Test 1

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EEEN 5338 Digital and DSP Based Control

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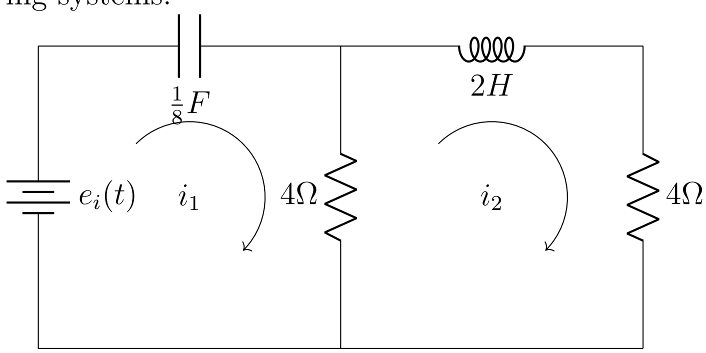
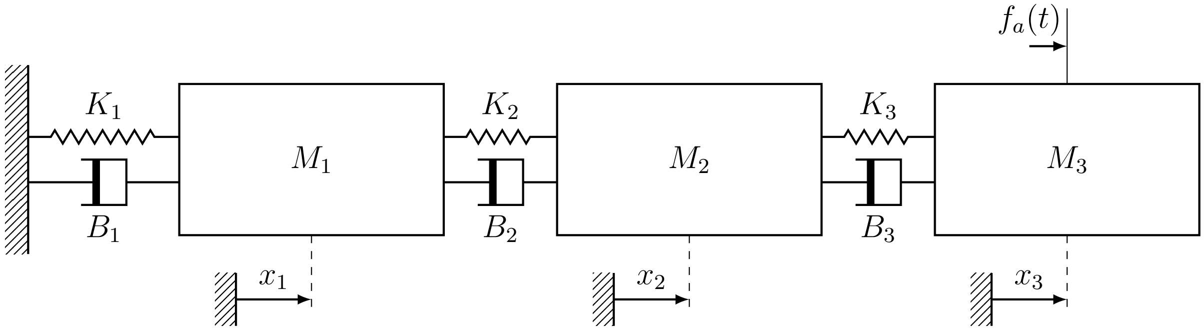
Problem 1. As per our review of pre-requisite materials in class and our brief discussions related to the following, find the mathematical models (differential equations) for the following systems:



$$
\begin{aligned}
e_i = \frac{1}{C}\int i_1 dt + R_1(i_1 - i_2) &= 8\int i_1 dt + 4(i_1 - i_2) \\
\implies \dot{e_i} &= 8i_1 + 4\frac{di_1}{dt} - 4\frac{di_2}{dt} \\
R_1(i_2 - i_1) + L\frac{di_2}{dt} + R_2(i_2) = 4(i_2 - i_1) + 2\frac{di_2}{dt} + 4i_2 &= 0 \\
\implies 8i_2 + 2\frac{di_2}{dt} = 4i_1 \implies 4i_2 + \frac{di_2}{dt} &= 2i_1 \\
e_A = R_1(i_2 - i_1) &= 4i_2 - 4i_1 \\
\implies \dot{e_A} &= 4\frac{di_2}{dt} - 4\frac{di_1}{dt} \\
\implies \dot{e_i} &= 8i_1 - \dot{e_A} \\
e_o = R_2(-i_2) &= -4i_2 \\
\implies 2i_1 &= -e_o - \frac{1}{4}\dot{e_o} \\
\implies \dot{e_i} &= -4e_o - \dot{e_o} - \dot{e_A}
\end{aligned}
$$



$$
\begin{aligned}
-B_1\dot{x_1} - K_1x_1 + K_2(x_2 - x_1) + B_2(\dot{x_2} - \dot{x_1}) &= M_1\ddot{x_1} \\
-B_2(\dot{x_2} - \dot{x_1}) - K_2(x_2 - x_1) + B_3(\dot{x_3} - \dot{x_2}) &= M_2\ddot{x_2} \\
-B_3(\dot{x_3} - \dot{x_2}) - K_3(x_3 - x_2) + f_a &= M_3\ddot{x_3}
\end{aligned}
$$

Problem 2. For the following equations, determine:
a) order?, b) linear?, c) time-invariant?, d) homogeneous?

i $y(k+3) + 0.2y(k+1) + 0.01y(k) = 0$
The equation is 3rd order from the difference of $k+3$ and k.
It is linear, since all the functions are linear.
It is time-invariant because all the coefficients are constant.
It is homogeneous since $u(k)$ is 0.

ii $y(k+2) + e^{-0.2k}y(k+1) + 0.1y(k) = u(k)$
The equation is 2nd order from the difference of $k+2$ and k.
It is linear, since all the functions are linear.
It is not time-invariant, since $e^{-0.2k}$ is not a constant coefficient.
It is not homogeneous, since $u(k)$ is not 0.

iii $y(k+5) + y(k+1) + 0.1y^3(k) = 0.1u(k)$
The equation is 5th order from the difference between $k+5$ and k.
It is not linear, since y^3 is not a linear function.
It is time-invariant because all the coefficients are constant.
It is not homogeneous, since $u(k)$ is not 0.

