QUEUEING MODELS

4.1 THE BIRTH AND DEATH PROCESS

4.1.0 Introduction

The intuitive idea behind a birth and death process is that of some population, that is gaining new members through births and losing old members through deaths-such as human population. But in many applications of this process in computer science, the population is the customers in a queueing system. The term customer may refer to a computer job to be processed, an I/O request, a message arrived to a communication system etc. Customer arrivals correspond to births and customer departures (after service) correspond to deaths.

The birth and death process has wide applications in a variety of fields such as queueing, traffic maintenance, population growth controlling epidemics etc. In the next section we shall use it to queueing theory.

4.2 QUEUEING THEORY

One of the most useful areas of application of probability theory is that of queueing theory or the study of waiting line phenomena. Queues are found everywhere in our day-to-day life. For example in industries, schools, colleges, hospitals, libraries, banks, post offices, theatres, ticket booking for trains and buses etc. Queues are also common in computer waiting systems, queues of enquiries waiting to be processed by an interactive computer system. Queues of data base requests, queues of input/output requests etc. Queueing problem arises in the following cases

(i) the demand for service is more than the capacity to provide service.

For example : Ticket booking counters in railway stations, queues are always formed.

(ii) The demand for service is less than the capacity to serve so that there is lot of idle facility time or too many facilities.

For example : In a petrol bunk, if there is no vehicle for refilling petrol then the system is idle, both the pump and the workers are idle.

4.2.1 Use of queueing theory

If customers are arriving to service facility in such a way that either the customer or the service facilities have to wait, then we have a queueing problem.

Queueing theory is used to achieve an optimum balance between the cost associated with waiting time of customers and idle time of service facilities so that the profit is maximized.

A customer may be a person, a machine, letter, a ship, a computer job to be processed etc.

4.2.2 Queueing system or Queueing Model

A queueing system can be described as consisting of customers arriving for service to a service facility, waiting for service, if it is not available immediately, and leaving the service centre after being serviced.

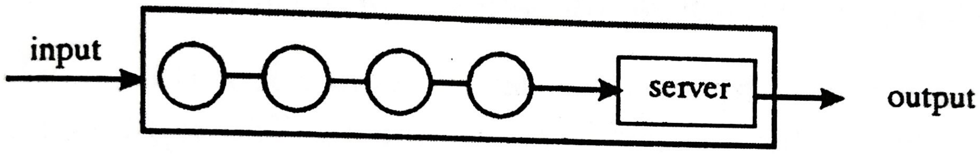
There are many types of queueing systems. But all of them can be completely described by the following characteristics.

4.2.3 Characteristics of queueing system or queueing model

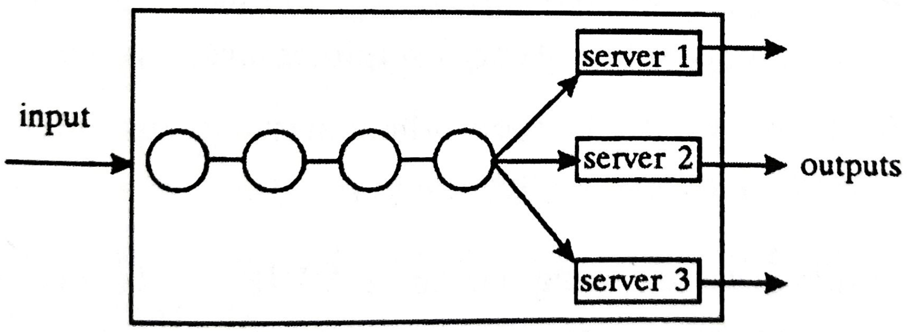
1. The input pattern or arrival pattern

The input pattern represents the manner in which customers arrive for service and join the queueing system. The actual time of arrival of a customer cannot be predicted or observed. The number of arrivals in a time period or the interval between two successive arrivals cannot be a constant, but a random variable. Hence the arrival pattern of customers is expressed by means of probability distribution of the number of arrivals per unit time or of inter arrival time. The number of customers arriving per unit time is called the **arrival rate,** which is a random variable.

The arrival pattern usually used is the Poisson distribution with parameter λ where λ is the average arrival rate. Then the time interval between consecutive arrivals follow an exponential distribution with mean $\frac{1}{\lambda}$.



Single server model



Multiservers in parallel queueing system

2. Service pattern (or departure pattern)

The service pattern is represented by the probability distribution of the number of customers serviced per unit of time (i.e. service rate) or the inter-service time. This rate assumes that the service channel to be busy always, that is no idle time is allowed.

A typical assumption used is that the service time is a random variable following exponential distribution with mean rate of service μ. Sometimes Poisson distribution is also used.

Note : Exponential distribution is usually used to describe random arrivals or departures because it has memoryless property and so one event does not influence the other.

3. Service channels

The queueing system may have single service channel. Arriving customers form one queue as in a doctor's clinic. This system may have more than one queue, arranged in parallel as in railway booking or in series. That is the system may have one server, or more than one server.

4. Service discipline (or queue discipline)

Service discipline or order of service is a rule by which customers are selected for service from the queue. The most common discipline is **'first come, first served' (FCFS)** or **'first in, first out' (FIFO)** according to which the customers are served in the order of their arrival. Example : cinema ticket counters, railway booking counters etc.

Another discipline is **'last come, first served' (LCFS)** or **'last in, first out' (LIFO)**. Example : In a big godown, the items arriving last are taken out first. Some other disciplines are **'random'** and **'priority'**.

Priority is said to happen when arriving customer is chosen for service ahead of others in the queue.

Example : In a voting booth a VVIP is chosen as 'priority' to vote as soon as he comes.

If **service** is given in **random order** then we have a SIRO discipline. In this case every customer in the queue has the same probability of being selected for service.

