Homework 7

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Stat 610 Distribution Theory

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Problem 1. A random sample of size n is obtained without replacement from a finite population of size N that has 3 separate categories A, B, C with sizes M_A, M_B, M_C, respectively, such that $M_A + M_B + M_C = N$.

(a) Let X_A, X_B, X_C be the numbers of individuals sampled from the three categories. Find the joint pmf for (X_A, X_B, X_C). Keep in mind that $X_A + X_B + X_C = n$.

(b) Now let $Y = X_A + X_B$. Find the joint pmf for (X_A, Y).

(c) What is the marginal pmf for Y? Hint: think about the sampling and what happens if you ignore the distinction between categories A and B.

(d) Find the conditional pmf for X_A, given $Y = y$. What distribution is this (name and parameters)?

(a) The joint pmf for (X_A, X_B, X_C) is

$$\mathsf{P}(X_A = x_a, X_B = x_b, X_C = x_c) = \frac{\binom{M_A}{x_a}\binom{M_B}{x_b}\binom{M_C}{x_c}}{\binom{N}{n}}$$

where $x_a + x_b + x_c = n$.

(b) The joint pmf for (X_A, Y) is

$$\mathsf{P}(X_A = x_a, Y = y) = \frac{\binom{M_A}{x_a}\binom{M_B}{y-x_a}\binom{M_C}{n-y}}{\binom{N}{n}}$$

where $x_a \leq y$ and $y \leq n$.

(c) The marginal pmf for Y is

$$\mathsf{P}(Y = y) = \frac{\binom{M_A+M_B}{y}\binom{M_C}{n-y}}{\binom{N}{n}}$$

where $y \leq n$.

(d) The conditional pmf for X_A, given $Y = y$ is

$$\mathsf{P}(X_A = x_a | Y = y) = \frac{\binom{M_A}{x_a}\binom{M_B}{y-x_a}}{\binom{M_A+M_B}{y}}$$

where $x_a \leq y$. This is a hypergeometric distribution with parameters $M_A + M_B, M_A, y$.

Problem 2. Let (X, Y) have joint cdf $F(x, y) = \mathsf{P}(X \leq x, Y \leq y)$.

(a) Show that, for $a < b$ and $c < d$,

$$\mathsf{P}(a < X \leq b, c < Y \leq d) = F(b, d) - F(b, c) - F(a, d) + F(a, c).$$

(b) Simplify the expression in (a) for the independence case: $F(x, y) = F_X(x)F_Y(y)$.

(a) We have 4 events to consider:

$A = \{X \leq b, Y \leq d\}, B = \{X \leq a, Y \leq d\}, C = \{X \leq b, Y \leq c\}, D = \{X \leq a, Y \leq c\}.$

Note that

$$\{a < X \leq b, c < Y \leq d\} = A \setminus (B \cup C) = A \cap B^c \cap C^c.$$

Also note that $B^c \cap C^c = (B \cup C)^c$. By inclusion-exclusion, we have

$$\mathsf{P}(B \cup C) = \mathsf{P}(B) + \mathsf{P}(C) - \mathsf{P}(B \cap C) = \mathsf{P}(B) + \mathsf{P}(C) - \mathsf{P}(D).$$

Therefore,

$$\begin{aligned}
&\mathsf{P}(a < X \leq b, c < Y \leq d) \\
=&\mathsf{P}(X \leq b, Y \leq d) - \mathsf{P}(X \leq a, Y \leq d) - \mathsf{P}(X \leq b, Y \leq c) + \mathsf{P}(X \leq a, Y \leq c) \\
=&F(b, d) - F(a, d) - F(b, c) + F(a, c).
\end{aligned}$$

(b) For the independence case, we have

$$\begin{aligned}
&F(b, d) - F(a, d) - F(b, c) + F(a, c) \\
=&F_X(b)F_Y(d) - F_X(a)F_Y(d) - F_X(b)F_Y(c) + F_X(a)F_Y(c) \\
=&(F_X(b) - F_X(a))(F_Y(d) - F_Y(c)).
\end{aligned}$$

Problem 3. Suppose T has negative binomial$(2, p)$ distribution and the conditional pmf for S, given $T = t$, is $f_{S|T}(s|t) = \frac{1}{t+1}$, $s \in \{0, ..., t\}$.

