Homework 5

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Stat 610 Distribution Theory

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Problem 1. Suppose Z has standard normal distribution.

(a) Use the mgf to find the third and fourth moments of Z. (Recall Slide 119.)

(b) Use part (a) to deduce $\mathsf{E}(X^3)$ and $\mathsf{E}(X^4)$ where $X \sim \text{normal}(\mu, \sigma^2)$ (e.g., $X = \mu + \sigma Z$).

(a) The mgf of Z is

$$M_Z(t) = \mathsf{E}(e^{tZ}) = e^{\frac{t^2}{2}}.$$

Thus, we take derivatives of $M_Z(t)$:

$$\begin{aligned} M_Z^{(1)}(t) &= te^{\frac{t^2}{2}}, \\ M_Z^{(2)}(t) &= (1+t^2)e^{\frac{t^2}{2}}, \\ M_Z^{(3)}(t) &= (3t+t^3)e^{\frac{t^2}{2}}, \\ M_Z^{(4)}(t) &= (3+6t^2+t^4)e^{\frac{t^2}{2}}. \end{aligned}$$

Evaluating these derivatives at $t = 0$, we have

$$\begin{aligned} \mathsf{E}(Z) &= M_Z^{(1)}(0) = 0, \\ \mathsf{E}(Z^2) &= M_Z^{(2)}(0) = 1, \\ \mathsf{E}(Z^3) &= M_Z^{(3)}(0) = 0, \\ \mathsf{E}(Z^4) &= M_Z^{(4)}(0) = 3. \end{aligned}$$

(b) Since $X = \mu + \sigma Z$, we have

$$\begin{aligned} \mathsf{E}(X^3) &= \mathsf{E}((\mu+\sigma Z)^3) \\ &= \mu^3 + 3\mu^2\sigma\mathsf{E}(Z) + 3\mu\sigma^2\mathsf{E}(Z^2) + \sigma^3\mathsf{E}(Z^3) \\ &= \mu^3 + 3\mu\sigma^2, \end{aligned}$$

$$\begin{aligned} \mathsf{E}(X^4) &= \mathsf{E}((\mu+\sigma Z)^4) \\ &= \mathsf{E}(\mu^4 + 4\mu^3\sigma Z + 6\mu^2\sigma^2 Z^2 + 4\mu\sigma^3 Z^3 + \sigma^4 Z^4) \\ &= \mu^4 + 4\mu^3\sigma\mathsf{E}(Z) + 6\mu^2\sigma^2\mathsf{E}(Z^2) + 4\mu\sigma^3\mathsf{E}(Z^3) + \sigma^4\mathsf{E}(Z^4) \\ &= \mu^4 + 6\mu^2\sigma^2 + 3\sigma^4. \end{aligned}$$

Problem 2. *Statistical Inference* by Casella and Berger, 2nd Edition, Chapter 2, Exercise 14.

14. (a) Let X be a continuous, nonnegative random variable [$f(x) = 0$ for $x < 0$]. Show that

$$\mathsf{E}(X) = \int_0^\infty [1 - F_X(x)]dx,$$

where $F_X(x)$ is the cdf of X.

(b) Let X be a discrete random variable whose range is the nonnegative integers. Show that

$$\mathsf{E}(X) = \sum_{k=0}^{\infty}[1 - F_X(k)],$$

where $F_X(k) = P(X \leq k)$. Compare this with part (a).

(c) Observe that

$$X = \int_0^X dx = \int_0^\infty 1_{X>x}dx,$$

and then, after taking expectation of the right- hand expression, exchange expectation and integral to prove that the expression in part (a) holds for any nonnegative random variable, regardless of the type of distribution. [Exchanging integration and expectation is like exchanging integration and sum or doing double integration in the other order, etc. The general result is *Fubini's Theorem* and is valid, at least, for nonnegative quantities like the example here.]

