Homework 6

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Stat 610 Distribution Theory

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Problem 1. Let X have the *Laplace* distribution (recalling Problem 3 of Assignment 4), with pdf

$$f(x) = \frac{\lambda}{2} e^{-\lambda|x|}, \quad \text{all } x.$$

Now suppose $a > 0$ and $b \in (-\infty, \infty)$. Let $Y = aX + b$.

(a) Find the pdf for Y.

(b) Show that X has mgf $M_X(t) = \frac{1}{1-\frac{t^2}{\lambda^2}}$. Hint: integrate two halves separately and then combine.

(c) Use the mgf and a property of mgfs (Theorem 2.28 in the notes) to obtain the mgf for Y.

(a) Using the mnemonic

$$f_Y(y) = f_X(x) \left| \frac{dx}{dy} \right|,$$

where $x = \frac{y-b}{a}$, we have

$$f_Y(y) = \frac{\lambda}{2a} e^{-\frac{\lambda}{a}|y-b|}, \quad \text{all } y.$$

(b) The mgf of X is

$$\begin{aligned}
M_X(t) = \mathsf{E}[e^{tX}] &= \int_{-\infty}^{\infty} e^{tx} f(x) dx \\
&= \int_{-\infty}^{0} e^{tx} \frac{\lambda}{2} e^{\lambda x} dx + \int_{0}^{\infty} e^{tx} \frac{\lambda}{2} e^{-\lambda x} dx \\
&= \frac{\lambda}{2} \int_{-\infty}^{0} e^{(t+\lambda)x} dx + \frac{\lambda}{2} \int_{0}^{\infty} e^{(t-\lambda)x} dx \\
&= \frac{\lambda}{2} \left[\frac{1}{t+\lambda} e^{(t+\lambda)x} \right]_{-\infty}^{0} + \frac{\lambda}{2} \left[\frac{1}{t-\lambda} e^{(t-\lambda)x} \right]_{0}^{\infty} \\
&= \frac{\lambda}{2} \left(\frac{1}{t+\lambda} - \frac{1}{t-\lambda} \right) = \frac{1}{1 - \frac{t^2}{\lambda^2}},
\end{aligned}$$

(c) **Theorem 2.28** Let X have mgf M_X and let $Y = aX + b$.
Then Y has mgf $M_Y(t) = e^{bt}M_X(at)$.

Thus, the mgf of Y is

$$M_Y(t) = e^{bt}M_X(at) = e^{bt}\frac{1}{1-\frac{a^2t^2}{\lambda^2}}.$$

Problem 2. Determine the hazard functions for each of the following (see Section 3.4 in the notes).

(a) The gamma(2,β) distribution. (The cdf can be expressed explicitly in this case.) Use $\beta = 1, 2, 5$ for the plot (all together in one plot).

(b) The distribution with pdf $f(x) = \frac{1}{3}e^{-x} + \frac{4}{3}e^{-2x}$ for $x > 0$. [This is the so-called *mixture* of two exponential pdfs.]

