Homework 3

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Stat 610 Distribution Theory

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Problem 1. Let Y have the Poisson(λ) distribution with pmf

$$f_Y(y) = \frac{\lambda^y e^{-\lambda}}{y!}, \quad y = 0, 1, 2, \ldots$$

Find $\mathsf{E}(Y)$ and $\mathsf{E}(Y(Y-1))$, and use these to get $\mathsf{Var}(Y)$. (Recall the binomial example in class.)

$$\begin{aligned}
\mathsf{E}(Y) &= \sum_{y=0}^{\infty} y \frac{\lambda^y e^{-\lambda}}{y!} \\
&= \lambda \sum_{y=1}^{\infty} \frac{\lambda^{y-1} e^{-\lambda}}{(y-1)!} \\
&= \lambda \sum_{k=0}^{\infty} \frac{\lambda^k e^{-\lambda}}{k!} \quad (k = y-1) \\
&= \lambda \quad \text{(since sum of pmf is just 1)}
\end{aligned}$$

$$\begin{aligned}
\mathsf{E}(Y(Y-1)) &= \sum_{y=0}^{\infty} y(y-1) \frac{\lambda^y e^{-\lambda}}{y!} \\
&= \lambda^2 \sum_{y=2}^{\infty} \frac{\lambda^{y-2} e^{-\lambda}}{(y-2)!} \\
&= \lambda^2 \sum_{k=0}^{\infty} \frac{\lambda^k e^{-\lambda}}{k!} \quad (k = y-2) \\
&= \lambda^2 \quad \text{(since sum of pmf is just 1)}
\end{aligned}$$

$$\begin{aligned}
\mathsf{Var}(Y) &= \mathsf{E}(Y(Y-1)) - \mathsf{E}(Y)(\mathsf{E}(Y) - 1) \\
&= \lambda^2 - \lambda(\lambda - 1) \\
&= \lambda
\end{aligned}$$

Problem 2. Find $\mathsf{E}(X)$ for the random variable in Problem 8 of Assignment 2.

From the assignment, we know that the pmf of X is

$$f_X(x) = \frac{1}{4}\left(\frac{2}{3}\right)^x + \left(\frac{1}{3}\right)^x \qquad x = 1, 2, \ldots$$

$$\begin{aligned}
\mathsf{E}(X) &= \sum_{x=1}^{\infty} x \left[\frac{1}{4}\left(\frac{2}{3}\right)^x + \left(\frac{1}{3}\right)^x\right] \\
&= \frac{1}{4}\sum_{x=1}^{\infty} x \left(\frac{2}{3}\right)^x + \sum_{x=1}^{\infty} x \left(\frac{1}{3}\right)^x \\
&\quad \left(\text{arithmetico-geometric series: } \sum_{x=1}^{\infty} x r^x = \frac{r}{(1-r)^2} \text{ for } |r| < 1\right) \\
&= \frac{1}{4} \cdot \frac{\frac{2}{3}}{(1-\frac{2}{3})^2} + \frac{\frac{1}{3}}{(1-\frac{1}{3})^2} \\
&= \frac{1}{4} \cdot 6 + \frac{3}{4} \\
&= \frac{9}{4}
\end{aligned}$$

Problem 3. *Statistical Inference* by Casella and Berger, 2nd Edition, Chapter 2, Exercise 4.

4. Let λ be a fixed positive constant, and define the function $f(x)$ by $f(x) = \frac{1}{2}\lambda e^{-\lambda x}$ if $x \geq 0$ and $f(x) = \frac{1}{2}\lambda e^{\lambda x}$ if $x < 0$.

   (a) Verify that $f(x)$ is a pdf.

   (b) If X is a random variable with pdf given by $f(x)$, find $\mathsf{P}(X < t)$ for all t. Evaluate all integrals.

   (c) Find $\mathsf{P}(|X| < t)$ for all t. Evaluate all integrals.

   (d) Find $\mathsf{E}(X)$, $\mathsf{Var}(X)$, and the standard deviation of X.

