ACOL 202

Lecture 3 (21st Jan)

Partitioning a set

$$S = \bigcup_{i=1}^{k(\geq 1)} A_i \quad \{A_1 \cup A_2 \cup A_3 \cup \ldots \cup A_k\}$$

for any i, j such that

$i \neq j$, $A_i \cap A_j = \{\}$

Cartesian product

The cartesian product of two sets A and B is the set $A \times B$

$$= \{ \langle a, b \rangle \mid a \in A, b \in B \}$$

containing all ordered pairs where the first component comes from A and the second component comes from B.

$\{1, 4\} \times \{2, 3\}$

$\{\langle 1,2\rangle, \langle 1,3\rangle, \langle 4,2\rangle, \langle 4,3\rangle\}$

Axioms

Set union and intersection are associative.

$A \cup (B \cup C) = (A \cup B) \cup C$

$A \cap (B \cap C) = (A \cap B) \cap C$

Theorem (Distributive Property)

i) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

Tutorial

ii) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

Proof (i) We first argue that

$$A \cup (B \cap C) \subseteq (A \cup B) \cap (A \cup C)$$

Suppose $x \in A \cup (B \cap C)$. We need to argue that $x \in (A \cup B) \cap (A \cup C)$

Therefore $x \in A$ or $x \in (B \cap C)$

If $x \in A$ then $x \in A \cup B$
and $x \in A \cup C$

$\therefore$ $x \in (A \cup B) \cap (A \cup C)$

If $x \in B \cap C$ then $x \in B$ and $x \in C$
From $x \in B$, we get $x \in A \cup B$
From $x \in C$, we get $x \in A \cup C$

$\therefore$ We have $x \in (A \cup B) \cap (A \cup C)$

In either case, $\therefore$ $x \in (A \cup B) \cap (A \cup C)$.

Next, we argue that

$$\underbrace{(A \cup B) \cap (A \cup C)}_{\text{LHS}} \subseteq A \cup (B \cap C).$$

Suppose $x \in \text{LHS}$

Clearly, $x \in (A \cup B)$ and $x \in (A \cup C)$

If $x \in A \cup B$, either $x \in A$ or $x \in B$ (or both)

If $x \in A$, we know that $x \in A \cup (B \cap C)$.

If $x \notin A$ then $x \in B$ and $x \in C$ Why? Because $x \notin A$ and $x \in (A \cup C)$

$\therefore x \in (B \cap C)$ $\therefore x \in A \cup (B \cap C)$.

Exercise

Suppose S is partitioned into $\{A_1, A_2, \ldots, A_K\}$.

Then $|S| = \sum_{i=1}^{K} |A_i|$

Functions

Programming
Algebra

function(input) = output

$$f(x) = x^2$$

$f(3) = 9$; $f(4) = 16$,

$$\text{sort}(\langle 1, 5, 3, 9, 2, 7 \rangle) = \langle 1, 2, 3, 5, 7, 9 \rangle$$

Let A and B be sets. A function f from A to B, written as $f: A \to B$, assigns to each input value $a \in A$, a unique output value $b \in B$.

$$\underbrace{\text{a unique output value } b \in B}_{f(a)}$$

symbolically

defn by defn

defn by algebraic expression

average : $\mathbb{R} \times \mathbb{R} \to \mathbb{R}$

$\text{average}(x, y) = \frac{x+y}{2}$

not : {True, False} → {True, False}

not (True) = False

not (False) = True

defn by explicit relationship

Domain / codomain

$\mathbb{Z} \to \{\text{True}, \text{False}\}$

For a function $f : A \to B$, the set A is called the domain of f, and the set B is called the codomain of f.

boolean isPrime (int n);

```
def square (n)
    return n x n
```

```
Def isPrime (n)
    :
    return true / false
```

Range / Image

The range (or image) of a function $f: A \to B$ is the set of all $b \in B$ such that $f(a) = b$ for some $a \in A$.

$\{ y \in B$: there exists at least one $a \in A$ such that $f(a) = y \}$

Square function

$\text{square} : \mathbb{R} \to \mathbb{R}$

The range of square function is the set of non-negative real numbers.

