Digital Signal Processing Circuits and Transforms

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Contents

1. Definitions

1. The unit step function is

$$u(t) = \begin{cases} 1 & t > 0 \\ \frac{1}{2} & t = 0 \\ 0 & t < 0 \end{cases} \tag{1.1}$$

2. The Laplace transform of $g(t)$ is defined as

$$G(s) = \int_{-\infty}^{\infty} g(t)e^{-st}\, dt \tag{1.2}$$

2. Laplace Transform

1. In the circuit, the switch S is connected to position P for a long time so that the charge on the capacitor becomes $q_1\,\mu C$. Then S is switched to position Q. After a long time, the charge on the capacitor is $q_2\,\mu C$.

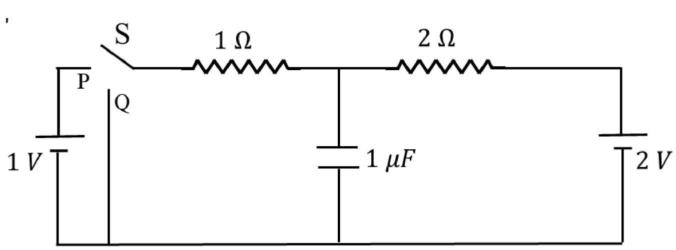

Fig. 2.1.

2. Draw the circuit using latex-tikz.
Solution: The following code yields Fig.2.2

```
wget 2.2.tex
```

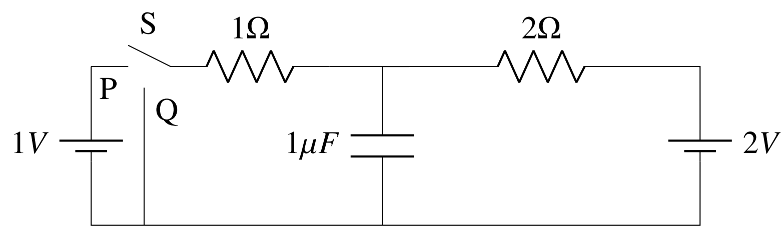

Fig. 2.2. Given Circuit

Fig. 2.3. Before switching S to Q

3. Find q_1.
Solution: Before switching S to Q: Calculating current,

$$1 - i - 2i - 2 = 0 \tag{2.1}$$

$$3i = -1 \Rightarrow i = \frac{-1}{3} \tag{2.2}$$

Potential Difference between capacitor at steady state is

$$1 - \left(\frac{-1}{3}\right) = \frac{4}{3} \tag{2.3}$$

$$q_1 = \frac{4}{3} \cdot 1 \tag{2.4}$$

$$= \frac{4}{3}\mu C \tag{2.5}$$

4. Show that the Laplace transform of $u(t)$ is $\frac{1}{s}$ and find the ROC.
Solution: We know that from definition of Laplace Transform,

$$F(s) = \int_{0}^{\infty} f(t)e^{-st}\, dt \tag{2.6}$$

for $u(t)$, we have

$$U(s) = \int_{0}^{\infty} u(t)e^{-st}\, dt \tag{2.7}$$

Using (1.1),

$$U(s) = \int_0^{\infty} u(t)e^{-st}\,dt \tag{2.8}$$
$$= \int_0^{\infty} e^{-st}\,dt \tag{2.9}$$
$$= -\left(0 - \frac{1}{s}\right) \tag{2.10}$$
$$= \frac{1}{s} \tag{2.11}$$

ROC is $Re(s) > 0$ since $e^{-st} < \infty$ for $t \to \infty$ The following command plots the ROC of above Laplace Transform.

```
wget 2.4.py
```

Fig. 2.4.

5. Show that

$$e^{-at}u(t) \overset{\mathcal{L}}{\longleftrightarrow} \frac{1}{s+a}, \quad a > 0 \tag{2.12}$$

and find the ROC. **Solution:** From (2.6),

$$F(s) = \int_0^{\infty} u(t)e^{-at}e^{-st}\,dt \tag{2.13}$$
$$= \int_0^{\infty} u(t)e^{-(s+a)t}\,dt \tag{2.14}$$
$$= \int_0^{\infty} e^{-(s+a)t}\,dt \tag{2.15}$$
$$= -\left(0 - \frac{1}{s+a}\right) \tag{2.16}$$
$$= \frac{1}{s+a} \tag{2.17}$$

ROC is

$$s + a > 0 \Rightarrow s > -a \tag{2.18}$$

The following command plots the ROC of above Laplace Transform.

```
wget 2.5.py
```

Fig. 2.5.

6. Now consider the following resistive circuit transformed from Fig. 2.8 where

$$u(t) \overset{\mathcal{L}}{\longleftrightarrow} V_1(s) \tag{2.19}$$
$$2u(t) \overset{\mathcal{L}}{\longleftrightarrow} V_2(s) \tag{2.20}$$

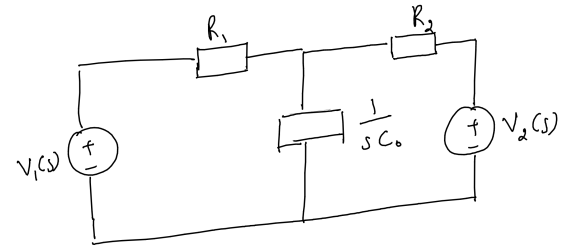

Fig. 2.6.