(a) To verify that $f(x)$ is a pdf, we need to show that $f(x) \geq 0$ for all x and $\int_{-\infty}^{\infty} f(x)dx = 1$. Since $\lambda > 0$ and e^x is positive for all $x \in \mathbb{R}$, $f(x) \geq 0$ for all x.

$$\begin{aligned}
\int_{-\infty}^{\infty} f(x)dx &= \int_{-\infty}^{0} \frac{1}{2}\lambda e^{\lambda x} dx + \int_{0}^{\infty} \frac{1}{2}\lambda e^{-\lambda x} dx \\
&= \frac{1}{2}e^{\lambda x}\Big|_{-\infty}^{0} + -\frac{1}{2}e^{-\lambda x}\Big|_{0}^{\infty} \\
&= \frac{1}{2} + \frac{1}{2} = 1.
\end{aligned}$$

(b) If $t \leq 0$, then

$$\begin{aligned}
\mathsf{P}(X < t) &= \int_{-\infty}^{t} f(x)dx = \int_{-\infty}^{t} \frac{1}{2}\lambda e^{\lambda x}dx \\
&= \frac{1}{2}e^{\lambda x}\Big|_{-\infty}^{t} \\
&= \frac{1}{2}e^{\lambda t}.
\end{aligned}$$

If $t > 0$, then

$$\begin{aligned}
\mathsf{P}(X < t) &= \int_{-\infty}^{t} f(x)dx = \int_{-\infty}^{0} \frac{1}{2}\lambda e^{\lambda x}dx + \int_{0}^{t} \frac{1}{2}\lambda e^{-\lambda x}dx \\
&= \frac{1}{2}e^{\lambda x}\Big|_{-\infty}^{0} + -\frac{1}{2}e^{-\lambda x}\Big|_{0}^{t} \\
&= \frac{1}{2} - \frac{1}{2}e^{-\lambda t} + \frac{1}{2} = 1 - \frac{1}{2}e^{-\lambda t}.
\end{aligned}$$

(c) This only makes sense for $t > 0$.

$$\begin{aligned}
\mathsf{P}(|X| < t) &= \mathsf{P}(-t < X < t) \\
&= \int_{-t}^{t} f(x)dx \\
&= \int_{-t}^{0} \frac{1}{2}\lambda e^{\lambda x}dx + \int_{0}^{t} \frac{1}{2}\lambda e^{-\lambda x}dx \\
&= \frac{1}{2}e^{\lambda x}\Big|_{-t}^{0} + -\frac{1}{2}e^{-\lambda x}\Big|_{0}^{t} \\
&= \frac{1}{2} - \frac{1}{2}e^{\lambda t} + \frac{1}{2} = 1 - \frac{1}{2}e^{\lambda t}.
\end{aligned}$$

(d)

$$\begin{aligned}
\mathsf{E}(X) &= \int_{-\infty}^{\infty} xf(x)dx \\
&= \int_{-\infty}^{0} x\frac{1}{2}\lambda e^{\lambda x}dx + \int_{0}^{\infty} x\frac{1}{2}\lambda e^{-\lambda x}dx \\
&\text{(using integration by parts with } u = x, dv = f(x)dx) \\
&= \frac{1}{2}xe^{\lambda x}\Big|_{-\infty}^{0} - \int_{-\infty}^{0} \frac{1}{2}e^{\lambda x}dx + -\frac{1}{2}xe^{-\lambda x}\Big|_{0}^{\infty} + \int_{0}^{\infty} \frac{1}{2}e^{-\lambda x}dx \\
&= 0 - \frac{1}{2}e^{\lambda x}\Big|_{-\infty}^{0} + 0 + -\frac{1}{2}e^{-\lambda x}\Big|_{0}^{\infty} \\
&= -\frac{1}{2} + \frac{1}{2} = 0.
\end{aligned}$$

