Homework 13

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Problem 1. Determine which rule of inference is used in each of the following arguments.

1. Kangaroos live in Australia and are marsupials. Therefore, kangaroos are marsupials.
Simplification: $(p \wedge q) \to p$

2. It is hotter than 100 degrees today or the pollution is dangerous. It is less than 100 degrees outside today. Therefore, the pollution is dangerous.
Disjunctive syllogism: $[(p \vee q) \wedge \neg p] \to q$

3. Linda is an excellent swimmer. If Linda is an excellent swimmer, then she can work as a lifeguard. Therefore, Linda can work as a lifeguard.
Modus ponens: $[p \wedge (p \to q)] \to q$

4. Steve will work at a computer company this summer. Therefore, this summer Steve will work at a computer company or he will be in beach bun.
Addition: $p \to (p \vee q)$

5. If I work all night on this homework, then I can answer all the exercises. If I answer all the exercises, I will understand the material. Therefore, if I work all night on this homework, then I will understand the material.
Hypothetical syllogism: $[(p \to q) \wedge (q \to r)] \to (p \to r)$

6. If George does not have eight legs, then he is not an insect. George is an insect. Therefore, George has eight legs.
Modus tollens: $[\neg q \wedge (p \to q)] \to \neg p$

Problem 2. Let P(n) be the statement that $1^2 + 2^2 + \ldots + n^2 = n(n+1)(2n+1)/6$ for the positive integer n.

1. What is the statement P(1)?
$1^2 = 1(1+1)(2+1)/6$

2. Show that P(1) is ture, completing the basis step of the proof.
$1^2 = 1(1) = 1$
$1(1+1)(2+1)/6 = 1(2)(3)/6 = 6/6 = 1$
$\therefore$ P(1) is true.

3. What is the inductive hypothesis?
Assume P(k) is true for some integer $k \geq 1$.

4. What do you need to prove in the inductive step?
Prove P($k+1$) is true using P(k).

5. Complete the inductive step.
P($k+1$): $1^2 + 2^2 + \ldots + k^2 + (k+1)^2 = (1^2 + 2^2 + \ldots + k^2) + (k+1)^2$

$$= \frac{k(k+1)(2k+1)}{6} + (k+1)^2 = (k+1)\left(\frac{2k^2+k}{6} + k + 1\right)$$

$$= (k+1)\left(\frac{2k^2+7k+6}{6}\right) = (k+1)\left(\frac{(k+2)(2k+3)}{6}\right)$$

$$= \frac{\Big(k+1\Big)\Big((k+1)+1\Big)\Big(2(k+1)+1\Big)}{6}$$

6. Explain why these steps show that this formula is true whenever n is a positive integer.
Since we have shown P(1) is true in step 2, we can let $k = 1$ in step 3 and show P(2) is true in step 5. We can then set $k = 2$ in step 3 and show P(3) is true in step 5, and so on for all the integers.

Problem 6. Prove each of the following statements.

1. 2 divides $n^2 + n$ whenever n is a positive integer.
Two cases for n:
Case 1: $n \equiv 0 \pmod 2 \to n^2 + n \equiv 0^2 + 0 \equiv 0 \pmod 2 \to 2|(n^2 + n)$
Case 2: $n \equiv 1 \pmod 2 \to n^2 + n \equiv 1^2 + 1 \equiv 2 \equiv 0 \pmod 2 \to 2|(n^2 + n)$

2. 3 divides $n^3 + 2n$ whenever n is a positive integer.
Three cases for n:
Case 1: $n \equiv 0 \pmod 3 \to n^3 + 2n \equiv 0^3 + 2(0) \equiv 0 \pmod 3 \to 3|(n^3 + 2n)$
Case 2: $n \equiv 1 \pmod 3 \to n^3 + 2n \equiv 1^3 + 2(1) \equiv 3 \equiv 0 \pmod 3 \to 3|(n^3 + 2n)$
Case 3: $n \equiv 2 \pmod 3 \to n^3 + 2n \equiv 2^3 + 2(2) \equiv 12 \equiv 0 \pmod 3 \to 3|(n^3 + 2n)$

3. 5 divides $n^5 - n$ whenever n is a nonnegative integer.
Five cases for n:
Case 1: $n \equiv 0 \pmod 5 \to n^5 - n \equiv 0^5 - 0 \equiv 0 \pmod 5 \to 5|(n^5 - n)$
Case 2: $n \equiv 1 \pmod 5 \to n^5 - n \equiv 1^5 - 1 \equiv 0 \pmod 5 \to 5|(n^5 - n)$
Case 3: $n \equiv 2 \pmod 5 \to n^5 - n \equiv 2^5 - 2 \equiv 30 \equiv 0 \pmod 5 \to 5|(n^5 - n)$
Case 4: $n \equiv 3 \pmod 5 \to n^5 - n \equiv n(n^4 - 1) \equiv n(n^2 - 1)(n^2 + 1)$
$\equiv 3(3^2 - 1)(3^2 + 1) \equiv 3(8)(10) \equiv 3(8)(0) \equiv 0 \pmod 5 \to 5|(n^5 - n)$
Case 5: $n \equiv 4 \pmod 5 \to n^5 - n \equiv n(n^4 - 1) \equiv n(n^2 - 1)(n^2 + 1)$
$\equiv n(n + 1)(n - 1)(n^2 + 1) \equiv 4(0)(3)(4^2 + 1) \equiv 0 \pmod 5 \to 5|(n^5 - n)$

4. 6 divides $n^3 - n$ whenever n is a nonnegative integer.
$n^3 - n = n(n^2 - 1) = n(n + 1)(n - 1) = (n - 1)(n)(n + 1)$
$\therefore n^3 - n$ is the product of 3 consecutive integers.
Given 3 consecutive integers, at least one must be a multiple of 3, and one (could be the same one) a multiple of 2 (pigeonhole principle). The product of the 3 integers therefore contains both 2 and 3 as factors, therefore the product must be divisible by 6.