Homework 10

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Stat 610 Distribution Theory

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Problem 1. Let $W_1, W_2, \ldots$ be an iid sequence of Weibull(2, 2) random variables, with pdf $f_W(w) = we^{-w^2/2}$, $w \geq 0$.

(a) Use the Strong Law of Large Numbers to determine $\lim_{n\to\infty} \overline{W}_n$. Be sure to evaluate the limit.

(b) Similarly, find $\lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^n W_i^2$.

(c) Based on the above, find $\lim_{n\to\infty} \hat{\sigma}_n^2$, where $\hat{\sigma}_n^2 = \frac{1}{n}\sum_{i=1}^n (W_i - \overline{W}_n)^2 = \frac{1}{n}\sum_{i=1}^n W_i^2 - \overline{W}_n^2$. Explain what theorem allows you to apply the results from (a) and (b)

(d) Similarly, does the sample coefficient of variation $\frac{\hat{\sigma}_n}{\overline{W}_n}$ have a limit? Explain.

(e) Apply the CLT to approximate $P(1.2 \leq \overline{W}_{50} \leq 1.4)$ for a random sample of size $n = 50$.

(a) By the Strong Law of Large Numbers, we have

$$\begin{aligned}
\overline{W}_n \to \mathsf{E}(W_1) &= \beta^{1/\gamma}\Gamma(1 + 1/\gamma) \\
&= (2)^{1/2}\Gamma(1 + 1/2) \\
&= \sqrt{\frac{\pi}{2}}.
\end{aligned}$$

(b) Similarly, by the Strong Law of Large Numbers, we have

$$\begin{aligned}
\frac{1}{n}\sum_{i=1}^n W_i^2 \to \mathsf{E}(W_1^2) &= \mathsf{Var}(W_1) + (\mathsf{E}(W_1))^2 \\
&= \beta^{2/\gamma}\Gamma(1 + 2/\gamma) \\
&= 2^{2/2}\Gamma(1 + 2/2) \\
&= 2.
\end{aligned}$$

(c) By the results from (a) and (b), we have

$$\begin{aligned}\hat{\sigma}_n^2 &\to \mathsf{E}(W_1^2) - (\mathsf{E}(W_1))^2 \\ &= 2 - \left(\sqrt{\frac{\pi}{2}}\right)^2 \\ &= 2 - \frac{\pi}{2} = \frac{4-\pi}{2}.\end{aligned}$$

The theorem that allows us to apply the results from (a) and (b) is Theorem 5.13.

(d) Yes, the sample coefficient of variation has a limit. By the results from (a) and (c), we have

$$\begin{aligned}\frac{\hat{\sigma}_n}{\overline{W}_n} &\to \frac{\sqrt{\mathsf{Var}(W_1)}}{\mathsf{E}(W_1)} \\ &= \frac{\sqrt{\frac{4-\pi}{2}}}{\sqrt{\frac{\pi}{2}}} \\ &= \sqrt{\frac{4-\pi}{\pi}}.\end{aligned}$$

(e) By the Central Limit Theorem, we have

$$\begin{aligned}\frac{\sqrt{n}(\overline{W}_n - \mathsf{E}(W_1))}{\sqrt{\mathsf{Var}(W_1)}} &\xrightarrow{D} Z \\ \frac{\sqrt{50}(\overline{W}_{50} - \sqrt{\frac{\pi}{2}})}{\sqrt{\frac{4-\pi}{2}}} &\xrightarrow{D} Z,\end{aligned}$$

The endpoints 1.2 and 1.4 can be standardized as follows:

$$\begin{aligned}z_1 &= \frac{\sqrt{50}(1.2 - \sqrt{\frac{\pi}{2}})}{\sqrt{\frac{4-\pi}{2}}} \approx -0.575, \\ z_2 &= \frac{\sqrt{50}(1.4 - \sqrt{\frac{\pi}{2}})}{\sqrt{\frac{4-\pi}{2}}} \approx 1.583.\end{aligned}$$