$$
\begin{aligned}
\mathsf{E}(X^2) &= \int_{-\infty}^{\infty} x^2 f(x)dx \\
&= \int_{-\infty}^{0} x^2 \frac{1}{2}\lambda e^{\lambda x}dx + \int_{0}^{\infty} x^2 \frac{1}{2}\lambda e^{-\lambda x}dx \\
&\text{(by symmetry)} \\
&= 2\int_{0}^{\infty} x^2 \frac{1}{2}\lambda e^{-\lambda x}dx \\
&= \lambda \int_{0}^{\infty} x^2 e^{-\lambda x}dx \\
&\left(\text{using } \int_{0}^{\infty} y^m e^{-y}dy = m! \text{ and } y = \lambda x\right) \\
&= \lambda \int_{0}^{\infty} \left(\frac{y}{\lambda}\right)^2 e^{-y}\frac{dy}{\lambda} \\
&= \frac{1}{\lambda^2}\int_{0}^{\infty} y^2 e^{-y}dy \\
&= \frac{2!}{\lambda^2} = \frac{2}{\lambda^2}.
\end{aligned}
$$

$$
\begin{aligned}
\mathsf{Var}(X) &= \mathsf{E}(X^2) - [\mathsf{E}(X)]^2 \\
&= \frac{2}{\lambda^2} - 0^2 = \frac{2}{\lambda^2}.
\end{aligned}
$$

$$
\sqrt{\mathsf{Var}(X)} = \sqrt{\frac{2}{\lambda^2}} = \frac{\sqrt{2}}{\lambda}.
$$

Problem 4. Let $f(x) = 4xe^{-2x}$ for $x > 0$ and $f(x) = 0$ otherwise. (Recall Problem 8 of Assignment 3.) Now find $\mathsf{E}(X)$ and $\mathsf{Var}(X)$. Hint: Theorem 2.16 in the notes.

Theorem 2.16. $\int_0^\infty y^m e^{-y}dy = m!$ for any nonnegative integer m.

$$
\begin{aligned}
\mathsf{E}(X) &= \int_{0}^{\infty} x f(x)dx \\
&= \int_{0}^{\infty} x4xe^{-2x}dx \\
&= 4\int_{0}^{\infty} x^2 e^{-2x}dx \\
&\left(\text{using } \int_{0}^{\infty} y^m e^{-y}dy = m! \text{ and } y = 2x\right) \\
&= 4\int_{0}^{\infty} \left(\frac{y}{2}\right)^2 e^{-y}\frac{dy}{2} \\
&= \frac{4}{8}\int_{0}^{\infty} y^2 e^{-y}dy \\
&= \frac{1}{2}\cdot 2! = 1.
\end{aligned}
$$

$$
\begin{aligned}
\mathsf{E}(X^2) &= \int_0^\infty x^2 f(x)dx \\
&= \int_0^\infty x^2 4xe^{-2x}dx \\
&= 4\int_0^\infty x^3 e^{-2x}dx \\
&\text{(using } \int_0^\infty y^m e^{-y}dy = m! \text{ and } y = 2x) \\
&= 4\int_0^\infty \left(\frac{y}{2}\right)^3 e^{-y}\frac{dy}{2} \\
&= \frac{4}{16}\int_0^\infty y^3 e^{-y}dy \\
&= \frac{1}{4}\cdot 3! = \frac{3}{2}.
\end{aligned}
$$

$$
\begin{aligned}
\mathsf{Var}(X) &= \mathsf{E}(X^2) - [\mathsf{E}(X)]^2 \\
&= \frac{3}{2} - 1^2 = \frac{1}{2}.
\end{aligned}
$$

Problem 5. Let T have pdf $f_T(v) = 60v^3(1-v)^2$, $0 < v < 1$.

(a) Verify that $f_T(v)$ is indeed a pdf.

(b) Find the mean of T.

(c) Find the pdf for $R = T/(1-T)$.

(d) Use the pdf for T to find $\mathsf{E}(R)$ (recalling what Theorem 2.17 in the notes says).

