Digital Signal Processing
Circuits and Transforms

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CONTENTS

1. DEFINITIONS

1. The unit step function is

$$u(t) = \begin{cases} 1 & t > 0 \\ \frac{1}{2} & t = 0 \\ 0 & t < 0 \end{cases} \qquad (1.1)$$

2. The Laplace transform of $g(t)$ is defined as

$$G(s) = \int_{-\infty}^{\infty} g(t)e^{-st}\, dt \qquad (1.2)$$

2. LAPLACE TRANSFORM

1. In the circuit, the switch S is connected to position P for a long time so that the charge on the capacitor becomes $q_1\,\mu C$. Then S is switched to position Q. After a long time, the charge on the capacitor is $q_2\,\mu C$.
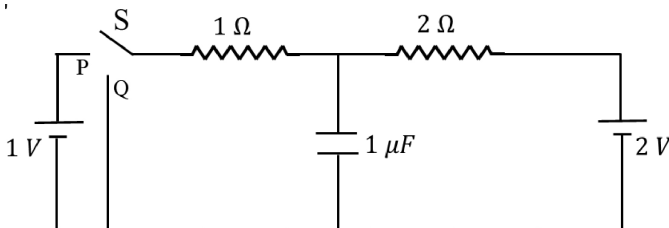

Fig. 2.1.

2. Draw the circuit using latex-tikz.
   Solution: The following code yields Fig.**??**

```
wget https://raw.githubusercontent.com/
   kumarsuraj151/EE3900/master/cktsig/Tikz
   %20Circuits/2_2.tex
```
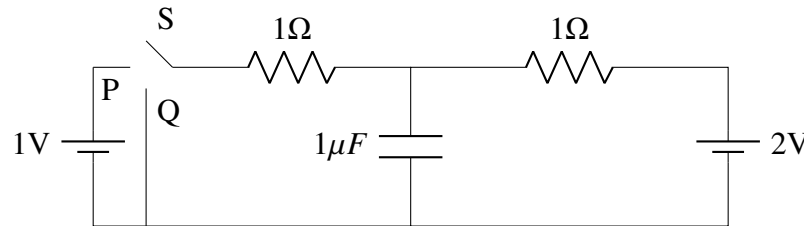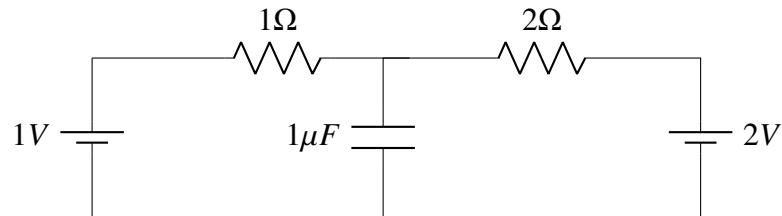

Fig. 2.2. Given Circuit

Fig. 2.3. Before switching S to Q

3. Find q_1.
   Solution: Before switching S to Q: Calculating current,

$$1 - i - 2i - 2 = 0 \qquad (2.1)$$

$$3i = -1 \Rightarrow i = \frac{-1}{3} \qquad (2.2)$$

Potential Difference between capacitor at steady state is

$$1 - \left(\frac{-1}{3}\right) = \frac{4}{3} \qquad (2.3)$$

$$q_1 = \frac{4}{3} \cdot 1 \qquad (2.4)$$

$$= \frac{4}{3}\mu C \qquad (2.5)$$

4. Show that the Laplace transform of $u(t)$ is $\frac{1}{s}$ and find the ROC.

Solution: We know that from definition of Laplace Transform,

$$F(s) = \int_0^\infty f(t)e^{-st}\,dt \qquad (2.6)$$

for $u(t)$, we have

$$U(s) = \int_0^\infty u(t)e^{-st}\,dt \qquad (2.7)$$

Using (**??**),

$$U(s) = \int_0^\infty u(t)e^{-st}\,dt \qquad (2.8)$$

$$= \int_0^\infty e^{-st}\,dt \qquad (2.9)$$

$$= -\left(0 - \frac{1}{s}\right) \qquad (2.10)$$

$$= \frac{1}{s} \qquad (2.11)$$

ROC is $Re(s) > 0$ since $e^{-st} < \infty$ for $t \to \infty$ The following command plots the ROC of above Laplace Transform.

