AGENDA

- Introduction to Linear Algebra
- Numpy Basics
- Vector Arithmetic
- Matrix Arithmetic

INTRODUCTION TO LINEAR ALGEBRA

A layman's definition:

- Linear algebra is the mathematics of data
- Matrices and vectors are the language of data
- Linear algebra is about linear combinations
- That is, using arithmetic on columns of numbers called vectors and arrays of numbers called matrices, to create new columns and arrays of numbers.

A linear equation

$$y = 3x + b$$

A system of linear equations

$$y = 5x_1 + 3x_2 + 7$$
$$y = 3x_1 + 8x_2 + 5$$
$$y = 4x_1 + 7x_2 + 9$$

Linear Algebra notation

$$y = A.b$$

Where,
A = Matrix
b = Vector

In this case:
3 Equations and 2 Unknowns
Usually, we may have the other way around!

LINEAR ALGEBRA FOR ML

3 Reasons not to learn it!

It's not required

- Having an appreciation for the abstract operations that underlie some machine learning algorithms is not required to use ML.

It's slow

- Taking months to years to study an entire related field before machine learning will delay your work goals.

It's a huge field.

- Not all of linear algebra is relevant to theoretical machine learning, let alone applied machine learning.

Why at all dig so deep like in the math below ?

Least Square Estimator

We have error as: $\hat{y} = x\beta \quad , \quad \hat{e}^2 = (Y - \hat{y})^2$

$$= [Y - x\beta]'[Y - x\beta]$$

Thus, $\hat{e}^2 = [Y'Y + (x\beta)'(x\beta) - (x\beta)'Y - Y'(x\beta)]$

$$\hat{e}^2 = [Y'Y + \beta'x'x\beta - \beta'x'Y - Y'x\beta]$$

(last 2 terms have same scalar product)

Now for minimising squared error:

$$\frac{\partial \hat{e}^2}{\partial \beta} = 0 \quad \Rightarrow \quad 2x'x\beta - (x'Y + Y'x) = 0$$

Now using $x'Y = Y'x \quad \Rightarrow \quad x'x\beta = x'Y$

$$\Rightarrow \beta = (x'x)^{-1}(x'Y)$$

Soln for $[\hat{\beta} = (x'x)^{-1}(x'Y)]$

PPT - Linear_Algebra Slide 6 formulas

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REASONS TO KNOW LINEAR ALGEBRA

Data and files

The famous 'iris' data set 

5.1,3.5,1.4,0.2,Iris-setosa
4.9,3.0,1.4,0.2,Iris-setosa
4.7,3.2,1.3,0.2,Iris-setosa
4.6,3.1,1.5,0.2,Iris-setosa
5.0,3.6,1.4,0.2,Iris-setosa

$$Y = [X]$$

Linear algebra helps convert this to the language of vectors and matrices; so ML models can be applied!

One hot encoding

Categorical variable column

Color
Red
Blue
Green

Red	Blue	Green
1	0	0
0	1	0
0	0	1

A sparse representation of data : a sub-field in Linear Algebra by itself

Principal Component Analysis

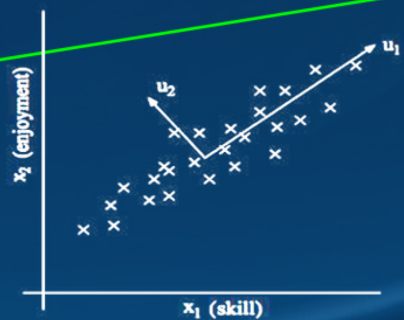
Principal Component Analysis/Multidimensional Scaling