(a) The pdf of gamma(2,β) distribution is

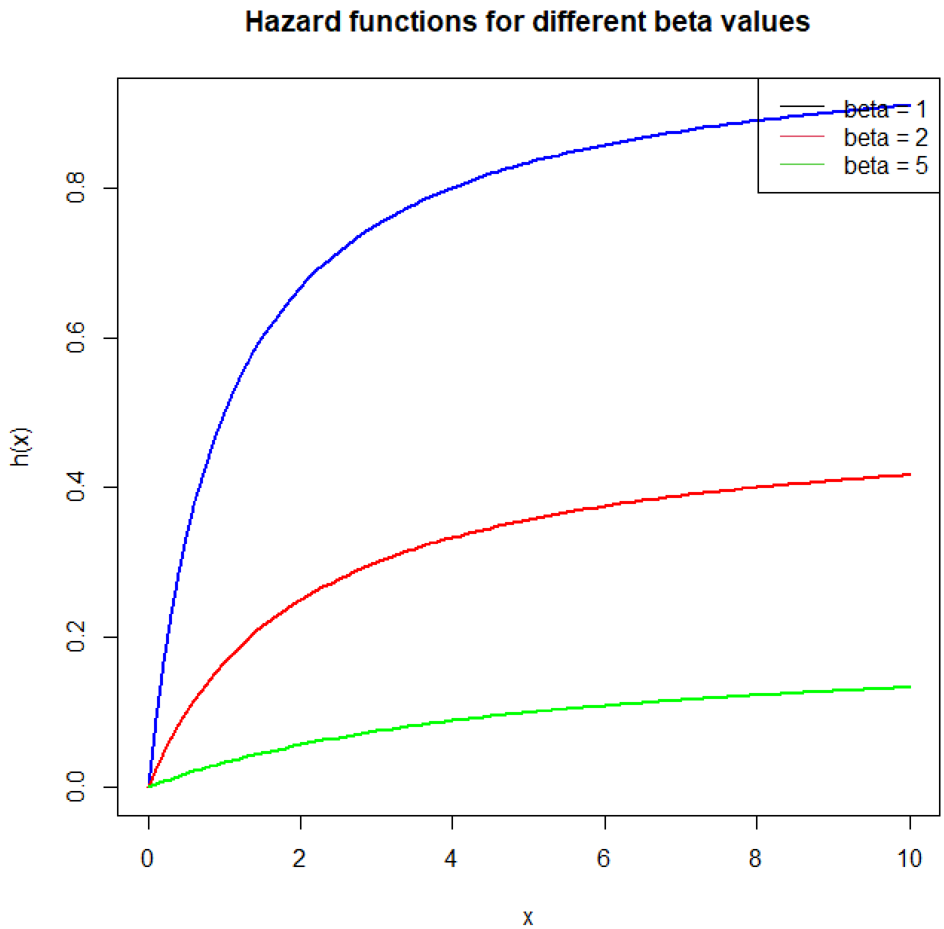
$$f(x) = \frac{1}{\beta^2}xe^{-\frac{x}{\beta}}1_{(0,\infty)}(x).$$

The cdf is

$$F(x) = 1 - e^{-\frac{x}{\beta}} - \frac{x}{\beta}e^{-\frac{x}{\beta}}1_{(0,\infty)}(x).$$

The hazard function is

$$h(x) = \frac{f(x)}{1-F(x)} = \frac{\frac{1}{\beta^2}xe^{-\frac{x}{\beta}}}{e^{-\frac{x}{\beta}} + \frac{x}{\beta}e^{-\frac{x}{\beta}}} = \frac{\frac{1}{\beta^2}x}{1+\frac{x}{\beta}} = \frac{x}{\beta(x+\beta)}.$$