wget https://raw.githubusercontent.com/
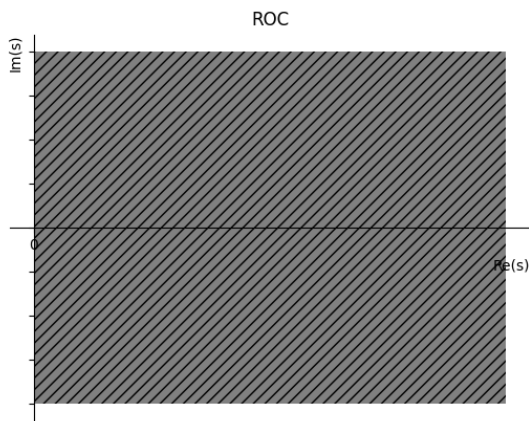   kumarsuraj151/EE3900/master/cktsig/code
    /2_4.py

Fig. 2.4.

5. Show that

$$e^{-at}u(t) \xleftrightarrow{\mathcal{L}} \frac{1}{s+a}, \quad a > 0 \qquad (2.12)$$

and find the ROC. **Solution:** From (**??**),

$$F(s) = \int_0^\infty u(t)e^{-at}e^{-st}\,dt \qquad (2.13)$$

$$= \int_0^\infty u(t)e^{-(s+a)t}\,dt \qquad (2.14)$$

$$= \int_0^\infty e^{-(s+a)t}\,dt \qquad (2.15)$$

$$= -\left(0 - \frac{1}{s+a}\right) \qquad (2.16)$$

$$= \frac{1}{s+a} \qquad (2.17)$$

ROC is

$$s + a > 0 \Rightarrow s > -a \qquad (2.18)$$

The following command plots the ROC of above Laplace Transform.

wget https://raw.githubusercontent.com/
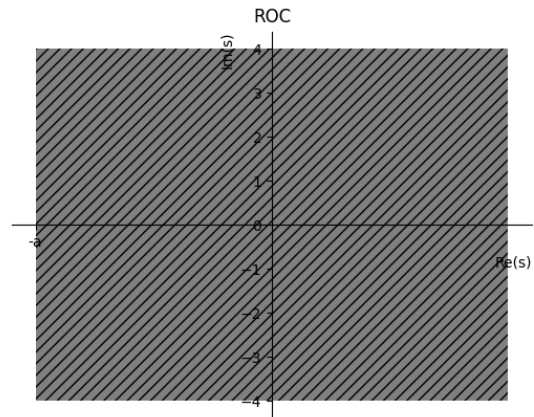   kumarsuraj151/EE3900/master/cktsig/code
    /2_5.py

Fig. 2.5.

6. Now consider the following resistive circuit transformed from Fig. **??** where

$$u(t) \xleftrightarrow{\mathcal{L}} V_1(s) \qquad (2.19)$$

$$2u(t) \xleftrightarrow{\mathcal{L}} V_2(s) \qquad (2.20)$$

Find the voltage across the capacitor $V_{C_0}(s)$.
Solution:

$$R_{eff} = \frac{1}{1 + \frac{1}{2}} = \frac{2}{3}\Omega \qquad (2.21)$$

$$V_{eff} = \frac{1}{1 + \frac{1}{2}} = \frac{2}{3}V \qquad (2.22)$$

Fig. 2.6.

