

Digital Signal Processing

Suraj kumar AI21BTECH11029

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Abstract—This manual provides a simple introduction to digital signal processing.

1 SOFTWARE INSTALLATION

Run the following commands

```
sudo apt-get update
sudo apt-get install libffi-dev libsndfile1 python3
    -scipy python3-numpy python3-matplotlib
sudo pip install cffi pysoundfile
```

2 DIGITAL FILTER

2.1 Download the sound file from

```
wget https://github.com/kumarsuraj151/
    EE3900/blob/master/codes/2_digital%20
    filter/Sound_Noise.wav
```

2.2 You will find a spectrogram at <https://academo.org/demos/spectrum-analyzer>. Upload the sound file that you downloaded in Problem 2.1 in the spectrogram and play. Observe the spectrogram. What do you find?

Solution: There are a lot of yellow lines between 440 Hz to 5.1 KHz. These represent the

synthesizer key tones. Also, the key strokes are audible along with background noise.

2.3 Write the python code for removal of out of band noise and execute the code.

Solution:

```
import soundfile as sf
from scipy import signal
#read .wav file
input_signal,fs = sf.read('Sound_Noise.wav'
    )
#sampling frequency of Input signal
sampl_freq=fs
#order of the filter
order=4
#cutoff frquency 4kHz
cutoff_freq=4000.0
#digital frequency
Wn=2*cutoff_freq/sampl_freq
# b and a are numerator and
    denominatorpolynomials respectively
b, a = signal.butter(order,Wn, 'low')
#filter the input signal with butterworth filter
output_signal = signal.filtfilt(b, a,
    input_signal)
#output signal = signal.lfilter(b, a,input
    signal)
#write the output signal into .wav file
sf.write('Sound_With_ReducedNoise.wav',
    output_signal, fs)
```

2.4 The output of the python script in Problem 2.3 is the audio file Sound_With_ReducedNoise.wav. Play the file in the spectrogram in Problem 2.2. What do you observe?

Solution: The key strokes as well as background noise is subdued in the audio. Also, the signal is blank for frequencies above 5.1 kHz.

3 DIFFERENCE EQUATION

3.1 Let

$$x(n) = \left\{ \underset{\uparrow}{1}, 2, 3, 4, 2, 1 \right\} \quad (3.1)$$

Sketch $x(n)$.

3.2 Let

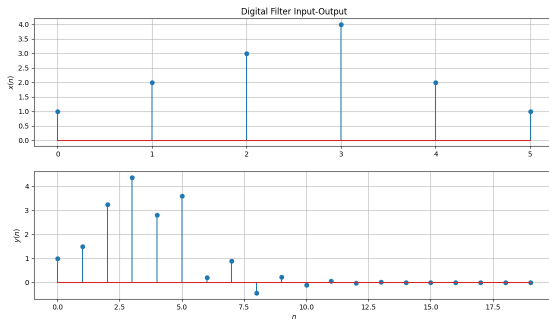
$$y(n) + \frac{1}{2}y(n-1) = x(n) + x(n-2),$$

$$y(n) = 0, n < 0 \quad (3.2)$$

Sketch $y(n)$.

Solution: The following code yields Fig. .

```
wget https://raw.githubusercontent.com/
kumarsuraj151/EE3900/master/codes/3
_difference%20equation/xnyn.py
```



and from (4.14),

$$U(z) = \sum_{n=0}^{\infty} z^{-n} \quad (4.17)$$

$$= \frac{1}{1 - z^{-1}}, \quad |z| > 1 \quad (4.18)$$

using the formula for the sum of an infinite geometric progression.

4.5 Show that

$$a^n u(n) \stackrel{Z}{\Leftrightarrow} \frac{1}{1 - az^{-1}} \quad |z| > |a| \quad (4.19)$$

Solution:

$$Z(a^n u(n)) = \sum_{n=-\infty}^{\infty} a^n u(n) z^{-n} \quad (4.20)$$

$$= \sum_{n=0}^{\infty} a^n z^{-n} \quad (4.21)$$

$$= \frac{1}{1 - az^{-1}}, \quad |az^{-1}| < 1 \quad (4.22)$$

$$= \frac{1}{1 - az^{-1}}, \quad |a| < |z| \quad (4.23)$$

using the formula for the sum of an infinite geometric progression.

4.6 Let

$$H(e^{j\omega}) = H(z = e^{j\omega}). \quad (4.24)$$

Plot $|H(e^{j\omega})|$. Comment. $H(e^{j\omega})$ is known as the *Discrete Time Fourier Transform* (DTFT) of $h(n)$.

Solution: The following code plots Fig. 4.6.

```
wget https://raw.githubusercontent.com/kumarsuraj151/EE3900/master/codes/4_Z%20transform/dtft.py
```

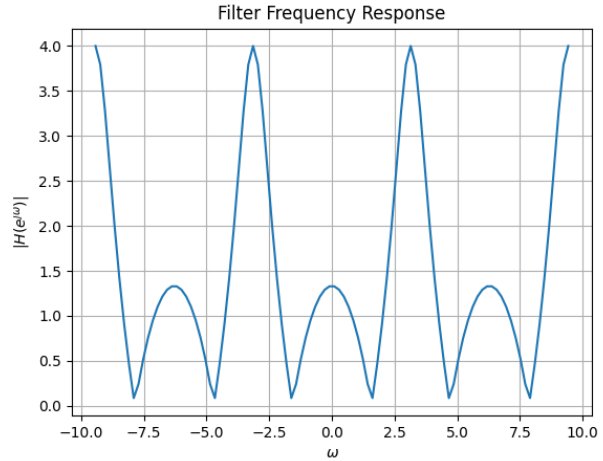


Fig. 4.6: $|H(e^{j\omega})|$

$$H(e^{j\omega}) = \frac{1 + e^{-2j\omega}}{1 + \frac{e^{-j\omega}}{2}} \quad (4.25)$$

