# Discrete Damage

Andres, Seb, ...

[a], , , , ,

---

**Abstract**

*Keywords:*

---

## 1. Variational formulation

A bar of length $L$ and section $S$ is stretched along the $x$ axis. The material of the bar has nominal Young's modulus $Y$. We note $u(x)$ the horizontal displacement. The bar is clamped at its left end, and a displacement $\Delta \geq 0$ is imposed of the right end

$$u(0) = 0\,, \quad u(L) = \Delta \tag{1}$$

The total energy of the bar is

$$\mathcal{E}(\epsilon(x), \alpha(x)) = \int_0^L \frac{1}{2} YS\, a(\alpha(x))\, \epsilon^2(x)\, \mathrm{d}x + \int_0^L WS\, w(\alpha(x))\, \mathrm{d}x \tag{2}$$

where the first integral is the strain energy and $W$ is the dissipated energy (per unit volume) due to damage and $\epsilon$ is the longitudinal strain. The functions $a(\alpha)$ and $w(\alpha)$ are non-dimensionalized functions, to be discussed later on. As the boundary conditions are written with $u(x)$, we need to include the constraint

$$\epsilon = u'(x) \tag{3}$$

that is, work with the Lagrangian

$$\mathcal{L}(\epsilon, \alpha, u) = \mathcal{E}(\epsilon, \alpha) - \int_0^L \sigma(x)\, S\, (\epsilon - u')\, \mathrm{d}x \tag{4}$$

where the continuous lagrange multiplier $\sigma(x)$ is identified with the axial stress in the bar.

## 2. Non-dimensionalisation

In this statics problem, we can freely (and with no loss of generality) choose a unit length and a unit force. We choose $L$ as unit length, and $YS$ as unit force. We introduce the 'hat' adim variables

$$\hat{x} = x/L\,,\ \hat{u} = u/L\,,\ \hat{\Delta} = \Delta/L\,,\ \hat{W} = W/Y\,,\ \hat{\sigma} = \sigma/Y\,,\ \hat{\mathcal{E}} = \frac{\mathcal{E}}{YSL}\,,\ \hat{\mathcal{L}} = \frac{\mathcal{L}}{YSL} \tag{5}$$

and simplify (4) to

$$\hat{\mathcal{E}}(\epsilon, \alpha) = \int_0^1 \frac{1}{2} a(\alpha)\, \epsilon^2(\hat{x})\, \mathrm{d}\hat{x} + \int_0^1 \hat{W} w(\alpha)\, \mathrm{d}\hat{x} \tag{6a}$$

$$\hat{\mathcal{L}}(\epsilon, \alpha, \hat{u}) = \hat{\mathcal{E}}(\epsilon, \alpha) - \int_0^1 \hat{\sigma}(\hat{x})\, (\epsilon - u'(\hat{x}))\, \mathrm{d}\hat{x} \tag{6b}$$

And from now on, we drop the 'hats' while keeping in mind that we deal with adim variables.

## 3. First variation

We seek extremal configurations of $\mathcal{E}(\epsilon, \alpha)$, under the constraints (3) and (1). Applying the lagrange multiplier rule, we work with $\mathcal{L}$ and look for the conditions under which the first variation of $\mathcal{L}$ vanishes, for the the two variables $\epsilon$ and $\hat{u}$. The minimisation with regard to the third variable $\alpha$, involving irreversibility conditions, will be considered in a second step. We introduce the variations $\bar{\epsilon}(x)$, $\bar{u}(x)$. The boundary conditions (1) imply that

$$\bar{u}(0) = 0 \text{ and } \bar{u}(1) = 0 \tag{7}$$

$$\left.\frac{\mathcal{L}(\epsilon + \eta\bar{\epsilon}, \alpha, u + \eta\bar{u}) - \mathcal{L}(\epsilon, \alpha, u)}{\eta}\right|_{\eta\to 0} = \int_0^1 a(\alpha)\,\epsilon\,\bar{\epsilon}\,\mathrm{d}x - \int_0^1 \sigma(x)\,(\bar{\epsilon} - \bar{u}')\,\mathrm{d}x = 0 \quad \forall\,\bar{\epsilon},\,\bar{u} \tag{8a}$$

$$= \int_0^1 (a(\alpha)\,\epsilon - \sigma(x))\,\bar{\epsilon} - \sigma'(x)\,\bar{u}\,\mathrm{d}x = 0 \quad \forall\,\bar{\epsilon},\,\bar{u} \tag{8b}$$

