Digital Signal Processing Circuits and Transforms

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CONTENTS

1. DEFINITIONS

1. The unit step function is

$$u(t) = \begin{cases} 1 & t > 0 \\ \frac{1}{2} & t = 0 \\ 0 & t < 0 \end{cases} \qquad (1.1)$$

2. The Laplace transform of $g(t)$ is defined as

$$G(s) = \int_{-\infty}^{\infty} g(t)e^{-st}\,dt \qquad (1.2)$$

2. LAPLACE TRANSFORM

1. In the circuit, the switch S is connected to position P for a long time so that the charge on the capacitor becomes $q_1\,\mu C$. Then S is switched to position Q. After a long time, the charge on the capacitor is $q_2\,\mu C$.
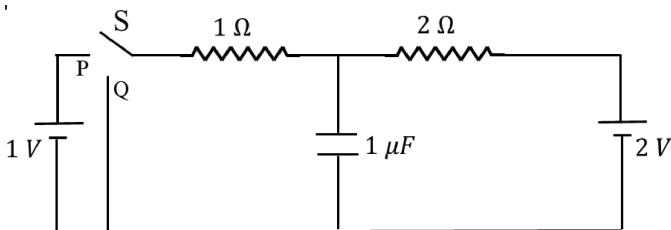

Fig. 2.1.

2. Draw the circuit using latex-tikz.
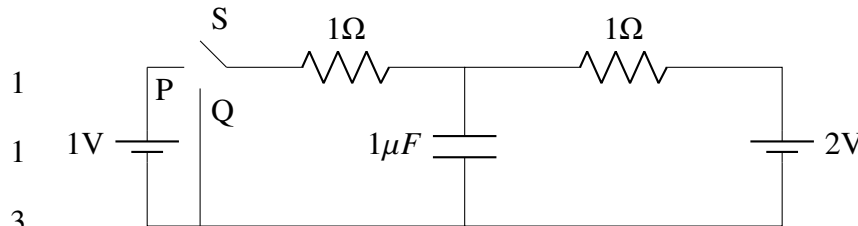   **Solution:** The following code yields Fig.2.2
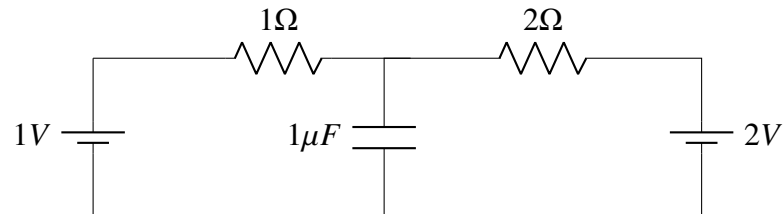

Fig. 2.2. Given Circuit

Fig. 2.3. Before switching S to Q

> wget 2.2.tex

3. Find q_1.
   Solution: Before switching S to Q: Calculating current,

$$1 - i - 2i - 2 = 0 \qquad (2.1)$$

$$3i = -1 \Rightarrow i = \frac{-1}{3} \qquad (2.2)$$

Potential Difference between capacitor at steady state is

$$1 - \left(\frac{-1}{3}\right) = \frac{4}{3} \qquad (2.3)$$

$$q_1 = \frac{4}{3} \cdot 1 \qquad (2.4)$$

$$= \frac{4}{3}\mu C \qquad (2.5)$$

4. Show that the Laplace transform of $u(t)$ is $\frac{1}{s}$ and find the ROC.

Solution: We know that from definition of Laplace Transform,

$$F(s) = \int_0^\infty f(t)e^{-st}\, dt \qquad (2.6)$$

for $u(t)$, we have

$$U(s) = \int_0^\infty u(t)e^{-st}\, dt \qquad (2.7)$$

Using (1.1),

$$U(s) = \int_0^\infty u(t)e^{-st}\, dt \qquad (2.8)$$

$$= \int_0^\infty e^{-st}\, dt \qquad (2.9)$$

$$= -\left(0 - \frac{1}{s}\right) \qquad (2.10)$$

$$= \frac{1}{s} \qquad (2.11)$$

ROC is $Re(s) > 0$ since $e^{-st} < \infty$ for $t \to \infty$ The following command plots the ROC of above Laplace Transform.

wget 2.4.py

Fig. 2.4.