$$x^8 - 12x^7 + 60x^6 - 160x^5 + 240x^4 - 192x^3 + 64x^2$$

$$\underbrace{\begin{pmatrix} x_{11} & x_{12} & \cdots & x_{1n} \\ x_{21} & x_{22} & \cdots & \\ \vdots & \vdots & \ddots & \\ x_{m1} & & & x_{mn} \end{pmatrix}}_{m \times n}^{\hat{X}} \approx \underbrace{\begin{pmatrix} u_{11} & \cdots & u_{1r} \\ \vdots & \ddots & \\ u_{m1} & & u_{mr} \end{pmatrix}}_{m \times r}^{U} \underbrace{\begin{pmatrix} s_{11} & 0 & \cdots \\ 0 & \ddots & \\ \vdots & & s_{rr} \end{pmatrix}}_{r \times r}^{S} \underbrace{\begin{pmatrix} v_{11} & \cdots & v_{1n} \\ \vdots & \ddots & \\ v_{r1} & & v_{rn} \end{pmatrix}}_{r \times n}^{V^T}$$

Singular Value Decomposition of A:

$$A = U\Sigma V^T$$



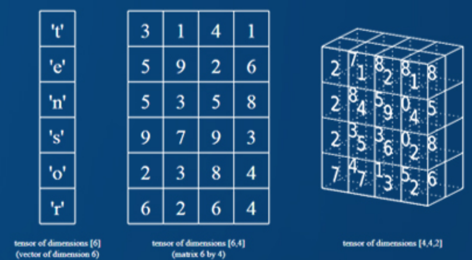
Singular Value Decomposition (SVD) sits at the heart of PCA!

ML APPLICATIONS OF LINEAR ALGEBRA

Deep Learning

- Vector: One-dimension
- Matrix: Two-dimensions
- Tensor: n-dimensions

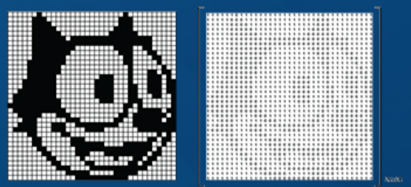
What is a tensor?



- Tensor Flow stores data in ensors

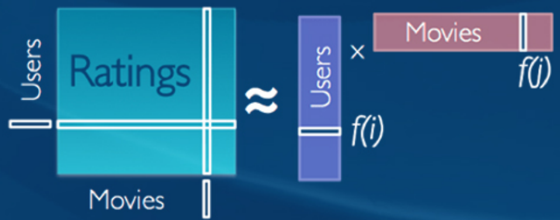
Linear Algebra

Image to Matrix



Recommender systems

Low-Rank Matrix Factorization:



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VECTOR MATH

Vectors

- Vectors are built from components, which are ordinary numbers.
- You can think of a vector as a list of numbers, and vector algebra as operations performed on the numbers in the list.

Example:

V is a vector of 3 scalar quantities v_1, v_2, v_3

Vector

$$v = (v_1; v_2; v_3)$$

Scalar

$$v = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}$$

Vector Addition

$$c = a + b$$

$$c = (a_1 + b_1, a_2 + b_2, a_3 + b_3)$$

$$c[0] = a[0] + b[0]$$
$$c[1] = a[1] + b[1]$$
$$c[2] = a[2] + b[2]$$

Vector Subtraction

$$c = a - b$$

$$c = (a_1 - b_1, a_2 - b_2, a_3 - b_3)$$

$$c[0] = a[0] - b[0]$$
$$c[1] = a[1] - b[1]$$
$$c[2] = a[2] - b[2]$$

VECTOR MATH (contd)

$$a = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \quad b = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$$

Vector Multiplication

$$c = a \times b$$
$$c = (a_1 \times b_1, a_2 \times b_2, a_3 \times b_3)$$
$$c = (a_1 b_1, a_2 b_2, a_3 b_3)$$
$$c[0] = a[0] \times b[0]$$
$$c[1] = a[1] \times b[1]$$
$$c[2] = a[2] \times b[2]$$

Vector Division

$$c = \frac{a}{b}$$
$$c = (\frac{a_1}{b_1}, \frac{a_2}{b_2}, \frac{a_3}{b_3})$$
$$c[0] = a[0]/b[0]$$
$$c[1] = a[1]/b[1]$$
$$c[2] = a[2]/b[2]$$