5. Maximum Queueing System Capacity

Maximum number of customers in the system can be either finite or infinite. In some facilities only limited number of customers are allowed in the system, the new arrivals are not allowed to join the queue. In some system the queue capacity is assumed to be infinite, if every arriving customer is allowed to wait until service is provided.

6. Calling source or population

The arrival pattern of the customers depends upon the source from which they come. An infinite source system is easier to describe mathematically than a finite source. In a finite source system, the number of customers in the system affects the arrival rate. For, if every potential customer is already in the queue the arrival rate drops to zero. Whereas for an infinite population system, the number of customers in the queue has no effect on the arrival pattern. So, if the customer population is finite but large, we assume it to be infinite. Infact, in practice if the number of potential customers is over 40 or 50 it is usually said to be infinite.

7. Customer behaviour

Generally a customer behaves in the following ways.

(i) **Balking** : A customer who refuses to enter queueing system because the queue is too long is said to be **balking.**

(ii) **Reneging** : A customer who leaves the queue without receiving service because of too much waiting (or due to impatience) is said to have **reneged.**

(iii) **Jockeying** : When there are parallel queues a customer who jumps from one queue to another with shorter length to reduce waiting time is said to be jockeying.

4.2.4 State of the System

In a queueing model, the probability distributions of arrivals, waiting time distribution and service time distribution of customers are in general, functions of time. In the long run it may happen that these characteristics are independent of time. Generally, the states of the queueing system are classified as

(i) Transient state, (ii) Steady state and (iii) Explosive state.

(i) Transient state : A queueing system is said to be in **transient state** when its operating characteristics like input, output, mean queue length etc. are dependent on time. This type of state always occur at the beginning of the functioning of the queueing system.

(ii) Steady state : A queueing system is said to be in steady state when its operating characteristics are independent of time. If $P_n(t)$ is the probability that there are n customers in the system at time t then the system reaches a steady state if

$$\lim_{t \to \infty} P_n(t) = P_n \quad \text{or} \quad \lim_{t \to \infty} P_n'(t) = 0$$

Note :

The system is in steady-state does not mean that the arrival rate and service rate are independent of the number of customers in the system.

(iii) Explosive state : If the arrival rate of customers is greater than the service rate, then the queue length will go on increasing with time and as $t \to \infty$, the length queue $\to \infty$. This state is said to be explosive state.

4.2.5 Kendal's notation for representing queueing models

David Kendal introduced a special notation given below to describe a queueing system. The notation has the form

$$(a \mid b \mid c) : (d \mid e)$$

where a = arrival (or inter arrival) probability law or distribution.

b = service time probability distribution

(i.e. the probability law according to which customers are served)

c = number of servers (or channels)

d = capacity of the system

e = queue discipline (or service discipline)

Symbols for a and b are the following :

M : Markov (Poisson) arrival and departure distribution or exponential distribution.

E_K : Erlangian (or gamma) inter arrival or service time distribution with parameter K.

G : General service time distribution or general departure distribution.

Symbol for e : FCFS = first come, first served.

or FIFO = first in, first out.

SIRO = service in random order.

We shall now consider some queueing models. We consider here only queueing models under steady state conditions.

The following are usual notations in the discussions.

n = **number of customers in the system (i.e. waiting for service in the queue + being served)**

λ = **mean arrival rate (i.e. average number of customers arriving per unit time)**

μ = **mean service rate per busy server (i.e. average number of customers served per unit time)**

ρ = $\frac{\lambda}{\mu}$ **is traffic intensity or utilization factor (i.e. the degree to which service station is utilized)**

c = **number of parallel service channels.**

L_q = **mean length of the queue (i.e. the average or expected number of customers waiting in the queue)**

L_s = mean length of the system

(i.e. the average or expected number of customers both waiting and in service)

W_q = mean waiting time in the queue (i.e. the expected waiting time before being served)

W_s = mean waiting time in the system (i.e. the expected waiting time in the system)

$P_n(t)$ = transient state probability of exactly n customers in the system

P_n = steady state probability of n customers in the system

4.3 Model - I (M|M|1) : (∞|FIFO) Model (i.e.) M|M|1 infinite capacity model

This model represents a queueing system with single server, Poisson arrivals, exponential service time and there is no limit on the system capacity and the customers are served on a first in first out (or first come first served) basis.

We assume that the arrival rate $\lambda_n = \lambda$ and $\mu_n = \mu$ for all n and $\frac{\lambda}{\mu} < 1$

∴ the probability of n customers P_n from formula (6) in 4.2.6 give

$$P_n = \left(\frac{\lambda}{\mu}\right)^n P_0$$

and

$$P_0 = \frac{1}{1 + \sum_{n=1}^{\infty} \left(\frac{\lambda}{\mu}\right)^n} = \frac{1}{1 + \frac{\lambda}{\mu} + \left(\frac{\lambda}{\mu}\right)^2 + \ldots}$$

$$= \frac{1}{\left(1 - \frac{\lambda}{\mu}\right)^{-1}}$$

$$= 1 - \frac{\lambda}{\mu} \qquad \left[\because \frac{\lambda}{\mu} < 1\right]$$

$$\therefore P_n = \left(\frac{\lambda}{\mu}\right)^n \left(1 - \frac{\lambda}{\mu}\right). \quad \forall\, n \geq 0$$

Little's formula : Relation between W_s, W_q, L_s, L_q

We have proved,

$$L_s = \frac{\lambda}{\mu - \lambda}, \qquad L_q = \frac{\lambda^2}{\mu(\mu - \lambda)}$$

$$W_s = \frac{1}{\mu - \lambda}, \qquad W_q = \frac{\lambda}{\mu(\mu - \lambda)}$$

$\therefore$ We have,

(i) $L_s = \lambda W_s$ **(ii)** $L_q = \lambda W_q$

(iii) $L_s = L_q + \frac{\lambda}{\mu}$ **(iv)** $W_q = W_s - \frac{1}{\mu}$

These are known as Little's formula.