Thus,

$$\begin{aligned}P(1.2 \le \overline{W}_{50} \le 1.4) &\approx P(z_1 \le Z \le z_2) \\ &= P(Z \le z_2) - P(Z \le z_1) \\ &\approx 0.943 - 0.282 = 0.661.\end{aligned}$$

Problem 2. Let $X_1, X_2, \ldots$ be iid Bernoulli(p) random variables. Provide consistent estimators for the odds $\frac{p}{1-p}$ and for the log-odds $\log\left(\frac{p}{1-p}\right)$. Explain your reasoning.

Since X_i are iid Bernoulli(p) random variables, by the Strong Law of Large Numbers, we have

$$\overline{X}_n \to \mathsf{E}(X_1) = p.$$

Therefore, a consistent estimator for the odds $\frac{p}{1-p}$ is

$$\frac{\overline{X}_n}{1 - \overline{X}_n},$$

and a consistent estimator for the log-odds $\log\left(\frac{p}{1-p}\right)$ is

$$\log\left(\frac{\overline{X}_n}{1 - \overline{X}_n}\right).$$

This is because both functions $\frac{x}{1-x}$ and $\log\left(\frac{x}{1-x}\right)$ are continuous for $x \in (0, 1)$, and we can apply theorem 5.13 to conclude the consistency of the estimators.

Problem 3. Suppose $X_1, X_2, \ldots$ is an iid sequence of gamma(3, β) random variables. Note that $\mathsf{E}(X_n) = 3\beta$, so a reasonable estimator for β is $\frac{1}{3}\overline{X}_n$.

(a) Use the central limit theorem to find the limiting distribution for $\sqrt{n}\left(\frac{1}{3}\overline{X}_n - \beta\right)$. (Hint: you will need $\mathsf{Var}\left(\frac{1}{3}X_i\right)$.)

(b) Let $\hat{\mu}_2' = \frac{1}{n}\sum_{i=1}^n X_i^2$ and show that $\mathsf{E}(\hat{\mu}_2') = 12\beta^2$.

(c) Apply the CLT to $Y_i = \frac{1}{12}X_i^2$ to determine the limiting distribution for $\sqrt{n}\left(\frac{1}{12}\hat{\mu}_2' - \beta^2\right)$.

(d) Now apply the delta method with $g(x) = \sqrt{x}$ to the result in part (c) in order to find the limiting distribution for

$$\sqrt{n}\left(\sqrt{\frac{1}{12}\hat{\mu}_2'} - \beta\right).$$

Which is smaller, the limiting variance for part (a) or for part (d)? [Smaller is better for estimation purposes.]

(a) By the Central Limit Theorem, we have

$$\frac{\sqrt{n}\left(\overline{X}_n - \mathsf{E}(X_1)\right)}{\sqrt{\mathsf{Var}(X_1)}} \xrightarrow{D} Z$$
$$\frac{\sqrt{n}\left(\overline{X}_n - 3\beta\right)}{\sqrt{3\beta^2}} \xrightarrow{D} Z.$$

Therefore,

$$\begin{aligned}\sqrt{n}\left(\frac{1}{3}\overline{X}_n - \beta\right) &= \frac{1}{3}\sqrt{n}\left(\overline{X}_n - 3\beta\right) \\ &\xrightarrow{D} N\left(0, \frac{\beta^2}{3}\right).\end{aligned}$$

(b) Note that

$$\begin{aligned}\mathsf{E}(X_i^2) &= \mathsf{Var}(X_i) + (\mathsf{E}(X_i))^2 \\ &= 3\beta^2 + (3\beta)^2 \\ &= 12\beta^2.\end{aligned}$$

