Digital Signal Processing

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Contents

Abstract—**This manual provides a simple introduction to digital signal processing.**

1 Software Installation

Run the following commands

```
sudo apt−get update
sudo apt−get install libffi−dev libsndfile1 python3
    −scipy python3−numpy python3−matplotlib
sudo pip install cffi pysoundfile
```

2 Digital Filter

2.1 Download the sound file from

```
wget https://github.com/kumarsuraj151/
    EE3900/blob/master/codes/2_digital%20
    filter/Sound_Noise.wav
```

2.2 You will find a spectrogram at https://academo.org/demos/spectrum-analyzer. Upload the sound file that you downloaded in Problem 2.1 in the spectrogram and play. Observe the spectrogram. What do you find?
Solution: There are a lot of yellow lines between 440 Hz to 5.1 KHz. These represent the synthesizer key tones. Also, the key strokes are audible along with background noise.

2.3 Write the python code for removal of out of band noise and execute the code.
Solution:

```
import soundfile as sf
from scipy import signal
#read .wav file
input_signal,fs = sf.read('Sound_Noise.wav'
    )
#sampling frequency of Input signal
sampl_freq=fs
#order of the filter
order=4
#cutoff frquency 4kHz
cutoff_freq=4000.0
#digital frequency
Wn=2*cutoff_freq/sampl_freq
# b and a are numerator and
    denominatorpolynomials respectively
b, a = signal.butter(order,Wn, 'low')
#filter the input signal with butterworth filter
output_signal = signal.filtfilt(b, a,
    input_signal)
#output signal = signal.lfilter(b, a,input
    signal)
#write the output signal into .wav file
sf.write('Sound With ReducedNoise.wav',
    output_signal, fs)
```

2.4 The output of the python script in Problem 2.3 is the audio file Sound_With_ReducedNoise.wav. Play the file in the spectrogram in Problem 2.2. What do you observe?
Solution: The key strokes as well as background noise is subdued in the audio. Also, the signal is blank for frequencies above 5.1 kHz.

3 Difference Equation

3.1 Let

$$x(n) = \left\{\underset{\uparrow}{1}, 2, 3, 4, 2, 1\right\} \tag{3.1}$$

Sketch $x(n)$.

3.2 Let

$$y(n) + \frac{1}{2}y(n-1) = x(n) + x(n-2),$$
$$y(n) = 0, n < 0 \tag{3.2}$$

Sketch $y(n)$.

Solution: The following code yields Fig. .

```
wget https://raw.githubusercontent.com/
    kumarsuraj151/EE3900/master/codes/3
    _diffrence%20equation/xnyn.py
```

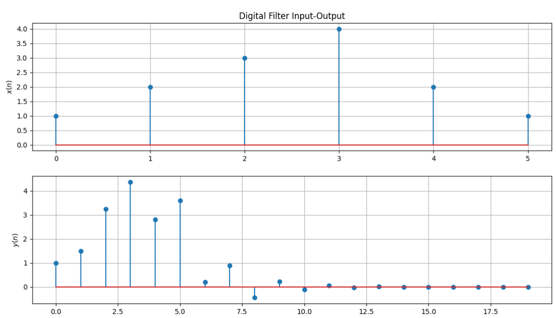

Fig. 3.2

3.3 Repeat the above exercise using c

4 Z-transform

4.1 The Z-transform of $x(n)$ is defined as

$$X(z) = \mathcal{Z}\{x(n)\} = \sum_{n=-\infty}^{\infty} x(n)z^{-n} \tag{4.1}$$

Show that

$$\mathcal{Z}\{x(n-1)\} = z^{-1}X(z) \tag{4.2}$$

and find

$$\mathcal{Z}\{x(n-k)\} \tag{4.3}$$

Solution: From (4.1),

$$\mathcal{Z}\{x(n-k)\} = \sum_{n=-\infty}^{\infty} x(n-1)z^{-n} \tag{4.4}$$

$$= \sum_{n=-\infty}^{\infty} x(n)z^{-n-1} = z^{-1}\sum_{n=-\infty}^{\infty} x(n)z^{-n} \tag{4.5}$$

resulting in (4.2). Similarly, it can be shown that

$$\mathcal{Z}\{x(n-k)\} = z^{-k}X(z) \tag{4.6}$$

4.2 Obtain $X(z)$ for $x(n)$ defined in problem (3.1).

Solution:

$$Z(x(n)) = \sum_{n=-\infty}^{\infty} x(n)z^{-n} \tag{4.7}$$

$$= x(0)z^0 + x(1)z^{-1} + x(2)z^{-2} + x(3)z^{-3} + \tag{4.8}$$

$$x(4)z^{-4} + x(5)z^{-5}$$

$$= 1 + 2z^{-1} + 3z^{-2} + 4z^{-3} + 2z^{-4} + z^{-5} \tag{4.9}$$

4.3 Find

$$H(z) = \frac{Y(z)}{X(z)} \tag{4.10}$$

from (3.2) assuming that the Z-transform is a linear operation.

Solution: Applying (4.6) in (3.2),

$$Y(z) + \frac{1}{2}z^{-1}Y(z) = X(z) + z^{-2}X(z) \tag{4.11}$$

$$\implies \frac{Y(z)}{X(z)} = \frac{1 + z^{-2}}{1 + \frac{1}{2}z^{-1}} \tag{4.12}$$

4.4 Find the Z transform of

$$\delta(n) = \begin{cases} 1 & n = 0 \\ 0 & \text{otherwise} \end{cases} \tag{4.13}$$

and show that the Z-transform of

$$u(n) = \begin{cases} 1 & n \geq 0 \\ 0 & \text{otherwise} \end{cases} \tag{4.14}$$

is

$$U(z) = \frac{1}{1 - z^{-1}}, \quad |z| > 1 \tag{4.15}$$

Solution: It is easy to show that

$$\delta(n) \overset{\mathcal{Z}}{\rightleftharpoons} 1 \tag{4.16}$$

and from (4.14),

$$U(z) = \sum_{n=0}^{\infty} z^{-n} \tag{4.17}$$

$$= \frac{1}{1 - z^{-1}}, \quad |z| > 1 \tag{4.18}$$

using the fomula for the sum of an infinite geometric progression.

4.5 Show that

$$a^n u(n) \overset{\mathcal{Z}}{\rightleftharpoons} \frac{1}{1 - az^{-1}} \quad |z| > |a| \tag{4.19}$$

Solution:

$$\mathcal{Z}(a^n u(n)) = \sum_{n=-\infty}^{\infty} a^n u(n) z^{-n} \tag{4.20}$$

$$= \sum_{n=0}^{\infty} a^n z^{-n} \tag{4.21}$$

$$= \frac{1}{1 - az^{-1}}, \quad \left|az^{-1}\right| < 1 \tag{4.22}$$

$$= \frac{1}{1 - az^{-1}}, \quad |a| < |z| \tag{4.23}$$

using the fomula for the sum of an infinite geometric progression.

4.6 Let

$$H\left(e^{j\omega}\right) = H\left(z = e^{j\omega}\right). \tag{4.24}$$

Plot $|H(e^{j\omega})|$. Comment. $H(e^{j\omega})$ is known as the *Discret Time Fourier Transform* (DTFT) of $h(n)$.