SUMMARY

1. First find $W_s = \frac{1}{\mu - \lambda}$, then $W_q = W_s - \frac{1}{\mu}$,

   $L_s = \lambda W_s$ and $L_q = \lambda W_q$ can be found.

2. $P_n = \left(\frac{\lambda}{\mu}\right)^n \left(1 - \frac{\lambda}{\mu}\right)$

3. Average length of non-empty queue $L_w = \mu . W_s$

4. $P(N \geq K) = \left(\frac{\lambda}{\mu}\right)^K$

   = Probability that the length of the queue system $\geq$ K.

5. P (Channel busy) $= \frac{\lambda}{\mu}$ and P (idle system) $= P_0 = 1 - \frac{\lambda}{\mu}$

6. P (Waiting time in the system > t) $= e^{-(\mu - \lambda)t}$

7. Probability density function of waiting time in the system is given by $f(w) = (\mu - \lambda)\, e^{-(\mu - \lambda)w}$, $w \geq 0$.

8. Probability density function of waiting time in the queue is given by $g(w) = \frac{\lambda}{\mu} (\mu - \lambda)\, e^{-(\mu - \lambda)w}$

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WORKED EXAMPLES

Example 1 : In the railway marshalling yard goods trains arrive at a rate of 30 trains per day. Assume that the inter arrival time follows exponential distribution and the service time distribution is also exponential with an average of 36 minutes. Calculate the following

(i) the mean queue size

(ii) the probability that the system size is atleast 10.

If the input of trains increases to an average of 33 per day, what will be the change in the above quantities ? [M.U 1990, A.U-2006]

Solution : Given mean arrival rate $\lambda = 30$ per day

$$= \frac{30}{24 \times 60} \text{ per minute}$$

$$\Rightarrow \lambda = \frac{1}{48} \text{ trains/minute}$$

Service time is exponential with mean equal to 36 minutes.

$$\therefore \mu = \frac{1}{36} \text{ per minute.} \qquad [\because \text{ mean } = \frac{1}{\mu}]$$

Then
$$W_s = \frac{1}{\mu - \lambda} = \frac{1}{\frac{1}{36} - \frac{1}{48}} = \frac{1}{\frac{4-3}{144}} = 144$$

$$W_q = W_s - \frac{1}{\mu} = 144 - 36 = 108$$

(i) Average queue size $L_q = \lambda W_q$

$$= \frac{1}{48} \times 108 = 2.25 \text{ trains.}$$

(ii)
$$P(N \geq 10) = \left(\frac{\lambda}{\mu}\right)^{10}$$

$$= \left(\frac{36}{48}\right)^{10} = (0.75)^{10} = 0.056$$

If the input increases to 33 trains per day, then

$$\lambda = \frac{33}{24 \times 60} = \frac{11}{480} \text{ trains/minute}$$

and $$\mu = \frac{1}{36} \text{ trains/minute}$$

$$\therefore \quad W_s = \frac{1}{\mu - \lambda} = \frac{1}{\frac{1}{36} - \frac{11}{480}}$$

$$= \frac{1440}{40 - 33} = \frac{1440}{7}$$

$$W_q = W_s - \frac{1}{\mu}$$

$$= \frac{1440}{7} - 36$$

$$= \frac{1440 - 252}{7} = \frac{1188}{7}$$

$$L_q = \lambda W_q = \frac{11}{480} \times \frac{1188}{7}$$

$$= 3.89 \text{ trains.}$$

$\therefore$ Change in the size of the queue $= 3.89 - 2.25$

$= 1.64$ trains

$$\text{Now } P(N \geq 10) = \left(\frac{\lambda}{\mu}\right)^{10}$$

$$= \left(\frac{11}{480} \times 36\right)^{10}$$

$$= \left(\frac{33}{40}\right)^{10} = 0.146$$

$\therefore$ Change in probability $= 0.146 - 0.056$

$= 0.09$

Example 2 : Customers arrive at a one-man barber-shop according to a poisson process with mean interval arrival time of 20 minutes. Customers spend an average of 15 minutes in the barber's chair. If an hour is used as unit of time, then

(i) What is the probability that a customer need not wait for a hair cut ?

(ii) What is the expected number of customers in the barber shop and in the queue ?

(iii) How much time can a customer expect to spend in the barber shop ?

(iv) Find the average time that the customer spend in the queue.

(v) What is the probability that there will be more than 3 customers in the system ?

[AU 2008, 2013]

Solution : Model $(M|M|1) : (\infty|FIFO)$

If the arrival is poisson with rate λ then the inter arrival is exponential with mean $\frac{1}{\lambda}$

Given $\frac{1}{\lambda} = 20 \Rightarrow \lambda = \frac{1}{20}$ per minute

$\frac{1}{\mu} = 15 \Rightarrow \mu = \frac{1}{15}$ per minute

(i) Probability that the queue is empty or the system is idle is

$$P_0 = 1 - \frac{\lambda}{\mu}$$

$$= 1 - \frac{15}{20} = \frac{5}{20} = 0.25$$

Now $W_s = \frac{1}{\mu - \lambda} = \frac{1}{\frac{1}{15} - \frac{1}{20}} = \frac{60}{4-3} = 60$ minutes.

(ii) Expected number of customers in the queue and the barber shop is

$$L_s = \lambda W_s$$

$$= \frac{1}{20} \times 60 = 3 \text{ customers}$$

$$L_q = L_s - \frac{\lambda}{\mu} = 3 - \frac{3}{4} = 2.25 \text{ customers}$$

(iii) Expected waiting time in the system is $W_s = 60$ minutes

(iv) Average time customer spends in the queue is $W_q = W_s - \frac{1}{\mu}$

$$= 60 - 15 = 45 \text{ minutes.}$$

(v) $P(N > 3) = P(N \geq 4)$

$$= \left(\frac{\lambda}{\mu}\right)^4 = \left(\frac{15}{20}\right)^4 = \left(\frac{3}{4}\right)^4 = 0.3164$$

Example 3 : Arrivals at a telephone booth are considered to be poisson with an average time 12 minutes between one arrival and the next. The length of telephone call is assumed to be distributed exponentially with mean 4 minutes.

