Final Review

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Problem 6. Let $P(n)$ be the statement:

$$\log n! > \frac{n \log n}{4}, \text{ for } n > 4$$

1. What is the statement $P(5)$?

$$P(5): \log 5! > \frac{5 \log 5}{4}$$

2. Show that $P(5)$ is true, completing the basis step of the proof.

$$\log 5! = \log(5 \times 4 \times 3 \times 2 \times 1) = \log(5 \times 24) > \log(5 \times 5) = \log 5^2 = 2 \log 5 > \frac{5 \log 5}{4}$$

3. What is the inductive hypothesis?

$$\text{Assume } P(k) \text{ is true: } \log k! > \frac{k \log k}{4}$$

4. What do you need to prove in the inductive step?

$$\text{Need to prove } P(k+1) \text{ is true: } \log(k+1)! > \frac{(k+1)\log(k+1)}{4}$$

5. Complete the inductive step. You must justify any single step in your proof. Otherwise, your answer is wrong. Show your work step by step.

$$\log(k+1)! = \log((k+1)k!) = \log(k+1) + \log k! > \log(k+1) + \frac{k \log k}{4}$$

$$\log(k+1) + \frac{k \log k}{4} = \frac{4\log(k+1) + k \log k}{4} = \frac{\log((k+1)^4 k^k)}{4}$$

Lemma Q. Let $Q(n)$ be the statement:

$$n^n > (n+1)^{n-1}, \text{ for } n > 2$$

Basis: $Q(3): \; 3^3 = 27 > (3+1)^{3-1} = 4^2 = 16$, so $Q(3)$ is true.
Inductive: Assume $Q(k)$ is true: $k^k > (k+1)^{k-1}$
Need to prove $Q(k+1)$: $(k+1)^{k+1} > (k+2)^k$

$$(k+1)^{k+1} = k^k \left(1 + \frac{1}{k}\right)^k (k+1) > (k+1)^{k-1}(k+1)\left(1 + \frac{1}{k}\right)^k$$

$$= (k+1)^k \left(1 + \frac{1}{k}\right)^k = \left(k + \frac{1}{k} + 2\right)^k > (k+2)^k \to \text{Q(k+1) is true.}$$

$$[Q(3) \wedge (Q(k) \to Q(k+1))] \to \forall n Q(n)$$

Using **Lemma Q**, we can proceed as follows:

$$\frac{\log((k+1)^4 k^k)}{4} > \frac{\log((k+1)^4 (k+1)^{k-1})}{4} = \frac{(k+3)\log(k+1)}{4} > \frac{(k+1)\log(k+1)}{4}$$

Therefore $P(k+1)$ is true.

$$[P(5) \wedge (P(k) \to P(k+1))] \to \forall n P(n)$$