Fig. 2.7. Plot of $V_{C_0}(t)$

$$V_{C_0}(s) = V_S(s)\frac{C_0}{C_0 + R_{eff}} \qquad (2.23)$$

$$= \left(\frac{4}{3s}\right)\left(\frac{\frac{1}{s}}{\frac{1}{s} + \frac{2}{3}}\right) \qquad (2.24)$$

$$= \frac{6}{s(4s + 9)} \qquad (2.25)$$

7. Find $v_{C_0}(t)$. Plot using python.
Solution: Using (**??**),

$$\frac{6}{s(4s + 9)} = \frac{4}{3s} - \frac{2}{9 + 4s} \qquad (2.26)$$

Apply inverse Laplacian Transform,

$$V_{C_0}(s) \xrightarrow{\mathcal{L}^{-\infty}} V_{C_0}(t) \qquad (2.27)$$

$$\mathcal{L}^{-1}[V_{C_0}(s)] = \mathcal{L}^{-1}\left[\frac{4}{3s} - \frac{2}{9 + 4s}\right] \qquad (2.28)$$

$$= \mathcal{L}^{-1}\left[\frac{4}{3s}\right] - \mathcal{L}^{-1}\left[\frac{2}{9 + 4s}\right] \qquad (2.29)$$

Since,

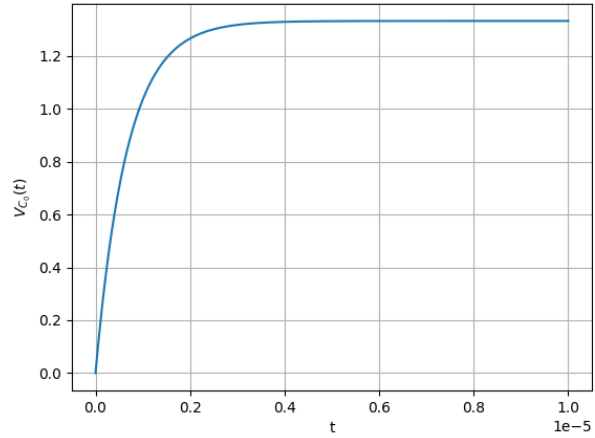$$\mathcal{L}^{-1}\left[\frac{1}{s}\right] = u(t) \qquad (2.30)$$

$$\mathcal{L}^{-1}\left[\frac{1}{s - a}\right] = e^{at}u(t) \qquad (2.31)$$

Using the above equations,

$$V_{C_0}(t) = \frac{4}{3}\left(1 - e^{\frac{-3}{2}t}\right)u(t) \qquad (2.32)$$

The following command plots the above equation.

```
wget https://raw.githubusercontent.com/
    kumarsuraj151/EE3900/master/cktsig/code
    /2_7.py
```

8. Verify your result using ngspice.
Solution: The following command plots the ROC of above Laplace Transform.

```
wget https://raw.githubusercontent.com/
    kumarsuraj151/EE3900/master/cktsig/code
    /2_8.cir
```
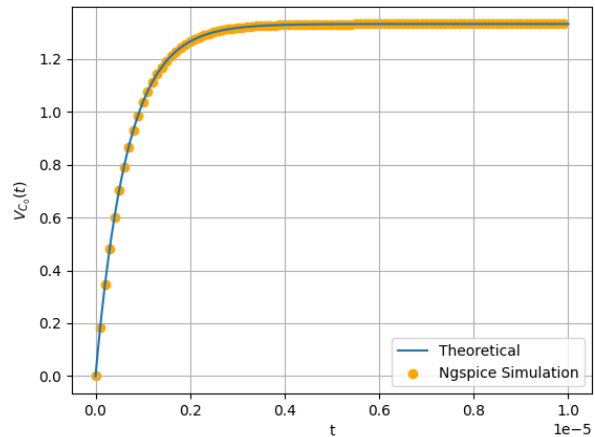

Fig. 2.8.

3. Initial Conditions

1. Find q_2 in Fig. **??**.
Solution: At steady state, $V_{C_0} = V_{1\Omega}$

$$V_{C_0} = \frac{q_2}{C} = V_{1\Omega} = \frac{2}{1 + 2} = \frac{2}{3}$$

$$q_2 = \frac{2}{3}\mu C$$

2. Draw the equivalent *s*-domain resistive circuit when S is switched to position Q. Use variables R_1, R_2, C_0 for the passive elements. Use latextikz.