(a) Find the joint pmf for (S, T). Be sure to indicate the range with any restrictions.

(b) Find the marginal pmf for S and the conditional pmf for T, given $S = s$.

(c) Show that S and $R = T - S$ are independent and have the same distribution.

(a) The joint pmf for (S, T) is

$$f_{S,T}(s,t) = f_{S|T}(s|t) f_T(t) = \frac{1}{t+1}\binom{t+1}{1} p^2(1-p)^{t-2} = p^2(1-p)^t$$

where $t \in \{0, 1, \ldots\}$ and $s \in \{0, 1, \ldots, t\}$.

(b) The marginal pmf for S is

$$\begin{aligned} f_S(s) &= \sum_{t=0}^{\infty} f_{S,T}(s,t) \\ &= \sum_{t=s}^{\infty} p^2(1-p)^t \\ &= p^2 \frac{(1-p)^s}{p} = p(1-p)^s \end{aligned}$$

The conditional pmf for T, given $S = s$ is

$$f_{T|S}(t|s) = \frac{f_{S,T}(s,t)}{f_S(s)} = \frac{p^2(1-p)^t}{p(1-p)^s} = p(1-p)^{t-s}.$$

(c) The joint pmf for (S, R) is

$$\begin{aligned} f_{S,R}(s,r) &= f_{S,T}(s, s+r) \\ &= p^2(1-p)^{s+r} \\ &= p(1-p)^s \cdot p(1-p)^r \\ &= f_S(s) \cdot f_R(r). \end{aligned}$$

Therefore, S and R are independent. Also, we have

$$f_R(r) = p(1-p)^r,$$

which is the same as the distribution of S.

Problem 4. Suppose $S \sim \text{binomial}(m, p)$ and $T \sim \text{binomial}(n, p)$ (same second parameter p), with S and T independent. Use the convolution formula to prove $S + T \sim \text{binomial}(m + n, p)$. Hint: factor out the result and observe that a hypergeometric pmf remains.

Let $X = S + T$. By the convolution formula, we have

$$\begin{aligned} f_X(x) &= \sum_s f_S(s) f_T(x - s) \\ &= \sum_s \binom{m}{s} p^s (1-p)^{m-s} \binom{n}{x-s} p^{x-s} (1-p)^{n-(x-s)} \\ &= p^x (1-p)^{m+n-x} \sum_s \binom{m}{s} \binom{n}{x-s} \\ &= p^x (1-p)^{m+n-x} \binom{m+n}{x} \end{aligned}$$

where the last equality follows from Vandermonde's identity:

$$\sum_s \binom{m}{s} \binom{n}{x-s} = \binom{m+n}{x}.$$

Therefore, $X \sim \text{binomial}(m + n, p)$.

Problem 5. *Statistical Inference* by Casella and Berger, 2nd Edition, Chapter 4, Exercise 4.

4. A pdf is defined by

$$f(x, y) = \begin{cases} C(x + 2y) & \text{if } 0 < y < 1 \text{ and } 0 < x < 2 \\ 0 & \text{otherwise.} \end{cases}$$

   (a) Find the value of C.

   (b) Find the marginal distribution of X.

   (c) Find the joint cdf of X and Y, be sure to give the joint cdf for all (x, y) in the real plane, and then check yourself by deriving the joint pdf for all (x, y).

   (d) Find $\mathsf{P}(Y^2 < X < \sqrt{Y})$.

   (e) Find $\mathsf{P}(X + 2Y \le t)$ for $t \in [0, 4]$ and deduce the pdf for $T = X + 2Y$. You will need to consider a couple cases of the double integral separately. Be sure to check that your pdf is valid.