Vector Dot product

$$c = a \cdot b$$
$$c = (a_1 \times b_1 + a_2 \times b_2 + a_3 \times b_3)$$
$$c = (a_1 b_1 + a_2 b_2 + a_3 b_3)$$

Scalar Vector product

$$c = s \times v$$
$$c = sv$$

Where,
s = scalar
v= vector

$$c = (s \times v_1, s \times v_2, s \times v_3)$$
$$c[0] = v[0] \times s$$
$$c[1] = v[1] \times s$$
$$c[2] = v[2] \times s$$

VECTOR NORMS

Vector norm:

- The length of a vector is a non-negative number that describes the extent of the vector in space, and is sometimes referred to as the vector's magnitude or the norm.
- Two types: L^1 and L^2

L^1 norm:

- The length of a vector can be calculated using the L^1 norm. The notation for the L^1 norm of a vector is $||v||_1$.
- This length is sometimes called the taxicab norm or the Manhattan norm.
- The L^1 norm is calculated as the sum of the absolute vector values, where the absolute value of a scalar uses the notation *|a|*.
- In effect, the norm is a calculation of the Manhattan distance from the origin of the vector space.

L^1 norm:

$$L^1(v) = ||v||_1$$

$$||v||_1 = |a_1| + |a_2| + |a_3|$$

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VECTOR NORMS

L^2 norm:

- The length of a vector can be calculated using the L^2 norm. The notation for the L^2 norm of a vector is $||v||_2$.
- The L^2 norm calculates the distance of the vector coordinate from the origin of the vector space. As such, it is also known as the Euclidean norm as it is calculated as the Euclidean distance from the origin. The result is a positive distance value.
- The L2 norm is calculated as the square root of the sum of the squared vector values.

$$L^2(v) = ||v||_2$$

$$||v||_2 = \sqrt{a_1^2 + a_2^2 + a_3^2}$$

MATRICES

Addition

$$C = A + B$$

$$C = \begin{pmatrix} a_{1,1} + b_{1,1} & a_{1,2} + b_{1,2} \\ a_{2,1} + b_{2,1} & a_{2,2} + b_{2,2} \\ a_{3,1} + b_{3,1} & a_{3,2} + b_{3,2} \end{pmatrix}$$

$$C[0,0] = A[0,0] + B[0,0]$$
$$C[1,0] = A[1,0] + B[1,0]$$
$$C[2,0] = A[2,0] + B[2,0]$$
$$C[0,1] = A[0,1] + B[0,1]$$
$$C[1,1] = A[1,1] + B[1,1]$$
$$C[2,1] = A[2,1] + B[2,1]$$

Subtraction

$$C = A - B$$

$$C = \begin{pmatrix} a_{1,1} - b_{1,1} & a_{1,2} - b_{1,2} \\ a_{2,1} - b_{2,1} & a_{2,2} - b_{2,2} \\ a_{3,1} - b_{3,1} & a_{3,2} - b_{3,2} \end{pmatrix}$$

$$C[0,0] = A[0,0] - B[0,0]$$
$$C[1,0] = A[1,0] - B[1,0]$$
$$C[2,0] = A[2,0] - B[2,0]$$
$$C[0,1] = A[0,1] - B[0,1]$$
$$C[1,1] = A[1,1] - B[1,1]$$
$$C[2,1] = A[2,1] - B[2,1]$$

Hadamard product

$$C = A \circ B$$

$$C = \begin{pmatrix} a_{1,1} \times b_{1,1} & a_{1,2} \times b_{1,2} \\ a_{2,1} \times b_{2,1} & a_{2,2} \times b_{2,2} \\ a_{3,1} \times b_{3,1} & a_{3,2} \times b_{3,2} \end{pmatrix}$$

$$C[0,0] = A[0,0] \times B[0,0]$$
$$C[1,0] = A[1,0] \times B[1,0]$$
$$C[2,0] = A[2,0] \times B[2,0]$$
$$C[0,1] = A[0,1] \times B[0,1]$$
$$C[1,1] = A[1,1] \times B[1,1]$$
$$C[2,1] = A[2,1] \times B[2,1]$$