(a) (i) Find the average number of persons waiting in the system.

(ii) What is the probability that a person arriving at the booth has to wait in the queue ?

(iii) Also estimate the fraction of the day when phone will be in use. **[A.U-2004, 2007]**

(b) What is the probability that it will take more than 10 minutes for a person to wait and complete his call ?

(c) The telephone department will install a second booth when convinced that an arrival would expect to wait atleast 3 minutes

for the phone. By how much should the flow of arrivals increase in order to justify a second booth ?

Solution : It is $(M|M|1) : (\infty|\text{ FIFO})$ Model :

Given mean inter arrival time $\frac{1}{\lambda} = 12$ minutes

$$\Rightarrow \lambda = \frac{1}{12} \text{ per minute}$$

Mean service time $\frac{1}{\mu} = 4$ minutes

$$\Rightarrow \mu = \frac{1}{4} \text{ per minute}$$

$$\text{Now } W_s = \frac{1}{\mu - \lambda} = \frac{1}{\frac{1}{4} - \frac{1}{12}} = \frac{12}{3-1} = 6$$

(a) (i) Average number of persons waiting in the system is

$$L_s = \lambda W_s$$

$$= \frac{1}{12} \times 6 = 0.5 \text{ person}$$

(ii) Probability of waiting in the system

$$= P \text{ (channel is busy)}$$

$$= \frac{\lambda}{\mu} = \frac{\frac{1}{12}}{\frac{1}{4}} = \frac{1}{3}$$

(iii) P (Phone in use) $= P$ (Phone is busy)

$$= \frac{\lambda}{\mu} = \frac{1}{3}$$

(b) A person takes more than 10 minutes to wait and complete his call means that he is in the queueing system for more than 10 minutes $= P(W_T > 10)$

We know $\quad P(W_T > t) = e^{-(\mu - \lambda)t}$

$$\therefore P(W_T > 10) = e^{-(\mu - \lambda)10}$$

$$= e^{-\left(\frac{1}{4} - \frac{1}{12}\right)10}$$

$$= e^{-\frac{5}{3}} = 0.1889$$

(c) The second phone will be installed if $E(W) > 3 \Rightarrow W_q > 3$

$$\text{i.e. if } \frac{\lambda}{\mu(\mu - \lambda)} > 3$$

$$\text{i.e. if } \quad \lambda > 3\mu(\mu - \lambda)$$

$$\text{i.e. if } \quad \lambda > 3\frac{1}{4}\left(\frac{1}{4} - \lambda\right)$$

$$> \frac{3}{16} - \frac{3\lambda}{4}$$

$$\text{i.e. if } \lambda + \frac{3\lambda}{4} > \frac{3}{16}$$

$$\text{i.e. if } \frac{7}{4}\lambda > \frac{3}{16}$$

$$\text{i.e. if } \quad \lambda > \frac{3}{16} \times \frac{4}{7}$$

$$\text{i.e. if } \quad \lambda > \frac{3}{28}$$

Hence the increase in arrival rate should be atleast $= \frac{3}{28} - \frac{1}{12}$

$$= \frac{1}{42} \text{ per minute}$$

So, the increase in arrival should be atleast $\frac{1}{42}$ per minute.

Example 4 : Customers arrive at a watch repair shop according to a poisson process at a rate of one per every 10 minutes, and the service time is an exponential random variable with mean 8 minutes.

(i) Find the average number of customers L_s in the shop.

(ii) Find the average time a customer spends in the shop W_s

(iii) Find the average number of customers in the queue L_q.

(iv) What is the probability that the server is idle ? [A.U-2006]

Solution : (M|M|1) : (∞ | FIFO) model.

Given that customer arrival is poisson process with

$$\text{rate } \lambda = \frac{1}{10} \text{ per minute}$$

service time is exponential with mean 8 minutes

$$\therefore \frac{1}{\mu} = 8$$

$$\Rightarrow \mu = \frac{1}{8} \text{ per minute.}$$

Now, $$W_s = \frac{1}{\mu - \lambda} = \frac{1}{\frac{1}{8} - \frac{1}{10}} = \frac{40}{5-4} = 40$$

(i) Average number of customers in the shop is

$$L_s = \lambda W_s = \frac{1}{10} \times 40 = 4$$

(ii) Average time a customer spends in the shop is

$$W_s = 40 \text{ minutes}$$

(iii) Now, $$W_q = W_s - \frac{1}{\mu} = 40 - 8 = 32$$

$$\therefore \quad L_q = \lambda W_q = \frac{1}{10} \times 32 = 3.2$$

(iii) Average number of customers in the queue $L_q = 3.2$

(iv) Probability that the server is idle $= P_0 = 1 - \frac{\lambda}{\mu}$

$$= 1 - \frac{8}{10} = \frac{1}{5} = 0.2$$

Example 5 : A repairman is to be hired to repair machines which breakdown at an average rate of 3 per hour. The breakdown follow poisson distribution. Non-productive time of machines is considered to cost Rs. 16 per hour. Two repairmen have been interviewed. One is slow but cheap while the other is fast and expensive. The slow repairman charges Rs. 8 per hour and he services machines at the rate of 4 per hour. The fast repairman demands Rs. 10 per hour and services at the average rate of 6 per hour. Which repairman should be hired ? **[A.U-2003]**

Solution : Given breakdown rate is $\lambda = 3$ per hour

Idle time cost Rs. 16 per hour.

For slow repairman :

Given average rate $\mu = 4$ per hour.