(a) To find C, we have

$$\begin{aligned}
1 &= \int_0^1 \int_0^2 C(x+2y)\,dx\,dy \\
&= C\int_0^1 \left[\frac{x^2}{2} + 2yx\right]_0^2 dy \\
&= C\int_0^1 (2+4y)dy \\
&= C\left[2y + 2y^2\right]_0^1 = 4C.
\end{aligned}$$

Therefore, $C = \frac{1}{4}$.

(b) The marginal distribution of X is

$$\begin{aligned}
f_X(x) &= \int_0^1 f(x,y)\,dy \\
&= \int_0^1 \frac{1}{4}(x+2y)\,dy \\
&= \frac{1}{4}\left[xy + y^2\right]_0^1 = \frac{x+1}{4}
\end{aligned}$$

where $0 < x < 2$.

(c) The joint cdf of X and Y is

$$F(x,y) = \begin{cases} 0 & \text{if } x \le 0 \text{ or } y \le 0 \\ \int_0^y \int_0^x \frac{1}{4}(u+2v)\,du\,dv & \text{if } 0 < x < 2 \text{ and } 0 < y < 1 \\ \int_0^1 \int_0^x \frac{1}{4}(u+2v)\,du\,dv & \text{if } 0 < x < 2 \text{ and } y \ge 1 \\ \int_0^y \int_0^2 \frac{1}{4}(u+2v)\,du\,dv & \text{if } x \ge 2 \text{ and } 0 < y < 1 \\ 1 & \text{if } x \ge 2 \text{ and } y \ge 1 \end{cases}$$

Calculating the integrals, we have

$$F(x,y) = \begin{cases} 0 & \text{if } x \le 0 \text{ or } y \le 0 \\ \frac{x^2 y}{8} + \frac{xy^2}{4} & \text{if } 0 < x < 2 \text{ and } 0 < y < 1 \\ \frac{x^2}{8} + \frac{x}{4} & \text{if } 0 < x < 2 \text{ and } y \ge 1 \\ \frac{y}{2} + \frac{y^2}{2} & \text{if } x \ge 2 \text{ and } 0 < y < 1 \\ 1 & \text{if } x \ge 2 \text{ and } y \ge 1 \end{cases}$$

Taking the partial derivatives with respect to both x and y, we have

$$\frac{\partial^2}{\partial x \partial y} F(x,y) = \begin{cases} \frac{1}{4}(x+2y) & \text{if } 0 < x < 2 \text{ and } 0 < y < 1 \\ 0 & \text{otherwise.} \end{cases}$$

This matches the original joint pdf.

(d) We have

$$
\begin{aligned}
\mathsf{P}(Y^2 < X < \sqrt{Y}) &= \int_0^1 \int_{y^2}^{\sqrt{y}} \frac{1}{4}(x+2y)\,dx\,dy \\
&= \int_0^1 \frac{1}{4}\left[\frac{x^2}{2} + 2yx\right]_{y^2}^{\sqrt{y}} dy \\
&= \int_0^1 \frac{1}{4}\left(\frac{y}{2} + 2y^{3/2} - \frac{y^4}{2} - 2y^3\right) dy \\
&= \frac{1}{4}\left[\frac{y^2}{4} + \frac{4y^{5/2}}{5} - \frac{y^5}{10} - \frac{y^4}{2}\right]_0^1 = \frac{7}{80}.
\end{aligned}
$$

(e) For $t \in [0,4]$, we have

$$
\begin{aligned}
\mathsf{P}(X+2Y \le t) &= \int_0^1 \int_0^{\min(2,t-2y)} \frac{1}{4}(x+2y)\,dx\,dy \\
&= \int_0^1 \frac{1}{4}\left[\frac{x^2}{2} + 2yx\right]_0^{\min(2,t-2y)} dy \\
&= \int_0^1 \frac{1}{4}\left(\frac{\min(2,t-2y)^2}{2} + 2y\min(2,t-2y)\right) dy.
\end{aligned}
$$

We need to consider two cases:

