Final Exam 2023

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Stat 610 Distribution Theory

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Problem 1. Suppose $X_1, \ldots, X_n$ are iid with pdf $f_X(x) = \frac{2x^3}{\beta^4} e^{-(x/\beta)^2}$ for $x > 0$. This has moments

$$\mathsf{E}(X) = \frac{3\sqrt{\pi}}{4}\beta, \quad \mathsf{E}(X^2) = 2\beta^2, \quad \mathsf{E}(X^3) = \frac{15\sqrt{\pi}}{8}\beta^3, \quad \mathsf{E}(X^4) = 6\beta^4.$$

(a) Determine C and V such that

$$\sqrt{n}\left(\frac{1}{2n}\sum_{i=1}^{n} X_i^2 - C\right) \xrightarrow{D} \text{normal}(0, V)$$

as $n \to \infty$, with explanation.

(b) Find the pdf for $Y = (X/\beta)^2$ and identify the distribution.

(a) By the Weak Law of Large Numbers (WLLN), we have

$$\frac{1}{n}\sum_{i=1}^{n} X_i^2 \xrightarrow{P} \mathsf{E}(X^2) = 2\beta^2.$$

Thus, we can choose $C = \beta^2$. By the Central Limit Theorem (CLT), we have

$$\sqrt{n}\left(\frac{1}{n}\sum_{i=1}^{n} X_i^2 - \mathsf{E}(X^2)\right) \xrightarrow{D} \text{normal}(0, \mathsf{Var}(X^2)).$$

Therefore,

$$\begin{aligned}
\sqrt{n}\left(\frac{1}{2n}\sum_{i=1}^{n} X_i^2 - C\right) &= \frac{1}{2}\sqrt{n}\left(\frac{1}{n}\sum_{i=1}^{n} X_i^2 - \mathsf{E}(X^2)\right) \\
&\xrightarrow{D} \text{normal}\left(0, \frac{1}{4}\mathsf{Var}(X^2)\right).
\end{aligned}$$

Next, we calculate $\mathsf{Var}(X^2)$:

$$\begin{aligned}
\mathsf{Var}(X^2) &= \mathsf{E}(X^4) - (\mathsf{E}(X^2))^2 \\
&= 6\beta^4 - (2\beta^2)^2 \\
&= 2\beta^4.
\end{aligned}$$

Thus, we can choose $V = \frac{1}{4}\mathsf{Var}(X^2) = \frac{1}{4} \cdot 2\beta^4 = \frac{1}{2}\beta^4$.

(b) To find the pdf of $Y = (X/\beta)^2$, we use the transformation method. The inverse transformation is $X = \beta\sqrt{Y}$. The Jacobian is

$$J = \left|\frac{dX}{dY}\right| = \left|\frac{\beta}{2\sqrt{Y}}\right| = \frac{\beta}{2\sqrt{Y}}.$$

The pdf of Y is given by

$$\begin{aligned} f_Y(y) &= f_X(\beta\sqrt{y}) \cdot J \\ &= \frac{2(\beta\sqrt{y})^3}{\beta^4} e^{-(\beta\sqrt{y}/\beta)^2} \cdot \frac{\beta}{2\sqrt{y}} \\ &= \frac{2\beta^3 y^{3/2}}{\beta^4} e^{-y} \cdot \frac{\beta}{2\sqrt{y}} \\ &= y e^{-y}, \quad y > 0. \end{aligned}$$

This is the pdf of a Gamma distribution with shape parameter $k = 2$ and scale parameter $\theta = 1$.

Problem 2. Assume $Y_1, \ldots, Y_n$ are iid with cdf $F_Y(y) = 1 - (1 + y)e^{-y}$.

(a) Define $C_1 = \sum_{i=1}^n 1_{[0,1]}(Y_i)$, $C_2 = \sum_{i=1}^n 1_{(1,3]}(Y_i)$ and $C_3 = \sum_{i=1}^n 1_{(3,\infty)}(Y_i)$, so that (C_1, C_2, C_3) has trinomial distribution. What is the mean and variance of each C_i?