   Solution: The following command plots the ROC of above Laplace Transform.

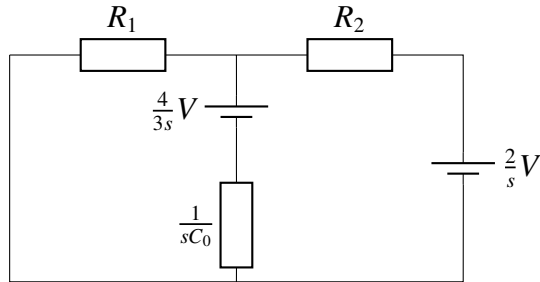   wget https://raw.githubusercontent.com/ kumarsuraj151/EE3900/master/cktsig/Tikz %20Circuits/3_2.tex

Fig. 3.1. After switching S to Q

3. $V_{C_0}(s) = ?$

   Solution: Using KCL at node in Fig. **??**

   $$\frac{V - 0}{R_1} + \frac{V - \frac{2}{s}}{R_2} + sC_0\left(V - \frac{4}{3s}\right) = 0 \quad (3.1)$$

   $$\implies V_{C_0}(s) = \frac{\frac{2}{sR_2} + \frac{4C_0}{3}}{\frac{1}{R_1} + \frac{2}{R_2} + sC_0} \quad (3.2)$$

4. $v_{C_0}(t) = ?$ Plot using python.

   Solution: From (**??**),

   $$V_{C_0}(s) = \frac{4}{3}\left(\frac{1}{\frac{1}{C_0}\left(\frac{1}{R_1} + \frac{1}{R_2}\right) + s}\right)$$

   $$+ \frac{2}{R_2\left(\frac{1}{R_1} + \frac{1}{R_2}\right)}\left(\frac{1}{s} - \frac{1}{\frac{1}{C_0}\left(\frac{1}{R_1} + \frac{1}{R_2}\right) + s}\right) \quad (3.3)$$

   Taking an inverse Laplace Transform,

   $$v_{C_0}(t) = \frac{4}{3}e^{-\left(\frac{1}{R_1} + \frac{1}{R_2}\right)\frac{t}{C_0}}u(t)$$

   $$+ \frac{2}{R_2\left(\frac{1}{R_1} + \frac{1}{R_2}\right)}\left(1 - e^{-\left(\frac{1}{R_1} + \frac{1}{R_2}\right)\frac{t}{C_0}}\right)u(t) \quad (3.4)$$

   Substituting values gives

   $$v_{C_0}(t) = \frac{2}{3}\left(1 + e^{-\left(1.5\times10^6\right)t}\right)u(t) \quad (3.5)$$
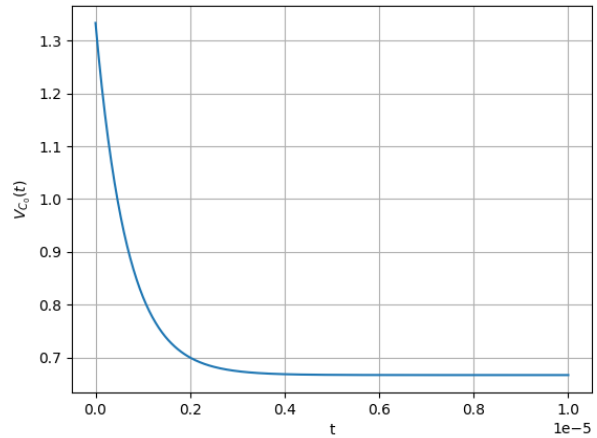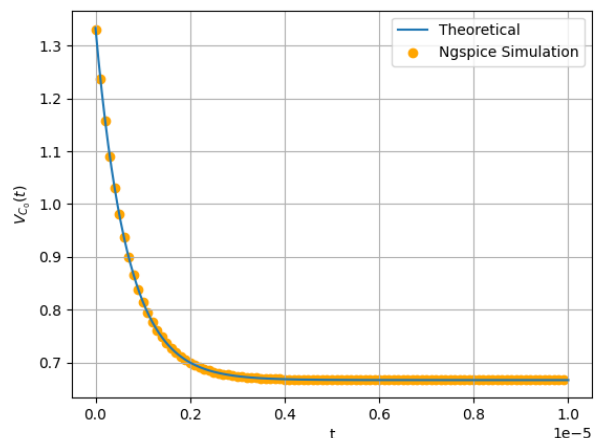
   The following command plots the above equation. wget