$\mathbb{R}^{\geq 0}$

$\boxed{\text{square}(\sqrt{x})} = x$

<u>Example</u> The smallest divisor function

$$sd : \mathbb{Z}^{\geq 2} \to \mathbb{Z}^{\geq 2}$$

where $sd(n)$ is the smallest integer ≥ 2 that evenly divides n.

Claim: The range of sd is the set of prime numbers.

Proof: We will prove it in two parts.

i) we will prove that all prime numbers must be in the range

ii) we will prove that no composite number is in the range.

i) Consider a prime number p.
From the definition of primes, we know that the smallest integer ≥ 2 that divides p must be p

$\therefore$ $sd(p) = p$ $\therefore$ p is in the range.

ii) Suppose k, a composite number, is in the range.

$$\boxed{sd(m) = k}$$

$\therefore \quad m = k \cdot n$

But k is composite, therefore it must have a divisor r where $r \geq 2$ and $r \leq k-1$

$\therefore \quad k = r \cdot s$

$\therefore \quad \boxed{m = r \cdot s \cdot n}$

This means that r evenly divides m

$\therefore \quad sd(m) \leq r$

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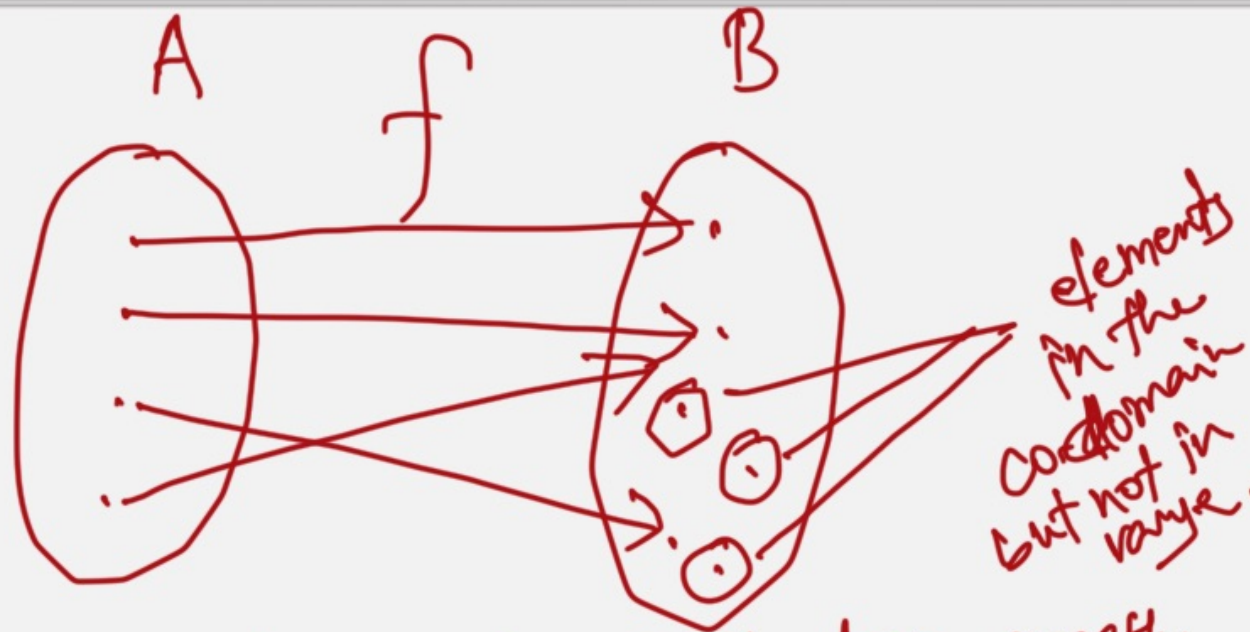
For a function $f : A \to B$ and a set $U \subseteq A$, we write

$\{f(x) : x \in U\}$ as a shorthand

for the set $\{b \in B :$ there exists some $u \in U$ for which $f(u) = b\}$

$f : A \to B$

range $\{f(a) : a \in A\}$



uniqueness:

every element on the left has exactly one outgoing arrow to an element on the right.