Solution: The following code plots Fig. 4.6.

```
wget https://raw.githubusercontent.com/
    kumarsuraj151/EE3900/master/codes/4_Z
    %20transform/dtft.py
```

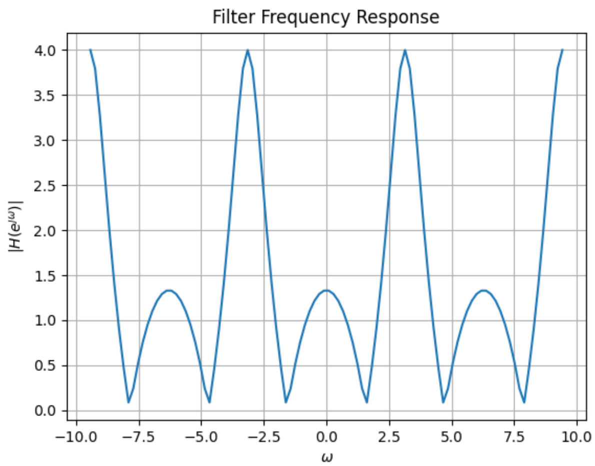

Fig. 4.6: $|H(e^{j\omega})|$

$$H\left(e^{j\omega}\right) = \frac{1 + e^{-2j\omega}}{1 + \frac{e^{-j\omega}}{2}} \tag{4.25}$$

$$\implies \left|H\left(e^{j\omega}\right)\right| = \frac{\left|1 + e^{-2j\omega}\right|}{\left|1 + \frac{e^{-j\omega}}{2}\right|} \tag{4.26}$$

$$= \frac{\left|1 + e^{2j\omega}\right|}{\left|e^{2j\omega} + \frac{e^{j\omega}}{2}\right|} \tag{4.27}$$

$$= \frac{|1 + \cos 2\omega + j \sin 2\omega|}{\left|e^{j\omega} + \frac{1}{2}\right|} \tag{4.28}$$

$$= \frac{\left|4\cos^2(\omega) + 4j\sin(\omega)\cos(\omega)\right|}{|2e^{j\omega} + 1|} \tag{4.29}$$

$$= \frac{|4\cos(\omega)|\,|\cos(\omega) + j\sin(\omega)|}{|2\cos(\omega) + 1 + 2j\sin(\omega)|} \tag{4.30}$$

$$\therefore \left|H\left(e^{j\omega}\right)\right| = \frac{|4\cos(\omega)|}{\sqrt{5 + 4\cos(\omega)}} \tag{4.31}$$

The period of $|\cos(\omega)|$ is π. The period of $5 + 4\cos(\omega)$ is 2π. Hence $|H(e^{j\omega})|$ is periodic with period π.(The LCM of the period of $|\cos(\omega)|$ and $5 + 4\cos(\omega)$ is π) The graph of $|H(e^{j\omega})|$ is symmetric with respect to y-axis. It is continuous over ω. The following code plots Fig. 4.6.

4.7 Express $h(n)$ in terms of $H(e^{j\omega})$.

Solution:

$$H(e^{j\omega}) = \sum_{k=-\infty}^{\infty} h(k) e^{-j\omega k} \tag{4.32}$$

and

$$h(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} H(e^{j\omega})e^{j\omega n}d\omega \tag{4.33}$$

Now,

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} H(e^{j\omega})e^{j\omega n}d\omega \tag{4.34}$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi}\sum_{k=-\infty}^{\infty} h(k)e^{-j\omega k}e^{j\omega n}d\omega \tag{4.35}$$

$$= \frac{1}{2\pi}\sum_{k=-\infty}^{\infty} h(k)\int_{-\pi}^{\pi} e^{j\omega(n-k)}d\omega \tag{4.36}$$

$$= \frac{1}{2\pi}\sum_{k=-\infty}^{\infty} h(k)\int_{-\pi}^{\pi}\cos w(n-k)d\omega \tag{4.37}$$

$$+ \int_{-\pi}^{\pi}\sin w(n-k)d\omega$$

$$= \frac{1}{2\pi}\sum_{k=-\infty}^{\infty} h(k)\int_{-\pi}^{\pi}\cos w(n-k) \tag{4.38}$$

$$= \frac{1}{2\pi}\sum_{k=-\infty}^{\infty} h(k)\frac{\sin w(n-k)}{n-k}\Bigg|_{-\pi}^{\pi} \tag{4.39}$$

$$= \frac{1}{2\pi}\sum_{k\neq n} h(n)\frac{\sin\pi(n-k)}{n-k} + \sum_{k=n} h(n)\frac{\sin\pi(n-k)}{n-k} \tag{4.40}$$

$$= \frac{0 + 2\pi h(n)}{2\pi} \tag{4.41}$$

$$= h(n) \tag{4.42}$$

5 Impulse Response

5.1 Using long division, find

$$h(n), \quad n < 5 \tag{5.1}$$

for H(z) in (4.12).
Solution: $H(z)$ is given by

$$H(z) = \frac{1 + z^{-2}}{1 + \frac{1}{2}z^{-1}} = \frac{2 + 2z^{-2}}{2 + z^{-1}} \tag{5.2}$$

$$2z^{-1} - 4 \tag{5.3}$$

$$z^{-1} + 2 \enclose{longdiv}{2z^{-2} + 2} \tag{5.4}$$

$$2z^{-2} + 4z^{-1} \tag{5.5}$$

$$-4z^{-1} + 2 \tag{5.6}$$

$$-4z^{-1} - 8 \tag{5.7}$$

$$10 \tag{5.8}$$

So,

$$H(z) = 2z^{-1} - 4 + \frac{10}{z^{-1} + 2} \tag{5.9}$$

$$= 2z^{-1} - 4 + \frac{5}{\frac{1}{2}z^{-1} + 1} \tag{5.10}$$

$$= 2z^{-1} - 4 + 5\sum_{n=0}^{\infty}\left(-\frac{z^{-1}}{2}\right)^n \tag{5.11}$$

$$= 1 - \frac{1}{2}z^{-1} + \sum_{n=2}^{\infty}\left(-\frac{1}{2}\right)^n z^{-n} \tag{5.12}$$

So,h(n) will be given by

$$h(n) = \begin{cases} 5 \times \left(-\frac{1}{2}\right)^n & n \geq 2 \\ \left(-\frac{1}{2}\right)^n & 2 > n \geq 0 \\ 0 & n < 0 \end{cases} \tag{5.13}$$

5.2 Find an expression for $h(n)$ using $H(z)$, given that

$$h(n) \overset{\mathcal{Z}}{\rightleftharpoons} H(z) \tag{5.14}$$

and there is a one to one relationship between $h(n)$ and $H(z)$. $h(n)$ is known as the *impulse response* of the system defined by (3.2).
Solution: From (4.12),

$$H(z) = \frac{1}{1 + \frac{1}{2}z^{-1}} + \frac{z^{-2}}{1 + \frac{1}{2}z^{-1}} \tag{5.15}$$

$$\implies h(n) = \left(-\frac{1}{2}\right)^n u(n) + \left(-\frac{1}{2}\right)^{n-2} u(n-2) \tag{5.16}$$

using (4.19) and (4.6).

5.3 Sketch $h(n)$. Is it bounded? Convergent?
Solution: The following code plots Fig. 5.3.

```
wget https://raw.githubusercontent.com/
    kumarsuraj151/EE3900/master/codes/5
    _impluse%20responce/hn.py
```

Yes, it is bounded and Convergent

5.4 Convergent? Justify using the ratio test.
Solution: We can say a given real sequence $\{x_n\}$ is convergent if

$$\lim_{n\to\infty}\left|\frac{x_{n+1}}{x_n}\right| < 1 \tag{5.17}$$

This is known as Ratio test.