Breakdown time of machine = Average waiting time spent in the system

$$= W_s = \frac{1}{\mu - \lambda} = \frac{1}{4-3} = 1 \text{ hr}$$

Non-productive cost per hour = Rs. 16

Idle cost per hour for 3 machines = $1 \times 3 \times 16$ = Rs. 48

Repair charges per hour = Rs. 8

In one hour he can repair 3 machines, so charge is for one hour.

$$\therefore \text{ Total cost} = \text{Rs. } (48 + 8) = 56$$

For fast repairman :

$$\mu = 6 \text{ per hour}$$

$$\text{Average breakdown time } W_s = \frac{1}{\mu - \lambda} = \frac{1}{6-3}$$

$$= \frac{1}{3} \text{ hour}$$

$$\text{Non-productive cost for 3 machines} = \frac{1}{3} \times 3 \times 16$$

$$= \text{Rs. } 16$$

$$\text{Repair charges} = 1 \times 10 = \text{Rs. } 10$$

$$\therefore \text{ Total cost} = 16 + 10 = \text{Rs. } 26$$

Since cost for fastman is less, the fast repair man should be engaged.

Example 8 : An automatic car wash facility operates with only one bay. Cars arrive according to a poisson distribution with a mean of 4 cars per hour and may wait in the facility's parking lot if the bay is busy. If the service time for all cars is constant and equal to 10 minutes, determine

(i) mean number of customers in the system L_s

(ii) mean waiting time of a customer in the system W_s

(iii) mean waiting time of a customer in the queue W_q

(iv) mean number of customers in the queue L_q

[A.U-2005, 2008, 2009]

Solution : It is M|M|1 model, infinite capacity.

Given arrival is poisson distribution with mean = 4 cars per hour

$$\Rightarrow \lambda = 4 \text{ cars per hour}$$

Mean service rate $\frac{1}{\mu}$ = 10 minutes per car

$$\Rightarrow \mu = \frac{1}{10} = \frac{60}{10} \text{ cars/hour} = 6 \text{ cars/hour}$$

Now waiting time is $W_s = \frac{1}{\mu - \lambda} = \frac{1}{6-4} = \frac{1}{2}$

(i) Mean number of customers in the system is $L_s = \lambda W_s = 4 \times \frac{1}{2} = 2$

(ii) Mean waiting time of a customer in the system is $W_s = \frac{1}{2}$

(iii) Mean waiting time of a customer in the queue is

$$W_q = W_s - \frac{1}{\mu} = \frac{1}{2} - \frac{1}{6} = \frac{1}{3}$$

(iv) Mean number of customers in the queue is

$$L_q = \lambda W_q = 4 \times \frac{1}{3} = \frac{4}{3}$$

show...

Example 10 : In a cinema theatre people arrive to purchase tickets at the average rate of 6 per minute and it takes 7.5 seconds on the average to purchase a ticket. If a person arrives just 2 minutes before the picture starts and it takes exactly 1.5 minutes to reach the correct seat after purchasing the ticket.

(i) Can he expect to be seated for the start of the picture ?

(ii) What is the probability that he will be seated when the film starts?

(iii) How early should he arrive to ensure a 99% chance of being seated at the start of the film ?

(iv) What is the probability that there is no customer in the system?

[A.U 2010]

Solution : Given arrival rate $\lambda = 6$ people per minute

Service rate is $\mu = \frac{1}{7.5}$ secs / ticket / person

$= \frac{60}{7.5}$ persons / minute

$= 8$ persons / minute

(i) Expected waiting time in the system is $W_s = \frac{1}{\mu - \lambda}$

$$= \frac{1}{8-6} = \frac{1}{2} \text{ minutes.}$$

$\therefore$ expected total time to purchase and reach the seat

$$= \frac{1}{2} + 1\frac{1}{2} \text{ minutes}$$

$$= 2 \text{ minutes.}$$

(It is given that to reach the correct seat time taken is 1.5 minutes)

$\therefore$ person who arrives just 2 minutes before can be seated at the start of the film.

(ii) P (total time $\leq$ 2 minutes) $= P\left(W \leq \frac{1}{2}\right)$

$$= 1 - P(W > 1/2)$$

$$= 1 - e^{-(\mu - \lambda)\,1/2}$$

$$= 1 - e^{-(8-6)\,1/2}$$

$$= 1 - e^{-1} = 0.6321$$

(iii) Suppose t minutes earlier be the time of arrival so that he is seated 99%, then

$$P(W \leq t) = 0.99$$

$$\Rightarrow \quad 1 - P(W > t) = .99$$

$$\Rightarrow \quad P(W > t) = 1 - .99 = 0.01$$

$$\Rightarrow \quad e^{-(\mu - \lambda)t} = 0.01$$

$$\Rightarrow \quad e^{-2t} = 0.01$$

$$\Rightarrow \quad -2t = \log(0.01) = -4.605$$

$$\Rightarrow \quad t = 2.3 \text{ minutes.}$$

This is the waiting time in the system. That is to purchase ticket. He takes 1.5 minutes to reach the seat after purchasing ticket.

$\therefore$ Total time $= 2.3 + 1.5 = 3.8$ minutes.

Hence he must arrive atleast 3.8 minutes earlier so as to be 99% sure of seeing the start of the film.

(iv) P (no customer in the system) $= P_0 = 1 - \frac{\lambda}{\mu} = 1 - \frac{6}{8} = 0.25.$

4.5 Model III : (M|M|1) : (K|FIFO) Model: ie M|M|1 model with finite capacity

In this model the capacity is k, finite, all other conditions are same as in model I. If the system has k customers, then the new arrivals cannot join the queue. Example parking of cars in areas with limited capacity, admission of patients in hospitals where a fixed number of rooms only are available.