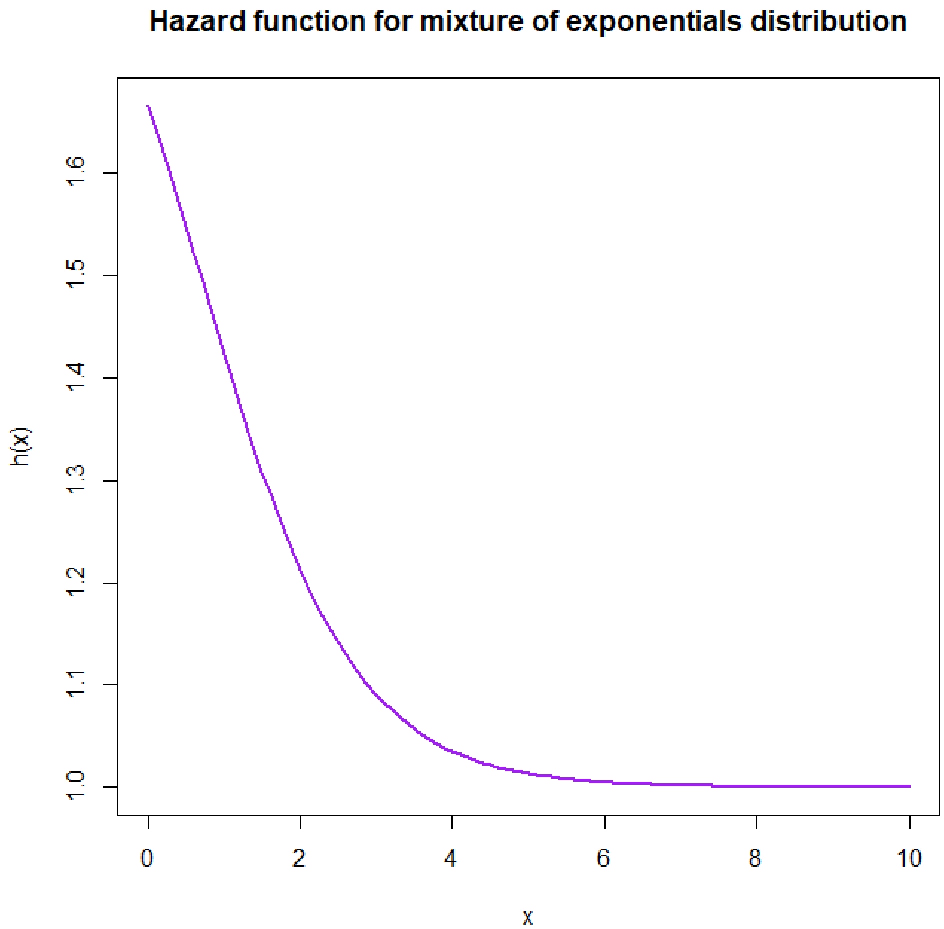


(b) The cdf is

$$F(x) = \int_0^x \left(\frac{1}{3}e^{-t} + \frac{4}{3}e^{-2t}\right) dt = \frac{1}{3}(1 - e^{-x}) + \frac{2}{3}(1 - e^{-2x}) = 1 - \frac{1}{3}e^{-x} - \frac{2}{3}e^{-2x}.$$

The hazard function is

$$h(x) = \frac{f(x)}{1 - F(x)} = \frac{\frac{1}{3}e^{-x} + \frac{4}{3}e^{-2x}}{\frac{1}{3}e^{-x} + \frac{2}{3}e^{-2x}} = \frac{1 + 4e^{-x}}{1 + 2e^{-x}}.$$



Problem 3. Let T be a positive random variable with hazard rate $h(t)$.

(a) Find the quantile function for T and identify $\mathsf{med}(T)$ in terms of $H(t) = \int_0^t h(x)dx$. Apply to Weibull(γ, β) which has cdf $F_T(t) = 1 - e^{-t^\gamma/\beta}$ for $t \geq 0$.

(b) Consider the "U-shaped" $h(t) = .5t^{-.5} + 6t^2$. When is the failure rate at its lowest? Find the pdf.

(a) We have

$$\begin{aligned} H(t) &= \int_0^t h(x)dx \\ &= \int_0^t \frac{f_T(x)}{1 - F_T(x)}dx \\ &= -\int_0^t \frac{d(1 - F_T(x))}{1 - F_T(x)} \\ &= -\left[\ln(1 - F_T(x))\right]_0^t \\ &= -\ln(1 - F_T(t)) \\ \Rightarrow F_T(t) &= 1 - e^{-H(t)}. \end{aligned}$$

The quantile function is

$$Q(p) = F_T^{-1}(p) = (-\ln(1-p))^{1/\gamma}\,\beta.$$

The median is

$$\mathsf{med}(T) = Q(0.5) = (\ln 2)^{1/\gamma}\beta.$$

(b) Since $h(t)$ is U-shaped, the failure rate is at its lowest when $h'(t) = 0$.

