Homework 1

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Stat 610 Distribution Theory

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Problem 1. *Statistical Inference* by Casella and Berger, 2nd Edition, Chapter 1, Exercise 4, 5, and 6.

4 For events A and B, find formulas for the probabilities of the following events in terms of the quantities $P(A)$, $P(B)$, and $P(A \cap B)$.

(a) either A or B or both
(b) either A or B but not both
(c) at least one of A or B
(d) at most one of A or B

(a) $P(A \cup B) = P(A) + P(B) - P(A \cap B)$
(b) $P(A \cup B) - P(A \cap B) = P(A) + P(B) - 2P(A \cap B)$
(c) $P(A \cup B) = P(A) + P(B) - P(A \cap B)$
(d) $1 - P(A \cap B)$

5 Approximately one-third of all human twins are identical (one-egg) and two-thirds are fraternal (two-egg) twins. Identical twins are necessarily the same sex, with male and female being equally likely. Among fraternal twins, approximately one-fourth are both female, one-fourth are both male, and half are one male and one female. Finally, among all U.S. births, approximately 1 in 90 is a twin birth. Define the following events:

- A = the birth results in twin females
- B = the twins are identical twins
- C = a U.S. birth results in twins

(a) State, in words, the event $A \cap B \cap C$.
(b) Find $P(A \cap B \cap C)$.

(a) The event that a U.S. birth results in identical twin females.
(b) From the given information, we have

* $P(C) = \frac{1}{90}$
* $P(B|C) = \frac{1}{3}$
* $P(A|B,C) = \frac{1}{2}$

So, $P(A \cap B \cap C) = P(C)P(B|C)P(A|B,C) = \frac{1}{90} \times \frac{1}{3} \times \frac{1}{2} = \frac{1}{540}$.

6 Two pennies, one with $P(\text{head}) = u$ and one with $P(\text{head}) = w$, are to be tossed together independently. Define

- $p_0 = P(0 \text{ heads occur})$,
- $p_1 = P(1 \text{ head occurs})$,
- $p_2 = P(2 \text{ heads occur})$.

Can u and w be chosen so that $p_0 = p_1 = p_2$? Prove your answer.

We have

- $p_0 = (1-u)(1-w)$,
- $p_1 = u(1-w) + w(1-u)$,
- $p_2 = uw$.

So, $p_0 = p_2$ implies $(1-u)(1-w) = uw$, which simplifies to $u + w = 1$. Also, $p_1 = p_2$ implies $u(1-w) + w(1-u) = uw$, which simplifies to $u + w = 3uw$. Combining these two equations, we have $3uw = 1$, or $uw = \frac{1}{3}$. However, since $u + w = 1$, by AM-GM inequality, we have $\frac{u+w}{2} = \frac{1}{2} \geq \sqrt{uw}$, which when squared gives $\frac{1}{4} \geq uw$. This contradicts $uw = \frac{1}{3}$. Therefore, there are no such u and w, assuming they are both in $[0,1]$.

Problem 2. Here is a sample space: $\mathcal{S} = \{a, b, c\}$.

(a) Explicitly provide the σ-algebra of all subsets.

(b) Suppose one may only observe whether the outcome is a or not. Explicitly provide the smallest relevant σ-algebra. Hint: what events may be obtained by complements, unions and intersections, starting only with a?

(a) The σ-algebra of all subsets is

$$\{\emptyset, \{a\}, \{b\}, \{c\}, \{a,b\}, \{a,c\}, \{b,c\}, \{a,b,c\}\}.$$

(b) Starting with $\{a\}$, we may obtain $\{b,c\}$ by taking its complement. Then, we may obtain $\{a,b,c\}$ by taking the union of $\{a\}$ and $\{b,c\}$. Finally, we may obtain $\emptyset$ by taking the complement of $\{a,b,c\}$. Thus the smallest relevant σ-algebra is

$$\{\emptyset, \{a\}, \{b,c\}, \{a,b,c\}\}.$$

Problem 3. For each of the following, be sure to explicitly provide the sample space and the event indicated.

(a) I hand out 4 pieces of candy at random, each to a child. There are three children and some will get more than others. Find the probability that each child gets at least one piece of candy. Find the probability that one child gets it all.

(b) The local Chevy dealer has 5 trucks to give away: two Silverados (S_1 and S_2), three Tahoes (T_1, T_2, and T_3). Suppose they randomly select a truck, give it to a lucky customer, and then randomly select another for a second lucky customer. Find the probability that x Silverados are selected, where $x = 0, 1, 2$.