Composition of function

$f : A \to B \qquad g : B \to C$

$g \circ f : A \to C$

maps an element $a \in A$

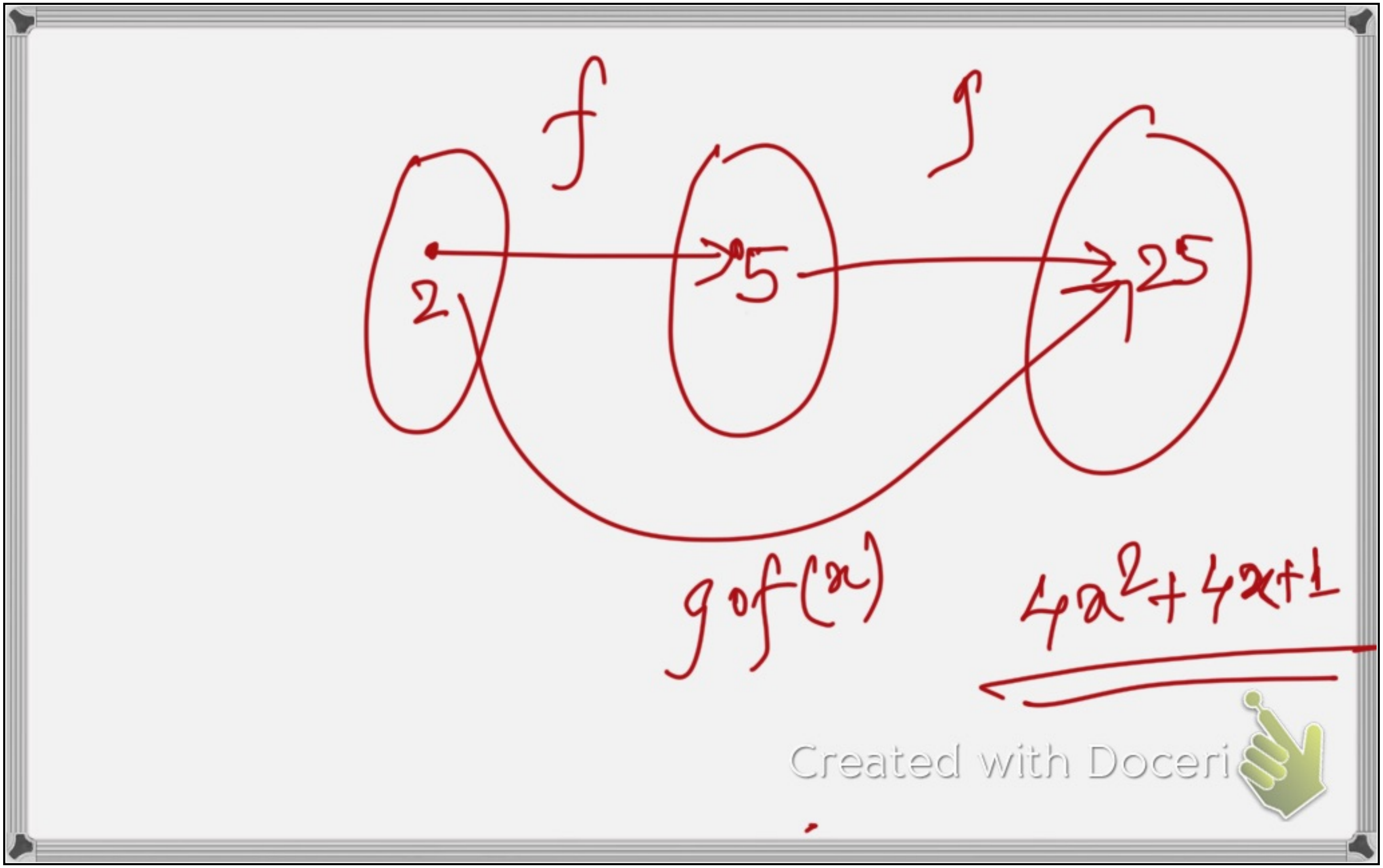
to $g(f(a))$ in C.

$f : \mathbb{R} \to \mathbb{R}$ $\quad f(x) = 2x+1$

$g : \mathbb{R} \to \mathbb{R}$ $\quad g(x) = x^2$

$$g \circ f(x) = g(f(x))$$
$$= g(2x+1)$$
$$= (2x+1)^2$$
$$= 4x^2 + 4x + 1$$

$$f \circ g(x)$$
$$= f(x^2)$$
$$= 2x^2 + 1$$


f
g
2
5
25
gof(x)
4x²+4x+1
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