$$h'(t) = -0.25t^{-1.5} + 12t = 0 \Rightarrow t = \left(\frac{1}{48}\right)^{1/2} = \frac{1}{4\sqrt{3}}.$$

The pdf is

$$f(t) = h(t)e^{-H(t)} = \left(0.5t^{-.5} + 6t^2\right)e^{-H(t)},$$

where

$$H(t) = \int_0^t h(x)dx = \int_0^t \left(0.5x^{-.5} + 6x^2\right)dx = t^{0.5} + 2t^3.$$

Problem 4. Identify each of the following as defining a location family, a scale family or a location-scale family (if any). (Note: the given parameters are not necessarily location or scale.) Determine the member of the family with mean = 0 (if location family), variance = 1 (if scale family) or both (if location-scale family).

(a) The uniform(a, b) distributions.

(b) The Laplace distributions of Problem 1(a).

(c) The Weibull$(\gamma,\ \beta)$ distributions with $\gamma = 2$ fixed.

(a) The uniform(a, b) distributions form a location-scale family, since if $X \sim$ uniform$(0, 1)$, then $Y = (b-a)X + a \sim$ uniform(a, b). The member of the family with mean = 0 and variance = 1 is uniform$(-\sqrt{3}, \sqrt{3})$. This is because

$$\mathsf{E}[X] = \frac{a+b}{2} = 0 \Rightarrow b = -a,$$

$$\mathsf{Var}(X) = \frac{(b-a)^2}{12} = 1 \Rightarrow b - a = \sqrt{12} = 2\sqrt{3}.$$

Solving these two equations gives $a = -\sqrt{3}$ and $b = \sqrt{3}$.

(b) The Laplace distributions form a location-scale family, since if $X \sim$ Laplace$(0, 1)$, then $Y = aX + b \sim$ Laplace(b, a). The member of the family with mean = 0 and variance = 1 is Laplace$(0, \frac{1}{\sqrt{2}})$. This is because

$$\mathsf{E}[X] = b = 0,$$

$$\mathsf{Var}(X) = 2a^2 = 1 \Rightarrow a = \frac{1}{\sqrt{2}}.$$

(c) The Weibull(γ, β) distributions with $\gamma = 2$ fixed form a scale family, since if $X \sim$ Weibull$(2, 1)$, then $Y = \beta X \sim$ Weibull$(2, \beta)$. The member of the family with variance $= 1$ is

$$\text{Weibull}\left(2, \frac{1}{\sqrt{\Gamma(2) - (\Gamma(1.5))^2}}\right).$$

This is because

$$\mathsf{Var}(X) = \beta^2\left(\Gamma\left(1 + \frac{2}{\gamma}\right) - \Gamma\left(1 + \frac{1}{\gamma}\right)^2\right) = 1 \Rightarrow \beta = \frac{1}{\sqrt{\Gamma(2) - (\Gamma(1.5))^2}}.$$

Problem 5. Suppose X has pdf $f_X(x) = \frac{1}{s}g((x-c)/s)$ from a location-scale family with location parameter c and scale parameter s (and "standard" pdf $g(y)$). Assume $\mathsf{E}(X^2) < \infty$.

(a) Show that $\mathsf{E}(X)$ is linear in c and s, and that $\mathsf{Var}(X)$ is proportional to s^2 but independent of c.

(b) How does the m-th *central moment* depend on c and s?

(a) We have

$$\begin{aligned}
\mathsf{E}(X) &= \int_{-\infty}^{\infty} x f_X(x)dx \\
&= \int_{-\infty}^{\infty} x\frac{1}{s}g\left(\frac{x-c}{s}\right)dx \\
&\text{Let } y = \frac{x-c}{s} \Rightarrow x = sy + c, dx = sdy \\
&= \int_{-\infty}^{\infty} (sy+c)g(y)dy \\
&= s\int_{-\infty}^{\infty} yg(y)dy + c\int_{-\infty}^{\infty} g(y)dy \\
&= s\mathsf{E}(Y) + c.
\end{aligned}$$

