Printed Name: ______________________________

STAT610 Exam II
Monday, 3 November 2023

Instructions.

- You have 60 minutes.
- You may use a stand-alone calculator, but no other resources.
- Please write your solutions on separate sheets of paper, and *return them in order* with this sheet on top.
- You may use any results from class or homework as long as you explain clearly the result you are using (by name or description).
- To ensure credit, explain or show all of your steps.
- There are 50 points total.
- Attempt all parts of a question; some may not rely on successful completion of earlier parts.

1. (10 points) X and Y are independent random variables with moment generating functions $M_X(t) = \frac{\mathrm{e}^t}{1-t^2}$ and $M_Y(t) = \mathrm{e}^{\mathrm{e}^t - t - 1}$, respectively.

   Find $\mathsf{Var}(X+Y)$.

2. (S,T) has joint pdf $f_{S,T}(s,t) = \frac{(s+t)^2}{6}\mathrm{e}^{-s-t}$ for $s > 0$, $t > 0$.

   (a) (10 points) Let $W = S + T$ and $Z = \frac{S}{S+T}$. Find the joint pdf for (W,Z) and identify.

   (b) (10 points) Accept as given that $\mathsf{E}(S) = \mathsf{E}(T) = 2$. Find $\mathsf{Cov}(S,T)$.

3. (10 points) Suppose X and Y are positive integer-valued random variables with joint pmf

$$f_{X,Y}(x,y) = C 1_{\{1,\ldots,10\}}(x) 1_{\{1,\ldots,x\}}(y)$$

   for some constant C.

   What is the best predictor of Y as a function of X? Explain.

4. (10 points) Show that the beta distributions form a two-parameter exponential family.

(distribution formulas on next page)

Some common probability density functions and probability mass functions are provided below.

beta(α, β) $f(x) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} x^{\alpha-1}(1-x)^{\beta-1}$, for $0 < x < 1$, $\alpha > 0$, $\beta > 0$.
$\mathsf{E}(X) = \frac{\alpha}{\alpha+\beta}$, $\mathsf{Var}(X) = \frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}$.

binomial(n, p) $f(x) = \binom{n}{x} p^x (1-p)^{n-x}$, for $x = 0, 1, \ldots, n$; $0 < p < 1$.
$\mathsf{E}(X) = np$, $\mathsf{Var}(X) = np(1-p)$.

exponential(β) $f(x) = \frac{1}{\beta} \mathrm{e}^{-x/\beta}$, for $x > 0$; $\beta > 0$.
$\mathsf{E}(X) = \beta$, $\mathsf{Var}(X) = \beta^2$.

gamma(α, β) $f(x) = \frac{1}{\beta^\alpha \Gamma(\alpha)} x^{\alpha-1} \mathrm{e}^{-x/\beta}$, for $x > 0$; $\alpha > 0$, $\beta > 0$.
$\mathsf{E}(X) = \alpha\beta$,$\mathsf{Var}(X) = \alpha\beta^2$.

geometric(p) $f(x) = p(1-p)^{x-1}$, for $x = 1, 2, \ldots$; $0 < p < 1$.
$\mathsf{E}(X) = \frac{1}{p}$, $\mathsf{Var}(X) = \frac{1-p}{p^2}$.

hypergeometric(N, M, n) $f(x) = \binom{M}{x}\binom{N-M}{n-x} / \binom{N}{n}$, for $x = 0, 1, \ldots, n$; $n > 0$, $M > 0$, $N > 0$.
$\mathsf{E}(X) = \frac{nM}{N}$, $\mathsf{Var}(X) = \frac{N-n}{N-1} \frac{nM}{N} (1 - \frac{M}{N})$.

negative binomial(k, p) $f(x) = \binom{k+x-1}{k-1} p^k (1-p)^x$, for $x = 0, 1, 2, \ldots$; $0 < p < 1$, $k > 0$.
$\mathsf{E}(X) = \frac{k(1-p)}{p}$, $\mathsf{Var}(X) = \frac{k(1-p)}{p^2}$.

normal(μ, σ^2) $f(x) = \frac{1}{\sqrt{2\pi}\sigma} \mathrm{e}^{-(x-\mu)^2/2\sigma^2}$, for $-\infty < x < \infty$; $-\infty < \mu < \infty$, $\sigma^2 > 0$.
$\mathsf{E}(X) = \mu$, $\mathsf{Var}(X) = \sigma^2$.

Poisson(λ) $f(x) = \frac{\lambda^x}{x!} \mathrm{e}^{-\lambda}$, for $x = 0, 1, 2, \ldots$; $\lambda > 0$.
$\mathsf{E}(X) = \lambda$, $\mathsf{Var}(X) = \lambda$.

uniform(a, b) $f(x) = \frac{1}{b-a}$, for $a < x < b$; $a < b$.
$\mathsf{E}(X) = \frac{a+b}{2}$, $\mathsf{Var}(X) = \frac{(b-a)^2}{12}$.

Weibull(γ, β) $f(x) = \frac{\gamma}{\beta} x^{\gamma-1} \mathrm{e}^{-x^\gamma/\beta}$, for $x > 0$; $\gamma > 0$, $\beta > 0$.
$\mathsf{E}(X) = \beta^{1/\gamma}\Gamma(1 + \frac{1}{\gamma})$, $\mathsf{Var}(X) = \beta^{2/\gamma}(\Gamma(1 + \frac{2}{\gamma}) - \Gamma(1 + \frac{1}{\gamma})^2)$.