$$\Rightarrow |H(e^{j\omega})| = \frac{|1 + e^{-2j\omega}|}{|1 + \frac{e^{-j\omega}}{2}|} \quad (4.26)$$

$$= \frac{|1 + e^{2j\omega}|}{|e^{2j\omega} + \frac{e^{j\omega}}{2}|} \quad (4.27)$$

$$= \frac{|1 + \cos 2\omega + j \sin 2\omega|}{|e^{j\omega} + \frac{1}{2}|} \quad (4.28)$$

$$= \frac{|4 \cos^2(\omega) + 4j \sin(\omega) \cos(\omega)|}{|2e^{j\omega} + 1|} \quad (4.29)$$

$$= \frac{|4 \cos(\omega)| |\cos(\omega) + j \sin(\omega)|}{|2 \cos(\omega) + 1 + 2j \sin(\omega)|} \quad (4.30)$$

$$\therefore |H(e^{j\omega})| = \frac{|4 \cos(\omega)|}{\sqrt{5 + 4 \cos(\omega)}} \quad (4.31)$$

The period of $|\cos(\omega)|$ is π . The period of $5 + 4 \cos(\omega)$ is 2π . Hence $|H(e^{j\omega})|$ is periodic with period π . (The LCM of the period of $|\cos(\omega)|$ and $5 + 4 \cos(\omega)$ is π) The graph of $|H(e^{j\omega})|$ is symmetric with respect to y-axis. It is continuous over ω . The following code plots Fig. 4.6.

4.7 Express $h(n)$ in terms of $H(e^{j\omega})$.

Solution:

$$H(e^{j\omega}) = \sum_{k=-\infty}^{\infty} h(k) e^{-j\omega k} \quad (4.32)$$

and

$$h(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} H(e^{j\omega}) e^{j\omega n} d\omega \quad (4.33)$$

Now,

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} H(e^{j\omega}) e^{j\omega n} d\omega \quad (4.34)$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} \sum_{k=-\infty}^{\infty} h(k) e^{-j\omega k} e^{j\omega n} d\omega \quad (4.35)$$

$$= \frac{1}{2\pi} \sum_{k=-\infty}^{\infty} h(k) \int_{-\pi}^{\pi} e^{j\omega(n-k)} d\omega \quad (4.36)$$

$$= \frac{1}{2\pi} \sum_{k=-\infty}^{\infty} h(k) \int_{-\pi}^{\pi} \cos \omega(n-k) d\omega \quad (4.37)$$

$$+ \int_{-\pi}^{\pi} \sin \omega(n-k) d\omega$$

$$= \frac{1}{2\pi} \sum_{k=-\infty}^{\infty} h(k) \int_{-\pi}^{\pi} \cos \omega(n-k) d\omega \quad (4.38)$$

$$= \frac{1}{2\pi} \sum_{k=-\infty}^{\infty} h(k) \left. \frac{\sin \omega(n-k)}{n-k} \right|_{-\pi}^{\pi} \quad (4.39)$$

$$= \frac{1}{2\pi} \sum_{k \neq n} h(n) \frac{\sin \pi(n-k)}{n-k} + \sum_{k=n} h(n) \frac{\sin \pi(n-k)}{n-k} \quad (4.40)$$

$$= \frac{0 + 2\pi h(n)}{2\pi} \quad (4.41)$$

$$= h(n) \quad (4.42)$$

5 IMPULSE RESPONSE

5.1 Using long division, find

$$h(n), \quad n < 5 \quad (5.1)$$

for $H(z)$ in (4.12).

Solution: $H(z)$ is given by

$$H(z) = \frac{1 + z^{-2}}{1 + \frac{1}{2}z^{-1}} = \frac{2 + 2z^{-2}}{2 + z^{-1}} \quad (5.2)$$

$$\frac{2z^{-1} - 4}{z^{-1} + 2} \quad (5.3)$$

$$2z^{-2} + 2 \quad (5.4)$$

$$\frac{2z^{-2} + 4z^{-1}}{-4z^{-1} + 2} \quad (5.5)$$

$$-4z^{-1} + 2 \quad (5.6)$$

$$-4z^{-1} - 8 \quad (5.7)$$

$$\frac{10}{10} \quad (5.8)$$

So,

$$H(z) = 2z^{-1} - 4 + \frac{10}{z^{-1} + 2} \quad (5.9)$$

$$= 2z^{-1} - 4 + \frac{5}{\frac{1}{2}z^{-1} + 1} \quad (5.10)$$

$$= 2z^{-1} - 4 + 5 \sum_{n=0}^{\infty} \left(-\frac{z^{-1}}{2}\right)^n \quad (5.11)$$

$$= 1 - \frac{1}{2}z^{-1} + \sum_{n=2}^{\infty} \left(-\frac{1}{2}\right)^n z^{-n} \quad (5.12)$$

So, $h(n)$ will be given by

$$h(n) = \begin{cases} 5 \times \left(-\frac{1}{2}\right)^n & n \geq 2 \\ \left(-\frac{1}{2}\right)^n & 2 > n \geq 0 \\ 0 & n < 0 \end{cases} \quad (5.13)$$

5.2 Find an expression for $h(n)$ using $H(z)$, given that

$$h(n) \stackrel{Z}{\rightleftharpoons} H(z) \quad (5.14)$$

and there is a one to one relationship between $h(n)$ and $H(z)$. $h(n)$ is known as the *impulse response* of the system defined by (3.2).

Solution: From (4.12),

$$H(z) = \frac{1}{1 + \frac{1}{2}z^{-1}} + \frac{z^{-2}}{1 + \frac{1}{2}z^{-1}} \quad (5.15)$$

$$\Rightarrow h(n) = \left(-\frac{1}{2}\right)^n u(n) + \left(-\frac{1}{2}\right)^{n-2} u(n-2) \quad (5.16)$$

using (4.19) and (4.6).