(a) The sample space is

$$\mathcal{S} = \{(c_1, c_2, c_3, c_4) : c_i \in \{1, 2, 3\}\}.$$

The event that each child gets at least one piece of candy is

$$\begin{aligned} A = \{&(1,1,2,3), (1,1,3,2), (1,2,1,3), (1,2,2,3), \\ &(1,2,3,1), (1,2,3,2), (1,2,3,3), (1,3,1,2), \\ &(1,3,2,1), (1,3,2,2), (1,3,2,3), (1,3,3,2), \\ &(2,1,1,3), (2,1,2,3), (2,1,3,1), (2,1,3,2), \\ &(2,1,3,3), (2,2,1,3), (2,2,3,1), (2,3,1,1), \\ &(2,3,1,2), (2,3,1,3), (2,3,2,1), (2,3,3,1), \\ &(3,1,1,2), (3,1,2,1), (3,1,2,2), (3,1,2,3), \\ &(3,1,3,2), (3,2,1,1), (3,2,1,2), (3,2,1,3), \\ &(3,2,2,1), (3,2,3,1), (3,3,1,2), (3,3,2,1)\}. \end{aligned}$$

The event that one child gets it all is

$$B = \{(1,1,1,1), (2,2,2,2), (3,3,3,3)\}.$$

Since each outcome in the sample space is equally likely, we have

$$P(A) = \frac{36}{3^4} = \frac{4}{9}, \quad P(B) = \frac{3}{3^4} = \frac{1}{27}.$$

(b) The sample space is

$$\mathcal{S} = \{(x, y) : x, y \in \{S_1, S_2, T_1, T_2, T_3\}, x \neq y\}.$$

The event that $x = 0$ Silverados are selected is

$$A = \{(T_1, T_2), (T_1, T_3), (T_2, T_1), (T_2, T_3), (T_3, T_1), (T_3, T_2)\}.$$

The event that $x = 1$ Silverados are selected is

$$\begin{aligned} B = \{&(S_1, T_1), (S_1, T_2), (S_1, T_3), \\ &(S_2, T_1), (S_2, T_2), (S_2, T_3), \\ &(T_1, S_1), (T_1, S_2), (T_2, S_1), \\ &(T_2, S_2), (T_3, S_1), (T_3, S_2)\}. \end{aligned}$$

The event that $x = 2$ Silverados are selected is

$$C = \{(S_1, S_2), (S_2, S_1)\}.$$

Since each outcome in the sample space is equally likely, we have

$$P(A) = \frac{6}{20} = \frac{3}{10}, \quad P(B) = \frac{12}{20} = \frac{3}{5}, \quad P(C) = \frac{2}{20} = \frac{1}{10}.$$

Problem 4. *Statistical Inference* by Casella and Berger, 2nd Edition, Chapter 1, Exercise 10. Use the case $n = 2$ (Thm. 1.1.4.d in the book) and mathematical induction to solvee this.

10 Formulate and prove a version of DeMorgan's Laws that applies to a finite collection of sets $A_1, A_2, \ldots, A_n$.

For a finite collection of sets $A_1, A_2, \ldots, A_n$, we conjecture that the formulation is

$$\left(\bigcap_{i=1}^{n} A_i\right)^c = \bigcup_{i=1}^{n} A_i^c.$$

The other law is simply setting $B_i = A_i^c$ for each i, and taking the complement of both sides, giving us

$$\bigcap_{i=1}^{n} B_i^c = \left(\bigcup_{i=1}^{n} B_i\right)^c.$$

Since taking complements on both sides of an equation preserves equality, we will only attempt to prove the first formulation by mathematical induction.
Base case: when $n = 2$, we have

$$\left(\bigcap_{i=1}^{2} A_i\right)^c = \bigcup_{i=1}^{2} A_i^c.$$

Proof of base case: An element x is in the left hand side if and only if x is not in the intersection of A_1 and A_2. This means x is not in A_1 or x is not in A_2. This is equivalent to saying x is in the complement of A_1 or x is in the complement of A_2, which is equivalent to saying x is in the union of the complements of A_1 and A_2.

Inductive hypothesis: Assume that for some $k \geq 2$,

$$\left(\bigcap_{i=1}^{k} A_i\right)^c = \bigcup_{i=1}^{k} A_i^c.$$

Inductive step: We want to show that

$$\left(\bigcap_{i=1}^{k+1} A_i\right)^c = \bigcup_{i=1}^{k+1} A_i^c.$$

Proof of inductive step: First we can rewrite the left hand side as

$$\left(\bigcap_{i=1}^{k+1} A_i\right)^c = \left(A_{k+1} \cap \bigcap_{i=1}^{k} A_i\right)^c.$$

By the case $n = 2$ of DeMorgan's Laws, we have

$$\left(A_{k+1} \cap \bigcap_{i=1}^{k} A_i\right)^c = A_{k+1}^c \cup \left(\bigcap_{i=1}^{k} A_i\right)^c.$$

By the inductive hypothesis, we have

$$A_{k+1}^c \cup \left(\bigcap_{i=1}^{k} A_i\right)^c = A_{k+1}^c \cup \bigcup_{i=1}^{k} A_i^c.$$

Finally, we have

$$A_{k+1}^c \cup \bigcup_{i=1}^{k} A_i^c = \bigcup_{i=1}^{k+1} A_i^c.$$

Problem 5. The lifetime X of a cell phone battery, inspected under stress, satisfies $\mathrm{P}(X > t) = e^{-t/10}$ for all $t > 0$.

(a) Show that, if $0 \le a < b$ then $\mathrm{P}(a < X \le b) = \frac{1}{10}\int_a^b e^{-t/10} dt$.

(b) Is there a similar internal representation for $\mathrm{P}(a < X \le b) = \int_a^b g(t)dt$ valid even if $a < 0 < b$ Hint: use an indicator function – it should not depend on either a or b?

(c) Let $a \to b$ to find $\mathrm{P}(X = b)$.