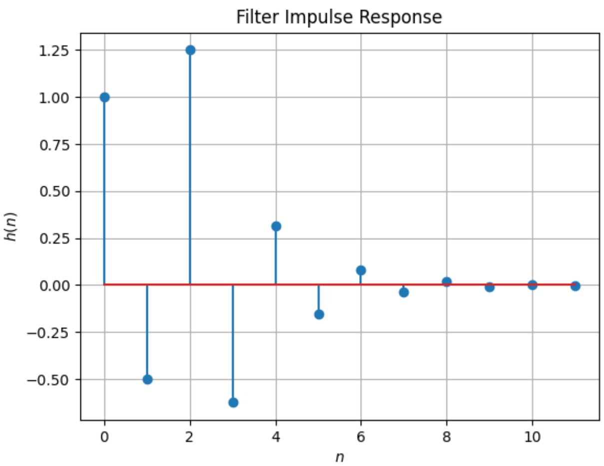

Fig. 5.3: $h(n)$ as the inverse of $H(z)$

In this case the limit will become,

$$\lim_{n\to\infty}\left|\frac{h\left(n+1\right)}{h\left(n\right)}\right| = \lim_{n\to\infty}\left|\frac{5\left(\frac{-1}{2}\right)^{n+1}}{5\left(\frac{-1}{2}\right)^{n}}\right| \tag{5.18}$$

$$= \lim_{n\to\infty}\left|\frac{-1}{2}\right| \tag{5.19}$$

$$= \frac{1}{2} \tag{5.20}$$

As $\frac{1}{2} < 1$, from root test we can say that $h\left(n\right)$ is convergent.

5.5 The system with $h(n)$ is defined to be stable if

$$\sum_{n=-\infty}^{\infty} h(n) < \infty \tag{5.21}$$

Is the system defined by (3.2) stable for the impulse response in (5.14)?
Solution:

$$\sum_{n=-\infty}^{\infty} h(n) = \sum_{n=-\infty}^{\infty}\left(-\frac{1}{2}\right)^{n} u(n) + \sum_{n=-\infty}^{\infty}\left(-\frac{1}{2}\right)^{n-2} u(n-2) \tag{5.22}$$

$$= \sum_{n=0}^{\infty}\left(-\frac{1}{2}\right)^{n} + \sum_{n=2}^{\infty}\left(-\frac{1}{2}\right)^{n-2} \tag{5.23}$$

$$= \frac{2}{3} + \frac{2}{3} = \frac{4}{3} \tag{5.24}$$

hence the system defined by (3.2) is stable for the impulse response in (5.14)?

5.6 Verify the above result using python code
Solution: the above result can be verify using following code

```
wget https://raw.githubusercontent.com/
    kumarsuraj151/EE3900/master/codes/5
    _impluse%20responce/hndef.py
```

5.7 Compute and sketch $h(n)$ using

$$h(n) + \frac{1}{2}h(n-1) = \delta(n) + \delta(n-2), \tag{5.25}$$

This is the definition of $h(n)$.
Solution: The following code plots Fig. 5.7. Note that this is the same as Fig. 5.3.

```
wget https://raw.githubusercontent.com/
    kumarsuraj151/EE3900/master/codes/5
    _impluse%20responce/hndef.py
```

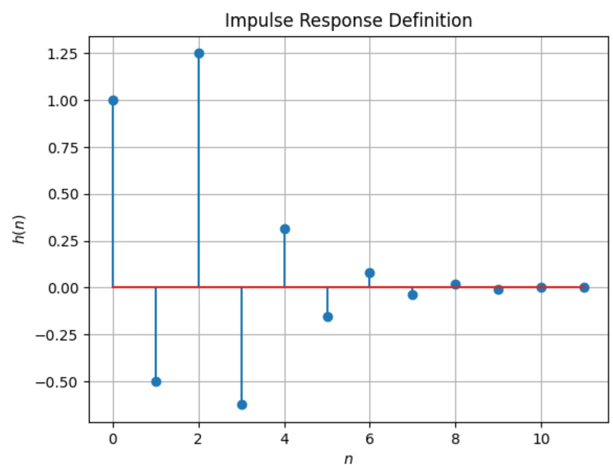

Fig. 5.7: $h(n)$ from the definition

5.8 Compute

$$y(n) = x(n) * h(n) = \sum_{n=-\infty}^{\infty} x(k)h(n-k) \tag{5.26}$$

Comment. The operation in (5.26) is known as *convolution*.
Solution: The following code plots Fig. 5.8. Note that this is the same as $y(n)$ in Fig. 3.2.

```
wget https://raw.githubusercontent.com/
    kumarsuraj151/EE3900/master/codes/5
    _impluse%20responce/ynconv.py
```

5.9 Express the above convolution using a Teoplitz matrix.

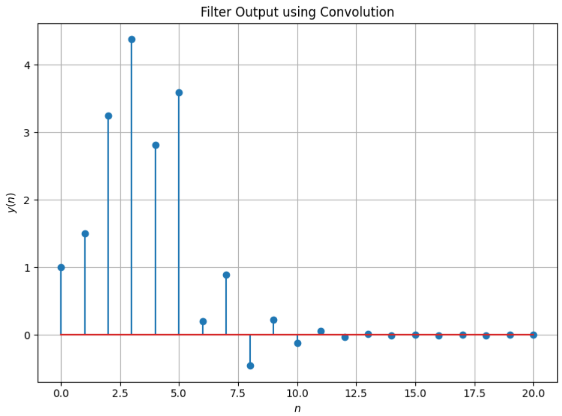

Fig. 5.8: $y(n)$ from the definition of convolution

Solution: From (5.26),we express $y(n)$ as

$$y(n) = \sum_{k=-\infty}^{\infty} x(k)\, h(n-k) \tag{5.27}$$

To understand how we can use a Toeplitz matrix

$$y(0) = x(0)\, h(0) \tag{5.28}$$

$$y(1) = x(0)\, h(1) + x(1)\, h(0) \tag{5.29}$$

$$y(2) = x(0)\, h(2) + x(1)\, h(1) + x(2)\, h(0) \tag{5.30}$$

.

.

The same thing can be written as,

$$y(0) = \begin{pmatrix} h(0) & 0 & 0 & . & . & .0 \end{pmatrix} \begin{pmatrix} x(0) \\ x(1) \\ x(2) \\ . \\ . \\ x(5) \end{pmatrix} \tag{5.31}$$

$$y(1) = \begin{pmatrix} h(1) & h(0) & 0 & 0 & . & . & .0 \end{pmatrix} \begin{pmatrix} x(0) \\ x(1) \\ x(2) \\ . \\ . \\ x(5) \end{pmatrix} \tag{5.32}$$

$$y(2) = \begin{pmatrix} h(2) & h(1) & h(0) & 0 & . & .0 \end{pmatrix} \begin{pmatrix} x(0) \\ x(1) \\ x(2) \\ . \\ . \\ x(5) \end{pmatrix} \tag{5.33}$$

.

.