Thus,

$$\mathsf{E}(\hat{\mu}_2') = \mathsf{E}\left(\frac{1}{n}\sum_{i=1}^n X_i^2\right) = \frac{1}{n}\sum_{i=1}^n \mathsf{E}(X_i^2) = 12\beta^2.$$

(c) Since $Y_i = \frac{1}{12}X_i^2$, we have

$$\begin{aligned}\mathsf{E}(Y_i) &= \frac{1}{12}\mathsf{E}(X_i^2) = \beta^2, \\ \mathsf{Var}(Y_i) &= \frac{1}{144}\mathsf{Var}(X_i^2) \\ &= \frac{1}{144}\left(\mathsf{E}(X_i^4) - (\mathsf{E}(X_i^2))^2\right) = \frac{1}{144}(360\beta^4 - (12\beta^2)^2) \\ &= \frac{216\beta^4}{144} = \frac{3\beta^4}{2}.\end{aligned}$$

By the Central Limit Theorem, we have

$$\begin{aligned}\frac{\sqrt{n}\left(Y_i - \mathsf{E}(Y_i)\right)}{\sqrt{\mathsf{Var}(Y_i)}} &\xrightarrow{D} Z \\ \frac{\sqrt{n}\left(\frac{1}{12}\hat{\mu}_2' - \beta^2\right)}{\sqrt{\frac{3\beta^4}{2}}} &\xrightarrow{D} Z.\end{aligned}$$

Therefore,

$$\sqrt{n}\left(\frac{1}{12}\hat{\mu}_2' - \beta^2\right) \xrightarrow{D} N\left(0, \frac{3\beta^4}{2}\right).$$

(d) Note that $g'(x) = \frac{1}{2\sqrt{x}}$. Thus, by the delta method, we have

$$\begin{aligned}\sqrt{n}\left(\sqrt{\frac{1}{12}\hat{\mu}_2'} - \beta\right) &\xrightarrow{D} N\left(0, \left(g'(\beta^2)\right)^2 \cdot \frac{3\beta^4}{2}\right) \\ &= N\left(0, \left(\frac{1}{2\beta}\right)^2 \cdot \frac{3\beta^4}{2}\right) \\ &= N\left(0, \frac{3\beta^2}{8}\right).\end{aligned}$$

The limiting variance for part (a) is $\frac{\beta^2}{3} \approx 0.333\beta^2$, and the limiting variance for part (d) is $\frac{3\beta^2}{8} = 0.375\beta^2$. Therefore, the limiting variance for part (a) is smaller.

Problem 4. Let $W_1, \ldots, W_n$ be a simple random sample from the logistic(μ, β) distribution with pdf

$$f(w) = \frac{1}{\beta\left(e^{\frac{w-\mu}{2\beta}} + e^{-\frac{w-\mu}{2\beta}}\right)^2}.$$

Note: the mgf is $M(t) = e^{\mu t}\frac{\pi\beta t}{\sin(\pi\beta t)}$ for $|t| < \frac{1}{\beta}$.

(a) Use the mgf to determine the mean and variance of this distribution. (You might want to use L'Hôpital's rule or Taylor's expansions.)

(b) Is $\overline{W}$ a consistent estimator for μ? In what senses? Explain which theorems you use.

(c) Identify the limit distribution of $\sqrt{n}(\overline{W} - \mu)$.

(d) Is $\hat{\mu}_2' = \frac{1}{n}\sum_{i=1}^n W_i^2$ consistent for any quantity (parameter)? Explain.

(e) Discuss the asymptotic distribution of $\hat{\mu}_2'$ in terms of the appropriate moments. That is, how can you center and re-scale the statistic so that it converges to some normal distribution? You do not need to compute the moments but explain how you could do so if needed.

