Homework 11

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Problem 1. Use one of the proof methods to prove the following results.

1. Prove that $n^2 + 1 \geq 2n$, where n is a positive integer with $1 \leq n \leq 4$.

Exhaustive proof: since n is an integer between 1 and 4, inclusive, there are only 4 different cases to consider.
$n = 1 \rightarrow n^2 + 1 = 1^2 + 1 = 2 \geq 2(1) = 2$
$n = 2 \rightarrow n^2 + 1 = 2^2 + 1 = 5 \geq 2(2) = 4$
$n = 3 \rightarrow n^2 + 1 = 3^2 + 1 = 10 \geq 2(3) = 6$
$n = 4 \rightarrow n^2 + 1 = 4^2 + 1 = 17 \geq 2(4) = 8 \quad \square$

2. Prove that if x and y are real numbers, then $max(x, y) + min(x, y) = x + y$.

Proof by cases: since it is either the case that $x \geq y$ or $x < y$, we have two cases to consider.
$x \geq y \rightarrow max(x, y) = x$ and $min(x, y) = y \rightarrow max(x, y) + min(x, y) = x + y$
$x < y \rightarrow max(x, y) = y$ and $min(x, y) = x \rightarrow max(x, y) + min(x, y) = y + x = x + y \quad \square$

Problem 2. Consider the set of integers.

1. Use a direct proof to show that the sum of two even integers is even

Let a and b be two even integers.
Since a is even, $a = 2m$ for some integer m.
Likewise $b = 2n$ for some integer n.
$a + b = 2m + 2n = 2(m + n) \quad \square$

2. Prove that if $m+n$ and $n+p$ are even integers, where m, n, and p are integers, then $m + p$ is even. What kind if proof did you use?

Direct proof: Since $m + n$ and $n + p$ are even integers, their sum must be even from the previous proof:
$(m + n) + (n + p) = m + 2n + p = 2k$ for some integer k
Subtract $2n$ from both sides we get $m + p = 2k - 2n = 2(k - n) \quad \square$

3. Use a direct proof to show that every odd integer is the difference of two squares

Every odd integer can be represented as $2k+1$ for some integer k.
$(k+1)^2 - k^2 = k^2 + 2k + 1 - k^2 = 2k + 1 \quad \Box$

4. Prove that if n is an integer and $3n+2$ is even, then n is even using
a) A proof by contraposition.

P: $3n+2$ is even.
Q: n is even.
$\neg Q$: n is odd, so $n = 2k+1$ for some integer k.
$3n+2 = 3(2k+1)+2 = 6k+3+2 = 6k+4+1 = 2(3k+2)+1 \equiv \neg P$
$\neg Q \rightarrow \neg P \quad \Box$

b) A proof by contradiction.

P: $3n+2$ is even.
Q: n is even.
$P \wedge \neg Q$: $3n+2$ is even and n is odd.
Since n is odd, $n = 2k+1$ for some integer k
$3n+2 = 3(2k+1)+2 = 6k+3+2 = 6k+4+1 = 2(3k+2)+1 \equiv \neg P$.
$P \wedge \neg P$ contradiction $\quad \Box$.

5. Prove that if n is an integer, the following four statements are equivalent:
a) n is even.
b) $n+1$ is odd.
c) $3n+1$ is odd.
d) $3n$ is even.

So we need to show $a \leftrightarrow b \leftrightarrow c \leftrightarrow d$.
$a \rightarrow b$: Direct: If n is even, $n = 2k$ for some integer k, so $n+1 = 2k+1$.
$b \rightarrow a$: Direct: If $n+1$ is odd, $n+1 = 2k+1$ for some integer k, so $n = 2k$.
$a \rightarrow c$: Direct: If n is even, $n = 2k$ for some integer k, so $3n+1 = 3(2k)+1 = 2(3k)+1$.
$c \rightarrow a$: Contraposition: If n is odd, $n = 2k+1$ for some integer k, so $3n+1 = 3(2k+1)+1 = 6k+3+1 = 6k+4 = 2(3k+2)$.
$a \rightarrow d$: Direct: If n is even, $n = 2k$ for some integer k, so $3n = 3(2k) = 2(3k)$.
$d \rightarrow a$: Contraposition: If n is odd, $n = 2k+1$ for some integer k, so $3n = 3(2k+1) = 6k+3 = 6k+2+1 = 2(3k+1)+1$.
The rest of missing implications can be found using the hypothetical syllogism inference rule (i.e. $(b \rightarrow a) \wedge (a \rightarrow c) \equiv b \rightarrow c$).

6. Find a counterexample to the statement “Every positive integer can be written as the sum of the squares of three integers”.

7 cannot be written as a sum of three squares since:
$0 + 0 + 0 = 0$
$0 + 0 + 1 = 1$
$0 + 1 + 1 = 2$
$1 + 1 + 1 = 3$
$0 + 0 + 4 = 4$
$0 + 1 + 4 = 5$
$1 + 1 + 4 = 6$
$0 + 4 + 4 = 8$
$1 + 4 + 4 = 9$
$4 + 4 + 4 = 12$
Since 0, 1, and 4 are the only squares less than 7, and all their combinations fail