For the Markovian queue, $P_n = P(N = n)$ in the steady-state is given by the equations.

$$\lambda_{n-1} P_{n-1} - (\lambda_n + \mu_n) P_n + \mu_{n+1} P_{n+1} = 0 \text{ if } n > 0$$

and $$-\lambda_0 P_0 + \mu_1 P_1 = 0 \qquad \text{if } n = 0$$

In this model arrivals $\lambda_n = \lambda$ for $n = 0, 1, 2, \ldots, k-1$

$$= 0 \quad \text{for } n \geq k$$

and $\mu_n = \mu$ for $n = 1, 2, 3, \ldots$

SUMMARY

1. $P_n = \begin{cases} \left(\frac{\lambda}{\mu}\right)^n P_0, & n=0,1,2,\dots k; \\ 0 & \text{if } n>k \end{cases}$ $P_0 = \begin{cases} \dfrac{1-\frac{\lambda}{\mu}}{1-\left(\frac{\lambda}{\mu}\right)^{k+1}} & \text{if } \lambda \neq \mu \\ \dfrac{1}{k+1} & \text{if } \lambda = \mu \end{cases}$

2. $$L_s = P_0 \sum_{n=0}^{k} n \left(\frac{\lambda}{\mu}\right)^n = \frac{\frac{\lambda}{\mu}}{\left(1-\frac{\lambda}{\mu}\right)} - \frac{(k+1)\left(\frac{\lambda}{\mu}\right)^{k+1}}{\left[1-\left(\frac{\lambda}{\mu}\right)^{k+1}\right]} \quad \text{if } \lambda \neq \mu$$
$$= \frac{k}{2} \quad \text{if } \lambda = \mu$$

3. Little's Formula $L_q = L_s - \frac{\lambda'}{\mu}$ where $\frac{\lambda'}{\mu} = (1 - P_0)$

4. $W_s = \frac{L_s}{\lambda'}$

5. $W_q = \frac{L_q}{\lambda'}$

6. P (a customer turned away) $= P_k = \left(\frac{\lambda}{\mu}\right)^k P_0$

WORKED EXAMPLES

Example 1: Assume that goods trains are coming in a yard at the rate of 30 trains per day and suppose that the inter arrival times follow an exponential distribution. The service time for each train is assumed to be exponential with an average of 36 minutes. If the yard can admit 9 trains at a time, calculate

(i) the probability that the yard is empty

(ii) the average queue length.

Solution : Here single service but finite capacity so it is (M|M|1) : (K|FIFO) model.

Here $\lambda = 30 / \text{day} = \dfrac{30}{24 \times 60} = \dfrac{1}{48} / \text{minute}.$

$$\mu = \frac{1}{36} / \text{minute} \; ; \; k = 9$$

$$\therefore \quad \frac{\lambda}{\mu} = \frac{36}{48} = 0.75 \; (\neq 1)$$

(i) P (empty queue) $= P_0 = \dfrac{1 - \dfrac{\lambda}{\mu}}{1 - \left(\dfrac{\lambda}{\mu}\right)^{k+1}} \quad (\because \lambda \neq \mu)$

$$= \frac{1 - 0.75}{1 - (0.75)^{10}}$$

$$= \frac{0.25}{1 - 0.0563} = \frac{0.25}{0.94}$$

$$= 0.2649$$

(ii) Average length of queue $L_q = L_s - \dfrac{\lambda'}{\mu}$

where $\dfrac{\lambda'}{\mu} = 1 - P_o = 1 - 0.2649 = 0.7351$

But $L_s = \dfrac{\dfrac{\lambda}{\mu}}{\left(1 - \dfrac{\lambda}{\mu}\right)} - \dfrac{(k+1)\left(\dfrac{\lambda}{\mu}\right)^{k+1}}{\left[1 - \left(\dfrac{\lambda}{\mu}\right)^{k+1}\right]}$

$$= \frac{0.75}{(1 - 0.75)} - \frac{10\,(.75)^{10}}{\left[1 - (.75)^{10}\right]}$$

$$= 3 - \frac{10 \times 0.0563}{1 - 0.0563} = 3 - \frac{0.563}{0.9437}$$

$$= 3 - 0.5966 = 2.4$$

$\therefore \quad L_q = L_s - (1 - P_0) = 2.4 - 0.7351 = 1.66$ trains.

Example 2 : A one-man barber shop can accomodate a maximum of 5 people at a time, 4 waiting and 1 getting hair cut. Customers arrive following poisson distribution with an average of 5 per hour and service is rendered according to exponential distribution at an average rate of 15 minutes,

(i) What is the percentage of idle time ?

(ii) Probability of a potential customer turned away.

(iii) Expected number of customers in the queue.

(iv) Expected time spent in the shop.

Solution : Since one-man shop with maximum capacity 5, it M|M|1 finite capacity, model - III.

Given : $\lambda = 5/\text{hr}, \quad \frac{1}{\mu} = 15$ minutes

$$\Rightarrow \quad \mu = \frac{1}{15} \text{ minutes} = \frac{60}{15} / \text{hr} = 4/\text{hr}$$

Capacity of system $k = 5$

$$\frac{\lambda}{\mu} = \frac{5}{4} \neq 1$$

$$\text{(i) P (idle time)} = P_0 = \frac{1 - \frac{\lambda}{\mu}}{1 - \left(\frac{\lambda}{\mu}\right)^{k+1}}$$

$$= \frac{1 - \frac{5}{4}}{1 - \left(\frac{5}{4}\right)^6} = \frac{.25}{2.8147} = 0.0888$$

