Discussion 14

Mengxiang Jiang
CSEN 5303 Foundations of Computer Science

November 19, 2022

Problem statement. Discuss the mathematical induction proof method. Provide an illustrative example.

Mathematical induction is a proof technique that applies to statements about integers. Suppose $P(n)$ is a statement about the integer n, and we want to prove it is true for all n in a certain iteratable domain (usually this is the positive integers or can be converted to such). The following are the steps of the proof:

1. Basis step: prove $P(1)$ is true.
2. Inductive hypothesis: assume $P(k)$ is true for any $k \geq 1$
3. Inductive step: Use $P(k)$ to prove $P(k+1)$. Once this is shown, the proof is complete.

The proof takes advantage of the fact that any positive integer is by definition either 1 or a successor of another positive integer, thereby covering all possible cases.

An illustrative example of proof by induction is the Fibonacci inequality: Let the *Fibonacci sequence* be defined as $F_0 = 0$, $F_1 = 1$ and all subsequent terms be $F_{n+2} = F_n + F_{n+1}$.
Let ϕ be $\frac{1+\sqrt{5}}{2}$.
Then $F_n \leq \phi^{n-1}$ for all positive integers n.

1. Basis step: $P(1)$ is the claim $F_1 = 1 \leq \phi^{1-1} = \phi^0 = 1$, so $P(1)$ is true.
However there is a second basis step required, namely $P(2)$:
$F_2 = F_0 + F_1 = 0 + 1 = 1 \leq \phi^{2-1} = \phi^1 = \frac{1+\sqrt{5}}{2}$.
This is true since $\sqrt{5} > \sqrt{4} = 2 \rightarrow \frac{1+\sqrt{5}}{2} > \frac{1+2}{2} > \frac{2}{2} = 1$.

2. Inductive hypothesis: we'll need to assume two inductive hypotheses, namely $P(k)$ and $P(k+1)$ is true, which is why we needed two base cases for $k \geq 1$ and $(k+1) \geq 2$.

3. Inductive step: $P(k+2)$: $F_{k+2} \leq \phi^{k+2-1}$. To show this we have:
$F_{k+2} = F_k + F_{k+1} \leq \phi^{k-1} + \phi^k$ from the inductive hypotheses.
One additional fact we have to make use of is the fact that ϕ is known as the golden ratio with the following property:
$1 + \phi = \phi^2$
We see that we can factor $\phi^{k-1} + \phi^k = \phi^{k-1}(1 + \phi)$
Substituting $1 + \phi = \phi^2$ in, we have $\phi^{k-1}(\phi^2) = \phi^{k-1+2} = \phi^{k+2-1}$ as desired.
$\therefore P(k+2) \equiv F_{k+2} \leq \phi^{k+2-1}$ is true $\quad\square$