(a) Since X is a continuous, nonnegative random variable, we have

$$\begin{aligned}
\mathsf{E}(X) &= \int_0^\infty xf(x)dx \\
&= \int_0^\infty \int_0^x f(x)dtdx \\
&= \int_0^\infty \int_t^\infty f(x)dxdt \\
&= \int_0^\infty \mathsf{P}(X > t)dt \\
&= \int_0^\infty [1 - \mathsf{P}(X \leq t)]dt \\
&= \int_0^\infty [1 - F_X(t)]dt.
\end{aligned}$$

(b) Since X is a discrete, nonnegative random variable, we have

$$\begin{aligned}
\mathsf{E}(X) &= \sum_{k=0}^{\infty} k\mathsf{P}(X = k) \\
&= \sum_{k=0}^{\infty}\sum_{j=1}^{k} \mathsf{P}(X = k) \\
&= \sum_{j=1}^{\infty}\sum_{k=j}^{\infty} \mathsf{P}(X = k) \\
&= \sum_{j=1}^{\infty} \mathsf{P}(X \geq j) \\
&= \sum_{j=1}^{\infty} [1 - \mathsf{P}(X < j)] \\
&= \sum_{j=1}^{\infty} [1 - F_X(j-1)] \\
&= \sum_{i=0}^{\infty} [1 - F_X(i)].
\end{aligned}$$

(c) Using the identity given, we have

$$\begin{aligned}
\mathsf{E}(X) &= \mathsf{E}\left(\int_0^{\infty} 1_{X>x}dx\right) \\
&= \int_0^{\infty} \mathsf{E}(1_{X>x})dx \\
&= \int_0^{\infty} \mathsf{P}(X > x)dx \\
&= \int_0^{\infty} [1 - \mathsf{P}(X \leq x)]dx \\
&= \int_0^{\infty} [1 - F_X(x)]dx.
\end{aligned}$$

Problem 3. Let X have the *Laplace distribution* (recalling Problem 3 of Assignment 4), with pdf

$$f(x) = \frac{\lambda}{2}e^{-\lambda|x|}, \quad \text{all } x.$$

(a) Show that X has mgf

$$M_X(t) = \frac{1}{1 - \frac{t^2}{\lambda^2}} \quad \text{for } |t| < \lambda.$$

Hint: integrate separately for $x < 0$ and $x > 0$, and then combine.

(b) Determine the quantile function for X. Again, you need to think about separate cases. Make sure you get a continuous increasing function.

(c) Find the median and the 25-th and 75-th percentiles, as functions of λ.

(a)

$$\begin{aligned} M_X(t) &= \mathsf{E}(e^{tX}) \\ &= \int_{-\infty}^{0} e^{tx}\frac{\lambda}{2}e^{\lambda x}dx + \int_{0}^{\infty} e^{tx}\frac{\lambda}{2}e^{-\lambda x}dx \\ &= \frac{\lambda}{2}\int_{-\infty}^{0} e^{(t+\lambda)x}dx + \frac{\lambda}{2}\int_{0}^{\infty} e^{(t-\lambda)x}dx \\ &= \frac{\lambda}{2(t+\lambda)} + \frac{\lambda}{2(\lambda-t)} \\ &= \frac{\lambda^2}{\lambda^2-t^2}, \quad |t|<\lambda. \end{aligned}$$

(b)

$$F_X(x) = \begin{cases} \frac{1}{2}e^{\lambda x}, & x<0, \\ 1-\frac{1}{2}e^{-\lambda x}, & x \geq 0. \end{cases}$$

By Theorem 2.35, the quantile function is

$$Q_X(p) = F_X^{-1}(p) = \begin{cases} \frac{1}{\lambda}\ln(2p), & 0<p<\frac{1}{2}, \\ -\frac{1}{\lambda}\ln[2(1-p)], & \frac{1}{2} \leq p < 1. \end{cases}$$