5.3 Sketch $h(n)$. Is it bounded? Convergent?

Solution: The following code plots Fig. 5.3.

```
wget https://raw.githubusercontent.com/kumarsuraj151/EE3900/master/codes/5_impluse%20responce/hn.py
```

Yes, it is bounded and Convergent

5.4 Convergent? Justify using the ratio test.

Solution: We can say a given real sequence $\{x_n\}$ is convergent if

$$\lim_{n \rightarrow \infty} \left| \frac{x_{n+1}}{x_n} \right| < 1 \quad (5.17)$$

This is known as Ratio test.

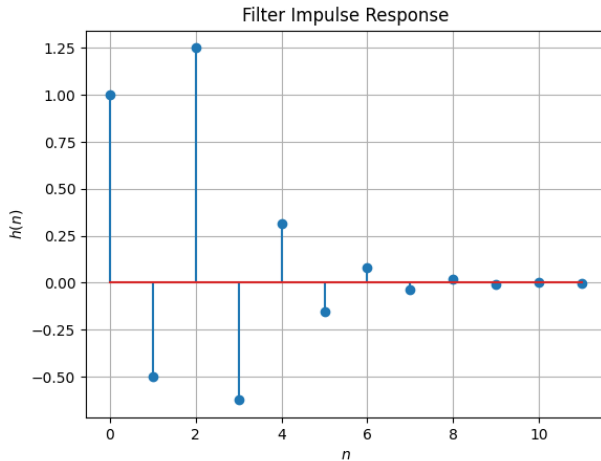


Fig. 5.3: $h(n)$ as the inverse of $H(z)$

In this case the limit will become,

$$\lim_{n \rightarrow \infty} \left| \frac{h(n+1)}{h(n)} \right| = \lim_{n \rightarrow \infty} \left| \frac{5 \left(\frac{-1}{2} \right)^{n+1}}{5 \left(\frac{-1}{2} \right)^n} \right| \quad (5.18)$$

$$= \lim_{n \rightarrow \infty} \left| \frac{-1}{2} \right| \quad (5.19)$$

$$= \frac{1}{2} \quad (5.20)$$

As $\frac{1}{2} < 1$, from root test we can say that $h(n)$ is convergent.

5.5 The system with $h(n)$ is defined to be stable if

$$\sum_{n=-\infty}^{\infty} h(n) < \infty \quad (5.21)$$

Is the system defined by (3.2) stable for the impulse response in (5.14)?

Solution:

$$\sum_{n=-\infty}^{\infty} h(n) = \sum_{n=-\infty}^{\infty} \left(-\frac{1}{2} \right)^n u(n) + \sum_{n=-\infty}^{\infty} \left(-\frac{1}{2} \right)^{n-2} u(n-2) \quad (5.22)$$

$$= \sum_{n=0}^{\infty} \left(-\frac{1}{2} \right)^n + \sum_{n=2}^{\infty} \left(-\frac{1}{2} \right)^{n-2} \quad (5.23)$$

$$= \frac{2}{3} + \frac{2}{3} = \frac{4}{3} \quad (5.24)$$

hence the system defined by (3.2) is stable for the impulse response in (5.14)?

5.6 Verify the above result using python code

Solution: the above result can be verify using

following code

```
wget https://raw.githubusercontent.com/
kumarsuraj151/EE3900/master/codes/5
_impluse%20response/hndef.py
```

5.7 Compute and sketch $h(n)$ using

$$h(n) + \frac{1}{2}h(n-1) = \delta(n) + \delta(n-2), \quad (5.25)$$

This is the definition of $h(n)$.

Solution: The following code plots Fig. 5.7. Note that this is the same as Fig. 5.3.

```
wget https://raw.githubusercontent.com/
kumarsuraj151/EE3900/master/codes/5
_impluse%20response/hndef.py
```

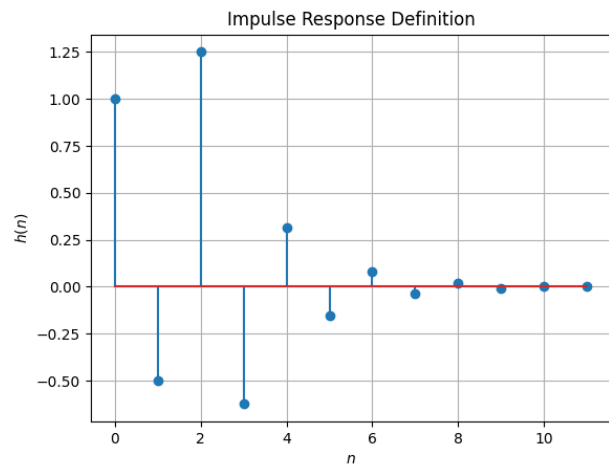


Fig. 5.7: $h(n)$ from the definition

5.8 Compute

$$y(n) = x(n) * h(n) = \sum_{n=-\infty}^{\infty} x(k)h(n-k) \quad (5.26)$$

Comment. The operation in (5.26) is known as *convolution*.

Solution: The following code plots Fig. 5.8. Note that this is the same as $y(n)$ in Fig. 3.2.

```
wget https://raw.githubusercontent.com/
kumarsuraj151/EE3900/master/codes/5
_impluse%20response/ynconv.py
```

