Homework 1

Mengxiang Jiang
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Problem 1. *Statistical Inference* by Casella and Berger, 2nd Edition, Chapter 1, Exercise 4, 5, and 6.

4 For events A and B, find formulas for the probabilities of the following events in terms of the quantities $P(A)$, $P(B)$, and $P(A \cap B)$.

(a) either A or B or both
(b) either A or B but not both
(c) at least one of A or B
(d) at most one of A or B

(a) $P(A \cup B) = P(A) + P(B) - P(A \cap B)$
(b) $P(A \cup B) - P(A \cap B) = P(A) + P(B) - 2P(A \cap B)$
(c) $P(A \cup B) = P(A) + P(B) - P(A \cap B)$
(d) $1 - P(A \cap B)$

5 Approximately one-third of all human twins are identical (one-egg) and two-thirds are fraternal (two-egg) twins. Identical twins are necessarily the same sex, with male and female being equally likely. Among fraternal twins, approximately one-fourth are both female, one-fourth are both male, and half are one male and one female. Finally, among all U.S. births, approximately 1 in 90 is a twin birth. Define the following events:

- A = the birth results in twin females
- B = the twins are identical twins
- C = a U.S. birth results in twins

(a) State, in words, the event $A \cap B \cap C$.
(b) Find $P(A \cap B \cap C)$.

(a) The event that a U.S. birth results in identical twin females.
(b) From the given information, we have

* $P(C) = \frac{1}{90}$
    * $P(B|C) = \frac{1}{3}$
    * $P(A|B, C) = \frac{1}{2}$

  So, $P(A \cap B \cap C) = P(C)P(B|C)P(A|B, C) = \frac{1}{90} \times \frac{1}{3} \times \frac{1}{2} = \frac{1}{540}$.

6 Two pennies, one with $P(\text{head}) = u$ and one with $P(\text{head}) = w$, are to be tossed together independently. Define

  - $p_0 = P(0 \text{ heads occur})$,
  - $p_1 = P(1 \text{ head occurs})$,
  - $p_2 = P(2 \text{ heads occur})$.

  Can u and w be chosen so that $p_0 = p_1 = p_2$? Prove your answer.

  We have

  - $p_0 = (1 - u)(1 - w)$,
  - $p_1 = u(1 - w) + w(1 - u)$,
  - $p_2 = uw$.

  So, $p_0 = p_2$ implies $(1 - u)(1 - w) = uw$, which simplifies to $u + w = 1$. Also, $p_1 = p_2$ implies $u(1 - w) + w(1 - u) = uw$, which simplifies to $u + w = 3uw$. Combining these two equations, we have $3uw = 1$, or $uw = \frac{1}{3}$. However, since $u + w = 1$, by AM-GM inequality, we have $\frac{u+w}{2} = \frac{1}{2} \geq \sqrt{uw}$, which when squared gives $\frac{1}{4} \geq uw$. This contradicts $uw = \frac{1}{3}$. Therefore, there are no such u and w, assuming they are both in $[0, 1]$.

Problem 2. Here is a sample space: $\mathcal{S} = \{a, b, c\}$.

(a) Explicitly provide the σ-algebra of all subsets.

(b) Suppose one may only observe whether the outcome is a or not. Explicitly provide the smallest relevant σ-algebra. Hint: what events may be obtained by complements, unions and intersections, starting only with a?