(e) What does Jensen's inequality say about the relationship between $\mathsf{E}(R)$ and $\mathsf{E}(T)/(1-\mathsf{E}(T))$? Confirm by evaluating both.

Theorem 2.17. Let $Y = h(X)$. The value of $\mathsf{E}(Y)$ (computed using F_Y) is the same as the value of $\mathsf{E}(h(X))$ (computed using F_X).

(a) We need to show that $f_T(v) \geq 0$ for all v and $\int_{-\infty}^{\infty} f_T(v)dv = 1$.
Since $0 < v < 1$, $v^3 > 0$ and $(1-v)^2 > 0$, so $f_T(v) \geq 0$ for all v.

$$
\begin{aligned}
\int_{-\infty}^{\infty} f_T(v)dv &= \int_0^1 60v^3(1-v)^2 dv \\
&= 60\int_0^1 v^3(1-2v+v^2)dv \\
&= 60\int_0^1 (v^3 - 2v^4 + v^5)dv \\
&= 60\left[\frac{v^4}{4} - \frac{2v^5}{5} + \frac{v^6}{6}\right]_0^1 \\
&= 60\left(\frac{1}{4} - \frac{2}{5} + \frac{1}{6}\right) = 60\cdot\frac{15-24+10}{60} = 60\cdot\frac{1}{60} = 1.
\end{aligned}
$$

(b)

$$\begin{aligned}
\mathsf{E}(T) &= 60\int_0^1 v^4(1-2v+v^2)dv \\
&= 60\int_0^1 (v^4-2v^5+v^6)dv \\
&= 60\left[\frac{v^5}{5}-\frac{2v^6}{6}+\frac{v^7}{7}\right]_0^1 \\
&= 60\left(\frac{1}{5}-\frac{1}{3}+\frac{1}{7}\right) = 60\cdot\frac{1}{105} = \frac{4}{7}.
\end{aligned}$$

(c) Since $R = \frac{T}{1-T}$, we have $T = \frac{R}{1+R}$. Using the mnemonic

$$f_Y(y) = f_X(x)\left|\frac{dx}{dy}\right|,$$

we have

$$\begin{aligned}
f_R(r) &= f_T\left(\frac{r}{1+r}\right)\left|\frac{d}{dr}\left(\frac{r}{1+r}\right)\right| \\
&= f_T\left(\frac{r}{1+r}\right)\cdot\frac{1}{(1+r)^2} \\
&= 60\left(\frac{r}{1+r}\right)^3\left(1-\frac{r}{1+r}\right)^2\cdot\frac{1}{(1+r)^2} \\
&= 60\cdot\frac{r^3}{(1+r)^3}\cdot\frac{1}{(1+r)^2}\cdot\frac{1}{(1+r)^2} = 60\cdot\frac{r^3}{(1+r)^7}, r>0.
\end{aligned}$$

(d)

$$\begin{aligned}
\mathsf{E}(R) &= \mathsf{E}(h(T)) \quad (h(T) = \frac{T}{1-T}) \\
&= \int_0^1 \frac{t}{1-t}60t^3(1-t)^2dt \\
&= 60\int_0^1 t^4(1-t)dt \\
&= 60\int_0^1 (t^4-t^5)dt \\
&= 60\left[\frac{t^5}{5}-\frac{t^6}{6}\right]_0^1 \\
&= 60\left(\frac{1}{5}-\frac{1}{6}\right) = 60\cdot\frac{1}{30} = 2.
\end{aligned}$$

(e) Since $R = \frac{T}{1-T}$ is a convex function for $0 < T < 1$, by Jensen's inequality, we have

$$\mathsf{E}(R) \geq \frac{\mathsf{E}(T)}{1-\mathsf{E}(T)}.$$

We have already evaluated $\mathsf{E}(R) = 2$ and $\mathsf{E}(T) = \frac{4}{7}$, so

$$\frac{\mathsf{E}(T)}{1 - \mathsf{E}(T)} = \frac{\frac{4}{7}}{1 - \frac{4}{7}} = \frac{\frac{4}{7}}{\frac{3}{7}} = \frac{4}{3} < 2 = \mathsf{E}(R).$$

Problem 6. Prove Theorem 2.21 in the notes.

