# Computing Integral Manually

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**1. Question:** If $\int \frac{2\sin x + 3\cos x}{\sin x + 4\cos x}\,dx = ax + b\log|\sin x + 4\cos x| + C$, then calculate the value of $a+b$.

**Answer:**

$$\text{Let,}\quad I = \int \frac{2\sin x + 3\cos x}{\sin x + 4\cos x}\,dx \tag{1}$$

Since the numerator and denominator consists of sin and cos terms, the derivative of denominator of the integration must be present in the numerator. Hence considering the numerator in the form of,

$$2\sin x + 3\cos x = A(\sin x + 4\cos x) + B\frac{d}{dx}(\sin x + 4\cos x) \tag{2}$$

$$2\sin x + 3\cos x = A\sin x + 4A\cos x + B(\cos x - 4\sin x)$$

$$2\sin x + 3\cos x = A\sin x + 4A\cos x + B\cos x - 4B\sin x$$

Combining the identical terms,

$$2\sin x + 3\cos x = (A - 4B)\sin x + (4A + B)\cos x \tag{3}$$

Comparing the coefficients of $\sin x$ and $\cos x$ from L.H.S and R.H.S in (3),

$$A - 4B = 2 \tag{4}$$

$$4A + B = 3 \tag{5}$$

Solving (4) and (5) simultaneously,

$$\boxed{A = \frac{14}{17} \quad \text{and} \quad B = -\frac{5}{17}} \tag{6}$$

Substituting the values from (6) in equation (2),

$$2\sin x + 3\cos x = \frac{14}{17}(\sin x + 4\cos x) + \left(-\frac{5}{17}\right)\frac{d}{dx}(\sin x + 4\cos x) \tag{7}$$

Plugging (7) in (2),

$$I = \int \frac{\frac{14}{17}(\sin x + 4\cos x) + \left(-\frac{5}{17}\right)\frac{d}{dx}(\sin x + 4\cos x)}{\sin x + 4\cos x}\,dx \tag{8}$$

$$I = \int \frac{14}{17}\,dx - \int \frac{5}{17}\frac{\frac{d}{dx}(\sin x + 4\cos x)}{\sin x + 4\cos x}\,dx$$

$$I = \int \frac{14}{17}\,dx - \int \frac{5}{17}\frac{(\cos x - 4\sin x)}{\sin x + 4\cos x}\,dx \qquad \left\{\because \frac{d}{dx}\sin x = \cos x \quad \text{and} \quad \frac{d}{dx}\cos x = -\sin x\right\}$$

$$I = \frac{14}{17}\int 1\,dx - \frac{5}{17}\int \frac{1}{t}\,dt$$

$$\text{where} \quad t = \sin x + 4\cos x$$

$$\text{differentiating w.r.t } x \implies \frac{dt}{dx} = \cos x - 4\sin x$$

$$\therefore\ dt = (\cos x - 4\sin x)\,dx$$

$$I = \frac{14}{17}x - \frac{5}{17}\log\mid t\mid +C \qquad \{\because \int \frac{1}{x}\,dx = \log\mid x\mid +c\}$$

Re-substituting the value of $t$ we get,

$$\therefore I = \frac{14}{17}x - \frac{5}{17}\log\mid \sin x + 4\cos x\mid +C \tag{9}$$

Comparing (9) with the main question we get,

$$a = \frac{14}{17} \quad \text{and} \quad b = -\frac{5}{17}$$

$$\therefore \boxed{a + b = \frac{9}{17}} \tag{10}$$