- Case 1: $0 \le t < 2$. In this case, we have

$$
\begin{aligned}
\mathsf{P}(X+2Y \le t) &= \int_0^{t/2} \int_0^{t-2y} \frac{1}{4}(x+2y)\,dx\,dy \\
&= \frac{t^3}{24}.
\end{aligned}
$$

- Case 2: $2 \le t \le 4$. In this case, we have

$$
\begin{aligned}
\mathsf{P}(X+2Y \le t) &= \int_0^{(t-2)/2} \frac{1}{4}(2+4y)dy + \int_{(t-2)/2}^1 \frac{1}{4}\left(\frac{(t-2y)^2}{2} + 2y(t-2y)\right) dy \\
&= -\frac{t^3}{24} + \frac{t^2}{4} - \frac{1}{3}.
\end{aligned}
$$

Therefore, the cdf for $T = X + 2Y$ is

$$
F_T(t) = \begin{cases} 0 & \text{if } t < 0 \\ \frac{t^3}{24} & \text{if } 0 \le t < 2 \\ -\frac{t^3}{24} + \frac{t^2}{4} - \frac{1}{3} & \text{if } 2 \le t \le 4 \\ 1 & \text{if } t > 4 \end{cases}
$$

Taking the derivative with respect to t, we have the pdf for T:

$$
f_T(t) = \begin{cases} \frac{t^2}{8} & \text{if } 0 \le t < 2 \\ -\frac{t^2}{8} + \frac{t}{2} & \text{if } 2 \le t \le 4 \\ 0 & \text{otherwise.} \end{cases}
$$

To check that the pdf is valid, we have

$$\begin{aligned}
\int_{-\infty}^{\infty} f_T(t)\,dt &= \int_0^2 \frac{t^2}{8}\,dt + \int_2^4 \left(-\frac{t^2}{8}+\frac{t}{2}\right)dt \\
&= \left[\frac{t^3}{24}\right]_0^2 + \left[-\frac{t^3}{24}+\frac{t^2}{4}\right]_2^4 \\
&= \frac{8}{24} + \left(-\frac{64}{24}+4-\left(-\frac{8}{24}+1\right)\right) \quad = \frac{8-64+96+8-24}{24} = \frac{24}{24} = 1.
\end{aligned}$$

Problem 6. Suppose (R, S) has joint pdf $f_{R,S}(r, s) = 8s^2e^{-2s}$ for $0 \le r \le s$.

(a) Find the marginal pdf for S and the conditional pdf for R, given $S = s$. Try to identify the distributions by name, giving appropriate values to the parameters.

(b) Find the marginal pdf for R and the conditional pdf for S, given $R = r$.

(a) The marginal pdf for S is

$$f_S(s) = \int_0^s 8s^2e^{-2s}\,dr = 8s^3e^{-2s}$$

where $s \ge 0$. This is a gamma distribution with parameters $\alpha = 3$ and $\beta = 2$. The conditional pdf for R, given $S = s$ is

$$f_{R|S}(r|s) = \frac{f_{R,S}(r,s)}{f_S(s)} = \frac{8s^2e^{-2s}}{8s^3e^{-2s}} = \frac{1}{s}$$

where $0 \le r \le s$. This is a uniform distribution on the interval $[0, s]$.

(b) The marginal pdf for R is

$$\begin{aligned}
f_R(r) &= \int_r^{\infty} 8s^2e^{-2s}\,ds \\
&= \left[-4s^2e^{-2s} - 4se^{-2s} - 2e^{-2s}\right]_r^{\infty} \\
&= 4r^2e^{-2r} + 4re^{-2r} + 2e^{-2r} \\
&= 2e^{-2r}(2r^2 + 2r + 1)
\end{aligned}$$

where $r \ge 0$. The conditional pdf for S, given $R = r$ is

$$f_{S|R}(s|r) = \frac{f_{R,S}(r,s)}{f_R(r)} = \frac{8s^2e^{-2s}}{2e^{-2r}(2r^2+2r+1)} = \frac{4s^2e^{-2(s-r)}}{2r^2+2r+1}$$

where $s \ge r$.