Using Toeplitz matrix of $h(n)$ we can simplify it as,

$$y(n) = \begin{pmatrix} h(0) & 0 & 0 & . & . & .0 \\ h(1) & h(0) & 0 & . & . & .0 \\ h(2) & h(1) & h(0) & . & . & .0 \\ & & .. & & & \\ & & .. & & & \\ 0 & 0 & 0 & . & . & .h(m-1) \end{pmatrix} \begin{pmatrix} x(0) \\ x(1) \\ x(2) \\ . \\ . \\ x(5) \end{pmatrix} \tag{5.34}$$

$$x(n) = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 2 \\ 1 \end{pmatrix} \tag{5.35}$$

And from (5.13) we will take some values of n,

$$..h(n) = \begin{pmatrix} 1 \\ -0.5 \\ 1.25 \\ . \\ . \end{pmatrix} \tag{5.36}$$

Now using (5.34),

$$y(n) = x(n) * h(n) \tag{5.37}$$

$$= \begin{pmatrix} 1 & 0 & 0 & . & . & .0 \\ -0.5 & 1 & 0 & . & . & .0 \\ 1.25 & -0.5 & 1 & . & . & .0 \\ & & .. & & & \\ & & .. & & & \\ 0 & 0 & 0 & . & . & . \end{pmatrix} \begin{pmatrix} x(0) \\ x(1) \\ x(2) \\ . \\ . \\ x(5) \end{pmatrix} \tag{5.38}$$

$$= \begin{pmatrix} 1 \\ 1.5 \\ 3.25 \\ . \\ . \\ . \end{pmatrix} \tag{5.39}$$

The above equation (5.39) is the convolution of $x(n)$ and $h(n)$

5.10 Show that

$$y(n) = \sum_{k=-\infty}^{\infty} x(n-k)h(k) \tag{5.40}$$

Solution: Substitute $k := n - k$ in (5.26), we will get

$$y(n) = \sum_{k=-\infty}^{\infty} x(k)h(n-k) \tag{5.41}$$

$$= \sum_{n-k=-\infty}^{\infty} x(n-k)h(k) \tag{5.42}$$

$$= \sum_{k=-\infty}^{\infty} x(n-k)h(k) \tag{5.43}$$

6 DFT

6.1 Compute

$$X(k) \triangleq \sum_{n=0}^{N-1} x(n)e^{-\jmath 2\pi kn/N}, \quad k = 0, 1, \dots, N-1 \tag{6.1}$$

and $H(k)$ using $h(n)$.

Solution: Downlode the python code for questions 6.1

```
wget https://raw.githubusercontent.com/
    kumarsuraj151/EE3900/master/codes/6
    _dft%20and%20fft/6_1.py
```

6.2 Compute

$$Y(k) = X(k)H(k) \tag{6.2}$$

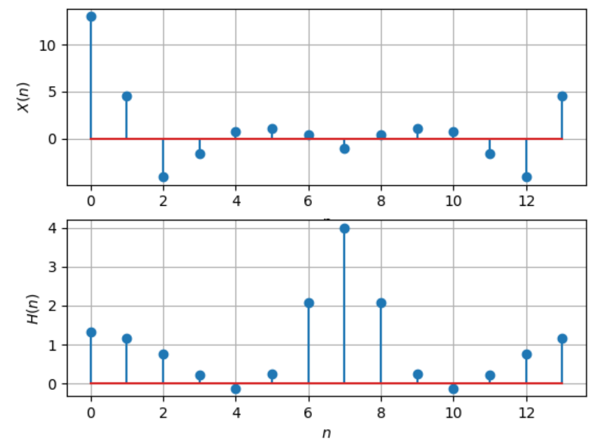

Fig. 6.1: Plot of real part of Discrete Fourier Transforms of $x(n)$ and $h(n)$

Solution: download the python code for questions 6.2

```
wget https://raw.githubusercontent.com/
    kumarsuraj151/EE3900/master/codes/6
    _dft%20and%20fft/6_2.py
```

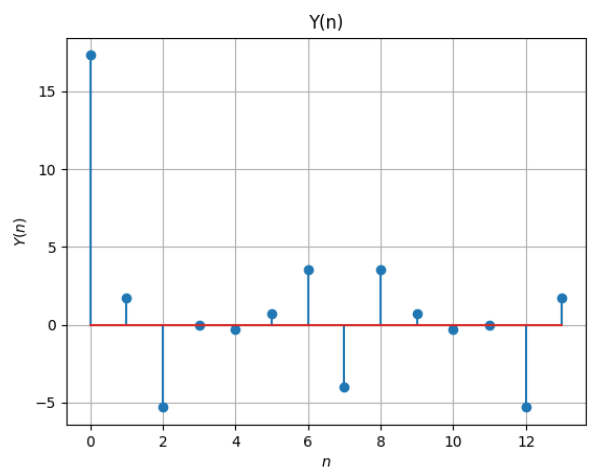

Fig. 6.2: $Y(k)$ as the product of $X(k)$ and $H(k)$

6.3 Compute

$$y(n) = \frac{1}{N}\sum_{k=0}^{N-1} Y(k)\cdot e^{\jmath 2\pi kn/N}, \quad n = 0, 1, \dots, N-1 \tag{6.3}$$

Solution: The following code plots Fig. 5.8. Note that this is the same as $y(n)$ in Fig. 3.2.

```
wget https://raw.githubusercontent.com/
    kumarsuraj151/EE3900/master/codes/6
    _dft%20and%20fft/yndft.py
```

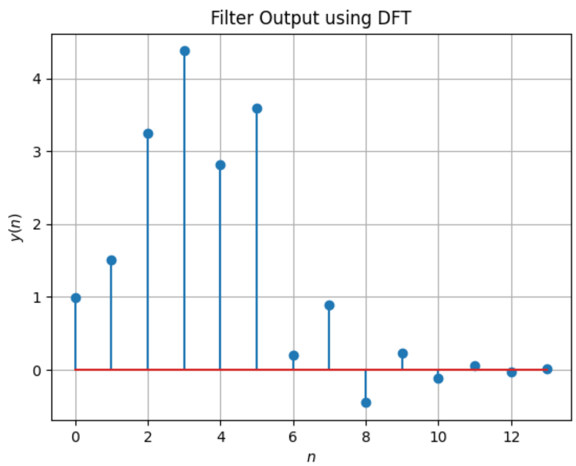

Fig. 6.3: $y(n)$ from the DFT

6.4 Repeat the previous exercise by computing $X(k), H(k)$ and $y(n)$ through FFT and IFFT.
Solution:

```
wget https://raw.githubusercontent.com/
    kumarsuraj151/EE3900/master/codes/6
    _dft%20and%20fft/yn_ifft.py
```