(c) The median is

$$F_X^{-1}\left(\frac{1}{2}\right) = \frac{1}{\lambda}\ln(1) = 0.$$

The 25-th percentile is

$$F_X^{-1}\left(\frac{1}{4}\right) = \frac{1}{\lambda}\ln\left(\frac{1}{2}\right) = -\frac{\ln 2}{\lambda}.$$

The 75-th percentile is

$$F_X^{-1}\left(\frac{3}{4}\right) = -\frac{1}{\lambda}\ln\left(\frac{1}{2}\right) = \frac{\ln 2}{\lambda}.$$

Problem 4. Suppose that the probability of being able to make a left turn on the first signal cycle of a very busy intersection is 32%. Assuming independent trips, let W be the number of times that one is not successful turning on the first cycle before the fifth time that one is successful.

(a) What is the distribution of W and its mean and variance?

(b) Determine the chance that W is no more than 10.

(c) Let Y be the number of successes in 15 trips. What is the chance that $Y \geq 5$?

(a) Since W is the number of failures before the fifth success, W has a negative binomial distribution with parameters $r = 5$ and $p = 0.32$. The mean and variance of a negative binomial distribution are given by

$$\mathsf{E}(W) = \frac{r(1-p)}{p} = \frac{5(1-0.32)}{0.32} = \frac{5 \cdot 0.68}{0.32} = 10.625,$$

$$\mathsf{Var}(W) = \frac{r(1-p)}{p^2} = \frac{5(1-0.32)}{0.32^2} = \frac{5 \cdot 0.68}{0.1024} \approx 33.203.$$

(b) We want to find $\mathsf{P}(W \leq 10)$. Using the negative binomial pmf, we have

$$\begin{aligned}\mathsf{P}(W \leq 10) &= \sum_{w=0}^{10} \mathsf{P}(W = w) \\ &= \sum_{w=0}^{10} \binom{w+5-1}{5-1} (0.32)^5 (0.68)^w \\ &= (0.32)^5 \sum_{w=0}^{10} \binom{w+4}{4} (0.68)^w.\end{aligned}$$

Using R, we find that $\mathsf{P}(W \leq 10) \approx 0.552$.

(c) Since Y is the number of successes in 15 trips, Y has a binomial distribution with parameters $n = 15$ and $p = 0.32$. We want to find $\mathsf{P}(Y \geq 5)$. Using the binomial pmf, we have

$$\begin{aligned}\mathsf{P}(Y \geq 5) &= 1 - \mathsf{P}(Y \leq 4) \\ &= 1 - \sum_{y=0}^{4} \mathsf{P}(Y = y) \\ &= 1 - \sum_{y=0}^{4} \binom{15}{y} (0.32)^y (0.68)^{15-y}.\end{aligned}$$

Using R, we find that $\mathsf{P}(Y \geq 5) \approx 0.552$.

Problem 5. *Statistical Inference* by Casella and Berger, 2nd Edition, Chapter 3, Exercise 24(b).

24. Many "named" distributions are special cases of the more common distributions already discussed. For each of the following named distributions derive the form of the pdf, verify that it is a pdf, and calculate the mean and variance.

    (b) If $X \sim \text{exponential}(\beta)$, then $Y = (2X/\beta)^{\frac{1}{2}}$ has the *Rayleigh distribution*.

(b) Since $X \sim \text{exponential}(\beta)$, the pdf of X is

$$f_X(x) = \frac{1}{\beta}e^{-\frac{x}{\beta}}, \quad x > 0.$$

The transformation is $Y = g(X) = (2X/\beta)^{\frac{1}{2}}$. The inverse transformation is $X = g^{-1}(Y) = \frac{\beta Y^2}{2}$. The derivative of the inverse transformation is

$$\frac{d}{dy}g^{-1}(Y) = \beta Y.$$

Thus, the pdf of Y is

$$\begin{aligned}
f_Y(y) &= f_X(g^{-1}(y)) \left| \frac{d}{dy}g^{-1}(y) \right| \\
&= f_X\left(\frac{\beta y^2}{2}\right)(\beta y) \\
&= \frac{1}{\beta}e^{-\frac{\beta y^2/2}{\beta}}(\beta y) \\
&= ye^{-\frac{y^2}{2}}, \quad y > 0.
\end{aligned}$$