Theorem 2.21. Suppose X is a rv with $\mathsf{E}(X^2) < \infty$. The value c that minimizes $\mathsf{E}((X-c)^2)$ is $c = \mu_X$.

$$\begin{aligned}\mathsf{E}((X-c)^2) &= \mathsf{E}(X^2 - 2cX + c^2) \\ &= \mathsf{E}(X^2) - 2c\mathsf{E}(X) + c^2 \\ &= \mathsf{E}(X^2) - 2c\mu_X + c^2 \\ &= c^2 - 2c\mu_X + \mathsf{E}(X^2).\end{aligned}$$

Taking the derivative with respect to c, we have

$$\frac{d}{dc}\mathsf{E}((X-c)^2) = 2c - 2\mu_X.$$

Setting the derivative to 0, we have

$$2c - 2\mu_X = 0 \implies c = \mu_X.$$

To confirm that this is a minimum, we take the second derivative with respect to c,

$$\frac{d^2}{dc^2}\mathsf{E}((X-c)^2) = 2 > 0.$$

Thus, $c = \mu_X$ is the value that minimizes $\mathsf{E}((X-c)^2)$.

Problem 7. Let W be a positive random variable with finite mean μ_W.

(a) Suppose $\alpha > 1$. Use Jensen's inequality to show that $\mathsf{E}(W^\alpha) > \mu_W^\alpha$. (Note: this is valid even if the left-hand side is infinite.)

(b) Suppose $\alpha < 0$ and show that $\mathsf{E}(W^\alpha) > \mu_W^\alpha$.

(c) Now suppose $0 < \alpha < 1$. What is true in this case? Hint: if $g(x)$ is concave then $-g(x)$ is convex.

(a) Since $\alpha > 1$, $g(x) = x^\alpha$ is a convex function for $x > 0$. By Jensen's inequality, we have

$$\mathsf{E}(W^\alpha) = \mathsf{E}(g(W)) \geq g(\mathsf{E}(W)) = \mu_W^\alpha.$$

(b) With $\alpha < 0$, $g(x) = x^\alpha$ is also convex for $x > 0$ (since $g''(x) = \alpha(\alpha-1)x^{\alpha-2} > 0$). By Jensen's inequality, we have

$$\mathsf{E}(W^\alpha) = \mathsf{E}(g(W)) \geq g(\mathsf{E}(W)) = \mu_W^\alpha.$$

(c) Since $0 < \alpha < 1$, $g(x) = x^\alpha$ is a concave function for $x > 0$. Thus, $-g(x) = -x^\alpha$ is a convex function for $x > 0$. By Jensen's inequality, we have

$$\mathsf{E}(-W^\alpha) = \mathsf{E}(-g(W)) \geq -g(\mathsf{E}(W)) = -\mu_W^\alpha,$$

which implies that

$$\mathsf{E}(W^\alpha) \leq \mu_W^\alpha.$$

Problem 8. Let $f(y) = \frac{\lambda^y e^{-\lambda}}{y!}$ for $y = 0, 1, 2, \ldots$ be the Poisson(λ) pmf, where $\lambda > 0$. (Recall Problem 1 above.) Now show that the mgf is $M(t) = e^{\lambda(e^t - 1)}$, and use the mgf to get the mean and variance.