Division

$$C = \frac{A}{B}$$

$$C = \begin{pmatrix} \frac{a_{1,1}}{b_{1,1}} & \frac{a_{1,2}}{b_{1,2}} \\ \frac{a_{2,1}}{b_{2,1}} & \frac{a_{2,2}}{b_{2,2}} \\ \frac{a_{3,1}}{b_{3,1}} & \frac{a_{3,2}}{b_{3,2}} \end{pmatrix}$$

$$C[0,0] = A[0,0]/B[0,0]$$
$$C[1,0] = A[1,0]/B[1,0]$$
$$C[2,0] = A[2,0]/B[2,0]$$
$$C[0,1] = A[0,1]/B[0,1]$$
$$C[1,1] = A[1,1]/B[1,1]$$
$$C[2,1] = A[2,1]/B[2,1]$$

MATRIX MATH

(Dot product)
Matrix Matrix Multiplication

$$C = A \cdot B$$
$$C(m,k) = A(m,n) \cdot B(n,k)$$

$$A = \begin{pmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \\ a_{3,1} & a_{3,2} \end{pmatrix}$$

$$B = \begin{pmatrix} b_{1,1} & b_{1,2} \\ b_{2,1} & b_{2,2} \end{pmatrix}$$

$$C = \begin{pmatrix} a_{1,1} \times b_{1,1} + a_{1,2} \times b_{2,1}, a_{1,1} \times b_{1,2} + a_{1,2} \times b_{2,2} \\ a_{2,1} \times b_{1,1} + a_{2,2} \times b_{2,1}, a_{2,1} \times b_{1,2} + a_{2,2} \times b_{2,2} \\ a_{3,1} \times b_{1,1} + a_{3,2} \times b_{2,1}, a_{3,1} \times b_{1,2} + a_{3,2} \times b_{2,2} \end{pmatrix}$$

$$C[0,0] = A[0,0] \times B[0,0] + A[0,1] \times B[1,0]$$
$$C[1,0] = A[1,0] \times B[0,0] + A[1,1] \times B[1,0]$$
$$C[2,0] = A[2,0] \times B[0,0] + A[2,1] \times B[1,0]$$
$$C[0,1] = A[0,0] \times B[0,1] + A[0,1] \times B[1,1]$$
$$C[1,1] = A[1,0] \times B[0,1] + A[1,1] \times B[1,1]$$
$$C[2,1] = A[2,0] \times B[0,1] + A[2,1] \times B[1,1]$$

Matrix Vector Multiplication

$$\boldsymbol{c} = \boldsymbol{A} \cdot \boldsymbol{v}$$

$$A = \begin{pmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \\ a_{3,1} & a_{3,2} \end{pmatrix}$$

$$v = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$$

$$c = \begin{pmatrix} a_{1,1} \times v_1 + a_{1,2} \times v_2 \\ a_{2,1} \times v_1 + a_{2,2} \times v_2 \\ a_{3,1} \times v_1 + a_{3,2} \times v_2 \end{pmatrix}$$

$$c[0] = A[0,0] \times v[0] + A[0,1] \times v[1]$$
$$c[1] = A[1,0] \times v[0] + A[1,1] \times v[1]$$
$$c[2] = A[2,0] \times v[0] + A[2,1] \times v[1]$$

MATRIX MATH (contd)

Matrix Scalar Multiplication

$C = A \cdot b$

$$A = \begin{pmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \\ a_{3,1} & a_{3,2} \end{pmatrix}$$

$$C = \begin{pmatrix} a_{1,1} \times b + a_{1,2} \times b \\ a_{2,1} \times b + a_{2,2} \times b \\ a_{3,1} \times b + a_{3,2} \times b \end{pmatrix}$$

$$C = \begin{pmatrix} a_{1,1}b + a_{1,2}b \\ a_{2,1}b + a_{2,2}b \\ a_{3,1}b + a_{3,2}b \end{pmatrix}$$

$$C[0,0] = A[0,0] \times b$$
$$C[1,0] = A[1,0] \times b$$
$$C[2,0] = A[2,0] \times b$$
$$C[0,1] = A[0,1] \times b$$
$$C[1,1] = A[1,1] \times b$$
$$C[2,1] = A[2,1] \times b$$

