Assignment 4

Partial solutions for selected problems

1. λ and λ^2, respectively. For example, letting $j = y - 2$,
$$\mathsf{E}(Y(Y-1)) = \sum_{y=0}^{\infty} y(y-1)\frac{\lambda^y \mathrm{e}^{-\lambda}}{y!} = \sum_{j=0}^{\infty} \frac{\lambda^{j+2}\mathrm{e}^{-\lambda}}{j!} = \cdots = \lambda^2.$$

2. There are multiple ways to do this; they primarily require you to split the sums into parts according to the two terms of the pmf. In particular (see the third method described on Slide 95 of the notes), you can use
$$\sum_{n=0}^{\infty} nq^{n-1} = \frac{\mathrm{d}}{\mathrm{d}q}\frac{1}{(1-q)} = \frac{1}{(1-q)^2}$$
to get
$$\mathsf{E}(X) = \cdots = \frac{1/3}{(2/3)^2} + \frac{2/3}{4(1/3)^2} = \frac{9}{4}.$$
[Take care with minus signs: you know that both sums must be positive.] (Try plotting the pmf to see where a "center of balance" ought to be.)

3. Exer. 2.4.
   (b) By continuity, $\mathsf{P}(X < t) = \mathsf{P}(X \le t)$ is the cdf for X: $\frac{1}{2}e^{\lambda t}$ for $t \le 0$ and $1 - \frac{1}{2}\mathrm{e}^{-\lambda t}$ for $t > 0$. [Check: $F_X(0) = .5$ by symmetry and $F_X(x)$ should increase from 0 (at $-\infty$) to 1 (at ∞).]
   (c) This turns out to be the exponential($\frac{1}{\lambda}$) cdf.
   (d) $\mathsf{E}(X) = 0$ and $\mathsf{E}(X^2) = \frac{2}{\lambda^2}$.

4. Use a change of variables $y = 2x$ and Theorem 2.16 in the notes. For example,
$$\mathsf{E}(X^2) = \int_0^{\infty} x^2\, 4x\mathrm{e}^{-2x}\, dx = \frac{1}{4}\int_0^{\infty} y^3 \mathrm{e}^{-y}\, dy = \frac{3!}{4}.$$

5. (b) $\mathsf{E}(T) = \frac{4}{7}$.
   (c) $T = \frac{R}{1+R}$ and R can be any positive value. $\frac{\mathrm{d}t}{\mathrm{d}r} = \frac{1}{(1+r)^2}$ so
$$f_R(r) = \frac{1}{(1+r)^2} \times 60\Big(\frac{r}{1+r}\Big)^3\Big(\frac{1}{1+r}\Big)^2 = \frac{60r^3}{(1+r)^7} \quad \text{for } r > 0.$$
   (d) $\mathsf{E}(R) = 2$.
   (e) $h(t) = \frac{t}{1-t}$ is <u>convex</u> on the interval $(0, 1)$ so $\mathsf{E}(R)$ must be <u>larger</u> than $\mathsf{E}(T)/(1 - \mathsf{E}(T))$.

7. (c) w^α is <u>concave</u> ($-w^\alpha$ is convex) for $0 < \alpha < 1$, so $\mathsf{E}(W^\alpha) < \mu_W^\alpha$.

8. The mean and variance are both equal to λ.

9. (a) Let $y = (1/\beta - t)x$ to compute
$$\begin{aligned}\mathsf{E}(\mathrm{e}^{tX}) &= \frac{1}{\beta^m(m-1)!}\int_0^{\infty} x^{m-1}\mathrm{e}^{-x(1/\beta - t)}\,\mathrm{d}x = \int_0^{\infty} \mathrm{e}^{tx}\,\frac{x^{m-1}\mathrm{e}^{-x/\beta}}{\beta^m(m-1)!}\,\mathrm{d}x \\ &= \frac{1}{(1-\beta t)^m(m-1)!}\int_0^{\infty} y^{m-1}\mathrm{e}^{-y}\,\mathrm{d}y = \frac{1}{(1-\beta t)^m}.\end{aligned}$$
   (b) $\mathsf{E}(X_m) = m\beta$ and $\mathsf{Var}(X_m) = m\beta^2$. [Take note of the "units", keeping in mind that β, as a scale parameter, has the same units as X_m.]