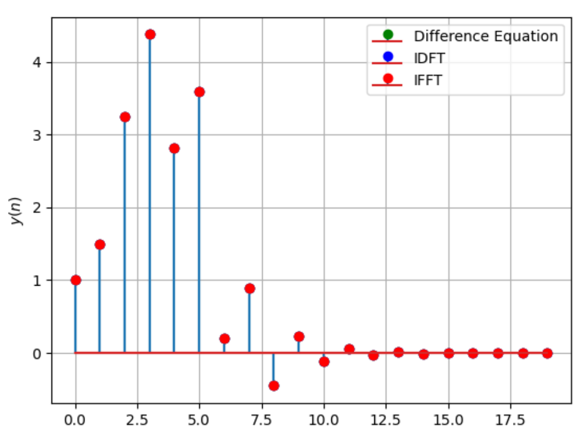

Fig. 6.4: The plot of $y(n)$ using IFFT

7 FFT

1. The DFT of $x(n)$ is given by
$$X(k) \triangleq \sum_{n=0}^{N-1} x(n)e^{-j2\pi kn/N}, \quad k = 0, 1, \dots, N-1 \tag{7.1}$$
2. Let
$$W_N = e^{-j2\pi/N} \tag{7.2}$$
Then the N-point *DFT matrix* is defined as
$$\vec{F}_N = [W_N^{mn}], \quad 0 \le m, n \le N-1 \tag{7.3}$$
where W_N^{mn} are the elements of $\vec{F}_N$.
3. Let
$$\vec{I}_4 = \begin{pmatrix} \vec{e}_4^1 & \vec{e}_4^2 & \vec{e}_4^3 & \vec{e}_4^4 \end{pmatrix} \tag{7.4}$$
be the 4×4 identity matrix. Then the 4 point *DFT permutation matrix* is defined as
$$\vec{P}_4 = \begin{pmatrix} \vec{e}_4^1 & \vec{e}_4^3 & \vec{e}_4^2 & \vec{e}_4^4 \end{pmatrix} \tag{7.5}$$
4. The 4 point *DFT diagonal matrix* is defined as
$$\vec{D}_4 = diag\begin{pmatrix} W_8^0 & W_8^1 & W_8^2 & W_8^3 \end{pmatrix} \tag{7.6}$$
5. Show that
$$W_N^2 = W_{N/2} \tag{7.7}$$
Solution: Given
$$W_N = e^{-j2\pi/N} \tag{7.8}$$
$$W_N^2 = e^{2(-j2\pi/N)} \tag{7.9}$$
$$= e^{-j2\pi/(N/2)} \tag{7.10}$$
$$= W_{N/2} \tag{7.11}$$
6. Show that
$$\vec{F}_4 = \begin{bmatrix} \vec{I}_2 & \vec{D}_2 \\ \vec{I}_2 & -\vec{D}_2 \end{bmatrix} \begin{bmatrix} \vec{F}_2 & 0 \\ 0 & \vec{F}_2 \end{bmatrix} \vec{P}_4 \tag{7.12}$$
Solution: $\vec{I}_2$ is 2×2 identity matrix
$$\begin{bmatrix} \vec{I}_2 & \vec{D}_2 \\ \vec{I}_2 & -\vec{D}_2 \end{bmatrix} \begin{bmatrix} \vec{F}_2 & 0 \\ 0 & \vec{F}_2 \end{bmatrix} = \begin{bmatrix} \vec{F}_2 & \vec{D}_2\vec{F}_2 \\ \vec{F}_2 & -\vec{D}_2\vec{F}_2 \end{bmatrix} \tag{7.13}$$
Given
$$\vec{F}_2 = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \tag{7.14}$$
$$\vec{D}_2 = diag(1, W_4) = \begin{bmatrix} 1 & 0 \\ 0 & -j \end{bmatrix} \tag{7.15}$$

$$\vec{P}_4 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{7.16}$$

$$\vec{F}_4 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix} \tag{7.17}$$

$$\begin{bmatrix} \vec{F}_2 & \vec{D}_2\vec{F}_2 \\ \vec{F}_2 & -\vec{D}_2\vec{F}_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & -j & j \\ 1 & 1 & -1 & -1 \\ 1 & -1 & j & -j \end{bmatrix} \tag{7.18}$$

$$= \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & -j & j \\ 1 & 1 & -1 & -1 \\ 1 & -1 & j & -j \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{7.19}$$

$$= \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix} \tag{7.20}$$

which is same as $\vec{F}_4$.

$$\therefore \vec{F}_4 = \begin{bmatrix} \vec{I}_2 & \vec{D}_2 \\ \vec{I}_2 & -\vec{D}_2 \end{bmatrix} \tag{7.21}$$

7. Show that

$$\vec{F}_N = \begin{bmatrix} \vec{I}_{N/2} & \vec{D}_{N/2} \\ \vec{I}_{N/2} & -\vec{D}_{N/2} \end{bmatrix} \begin{bmatrix} \vec{F}_{N/2} & 0 \\ 0 & \vec{F}_{N/2} \end{bmatrix} \vec{P}_N \tag{7.22}$$

Solution:

$$\begin{bmatrix} \vec{I}_{N/2} & \vec{D}_{N/2} \\ \vec{I}_{N/2} & -\vec{D}_{N/2} \end{bmatrix} \begin{bmatrix} \vec{F}_{N/2} & 0 \\ 0 & \vec{F}_{N/2} \end{bmatrix} \tag{7.23}$$

$$\tag{7.24}$$

$$= \begin{bmatrix} \vec{F}_{N/2} & \vec{D}_{N/2}\vec{F}_{N/2} \\ \vec{F}_{N/2} & -\vec{D}_{N/2}\vec{F}_{N/2} \end{bmatrix} \tag{7.25}$$

Now

$$\vec{D}_{N/2}\vec{F}_{N/2} \tag{7.26}$$

$$= \begin{bmatrix} W_N^0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & W_N^{N/2-1} \end{bmatrix} \begin{bmatrix} W_{N/2}^0 & \cdots & W_{N/2}^0 \\ \vdots & \ddots & \vdots \\ W_{N/2}^0 & \cdots & W_{N/2}^{(N/2-1)^2} \end{bmatrix} \tag{7.27}$$

$$= \begin{bmatrix} W_N^0 W_{N/2}^0 & \cdots & W_N^0 W_{N/2}^0 \\ \vdots & \ddots & \vdots \\ W_N^{N/2-1} W_{N/2}^0 & \cdots & W_N^{N/2-1} W_{N/2}^{(N/2-1)^2} \end{bmatrix} \tag{7.28}$$

Thus

$$\left(\vec{D}_{N/2}\vec{F}_{N/2}\right)_{ij} = W_N^i W_{N/2}^{ij} \tag{7.29}$$

$$= W_N^i W_N^{2ij} \tag{7.30}$$

$$= W_N^{i(2j+1)} \tag{7.31}$$

where $i, j = 0, \ldots, N/2 - 1$
Therefore, $\vec{D}_{N/2}\vec{F}_{N/2}$ forms the first $N/2$ rows of the odd-indexed columns of $\vec{F}_N$

$$W_N^{(i+N/2)(2j+1)} = \exp\left(-\jmath \frac{2\pi}{N}(2j+1)\left(i + \frac{N}{2}\right)\right) \tag{7.32}$$

$$= \exp\left(-\jmath\left(\frac{2\pi}{N}(2j+1)i + (2j+1)\pi\right)\right) \tag{7.33}$$

$$= -\exp\left(-\jmath \frac{2\pi}{N}(2j+1)i\right) \tag{7.34}$$

$$= -W_N^{i(2j+1)} \tag{7.35}$$

Thus, the remaining $N/2$ rows will be the negatives of the first $N/2$ rows

$$\left(\vec{F}_{N/2}\right)_{ij} = W_{N/2}^{ij} \tag{7.36}$$

$$= W_N^{i(2j)} \tag{7.37}$$

where $i, j = 0, \ldots, N/2 - 1$
Therefore, $\vec{F}_{N/2}$ forms the first $N/2$ rows of the even-indexed columns of $\vec{F}_N$

$$W_N^{(i+N/2)(2j)} = \exp\left(-\jmath \frac{2\pi}{N}(2j)\left(i + \frac{N}{2}\right)\right) \tag{7.38}$$

$$= \exp\left(-\jmath\left(\frac{2\pi}{N}(2j)i + (2j)\pi\right)\right) \tag{7.39}$$

$$= \exp\left(-\jmath \frac{2\pi}{N}(2j)i\right) \tag{7.40}$$

$$= W_N^{i(2j)} \tag{7.41}$$

Thus, the remaining $N/2$ rows will be the same as the first $N/2$ rows

Therefore

$$\begin{bmatrix}\vec{F}_{N/2} & \vec{D}_{N/2}\vec{F}_{N/2}\\ \vec{F}_{N/2} & -\vec{D}_{N/2}\vec{F}_{N/2}\end{bmatrix} = \vec{F}_N\vec{P}_N \tag{7.42}$$

where

$$\vec{P}_N = \begin{pmatrix}\vec{e}_N^1 & \vec{e}_N^3 & \cdots & \vec{e}_N^{N-1} & \vec{e}_N^2 & \vec{e}_N^4 & \cdots & \vec{e}_N^N\end{pmatrix} \tag{7.43}$$