Problem 7. (a) Prove Corollary 4.19.ii in the notes, by applying Theorem 4.18.

(b) Suppose X, Y are independent exponential(1) random variables and $W = X + Y$, $Z = X - Y$. Find the joint pdf for (W, Z). Be aware of the range for the random pair: Z can be positive or negative but it also is restricted by W.

(c) Find the marginal distributions for W and Z. (Note: W is a special case of Example 4.10 in the notes.)

Corollary 4.19.ii. Suppose (X,Y) has joint pdf $f_{X,Y}(x,y)$. Let $W = aX + bY$ and $Z = cX + dY$. If $ad \neq bc$ then (W, Z) has joint pdf

$$f_{W,Z}(w,z) = \frac{1}{|ad-bc|} f_{X,Y}\left(\frac{dw-bz}{ad-bc}, \frac{az-cw}{ad-bc}\right).$$

Theorem 4.18. Suppose (X,Y) has pdf $f_{X,Y}(x,y)$, and let $U = g(X,Y)$ and $V = h(X,Y)$. Assume the transformation $(x,y) \to (g(x,y), h(x,y))$ is 1-1 and differentiable on a set A such that $\mathsf{P}((X,Y) \in A) = 1$. Then (U,V) has pdf satisfying

$$f_{U,V}(u,v) = f_{X,Y}(x,y) \frac{dx\,dy}{du\,dv}$$

expressed as a function of (u,v).

(a) By Theorem 4.18, we have

$$f_{W,Z}(w,z) = f_{X,Y}(x,y) \frac{dx\,dy}{dw\,dz}.$$

Note that

$$\begin{bmatrix} dw \\ dz \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} dx \\ dy \end{bmatrix}.$$

Therefore,

$$\frac{dw\,dz}{dx\,dy} = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc.$$

Since $ad \neq bc$, we have

$$\frac{dx\,dy}{dw\,dz} = \frac{1}{|ad-bc|}.$$

Also, solving for x and y, we have

$$\begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \begin{bmatrix} w \\ z \end{bmatrix} = \begin{bmatrix} \frac{dw-bz}{ad-bc} \\ \frac{az-cw}{ad-bc} \end{bmatrix}.$$

Combining the above results, we have

$$f_{W,Z}(w,z) = \frac{1}{|ad-bc|} f_{X,Y}\left(\frac{dw-bz}{ad-bc}, \frac{az-cw}{ad-bc}\right).$$

(b) The joint pdf for (W,Z) is

$$f_{W,Z}(w,z) = \frac{1}{|ad-bc|} f_{X,Y}\left(\frac{dw-bz}{ad-bc}, \frac{az-cw}{ad-bc}\right).$$

Here, $a = 1, b = 1, c = 1, d = -1$. Therefore,

$$f_{W,Z}(w,z) = \frac{1}{2} f_{X,Y}\left(\frac{w+z}{2}, \frac{w-z}{2}\right).$$

Since X and Y are independent exponential(1) random variables, we have

$$f_{X,Y}(x,y) = e^{-x}e^{-y} = e^{-(x+y)}.$$

Therefore,

$$f_{W,Z}(w,z) = \frac{1}{2} e^{-\left(\frac{w+z}{2}+\frac{w-z}{2}\right)} = \frac{1}{2}e^{-w}$$

where $w \geq 0$ and $-w \leq z \leq w$.

(c) The marginal distribution for W is

$$f_W(w) = \int_{-w}^{w} f_{W,Z}(w,z)\, dz = \int_{-w}^{w} \frac{1}{2}e^{-w}\, dz = \frac{1}{2}e^{-w}(2w) = we^{-w}.$$

This is a gamma distribution with parameters $\alpha = 2$ and $\beta = 1$. The marginal distribution for Z is

$$f_Z(z) = \int_0^{\infty} f_{W,Z}(w,z)\, dw = \int_{|z|}^{\infty} \frac{1}{2}e^{-w}\, dw = \frac{1}{2}e^{-|z|}.$$

This is a Laplace distribution with parameters $\mu = 0$ and $b = 1$.