$$\begin{aligned}
M(t) = \mathsf{E}(e^{tY}) &= \sum_{y=0}^{\infty} e^{ty} \frac{\lambda^y e^{-\lambda}}{y!} \\
&= e^{-\lambda} \sum_{y=0}^{\infty} \frac{(\lambda e^t)^y}{y!} \\
&= e^{-\lambda} e^{\lambda e^t} \quad \text{(by Taylor series of } e^x\text{)} \\
&= e^{\lambda(e^t - 1)}.
\end{aligned}$$

$$\begin{aligned}
M'(t) &= \lambda e^t e^{\lambda(e^t - 1)} \\
M''(t) &= \lambda e^t e^{\lambda(e^t - 1)} + \lambda^2 e^{2t} e^{\lambda(e^t - 1)}.
\end{aligned}$$

$$\begin{aligned}
\mathsf{E}(Y) &= M'(0) = \lambda e^0 e^{\lambda(e^0 - 1)} = \lambda. \\
\mathsf{E}(Y^2) &= M''(0) = \lambda e^0 e^{\lambda(e^0 - 1)} + \lambda^2 e^0 e^{\lambda(e^0 - 1)} = \lambda + \lambda^2. \\
\mathsf{Var}(Y) &= \mathsf{E}(Y^2) - [\mathsf{E}(Y)]^2 = (\lambda + \lambda^2) - \lambda^2 = \lambda.
\end{aligned}$$

Problem 9. Let X_m have pdf

$$f_{X_m}(x) = \frac{x^{m-1} e^{-x/\beta}}{\beta^m (m-1)!}$$

for $m = 1, 2, \ldots$ and $x > 0, \beta > 0$.

(a) Show that the mgf for X_m is $M_{X_m}(t) = (1 - \beta t)^{-m}$ for $t < 1/\beta$. Hint: use a linear change of variables.

(b) Use the mgf to derive the mean and variance.

(c) Show that for any m, n, $M_{X_m}(t) M_{X_n}(t) = M_{X_{n+m}}(t)$. As we shall see, this has a probability interpretation in terms of sums of independent random variables.

(a)

$$\begin{aligned}
M_{X_m}(t) = \mathsf{E}(e^{tX_m}) &= \int_0^\infty e^{tx}\frac{x^{m-1}e^{-x/\beta}}{\beta^m(m-1)!}dx \\
&= \frac{1}{\beta^m(m-1)!}\int_0^\infty x^{m-1}e^{-(\frac{1}{\beta}-t)x}dx \\
&\text{(let } y=(\frac{1}{\beta}-t)x, dy=(\frac{1}{\beta}-t)dx) \\
&= \frac{1}{\beta^m(m-1)!}\int_0^\infty \left(\frac{y}{\frac{1}{\beta}-t}\right)^{m-1} e^{-y}\frac{dy}{\frac{1}{\beta}-t} \\
&= \frac{1}{\beta^m(m-1)!(\frac{1}{\beta}-t)^m}\int_0^\infty y^{m-1}e^{-y}dy \\
&= \frac{(m-1)!}{\beta^m(m-1)!(\frac{1}{\beta}-t)^m} \quad \text{(by Theorem 2.16)} \\
&= (1-\beta t)^{-m}.
\end{aligned}$$

(b)

$$\begin{aligned}
M'_{X_m}(t) &= m\beta(1-\beta t)^{-m-1} \\
M''_{X_m}(t) &= m\beta^2(m+1)(1-\beta t)^{-m-2}.
\end{aligned}$$

$$\begin{aligned}
\mathsf{E}(X_m) &= M'_{X_m}(0) = m\beta(1-0)^{-m-1} = m\beta. \\
\mathsf{E}(X_m^2) &= M''_{X_m}(0) = m\beta^2(m+1)(1-0)^{-m-2} = m(m+1)\beta^2. \\
\mathsf{Var}(X_m) &= \mathsf{E}(X_m^2) - [\mathsf{E}(X_m)]^2 = m(m+1)\beta^2 - (m\beta)^2 = m\beta^2.
\end{aligned}$$

(c)

$$\begin{aligned}
M_{X_m}(t)M_{X_n}(t) &= (1-\beta t)^{-m}(1-\beta t)^{-n} \\
&= (1-\beta t)^{-(m+n)} = M_{X_{m+n}}(t).
\end{aligned}$$