The boundary term involved in the integration by part on $\bar{u}'(x)$ identically vanishes because of (7). Condition (8b) consequently yield

$$\sigma'(x) = 0 \tag{9a}$$

$$\sigma = a(\alpha(x))\;\epsilon(x) \tag{9b}$$

We find that the axial stress in the beam $\sigma$ is uniform. As the damage field $\alpha(x)$ might not be uniform, the longitudinal strain $\epsilon(x)$ still generically depend on $x$. We now seek to minimise $\mathcal{E}$ using the equilibrium conditions we have just found. First we discard the variable $u(x)$ and replace the imposed displacement condition (1) with

$$u(1) - u(0) = \int_0^1 u'(x)\,\mathrm{d}x = \int_0^1 \epsilon(x)\,\mathrm{d}x = \Delta \tag{10}$$

Consequently we now work with the Lagrangian

$$\mathcal{L}(\epsilon(x), \alpha(x)) = \int_0^1 \frac{1}{2}a(\alpha)\,\epsilon^2\,\mathrm{d}x + \int_0^1 Ww(\alpha)\,\mathrm{d}x - \sigma\int_0^1 \epsilon(x)\,\mathrm{d}x \tag{11}$$

where the lagrange multiplier associated the the displacement condition (10) is directly identified with $\sigma$, the axial stress in the bar which is also the applied external tension. Extremizing with regard to $\epsilon(x)$ leads to (9b) which we use to rewrite (10) as

$$\sigma = \frac{\Delta}{\int_0^1 a^{-1}(\alpha)\,\mathrm{d}x} \tag{12}$$

which enable us to rewrite the strain energy as

$$\int_0^1 \frac{1}{2}a(\alpha)\,\epsilon^2\,\mathrm{d}x = \int_0^1 \frac{1}{2}\frac{\sigma^2}{a(\alpha)}\,\mathrm{d}x = \frac{1}{2}\frac{\Delta^2}{\int_0^1 a^{-1}(\alpha)\,\mathrm{d}x} \tag{13}$$

We finally obtain an energy which only depends on $\alpha(x)$

$$\mathcal{E}(\alpha(x)) = \frac{1}{2}\frac{\Delta^2}{\int_0^1 a^{-1}(\alpha)\,\mathrm{d}x} + \int_0^1 Ww(\alpha)\,\mathrm{d}x \tag{14}$$

During the loading process, $\Delta = \Delta(t)$, the field $\alpha(x, t)$ cannot decrease. A necessary condition is that, at all time

$$\forall x: \quad \dot{\alpha}(x,t) \geq 0 \;\text{ and }\; \mu(x) \geq 0 \;\text{ and }\; \mu(x)\,\dot{\alpha}(x,t) = 0 \tag{15a}$$

$$\text{with } \left.\frac{\mathcal{E}(\alpha + \eta\bar{\alpha}) - \mathcal{E}(\alpha)}{\eta}\right|_{\eta\to 0} = \int_0^1 \mu(x)\,\bar{\alpha}\;\mathrm{d}x\;\forall\bar{\alpha} \tag{15b}$$

$$\text{hence } \mu(x) = W\,w'(\alpha) + \frac{1}{2}\frac{a'(\alpha)}{a^2}\frac{\Delta^2}{\left(\int_0^1 a^{-1}(\alpha)\,\mathrm{d}x\right)^2} \tag{15c}$$

And a sufficient condition is

$$\forall \bar\alpha \geq 0 : \int_0^1 W w''(\alpha)\bar\alpha^2 \, \mathrm{d}x + \frac{1}{2}\frac{\Delta^2}{\left(\int_0^1 a^{-1}(\alpha)\,\mathrm{d}x\right)^2}\int_0^1\left(\frac{a''(\alpha)}{a^2} - 2\frac{a'(\alpha)^2}{a^3}\right)\bar\alpha^2\,\mathrm{d}x + \Delta^2\frac{\left(\int_0^1 \frac{a'(\alpha)}{a^2}\,\bar\alpha\,\mathrm{d}x\right)^2}{\left(\int_0^1 a^{-1}(\alpha)\,\mathrm{d}x\right)^3} > 0 \tag{16}$$

or, setting $s(\alpha) = 1/a(\alpha)$

$$\forall \bar\alpha \geq 0 : \int_0^1 W w''(\alpha)\bar\alpha^2 \, \mathrm{d}x - \frac{1}{2}\frac{\Delta^2}{\left(\int_0^1 s(\alpha)\,\mathrm{d}x\right)^2}\int_0^1 s''(\alpha)\,\bar\alpha^2\,\mathrm{d}x + \Delta^2\frac{\left(\int_0^1 s'(\alpha)\,\bar\alpha\,\mathrm{d}x\right)^2}{\left(\int_0^1 s(\alpha)\,\mathrm{d}x\right)^3} > 0 \tag{17}$$

## 4. Our specific material model

**seb: notations: $Y \equiv E_0$, $W = w_1$, and $\varepsilon \equiv \epsilon$**

To analyze the effect of softening on the strength, we consider a specific parametric material model characterized by the degradation functions

$$a(\alpha) := \frac{1 - w(\alpha)}{1 + (\gamma - 1)\, w(\alpha)}\,, \qquad w(\alpha) := 1 - (1-\alpha)^2 \tag{18}$$

which we call the Linear Softening (LS) model (Le et al., 2018).