To verify that $f_Y(y)$ is a pdf, we check that

$$\int_0^\infty f_Y(y)dy = \int_0^\infty ye^{-\frac{y^2}{2}}dy.$$

Let $u = -\frac{y^2}{2}$, then $du = -ydy$. Thus,

$$\int_0^\infty ye^{-\frac{y^2}{2}}dy = -\int_0^{-\infty} e^u du = -[e^u]_0^{-\infty} = 1.$$

Therefore, $f_Y(y)$ is a valid pdf.

Next, we calculate the mean of Y:

$$\begin{aligned}
\mathsf{E}(Y) &= \int_0^\infty yf_Y(y)dy \\
&= \int_0^\infty y^2e^{-\frac{y^2}{2}}dy.
\end{aligned}$$

Using the gamma function, we have

$$\begin{aligned}
\mathsf{E}(Y) &= \int_0^\infty y^2e^{-\frac{y^2}{2}}dy \\
&= 2^{\frac{3}{2}-1}\int_0^\infty t^{\frac{3}{2}-1}e^{-t}dt \\
&= 2^{\frac{3}{2}-1}\Gamma\left(\frac{3}{2}\right) \\
&= \sqrt{2}\cdot\frac{1}{2}\sqrt{\pi} \\
&= \sqrt{\frac{\pi}{2}}.
\end{aligned}$$

Finally, we calculate the variance of Y:

$$\begin{aligned}
\mathsf{E}(Y^2) &= \int_0^\infty y^2 f_Y(y)dy \\
&= \int_0^\infty y^3 e^{-\frac{y^2}{2}} dy \\
&= 2^{\frac{4}{2}-1} \int_0^\infty t^{\frac{4}{2}-1} e^{-t} dt \\
&= 2^{\frac{4}{2}-1} \Gamma\left(\frac{4}{2}\right) \\
&= 2^{\frac{4}{2}-1} \cdot 1! \\
&= 2.
\end{aligned}$$

Thus,

$$\mathsf{Var}(Y) = \mathsf{E}(Y^2) - [\mathsf{E}(Y)]^2 = 2 - \frac{\pi}{2} = \frac{4-\pi}{2}.$$

Problem 6. Recall Theorem 3.4 in the notes.

(a) Analytically prove that

$$\int_0^t \frac{\lambda^n u^{n-1} e^{-\lambda u}}{(n-1)!} du = 1 - \sum_{j=0}^{n-1} \frac{(\lambda t)^j e^{-\lambda t}}{j!}, \quad t \geq 0,$$

by showing (i) that both sides have the same derivative with respect to t and (ii) that both have the same value when $t = 0$.

(b) Let $\lambda = 2.5$, $n = 5$ and $t = 3$, and use the `ppois` and `pgamma` functions in R (the Poisson and gamma cdfs, respectively) to find $\mathsf{P}(Y \geq 5)$ and $\mathsf{P}(T \leq 3)$ for $Y \sim \text{Poisson}(7.5)$ and $T \sim \text{gamma}(5, 0.4)$.

Theorem 3.4. Let X_k be the time until the k-th occurrence for a Poisson process and let Y_t be the number of occurrences in the interval $[0, t]$. Then $X_k \sim \text{gamma}(k, 1/\lambda)$ and $Y_t \sim \text{Poisson}(\lambda t)$. Furthermore, $\mathsf{P}(X_k \leq t) = \mathsf{P}(Y_t \geq k)$. Specifically,

$$\int_0^t \frac{\lambda^k u^{k-1} e^{-\lambda u}}{(k-1)!} du = 1 - \sum_{j=0}^{k-1} \frac{(\lambda t)^j e^{-\lambda t}}{j!}, \quad t \geq 0.$$

