Final Exam 2024

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Stat 610 Distribution Theory

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Problem 1. Assume $X \sim \text{gamma}(6,1)$ and the conditional distribution of Y, given X, is Poisson(X). Find the conditional distribution of X, given Y, and $\mathsf{E}(X|Y)$. Hint: factor out the part that depends on x and see if you can recognize what it is proportional to.

The joint distribution of X and Y is

$$\begin{aligned} f_{X,Y}(x,y) &= f_{Y|X}(y|x)f_X(x) \\ &= \frac{e^{-x}x^y}{y!} \cdot \frac{1}{\Gamma(6)}x^{6-1}e^{-x} \\ &= \frac{1}{y!\Gamma(6)}x^{y+5}e^{-2x}. \end{aligned}$$

Thus, the conditional distribution of X given $Y = y$ is

$$\begin{aligned} f_{X|Y}(x|y) &= \frac{f_{X,Y}(x,y)}{f_Y(y)} \\ &\propto x^{y+5}e^{-2x}. \end{aligned}$$

Recognizing this as the kernel of a gamma distribution, we have

$$X|Y = y \sim \text{gamma}(y+6, 1/2).$$

Therefore,

$$\mathsf{E}(X|Y=y) = \frac{y+6}{2}.$$

Problem 2. Suppose $T_1, T_2, \ldots$ are iid Laplace(μ, β) (see the formula sheet), and let $\overline{T}_n = \frac{1}{n}\sum_{i=1}^n T_i$.

(a) Identify values of a and b such that $\sqrt{n}b(\overline{T}_n - a)$ converges in distribution as $n \to \infty$. What is the limit distribution? Explain what theory you are using.

(b) Assume $\mu \neq 0$. Use the delta method to show that $\sqrt{n}(\overline{T}_n^2 - \mu^2) \xrightarrow{D} \text{normal}(0, \gamma)$ for some $\gamma > 0$. What is γ?

(a) We have $\mathsf{E}(T_i) = \mu$ and $\mathsf{Var}(T_i) = 2\beta^2$. By the Central Limit Theorem,

$$\sqrt{n}(\overline{T}_n - \mu) \xrightarrow{D} \text{normal}(0, 2\beta^2).$$

Thus, we can take $a = \mu$ and $b = 1$. The limit distribution is normal$(0, 2\beta^2)$.

(b) Let $g(x) = x^2$. Then, $g'(\mu) = 2\mu$. By the delta method,

$$\sqrt{n}(\overline{T}_n^2 - \mu^2) \xrightarrow{D} \text{normal}(0, (g'(\mu))^2\mathsf{Var}(T_i)) = \text{normal}(0, 4\mu^2 \cdot 2\beta^2) = \text{normal}(0, 8\mu^2\beta^2).$$

Thus, $\gamma = 8\mu^2\beta^2$.

Problem 3. $\tilde{\theta}$ is an estimator of a parameter θ such that $\mathsf{E}(\tilde{\theta}) = \theta$ and $\mathsf{Var}(\tilde{\theta}) = \frac{\theta^2}{n}$. T is another statistic that has mean 0 and variance $\frac{\theta^2}{2n}$. Also, $\mathsf{Cov}(T, \tilde{\theta}) = -\frac{\theta^2}{2n}$. What are the mean and variance of $\tilde{\theta}^* = \tilde{\theta} + T$?

We have

$$\mathsf{E}(\tilde{\theta}^*) = \mathsf{E}(\tilde{\theta} + T) = \mathsf{E}(\tilde{\theta}) + \mathsf{E}(T) = \theta + 0 = \theta,$$

and

$$\mathsf{Var}(\tilde{\theta}^*) = \mathsf{Var}(\tilde{\theta}+T) = \mathsf{Var}(\tilde{\theta})+\mathsf{Var}(T)+2\mathsf{Cov}(\tilde{\theta}, T) = \frac{\theta^2}{n}+\frac{\theta^2}{2n}+2\left(-\frac{\theta^2}{2n}\right) = \frac{\theta^2}{n}+\frac{\theta^2}{2n}-\frac{\theta^2}{n} = \frac{\theta^2}{2n}.$$

Problem 4. $X_1, \ldots, X_n$ are iid normal(μ_X, σ_X^2) and, independent of those, $Y_1, \ldots, Y_n$ are iid normal(μ_Y, σ_Y^2). What is the sampling distribution of $\overline{X} - \overline{Y}$? Explain.