5.9 Express the above convolution using a Teoplitz matrix.

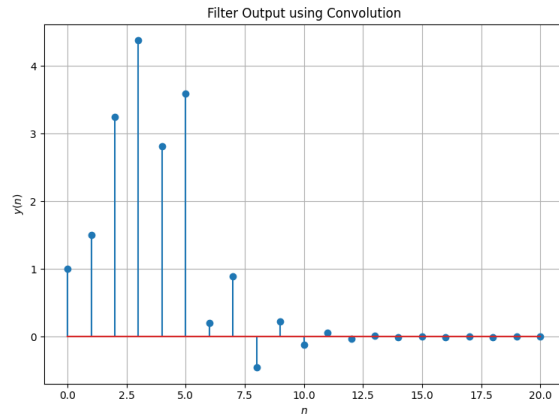


Fig. 5.8: $y(n)$ from the definition of convolution

The same thing can be written as,

$$y(0) = \begin{pmatrix} h(0) & 0 & 0 & . & . & .0 \end{pmatrix} \begin{pmatrix} x(0) \\ x(1) \\ x(2) \\ . \\ . \\ x(5) \end{pmatrix} \quad (5.31)$$

$$y(1) = \begin{pmatrix} h(1) & h(0) & 0 & 0 & . & . & .0 \end{pmatrix} \begin{pmatrix} x(0) \\ x(1) \\ x(2) \\ . \\ . \\ x(5) \end{pmatrix} \quad (5.32)$$

$$y(2) = \begin{pmatrix} h(2) & h(1) & h(0) & 0 & . & .0 \end{pmatrix} \begin{pmatrix} x(0) \\ x(1) \\ x(2) \\ . \\ . \\ x(5) \end{pmatrix} \quad (5.33)$$

·
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Using Toeplitz matrix of $h(n)$ we can simplify it as,

$$y(n) = \begin{pmatrix} h(0) & 0 & 0 & . & . & .0 \\ h(1) & h(0) & 0 & . & . & .0 \\ h(2) & h(1) & h(0) & . & . & .0 \\ . & . & . & . & . & . \\ 0 & 0 & 0 & . & . & h(m-1) \end{pmatrix} \begin{pmatrix} x(0) \\ x(1) \\ x(2) \\ . \\ . \\ x(5) \end{pmatrix} \quad (5.34)$$

Solution: From (5.26), we express $y(n)$ as

$$y(n) = \sum_{k=-\infty}^{\infty} x(k) h(n-k) \quad (5.27)$$

To understand how we can use a Toeplitz matrix

$$y(0) = x(0) h(0) \quad (5.28)$$

$$y(1) = x(0) h(1) + x(1) h(0) \quad (5.29)$$

$$y(2) = x(0) h(2) + x(1) h(1) + x(2) h(0) \quad (5.30)$$

·
·

$$x(n) = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 2 \\ 1 \end{pmatrix} \quad (5.35)$$

And from (5.13) we will take some values of n ,

$$..h(n) = \begin{pmatrix} 1 \\ -0.5 \\ 1.25 \\ . \\ . \end{pmatrix} \quad (5.36)$$

Now using (5.34),

$$y(n) = x(n) * h(n) \quad (5.37)$$

$$= \begin{pmatrix} 1 & 0 & 0 & . & . & .0 \\ -0.5 & 1 & 0 & . & . & .0 \\ 1.25 & -0.5 & 1 & . & . & .0 \\ & & .. & & & \\ & & .. & & & \\ 0 & 0 & 0 & . & . & . \end{pmatrix} \begin{pmatrix} x(0) \\ x(1) \\ x(2) \\ . \\ . \\ x(5) \end{pmatrix} \quad (5.38)$$

$$= \begin{pmatrix} 1 \\ 1.5 \\ 3.25 \\ . \\ . \\ . \end{pmatrix} \quad (5.39)$$

The above equation (5.39) is the convolution of $x(n)$ and $h(n)$

5.10 Show that

$$y(n) = \sum_{k=-\infty}^{\infty} x(n-k)h(k) \quad (5.40)$$

Solution: Substitute $k := n - k$ in (5.26), we will get

$$y(n) = \sum_{k=-\infty}^{\infty} x(k)h(n-k) \quad (5.41)$$

$$= \sum_{n-k=-\infty}^{\infty} x(n-k)h(k) \quad (5.42)$$

$$= \sum_{k=-\infty}^{\infty} x(n-k)h(k) \quad (5.43)$$

6 DFT

6.1 Compute

$$X(k) \triangleq \sum_{n=0}^{N-1} x(n)e^{-j2\pi kn/N}, \quad k = 0, 1, \dots, N-1 \quad (6.1)$$

and $H(k)$ using $h(n)$.

Solution: Downloade the python code for questions 6.1

```
wget https://raw.githubusercontent.com/kumarsuraj151/EE3900/master/codes/6_dft%20and%20fft/6_1.py
```

6.2 Compute

$$Y(k) = X(k)H(k) \quad (6.2)$$

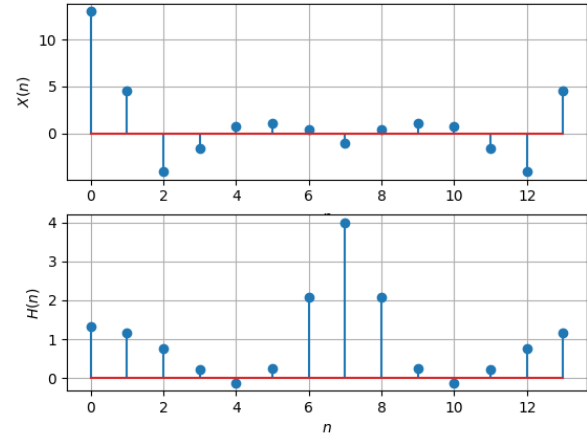


Fig. 6.1: Plot of real part of Discrete Fourier Transforms of $x(n)$ and $h(n)$

Solution: download the python code for questions 6.2

```
wget https://raw.githubusercontent.com/kumarsuraj151/EE3900/master/codes/6_dft%20and%20fft/6_2.py
```