$$\begin{aligned}
\mathsf{Var}(X) &= \mathsf{E}(X^2) - (\mathsf{E}(X))^2 \\
&= \int_{-\infty}^{\infty} x^2 f_X(x)dx - (s\mathsf{E}(Y) + c)^2 \\
&= \int_{-\infty}^{\infty} x^2\frac{1}{s}g\left(\frac{x-c}{s}\right)dx - (s\mathsf{E}(Y) + c)^2 \\
&\text{Let } y = \frac{x-c}{s} \Rightarrow x = sy + c, dx = sdy \\
&= \int_{-\infty}^{\infty} (sy+c)^2 g(y)dy - (s\mathsf{E}(Y) + c)^2 \\
&= s^2\int_{-\infty}^{\infty} y^2 g(y)dy + 2sc\int_{-\infty}^{\infty} yg(y)dy + c^2\int_{-\infty}^{\infty} g(y)dy - (s\mathsf{E}(Y) + c)^2 \\
&= s^2\mathsf{E}(Y^2) + 2sc\mathsf{E}(Y) + c^2 - (s\mathsf{E}(Y) + c)^2 \\
&= s^2(\mathsf{E}(Y^2) - (\mathsf{E}(Y))^2) = s^2\mathsf{Var}(Y).
\end{aligned}$$

(b) The m-th central moment is

$$\begin{aligned}
\mu'_m &= \mathsf{E}[(X-\mathsf{E}(X))^m] \\
&= \mathsf{E}[(X-(s\mathsf{E}(Y)+c))^m] \\
&\text{Let } Y = \frac{X-c}{s} \Rightarrow X = sY + c \\
&\Rightarrow \mathsf{E}(X) = s\mathsf{E}(Y) + c \\
&\Rightarrow X - \mathsf{E}(X) = sY + c - (s\mathsf{E}(Y)+c) \\
&= sY - s\mathsf{E}(Y) = s(Y - \mathsf{E}(Y)).
\end{aligned}$$

Thus,

$$\mu'_m = \mathsf{E}[(s(Y-\mathsf{E}(Y)))^m] = s^m \mathsf{E}[(Y-\mathsf{E}(Y))^m] = s^m \mu'_{m,Y}.$$

Problem 6. *Statistical Inference* by Casella and Berger, 2nd Edition, Chapter 3, Exercise 28(c-e).

28. Show that each of the following families is an exponential family.

   (c) beta family with either parameter α or β known or both unknown

   (d) Poisson family

   (e) negative binomial family with r known, $0 < p < 1$

Definition 3.16 A family of pdfs or pmfs, with parameter θ, is a one-parameter exponential family if

   i. the set $A = \{x : f(x) > 0\}$ (the support of f) is the same for all f in the family, and

   ii. $f(x) = c(\theta)h(x)e^{w(\theta)t(x)}$ for some functions, c, h, w and t.

(c) The pdf of beta family is

$$f(x) = \frac{x^{\alpha-1}(1-x)^{\beta-1}}{B(\alpha,\beta)}, \quad 0 < x < 1.$$

* If α is known, then

$$\begin{aligned}
f(x) &= \frac{x^{\alpha-1}(1-x)^{\beta-1}}{B(\alpha,\beta)} \\
&= \exp\left((\beta-1)\ln(1-x) - \ln(B(\alpha,\beta)) + (\alpha-1)\ln(x)\right) \\
&\text{Let } c(\beta) = \frac{1}{B(\alpha,\beta)}, h(x) = x^{\alpha-1}, w(\beta) = \beta - 1, t(x) = \ln(1-x) \\
&\Rightarrow f(x) = c(\beta)h(x)e^{w(\beta)t(x)}.
\end{aligned}$$

Thus, the beta family with α known is an exponential family.

