1. The inequality of arithmetic and geometric means, also known as the AM–GM inequality, states that the arithmetic mean of a list of non-negative real numbers is greater than or equal to the geometric mean of the same list. We would like to prove this for the case when the list has only two numbers. In other words, prove that for $x, y \in \mathbb{R}^{\geq 0}$, we have $\sqrt{xy} \leq (x+y)/2$.

2. Recall the proof done in the class to show that $\sqrt{2}$ is irrational.

   (a) Give a similar proof to argue that $\sqrt{3}$ is irrational.

   (b) What goes wrong if you try to prove similarly that $\sqrt{4}$ is irrational?

3. Recall the question from tutorial 4, where we had used predicates $B(x)$ for *x is a barber* and $S(x, y)$ for *x shaves y*. Consider the statement given below, and the predicate logic encodings that follow.

   Every barber shaves all persons who do not shave themselves.
   $\forall x\ \forall y\ ((B(x) \land \neg S(y, y)) \rightarrow S(x, y))$
   $\forall x\ \forall y\ ((B(x) \land \neg S(x, y)) \rightarrow S(y, y))$

   For each of the two encodings, state whether they correctly capture the statement given above, and justify your answer. If both the encodings are correct, argue why they are logically the same.

4. Consider $f : \mathbb{R} \rightarrow \mathbb{R}$ given by $f(x) = x^2 + 2$. Is f injective (one-to-one)? Is f surjective (onto)?

5. Let n be any positive integer, and let p_n denote the smallest prime number that evenly divides n. Prove that there are infinite number of integers n such that $p_n \geq \sqrt{n}$.

6. Consider the following *bogus* proof of the false claim that $3 \leq 2$. Explain why this proof is incorrect.

   Let x and y be arbitrary non-negative numbers. Because $y \geq 0$ implies $-y \leq y$, we can add x to both sides of this inequality to get
   $$x - y \leq x + y \tag{1}$$
   Similarly, adding $y - 3x$ to both sides of $-x \leq x$ yields
   $$y - 4x \leq y - 2x \tag{2}$$
   Observe that whenever $a \leq b$ and $c \leq d$, we know that $ac \leq bd$. So we can combine equations 1 and 2 to get,
   $$(x - y) \cdot (y - 4x) \leq (x + y) \cdot (y - 2x) \tag{3}$$
   Multiplying out and combining common terms, we get
   $$xy - 4x^2 - y^2 + 4xy \leq xy - 2x^2 + y^2 - 2xy \tag{4}$$
   Upon simplification, we get
   $$6xy \leq 2x^2 + 2y^2 \tag{5}$$
   Since equation 5 holds for any non-negative x and y, let us put $x = y = \sqrt{1/2}$. In that case, $xy = x^2 = y^2 = 1/2$. So, we get
   $$\frac{6}{2} \leq \frac{2}{2} + \frac{2}{2} \tag{6}$$
   which proves that $3 \leq 2$.