Hence

$$\begin{bmatrix}\vec{F}_{N/2} & \vec{D}_{N/2}\vec{F}_{N/2}\\ \vec{F}_{N/2} & -\vec{D}_{N/2}\vec{F}_{N/2}\end{bmatrix}\vec{P}_N = \vec{F}_N\vec{P}_N^2 = \vec{F}_N \tag{7.44}$$

$$\therefore \vec{F}_N = \begin{bmatrix}\vec{I}_{N/2} & \vec{D}_{N/2}\\ \vec{I}_{N/2} & -\vec{D}_{N/2}\end{bmatrix}\begin{bmatrix}\vec{F}_{N/2} & 0\\ 0 & \vec{F}_{N/2}\end{bmatrix}\vec{P}_N \tag{7.45}$$

for even N

8. Find

$$\vec{P}_4\vec{x} \tag{7.46}$$

Solution: From (7.16),

$$\vec{P}_4 = \begin{bmatrix}1&0&0&0\\0&0&1&0\\0&1&0&0\\0&0&0&1\end{bmatrix} \tag{7.47}$$

$$\vec{x} = \begin{pmatrix}1\\2\\3\\4\\2\\1\end{pmatrix} \tag{7.48}$$

After proper zero padding of $\vec{P}_4$,

$$\vec{P}_4 = \begin{bmatrix}1&0&0&0\\0&0&1&0\\0&1&0&0\\0&0&0&1\end{bmatrix} \tag{7.49}$$

$$\vec{P}_4\vec{x} = \begin{bmatrix}1&0&0&0\\0&0&1&0\\0&1&0&0\\0&0&0&1\end{bmatrix}\begin{pmatrix}1\\2\\3\\4\end{pmatrix} \tag{7.50}$$

$$= \begin{pmatrix}1\\3\\2\\4\end{pmatrix} \tag{7.51}$$

9. Show that

$$\vec{X} = \vec{F}_N\vec{x} \tag{7.52}$$

where $\vec{x}, \vec{X}$ are the vector representations of $x(n), X(k)$ respectively.

Solution: Given $\vec{x}, \vec{X}$ are the vector representations of $x(n), X(k)$ respectively.

$$\vec{x} = \begin{bmatrix}x(0)\\x(1)\\\vdots\\x(N-1)\end{bmatrix} \tag{7.53}$$

$$\vec{X} = \begin{bmatrix}X(0)\\X(1)\\\vdots\\X(N-1)\end{bmatrix} \tag{7.54}$$

$$\vec{F}_N = \begin{bmatrix}1 & 1 & 1 & \cdots & 1\\ 1 & W_N & W_N^2 & \cdots & W_N^{(N-1)}\\ 1 & W_N^2 & W_N^4 & \cdots & W_N^{2(N-1)}\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ 1 & W_N^{N-1} & W_N^{2(N-1)} & \cdots & W_N^{(N-1)(N-1)}\end{bmatrix} \tag{7.55}$$

As

$$X(k) = \sum_{n=0}^{N-1} x(n)e^{-j2\pi kn/N} \tag{7.56}$$

Upon linear transformation over k,

$$\begin{bmatrix}X(0)\\X(1)\\\vdots\\X(N-1)\end{bmatrix} = \begin{bmatrix}1 & 1 & 1 & \cdots & 1\\ 1 & W_N & W_N^2 & \cdots & W_N^{(N-1)}\\ 1 & W_N^2 & W_N^4 & \cdots & W_N^{2(N-1)}\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ 1 & W_N^{N-1} & W_N^{2(N-1)} & \cdots & W_N^{(N-1)(N-1)}\end{bmatrix}\begin{bmatrix}x(0)\\x(1)\\\vdots\\x(N-1)\end{bmatrix} \tag{7.57}$$

$$\therefore \vec{X} = \vec{F}_N\vec{x} \tag{7.58}$$

10. Derive the following Step-by-step visualisation of 8-point FFTs into 4-point FFTs and so on

$$\begin{bmatrix}X(0)\\X(1)\\X(2)\\X(3)\end{bmatrix} = \begin{bmatrix}X_1(0)\\X_1(1)\\X_1(2)\\X_1(3)\end{bmatrix} + \begin{bmatrix}W_8^0 & 0 & 0 & 0\\ 0 & W_8^1 & 0 & 0\\ 0 & 0 & W_8^2 & 0\\ 0 & 0 & 0 & W_8^3\end{bmatrix}\begin{bmatrix}X_2(0)\\X_2(1)\\X_2(2)\\X_2(3)\end{bmatrix} \tag{7.59}$$

$$\begin{bmatrix} X(4) \\ X(5) \\ X(6) \\ X(7) \end{bmatrix} = \begin{bmatrix} X_1(0) \\ X_1(1) \\ X_1(2) \\ X_1(3) \end{bmatrix} - \begin{bmatrix} W_8^0 & 0 & 0 & 0 \\ 0 & W_8^1 & 0 & 0 \\ 0 & 0 & W_8^2 & 0 \\ 0 & 0 & 0 & W_8^3 \end{bmatrix} \begin{bmatrix} X_2(0) \\ X_2(1) \\ X_2(2) \\ X_2(3) \end{bmatrix} \tag{7.60}$$

4-point FFTs into 2-point FFTs

$$\begin{bmatrix} X_1(0) \\ X_1(1) \end{bmatrix} = \begin{bmatrix} X_3(0) \\ X_3(1) \end{bmatrix} + \begin{bmatrix} W_4^0 & 0 \\ 0 & W_4^1 \end{bmatrix} \begin{bmatrix} X_4(0) \\ X_4(1) \end{bmatrix} \tag{7.61}$$

$$\begin{bmatrix} X_1(2) \\ X_1(3) \end{bmatrix} = \begin{bmatrix} X_3(0) \\ X_3(1) \end{bmatrix} - \begin{bmatrix} W_4^0 & 0 \\ 0 & W_4^1 \end{bmatrix} \begin{bmatrix} X_4(0) \\ X_4(1) \end{bmatrix} \tag{7.62}$$

$$\begin{bmatrix} X_2(0) \\ X_2(1) \end{bmatrix} = \begin{bmatrix} X_5(0) \\ X_5(1) \end{bmatrix} + \begin{bmatrix} W_4^0 & 0 \\ 0 & W_4^1 \end{bmatrix} \begin{bmatrix} X_6(0) \\ X_6(1) \end{bmatrix} \tag{7.63}$$

$$\begin{bmatrix} X_2(2) \\ X_2(3) \end{bmatrix} = \begin{bmatrix} X_5(0) \\ X_5(1) \end{bmatrix} - \begin{bmatrix} W_4^0 & 0 \\ 0 & W_4^1 \end{bmatrix} \begin{bmatrix} X_6(0) \\ X_6(1) \end{bmatrix} \tag{7.64}$$

$$P_8 \begin{bmatrix} x(0) \\ x(1) \\ x(2) \\ x(3) \\ x(4) \\ x(5) \\ x(6) \\ x(7) \end{bmatrix} = \begin{bmatrix} x(0) \\ x(2) \\ x(4) \\ x(6) \\ x(1) \\ x(3) \\ x(5) \\ x(7) \end{bmatrix} \tag{7.65}$$

$$P_4 \begin{bmatrix} x(0) \\ x(2) \\ x(4) \\ x(6) \end{bmatrix} = \begin{bmatrix} x(0) \\ x(4) \\ x(2) \\ x(6) \end{bmatrix} \tag{7.66}$$

$$P_4 \begin{bmatrix} x(1) \\ x(3) \\ x(5) \\ x(7) \end{bmatrix} = \begin{bmatrix} x(1) \\ x(5) \\ x(3) \\ x(7) \end{bmatrix} \tag{7.67}$$