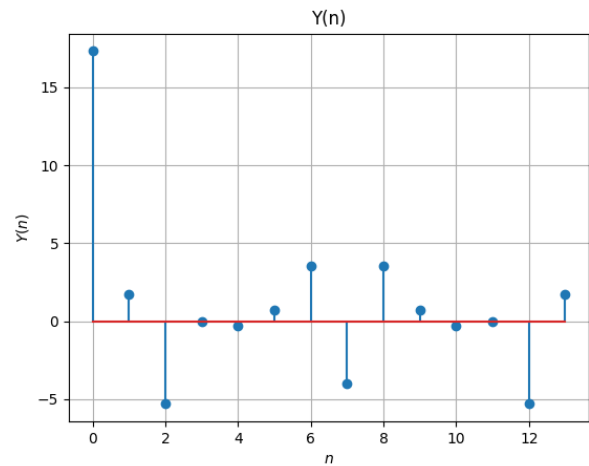


Fig. 6.2: $Y(k)$ as the product of $X(k)$ and $H(k)$

6.3 Compute

$$y(n) = \frac{1}{N} \sum_{k=0}^{N-1} Y(k) \cdot e^{j2\pi kn/N}, \quad n = 0, 1, \dots, N-1 \quad (6.3)$$

Solution: The following code plots Fig. 5.8. Note that this is the same as $y(n)$ in Fig. 3.2.

```
wget https://raw.githubusercontent.com/kumarsuraj151/EE3900/master/codes/6_dft%20and%20fft/yn_dft.py
```

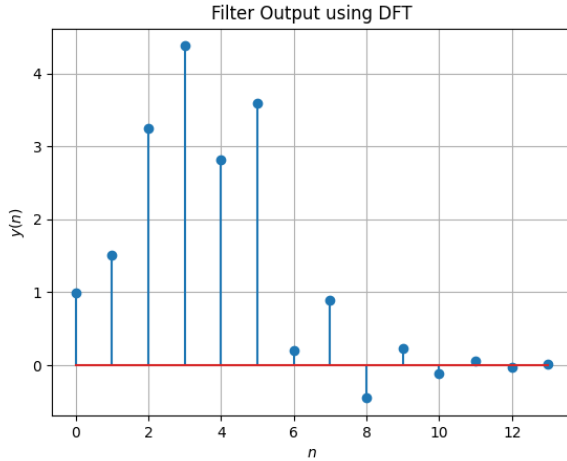


Fig. 6.3: $y(n)$ from the DFT

6.4 Repeat the previous exercise by computing $X(k)$, $H(k)$ and $y(n)$ through FFT and IFFT.

Solution:

```
wget https://raw.githubusercontent.com/kumarsuraj151/EE3900/master/codes/6_dft%20and%20fft/yn_ifft.py
```

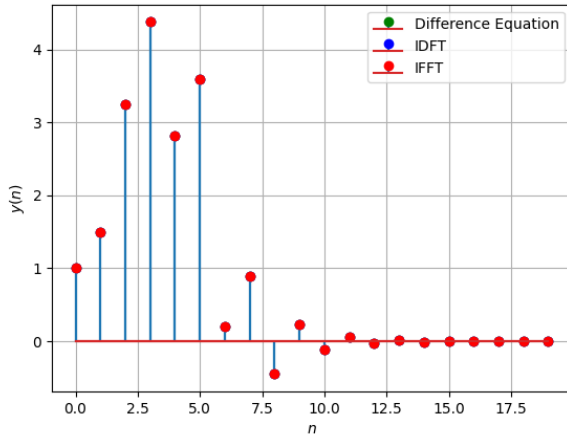


Fig. 6.4: The plot of $y(n)$ using IFFT

7 FFT

1. The DFT of $x(n)$ is given by

$$X(k) \triangleq \sum_{n=0}^{N-1} x(n)e^{-j2\pi kn/N}, \quad k = 0, 1, \dots, N-1 \quad (7.1)$$

2. Let

$$W_N = e^{-j2\pi/N} \quad (7.2)$$

Then the N -point DFT matrix is defined as

$$\vec{F}_N = [W_N^{mn}], \quad 0 \leq m, n \leq N-1 \quad (7.3)$$

where W_N^{mn} are the elements of \vec{F}_N .

3. Let

$$\vec{I}_4 = \begin{pmatrix} \vec{e}_4^1 & \vec{e}_4^2 & \vec{e}_4^3 & \vec{e}_4^4 \end{pmatrix} \quad (7.4)$$

be the 4×4 identity matrix. Then the 4 point DFT permutation matrix is defined as

$$\vec{P}_4 = \begin{pmatrix} \vec{e}_4^1 & \vec{e}_4^3 & \vec{e}_4^2 & \vec{e}_4^4 \end{pmatrix} \quad (7.5)$$

4. The 4 point DFT diagonal matrix is defined as

$$\vec{D}_4 = \text{diag}(W_8^0, W_8^1, W_8^2, W_8^3) \quad (7.6)$$

5. Show that

$$W_N^2 = W_{N/2} \quad (7.7)$$

Solution: Given

$$W_N = e^{-j2\pi/N} \quad (7.8)$$

$$W_N^2 = e^{2(-j2\pi/N)} \quad (7.9)$$

$$= e^{-j2\pi/(N/2)} \quad (7.10)$$

$$= W_{N/2} \quad (7.11)$$

6. Show that

$$\vec{F}_4 = \begin{bmatrix} \vec{I}_2 & \vec{D}_2 \\ \vec{I}_2 & -\vec{D}_2 \end{bmatrix} \begin{bmatrix} \vec{F}_2 & 0 \\ 0 & \vec{F}_2 \end{bmatrix} \vec{P}_4 \quad (7.12)$$

Solution: \vec{I}_2 is 2×2 identity matrix

$$\begin{bmatrix} \vec{I}_2 & \vec{D}_2 \\ \vec{I}_2 & -\vec{D}_2 \end{bmatrix} \begin{bmatrix} \vec{F}_2 & 0 \\ 0 & \vec{F}_2 \end{bmatrix} = \begin{bmatrix} \vec{F}_2 & \vec{D}_2 \vec{F}_2 \\ \vec{F}_2 & -\vec{D}_2 \vec{F}_2 \end{bmatrix} \quad (7.13)$$