5. Show that

$$e^{-at}u(t) \xleftrightarrow{\mathcal{L}} \frac{1}{s+a}, \quad a > 0 \qquad (2.12)$$

and find the ROC. **Solution:** From (2.6),

$$F(s) = \int_0^\infty u(t)e^{-at}e^{-st}\, dt \qquad (2.13)$$

$$= \int_0^\infty u(t)e^{-(s+a)t}\, dt \qquad (2.14)$$

$$= \int_0^\infty e^{-(s+a)t}\, dt \qquad (2.15)$$

$$= -\left(0 - \frac{1}{s+a}\right) \qquad (2.16)$$

$$= \frac{1}{s+a} \qquad (2.17)$$

ROC is

$$s + a > 0 \Rightarrow s > -a \qquad (2.18)$$

The following command plots the ROC of above Laplace Transform.

Fig. 2.5.

wget 2.5.py

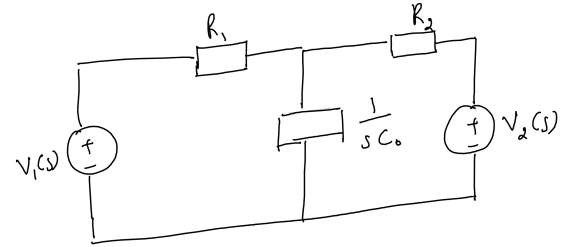6. Now consider the following resistive circuit transformed from Fig. 2.8 where

Fig. 2.6.

$$u(t) \xleftrightarrow{\mathcal{L}} V_1(s) \qquad (2.19)$$

$$2u(t) \xleftrightarrow{\mathcal{L}} V_2(s) \qquad (2.20)$$

Find the voltage across the capacitor $V_{C_0}(s)$.

Solution:

$$R_{eff} = \frac{1}{1 + \frac{1}{2}} = \frac{2}{3}\Omega \qquad (2.21)$$

$$V_{eff} = \frac{1}{1 + \frac{1}{2}} = \frac{2}{3}V \qquad (2.22)$$

$$V_{C_0}(s) = V_S(s)\frac{C_0}{C_0 + R_{eff}} \qquad (2.23)$$

$$= \left(\frac{4}{3s}\right)\left(\frac{\frac{1}{s}}{\frac{1}{s} + \frac{2}{3}}\right) \qquad (2.24)$$

$$= \frac{6}{s(4s + 9)} \qquad (2.25)$$

7. Find $v_{C_0}(t)$. Plot using python.

Solution: Using (2.25),

$$\frac{6}{s(4s + 9)} = \frac{4}{3s} - \frac{2}{9 + 4s} \qquad (2.26)$$

Apply inverse Laplacian Transform,

$$V_{C_0}(s) \xleftrightarrow{\mathcal{L}^{-\infty}} V_{C_0}(t) \qquad (2.27)$$

$$\mathcal{L}^{-1}\left[V_{C_0}(s)\right] = \mathcal{L}^{-1}\left[\frac{4}{3s} - \frac{2}{9 + 4s}\right] \qquad (2.28)$$

$$= \mathcal{L}^{-1}\left[\frac{4}{3s}\right] - \mathcal{L}^{-1}\left[\frac{2}{9 + 4s}\right] \qquad (2.29)$$

Since,

$$\mathcal{L}^{-1}\left[\frac{1}{s}\right] = u(t) \qquad (2.30)$$

$$\mathcal{L}^{-1}\left[\frac{1}{s-a}\right] = e^{at}u(t) \qquad (2.31)$$

Using the above equations,

$$V_{C_0}(t) = \frac{4}{3}\left(1 - e^{\frac{-3}{2}t}\right)u(t) \qquad (2.32)$$

The following command plots the above equation.

```
wget 2.7.py
```

Fig. 2.7. Plot of $V_{C_0}(t)$

8. Verify your result using ngspice.
   Solution: The following command plots the ROC of above Laplace Transform.

```
wget
```

Fig. 2.8.