TYPES OF MATRICES I

Square Matrix

A square matrix is a matrix where the number of rows (n) is equivalent to the number of columns (m).

$$M = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}$$

Symmetric Matrix

A symmetric matrix is a type of square matrix where the top-right triangle is the same as the bottom-left triangle.

$$M = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 1 & 2 & 3 & 4 \\ 3 & 2 & 1 & 2 & 3 \\ 4 & 3 & 2 & 1 & 2 \\ 5 & 4 & 3 & 2 & 1 \end{pmatrix}$$

TYPES OF MATRICES II

Triangular Matrix

A triangular matrix is a type of square matrix that has all values in the upper-right or lower-left of the matrix with the remaining elements filled with zero values

Upper Triangular Matrix

$$M = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 2 & 3 \\ 0 & 0 & 3 \end{pmatrix}$$

Lower Triangular Matrix

$$M = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 2 & 0 \\ 1 & 2 & 3 \end{pmatrix}$$

TYPES OF MATRICES III

Diagonal Matrix

A diagonal matrix is one where values outside of the main diagonal have a zero value, where the main diagonal is taken from the top left of the matrix to the bottom right.

$$D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix} \rightarrow d = \begin{pmatrix} d_{1,1} \\ d_{2,2} \\ d_{3,3} \end{pmatrix} \rightarrow d = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$$

Non Square: Diagonal Matrix

$$D = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 4 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

- A diagonal matrix does not have t be square
- In the case of a rectangular matri the diagonal would cover the dimension with the smallest

TYPES OF MATRICES IV

Identity Matrix

An identity matrix is a matrix that does not change any vector when we multiply that vector by that matrix.

$$I = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Orthogonal Matrix

Two vectors are orthogonal when their dot product equals zero.

If the length of each vector is 1 then the vectors are called orthonormal because they are both orthogonal and normalized.

$$v \cdot w = 0 \quad \textbf{OR} \quad v \cdot w^T = 0$$

Formal definition

$$Q^T \cdot Q = Q \cdot Q^T = I$$

Example

$$Q = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

MATRIX OPERATIONS

Transpose

A defined matrix can be transposed, which creates a new matrix with the number of columns and rows flipped.

$$A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{pmatrix} \qquad A^T = \begin{pmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{pmatrix}$$

Inverse

Matrix inversion is a process that finds another matrix that when multiplied with the matrix, results in an identity matrix. Given a matrix A, and matrix B, such that $AB = I_n$ or $BA = I_n$

We have $AB = BA = I_n$ where, $B = A^{-1}$

INVERSE OF A MATRIX (contd)

How do we find inverse ?

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad-bc} \begin{bmatrix} a & -b \\ -c & d \end{bmatrix}$$

determinant

A simple 2x2 square matrix example

$$\begin{bmatrix} 4 & 7 \\ 2 & 6 \end{bmatrix}^{-1} = \frac{1}{4\times6-7\times2} \begin{bmatrix} 6 & -7 \\ -2 & 4 \end{bmatrix}$$

$$= \frac{1}{10} \begin{bmatrix} 6 & -7 \\ -2 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 0.6 & -0.7 \\ -0.2 & 0.4 \end{bmatrix}$$

INVERSE OF A MATRIX (contd)

Reality is not square always!

A Real Life Example: Bus and Train

A group took a trip on a **bus,** at \$3 per child, and \$3.20 per adult for a total of \$118.40.

They took the **train** back at \$3.50 per child, and \$3.60 per adult for a total of \$135.20.

How many children, and how many adults?