Since $\overline{X} \sim$ normal$(\mu_X, \frac{\sigma_X^2}{n})$ and $\overline{Y} \sim$ normal$(\mu_Y, \frac{\sigma_Y^2}{n})$, and $\overline{X}$ and $\overline{Y}$ are independent, we have

$$\overline{X} - \overline{Y} \sim \text{normal}\left(\mu_X - \mu_Y, \frac{\sigma_X^2}{n} + \frac{\sigma_Y^2}{n}\right) = \text{normal}\left(\mu_X - \mu_Y, \frac{\sigma_X^2 + \sigma_Y^2}{n}\right).$$

Problem 5. Suppose $V_{1,2}, \ldots, V_n$ are iid with cdf $F_V(v) = e^{-1/v}$ for $v > 0$.

(a) Let $V(n) = \max(V_1, \ldots, V_n)$. Prove that $W = \frac{V(n)}{n}$ has the same distribution as V_i.

(b) We know that $\frac{1}{n}\sum_{i=1}^n \sqrt{V_i} \to E(\sqrt{V})$, as $n \to \infty$, with probability 1. What is the value of $E(\sqrt{V})$? Hint: use an appropriate change of variables to convert the expectation to something recognizable.

(a) We have

$$\begin{aligned}
F_W(w) &= P\left(W \le w\right) = P\left(\frac{V(n)}{n} \le w\right) = P\left(V(n) \le nw\right) \\
&= P\left(V_1 \le nw, V_2 \le nw, \ldots, V_n \le nw\right) \\
&= \prod_{i=1}^{n} P\left(V_i \le nw\right) = \left(F_V(nw)\right)^n = \left(e^{-1/(nw)}\right)^n = e^{-1/w} = F_V(w).
\end{aligned}$$

Thus, W has the same distribution as V_i.

(b) We have

$$E(\sqrt{V}) = \int_0^\infty \sqrt{v} f_V(v) dv = \int_0^\infty \sqrt{v} \frac{d}{dv}\left(e^{-1/v}\right) dv$$
$$= \int_0^\infty \sqrt{v} \cdot \frac{1}{v^2} e^{-1/v} dv = \int_0^\infty v^{-3/2} e^{-1/v} dv.$$

Let $u = 1/v$, then $du = -1/v^2 dv$ and $dv = -v^2 du = -\frac{1}{u^2} du$. When $v \to 0$, $u \to \infty$, and when $v \to \infty$, $u \to 0$. Thus,

$$E(\sqrt{V}) = \int_\infty^0 (1/u)^{-3/2} e^{-u} \left(-\frac{1}{u^2}\right) du = \int_0^\infty u^{3/2} e^{-u} \cdot \frac{1}{u^2} du$$
$$= \int_0^\infty u^{-1/2} e^{-u} du = \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}.$$

Problem 6. Z has quantile function $Q_Z(p) = 1 - (1-p)^{1/3}$, which takes values in (0, 1). What are the cdf and pdf for Z?

We have
$$F_Z(z) = P(Z \le z) = p \text{ such that } Q_Z(p) = z.$$
Solving for p, we have

$$\begin{aligned} z &= 1 - (1-p)^{1/3} \\ (1-p)^{1/3} &= 1 - z \\ 1 - p &= (1-z)^3 \\ p &= 1 - (1-z)^3. \end{aligned}$$

Thus,
$$F_Z(z) = 1 - (1-z)^3, \text{ for } 0 < z < 1.$$
Taking the derivative, we have

$$f_Z(z) = F_Z'(z) = 3(1-z)^2, \text{ for } 0 < z < 1.$$