(b) Find the cdf and pdf for the minimum order statistic $Y_{(1)} = \min_{i \le n} Y_i$. Hint: $Y_{(1)} > y$ iff every $Y_i > y$

(a) We have

$$p_1 = \mathsf{P}(Y_i \in [0, 1]) = F_Y(1) - F_Y(0) = \left(1 - 2e^{-1}\right) - 0 = 1 - 2e^{-1},$$

$$p_2 = \mathsf{P}(Y_i \in (1, 3]) = F_Y(3) - F_Y(1) = \left(1 - 4e^{-3}\right) - \left(1 - 2e^{-1}\right) = 2e^{-1} - 4e^{-3},$$

$$p_3 = \mathsf{P}(Y_i \in (3, \infty)) = 1 - F_Y(3) = 4e^{-3}.$$

The mean and variance of each C_i are given by

$$\mathsf{E}(C_i) = np_i, \quad \mathsf{Var}(C_i) = np_i(1 - p_i).$$

(b) To find the cdf and pdf of the minimum order statistic $Y_{(1)}$, we use the fact that

$$\mathsf{P}(Y_{(1)} > y) = \mathsf{P}(Y_1 > y, \ldots, Y_n > y) = (1 - F_Y(y))^n.$$

Therefore, the cdf of $Y_{(1)}$ is

$$F_{Y_{(1)}}(y) = 1 - (1 - F_Y(y))^n,$$

and the pdf is

$$f_{Y_{(1)}}(y) = \frac{d}{dy} F_{Y_{(1)}}(y) = n\,(1 - F_Y(y))^{n-1} f_Y(y).$$

Problem 3. The conditional distribution of T , given $U = u$, has pdf

$$f_{T|U}(t|u) = \frac{1}{\sqrt{\pi}} u^{1/2} e^{-ut^2}$$

for all real t, and $U \sim \text{gamma}(3, 1/2)$. Find the conditional pdf for U, given $T = t$, and identify the marginal pdf for T.

$$f_U(u) = \frac{1}{\Gamma(3)(1/2)^3} u^{3-1} e^{-u/(1/2)} = \frac{8}{\Gamma(3)} u^2 e^{-2u} = \frac{8}{2} u^2 e^{-2u} = 4u^2 e^{-2u}$$

The joint pdf of (T, U) is

$$\begin{aligned} f_{T,U}(t,u) &= f_{T|U}(t|u) f_U(u) \\ &= \frac{1}{\sqrt{\pi}} u^{1/2} e^{-ut^2} \cdot 4u^2 e^{-2u} \\ &= \frac{4}{\sqrt{\pi}} u^{5/2} e^{-u(t^2+2)}. \end{aligned}$$

Thus the conditional pdf of U given $T = t$ is a gamma distribution with shape parameter $7/2$ and scale parameter $1/(t^2 + 2)$:

$$f_{U|T}(u|t) = \frac{(t^2+2)^{7/2}}{\Gamma(7/2)} u^{5/2} e^{-u(t^2+2)}.$$

To find the marginal pdf of T, we integrate out U:

$$\begin{aligned} f_T(t) &= \int_0^\infty f_{T,U}(t,u)\, du \\ &= \int_0^\infty \frac{4}{\sqrt{\pi}} u^{5/2} e^{-u(t^2+2)}\, du \\ &= \frac{4}{\sqrt{\pi}} \cdot \frac{\Gamma(7/2)}{(t^2+2)^{7/2}} \\ &= \frac{4}{\sqrt{\pi}} \cdot \frac{15\sqrt{\pi}/8}{(t^2+2)^{7/2}} \\ &= \frac{15/2}{(t^2+2)^{7/2}}. \end{aligned}$$

This is the pdf of a Student's t-distribution.

Problem 4. Let $T_1, T_2, \ldots$ be iid geometric(p) random variables, and set $\overline{T}_n = \frac{1}{n}\sum_{i=1}^n T_i$.

(a) Does $-\log(\overline{T}_n)$ converge with probability 1, and, if so, then to what? Explain fully.

(b) Use the delta method to show that $\frac{1}{\overline{T}_n}$ is asymptotically normal, giving particulars.