A close inspection reveals that this model enjoys the following properties:

- Strain hardening if and only if $\gamma > 1$;
- Stress softening for any $\gamma > 0$;
- Maximum allowable stress $\sigma_c$ and strain for elastic limit $\varepsilon_c$ are given by:

$$\sigma_c := \sqrt{\frac{2E_0 w_1}{\gamma}}\,, \quad \varepsilon_c := \sqrt{\frac{2w_1}{E_0\gamma}} \tag{19}$$

NB: we will choose $w_1 = \gamma E_0/2$ with no loss of generality.

### 4.1. Homogenous evolutions

The evolution of stress, strain, and damage under monotonically increasing displacement-controlled loading, where the response is uniform (homogenous) $\varepsilon L = \Delta$, the response is purely elastic for $\Delta \leq \varepsilon_c = \sqrt{2w_1/(E_0\,\gamma)}$, where the damage criterion is not still attained. For $\Delta \geq \varepsilon_\infty := \sqrt{2w_1\gamma/E_0}$ the full damage level is attained and the stress is zero. For intermediate value, the damage is found by solving for the damage criterion as an equality. This gives the following strain-stress relations

$$\begin{cases} \sigma = E_0\,\varepsilon, & \text{if } \varepsilon \leq \varepsilon_c\,, \\ \sigma = E_0\left(\varepsilon - \sqrt{\frac{2w_1\gamma}{E_0}}\right), & \text{if } \varepsilon_c \leq \varepsilon \leq \varepsilon_\infty, \\ \sigma = 0, & \text{if } \varepsilon_\infty \leq \varepsilon. \end{cases} \tag{20}$$

During the second phase ($\varepsilon_c \leq \varepsilon \leq \varepsilon_\infty$) we have

$$\varepsilon_*(\alpha) = \sqrt{\frac{2w_1}{E_0\gamma}}\,(1 + \alpha\,(\gamma - 1)(2 - \alpha)) \tag{21a}$$

$$\sigma_*(\alpha) = \sqrt{\frac{2E_0 w_1}{\gamma}}(1-\alpha)^2 \tag{21b}$$

NB: once the adim variables are used, $\sigma_c = 1$, $\varepsilon_c = 1$, and $\varepsilon_\infty = \gamma$.

## 5. Discretisation

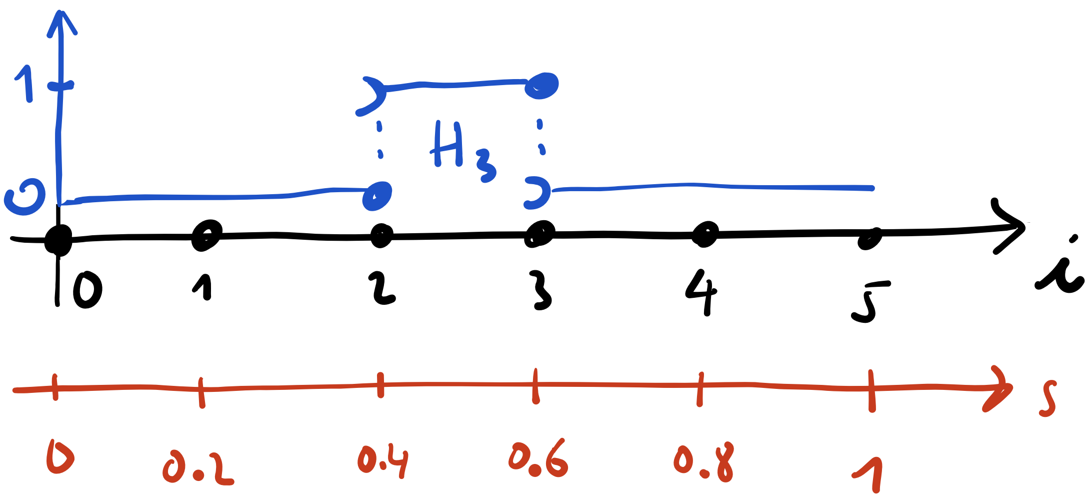


Figure 1: —.

