Homework 2

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Problem 1. For each of the following equations, determine the order of the equation and then test it for (i) linearity, (ii) time invariance, and (iii) homogeneity.

a. $y(k+2) = y(k+1)y(k) + u(k)$
This equation is 2nd order from the difference of $k+2$ and k.
It is not linear since $y(k+1)y(k)$ is not linear.
It is time invariant because all the coefficients are constant.
Since $u(k)$ is not 0, it is not homogeneous.

b. $y(k+3) + 2y(k) = 0$
This equation is 3rd order from the difference of $k+3$ and k.
It is linear, since all the functions are linear.
It is time invariant because all the coefficients are constant.
Since $u(k)$ is 0, it is homogeneous.

c. $y(k+4) + y(k-1) = u(k)$
This equation is 5th order from the difference of $k+4$ and $k-1$.
It is linear, since all the functions are linear.
It is time invariant because all the coefficients are constant.
Since $u(k)$ is not 0, it is not homogeneous.

d. $y(k+5) = y(k+4) + u(k+1) - u(k)$
This equation is 5th order from the difference of $k+5$ and k.
It is linear, since all the functions are linear.
It is time invariant because all the coefficients are constant.
Since $u(k)$ is not 0, it is not homogeneous.

e. $y(k+2) = y(k)u(k)$
This equation is 2nd order from the difference of $k+2$ and k.
It is not linear, since $y(k)u(k)$ is not linear.
It is time invariant because all the coefficients are constant.
Since $u(k)$ is not 0, it is not homogeneous.

Problem 2. Solve the following difference equations.

a. $y(k+1) - 0.8y(k) = 0,\ y(0) = 1$

$$\begin{aligned} zY(z) - z - 0.8Y(z) &= 0 \\ (z-0.8)Y(z) &= z \\ Y(z) &= \frac{z}{z-0.8} \\ f(k) &= (0.8)^k,\ k \in \mathbb{N} \end{aligned}$$

b. $y(k+1) - 0.8y(k) = 1(k),\ y(0) = 0$

$$\begin{aligned} zY(z) - 0.8Y(z) &= \frac{z}{z-1} \\ Y(z) &= \frac{z}{(z-0.8)(z-1)} \\ z\left(\frac{1}{(z-0.8)(z-1)}\right) &= z\left(\frac{A}{z-0.8} + \frac{B}{z-1}\right) \\ 1 &= (z-1)A + (z-0.8)B \\ z = 1 \implies 1 &= 0.2B \implies B = 5 \\ z = 0.8 \implies 1 &= -0.2A \implies A = -5 \\ Y(z) &= \frac{-5z}{z-0.8} + \frac{5z}{z-1} \\ f(k) &= -5(0.8)^k + 5,\ k \in \mathbb{N} \end{aligned}$$

c. $y(k+1) - 0.8y(k) = 1(k),\ y(0) = 1$

$$\begin{aligned} zY(z) - z - 0.8Y(z) &= \frac{z}{z-1} \\ Y(z) &= \frac{z}{z-0.8} + \frac{z}{(z-0.8)(z-1)} \\ &\text{from partial fraction decomposition of part b we get:} \\ Y(z) = \frac{z}{z-0.8} + \frac{-5z}{z-0.8} + \frac{5z}{z-1} &= \frac{-4z}{z-0.8} + \frac{5z}{z-1} \\ f(k) &= -4(0.8)^k + 5,\ k \in \mathbb{N} \end{aligned}$$

d. $y(k+2)+0.7y(k+1)+0.06y(k)=\delta(k),\ y(0)=0,\ y(1)=2$

$$z^2Y(z)+0.7zY(z)+0.06Y(z)=1+2z$$

$$Y(z)=\frac{2z+1}{z^2+0.7z+0.06}=\frac{2z+1}{(z+0.1)(z+0.6)}$$

$$\frac{2z+1}{(z+0.1)(z+0.6)}=\frac{A}{z+0.1}+\frac{B}{z+0.6}$$

$$2z+1=(z+0.6)A+(z+0.1)B$$

$$z=-0.6 \implies -0.2=-0.5B \implies B=0.4$$

$$z=-0.1 \implies 0.8=0.5A \implies A=1.6$$

$$Y(z)=\frac{1.6}{z+0.1}+\frac{0.4}{z+0.6}$$

this is not in the table, but we can try decomposing $\frac{Y(z)}{z}$ instead:

$$\frac{Y(z)}{z}=\frac{2z+1}{z(z+0.1)(z+0.6)}$$

$$\frac{2z+1}{z(z+0.1)(z+0.6)}=\frac{A}{z}+\frac{B}{z+0.1}+\frac{C}{z+0.6}$$

$$2z+1=(z+0.1)(z+0.6)A+z(z+0.6)B+z(z+0.1)C$$

$$z=0 \implies 1=0.06A \implies A=\frac{100}{6}=\frac{50}{3}$$

$$z=-0.1 \implies 0.8=-0.05B \implies B=-16$$

$$z=-0.6 \implies -0.2=0.3C \implies C=-\frac{2}{3}$$

$$\frac{Y(z)}{z}=\frac{\frac{50}{3}}{z}+\frac{-16}{z+0.1}+\frac{-\frac{2}{3}}{z+0.6}$$