Find the voltage across the capacitor $V_{C_0}(s)$. **Solution:**

$$R_{eff} = \frac{1}{1+\frac{1}{2}} = \frac{2}{3}\Omega \tag{2.21}$$
$$V_{eff} = \frac{1}{1+\frac{1}{2}} = \frac{2}{3}V \tag{2.22}$$

$$V_{C_0}(s) = V_S(s)\frac{C_0}{C_0 + R_{eff}} \tag{2.23}$$
$$= \left(\frac{4}{3s}\right)\left(\frac{\frac{1}{s}}{\frac{1}{s}+\frac{2}{3}}\right) \tag{2.24}$$
$$= \frac{6}{s(4s+9)} \tag{2.25}$$

7. Find $v_{C_0}(t)$. Plot using python. **Solution:** Using (2.25),

$$\frac{6}{s(4s+9)} = \frac{4}{3s} - \frac{2}{9+4s} \tag{2.26}$$

Apply inverse Laplacian Transform,

$$V_{C_0}(s) \overset{\mathcal{L}^{-\infty}}{\longleftrightarrow} V_{C_0}(t) \tag{2.27}$$
$$\mathcal{L}^{-1}\left[V_{C_0}(s)\right] = \mathcal{L}^{-1}\left[\frac{4}{3s} - \frac{2}{9+4s}\right] \tag{2.28}$$
$$= \mathcal{L}^{-1}\left[\frac{4}{3s}\right] - \mathcal{L}^{-1}\left[\frac{2}{9+4s}\right] \tag{2.29}$$

Since,

$$\mathcal{L}^{-1}\left[\frac{1}{s}\right] = u(t) \tag{2.30}$$
$$\mathcal{L}^{-1}\left[\frac{1}{s-a}\right] = e^{at}u(t) \tag{2.31}$$

Using the above equations,

$$V_{C_0}(t) = \frac{4}{3}\left(1 - e^{\frac{-3}{2}t}\right)u(t) \tag{2.32}$$

The following command plots the above equation.

```
wget 2.7.py
```

Fig. 2.7. Plot of $V_{C_0}(t)$

8. Verify your result using ngspice. **Solution:** The following command plots the ROC of above Laplace Transform.

```
wget
```

Fig. 2.8.

3. Initial Conditions

1. Find q_2 in Fig. 2.8.
Solution: At steady state, $V_{C_0} = V_{1\Omega}$

$$V_{C_0} = \frac{q_2}{C} = V_{1\Omega} = \frac{2}{1+2} = \frac{2}{3}$$
$$q_2 = \frac{2}{3}\mu C$$

2. Draw the equivalent s-domain resistive circuit when S is switched to position Q. Use variables R_1, R_2, C_0 for the passive elements. Use latex-tikz. **Solution:** The following command plots the ROC of above Laplace Transform.

```
wget https://raw.githubusercontent.com/
    LokeshBadisa/EE3900−Linear−Systems−
    and−Signal−Processing/main/Circuits/
    Tikz Circuits/3.2.tex
```

Fig. 3.1. After switching S to Q

3. $V_{C_0}(s) = ?$
Solution: Using KCL at node in Fig. 3.1

$$\frac{V-0}{R_1} + \frac{V-\frac{2}{s}}{R_2} + sC_0\left(V - \frac{4}{3s}\right) = 0 \tag{3.1}$$

$$\implies V_{C_0}(s) = \frac{\frac{2}{sR_2} + \frac{4C_0}{3}}{\frac{1}{R_1} + \frac{2}{R_2} + sC_0} \tag{3.2}$$

4. $v_{C_0}(t) = ?$ Plot using python.
Solution: From (3.2),

$$V_{C_0}(s) = \frac{4}{3}\left(\frac{1}{\frac{1}{C_0}\left(\frac{1}{R_1}+\frac{1}{R_2}\right)+s}\right) + \frac{2}{R_2\left(\frac{1}{R_1}+\frac{1}{R_2}\right)}\left(\frac{1}{s} - \frac{1}{\frac{1}{C_0}\left(\frac{1}{R_1}+\frac{1}{R_2}\right)+s}\right) \tag{3.3}$$

Taking an inverse Laplace Transform,

$$v_{C_0}(t) = \frac{4}{3}e^{-\left(\frac{1}{R_1}+\frac{1}{R_2}\right)\frac{t}{C_0}}u(t) + \frac{2}{R_2\left(\frac{1}{R_1}+\frac{1}{R_2}\right)}\left(1 - e^{-\left(\frac{1}{R_1}+\frac{1}{R_2}\right)\frac{t}{C_0}}\right)u(t) \tag{3.4}$$

Substituting values gives

$$v_{C_0}(t) = \frac{2}{3}\left(1 + e^{-\left(1.5\times10^6\right)t}\right)u(t) \tag{3.5}$$

The following command plots the above equation.

```
wget https://raw.githubusercontent.com/
    LokeshBadisa/EE3900−Linear−Systems−
    and−Signal−Processing/main/Circuits/
    codes/3.4.py
python3 3.4.py
```

Fig. 3.2. Plot of $V_{C_0}(t)$

5. Verify your result using ngspice. **Solution:** The following command plots Fig.3.3

```
wget
```

Fig. 3.3.

6. Find $v_{C_0}(0-), v_{C_0}(0+)$ and $v_{C_0}(\infty)$.
Solution: From the initial conditions,

$$v_{C_0}(0-) = \frac{q_1}{C} = \frac{4}{3}V \tag{3.6}$$

Using (3.5),

$$v_{C_0}(0+) = \lim_{t\to0+} v_{C_0}(t) = \frac{4}{3}V \tag{3.7}$$

$$v_{C_0}(\infty) = \lim_{t\to\infty} v_{C_0}(t) = \frac{2}{3}V \tag{3.8}$$