Therefore,

$$\begin{bmatrix} X_3(0) \\ X_3(1) \end{bmatrix} = F_2 \begin{bmatrix} x(0) \\ x(4) \end{bmatrix} \tag{7.68}$$

$$\begin{bmatrix} X_4(0) \\ X_4(1) \end{bmatrix} = F_2 \begin{bmatrix} x(2) \\ x(6) \end{bmatrix} \tag{7.69}$$

$$\begin{bmatrix} X_5(0) \\ X_5(1) \end{bmatrix} = F_2 \begin{bmatrix} x(1) \\ x(5) \end{bmatrix} \tag{7.70}$$

$$\begin{bmatrix} X_6(0) \\ X_6(1) \end{bmatrix} = F_2 \begin{bmatrix} x(3) \\ x(7) \end{bmatrix} \tag{7.71}$$

Solution: We write out the values of performing an 8-point FFT on $\vec{x}$ as follows.

$$X(k) = \sum_{n=0}^{7} x(n) e^{-\frac{\jmath 2kn\pi}{8}} \tag{7.72}$$

$$= \sum_{n=0}^{3} \left(x(2n) e^{-\frac{\jmath 2kn\pi}{4}} + e^{-\frac{\jmath 2k\pi}{8}} x(2n+1) e^{-\frac{\jmath 2kn\pi}{4}} \right) \tag{7.73}$$

$$= X_1(k) + e^{-\frac{\jmath 2k\pi}{4}} X_2(k) \tag{7.74}$$

where $\vec{X}_1$ is the 4-point FFT of the even-numbered terms and $\vec{X}_2$ is the 4-point FFT of the odd numbered terms. Noticing that for $k \geq 4$,

$$X_1(k) = X_1(k-4) \tag{7.75}$$

$$e^{-\frac{\jmath 2k\pi}{8}} = -e^{-\frac{\jmath 2(k-4)\pi}{8}} \tag{7.76}$$

we can now write out $X(k)$ in matrix form as in (7.59) and (7.60). We also need to solve the two 4-point FFT terms so formed.

$$X_1(k) = \sum_{n=0}^{3} x_1(n) e^{-\frac{\jmath 2kn\pi}{8}} \tag{7.77}$$

$$= \sum_{n=0}^{1} \left(x_1(2n) e^{-\frac{\jmath 2kn\pi}{4}} + e^{-\frac{\jmath 2k\pi}{8}} x_2(2n+1) e^{-\frac{\jmath 2kn\pi}{4}} \right) \tag{7.78}$$

$$= X_3(k) + e^{-\frac{\jmath 2k\pi}{4}} X_4(k) \tag{7.79}$$

using $x_1(n) = x(2n)$ and $x_2(n) = x(2n+1)$. Thus we can write the 2-point FFTs

$$\begin{bmatrix} X_3(0) \\ X_3(1) \end{bmatrix} = F_2 \begin{bmatrix} x(0) \\ x(4) \end{bmatrix} \tag{7.80}$$

$$\begin{bmatrix} X_4(0) \\ X_4(1) \end{bmatrix} = F_2 \begin{bmatrix} x(2) \\ x(6) \end{bmatrix} \tag{7.81}$$

Using a similar idea for the terms X_2,

$$\begin{bmatrix} X_5(0) \\ X_5(1) \end{bmatrix} = F_2 \begin{bmatrix} x(1) \\ x(5) \end{bmatrix} \tag{7.82}$$

$$\begin{bmatrix} X_6(0) \\ X_6(1) \end{bmatrix} = F_2 \begin{bmatrix} x(3) \\ x(7) \end{bmatrix} \tag{7.83}$$

But observe that ,

$$\vec{P}_8\vec{x} = \begin{pmatrix}\vec{x}_1\\ \vec{x}_2\end{pmatrix} \tag{7.84}$$

$$\vec{P}_4\vec{x}_1 = \begin{pmatrix}\vec{x}_3\\ \vec{x}_4\end{pmatrix} \tag{7.85}$$

$$\vec{P}_4\vec{x}_2 = \begin{pmatrix}\vec{x}_5\\ \vec{x}_6\end{pmatrix} \tag{7.86}$$

where we define $x_3(k) = x(4k)$, $x_4(k) = x(4k+2)$, $x_5(k) = x(4k+1)$, and $x_6(k) = x(4k+3)$ for $k = 0, 1$.

11. For

$$\vec{x} = \begin{pmatrix}1\\2\\3\\4\\2\\1\end{pmatrix} \tag{7.87}$$

compte the DFT using (7.52)

Solution:

$$\vec{F}_6 = \begin{bmatrix}1 & 1 & 1 & 1 & 1 & 1\\ 1 & e^{-j\pi/3} & e^{-j2\pi/3} & e^{-j\pi} & e^{-j4\pi/3} & e^{-j5\pi/3}\\ 1 & e^{-j2\pi/3} & e^{-j4\pi/3} & e^{-j2\pi} & e^{-j8\pi/3} & e^{-j10\pi/3}\\ 1 & e^{-j\pi} & e^{-j2\pi} & e^{-j3\pi} & e^{-j4\pi} & e^{-j5\pi}\\ 1 & e^{-j4\pi/3} & e^{-j8\pi/3} & e^{-j4\pi} & e^{-j16\pi/3} & e^{-j20\pi/3}\\ 1 & e^{-j5\pi/3} & e^{-j10\pi/3} & e^{-j5\pi} & e^{-j20\pi/3} & e^{-j25\pi/3}\end{bmatrix} \tag{7.88}$$

Using (7.87),

$$\vec{X} = \vec{F}_6\vec{x} \tag{7.89}$$

$$\vec{X} = \begin{bmatrix}1 & 1 & 1 & 1 & 1 & 1\\ 1 & e^{-j\pi/3} & e^{-j2\pi/3} & e^{-j\pi} & e^{-j4\pi/3} & e^{-j5\pi/3}\\ 1 & e^{-j2\pi/3} & e^{-j4\pi/3} & e^{-j2\pi} & e^{-j8\pi/3} & e^{-j10\pi/3}\\ 1 & e^{-j\pi} & e^{-j2\pi} & e^{-j3\pi} & e^{-j4\pi} & e^{-j5\pi}\\ 1 & e^{-j4\pi/3} & e^{-j8\pi/3} & e^{-j4\pi} & e^{-j16\pi/3} & e^{-j20\pi/3}\\ 1 & e^{-j5\pi/3} & e^{-j10\pi/3} & e^{-j5\pi} & e^{-j20\pi/3} & e^{-j25\pi/3}\end{bmatrix}\begin{pmatrix}1\\2\\3\\4\\2\\1\end{pmatrix} \tag{7.90}$$

$$= \begin{bmatrix}13\\ -4-\sqrt{3}j\\ 1\\ -1\\ 1\\ -4+\sqrt{3}j\end{bmatrix} \tag{7.91}$$