Fig. 3.2. Plot of $V_{C_0}(t)$

https://raw.githubusercontent.com/kumarsuraj151/EE3900

5. Verify your result using ngspice.

   Solution: The following command plots Fig.3.3

   wget https://raw.githubusercontent.com/ kumarsuraj151/EE3900/master/cktsig/code /3.5.cir

6. Find $v_{C_0}(0-)$, $v_{C_0}(0+)$ and $v_{C_0}(\infty)$.

   Solution: From the initial conditions,

   $$v_{C_0}(0-) = \frac{q_1}{C} = \frac{4}{3}V \quad (3.6)$$

Using (??),

$$v_{C_0}(0+) = \lim_{t \to 0+} v_{C_0}(t) = \frac{4}{3}V \qquad (3.7)$$

$$v_{C_0}(\infty) = \lim_{t \to \infty} v_{C_0}(t) = \frac{2}{3}V \qquad (3.8)$$

7. Obtain Fig. ?? using the equivalent differential equation
   Solution: Using Kirchoff's junction law

$$\frac{v_c(t) - 0}{R_1} + \frac{v_c(t) - v_2(t)}{R_2} + \frac{dq}{dt} = 0 \qquad (3.9)$$

where $q(t)$ is the charge on the capacitor
On taking the Laplace transform on both sides of this equation

$$\frac{V_c(s) - 0}{R_1} + \frac{V_c(s) - V_2(s)}{R_2} + (sQ(s) - q(0^-)) = 0 \qquad (3.10)$$

But $q(0^-) = \frac{4}{3}C_0$ and

$$q(t) = C_0 v_c(t) \qquad (3.11)$$

$$\implies Q(s) = C_0 V_c(s) \qquad (3.12)$$

Thus

$$\frac{V_c(s) - 0}{R_1} + \frac{V_c(s) - V_2(s)}{R_2} + \left(sC_0 V_c(s) - \frac{4}{3}C_0\right) = 0 \qquad (3.13)$$

$$\implies \frac{V_c(s) - 0}{R_1} + \frac{V_c(s) - V_2(s)}{R_2} + \frac{V_c(s) - \frac{4}{3s}}{\frac{1}{sC_0}} = 0 \qquad (3.14)$$

which is the same equation as the one we obtained from Fig. ??

4. BILINEAR TRANSFORM

1. In Fig. **??**, consider the case when S is switched to Q right in the beginning. Formulate the differential equation.
   Solution: The differential equation is the same as before

$$\frac{v_c(t) - 0}{R_1} + \frac{v_c(t) - v_2(t)}{R_2} + \frac{dq}{dt} = 0 \quad (4.1)$$

$$\text{i.e., } \frac{v_c(t)}{R_1} + \frac{v_c(t) - v_2(t)}{R_2} + C_0 \frac{dv_c}{dt} = 0 \quad (4.2)$$

$$q(0^-) = q(0) = 0 \qquad (4.3)$$

2. Find $H(s)$ considering the ouput voltage at the capacitor.

Solution: On taking the Laplace transform on both sides of this equation

$$\frac{V_c(s)}{R_1} + \frac{V_c(s) - V_2(s)}{R_2} + sQ(s) - 0 = 0 \qquad (4.4)$$

