Property : (1)

Composition of functions is associative.

If $f : A \to B$, $g : B \to C$ and $h : C \to D$, then

$$h \circ (g \circ f) = (h \circ g) \circ f$$

Proof: Given $f : A \to B$ and $g : B \to C$ so that $f : A \to B$

Let $x \in A$, $y \in B$ and $z \in C$ implies $y = f(x)$; $g : B \to C$ implies $z = g(y)$

$\{h \circ (g \circ f)\} x = h \circ (g(f(x))) = h \circ (g(y)) = h\{g(y)\} = h(z)$

$\{(h \circ g) \circ f\} x = (h \circ g)\{f(x)\} = (h \circ g)(y) = h(g(y)) = h(z)$

$\therefore\ h \circ (g \circ f) = (h \circ g) \circ f$

Property (2):-

When $f : A \to B$ & $g : B \to C$ are fns, then $g \circ f : A \to C$ is an injection, surjection or bijection according as f and g are injections, surjections or bijections.

Proof: Let $a_1, a_2 \in A$. Then to prove $g \circ f$ is 1-1

(i) $(g \circ f) a_1 = (g \circ f) a_2$

$g(f(a_1)) = g(f(a_2))$ ($\because$ g is injective)

$f(a_1) = f(a_2)$ ($\because$ f is injective

$a_1 = a_2$

(ii) Let $c \in C$.

Since g is onto, there is an element $b \in B \ni c = g(b)$. Since f is onto, there is an element $a \in A \ni b = f(a)$

Now $(g \circ f)(a) = g(f(a)) = g(b) = c$

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$\Rightarrow$ for every elt $c \in C$, there is an elt $a \in A \ni$ $(g \circ f)\, a = c$ $\therefore$ $g \circ f : A \to C$ is onto

(iii) From (i) & (ii), $g \circ f$ is bijective when f and g are bijective.

Property (3):- The necessary and sufficient condition for the fun. $f : A \to B$ to be invertible is that f is 1-1 and onto

Proof :- case(i) If f is invertible, then to prove f is 1-1 and onto

To prove f is 1-1 :- Let $f : A \to B$ be invertible. Then there exists a unique fun. $g : B \to A$ such that

$$g \circ f = I_A, \; f \circ g = I_B \quad \text{--- (1)}$$

Now $f(a_1) = f(a_2)$

$g(f(a_1)) = g(f(a_2))$

$g \circ f(a_1) = g \circ f(a_2) \Rightarrow a_1 = a_2$ {using (1)}

$\therefore f$ is 1-1

To prove f is onto :- Since g is a fun. $g(b) \in A$ for $b \in B$. Now $b = I_B(b) = f \circ g(b) = f(g(b))$

$\therefore$ for every $b \in B$, there exist an element $g(b) \in$ such that $f(g(b)) = b$ $\therefore f$ is onto.

Case(ii) :- If f is 1-1 & onto, then to prove f is invertible.

For each $b \in B$, $\exists\ a \in A \ni f(a) = b$
Hence we define $g: B \to A$ by $g(b) = a$ — (2)
where $f(a) = b$
If possible, let $g(b) = a_1$ & $g(b) = a_2$ where $a_1 \neq a_2$. $\Rightarrow f(a_1) = b$, $f(a_2) = b$
$\Rightarrow f(a_1) = f(a_2)$ where $a_1 \neq a_2$
$\Rightarrow f$ is not 1-1, a contradiction $\therefore g$ is a unique fn.
Hence from (2) $g \circ f = I_A$, $f \circ g = I_B \Rightarrow f$ is invertible

Property:-
The inverse of a fn f, if exists, is unique.
proof:- Let h and g be inverses of f.
(i.e) If $f: A \to B$ then $g: B \to A$ and also $h: B \to A$
By definition $g \circ f = I_A$, $h \circ f = I_A$
$f \circ g = I_B$ and $f \circ h = I_B$

Now $h = h \circ I_B = h \circ (f \circ g) = (h \circ f) \circ g = I_A \circ g = g$
$\therefore h = g$

property:- If $f: A \to B$, $g: B \to C$ are invertible (inverse exists) fns, then $g \circ f: A \to C$ is also invertible and $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$
proof:- Given f and g are 1-1 and onto
$\Rightarrow g \circ f$ is also bijective $\Rightarrow g \circ f$ is invertible

Since $f: A \to B$ & $g: B \to C$

we have $f^{-1}: B \to A$ and $g^{-1}: C \to B$

For any $a \in A$, let $b = f(a)$ and $c = g(b)$

$\Rightarrow f^{-1}(b) = a$ and $g^{-1}(c) = b$

$(g \circ f)(a) = g\{f(a)\} = g\{b\} = c \Rightarrow a = (g \circ f)^{-1}(c)$ — ①

and $(f^{-1} \circ g^{-1})(c) = f^{-1}[g^{-1}(c)] = f^{-1}(b) = a$

$\Rightarrow a = (f^{-1} \circ g^{-1})(c)$ — ② so ① & ② $\Rightarrow \boxed{(g \circ f)^{-1} = f^{-1} \circ g^{-1}}$