First, let us set up the matrices
(be careful to get rows and columns correct!):

$$\begin{array}{cc} \text{Child} & \text{Adult} \\ \end{array} \quad \begin{array}{cc} \text{Bus} & \text{Train} \\ \end{array} \quad \begin{array}{cc} \text{Bus} & \text{Train} \\ \end{array}$$

$$\begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} 3 & 3.5 \\ 3.2 & 3.6 \end{bmatrix} = \begin{bmatrix} 118.4 & 135.2 \end{bmatrix}$$

INVERSE OF A MATRIX (contd)

The problem as it is: $XA = B$

So, to solve it we need the inverse of A:

$$\begin{bmatrix} 3 & 3.5 \\ 3.2 & 3.6 \end{bmatrix}^{-1} = \frac{1}{3 \times 3.6 - 3.5 \times 3.2} \begin{bmatrix} 3.6 & -3.5 \\ -3.2 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} -9 & 8.75 \\ 8 & -7.5 \end{bmatrix}$$

Now, we have the inverse we can solve using: $X = BA^{-1}$

$$\begin{bmatrix} x_1 & x_2 \end{bmatrix} = \begin{bmatrix} 118.4 & 135.2 \end{bmatrix} \begin{bmatrix} -9 & 8.75 \\ 8 & -7.5 \end{bmatrix}$$

$$= \begin{bmatrix} 118.4 \times -9 + 135.2 \times 8 & 118.4 \times 8.75 + 135.2 \times -7.5 \end{bmatrix}$$

$$= \begin{bmatrix} 16 & 22 \end{bmatrix}$$

This is what it looks like as: $AX=B$

$$\begin{bmatrix} 3 & 3.2 \\ 3.5 & 3.6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 118.4 \\ 135.2 \end{bmatrix}$$

It looks so neat! I think I prefer it like this.

Also, note how the rows and columns are swapped over ("Transposed") compared to the previous example.

To solve it, we need the inverse of A:

$$\begin{bmatrix} 3 & 3.2 \\ 3.5 & 3.6 \end{bmatrix}^{-1} = \frac{1}{3 \times 3.6 - 3.2 \times 3.5} \begin{bmatrix} 3.6 & -3.2 \\ -3.5 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} -9 & 8 \\ 8.75 & -7.5 \end{bmatrix}$$

It is like the inverse, we got before, but Transposed (rows and columns swapped over).

Now, we can solve using: $X = A^{-1}B$

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ANALYTICAL STEPS TO COMPUTE INVERSE

Step 1: Calculating the Matrix of Minors

Step 2: Turn Matrix of Minors into the Matrix of Cofactors

Step 3: Adjugate

Step 4: Multiply by 1/Determinant.

Example: Find Inverse of A:

$$A = \begin{bmatrix} 3 & 0 & 2 \\ 2 & 0 & -2 \\ 0 & 1 & 1 \end{bmatrix}$$

It needs 4 step. It is all simple arithmetic, but there is lot of it, so try not to make a mistake!

ANALYTICAL STEPS TO COMPUTE INVERSE

Step1: Calculating the Matrix of Minors

For each element of the matrix:

- Ignore the values on the current row and column
- Calculate the determinant of the remaining values

$$A = \begin{bmatrix} 3 & 0 & 2 \\ 2 & 0 & -2 \\ 0 & 1 & 1 \end{bmatrix}$$

$$\begin{bmatrix} \bullet & 0 & 2 \\ 2 & 0 & -2 \\ 0 & 1 & 1 \end{bmatrix} \quad 0\times1 - (-2)\times1 = 2$$

$$\begin{bmatrix} 3 & \bullet & 2 \\ 2 & 0 & -2 \\ 0 & 1 & 1 \end{bmatrix} \quad 2\times1 - (-2)\times0 = 2$$

$$\begin{bmatrix} 3 & 0 & 2 \\ 2 & 0 & -2 \\ 0 & \bullet & 1 \end{bmatrix} \quad 3\times-2 - 2\times2 = -10$$

$$\begin{bmatrix} 3 & 0 & 2 \\ 2 & 0 & -2 \\ 0 & 1 & \bullet \end{bmatrix} \quad 3\times0 - 0\times2 = 0$$