$$\implies V_c(s)\left(\frac{1}{R_1} + \frac{1}{R_2}\right) + sC_0 V_c(s) = \frac{V_2(s)}{R_2} \qquad (4.5)$$

$$\implies \frac{V_c(s)}{V_2(s)} = \frac{\frac{1}{R_2}}{\frac{1}{R_1} + \frac{1}{R_2} + sC_0} \qquad (4.6)$$

$$H(s) = \frac{V_c(s)}{V_2(s)} = \frac{\frac{1}{R_2 C_0}}{s + \frac{1}{R_1 C_0} + \frac{1}{R_2 C_0}} \qquad (4.7)$$

$$H(s) = \frac{5 \times 10^5}{s + 1.5 \times 10^6} \qquad (4.8)$$

3. Plot $H(s)$. What kind of filter is it? **Solution:** Download the following Python code that plots Fig. **??**

wget https://raw.githubusercontent.com/
    kumarsuraj151/EE3900/master/cktsig/Tikz
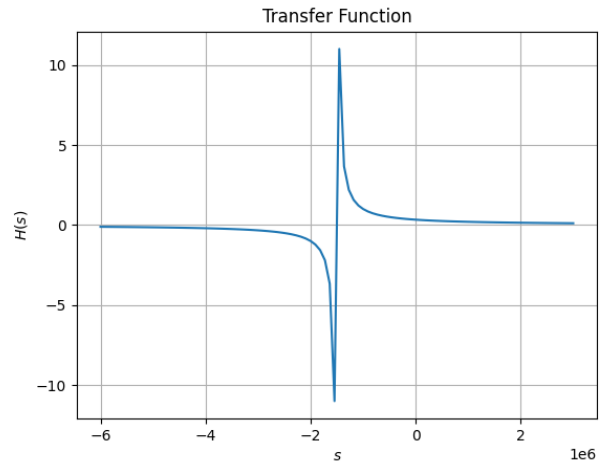   %20Circuits/4.3.tex

Fig. 4.1. Plot of $H(s)$

$$H(\jmath\omega) = \frac{5 \times 10^5}{\jmath\omega + 1.5 \times 10^6} \qquad (4.9)$$

$$\implies |H(\jmath\omega)| = \frac{5 \times 10^5}{\sqrt{\omega^2 + 2.25 \times 10^{12}}} \qquad (4.10)$$

As ω increases, $|H(\jmath\omega)|$ decreases.

In other words, the amplitude of high-frequency signals gets diminished and they get filtered out.

Therefore, this is a low-pass filter.

4. Using trapezoidal rule for integration, formulate the difference equation by considering

$$y(n) = y(t)|_{t=n} \qquad (4.11)$$

Solution:

$$\frac{v_c(t)}{R_1} + \frac{v_c(t) - v_2(t)}{R_2} + C_0\frac{dv_c}{dt} = 0 \quad (4.12)$$

$$\implies C_0\frac{dv_c}{dt} = \frac{2u(t) - v_c(t)}{R_2} - \frac{v_c(t)}{R_1} \qquad (4.13)$$

$$\implies v_c(t)|_{t=n}^{n+1} = \int_n^{n+1}\left(\frac{2u(t) - v_c(t)}{R_2C_0} - \frac{v_c(t)}{R_1C_0}\right)dt \qquad (4.14)$$

By the trapezoidal rule of integration

$$\int_a^b f(t)dt \approx \frac{b-a}{2}(f(a) + f(b)) \qquad (4.15)$$

Consider $y(t) = v_c(t)$

$$y(n+1) - y(n) = \frac{1}{R_2C_0}(u(n) + u(n+1))$$
$$- \frac{1}{2}(y(n+1) + y(n))\left(\frac{1}{R_1C_0} + \frac{1}{R_2C_0}\right) \quad (4.16)$$

Thus, the difference equation is

$$y(n+1)\left(1 + \frac{1}{2R_1C_0} + \frac{1}{2R_2C_0}\right)$$
$$= y(n)\left(1 - \frac{1}{2R_1C_0} - \frac{1}{2R_2C_0}\right)$$
$$+ \frac{1}{R_2C_0}(u(n) + u(n+1)) \quad (4.17)$$

