Mid-term exam (*25 Marks, 26th Feb, 2 hours*) **ACOL202, IITD-AD, Jan-May 2025**

Name: **Entry No.:**

1. **[8 marks]** Consider a new logic with two connectives $\otimes$ and $\odot$ in which well-formed formulas are formed as per the following syntax rules (also called a *grammar*)

$$\alpha ::= p \mid \otimes(\alpha, \alpha, \alpha) \mid \odot(\alpha, \alpha)$$

where p is a propositional variable that can either be *true* or *false*.

Thus, a well-formed formula in this logic is either a propositional variable, or the ternary (3-argument) operation $\otimes$ applied on three already well-formed formulas, or the binary (2-argument) operation $\odot$ applied on two already well-formed formulas.

Consider the semantics of $\otimes$ and $\odot$ as defined by the following truth tables.

p	q	r	$\otimes(p, q, r)$
F	F	F	T
F	F	T	F
F	T	F	F
F	T	T	T
T	F	F	F
T	F	T	T
T	T	F	T
T	T	T	F

p	q	$(p \odot q)$
F	F	F
F	T	T
T	F	T
T	T	F

(a) **[2 marks]** Answer whether the following formulas are well-formed according to the syntax rules given above. For the formulas that are well-formed, also give the tree representation of the formula.

i. $((\otimes(p, q)) \odot q)$

ii. $((p \odot p) \odot (\otimes(q, (p \odot q), r)))$

(b) **[3 marks]** Construct formulas using the operators $\otimes$ and $\odot$ that are semantically equivalent to the following propositional logic formulas. Thus, in each case, the semantics of your constructed formula, as per the truth tables given above, must exactly match the semantics of the given propositional logic formula, as per the semantics of propositional logic connectives.

i. $\bot$

ii. $\neg p$

iii. $p \leftrightarrow q$

(c) **[3 marks]** Consider another ternary operator `ite` (short form for if-then-else), for which you are told that

$$\texttt{ite}(\alpha, \beta, \gamma) \quad \leftrightarrow \quad ((\alpha \wedge \beta) \vee (\neg\alpha \wedge \gamma))$$

is valid for every propositional logic formulas α, β, and γ.

You are required to write a formula using only `ite` (and no other propositional operators) that is semantically equivalent to $\otimes(p, q, r)$. You may use $\top$ and $\bot$ in your formula if you need to.

2. **[4 marks]** Suppose we would like to prove that $\sqrt{3}+\sqrt{2}$ is irrational. We do it in the following steps.

   (a) **[2 marks]** We claim that $\sqrt{3}+\sqrt{2}$ is rational if and only if $\sqrt{3}-\sqrt{2}$ is rational. Prove this claim.

   (b) **[1 marks]** We further claim that the sum of two rational numbers cannot be irrational. Prove this claim.

   (c) **[1 marks]** Use the above two claims to give a proof by contradiction that $\sqrt{3}+\sqrt{2}$ is irrational. If you need to, you can assume that $\sqrt{2}$ and $\sqrt{3}$ are irrational (without proving it).

3. **[4 marks]** A device consists of a thermostat, a pump, and a warning light. Suppose we are told the following four facts about the device.

   - The thermostat or the pump (or both) are broken.
   - If the thermostat is broken then the pump is also broken.
   - If the pump is broken and the warning light is on then the thermostat is not broken.
   - The warning light is on.

   We want to find out if it is possible for the above sentences to all be true at the same time. To this end let us introduce atomic propositions t (*the thermostat is broken*), p (*the pump is broken*), and w (*the warning light is on*).

   (a) **[2 marks]** Write down a propositional logic formula (call it α), using t, p, and w, that expresses all the four facts given above.

   (b) **[2 marks]** Return all possible valuations (every valuation here would be an assignment of truth values to t, p, and w) that make α true.

4. **[5 marks]** Let the predicate $F(x, y)$ denote that "x can fool y", where the domain consists of all the people in this world. Express each of the following sentences in predicate logic.

   (a) Everybody can fool somebody.

   (b) There is no one who can fool everybody.

   (c) Everyone can be fooled by somebody.

   (d) No one can fool both Alice and Bob.

   (e) Charlie can fool exactly two people.

5. **[4 marks]** Prove that there is an irrational number a such that $a^{\sqrt{3}}$ is rational. (*Hint:* Consider $\sqrt[3]{2}^{\sqrt{3}}$ and argue by cases.)