Homework 3

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Problem 1. Many chemical processes can be modeled by the following transfer function:

$$G(s) = \frac{K}{\tau s + 1} e^{-T_d s}$$

where K is the gain, τ is the time constant, and T_d is the time delay. Obtain the transfer function $G_{ZAS}(z)$ for the system in terms of the system parameters. Assume that the time delay T_d is a multiple of the sampling period T.

a. Partial fraction expansion:

$$\begin{aligned} \frac{G(s)}{s} = \frac{K}{s(\tau s + 1)} e^{-T_d s} &= \left(\frac{A}{s} + \frac{B}{\tau s + 1} \right) e^{-T_d s} \\ K &= (\tau s + 1)A + sB \\ s = 0 &\implies K = A \\ s = -\frac{1}{\tau} \implies K = -\frac{B}{\tau} &\implies B = -\tau K \\ \implies \frac{G(s)}{s} = \left(\frac{K}{s} - \frac{\tau K}{\tau s + 1} \right) e^{-T_d s} &= \left(\frac{K}{s} - \frac{K}{s + 1/\tau} \right) e^{-T_d s} \end{aligned}$$

b. z-transfer function:

$$\begin{aligned} G_{ZAS}(z) = (1 - z^{-1}) \mathcal{Z} \left\{ \frac{G(s)}{s} \right\} &= (1 - z^{-1}) \mathcal{Z} \left\{ \left(\frac{K}{s} - \frac{K}{s + 1/\tau} \right) e^{-T_d s} \right\} \\ &= (1 - z^{-1}) \left(\frac{Kz}{z - 1} - \frac{Kz}{z - e^{-T/\tau}} \right) z^{-T_d/T} \end{aligned}$$

using Wolfram Alpha to simplify the algebra:

$$G_{ZAS}(z) = \frac{(e^{-T/\tau} - 1)K}{e^{-T/\tau} - z} z^{-T_d/T}$$

Problem 2. For an internal combustion engine, the transfer function with injected fuel rate as input and fuel flow rate into the cylinder as output is given by:

$$G(s) = \frac{\varepsilon\tau s + 1}{\tau s + 1}$$

where τ is a time constant and ε is known as the fuel split parameter. Obtain the transfer function $G_{ZAS}(z)$ for the system in terms of the system parameters.

a. Partial fraction expansion:

$$\begin{aligned} \frac{G(s)}{s} = \frac{\varepsilon\tau s + 1}{s(\tau s + 1)} &= \frac{A}{s} + \frac{B}{\tau s + 1} \\ \varepsilon\tau s + 1 &= (\tau s + 1)A + sB \\ s = 0 &\implies 1 = A \\ s = -\frac{1}{\tau} \implies -\varepsilon + 1 = -\frac{B}{\tau} &\implies B = \varepsilon\tau - \tau \\ \implies \frac{G(s)}{s} = \frac{1}{s} + \frac{\varepsilon\tau - \tau}{\tau s + 1} &= \frac{1}{s} + \frac{\varepsilon - 1}{s + 1/\tau} \end{aligned}$$

b. z-transfer function:

$$\begin{aligned} G_{ZAS}(z) = (1 - z^{-1})\mathcal{Z}\left\{\frac{G(s)}{s}\right\} &= (1 - z^{-1})\mathcal{Z}\left\{\frac{1}{s} + \frac{\varepsilon - 1}{s + 1/\tau}\right\} \\ &= (1 - z^{-1})\left(\frac{z}{z-1} + \frac{z(\varepsilon - 1)}{z - e^{-T/\tau}}\right) \end{aligned}$$

using Wolfram Alpha to simplify the algebra:

$$G_{ZAS}(z) = 1 + \frac{(\varepsilon - 1)(z - 1)}{z - e^{-T/\tau}}$$

Problem 3. Repeat **Problem 2** including a delay of 25 ms in the transfer function with a sampling period of 10 ms.

a. Transport delay:

$$
\begin{aligned}
T_d = lT - mT, \text{ where } l \text{ is a positive integer and } 0 \leq m < 1 \\
25 = l(10) - m(10) = 3(10) - 0.5(10) \\
\implies l = 3,\ m = 0.5
\end{aligned}
$$

b. z-transfer function:

$$
G_{ZAS}(z) = (1 - z^{-1}) \left(\frac{z}{z-1} + \frac{z(\varepsilon - 1)e^{-5/\tau}}{z - e^{-T/\tau}} \right) z^{-3}
$$

using Wolfram Alpha to simplify the algebra:

$$
G_{ZAS}(z) = \left(1 + \frac{(\varepsilon - 1)(z-1)e^{-5/\tau}}{z - e^{-T/\tau}} \right) z^{-3}
$$

Problem 4. Find the equivalent sampled impulse response sequence and the equivalent z-transfer function for the cascade of the two analog systems with sampled input

$$H_1(s) = \frac{1}{s+6}$$

$$H_2(s) = \frac{10}{s+1}$$

a. If the systems are directly connected

$$H(s) = H_1(s)H_2(s) = \frac{10}{(s+6)(s+1)} = \frac{A}{s+6} + \frac{B}{s+1}$$

$$10 = (s+1)A + (s+6)B$$

$$s = -6 \implies 10 = -5A \implies A = -2$$

$$s = -1 \implies 10 = 5B \implies B = 2$$

$$H(s) = \frac{-2}{s+6} + \frac{2}{s+1}$$

$$h(t) = -2e^{-6t} + 2e^{-t}$$

$$\text{sampled impulse response: } h(kT) = -2e^{-6kT} + 2e^{-kT},\ k \in \mathbb{N}$$

$$z\text{-transfer function: } H(z) = \frac{-2z}{z-e^{-6T}} + \frac{2z}{z-e^{-T}} = \frac{2z(e^{-T}-e^{-6T})}{(z-e^{-6T})(z-e^{-T})}$$

b. If the systems are separated by a sampler

$$H_1(z) = \frac{z}{z-e^{-6T}}$$

$$H_2(z) = \frac{10z}{z-e^{-T}}$$

$$z\text{-transfer function: } H(z) = \frac{10z^2}{(z-e^-6T)(z-e^{-T})}$$

$$\text{partial fraction expansion: } \frac{10z^2}{(z-e^{-6T})(z-e^{-T})} = \frac{Az}{z-e^{-6T}} + \frac{Bz}{z-e^{-T}}$$

$$10z = (z-e^{-T})A + (z-e^{-6T})B$$

$$z = e^{-6T} \implies 10e^{-6T} = (e^{-6T}-e^{-T})A \implies A = \frac{10e^{-6T}}{e^{-6T}-e^{-T}}$$

$$z = e^{-T} \implies 10e^{-T} = (e^{-T}-e^{-6T})B \implies B = \frac{10e^{-T}}{e^{-T}-e^{-6T}}$$

$$H(z) = \frac{10e^{-6T}z}{(e^{-6T}-e^{-T})(z-e^{-6T})} + \frac{10e^{-T}z}{(e^{-T}-e^{-6T})(z-e^{-T})}$$

$$= \frac{10}{e^{-6T}-e^{-T}}\left(\frac{e^{-6T}z}{z-e^{-6T}} - \frac{e^{-T}z}{z-e^{-T}}\right)$$

$$\text{sampled impulse response: } h(kT) = \frac{10}{e^{-6T}-e^{-T}}(e^{-6(k+1)T} - e^{-(k+1)T}),\ k \in \mathbb{N}$$