \neg\alpha)$

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- Show that if $\alpha \wedge \beta$ is consistent then both α and β are consistent.
- What about the converse?

Completeness Proof

β consistent $\quad\rightarrow\quad$ β satisfiable

- every consistent set can be extended to a maximal consistent set (MCS)
- let X be an MCS; for all formulas α, $\quad v_X \models \alpha$ *iff* $\alpha \in X$
  (where v_X is the valuation that every atomic proposition in X to *true*)

Next week

- Compactness and Strong Completeness
- Horn-SAT, 2-SAT, DPLL
- Binary Decision Diagrams

Thank you!