Given

$$\vec{F}_2 = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \quad (7.14)$$

$$\vec{D}_2 = \text{diag}(1, W_4) = \begin{bmatrix} 1 & 0 \\ 0 & -j \end{bmatrix} \quad (7.15)$$

$$\vec{P}_4 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad (7.16)$$

$$\vec{F}_4 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix} \quad (7.17)$$

$$\begin{bmatrix} \vec{F}_2 & \vec{D}_2 \vec{F}_2 \\ \vec{F}_2 & -\vec{D}_2 \vec{F}_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & -j & j \\ 1 & 1 & -1 & -1 \\ 1 & -1 & j & -j \end{bmatrix} \quad (7.18)$$

$$= \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & -j & j \\ 1 & 1 & -1 & -1 \\ 1 & -1 & j & -j \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad (7.19)$$

$$= \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix} \quad (7.20)$$

which is same as \vec{F}_4 .

$$\therefore \vec{F}_4 = \begin{bmatrix} \vec{I}_2 & \vec{D}_2 \\ \vec{I}_2 & -\vec{D}_2 \end{bmatrix} \quad (7.21)$$

7. Show that

$$\vec{F}_N = \begin{bmatrix} \vec{I}_{N/2} & \vec{D}_{N/2} \\ \vec{I}_{N/2} & -\vec{D}_{N/2} \end{bmatrix} \begin{bmatrix} \vec{F}_{N/2} & 0 \\ 0 & \vec{F}_{N/2} \end{bmatrix} \vec{P}_N \quad (7.22)$$

8. Find

$$\vec{P}_4 \vec{x} \quad (7.23)$$

Solution: From (7.16),

$$\vec{P}_4 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad (7.24)$$

$$\vec{x} = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 2 \\ 1 \end{pmatrix} \quad (7.25)$$

After proper zero padding of \vec{P}_4 ,

$$\vec{P}_4 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad (7.26)$$

$$\vec{P}_4 \vec{x} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 2 \\ 1 \end{pmatrix} \quad (7.27)$$

$$= \begin{pmatrix} 1 \\ 3 \\ 2 \\ 4 \\ 0 \\ 0 \end{pmatrix} \quad (7.28)$$

9. Show that

$$\vec{X} = \vec{F}_N \vec{x} \quad (7.29)$$

where \vec{x}, \vec{X} are the vector representations of $x(n), X(k)$ respectively.

Solution: Given \vec{x}, \vec{X} are the vector representations of $x(n), X(k)$ respectively.

$$\vec{x} = \begin{bmatrix} x(0) \\ x(1) \\ \vdots \\ x(N-1) \end{bmatrix} \quad (7.30)$$

$$\vec{X} = \begin{bmatrix} X(0) \\ X(1) \\ \vdots \\ X(N-1) \end{bmatrix} \quad (7.31)$$

$$\vec{F}_N = \begin{bmatrix} 1 & 1 & 1 & \dots & 1 \\ 1 & W_N & W_N^2 & \dots & W_N^{(N-1)} \\ 1 & W_N^2 & W_N^4 & \dots & W_N^{2(N-1)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & W_N^{N-1} & W_N^{2(N-1)} & \dots & W_N^{(N-1)(N-1)} \end{bmatrix} \quad (7.32)$$

As

$$X(k) = \sum_{n=0}^{N-1} x(n) e^{-j2\pi kn/N} \quad (7.33)$$

Upon linear transformation over k,

$$\begin{bmatrix} X(0) \\ X(1) \\ \vdots \\ X(N-1) \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & \dots & 1 \\ 1 & W_N & W_N^2 & \dots & W_N^{(N-1)} \\ 1 & W_N^2 & W_N^4 & \dots & W_N^{2(N-1)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & W_N^{N-1} & W_N^{2(N-1)} & \dots & W_N^{(N-1)(N-1)} \end{bmatrix} \begin{bmatrix} x(0) \\ x(1) \\ \vdots \\ x(N-1) \end{bmatrix} \quad P_4 \begin{bmatrix} x(0) \\ x(2) \\ x(4) \\ x(6) \end{bmatrix} = \begin{bmatrix} x(0) \\ x(4) \\ x(2) \\ x(6) \end{bmatrix} \quad (7.43)$$

$$P_4 \begin{bmatrix} x(1) \\ x(3) \\ x(5) \\ x(7) \end{bmatrix} = \begin{bmatrix} x(1) \\ x(5) \\ x(3) \\ x(7) \end{bmatrix} \quad (7.44)$$

$$\therefore \vec{X} = \vec{F}_N \vec{x} \quad (7.35)$$

10. Derive the following Step-by-step visualisation of 8-point FFTs into 4-point FFTs and so on

$$\begin{bmatrix} X(0) \\ X(1) \\ X(2) \\ X(3) \end{bmatrix} = \begin{bmatrix} X_1(0) \\ X_1(1) \\ X_1(2) \\ X_1(3) \end{bmatrix} + \begin{bmatrix} W_8^0 & 0 & 0 & 0 \\ 0 & W_8^1 & 0 & 0 \\ 0 & 0 & W_8^2 & 0 \\ 0 & 0 & 0 & W_8^3 \end{bmatrix} \begin{bmatrix} X_2(0) \\ X_2(1) \\ X_2(2) \\ X_2(3) \end{bmatrix} \quad (7.36)$$

$$\begin{bmatrix} X(4) \\ X(5) \\ X(6) \\ X(7) \end{bmatrix} = \begin{bmatrix} X_1(0) \\ X_1(1) \\ X_1(2) \\ X_1(3) \end{bmatrix} - \begin{bmatrix} W_8^0 & 0 & 0 & 0 \\ 0 & W_8^1 & 0 & 0 \\ 0 & 0 & W_8^2 & 0 \\ 0 & 0 & 0 & W_8^3 \end{bmatrix} \begin{bmatrix} X_2(0) \\ X_2(1) \\ X_2(2) \\ X_2(3) \end{bmatrix} \quad (7.37)$$