(ii) P (a potential customer turned away) $= P(N = 5)$

$$= P_5 = \left(\frac{\lambda}{\mu}\right)^5 P_0$$

$$= \left(\frac{5}{4}\right)^5 \times 0.0888 = 0.271$$

(iii) Expected number of customers in the queue is

$$L_q = L_s - \frac{\lambda'}{\mu}$$

where $\frac{\lambda'}{\mu} = 1 - P_0 = 1 - 0.0888 = 0.9112$

But $$L_s = \frac{\frac{\lambda}{\mu}}{\left(1 - \frac{\lambda}{\mu}\right)} - (k+1)\frac{\left(\frac{\lambda}{\mu}\right)^{k+1}}{1 - \left(\frac{\lambda}{\mu}\right)^{k+1}}$$

$$= \frac{1.25}{1 - 1.25} - 6 \times \frac{(1.25)^6}{1 - (1.25)^6} \qquad \left[\because \frac{\lambda}{\mu} = \frac{5}{4} = 1.24\right]$$

$$= \frac{1.25}{1 - .25} - 6 \times \frac{3.8147}{1 - 3.8147}$$

$$= -5 + 8.1317 = 3.1317$$

$$\therefore L_q = 3.1317 - 0.9112$$

$$= 2.22 \text{ customers.}$$

(iv) Expected time spend in the shop is

$$\frac{L_s}{\lambda'} = \frac{3.1317}{\mu(1 - P_0)} = \frac{3.1317}{4 \times 0.9112} = \frac{3.1317}{3.6448}$$

$$= 0.8592 \text{ hrs} = 0.859 \times 60 \text{ minutes}$$

$$= 51.55 \text{ minutes.}$$

Example 3 : At a railway station, only one train is handled at a time. The railway yard is sufficient only for 2 trains to wait while other is given signal to leave the station. Trains arrive at the station at an average rate of 6 per hour and the railway station can handle them on an average of 12 per hour. Assuming poisson arrivals and exponential service distribution.

(i) Find the steady state probabilities of the various number of trains in the system, (ii) Also find the average number of trains in the system and (iii) the average waiting time.

Solution : One channel [$\because$ only one train is handled at a time]

finite capacity model.

$$\lambda = 6 \text{ per hour} \quad \mu = 12 \text{ per hour, } k = 2 + 1 = 3$$

$$\frac{\lambda}{\mu} = \frac{6}{12} = 0.5$$

$$P_n = \left(\frac{\lambda}{\mu}\right)^n P_0 = (0.5)^n P_0$$

Now

$$P_0 = \frac{1 - \frac{\lambda}{\mu}}{1 - \left(\frac{\lambda}{\mu}\right)^{k+1}} = \frac{1 - 0.5}{1 - (0.5)^4} = \frac{0.5}{1 - 0.0625}$$

$$= \frac{0.5}{0.9375} = 0.53$$

$$\therefore P_n = (0.5)^n \times 0.53, \; n = 0, 1, 2, 3...$$

$$\therefore P_0 = 0.53$$

$$P_1 = 0.5 \times 0.53 = 0.266$$

$$P_2 = (0.5)^2 \times 0.53 = 0.133$$

$$P_3 = (0.5)^2 \times 0.53 = 0.066$$

$$L_s = \sum_{n=0}^{3} n\,P_n$$

$$= P_1 + 2.P_2 + 3\,P_3$$

$$= 0.266 + 2 \times 0.133 + 3 \times 0.066$$

$$= 0.266 + 0.266 + 0.198 = 0.73$$

(iii) Average waiting time $W_s = \dfrac{L_s}{\lambda'}$

where $\dfrac{\lambda'}{\mu} = 1 - P_0$

$$\Rightarrow \lambda' = \mu\,(1 - P_0)$$

$$= 12\,(1 - 0.53) = 12 \times 0.47 = 5.64$$

$$\therefore\ W_s = \frac{0.73}{5.64}\ \text{hrs} = \frac{0.73 \times 60}{5.64}\ \text{minutes}$$

$$= 7.77\ \text{minutes.}$$

Example 4 : A car park contains 5 cars. The arrival of cars is poisson at a mean rate of 10 per hour. The length of time each car spends in the car park has exponential distribution with mean 2 minutes. (i) How many cars are there in the car park on the average? (ii) What is the probability of a newly arriving has to go away ?

Solution : It is M|M|1 ; K | FIFO model.

$\lambda = 10$ per hour ; $\mu = \dfrac{1}{2}$ / minutes $= \dfrac{60}{2}$ per hour $=$ 30 per hour

$k = 5$ $\qquad \therefore\ \dfrac{\lambda}{\mu} = \dfrac{1}{3}$

(i) Number of cars in the park is

$$L_s = \frac{\frac{\lambda}{\mu}}{\left[1-\frac{\lambda}{\mu}\right]} - \frac{(k+1)\left(\frac{\lambda}{\mu}\right)^{k+1}}{\left[1-\left(\frac{\lambda}{\mu}\right)^{k+1}\right]}$$

$$= \frac{\frac{1}{3}}{\left[1-\frac{1}{3}\right]} - \frac{6\left(\frac{1}{3}\right)^{6}}{1-\left(\frac{1}{3}\right)^{6}} = \frac{1}{2} - \frac{6\,(0.00137)}{1-0.00137}$$

$$= 0.5 - 0.0082 = 0.49$$

(ii) If the car park is full, then the newly arrived car has to go away.

$\therefore$ P (new car going away) $= P\,(N = 5)$

$$= P_5 = \left(\frac{\lambda}{\mu}\right)^5 \times P_0$$

$$= \left(\frac{\lambda}{\mu}\right)^5 \frac{\left(1-\frac{\lambda}{\mu}\right)}{1-\left(\frac{\lambda}{\mu}\right)^6}$$

$$= \left(\frac{1}{3}\right)^5 \frac{\left(1-\frac{1}{3}\right)}{1-\left(\frac{1}{3}\right)^6} = \frac{2\left(\frac{1}{3}\right)^6}{1-\left(\frac{1}{3}\right)^6}$$

$$= \frac{2 \times 0.00137}{1-0.00137} = \frac{0.00274}{0.99863} = 0.0027$$

Example 5 : Patients arrive at a clinic according to poisson distribution at a rate of 30 patients per hour. The waiting room cannot accomodate more than 14 patients. Examination time per patient is exponential at the rate of 20 per hour.

