Homework 15

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Problem 1. What is wrong with the following "proof" by mathematical induction?

> We will prove that all computers are built by the same manufacturer. In particular, we will prove that in any collection of n computers, where n is a positive integer, all of the computers are built by the same manufacturer. We first prove $P(1)$. This is a trivial process because in any collection consisting of one computer, there is only one manufacturer. Now, we assume $P(k)$, i.e., the inductive hypothesis. That is, in any collection of k computers, all the computers are built by the same manufacturer. To prove $P(k+1)$, we consider any collection of $k+1$ computers. Pull one of these $k+1$ computers (call it HAL) out of the collection. By our assumption, the remaining k computers all have the same manufacturer. Let HAL change places with one of these k computers. In the new group of k computers, all have the same manufacturer. Thus, HAL's manufacturer is the same one that produced all the other computers, and all $k+1$ computers have the same manufacturer.

The issue is that during the inductive step, proving $P(k+1)$ does not immediately follow from $P(k)$ when $k = 1$. This is because when you have 2 computers, and you take 1 out, the remaining computer will trivially be from the same manufacturer as itself. When you put the taken out computer back in and take out the one that wasn't taken, again, you have one computer from the same manufacturer as itself, but not necessarily the same as the one taken out.

Problem 2. An obscure tribe has only three words in its language, *moon*, *noon*, and *soon*. New words are composed by juxtaposing these words in any order, as in *soonnoonmoonnoon*. Any such juxtaposition is a legal word.

1. Use mathematical induction (also called *first principle of induction*) on the number of subwords in the word to prove that any word in this language has an even number of o's.

   Basis step: $P(1)$: when there is only one word in the juxtaposition, it has to be either *moon*, *noon*, or *soon*, all of which have exactly 2 o's, therefore even and $P(1)$ is true.
   Inductive hypothesis: Assume $P(k)$ is true, namely any juxtaposition of k words has an even number of o's. Namely any juxtaposition $w_1 w_2 \dots w_k$ has $2m$ o's, for some integer m.
   Inductive step: Need to show $P(k+1)$ is true using $P(k)$. For any juxtaposition of $k+1$ words, $w_1 w_2 \dots w_k w_{k+1}$, the first k words, $w_1 w_2 \dots w_k$, has an even number, $2m$, of o's ($P(k)$). Since w_{k+1} is one word, it has exactly 2 o's, so when appended to the end of $w_1 w_2 \dots w_k$, we have a total of $2m + 2 = 2(m+1)$ o's, which is even. Therefore $P(k+1)$ is true. $\square$

2. Use strong induction (also called *second principle of induction*) on the number of subwords in the word to prove that any word in this language has an even number of o's.

   Basis step: $P(1)$: when there is only one word in the juxtaposition, it has to be either *moon*, *noon*, or *soon*, all of which have exactly 2 o's, therefore even and $P(1)$ is true.
   Inductive hypotheses: Assume $P(1), P(2), \dots, P(k)$ is true, namely any juxtaposition of 1 to k words has an even number of o's.
   Inductive step: Need to show $P(k+1)$ is true using $P(1), P(2), \dots P(k)$. For any juxtaposition of $k+1$ words, $w_1 w_2 \dots w_k w_{k+1}$, we can break it into the concatenation of two smaller juxtapositions, namely $jux_a = w_1 w_2 \dots w_i$ and $jux_b = w_{i+1} \dots w_k w_{k+1}$. The number of words in jux_a is less than or equal to k, and the same for jux_b. So the inductive hypotheses apply to both, so let's call $2l$ the number of o's for jux_a and $2m$ for jux_b. Therefore the original juxtaposition with $k+1$ words has $2l + 2m = 2lm$ o's, an even number. Therefore $P(k+1)$ is true. $\square$

Problem 3. Your bank ATM delivers cash using only \$20 and \$50 bills. Prove that you can collect, in addition to \$20, any multiple of \$10 that is \$40 or greater.

Basis step: $P(40)$: $40 = 20 + 20$, so \$40 is an amount that is collectable, so $P(40)$ is true. $P(50)$: $50 = 50$, so \$50 is an amount that is also collectable, so $P(50)$ is also true.
Inductive hypotheses: Assume $P(40), P(50), \ldots P(10k)$ is true, namely any multiple of \$10 greater than \$40 and less than \$10k is collectable.
Inductive step: Need to show $P(10(k+1))$ is true using $P(40), P(50), \ldots P(10k)$. Since \$10(k-1) is collectable ($P(10(k-1))$) and the difference between \$10(k+1) and \$10(k-1) is \$20, we can simply add an additional \$20 to the \$10(k-1) solution for a solution to \$10(k+1), therefore $P(10(k+1))$ is true when $k > 4$. The corner case when $k = 4$ is handled by the second basis step of $P(50)$, since \$30 is not collectable but \$50 is. $\square$