4-point FFTs into 2-point FFTs

$$\begin{bmatrix} X_1(0) \\ X_1(1) \end{bmatrix} = \begin{bmatrix} X_3(0) \\ X_3(1) \end{bmatrix} + \begin{bmatrix} W_4^0 & 0 \\ 0 & W_4^1 \end{bmatrix} \begin{bmatrix} X_4(0) \\ X_4(1) \end{bmatrix} \quad (7.38)$$

$$\begin{bmatrix} X_1(2) \\ X_1(3) \end{bmatrix} = \begin{bmatrix} X_3(0) \\ X_3(1) \end{bmatrix} - \begin{bmatrix} W_4^0 & 0 \\ 0 & W_4^1 \end{bmatrix} \begin{bmatrix} X_4(0) \\ X_4(1) \end{bmatrix} \quad (7.39)$$

$$\begin{bmatrix} X_2(0) \\ X_2(1) \end{bmatrix} = \begin{bmatrix} X_5(0) \\ X_5(1) \end{bmatrix} + \begin{bmatrix} W_4^0 & 0 \\ 0 & W_4^1 \end{bmatrix} \begin{bmatrix} X_6(0) \\ X_6(1) \end{bmatrix} \quad (7.40)$$

$$\begin{bmatrix} X_2(2) \\ X_2(3) \end{bmatrix} = \begin{bmatrix} X_5(0) \\ X_5(1) \end{bmatrix} - \begin{bmatrix} W_4^0 & 0 \\ 0 & W_4^1 \end{bmatrix} \begin{bmatrix} X_6(0) \\ X_6(1) \end{bmatrix} \quad (7.41)$$

$$P_8 \begin{bmatrix} x(0) \\ x(1) \\ x(2) \\ x(3) \\ x(4) \\ x(5) \\ x(6) \\ x(7) \end{bmatrix} = \begin{bmatrix} x(0) \\ x(2) \\ x(4) \\ x(6) \\ x(1) \\ x(3) \\ x(5) \\ x(7) \end{bmatrix} \quad (7.42)$$

Therefore,

$$\begin{bmatrix} X_3(0) \\ X_3(1) \end{bmatrix} = F_2 \begin{bmatrix} x(0) \\ x(4) \end{bmatrix} \quad (7.45)$$

$$\begin{bmatrix} X_4(0) \\ X_4(1) \end{bmatrix} = F_2 \begin{bmatrix} x(2) \\ x(6) \end{bmatrix} \quad (7.46)$$

$$\begin{bmatrix} X_5(0) \\ X_5(1) \end{bmatrix} = F_2 \begin{bmatrix} x(1) \\ x(5) \end{bmatrix} \quad (7.47)$$

$$\begin{bmatrix} X_6(0) \\ X_6(1) \end{bmatrix} = F_2 \begin{bmatrix} x(3) \\ x(7) \end{bmatrix} \quad (7.48)$$

11. For

$$\vec{x} = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 2 \\ 1 \end{pmatrix} \quad (7.49)$$

compute the DFT using (7.29)

Solution:

$$\vec{F}_6 = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & e^{-j\pi/3} & e^{-j2\pi/3} & e^{-j\pi} & e^{-j4\pi/3} & e^{-j5\pi/3} \\ 1 & e^{-j2\pi/3} & e^{-j4\pi/3} & e^{-j2\pi} & e^{-j8\pi/3} & e^{-j10\pi/3} \\ 1 & e^{-j\pi} & e^{-j2\pi} & e^{-j3\pi} & e^{-j4\pi} & e^{-j5\pi} \\ 1 & e^{-j4\pi/3} & e^{-j8\pi/3} & e^{-j4\pi} & e^{-j16\pi/3} & e^{-j20\pi/3} \\ 1 & e^{-j5\pi/3} & e^{-j10\pi/3} & e^{-j5\pi} & e^{-j20\pi/3} & e^{-j25\pi/3} \end{bmatrix} \quad (7.50)$$

Using (7.49),

$$\vec{X} = \vec{F}_6 \vec{x} \quad (7.51)$$

$$\vec{X} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & e^{-j\pi/3} & e^{-j2\pi/3} & e^{-j\pi} & e^{-j4\pi/3} & e^{-j5\pi/3} \\ 1 & e^{-j2\pi/3} & e^{-j4\pi/3} & e^{-j2\pi} & e^{-j8\pi/3} & e^{-j10\pi/3} \\ 1 & e^{-j\pi} & e^{-j2\pi} & e^{-j3\pi} & e^{-j4\pi} & e^{-j5\pi} \\ 1 & e^{-j4\pi/3} & e^{-j8\pi/3} & e^{-j4\pi} & e^{-j16\pi/3} & e^{-j20\pi/3} \\ 1 & e^{-j5\pi/3} & e^{-j10\pi/3} & e^{-j5\pi} & e^{-j20\pi/3} & e^{-j25\pi/3} \end{bmatrix} \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 2 \\ 1 \end{pmatrix} \quad (7.52)$$

$$= \begin{bmatrix} 13 \\ -4 - \sqrt{3}j \\ 1 \\ -1 \\ 1 \\ -4 + \sqrt{3}j \end{bmatrix} \quad (7.53)$$

12. Repeat the above exercise using the FFT after zero padding \vec{x} .

Solution: \vec{x} after padding is

$$\begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 2 \\ 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad (7.54)$$

