Discrete Damage

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Abstract

Keywords:

1. Variational formulation

A bar of length L and section S is stretched along the x axis. The material of the bar has nominal Young's modulus Y. We note $u(x)$ the horizontal displacement. The bar is clamped at its left end, and a displacement $\Delta \geq 0$ is imposed of the right end

$$u(0) = 0\,, \quad u(L) = \Delta \tag{1}$$

The total energy of the bar is

$$\mathcal{E}(\epsilon(x), \alpha(x)) = \int_0^L \frac{1}{2} YS\, a(\alpha(x))\, \epsilon^2(x)\, \mathrm{d}x + \int_0^L WS\, w(\alpha(x))\, \mathrm{d}x \tag{2}$$

where the first integral is the strain energy and W is the dissipated energy (per unit volume) due to damage and ϵ is the longitudinal strain. The functions $a(\alpha)$ and $w(\alpha)$ are non-dimensionalized functions, to be discussed later on. As the boundary conditions are written with $u(x)$, we need to include the constraint

$$\epsilon = u'(x) \tag{3}$$

that is, work with the Lagrangian

$$\mathcal{L}(\epsilon, \alpha, u) = \mathcal{E}(\epsilon, \alpha) - \int_0^L \sigma(x)\, S\, (\epsilon - u')\, \mathrm{d}x \tag{4}$$

where the continuous lagrange multiplier $\sigma(x)$ is identified with the axial stress in the bar.

2. Non-dimensionalisation

In this statics problem, we can freely (and with no loss of generality) choose a unit length and a unit force. We choose L as unit length, and YS as unit force. We introduce the 'hat' adim variables

$$\hat{x} = x/L\,,\ \hat{u} = u/L\,,\ \hat{\Delta} = \Delta/L\,,\ \hat{W} = W/Y\,,\ \hat{\sigma} = \sigma/Y\,,\ \hat{\mathcal{E}} = \frac{\mathcal{E}}{YSL}\,,\ \hat{\mathcal{L}} = \frac{\mathcal{L}}{YSL} \tag{5}$$

and simplify (4) to

$$\hat{\mathcal{E}}(\epsilon, \alpha) = \int_0^1 \frac{1}{2} a(\alpha)\, \epsilon^2(\hat{x})\, \mathrm{d}\hat{x} + \int_0^1 \hat{W} w(\alpha)\, \mathrm{d}\hat{x} \tag{6a}$$

$$\hat{\mathcal{L}}(\epsilon, \alpha, \hat{u}) = \hat{\mathcal{E}}(\epsilon, \alpha) - \int_0^1 \hat{\sigma}(\hat{x})\, (\epsilon - u'(\hat{x}))\, \mathrm{d}\hat{x} \tag{6b}$$

And from now on, we drop the 'hats' while keeping in mind that we deal with adim variables.

3. First variation

We seek extremal configurations of $\mathcal{E}(\epsilon,\alpha)$, under the constraints (3) and (1). Applying the lagrange multiplier rule, we work with $\mathcal{L}$ and look for the conditions under which the first variation of $\mathcal{L}$ vanishes, for the the two variables ϵ and $\hat{u}$. The minimisation with regard to the third variable α, involving irreversibility conditions, will be considered in a second step. We introduce the variations $\bar{\epsilon}(x)$, $\bar{u}(x)$. The boundary conditions (1) imply that

$$\bar{u}(0) = 0 \text{ and } \bar{u}(1) = 0 \tag{7}$$

$$\left.\frac{\mathcal{L}(\epsilon+\eta\bar{\epsilon},\alpha,u+\eta\bar{u})-\mathcal{L}(\epsilon,\alpha,u)}{\eta}\right|_{\eta\to 0} = \int_0^1 a(\alpha)\,\epsilon\,\bar{\epsilon}\,\mathrm{d}x - \int_0^1 \sigma(x)\,(\bar{\epsilon}-\bar{u}')\,\mathrm{d}x = 0 \quad \forall\,\bar{\epsilon},\,\bar{u} \tag{8a}$$

$$= \int_0^1 (a(\alpha)\,\epsilon - \sigma(x))\,\bar{\epsilon} - \sigma'(x)\,\bar{u}\,\mathrm{d}x = 0 \quad \forall\,\bar{\epsilon},\,\bar{u} \tag{8b}$$