(a) We have the Taylor expansions of e^x and $\sin x$ as follows:

$$\begin{aligned} e^x &= 1 + x + \frac{x^2}{2} + O(x^3), \\ \sin x &= x - \frac{x^3}{3!} + O(x^5). \end{aligned}$$

Thus,

$$\begin{aligned} M(t) &= e^{\mu t}\frac{\pi\beta t}{\sin(\pi\beta t)} \\ &= e^{\mu t}\frac{\pi\beta t}{\pi\beta t - \frac{(\pi\beta t)^3}{3!} + O(t^5)} \\ &= e^{\mu t}\left(1 + \frac{(\pi\beta t)^2}{6} + O(t^4)\right) \\ &= \left(1 + \mu t + \frac{\mu^2 t^2}{2} + O(t^3)\right)\left(1 + \frac{\pi^2\beta^2 t^2}{6} + O(t^4)\right) \\ &= 1 + \mu t + \left(\frac{\mu^2}{2} + \frac{\pi^2\beta^2}{6}\right)t^2 + O(t^3). \end{aligned}$$

Therefore, we have

$$\begin{aligned} \mathsf{E}(W) &= M'(0) = \mu, \\ \mathsf{Var}(W) &= M''(0) - (M'(0))^2 = \left(\mu^2 + \frac{\pi^2\beta^2}{3}\right) - \mu^2 = \frac{\pi^2\beta^2}{3}. \end{aligned}$$

(b) By the Strong Law of Large Numbers, we have

$$\overline{W} \to \mathsf{E}(W_1) = \mu, \quad \text{with probability 1 as } n \to \infty.$$

Therefore, $\overline{W}$ is a consistent estimator for μ with probability 1. We used the Strong Law of Large Numbers theorem.

(c) By the Central Limit Theorem, we have

$$\frac{\sqrt{n}(\overline{W} - \mathsf{E}(W_1))}{\sqrt{\mathsf{Var}(W_1)}} \xrightarrow{D} Z$$
$$\frac{\sqrt{n}(\overline{W} - \mu)}{\sqrt{\frac{\pi^2\beta^2}{3}}} \xrightarrow{D} Z.$$

Therefore,

$$\sqrt{n}(\overline{W} - \mu) \xrightarrow{D} N\left(0, \frac{\pi^2\beta^2}{3}\right).$$

(d) Note that

$$\begin{aligned}\mathsf{E}(W_i^2) &= \mathsf{Var}(W_i) + (\mathsf{E}(W_i))^2 \\ &= \frac{\pi^2\beta^2}{3} + \mu^2.\end{aligned}$$

By the Strong Law of Large Numbers, we have

$$\hat{\mu}_2' = \frac{1}{n}\sum_{i=1}^{n} W_i^2 \to \mathsf{E}(W_1^2) = \frac{\pi^2\beta^2}{3} + \mu^2, \quad \text{with probability 1 as } n \to \infty.$$

Therefore, $\hat{\mu}_2'$ is a consistent estimator for $\mathsf{E}(W_1^2) = \frac{\pi^2\beta^2}{3} + \mu^2$.

(e) Let $Y_i = W_i^2$. Then, we have

$$\begin{aligned}\mathsf{E}(Y_i) &= \mathsf{E}(W_i^2) = \frac{\pi^2\beta^2}{3} + \mu^2, \\ \mathsf{Var}(Y_i) &= \mathsf{E}(W_i^4) - (\mathsf{E}(W_i^2))^2.\end{aligned}$$

By the Central Limit Theorem, we have

$$\frac{\sqrt{n}(\hat{\mu}_2' - \mathsf{E}(Y_i))}{\sqrt{\mathsf{Var}(Y_i)}} \xrightarrow{D} Z$$
$$\frac{\sqrt{n}\left(\hat{\mu}_2' - \left(\frac{\pi^2\beta^2}{3} + \mu^2\right)\right)}{\sqrt{\mathsf{Var}(Y_i)}} \xrightarrow{D} Z.$$

Therefore,

$$\sqrt{n}\left(\hat{\mu}_2' - \left(\frac{\pi^2\beta^2}{3} + \mu^2\right)\right) \xrightarrow{D} N(0, \mathsf{Var}(Y_i)).$$

To compute $\mathsf{Var}(Y_i)$, we can use the mgf with higher order expansions to find $\mathsf{E}(W_i^4)$.