We introduce $N$ segments of equal size $h = 1/N$, with $s_i = i\,h$ and $i \in (1;N)$, and use the following base functions, see Figure 1

$$H_i(x) = 1 \;\text{ if }\; x_{i-1} \le x \le x_i \tag{22a}$$

$$H_i(x) = 0 \;\text{ otherwise} \tag{22b}$$

The variation $\bar{\alpha}(x)$ and $\alpha(x)$ are represented as

$$\bar{\alpha}(s) = \sum_{i=1}^{N} \bar{\alpha}_i\, H_i(s) \tag{23a}$$

$$\alpha(s) = \sum_{i=1}^{N} \alpha_i\, H_i(s) \tag{23b}$$

We note $S = \int_0^1 s(\alpha)\,\mathrm{d}x = h\,\sum_i s(\alpha_i)$. Equation (15c) reads

$$\mu_i = W\,w'(\alpha_i) + \frac{1}{2}\,\frac{a'(\alpha_i)}{a^2}\,\frac{\Delta^2}{S^2} \tag{24}$$

The second variation in (17) can then be written in matrix form as

$$\text{for } i \neq j:\; H_{ij} = \Delta^2\,h^2\,\frac{s'(\alpha_i)\,s'(\alpha_j)}{S^3} \tag{25a}$$

$$H_{ii} = h\,W w''(\alpha_i) - \frac{\Delta^2}{2}\,h\,\frac{s''(\alpha_i)}{S^2} + \Delta^2\,h^2\,\frac{s'(\alpha_i)^2}{S^3} \tag{25b}$$

And the condition (17) can be written as

$$\forall \bar{\alpha}_i > 0\,, \forall \bar{\alpha}_j > 0:\; \sum_{i,j} H_{ij}\,\bar{\alpha}_i\,\bar{\alpha}_j > 0 \tag{26}$$

this condition is fulfilled iff

$$\forall i : 0 < H_{ii} \tag{27a}$$

$$\text{and } \forall i \neq j : -\sqrt{H_{ii}H_{jj}} < H_{ij} \tag{27b}$$

which seems to be related to Perron-Froebenius theorem.

Perron-Froebenius theorem says that if a matrix has all its entries > 0, then it admits an eigenvector with only > 0 components. The associated eigenvalue is > 0 and is equal to the spectral radius of the matrix (max of the module of all eigenvalues).

## 6. Loss of cone-stability

In this section, we show that *all* equilibrium branches loose stability when $\Delta = h\,\gamma$. Please note that for an index $i$ experiencing damage, we have $\mu_i = 0$ in (24), that is

$$\mu_i = \frac{\gamma}{2}\left(1 - \frac{1}{(1-\alpha)^2}\,\frac{\Delta^2}{S^2}\right) = 0 \tag{28a}$$

$$\text{that is } \Delta = (1-\alpha_i)\,S\,(\alpha_1, \alpha_2, \ldots, \alpha_N) \tag{28b}$$

Please also note that (25b) writes

$$H_{ii} = \frac{h\,\Delta^2\gamma}{S^3(1-\alpha_i)^4}\,(-(1-\alpha_i)S\,(\alpha_1, \alpha_2, \ldots, \alpha_N) + h\gamma) \tag{29}$$

### 6.1. *The branch of homogeneous solutions*

On this branch we have $\alpha_i = \alpha_H$ for all $i$. Hence $S = h\sum_i s(\alpha_i) = hNs(\alpha_H) = s(\alpha_H)$. For each $i$, (28a) holds. All diagonal terms in the Hessian matrix are the same, and (29) with (28a) yields

$$H_{ii} = \frac{h\,\Delta^2\gamma}{S^3(1-\alpha_H)^4}\,(-\Delta + h\gamma) \tag{30}$$

So the loss of stability happens when $\Delta = h\gamma$.

### 6.2. *Any branch with several i experiencing damage*

On a branche with indices $i$ damaging and indices $j$ passives, we have that all the $\alpha_i$ share the same value (because of history), and all the $\mu_j > 0$ (because the value of $\sigma = \Delta/S$ has decreased since this $j$ was damaging, see (28a)). Hence the stability test has only to be carried for the indices $i$, and consequently yields the same result as in (30).

### 6.3. *Remark for AT1*

For AT1 model, (28b) reads $\Delta = S\,(1-\alpha_i)^{3/2}$ and (29) reads $H_{ii} = c\,(-(1-\alpha_i)^2\,S + (4/3)h)$. For the branch of homogeneous solutions, we can evaluate $S = (1-\alpha_H)^{-2}$ and hence $H_{ii} = c\,(-1 + (4/3)h)$ (stable only for $N < 4/3$ !). But for other branches, a formula for $H_{ii}$ involving only $\Delta$, as (30), does not exist.

## References


Le, D.T., Marigo, J.J., Maurini, C., Vidoli, S., 2018. Strain-gradient vs damage-gradient regularizations of softening damage models. Computer Methods in Applied Mechanics and Engineering 340, 424–450.