The boundary term involved in the integration by part on $\bar{u}'(x)$ identically vanishes because of (7). Condition (8b) consequently yield

$$\sigma'(x) = 0 \tag{9a}$$

$$\sigma = a(\alpha(x))\;\epsilon(x) \tag{9b}$$

We find that the axial stress in the beam σ is uniform. As the damage field $\alpha(x)$ might not be uniform, the longitudinal strain $\epsilon(x)$ still generically depend on x. We now seek to minimise $\mathcal{E}$ using the equilibrium conditions we have just found. First we discard the variable $u(x)$ and replace the imposed displacement condition (1) with

$$u(1) - u(0) = \int_0^1 u'(x)\,\mathrm{d}x = \int_0^1 \epsilon(x)\,\mathrm{d}x = \Delta \tag{10}$$

Consequently we now work with the Lagrangian

$$\mathcal{L}(\epsilon(x),\alpha(x)) = \int_0^1 \frac{1}{2}a(\alpha)\,\epsilon^2\,\mathrm{d}x + \int_0^1 W w(\alpha)\,\mathrm{d}x - \sigma\int_0^1 \epsilon(x)\,\mathrm{d}x \tag{11}$$

where the lagrange multiplier associated the the displacement condition (10) is directly identified with σ, the axial stress in the bar which is also the applied external tension. Extremizing with regard to $\epsilon(x)$ leads to (9b) which we use to rewrite (10) as

$$\sigma = \frac{\Delta}{\int_0^1 a^{-1}(\alpha)\,\mathrm{d}x} \tag{12}$$

which enable us to rewrite the strain energy as

$$\int_0^1 \frac{1}{2}a(\alpha)\,\epsilon^2\,\mathrm{d}x = \int_0^1 \frac{1}{2}\frac{\sigma^2}{a(\alpha)}\,\mathrm{d}x = \frac{1}{2}\frac{\Delta^2}{\int_0^1 a^{-1}(\alpha)\,\mathrm{d}x} \tag{13}$$

We finally obtain an energy which only depends on $\alpha(x)$

$$\mathcal{E}(\alpha(x)) = \frac{1}{2}\frac{\Delta^2}{\int_0^1 a^{-1}(\alpha)\,\mathrm{d}x} + \int_0^1 W w(\alpha)\,\mathrm{d}x \tag{14}$$

During the loading process, $\Delta = \Delta(t)$, the field $\alpha(x,t)$ cannot decrease. A necessary condition is that, at all time

$$\forall x: \quad \dot{\alpha}(x,t) \geq 0 \;\text{ and }\; \mu(x) \geq 0 \;\text{ and }\; \mu(x)\,\dot{\alpha}(x,t) = 0 \tag{15a}$$

$$\text{with } \left.\frac{\mathcal{E}(\alpha+\eta\bar{\alpha})-\mathcal{E}(\alpha)}{\eta}\right|_{\eta\to 0} = \int_0^1 \mu(x)\,\bar{\alpha}\;\mathrm{d}x\;\forall\bar{\alpha} \tag{15b}$$

$$\text{hence } \mu(x) = W\,w'(\alpha) + \frac{1}{2}\frac{a'(\alpha)}{a^2}\frac{\Delta^2}{\left(\int_0^1 a^{-1}(\alpha)\,\mathrm{d}x\right)^2} \tag{15c}$$

And a sufficient condition is

$$\forall \bar{\alpha} \geq 0 : \int_0^1 W w''(\alpha)\bar{\alpha}^2 \,\mathrm{d}x + \frac{1}{2} \frac{\Delta^2}{\left(\int_0^1 a^{-1}(\alpha)\,\mathrm{d}x\right)^2} \int_0^1 \left(\frac{a''(\alpha)}{a^2} - 2\frac{a'(\alpha)^2}{a^3}\right)\bar{\alpha}^2 \,\mathrm{d}x + \Delta^2 \frac{\left(\int_0^1 \frac{a'(\alpha)}{a^2}\,\bar{\alpha}\,\mathrm{d}x\right)^2}{\left(\int_0^1 a^{-1}(\alpha)\,\mathrm{d}x\right)^3} > 0 \tag{16}$$