(a) By the Strong Law of Large Numbers (SLLN), we have

$$\overline{T}_n \xrightarrow{w.p.1} \mathsf{E}(T_1) = \frac{1}{p}.$$

Since the function $g(x) = -\log(x)$ is continuous, by the Continuous Mapping Theorem, we have

$$-\log(\overline{T}_n) \xrightarrow{w.p.1} -\log\left(\frac{1}{p}\right) = \log(p).$$

(b) By the Central Limit Theorem (CLT), we have

$$\sqrt{n}\left(\overline{T}_n - \mathsf{E}(T_1)\right) \xrightarrow{D} \text{normal}\left(0, \mathsf{Var}(T_1)\right).$$

The mean and variance of a geometric(p) random variable are

$$\mathsf{E}(T_1) = \frac{1}{p}, \quad \mathsf{Var}(T_1) = \frac{1-p}{p^2}.$$

Thus,

$$\sqrt{n}\left(\overline{T}_n - \frac{1}{p}\right) \xrightarrow{D} \text{normal}\left(0, \frac{1-p}{p^2}\right).$$

Now, we apply the delta method with $g(x) = \frac{1}{x}$, which has derivative

$$g'(x) = -\frac{1}{x^2}.$$

Evaluating at $x = \mathsf{E}(T_1) = \frac{1}{p}$, we have

$$g'\left(\frac{1}{p}\right) = -p^2.$$

Therefore, by the delta method,

$$\sqrt{n}\left(\frac{1}{\overline{T}_n} - p\right) \xrightarrow{D} \text{normal}\left(0, \left(-p^2\right)^2 \cdot \frac{1-p}{p^2}\right) = \text{normal}\left(0, p^2(1-p)\right).$$

Problem 5. Assume (X, Y) is bivariate normal with $\mathsf{E}(X) = \mathsf{E}(Y) = 0$, $\mathsf{Var}(X) = \mathsf{Var}(Y) = 1$ and $\mathrm{Corr}(X, Y) = \rho$. Determine the joint distribution of $(2X + Y, X - 2Y)$ by name and parameter values. (There is no need to use pdfs.)

Let $Z = (2X + Y, X - 2Y)^T$. Since (X, Y) is bivariate normal, any linear combination of X and Y is also normally distributed. Thus, Z is bivariate normal. Next, we calculate the mean vector and covariance matrix of Z. The mean vector is

$$\mathsf{E}(Z) = \begin{pmatrix} \mathsf{E}(2X+Y) \\ \mathsf{E}(X-2Y) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

The covariance matrix is

$$\mathrm{Cov}(Z) = \begin{pmatrix} \mathsf{Var}(2X+Y) & \mathrm{Cov}(2X+Y, X-2Y) \\ \mathrm{Cov}(X-2Y, 2X+Y) & \mathsf{Var}(X-2Y) \end{pmatrix}.$$

Calculating each element, we have

$$\begin{aligned} \mathsf{Var}(2X+Y) &= 4\mathsf{Var}(X) + \mathsf{Var}(Y) + 4\mathrm{Cov}(X,Y) = 4 + 1 + 4\rho = 5 + 4\rho, \\ \mathsf{Var}(X-2Y) &= \mathsf{Var}(X) + 4\mathsf{Var}(Y) - 4\mathrm{Cov}(X,Y) = 1 + 4 - 4\rho = 5 - 4\rho, \\ \mathrm{Cov}(2X+Y, X-2Y) &= 2\mathsf{Var}(X) - 4\mathrm{Cov}(X,Y) + \mathrm{Cov}(Y,X) - 2\mathsf{Var}(Y) \\ &= 2 - 4\rho + \rho - 2 = -3\rho. \end{aligned}$$

Thus, the covariance matrix is

$$\mathrm{Cov}(Z) = \begin{pmatrix} 5+4\rho & -3\rho \\ -3\rho & 5-4\rho \end{pmatrix}.$$

Therefore, the joint distribution of $(2X + Y, X - 2Y)$ is bivariate normal with mean vector $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$ and covariance matrix $\begin{pmatrix} 5+4\rho & -3\rho \\ -3\rho & 5-4\rho \end{pmatrix}$.