COL703: Logic for Computer Science (Aug-Nov 2022)

Lectures 17 & 18 (Herbrand's Theorem, Ground Resolution)

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October 13th and 17th, 2022

Proofs for recap: Logical equivalence

$(\forall x\ F \wedge G) \equiv \forall x\ (F \wedge G)$ (if x does not occur free in G)

Proof:

Proofs for recap: Renaming bound variables

Let F denote the formula $Qx\ G$ where Q is a quantifier. Let y be a variable that does not occur in G.

Then $F \equiv Qy\ (G[y/x])$.

Proof:

Proofs for recap: Skolem Form

Let $F = \forall y_1 \, \forall y_2 \; \ldots \; \forall y_n \, \exists z \; G$ be a rectified formula. Given a function symbol f of arity n that does not appear in F, write

$F' = \forall y_1 \, \forall y_2 \; \ldots \; \forall y_n \; G[f(y_1, y_2, \ldots, y_n)/z]$.

Then F and F' are equisatisfiable.

Proof:

Translation Lemma

If t is a term and F is a formula such that no variable in t occurs bound in F,

then $\mathcal{A} \models F[t/x]$ iff $\mathcal{A}_{[x \mapsto \mathcal{A}(t)]} \models F$.

Proof: reading exercise

Herbrand structure

universe is the set of ground terms

terms and function symbols being interpreted ”as themselves”

built from syntax

Herbrand structure

Definition 1. Let σ be a signature with at least one constant symbol. A σ-structure $\mathcal{H}$ is called a *Herbrand structure* if the following hold:

1. The universe $U_{\mathcal{H}}$ is the set of ground terms over σ.
2. For every constant symbol c in σ we have $c_{\mathcal{H}} = c$.
3. For every k-ary function symbol f in σ and for all ground terms $t_1, t_2 \ldots, t_n \in U_{\mathcal{H}}$ we have $f_{\mathcal{H}}(t_1, \ldots, t_k) = f(t_1, \ldots, t_k)$.

Interpretation of a ground term

Let $\mathcal{H}$ be a Herbrand structure, and t be a ground term.

Then, $\mathcal{H}[\![t]\!] = t$.

Translation Lemma for Herbrand structures

Let $\mathcal{H}$ be a Herbrand structure, F be a formula, and t be a ground term.

Then $\mathcal{H} \models F[t/x]$ if and only if $\mathcal{H}_{[x \mapsto t]} \models F$.

Herbrand's Theorem and Proof

Let $F := \forall x_1 \dots \forall x_n \; F^*$ be a closed formula in Skolem form.

Then F is satisfiable iff it has a Herbrand model.

Proof:

Example

Is the following formula satisfiable?

$$F := \exists x_1 \exists x_2 \exists x_3 \; (\neg(\neg P(x_1) \rightarrow P(x_2)) \wedge \neg(\neg P(x_1) \rightarrow \neg P(x_3)))$$

Finite model

- $\exists x_1 \exists x_2 \ldots \exists x_n \; F^*$, where the matrix F^* does not contain any function symbol

Finite model

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- does not work for $\forall x_1 \exists x_2 \; F^*$

Finite model

- $\exists x_1 \exists x_2 \ldots \exists x_n \; F^*$, where the matrix F^* does not contain any function symbol
- does not work for $\forall x_1 \exists x_2 \; F^*$
- the presence of a function symbols in its Skolem form makes each Herbrand structure infinite

Herbrand expansion

Let $F := \forall x_1 \ldots \forall x_n\ F^*$ be a closed formula in Skolem form with matrix F^*.

$E(F) := \{F^*[t_1/x_1] \ldots [t_n/x_n] \mid t_1 \ldots t_n$ are ground terms $\}$

Herbrand expansion

Let $F := \forall x_1 \ldots \forall x_n \; F^*$ be a closed formula in Skolem form with matrix F^*.

$E(F) := \{ F^*[t_1/x_1] \ldots [t_n/x_n] \mid t_1 \ldots t_n \text{ are ground terms } \}$

A closed formula F in Skolem form is satisfiable iff $E(F)$ is satisfiable when considered as a set of propositional formulas.

Proof:

Ground resolution

A closed formula F in Skolem form is unsatisfiable iff there is a propositional resolution proof of $\Box$ from $E(F)$.

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Ground resolution

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Proof:
$E(F)$ is unsat iff some finite subset of $E(F)$ is unsat. (Compactness theorem)

Ground resolution

A closed formula F in Skolem form is unsatisfiable iff there is a propositional resolution proof of $\square$ from $E(F)$.

Proof:
$E(F)$ is unsat iff some finite subset of $E(F)$ is unsat. (Compactness theorem)

Soundness and completeness of propositional resolution says that we can derive $\square$ from $E(F)$ using resolution.

Generalized version of Ground Resolution Theorem

Let $F_1, F_2, \ldots, F_n$ be closed formulas in Skolem form

whose respective matrices $F_1^*, F_2^*, \ldots, F_n^*$ are in CNF.

$F_1 \wedge F_2 \wedge \ldots \wedge F_n$ is unsatisfiable iff there is a propositional resolution proof of $\Box$ from the ground instances[1] of clauses from $F_1^*, F_2^*, \ldots, F_n^*$.

[1] a ground instance of F is a formula obtained by replacing all variables in F with ground terms

Example

Let us use ground resolution to show that (a), (b), and (c) together entail (d).

(a) Everyone in the class is either sleepy, bored, or day-dreaming.

(b) All those who are bored are sleepy.

(c) Someone in the class is not day-dreaming.

(d) Someone in the class is sleepy.

Example

Show that $\forall x\ \exists y\ (P(x) \to Q(y)) \to \exists y\ \forall x\ (P(x) \to Q(y))$.

Semi-decidability of validity

Validity of first-order formulas is semi-decidable.

Next week

- Undecidability of satisfiability
- Resolution for Predicate Logic
- Soundness and Completeness

Thank you!