And here is the calculation for the whole matrix:

$$\begin{bmatrix} 0\times1 - (-2)\times1 & 2\times1 - (-2)\times0 & 2\times1 - 0\times0 \\ 0\times1 - 2\times1 & 3\times1 - 2\times0 & 3\times1 - 0\times0 \\ 0\times(-2) - 2\times0 & 3\times(-2) - 2\times2 & 3\times0 - 0\times2 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 2 & 2 \\ -2 & 3 & 3 \\ 0 & -10 & 0 \end{bmatrix}$$ **Matrix of Minors**

ANALYTICAL STEPS TO COMPUTE INVERSE

Step 2: Turn Matrix of Minors into the Matrix of Cofactors

This is easy! Just apply a "checkerboard" of minuses to the "Matrix of Minors". In other words, we need to change the sign of alternate cells, like this,

$$\underbrace{\begin{bmatrix} 2 & 2 & 2 \\ -2 & 3 & 3 \\ 0 & -10 & 0 \end{bmatrix}}_{\text{Matrix of Minors}} \rightarrow \begin{bmatrix} + & - & + \\ - & + & - \\ + & - & + \end{bmatrix} \rightarrow \underbrace{\begin{bmatrix} 2 & -2 & 2 \\ +2 & 3 & -3 \\ 0 & +10 & 0 \end{bmatrix}}_{\text{Matrix of Cofactors}}$$

Step 3: Adjugate or Adjoint

Now, "Transpose" all elements of the previous matrix. In other words, swap their positions over the diagonal (the diagonal stays the same),

$$\begin{bmatrix} 2 & 2 & 0 \\ -2 & 3 & 10 \\ 2 & -3 & 0 \end{bmatrix}$$

ANALYTICAL STEPS TO COMPUTE INVERSE

Step 4 : Multiply by 1/determinant

Let us find the determinant first!

$$\left[a \begin{vmatrix} e & f \\ h & i \end{vmatrix}\right] - \left[b \begin{vmatrix} d & f \\ g & i \end{vmatrix}\right] + \left[c \begin{vmatrix} d & e \\ g & h \end{vmatrix}\right]$$

In practice, we can just multiply each of the top row elements by the cofactor for the same location:

Elements of top row: 3, 0, 2

Cofactors for top row: 2, –2, 2

Determinant $= 3 \times 2 + 0 \times (-2) + 2 \times 2 = 10$

Now, multiply the Adjugate by 1/determinant.

$$A^{-1} = \frac{1}{10}\begin{bmatrix} 2 & 2 & 0 \\ -2 & 3 & 10 \\ 2 & -3 & 0 \end{bmatrix} = \begin{bmatrix} 0.2 & 0.2 & 0 \\ -0.2 & 0.3 & 1 \\ 0.2 & -0.3 & 0 \end{bmatrix}$$

Adjugate **Inverse**

And we are done!

TRACE

The trace operator gives the sum of all of the diagonal entries of a matrix.

The operation of calculating a trace on a square matrix is described using the notation *tr(A)* where, *A* is the square matrix on which the operation is being performed.

Thus, for a 3x3 matrix we have,

$$tr(A) = a_{1,1} + a_{2,2} + a_{3,3}$$

$$tr(A) = A[0,0] + A[1,1] + A[2,2]$$

RANK

The rank of a matrix is the estimate of the number of linearly independent rows or columns in a matrix.

The rank of a matrix, M, is often denoted as the function rank().

A 2x2 matrix of Rank 0

$$\begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix}$$

A 2x2 matrix of Rank 1

$$\begin{matrix} 1 & 2 \\ 1 & 2 \end{matrix}$$

A 2x2 matrix of Rank 2

$$\begin{matrix} 1 & 2 \\ 3 & 4 \end{matrix}$$