12. Repeat the above exercise using the FFT after zero padding $\vec{x}$.

Solution: $\vec{x}$ after padding is

$$\begin{pmatrix}1\\2\\3\\4\\2\\1\\0\\0\end{pmatrix} \tag{7.92}$$

Using (7.12),

$$\vec{F}_8 = \begin{bmatrix}\vec{I}_4 & \vec{D}_4\\ \vec{I}_4 & -\vec{D}_4\end{bmatrix}\begin{bmatrix}\vec{F}_4 & 0\\ 0 & \vec{F}_4\end{bmatrix}\vec{P}_8 \tag{7.93}$$

$$\vec{F}_4 = \begin{bmatrix}\vec{I}_2 & \vec{D}_2\\ \vec{I}_2 & -\vec{D}_2\end{bmatrix}\begin{bmatrix}\vec{F}_2 & 0\\ 0 & \vec{F}_2\end{bmatrix}\vec{P}_4 \tag{7.94}$$

$$\vec{F}_2 = \begin{bmatrix}\vec{I}_1 & \vec{D}_1\\ \vec{I}_1 & -\vec{D}_1\end{bmatrix}\begin{bmatrix}\vec{F}_1 & 0\\ 0 & \vec{F}_1\end{bmatrix}\vec{P}_2 \tag{7.95}$$

$$\vec{F}_1 = [1] \tag{7.96}$$

Calculating $\vec{F}_2$,

$$\vec{F}_2 = \begin{bmatrix}\vec{F}_1 & D_1\vec{F}_1\\ \vec{F}_1 & -D_1\vec{F}_1\end{bmatrix}\vec{P}_2 \tag{7.97}$$

$$= \begin{bmatrix}1 & 1\\ 1 & -1\end{bmatrix} \tag{7.98}$$

Calculating $\vec{F}_4$,

$$\vec{D}_2 = diag(1, W_4) = \begin{bmatrix}1 & 0\\ 0 & -j\end{bmatrix} \tag{7.99}$$

$$D_2\vec{F}_2 = \begin{bmatrix}1 & 0\\ 0 & -j\end{bmatrix}\begin{bmatrix}1 & 1\\ 1 & -1\end{bmatrix} \tag{7.100}$$

$$= \begin{bmatrix}1 & 1\\ -j & j\end{bmatrix} \tag{7.101}$$

$$\vec{F}_4 = \begin{bmatrix}\vec{F}_2 & D_2\vec{F}_2\\ \vec{F}_2 & -D_2\vec{F}_2\end{bmatrix}\vec{P}_4 \tag{7.102}$$

$$\vec{F}_4 = \begin{bmatrix}1 & 0 & 1 & 1\\ 0 & 1 & -j & j\\ 1 & 0 & -1 & -1\\ 0 & 1 & j & -j\end{bmatrix}\begin{bmatrix}1 & 0 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 0 & 1\end{bmatrix} \tag{7.103}$$

$$= \begin{bmatrix}1 & 1 & 0 & 1\\ 0 & -j & 1 & j\\ 1 & -1 & 0 & j\\ 0 & j & 1 & -j\end{bmatrix} \tag{7.104}$$

Calculating $\vec{F}_8$,

$$\vec{D}_4 = diag\left(1, W_8, W_8^2, W_8^3\right) \tag{7.105}$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1-j}{\sqrt{2}} & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & \frac{-1-j}{\sqrt{2}} \end{bmatrix} \tag{7.106}$$

$$D_4\vec{F}_4 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1-j}{\sqrt{2}} & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & \frac{-1-j}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & -j & 1 & j \\ 1 & -1 & 0 & j \\ 0 & j & 1 & -j \end{bmatrix} \tag{7.107}$$

$$= \begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & \frac{-1-j}{\sqrt{2}} & \frac{1-j}{\sqrt{2}} & \frac{1+j}{\sqrt{2}} \\ -1 & 1 & 0 & -j \\ 0 & \frac{1-j}{\sqrt{2}} & \frac{-1-j}{\sqrt{2}} & \frac{-1+j}{\sqrt{2}} \end{bmatrix} \tag{7.108}$$

$F_8 = A\vec{B}P_8$ where

$$\vec{A} = \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & \frac{-1-j}{\sqrt{2}} & \frac{1-j}{\sqrt{2}} & \frac{1+j}{\sqrt{2}} \\ 0 & 0 & 1 & 0 & -1 & 1 & 0 & -j \\ 0 & 0 & 0 & 1 & 0 & \frac{1-j}{\sqrt{2}} & \frac{-1-j}{\sqrt{2}} & \frac{-1+j}{\sqrt{2}} \\ 1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & \frac{1+j}{\sqrt{2}} & \frac{-1+j}{\sqrt{2}} & \frac{-1-j}{\sqrt{2}} \\ 0 & 0 & 1 & 0 & 1 & -1 & 0 & j \\ 0 & 0 & 0 & 1 & 0 & \frac{-1+j}{\sqrt{2}} & \frac{1+j}{\sqrt{2}} & \frac{1-j}{\sqrt{2}} \end{bmatrix} \tag{7.109}$$

$$\vec{B} = \begin{bmatrix} 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & -j & 1 & j & 0 & 0 & 0 & 0 \\ 1 & -1 & 0 & j & 0 & 0 & 0 & 0 \\ 0 & j & 1 & -j & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & -1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & j & -1 & j \\ 0 & 0 & 0 & 0 & -1 & 1 & 0 & -j \\ 0 & 0 & 0 & 0 & 0 & -j & -1 & j \end{bmatrix} \tag{7.110}$$

And $\vec{P}_8$ is a permutation matrix.

$$\vec{F}_8 = \begin{bmatrix} 13 \\ -3.12 - 6.53j \\ j \\ 1.12 - 0.53j \\ -1 \\ 1.12 + 0.53j \\ -j \\ -3.12 + 6.5355 \end{bmatrix} \tag{7.111}$$

13. Write a C program to compute the 8-point FFT.
Solution: Downlode the C code

```
wget https://raw.githubusercontent.com/
    kumarsuraj151/EE3900/master/codes/7
    _FFt/7_13.c
```

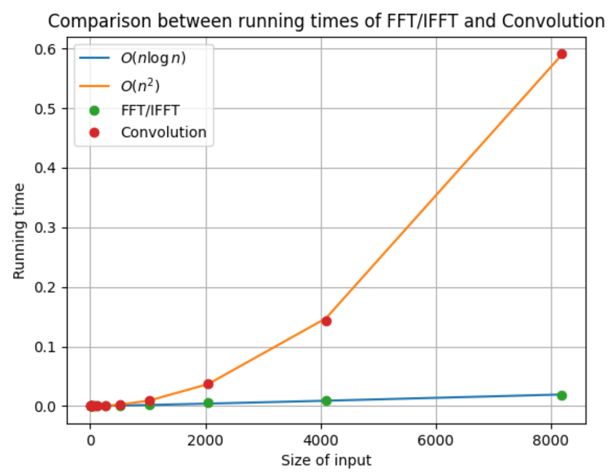

Fig. 13

8 Exercises

Answer the following questions by looking at the python code in Problem 2.3.

8.1 The command

```
output_signal = signal.lfilter(b, a,
    input_signal)
```

in Problem 2.3 is executed through the following difference equation

$$\sum_{m=0}^{M} a(m)\, y(n-m) = \sum_{k=0}^{N} b(k)\, x(n-k) \tag{8.1}$$

where the input signal is $x(n)$ and the output signal is $y(n)$ with initial values all 0. Replace

signal.filtfilt with your own routine and verify.
Solution: On taking the Z-transform on both sides of the difference equation

$$\sum_{m=0}^{M} a(m) z^{-m} Y(z) = \sum_{k=0}^{N} b(k) z^{-k} X(z) \quad (8.2)$$

$$\implies H(z) = \frac{Y(z)}{X(z)} = \frac{\sum_{k=0}^{N} b(k) z^{-k}}{\sum_{m=0}^{M} a(m) z^{-m}} \quad (8.3)$$

For obtaining the discrete Fourier transform, put $z = \jmath\frac{2\pi i}{I}$ where I is the length of the input signal and $i = 0, 1, \ldots, I-1$
Download the following Python code that does the above

```
wget https://github.com/karthik6281/Signal−
    Processing/tree/main/Assignment1/codes/8
    _1.c
```

Run the code by executing

```
python3 8_1.py
```

8.2 Repeat all the exercises in the previous sections for the above a and b.
8.3 What is the sampling frequency of the input signal?
Solution: Sampling frequency(fs)=44.1kHZ.
8.4 What is type, order and cutoff-frequency of the above butterworth filter
Solution: The given butterworth filter is low pass with order=2 and cutoff-frequency=4kHz.
8.5 Modifying the code with different input parameters and to get the best possible output.