(i) Find the effective arrival rate at the clinic.

(ii) What is the probability that an arriving patient will not wait ?

(iii) What is the expected waiting time until a patient is discharged from the clinic ?

[AU 2007, 2009]

Solution : Given $\lambda = 30/\text{hr}, \quad \mu = 20/\text{hr} \; ; \; k = 14 + 1 = 15$

$$\frac{\lambda}{\mu} = \frac{30}{20} = 1.5$$

We know P_0 $$= \frac{1 - \frac{\lambda}{\mu}}{1 - \left(\frac{\lambda}{\mu}\right)^{k+1}} = \frac{1 - 1.5}{1 - (1.5)^{16}}$$

$$= \frac{-0.5}{1 - 656.93} = \frac{0.5}{655.93} = 0.00076$$

(i) Effective arrival rate λ' $= \mu (1 - P_0)$

$$= 20\,(1 - 0.00076) = 19.98$$

$$= 20 \text{ per hour, nearly}$$

(ii) P (a patient will not wait) $= P_0 = 0.00076$

(iii) Expected waiting time $W_s = \dfrac{L_s}{\lambda'}$

But L_s $$= \frac{\frac{\lambda}{\mu}}{1 - \frac{\lambda}{\mu}} - \frac{(k+1)\left(\frac{\lambda}{\mu}\right)^{k+1}}{1 - \left(\frac{\lambda}{\mu}\right)^{k+1}}$$

$$= \frac{1.5}{1 - 1.5} - \frac{16\,(1.5)^{16}}{1 - (1.5)^{16}}$$

$$= -3 - 16 \times \frac{656.93}{-(655.93)} = -3 + 16.02 = 13.02$$

$$W_s = \frac{13.02}{20} \text{ hrs} = \frac{13.02}{20} \times 60 \text{ min} = 39 \text{ minutes}$$

Example 6 : Consider a single server queueing system with a poisson input, exponential service times. Suppose the mean arrival rate is 3 calling units per hour, the expected service time is 0.25 hours, and the maximum permissible number calling units in the system is 2. Calculate (i) the expected number in the system (ii) the average waiting time in the queue.

Solution : Given $\lambda = 3$ per hour, $\frac{1}{\mu} = 0.25 \Rightarrow \mu = 4$ per hour

$$k = 2, \frac{\lambda}{\mu} = \frac{3}{4} = 0.75$$

$$P_0 = \frac{1 - \frac{\lambda}{\mu}}{1 - \left(\frac{\lambda}{\mu}\right)^{k+1}} = \frac{1 - 0.75}{1 - (0.75)^3}$$

$$= \frac{0.25}{1 - 0.42} = \frac{0.25}{0.58} = 0.43$$

(i) Expected number in the system is

$$L_s = \frac{\frac{\lambda}{\mu}}{1 - \frac{\lambda}{\mu}} - \frac{(k+1)\left(\frac{\lambda}{\mu}\right)^{k+1}}{1 - \left(\frac{\lambda}{\mu}\right)^{k+1}}$$

$$= \frac{0.75}{1 - 0.75} - \frac{3\,(0.75)^3}{1 - (0.75)^3}$$

$$= 3 - \frac{1.266}{0.578} = 3 - 2.19 = 0.81$$

(ii) Average waiting time in the queue is

$$W_q = \frac{L_q}{\lambda'} \quad \text{and} \quad \frac{\lambda'}{\mu} = (1 - P_0).$$

$$\text{But } L_q = L_s - \frac{\lambda'}{\mu} = L_s - (1 - P_0)$$

$$= 0.81 - (1 - 0.43)$$

$$= 0.81 - 0.57 = 0.24$$

$$\lambda' = 4\ (1 - 0.43) = 4\ (0.5) = 2.28$$

$$\therefore W_q = \frac{0.24}{2.28}$$

$$= 0.11 \text{ hr} = 0.11 \times 60 = 6.6 \text{ minutes.}$$

EXERCISE 4.3

1. If for a period of 2 hours in a day 9 AM to 11 AM, trains arrive at every 20 minutes but the service time is 36 minutes. Assuming poisson arrival and exponential service and capacity 4 trains only, calculate for this period, (ii) the probability the yard is empty, (ii) the average queue length. **[Ans. (i) 0.04, (ii) 3]**

2. In an office a stenographer gets works in a poisson rate of one in 15 minutes and the steno takes an average of 10 minutes to complete a work. Suppose the steno has the capacity of a maximum of 20 works in a day, find (i) the effective arrival rate, (ii) the fraction of jobs could not be undertaken, (iii) the percentage of time the steno is idle, (iv) the average length of the queue. **[Ans. (i) 4/hr, (ii) 0.01%, (iii) 33.3%, (iv) 1.3]**

3. A petrol station has a single pump and space for not more than 3 ears (2 waiting, being served). A car arriving to the station when it is full is turned away. Cars arrive at a poisson rate of one in every 8 minutes and service time is exponential with a mean of 4 minutes. The proprietor has the opportunity of renting an adjacent piece of land, which would provide space for another car to wait. The rent would be Rs. 10 per week. The expected net profit from each customer is Re. 0.50 and the station is open for 10 hours everyday. Would it be profitable to rent the additional space ?

[**Hint :** If $k = 3$, $P_3 = 0.067$ (i.e., the prob. of the system in maximum state. If $k = 4$, $P_4 = 0.032$; Increase in cars served per hour $= \lambda\,(0.067 - 0.032) = \frac{1}{8} \times 0.035 \times 60 = 0.262$ cars per hour.

$\therefore$ Increase per week $= 0.262 \times 10 \times 7 = 18.34$

Increase in profit per week $= 0.5 \times 18.34 =$ Rs. 9.17

But rent per week Rs. 10 > 9.17. So it is not economical to increase the existing space.]

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