SIMPLICIAL HOMOLOGY

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1 Simplices

1. Simplicial homology is defined for a certain class of spaces, which is the class of all polyhedra.

Definition 1.1. Given a set $\{a_0, ..., a_n\}$ of points of $\mathbb{R}^N$, this set is said to be **geometrically independent** if for any real scalars t_i, the equations

$$\sum_{i=0}^{n} t_i = 0 \quad \text{and} \quad \sum_{i=0}^{n} t_i a_i = 0$$

imply that $t_0 = ... = t_n = 0$.

2. One-point set is always geometrically independent.
3. It is easy to see that in general $\{a_0, ..., a_n\}$ is geometrically independent iff the vectors

$$a_1 - a_0, \cdots, a_n - a_0$$

are linearly independent in the sense of ordinary linear algebra. This is true for any a_i in the place of a_0.
4. The **n-plane** spanned by these points consist of all points x of $\mathbb{R}^N$ such that

$$x = \sum_{i=0}^{n} t_i a_i$$

for some scalars t_i with $\sum_{i=0}^{n} t_i = 1$.

Lemma 1.2. *If* $\{a_0, ..., a_n\}$ *is geometrically independent, and if* w *lies outside the plane that these points span, then* $\{w, a_0, ..., a_n\}$ *is geometrically independent.*

5. An **affine transformation** T of $\mathbb{R}^N$ is a map that is a composition of translations and non-singular linear maps.
6. An affine transformations preserves geometric independent sets and carries the plane P spanned by $\{a_0, ..., a_n\}$ onto the plane spanned by $\{Ta_0, ..., Ta_n\}$.
7. The translation $T(x) = x - a_0$ carries P onto the vector subspace of $\mathbb{R}^N$ having $a_1 - a_0, \cdots, a_n - a_0$ as a basis; if we follow T by a linear transformation of $\mathbb{R}^N$ carrying $a_1 - a_0, \cdots, a_n - a_0$ to the first n unit basis vectors of $\mathbb{R}^N$, we obtain an affine transformation S of $\mathbb{R}^N$ that carries P onto the plane $\mathbb{R}^n \times 0$ of the first n coordinates in $\mathbb{R}^N$.

Definition 1.3. Let $\{a_0, ..., a_n\}$ be a geometrically independent set in $\mathbb{R}^N$, We define **n-simplex** σ spanned by $a_0, \cdots, a_n$ to be the set of all points x in $\mathbb{R}^N$ such that

$$x = \sum_{i=0}^{n} t_i a_i \quad \text{where} \quad \sum_{i=0}^{n} t_i = 1$$

and $t_i \geq 0$for all i. The numbers t_i are uniquely determined by x; they are called the **barycentric coordinates** of the point x of σ with respect to $a_0, \cdots, a_n$.

9. The points $a_0, \cdots, a_n$. that span σ are called **vertices** of σ; the number n is called the dimension of σ. Any simplex spanned by a subset of $\{a_0, \cdots, a_n\}$ is called a **face** of σ. In particular, the face of σ spanned by $a_1, \cdots, a_n$ is called the face opposite a_0.

10. The faces of σ different from σ itself are called **proper faces** of σ; their union is called the **boundary** of σ and denoted Bdσ. The **interior** of σ is defined by the equation Intσ = $\sigma - Bd\sigma$; the set Intσ is sometimes called an **open simplex**.
11. Bd σ consists of all points x of σ such that at least one of the barycentric coordinates $t_i(x)$ is zero. Int σ consists of those points of σ for which $t_i(x) > 0$ for all i. It follows that, given $x \in \sigma$, there is exactly one face s of σ such that $x \in Ints$, for s must be the face of σ spanned by those a_i for which $t_i(x)$ is positive.
12. Let us list some basic properties of simplices.

- i The barycentric coordinates $t_i(x)$ of x with respect to $a_0, \cdots, a_n$ are continuous functions of x.
- ii σ equals the union of all line segments joining a_0 to points of the simplex s spanned by $a_0, \cdots, a_n$. Two such line segments intersect only in the point a_0.
- iii σ is a compact , convex set in $\mathbb{R}^N$, which equals the intersection of all convex sets in $\mathbb{R}^N$ containing $a_0, \cdots, a_n$. It means it is the **convex hull** of $a_0, \cdots, a_n$.
- iv Given a simplex σ there is one and only one geometrically independent set of points spanning σ.
- v Int σ is convex and is open in the plane P; its closure is σ. Furthermore, Int σ equals the union of all open line segments joining a_0 to points of Int s, where s the face opposite σ opposite a_0.
- vi There is a homeomorphism of σ with the unit ball B^n that carries Bd σ onto the unit sphere S^{n-1}.

Lemma 1.4. *Let U be a bounded, convex, open sets in* $\mathbb{R}^n$*; let* $w \in U$*.*

1. *Each ray emanating from w intersects Bd* $U = \bar{U} - U$ *in precisely one point.*
2. *There is a homeomorphism of* $\bar{U}$ *with* B^n *carrying Bd U onto* S^{n-1}*.*