Numerical method

In numerical analysis, a **numerical method** is a mathematical tool designed to solve numerical problems. The implementation of a numerical method with an appropriate convergence check in a programming language is called a numerical algorithm.

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Mathematical definition

Let $F(x,y)=0$ be a well-posed problem, i.e. $F:X\times Y\rightarrow\mathbb{R}$ is a real or complex functional relationship, defined on the cross-product of an input data set X and an output data set Y, such that exists a locally lipschitz function $g:X\rightarrow Y$ called resolvent, which has the property that for every root (x,y) of F, $y=g(x)$. We define **numerical method** for the approximation of $F(x,y)=0$, the sequence of problems

$$\{M_n\}_{n\in\mathbb{N}}=\{F_n(x_n,y_n)=0\}_{n\in\mathbb{N}},$$

with $F_n:X_n\times Y_n\rightarrow\mathbb{R}$, $x_n\in X_n$ and $y_n\in Y_n$ for every $n\in\mathbb{N}$. The problems of which the method consists need not be well-posed. If they are, the method is said to be *stable* or *well-posed*.[1]

Consistency

Necessary conditions for a numerical method to effectively approximate $F(x,y)=0$ are that $x_n\rightarrow x$ and that F_n behaves like F when $n\rightarrow\infty$. So, a numerical method is called *consistent* if and only if the sequence of functions $\{F_n\}_{n\in\mathbb{N}}$ pointwise converges to F on the set S of its solutions:

$$\lim F_n(x,y+t)=F(x,y,t)=0,\qquad\forall(x,y,t)\in S.$$

When $F_n=F,\forall n\in\mathbb{N}$ on S the method is said to be *strictly consistent*.[1]

Convergence

Denote by ℓ_n a sequence of *admissible perturbations* of $x\in X$ for some numerical method M (i.e. $x+\ell_n\in X_n\forall n\in\mathbb{N}$) and with $y_n(x+\ell_n)\in Y_n$ the value such that $F_n(x+\ell_n,y_n(x+\ell_n))=0$. A condition which the method has to satisfy to be a meaningful tool for solving the problem $F(x,y)=0$ is *convergence*:

$$\begin{aligned}&\forall\varepsilon>0,\exists n_0(\varepsilon)>0,\exists\delta_{\varepsilon,n_0}\text{ such that}\\&\forall n>n_0,\forall\ell_n:\|\ell_n\|<\delta_{\varepsilon,n_0}\Rightarrow\|y_n(x+\ell_n)-y\|\leq\varepsilon.\end{aligned}$$

One can easily prove that the point-wise convergence of $\{y_n\}_{n\in\mathbb{N}}$ to y implies the convergence of the associated method.[1]

References

1. Quarteroni, Sacco, Saleri (2000). *Numerical Mathematics* (http://www.techmat.vgtu.lt/~inga/Files/Quarteroni-SkaitMetod.pdf) (PDF). Milano: Springer. p. 33.