Problem 3. Solve the following difference equations.

i $y(k+1) - 0.8y(k) = 1(k),\ y(0) = 0$

$$
\begin{aligned}
zY(z) - 0.8Y(z) &= \frac{z}{z-1} \\
Y(z) &= \frac{z}{(z-0.8)(z-1)} \\
z\left(\frac{1}{(z-0.8)(z-1)}\right) &= z\left(\frac{A}{z-0.8} + \frac{B}{z-1}\right) \\
1 &= (z-1)A + (z-0.8)B \\
z = 1 \implies 1 = 0.2B &\implies B = 5 \\
z = 0.8 \implies 1 = -0.2A &\implies A = -5 \\
Y(z) &= \frac{-5z}{z-0.8} + \frac{5z}{z-1} \\
f(k) &= -5(0.8)^k + 5,\ k \in \mathbb{N}
\end{aligned}
$$

ii $y(k+1) - 0.8y(k) = 1(k),\ y(0) = 1$

$$
\begin{aligned}
zY(z) - z - 0.8Y(z) &= \frac{z}{z-1} \\
Y(z) &= \frac{z}{z-0.8} + \frac{z}{(z-0.8)(z-1)} \\
\text{from partial fraction decomposition of part i we get:} & \\
Y(z) = \frac{z}{z-0.8} + \frac{-5z}{z-0.8} + \frac{5z}{z-1} &= \frac{-4z}{z-0.8} + \frac{5z}{z-1} \\
f(k) &= -4(0.8)^k + 5,\ k \in \mathbb{N}
\end{aligned}
$$

iii $y(k+2)+0.7y(k+1)+0.06y(k)=\delta(k),\ y(0)=0,\ y(1)=2$

$$\begin{aligned}
z^2Y(z)+0.7zY(z)+0.06Y(z) &= 1+2z \\
\frac{Y(z)}{z} &= \frac{2z+1}{z(z+0.1)(z+0.6)} \\
\frac{2z+1}{z(z+0.1)(z+0.6)} &= \frac{A}{z}+\frac{B}{z+0.1}+\frac{C}{z+0.6} \\
2z+1 &= (z+0.1)(z+0.6)A+z(z+0.6)B+z(z+0.1)C \\
z=0 \implies 1=0.06A \implies A &= \frac{100}{6}=\frac{50}{3} \\
z=-0.1 \implies 0.8=-0.05B \implies B &= -16 \\
z=-0.6 \implies -0.2=0.3C \implies C &= -\frac{2}{3} \\
\frac{Y(z)}{z} &= \frac{\frac{50}{3}}{z}+\frac{-16}{z+0.1}+\frac{-\frac{2}{3}}{z+0.6} \\
Y(z) &= \frac{50}{3}+\frac{-16z}{z+0.1}+\frac{-\frac{2}{3}z}{z+0.6} \\
f(k) &= \frac{50}{3}\delta(k)-16(-0.1)^k-\frac{2}{3}(-0.6)^k,\ k\in\mathbb{N}
\end{aligned}$$

Problem 4. Given:

$$y(k+1) - ay(k) = u(k),\ y(0) = 0$$

Find the impulse response h(k):

a) From the difference equation

$$k = 0 \implies y(1) - ay(0) = 1 \implies y(1) = 1$$
$$k = 1 \implies y(2) - ay(1) = 0 \implies y(2) = ay(1) = a$$
$$k = 2 \implies y(3) - ay(2) = 0 \implies y(3) = ay(2) = a^2$$
$$\implies y(k) = a^{k-1}$$
$$g(k) = \begin{cases} a^{k-1} & k \geq 1 \\ 0 & k < 1 \end{cases}$$

b) Using Z-transformation

$$zY(z) - aY(z) = U(z)$$
$$G(z) = \frac{Y(z)}{U(z)} = \frac{1}{z-a} = z^{-1}\frac{z}{z-a}$$
$$g(k) = \begin{cases} a^{k-1} & k \geq 1 \\ 0 & k < 1 \end{cases}$$

Problem 5. Given:

$$y(k+1) - y(k) = u(k+1),\ y(0) = 0$$

Find the system transfer function and its response to a sampled unit step.

$$zY(z) - Y(z) = zU(z)$$

$$\text{transfer function: } H(z) = \frac{Y(z)}{U(z)} = \frac{z}{z-1}$$

$$\text{The } z\text{-transform of the sampled unit step } 1(k) \text{ is also: } \frac{z}{z-1}$$

$$\text{We multiply both to get: } \frac{z^2}{(z-1)^2} = z\frac{z}{(z-1)^2}$$

Applying the inverse z-transform:

$$y(i) = \begin{cases} i+1 & i \geq 0 \\ 0 & i < 0 \end{cases}$$