STAT610 Assignment 5
due Monday, 6 October 2025

1. Suppose Z has standard normal distribution.
   (a) Use the mgf to find the third and fourth moments of Z. (Recall Slide 119.)
   (b) Use part (a) to deduce $\mathsf{E}(X^3)$ and $\mathsf{E}(X^4)$ where $X \sim \text{normal}(\mu, \sigma^2)$ (e.g., $X = \mu + \sigma Z$).

2. Exercise 2.14. Add
   (c) Observe that $X = \int_0^X \mathrm{d}x = \int_0^\infty 1_{X>x}\,\mathrm{d}x$, and then, after taking expectation of the right-hand expression, exchange expectation and integral to prove that the expression in part (a) holds for *any* nonnegative random variable, regardless of the type of distribution. [Exchanging integration and expectation is like exchanging integration and sum or doing double integration in the other order, etc. The general result is *Fubini's Theorem* and is valid, at least, for nonnegative quantities like the example here.]

3. Let X have the *Laplace* distribution (recalling Problem 3 of Assignment 4), with pdf

$$f(x) = \frac{\lambda}{2}\mathrm{e}^{-\lambda|x|}, \quad \text{all } x.$$

   (a) Show that X has mgf $M_X(t) = \frac{1}{1-t^2/\lambda^2}$ for $|t| < \lambda$. Hint: integrate separately for $x < 0$ and $x > 0$, and then combine.
   (b) Determine the quantile function for X. Again, you need to think about separate cases. Make sure you get a continuous increasing function.
   (c) Find the median and the 25-th and 75-th percentiles, as functions of λ.

4. Suppose that the probability of being able to make a left turn on the first signal cycle of a very busy intersection is 32%. Assuming independent trips, let W be the number of times that one is not successful turning on the first cycle before the fifth time that one is successful.
   (a) What is the distribution of W and its mean and variance?
   (b) Determine the chance the W is no more than 10.
   (c) Let Y be the number of successes in 15 trips. What is the chance that $Y \geq 5$?

5. Chapter 3 Exercise 24(b,e). Determine the medians and 90-th percentiles of each distribution, and find the mean and variance only for part (b).

6. Recall Theorem 3.4 in the notes.
   (a) Analytically prove that

$$\int_0^t \frac{\lambda^n u^{n-1}\mathrm{e}^{-\lambda u}}{(n-1)!}\,\mathrm{d}u = 1 - \sum_{j=0}^{n-1} \frac{(\lambda t)^j \mathrm{e}^{-\lambda t}}{j!}, \quad t \geq 0,$$

   by showing (i) that both sides have the same derivative with respect to t and (ii) that both have the same value when $t = 0$.
   (b) Let $\lambda = 2.5$, $n = 5$ and $t = 3$, and use the `ppois` and `pgamma` functions in R (the Poisson and gamma cdfs, respectively) to find $\mathsf{P}(Y \geq 5)$ and $\mathsf{P}(T \leq 3)$ for $Y \sim \text{Poisson}(7.5)$ and $T \sim \text{gamma}(5, 0.4)$.

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7. Let $Z \sim \text{normal}(0, 1)$. Prove $Z^2 \sim \text{chi-square}(1)$. This is Theorem 3.11 in the notes. Hint: express the event $Z^2 \leq y$ as an interval of values for Z, keeping in mind that Z can be negative and positive.