5. Find $H(z)$.

Solution: Let $\mathcal{Z}\{y(n)\} = Y(z)$

On taking the Z-transform on both sides of the difference equation

$$zY(z)\left(1 + \frac{1}{2R_1C_0} + \frac{1}{2R_2C_0}\right)$$
$$= Y(z)\left(1 - \frac{1}{2R_1C_0} - \frac{1}{2R_2C_0}\right)$$
$$+ \frac{1}{R_2C_0}\left(\frac{1}{1-z^{-1}} + \frac{z}{1-z^{-1}}\right) \quad (4.18)$$

$$Y(z)\left(z + \frac{z}{2R_1C_0} + \frac{z}{2R_2C_0} - 1 + \frac{1}{2R_1C_0} + \frac{1}{2R_2C_0}\right)$$
$$= \frac{1}{R_2C_0}\frac{1+z}{1-z^{-1}} \quad (4.19)$$

Also

$$v_2(t) = 2 \qquad \forall t \geq 0 \qquad (4.20)$$
$$\implies x(n) = 2u(n) \qquad (4.21)$$
$$\implies X(z) = \frac{2}{1-z^{-1}} \qquad |z| > 1 \qquad (4.22)$$

Thus, the transfer function in z-domain is

$$H(z) = \frac{Y(z)}{X(z)} \qquad (4.23)$$

$$= \frac{\frac{1+z}{2R_2C_0}}{z + \frac{z}{2R_1C_0} + \frac{z}{2R_2C_0} - 1 + \frac{1}{2R_1C_0} + \frac{1}{2R_2C_0}} \quad (4.24)$$

$$= \frac{\frac{1+z^{-1}}{2R_2C_0}}{1 + \frac{1}{2R_1C_0} + \frac{1}{2R_2C_0} - z^{-1} + \frac{z^{-1}}{2R_1C_0} + \frac{z^{-1}}{2R_2C_0}} \quad (4.25)$$

On substituting the values

$$H(z) = \frac{2.5 \times 10^5(1 + z^{-1})}{7.5 \times 10^5 + 1 + (7.5 \times 10^5 - 1)z^{-1}} \quad (4.26)$$

with the ROC being

$$|z| > \max\left(1, \left|\frac{7.5 \times 10^5 - 1}{7.5 \times 10^5 + 1}\right|\right) \qquad (4.27)$$
$$\implies |z| > 1 \qquad (4.28)$$

6. How can you obtain $H(z)$ from $H(s)$?

f **Solution:** The Z-transform can be obtained from the Laplace transform by the substitution

$$s = \frac{2}{T}\frac{1 - z^{-1}}{1 + z^{-1}} \qquad (4.29)$$

where T is the step size of the trapezoidal rule (1 in our case)

This is known as the bilinear transform

Thus

$$H(z) = \frac{\frac{1}{R_2 C_0}}{2\frac{1-z^{-1}}{1+z^{-1}} + \frac{1}{R_1 C_0} + \frac{1}{R_2 C_0}} \qquad (4.30)$$

$$= \frac{\frac{1+z^{-1}}{2R_2 C_0}}{1 - z^{-1} + \left(\frac{1}{2R_1 C_0} + \frac{1}{2R_2 C_0}\right)(1 + z^{-1})} \qquad (4.31)$$

$$= \frac{\frac{1+z^{-1}}{2R_2 C_0}}{1 + \frac{1}{2R_1 C_0} + \frac{1}{2R_2 C_0} - z^{-1} + \frac{z^{-1}}{2R_1 C_0} + \frac{z^{-1}}{2R_2 C_0}} \qquad (4.32)$$

$$= \frac{2.5 \times 10^5 (1 + z^{-1})}{7.5 \times 10^5 + 1 + (7.5 \times 10^5 - 1)z^{-1}} \qquad (4.33)$$

which is the same as what we obtained earlier