Problem 5. Suppose $Y_1, Y_2, \ldots$ are iid $\sim$ Poisson(λ). The best estimator for λ is known to be the sample mean $\overline{Y}_n$. Suppose we want to estimate $g(\lambda) = P(Y_i = 0) = e^{-\lambda}$ using $g(\overline{Y}_n) = e^{-\overline{Y}_n}$.

(a) Find $\mathsf{E}(e^{-\overline{Y}_n})$ and compare it to $e^{-\lambda}$. Hint: think about mgfs.

(b) Use the delta method to determine a limit distribution for $\sqrt{n}(e^{-\overline{Y}_n} - e^{-\lambda})$.

(c) (Rao-Blackwell) Let $T = Y_1 + \cdots + Y_n$, which has Poisson$(n\lambda)$ distribution. Use the fact (which you can assume) that the conditional distribution of Y_1, given T, is binomial$(T, \frac{1}{n})$ to find $h(T) = \mathsf{E}(1_{\{Y_1=0\}}|T) = P(Y_1 = 0|T)$, which must be a function solely of T. Next, use iterated expectation to confirm that $\mathsf{E}(\mathsf{E}(1_{\{Y_1=0\}}|T)) = e^{-\lambda}$. [Observe that $h(T)$ is a statistic since it does not depend on λ. Therefore, $h(T)$ is an unbiased estimator of $e^{-\lambda}$.]

(a) Note that the mgf of Y_i is $M_{Y_i}(t) = e^{\lambda(e^t-1)}$. Thus, the mgf of $\overline{Y}_n$ is

$$M_{\overline{Y}_n}(t) = \left(M_{Y_i}\left(\frac{t}{n}\right)\right)^n = e^{n\lambda\left(e^{\frac{t}{n}}-1\right)}.$$

Therefore,

$$\mathsf{E}(e^{-\overline{Y}_n}) = M_{\overline{Y}_n}(-1) = e^{n\lambda\left(e^{-\frac{1}{n}}-1\right)}.$$

Using the Taylor expansion of e^x, we have

$$e^{-\frac{1}{n}} = 1 - \frac{1}{n} + \frac{1}{2n^2} - \frac{1}{6n^3} + O\left(\frac{1}{n^4}\right).$$

Thus,

$$\begin{aligned}\mathsf{E}(e^{-\overline{Y}_n}) &= e^{n\lambda\left(-\frac{1}{n}+\frac{1}{2n^2}-\frac{1}{6n^3}+O\left(\frac{1}{n^4}\right)\right)} \\ &= e^{-\lambda+\frac{\lambda}{2n}-\frac{\lambda}{6n^2}+O\left(\frac{1}{n^3}\right)} \\ &= e^{-\lambda}e^{\frac{\lambda}{2n}-\frac{\lambda}{6n^2}+O\left(\frac{1}{n^3}\right)}.\end{aligned}$$

Since $e^{\frac{\lambda}{2n}-\frac{\lambda}{6n^2}+O\left(\frac{1}{n^3}\right)} > 1$ for all n, we have

$$\mathsf{E}(e^{-\overline{Y}_n}) > e^{-\lambda}.$$

As $n \to \infty$, we have

$$\mathsf{E}(e^{-\overline{Y}_n}) \to e^{-\lambda}.$$

(b) By the Central Limit Theorem, we have

$$\frac{\sqrt{n}(\overline{Y}_n - \lambda)}{\sqrt{\lambda}} \xrightarrow{D} Z.$$

Let $g(x) = e^{-x}$, then $g'(\lambda) = -e^{-\lambda}$. By the delta method, we have

$$\begin{aligned}\sqrt{n}(e^{-\overline{Y}_n} - e^{-\lambda}) &\xrightarrow{D} N\left(0, (g'(\lambda))^2 \cdot \lambda\right) \\ &= N\left(0, e^{-2\lambda} \cdot \lambda\right).\end{aligned}$$