Using ,

$$\vec{F}_8 = \begin{bmatrix} \vec{I}_4 & \vec{D}_4 \\ \vec{I}_4 & -\vec{D}_4 \end{bmatrix} \begin{bmatrix} \vec{F}_4 & 0 \\ 0 & \vec{F}_4 \end{bmatrix} \vec{P}_8 \quad (7.55)$$

$$\vec{F}_4 = \begin{bmatrix} \vec{I}_2 & \vec{D}_2 \\ \vec{I}_2 & -\vec{D}_2 \end{bmatrix} \begin{bmatrix} \vec{F}_2 & 0 \\ 0 & \vec{F}_2 \end{bmatrix} \vec{P}_4 \quad (7.56)$$

$$\vec{F}_2 = \begin{bmatrix} \vec{I}_1 & \vec{D}_1 \\ \vec{I}_1 & -\vec{D}_1 \end{bmatrix} \begin{bmatrix} \vec{F}_1 & 0 \\ 0 & \vec{F}_1 \end{bmatrix} \vec{P}_2 \quad (7.57)$$

$$\vec{F}_1 = [1] \quad (7.58)$$

Calculating \vec{F}_2 ,

$$\vec{F}_2 = \begin{bmatrix} \vec{F}_1 & D_1 \vec{F}_1 \\ \vec{F}_1 & -D_1 \vec{F}_1 \end{bmatrix} \vec{P}_2 \quad (7.59)$$

$$= \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \quad (7.60)$$

Calculating \vec{F}_4 ,

$$\vec{D}_2 = \text{diag}(1, W_4) = \begin{bmatrix} 1 & 0 \\ 0 & -j \end{bmatrix} \quad (7.61)$$

$$D_2 \vec{F}_2 = \begin{bmatrix} 1 & 0 \\ 0 & -j \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \quad (7.62)$$

$$= \begin{bmatrix} 1 & 1 \\ -j & j \end{bmatrix} \quad (7.63)$$

$$\vec{F}_4 = \begin{bmatrix} \vec{F}_2 & D_2 \vec{F}_2 \\ \vec{F}_2 & -D_2 \vec{F}_2 \end{bmatrix} \vec{P}_4 \quad (7.64)$$

$$\vec{F}_4 = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & -j & j \\ 1 & 0 & -1 & -1 \\ 0 & 1 & j & -j \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad (7.65)$$

$$= \begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & -j & 1 & j \\ 1 & -1 & 0 & j \\ 0 & j & 1 & -j \end{bmatrix} \quad (7.66)$$

Calculating \vec{F}_8 ,

$$\vec{D}_4 = \text{diag}(1, W_8, W_8^2, W_8^3) \quad (7.67)$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1-j}{\sqrt{2}} & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & \frac{-1-j}{\sqrt{2}} \end{bmatrix} \quad (7.68)$$

$$D_4 \vec{F}_4 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1-j}{\sqrt{2}} & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & \frac{-1-j}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & -j & 1 & j \\ 1 & -1 & 0 & j \\ 0 & j & 1 & -j \end{bmatrix} \quad (7.69)$$

$$= \begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & \frac{-1-j}{\sqrt{2}} & \frac{1-j}{\sqrt{2}} & \frac{1+j}{\sqrt{2}} \\ -1 & 1 & 0 & -j \\ 0 & \frac{1-j}{\sqrt{2}} & \frac{-1-j}{\sqrt{2}} & \frac{-1+j}{\sqrt{2}} \end{bmatrix} \quad (7.70)$$

$F_8 = A\vec{B}P_8$ where

$$\vec{A} = \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & \frac{-1-j}{\sqrt{2}} & \frac{1-j}{\sqrt{2}} & \frac{1+j}{\sqrt{2}} \\ 0 & 0 & 1 & 0 & -1 & 1 & 0 & -j \\ 0 & 0 & 0 & 1 & 0 & \frac{1-j}{\sqrt{2}} & \frac{-1-j}{\sqrt{2}} & \frac{-1+j}{\sqrt{2}} \\ 1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & \frac{1+j}{\sqrt{2}} & \frac{-1+j}{\sqrt{2}} & \frac{-1-j}{\sqrt{2}} \\ 0 & 0 & 1 & 0 & 1 & -1 & 0 & j \\ 0 & 0 & 0 & 1 & 0 & \frac{-1+j}{\sqrt{2}} & \frac{1+j}{\sqrt{2}} & \frac{1-j}{\sqrt{2}} \end{bmatrix} \quad (7.71)$$

$$\vec{B} = \begin{bmatrix} 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & -j & 1 & j & 0 & 0 & 0 & 0 \\ 1 & -1 & 0 & j & 0 & 0 & 0 & 0 \\ 0 & j & 1 & -j & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & -1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & j & -1 & j \\ 0 & 0 & 0 & 0 & -1 & 1 & 0 & -j \\ 0 & 0 & 0 & 0 & 0 & -j & -1 & j \end{bmatrix} \quad (7.72)$$

$$\vec{F}_8 = \begin{bmatrix} 13 \\ -3.12 - 6.53j \\ j \\ 1.12 - 0.53j \\ -1 \\ 1.12 + 0.53j \\ -j \\ -3.12 + 6.5355 \end{bmatrix} \quad (7.73)$$

And P_8 is a permutation matrix.

$$\vec{X} = \begin{bmatrix} 13 \\ -4 - 8j \\ j \\ 2 - 2j \\ -1 \\ 2 + 2j \\ -j \\ -4 + 8j \end{bmatrix} \quad (7.74)$$

13. Write a C program to compute the 8-point FFT.

8 EXERCISES

Answer the following questions by looking at the python code in Problem 2.3.

8.1 The command

```
output_signal = signal.lfilter(b, a,
                                input_signal)
```

in Problem 2.3 is executed through the following difference equation

$$\sum_{m=0}^M a(m)y(n-m) = \sum_{k=0}^N b(k)x(n-k) \quad (8.1)$$

where the input signal is $x(n)$ and the output signal is $y(n)$ with initial values all 0. Replace **signal.filtfilt** with your own routine and verify.

8.2 Repeat all the exercises in the previous sections for the above a and b .

8.3 What is the sampling frequency of the input signal?

Solution: Sampling frequency(fs)=44.1kHz.

8.4 What is type, order and cutoff-frequency of the above butterworth filter

Solution: The given butterworth filter is low pass with order=2 and cutoff-frequency=4kHz.

8.5 Modifying the code with different input parameters and to get the best possible output.