or, setting $s(\alpha) = 1/a(\alpha)$

$$\forall \bar{\alpha} \geq 0 : \int_0^1 W w''(\alpha)\bar{\alpha}^2 \,\mathrm{d}x - \frac{1}{2} \frac{\Delta^2}{\left(\int_0^1 s(\alpha)\,\mathrm{d}x\right)^2} \int_0^1 s''(\alpha)\,\bar{\alpha}^2 \,\mathrm{d}x + \Delta^2 \frac{\left(\int_0^1 s'(\alpha)\,\bar{\alpha}\,\mathrm{d}x\right)^2}{\left(\int_0^1 s(\alpha)\,\mathrm{d}x\right)^3} > 0 \tag{17}$$

4. Discretisation

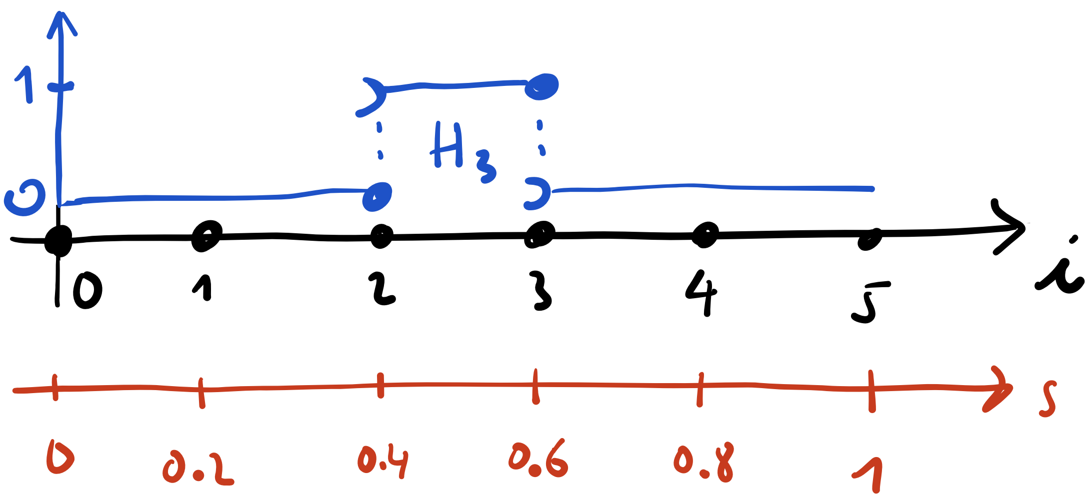

Figure 1: —.

We introduce N segments of equal size $h = 1/N$, with $s_i = i\,h$ and $i \in (1; N)$, and use the following base functions, see Figure 1

$$H_i(x) = 1 \;\text{ if }\; x_{i-1} \leq x \leq x_i \tag{18a}$$
$$H_i(x) = 0 \;\text{ otherwise} \tag{18b}$$

The variation $\bar{\alpha}(x)$ and $\alpha(x)$ are represented as

$$\bar{\alpha}(s) = \sum_{i=1}^{N} \bar{\alpha}_i\, H_i(s) \tag{19a}$$
$$\alpha(s) = \sum_{i=1}^{N} \alpha_i\, H_i(s) \tag{19b}$$

We note $S = \int_0^1 s(\alpha)\,\mathrm{d}x$. The second variation in (17) can then be written in matrix form as

$$\text{for } i \neq j : \; H_{ij} = \Delta^2\, h^2\, \frac{s'(\alpha_i)\, s'(\alpha_j)}{S^3} \tag{20a}$$
$$H_{ii} = h\, W w''(\alpha_i) - \frac{\Delta^2}{2}\, h\, \frac{s''(\alpha_i)}{S^2} + \Delta^2\, h^2\, \frac{s'(\alpha_i)^2}{S^3} \tag{20b}$$

And the condition (17) can be written as

$$\forall \bar{\alpha}_i > 0\,, \forall \bar{\alpha}_j > 0 : \sum_{i,j} H_{ij}\, \alpha_i\, \alpha_j > 0 \tag{21}$$

this condition is fulfilled if

$$\forall i, j : H_{ij} > 0 \tag{22}$$