(a) Let

$$\begin{aligned}
L(t) &= \int_0^t \frac{\lambda^n u^{n-1} e^{-\lambda u}}{(n-1)!} du, \\
R(t) &= 1 - \sum_{j=0}^{n-1} \frac{(\lambda t)^j e^{-\lambda t}}{j!}.
\end{aligned}$$

Then,

$$
\begin{aligned}
L'(t) &= \frac{\lambda^n t^{n-1} e^{-\lambda t}}{(n-1)!}, \\
R'(t) &= -\sum_{j=0}^{n-1} \left[\frac{j(\lambda t)^{j-1} e^{-\lambda t}}{j!} + \frac{(\lambda t)^j (-\lambda) e^{-\lambda t}}{j!} \right] \\
&= -\sum_{j=0}^{n-1} \left[\frac{\lambda^j t^{j-1} e^{-\lambda t}}{(j-1)!} - \frac{\lambda^{j+1} t^j e^{-\lambda t}}{j!} \right] \\
&= -\left[\sum_{j=1}^{n-1} \frac{\lambda^j t^{j-1} e^{-\lambda t}}{(j-1)!} - \sum_{j=0}^{n-2} \frac{\lambda^{j+1} t^j e^{-\lambda t}}{j!} - \frac{\lambda^n t^{n-1} e^{-\lambda t}}{(n-1)!} \right] \\
&= -\left[\sum_{i=0}^{n-2} \frac{\lambda^{i+1} t^i e^{-\lambda t}}{i!} - \sum_{i=0}^{n-2} \frac{\lambda^{i+1} t^i e^{-\lambda t}}{i!} - \frac{\lambda^n t^{n-1} e^{-\lambda t}}{(n-1)!} \right] \\
&= \frac{\lambda^n t^{n-1} e^{-\lambda t}}{(n-1)!}.
\end{aligned}
$$

Thus, $L'(t) = R'(t)$. Also, $L(0) = 0$ and $R(0) = 1 - 1 = 0$. Therefore, $L(t) = R(t)$.

(b) Using R, we have

$$\mathsf{P}(Y \geq 5) = 1 - \texttt{ppois}(4, 7.5) \approx 0.868,$$

and

$$\mathsf{P}(T \leq 3) = \texttt{pgamma}(3, 5, 0.4) \approx 0.868.$$

Problem 7. Let $Z \sim \text{normal}(0, 1)$. Prove $Z^2 \sim \chi^2(1)$. This is Theorem 3.11 in the notes. Hint: express the event $Z^2 \leq y$ as an interval of values for Z, keeping in mind that Z can be negative and positive.

Theorem 3.11. Let $Z \sim \text{normal}(0, 1)$. Then $Z^2 \sim \chi^2(1)$.

Let $Y = Z^2$. Then, for $y \geq 0$,

$$
\begin{aligned}
F_Y(y) &= \mathsf{P}(Y \leq y) \\
&= \mathsf{P}(Z^2 \leq y) \\
&= \mathsf{P}(-\sqrt{y} \leq Z \leq \sqrt{y}) \\
&= F_Z(\sqrt{y}) - F_Z(-\sqrt{y}) \\
&= F_Z(\sqrt{y}) - [1 - F_Z(\sqrt{y})] \\
&= 2F_Z(\sqrt{y}) - 1.
\end{aligned}
$$

Thus, the pdf of Y is

$$
\begin{aligned}
f_Y(y) &= F_Y'(y) \\
&= 2f_Z(\sqrt{y}) \cdot \frac{1}{2\sqrt{y}} \\
&= \frac{1}{\sqrt{2\pi y}} e^{-\frac{y}{2}}, \quad y > 0.
\end{aligned}
$$

This is the pdf of a $\chi^2(1)$ distribution. Therefore, $Z^2 \sim \chi^2(1)$.