STAT610 Assignment 8
due Friday, 31 October 2025

1. Let X and Y be independent random variables with the *same* cdf $F(x)$. Define $U = \min(X, Y)$ and $V = \max(X, Y)$.
   (a) Show that $\mathsf{P}(u < U \leq V \leq v) = (F(v) - F(u))^2$, if $u < v$. Hint: think about the event in terms of the values of X and Y.
   (b) Explain why $\mathsf{P}(U \leq u, V \leq v) = \mathsf{P}(V \leq v) - \mathsf{P}(u < U \leq V \leq v)$, and then deduce the joint cdf for (U, V).
   (c) What are the marginal cdfs?
   (d) Now assume $F(x)$ has pdf $f(x)$. What are the joint and marginal pdfs for (U, V)?

2. Suppose $R \sim$ exponential(2) (same as chi-square(2)) and $\Theta \sim$ uniform$(-\pi, \pi)$, independent.
   (a) Find the joint pdf for $(X_1, X_2) = (R\sin(\Theta), R\cos(\Theta))$. (This is a 1-1 transformation.)
   (b) Are X_1 and X_2 dependent or independent? What are their marginal distributions?

3. Recall Problem 6 of Assignment 7, where (R, S) has joint pdf $f_{R,S}(r, s) = \frac{8}{3}s^2\mathrm{e}^{-2s}$ for $0 \leq r \leq s$. You may use the solutions from that problem.
   (a) Find the mean and variance of S.
   (b) Find $\mathsf{E}(R|S = s)$ and $\mathsf{E}(R^2|S = s)$. Then use iterated expectation and the variance partition formula to compute the mean and variance of R.
   (c) Use iterated expectation to find $\mathsf{E}(RS) = \mathsf{E}(\mathsf{E}(RS|S))$.
   (d) Now use the above to determine $\mathsf{Cov}(R, S)$.

4. Suppose $W \sim$ Poisson(λ) and the conditional distribution of X, given $W = w$, is exponential(w^2) if $w > 0$ and $X = 0$ if $w = 0$. So, in particular, $\mathsf{E}(X|W) = W^2$.

   Find the mean and variance of X. You may use (without proof) the following fact: $\mathsf{E}(W(W-1) \times \cdots \times (W-k)) = \lambda^{k+1}$, $k = 1, 2 \ldots$.

5. Recall Exercise 4.4 (Problem 5 of Assignment 7). Now do the following.
   (a) Determine the conditional pdf for Y, given $X = x$, and the conditional pdf for X, given $Y = y$.
   (b) Find $g(x) = \mathsf{E}(Y|X = x)$ and $h(y) = \mathsf{E}(X|Y = y)$. Are either of these linear? Are they inverses of each other?
   (c) Plot $y = g(x)$ versus x and y versus $x = h(y)$ on the same graph. Comment on what they indicate about predicting one random variable from the other.
   (d) Compute $\mathsf{E}(X)$, $\mathsf{E}(Y)$ and $\mathsf{E}(XY)$ by whatever method(s) you prefer.
   (e) Compute $\mathsf{Cov}(X, Y)$.

6. Assume the random pair (S, T) has joint distribution

$$f_{S,T}(s, t) = (s + t)\mathrm{e}^{-(s+t)}, \quad s > 0, t > 0.$$

   (a) Find $\mathsf{E}(S)$, $\mathsf{E}(S^2)$, $\mathsf{E}(T)$, $\mathsf{E}(T^2)$ and $\mathsf{E}(ST)$. Hint: you can express each of these in terms of some gamma function values.
   (b) Determine $\mathsf{Corr}(S, T)$.

(continued next page)

7. Suppose X and Y are *integer-valued* rvs with joint pmf

$$f_{X,Y}(x,y) = \frac{1}{13}, \quad \text{if } |x+y| \leq 2 \text{ and } |x-y| \leq 2.$$

(It may help to graph the possible points (x, y).)

(a) Are X and Y independent? What are their marginal pmfs?

(b) What is $\mathsf{Cov}(X, Y)$?

8. (a) Use Theorem 4.22 in the notes to prove that if $X_1, \ldots, X_k$ are independent random variables with respective mgfs $M_1(t), \ldots, M_k(t)$ then the mgf for $S = X_1 + \cdots + X_k$ is

$$M_S(t) = \prod_{i=1}^{k} M_i(t).$$

(b) Use the result in part (a) to confirm the following.

i. If $S \sim$ binomial(m, p) and $T \sim$ binomial(n, p), independent, then $S + T \sim$ binomial$(m+n, p)$. (This was done by use of a convolution in Assignment 6 Problem 4.)

ii. If T and U are independent with $T \sim$ gamma(α, γ) and $U \sim$ gamma(β, γ) then $T + U \sim$ gamma$(\alpha + \beta, \gamma)$. (You also showed this in Assignment 7 Problem 8 using a different method.)

iii. If $X \sim$ Poisson(λ) and $Y \sim$ Poisson(μ), independent, then $X + Y \sim$ Poisson$(\lambda + \mu)$. (This was also done by convolution in Theorem 4.10 of the notes.)