3. INITIAL CONDITIONS

1. Find q_2 in Fig. 2.8.
   Solution: At steady state, $V_{C_0} = V_{1\Omega}$

$$V_{C_0} = \frac{q_2}{C} = V_{1\Omega} = \frac{2}{1+2} = \frac{2}{3}$$

$$q_2 = \frac{2}{3}\mu C$$

2. Draw the equivalent s-domain resistive circuit when S is switched to position Q. Use variables R_1, R_2, C_0 for the passive elements. Use latex-tikz. **Solution:** The following command plots the ROC of above Laplace Transform.

```
wget
```

Fig. 3.1. After switching S to Q

3. $V_{C_0}(s) = ?$
   Solution: Using KCL at node in Fig. 3.1

$$\frac{V-0}{R_1} + \frac{V-\frac{2}{s}}{R_2} + sC_0\left(V - \frac{4}{3s}\right) = 0 \qquad (3.1)$$

$$\implies V_{C_0}(s) = \frac{\frac{2}{sR_2} + \frac{4C_0}{3}}{\frac{1}{R_1} + \frac{2}{R_2} + sC_0} \qquad (3.2)$$

4. $v_{C_0}(t) = ?$ Plot using python.
   Solution: From (3.2),

$$V_{C_0}(s) = \frac{4}{3}\left(\frac{1}{\frac{1}{C_0}\left(\frac{1}{R_1} + \frac{1}{R_2}\right) + s}\right)$$

$$+ \frac{2}{R_2\left(\frac{1}{R_1} + \frac{1}{R_2}\right)}\left(\frac{1}{s} - \frac{1}{\frac{1}{C_0}\left(\frac{1}{R_1} + \frac{1}{R_2}\right) + s}\right) \qquad (3.3)$$

Taking an inverse Laplace Transform,

$$v_{C_0}(t) = \frac{4}{3}e^{-\left(\frac{1}{R_1} + \frac{1}{R_2}\right)\frac{t}{C_0}}u(t)$$

$$+ \frac{2}{R_2\left(\frac{1}{R_1} + \frac{1}{R_2}\right)}\left(1 - e^{-\left(\frac{1}{R_1} + \frac{1}{R_2}\right)\frac{t}{C_0}}\right)u(t) \qquad (3.4)$$

Substituting values gives

$$v_{C_0}(t) = \frac{2}{3}\left(1 + e^{-\left(1.5\times10^6\right)t}\right)u(t) \qquad (3.5)$$

The following command plots the above equation.

```
wget
```

Fig. 3.2. Plot of $V_{C_0}(t)$

5. Verify your result using ngspice. **Solution:** The following command plots Fig.3.3

```
wget
```

Fig. 3.3.

6. Find $v_{C_0}(0-)$, $v_{C_0}(0+)$ and $v_{C_0}(\infty)$.
   Solution: From the initial conditions,

$$v_{C_0}(0-) = \frac{q_1}{C} = \frac{4}{3}V \qquad (3.6)$$

Using (3.5),

$$v_{C_0}(0+) = \lim_{t\to0+} v_{C_0}(t) = \frac{4}{3}V \qquad (3.7)$$

$$v_{C_0}(\infty) = \lim_{t\to\infty} v_{C_0}(t) = \frac{2}{3}V \qquad (3.8)$$

4. Bilinear Transform

1. In Fig. 2.8, consider the case when S is switched to Q right in the beginning. Formulate the differential equation.
2. Find $H(s)$ considering the ouput voltage at the capacitor.
3. Plot $H(s)$. What kind of filter is it?
4. Using trapezoidal rule for integration, formulate the difference equation by considering

$$y(n) = y(t)|_{t=n} \qquad (4.1)$$

5. Find $H(z)$.
6. How can you obtain $H(z)$ from $H(s)$?