$$Y(z)=\frac{50}{3}+\frac{-16z}{z+0.1}+\frac{-\frac{2}{3}z}{z+0.6}$$

$$f(k)=\frac{50}{3}\delta(k)-16(-0.1)^k-\frac{2}{3}(-0.6)^k,\ k\in\mathbb{N}$$

Problem 3. Given the discrete-time system

$$y(k+2) - y(k) = 2u(k)$$

find the impulse response of the system $g(k)$:

a. From the difference equation

$$\begin{aligned} k = -2 \implies y(0) - y(-2) = 0, \text{ assuming y is causal } \implies y(0) = 0 \\ k = -1 \implies y(1) - y(-1) = 0 \implies y(1) = 0 \\ k = 0 \implies y(2) - y(0) = 2 \implies y(2) = 2 \\ k = 1 \implies y(3) - y(1) = 0 \implies y(3) = y(1) = 0 \\ k = 2 \implies y(4) - y(2) = 0 \implies y(4) = y(2) = 2 \\ g(k) = \begin{cases} 2 & \text{if } k > 0 \text{ and } k \text{ is even} \\ 0 & \text{otherwise} \end{cases} \end{aligned}$$

b. Using z-transformation

$$\begin{aligned} z^2Y(z) - Y(z) = 2U(z) \\ G(z) = \frac{Y(z)}{U(z)} = \frac{2}{z^2-1} = \frac{2}{(z+1)(z-1)} \\ \frac{G(z)}{z} = \frac{2}{z(z+1)(z-1)} = \frac{A}{z} + \frac{B}{z+1} + \frac{C}{z-1} \\ 2 = (z+1)(z-1)A + z(z-1)B + z(z+1)C \\ z = 0 \implies 2 = -A \implies A = -2 \\ z = -1 \implies 2 = 2B \implies B = 1 \\ z = 1 \implies 2 = 2C \implies C = 1 \\ \frac{G(z)}{z} = \frac{-2}{z} + \frac{1}{z+1} + \frac{1}{z-1} \\ G(z) = -2 + \frac{z}{z+1} + \frac{z}{z-1} \\ g(k) = \begin{cases} -2\delta(k) + (-1)^k + 1 & k \in \mathbb{N} \\ 0 & k < 0 \end{cases} \end{aligned}$$

Problem 4. Find the impulse response functions for the systems governed by the following difference equations.

a. $y(k+1) - 0.5y(k) = u(k)$

$$zY(z) - 0.5Y(z) = U(z)$$
$$G(z) = \frac{Y(z)}{U(z)} = \frac{1}{z-0.5} = z^{-1}\frac{z}{z-0.5}$$
$$g(k) = \begin{cases} (0.5)^{k-1} & k \geq 1 \\ 0 & k < 1 \end{cases}$$

b. $y(k+2) - 0.1y(k+1) + 0.8y(k) = u(k)$

$$z^2Y(z) - 0.1zY(z) + 0.8Y(z) = U(z)$$
$$G(z) = \frac{Y(z)}{U(z)} = \frac{1}{z^2 - 0.1z + 0.8}$$

does not look easily factorable, using the quadratic formula to find roots:

$$z = \frac{0.1 \pm \sqrt{0.01 - 3.2}}{2} = \frac{0.1 \pm j\sqrt{3.19}}{2}$$

the roots are complex conjugates, using $a^k \sin bk$ z-transform:

$$\mathcal{Z}\{a^k \sin bk\} = \frac{az \sin b}{z^2 - 2az\cos b + a^2}$$
$$G(z) = z^{-1}\frac{z}{z^2 - 0.1z + 0.8}$$

and equating coefficients:

$$a = \sqrt{0.8} \approx 0.894$$
$$\cos b = \frac{0.05}{\sqrt{0.8}} \approx 0.056$$
$$b = \cos^{-1}\left(\frac{0.05}{\sqrt{0.8}}\right) \approx 1.515$$
$$\sin b \approx 0.998$$
$$\frac{1}{a \sin b} \approx 1.12$$
$$\implies g(k) = \begin{cases} 1.12(0.894)^{k-1}\sin(1.515(k-1)) & k \geq 1 \\ 0 & k < 1 \end{cases}$$

Problem 5. Find the steady-state response of the systems resulting from the sinusoidal input $u(k) = 0.5\sin(0.4k)$.

a.

$$H(z) = \frac{z}{z - 0.4}$$

$$\begin{aligned} H(e^{0.4j}) &= \frac{e^{0.4j}}{e^{0.4j} - 0.4} \approx 1.537e^{-0.242j} \\ y_{ss}(k) &\approx 0.5(1.537)\sin(0.4k - 0.242) \approx 0.769\sin(0.4k - 0.242) \end{aligned}$$

b.

$$H(z) = \frac{z}{z^2 + 0.4z + 0.03}$$

$$\begin{aligned} H(e^{0.4j}) &= \frac{e^{0.4j}}{e^{0.8j} + 0.4e^{0.4j} + 0.03} \approx 0.714e^{-0.273j} \\ y_{ss}(k) &\approx 0.5(0.714)\sin(0.4k - 0.273) \approx 0.357\sin(0.4k - 0.273) \end{aligned}$$