(c) Since the conditional distribution of Y_1 given T is binomial$(T, \frac{1}{n})$, we have

$$h(T) = \mathsf{E}(1_{\{Y_1=0\}}|T) = P(Y_1 = 0|T) = \left(1 - \frac{1}{n}\right)^T.$$

Using iterated expectation, we have

$$\mathsf{E}(h(T)) = \mathsf{E}(\mathsf{E}(1_{\{Y_1=0\}}|T)) = \mathsf{E}(1_{\{Y_1=0\}}) = P(Y_1 = 0) = e^{-\lambda}.$$

Therefore, $h(T)$ is an unbiased estimator of $e^{-\lambda}$.

Problem 6. Suppose $(X_1, \ldots, X_m)$ are iid normal(μ_X, σ^2) and $(Y_1, \ldots, Y_n)$ are iid normal(μ_Y, σ^2) with the two samples independent. Note the same variance σ^2 for each. Let $\overline{X}$ and $\overline{Y}$ be the respective sample means.

(a) By first noting the joint distribution of $\overline{X}$ and $\overline{Y}$, determine the distribution of $\overline{X} - \overline{Y}$ and of

$$Z = \frac{\overline{X} - \overline{Y} - (\mu_X - \mu_Y)}{\sigma\sqrt{\frac{1}{m} + \frac{1}{n}}}.$$

(b) Let S_X^2 and S_Y^2 be the conventional sample variances, respectively. Show that

$$U = \frac{(m-1)S_X^2}{\sigma^2} + \frac{(n-1)S_Y^2}{\sigma^2}$$

has a chi-square distribution. What is the degrees of freedom parameter?

(c) Observe that U is independent of $\overline{X} - \overline{Y}$ (why?), and deduce the distribution of

$$T = \frac{\sqrt{m+n-2}Z}{\sqrt{U}}.$$

(a) Since $\overline{X} \sim$ normal$(\mu_X, \frac{\sigma^2}{m})$ and $\overline{Y} \sim$ normal$(\mu_Y, \frac{\sigma^2}{n})$, we have

$$\overline{X} - \overline{Y} \sim \text{normal}\left(\mu_X - \mu_Y, \sigma^2\left(\frac{1}{m} + \frac{1}{n}\right)\right).$$

Therefore,

$$Z = \frac{\overline{X} - \overline{Y} - (\mu_X - \mu_Y)}{\sigma\sqrt{\frac{1}{m} + \frac{1}{n}}} \sim \text{normal}(0, 1).$$

(b) Note that

$$\frac{(m-1)S_X^2}{\sigma^2} = \frac{1}{\sigma^2}\sum_{i=1}^m (X_i - \overline{X})^2 \sim \chi^2_{m-1},$$
$$\frac{(n-1)S_Y^2}{\sigma^2} = \frac{1}{\sigma^2}\sum_{i=1}^n (Y_i - \overline{Y})^2 \sim \chi^2_{n-1}.$$

Since the two samples are independent, we have

$$U = \frac{(m-1)S_X^2}{\sigma^2} + \frac{(n-1)S_Y^2}{\sigma^2} \sim \chi^2_{m+n-2}.$$

So the degrees of freedom parameter is $m + n - 2$.

(c) Since $\overline{X} - \overline{Y}$ is independent of S_X^2 and S_Y^2 because they are based on independent components of the samples, we have Z is independent of U. Therefore,

$$T = \frac{\sqrt{m+n-2}Z}{\sqrt{U}} = \frac{Z}{\sqrt{U/(m+n-2)}} \sim t_{m+n-2},$$

where t_{m+n-2} is the t-distribution with $m + n - 2$ degrees of freedom.