AGENDA

- Derivatives
- Chain rule
- Gradient
- Directional Derivatives
- Integrals

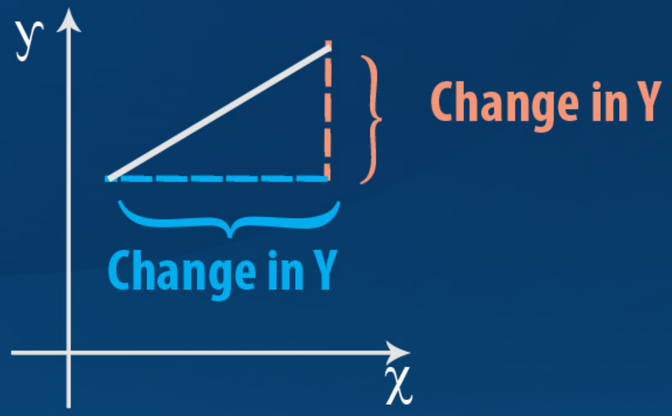
DERIVATIVES

Definition

- Instantaneous rate of change - From Physics
- Slope of a line at a point - From Geometry

Geometry view

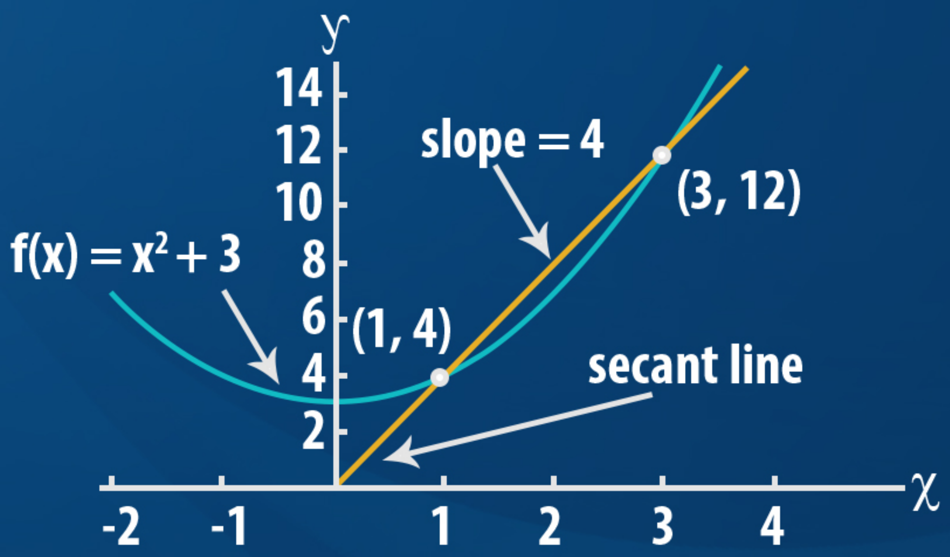
$$\text{Slope} = \frac{\text{Change in Y}}{\text{Change in X}}$$



DERIVATIVES

Example

Consider the graph below, where $f(x) = x^2 + 3$



The slope between (1,4) and (3,12) would be:

$$slope = \frac{y^2 - y^1}{x^2 - x^1} = \frac{12 - 4}{3 - 1} = 4$$

What would be the slope at a specified point?

Hint: We would use derivatives!

TAKING THE DERIVATIVE

Logic

Calculating the derivative is the same as calculating normal slope, however in this case, we calculate the slope between our point and a point infinitesimally close to it.

Step 1 **Define function** $f(x) = x^2$

Step 2 **Increment** x **by an infinitesimally small value** $h, \ (h = \Delta x)$

Step 3 **Compute slope** $\dfrac{f(x+h) - f(x)}{h} = 2x + h$

Step 3 **Apply limit on h** $\lim\limits_{0} h(2x + h) = 2x$

Thus, we have the slope formula,

$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

CHAIN RULE

The chain rule is a formula for calculating the derivatives of composite functions. Composite functions are functions composed of functions inside another function(s).

Generic rule

Composite function derivative = outer function derivative * inner function derivative

Consider the following case,

$$f(x) = A(B(C(x)))$$

Where say: $A(x) = sin(x)$

$$B(x) = x^2$$

$$C(x) = 4x$$

Chain rule applied here, would mean,

$$\frac{df}{dx} = \frac{dA}{dB}\frac{dB}{dC}\frac{dC}{dx}$$

Individual

$$A'(x) = cos(x)$$

$$B'(x) = 2x$$

$$C'(x) = 4$$

The derivatives then maybe computed as,

$$f'(x) = A'((4x)^2) \cdot B'(4x) \cdot C'(x)$$

Putting them all together !

$$\begin{aligned} f'(x) &= cos((4x)^2) \cdot 2(4x) \cdot 4 \\ &= cos(16x^2) \cdot 8x \cdot 4 \\ &= cos(16x^2)32x \end{aligned}$$

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Gradient

A gradient is a vector that stores the partial derivatives of multivariable functions. It helps us calculate the slope at a specific point on a curve for functions with multiple independent variables.