Problem 8. (a) Let T and U be independent with $T \sim \text{gamma}(\alpha, \gamma)$ and $U \sim \text{gamma}(\beta, \gamma)$ (with the same scale parameter γ). Let $X = T + U$ and $Y = T/(T+U)$. Determine the joint pdf for (X, Y). Identify the marginal distributions by name and describe, in words, what the joint distribution of (X, Y) is. Hint: review distributions defined in Section 3.3 of the notes.

(b) Using the result for the distribution of X in part (a), prove by mathematical induction (iteration) that if $T_1, \ldots, T_k$ are independent random variables such that $T_i \sim \text{gamma}(\alpha_i, \gamma)$, then $T_1 + \cdots + T_k \sim \text{gamma}(\alpha_1 + \cdots + \alpha_k, \gamma)$.

(a) $T = XY$ and $U = X(1-Y)$. The Jacobian determinant is

$$J = \begin{vmatrix} Y & X \\ 1-Y & -X \end{vmatrix} = -X.$$

Therefore, by the change of variables formula, we have

$$\begin{aligned}
f_{X,Y}(x,y) &= f_{T,U}(xy, x(1-y))|J| \\
&= f_T(xy) f_U(x(1-y))x \\
&= \frac{1}{\gamma^\alpha \Gamma(\alpha)}(xy)^{\alpha-1}e^{-\frac{xy}{\gamma}} \cdot \frac{1}{\gamma^\beta \Gamma(\beta)}(x(1-y))^{\beta-1}e^{-\frac{x(1-y)}{\gamma}} \cdot x \\
&= \frac{x^{\alpha+\beta-1}y^{\alpha-1}(1-y)^{\beta-1}}{\gamma^{\alpha+\beta}\Gamma(\alpha)\Gamma(\beta)}e^{-\frac{x}{\gamma}}
\end{aligned}$$

where $x > 0$ and $0 < y < 1$. We can factor the joint pdf as

$$f_{X,Y}(x,y) = \left(\frac{x^{\alpha+\beta-1}}{\gamma^{\alpha+\beta}\Gamma(\alpha+\beta)}e^{-\frac{x}{\gamma}}\right)\left(\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}y^{\alpha-1}(1-y)^{\beta-1}\right).$$

Notice the left factor is the marginal distribution for X

$$f_X(x) = \frac{x^{\alpha+\beta-1}}{\gamma^{\alpha+\beta}\Gamma(\alpha+\beta)}e^{-\frac{x}{\gamma}},$$

which is a gamma distribution with parameters $\alpha+\beta$ and γ.
The right is the marginal distribution for Y

$$f_Y(y) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}y^{\alpha-1}(1-y)^{\beta-1},$$

which is a beta distribution with parameters α and β.

(b) Base case: for $k = 2$, the result holds by part (a).
Inductive hypothesis: assume the result holds for $k = n$, i.e.,

$$T_1 + \cdots + T_n \sim \text{gamma}(\alpha_1 + \cdots + \alpha_n, \gamma).$$

Inductive step: for $k = n+1$, we have

$$T_1 + \cdots + T_n + T_{n+1} = (T_1 + \cdots + T_n) + T_{n+1}.$$

By the inductive hypothesis, $(T_1 + \cdots + T_n) \sim \text{gamma}(\alpha_1 + \cdots + \alpha_n, \gamma)$ and $T_{n+1} \sim \text{gamma}(\alpha_{n+1}, \gamma)$. By part (a), we have

$$\text{gamma}(\alpha_1 + \cdots + \alpha_n, \gamma) + \text{gamma}(\alpha_{n+1}, \gamma) \sim \text{gamma}(\alpha_1 + \cdots + \alpha_n + \alpha_{n+1}, \gamma).$$