PARTIAL DERIVATIVES

Consider the multi-variable function,

$$f(x, z) = 2z^3x^2$$

How do we calculate partial derivative say wrt *x* ?

① **Swap** $2z^3$ **with a constant 'b'**

$$f(x, z) = bx^2$$

② **Calculate the derivative with 'b' constant**

$$\frac{df}{dx} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{b(x+h)^2 - b(x^2)}{h}$$

$$= \lim_{h \to 0} \frac{b((x+h)(x+h)) - bx^2}{h}$$

$$= \lim_{h \to 0} \frac{b((x^2 + xh + hx + h^2)) - bx^2}{h}$$

$$= \lim_{h \to 0} \frac{bx^2 + 2bxh + bh^2 - bx^2}{h}$$

$$= \lim_{h \to 0} \frac{2bxh + bh^2}{h}$$

$$= \lim_{h \to 0} \frac{2bxh + bh^2}{h}$$

$$= \lim_{h \to 0} 2bx + bh$$

As h—> 0...

2bx + 0

Plug back for 'b'

$$\frac{df}{dx}(x, z) = 2(2z^3)x$$
$$= 4z^3x$$

Similarly the derivative for *z* would give keeping *x* constant

$$\frac{df}{dz}(x, z) = 6x^2z^2$$

GRADIENT AS A VECTOR AND DIRECTIONAL DERIVATIVES

Storing the partial derivatives in a gradient,

$$\nabla f(\chi, Z) = \begin{bmatrix} \dfrac{\partial f}{\partial \chi} \\ \dfrac{\partial f}{\partial Z} \end{bmatrix} = \begin{bmatrix} 4Z^3\chi \\ 6\chi^2 Z^2 \end{bmatrix}$$

In cartesian system, gradient is given as in 3 Dimensions,

$$\nabla f = \frac{\partial f}{\partial \chi}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} + \frac{\partial f}{\partial z}\mathbf{k},$$

GRADIENT AS A VECTOR AND DIRECTIONAL DERIVATIVES

The directional derivative is computed by taking the dot product of the gradient of f and a unit vector $\vec{v}$ representing the direction. The unit vector describes the proportions we want to move in each direction.

Say we want to compute the directional derivative of $f(x,y,z)$ along the following vector

$$\vec{v} = \begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix}$$

We have the dot product thus,

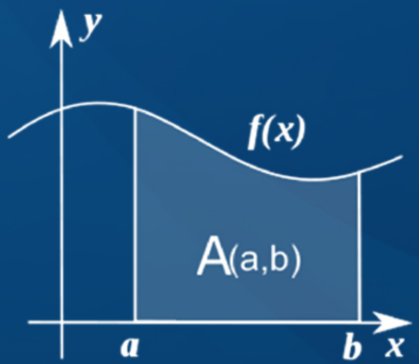
$$\begin{bmatrix} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \\ \frac{\partial f}{\partial z} \end{bmatrix} \cdot \begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix} \quad \nabla_v f = 2\frac{\partial f}{\partial x} + 3\frac{\partial f}{\partial y} - 1\frac{\partial f}{\partial z}$$

INTEGRALS

The integral of $f(x)$ corresponds to the computation of the area under the graph of $f(x)$. The area under $f(x)$ between the points $x = a$ and $x = b$ is denoted as follows:

Area under curve

$$A(a,b) = \int_a^b f(x)\,dx.$$

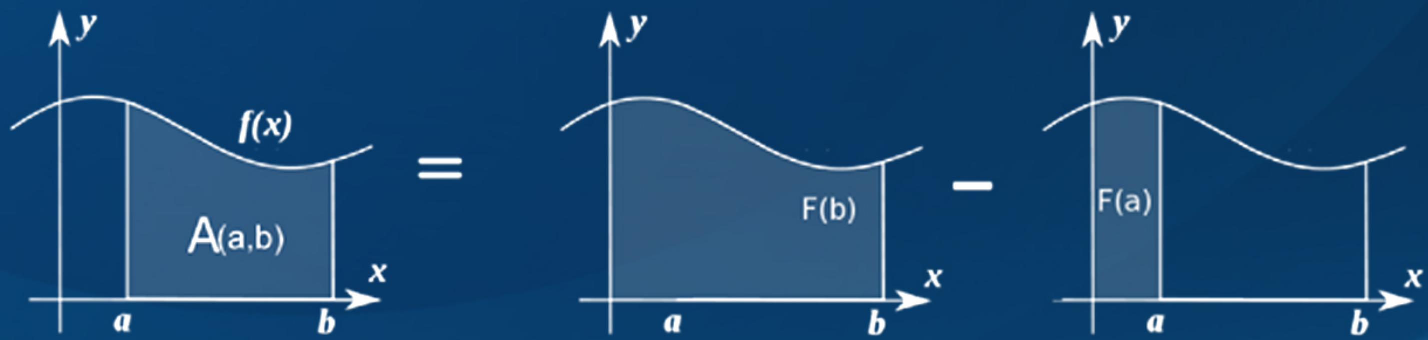


Integral function

$$F(c) \equiv \int_0^c f(x)\,dx$$

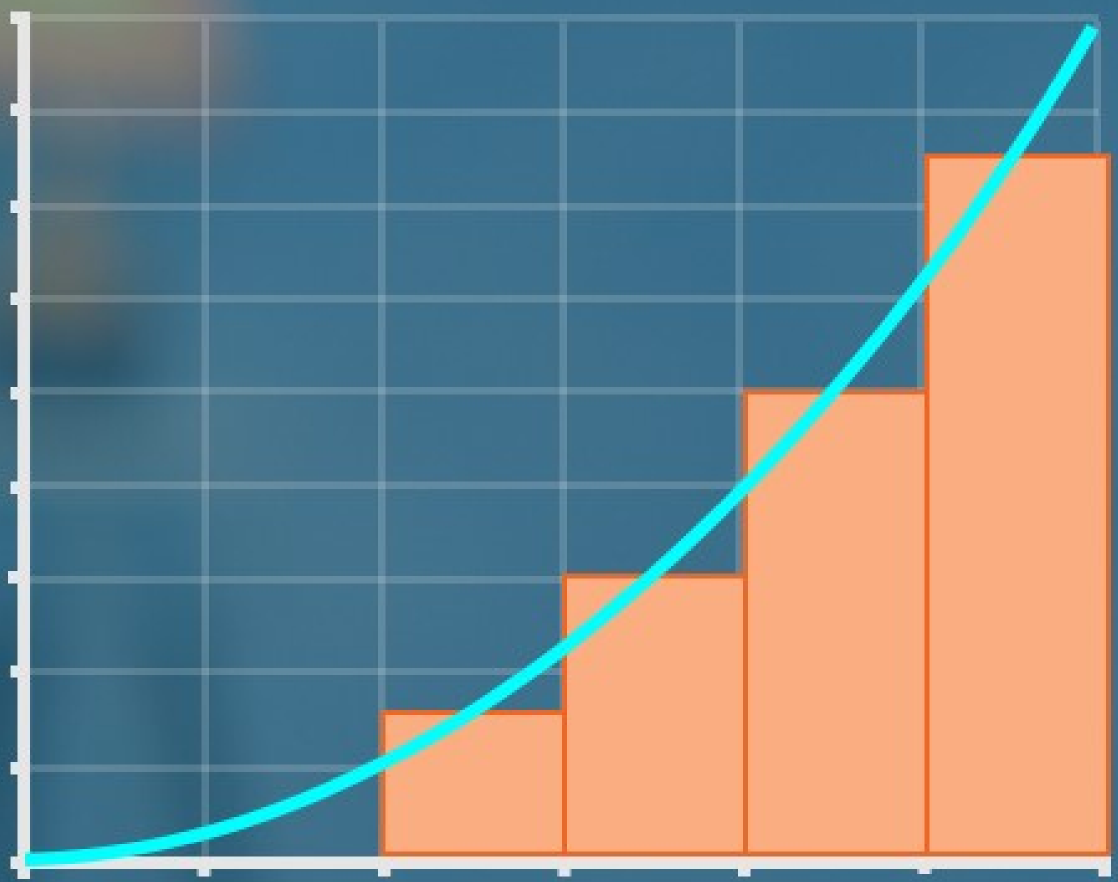
Area under curve as change in integral function

$$A(a,b) = \int_a^b f(x)\,dx = F(b) - F(a).$$



COMPUTING INTEGRALS

Approximate Area under curve



We can approximate the total area under the function f(x) between x=a and x=b by splitting the region into tiny vertical strips of width h, then adding up the areas of the rectangular strips.

COMPUTING INTEGRALS

Finding integral functions

$$F(\chi) = \int f(\chi)\, \mathrm{d}\chi$$

Equivalent problem: Find a function whose derivative is the function we are trying to integrate!

$$F'(\chi) = f(\chi)$$

COMPUTING INTEGRALS

Example

$\int \chi^2$ We note its quadratic in nature: thus a cubic function is what we are looking for !

$$F(\chi) = c\chi^3 \Rightarrow F'(\chi) = 3c\chi^3 = \chi^2$$

Solving for c ; we finally have:

$$F(\chi) = \int \chi^2 \, d\chi = \frac{1}{3}\chi^3 + C.$$

$$\text{Slope} = \frac{\text{Change in Y}}{\text{Change in X}}$$