* If β is known, then

$$
\begin{aligned}
f(x) &= \frac{x^{\alpha-1}(1-x)^{\beta-1}}{B(\alpha,\beta)} \\
&= \exp\left((\alpha-1)\ln(x) - \ln(B(\alpha,\beta)) + (\beta-1)\ln(1-x)\right) \\
&\text{Let } c(\alpha) = \frac{1}{B(\alpha,\beta)}, h(x) = (1-x)^{\beta-1}, w(\alpha) = \alpha - 1, t(x) = \ln(x) \\
&\Rightarrow f(x) = c(\alpha)h(x)e^{w(\alpha)t(x)}.
\end{aligned}
$$

Thus, the beta family with β known is an exponential family.

* If both α and β are unknown, then

$$
\begin{aligned}
f(x) &= \frac{x^{\alpha-1}(1-x)^{\beta-1}}{B(\alpha,\beta)} \\
&= \exp\left((\beta-1)\ln(1-x) - \ln(B(\alpha,\beta)) + (\alpha-1)\ln(x)\right) \\
&\text{Let } c(\alpha,\beta) = \frac{1}{B(\alpha,\beta)}, h(x) = (1-x)^{\beta-1}, w(\alpha) = \alpha - 1, t(x) = \ln(x) \\
&\Rightarrow f(x) = c(\alpha,\beta)h(x)e^{w(\alpha)t(x)}.
\end{aligned}
$$

Thus, the beta family with both α and β unknown is an exponential family.

(d) The pmf of Poisson family is

$$f(x) = \frac{e^{-\lambda}\lambda^x}{x!}, \quad x = 0, 1, 2, \dots$$

$$
\begin{aligned}
f(x) &= \frac{e^{-\lambda}\lambda^x}{x!} \\
&= \exp\left(x\ln(\lambda) - \lambda - \ln(x!)\right) \\
&\text{Let } c(\lambda) = \frac{1}{x!}, h(x) = 1, w(\lambda) = \ln(\lambda), t(x) = x \\
&\Rightarrow f(x) = c(\lambda)h(x)e^{w(\lambda)t(x)-\lambda}.
\end{aligned}
$$

Thus, the Poisson family is an exponential family.

(e) The pmf of negative binomial family is

$$f(x) = \binom{x+r-1}{r-1} p^r(1-p)^x, \quad x = 0, 1, 2, \dots$$

$$
\begin{aligned}
f(x) &= \binom{x+r-1}{r-1} p^r(1-p)^x \\
&= \exp\left(x\ln(1-p) + r\ln(p) + \ln\left(\binom{x+r-1}{r-1}\right)\right) \\
&\text{Let } c(p) = \binom{x+r-1}{r-1}, h(x) = 1, w(p) = \ln(1-p), t(x) = x \\
&\Rightarrow f(x) = c(p)h(x)e^{w(p)t(x)-A(p)}.
\end{aligned}
$$

Thus, the negative binomial family with r known is an exponential family.

Problem 7. *Statistical Inference* by Casella and Berger, 2nd Edition, Chapter 3, Exercise 33(b). Note: $\theta \in (-\infty, \infty)$. Also, plot $w_2(\theta)$ versus $w_1(\theta)$.

33. For each of the following families:

    (i) Verify that it is an exponential family.

    (ii) Describe the curve on which the θ parameter vector lies.

    (iii) Sketch a graph of the curved parameter space.

    (b) $n(\theta, a\theta^2)$, a known.

Definition 3.17 Suppose $\theta \in \mathbb{R}^d$, $1 \le d \le k$. A family of pmfs or pdfs with parameter vector θ form an exponential family if

- the support of f is the same for all f in the family, and
- $f(x) = c(\theta)h(x)e^{w_1(\theta)t_1(x)+\cdots+w_k(\theta)t_k(x)}$ for some functions, $c, h, w_1, \ldots, w_k$ and $t_1, \ldots, t_k$.

(b) Since a is known, we can treat it as a constant.

(i) The pdf of $n(\theta, a\theta^2)$ is

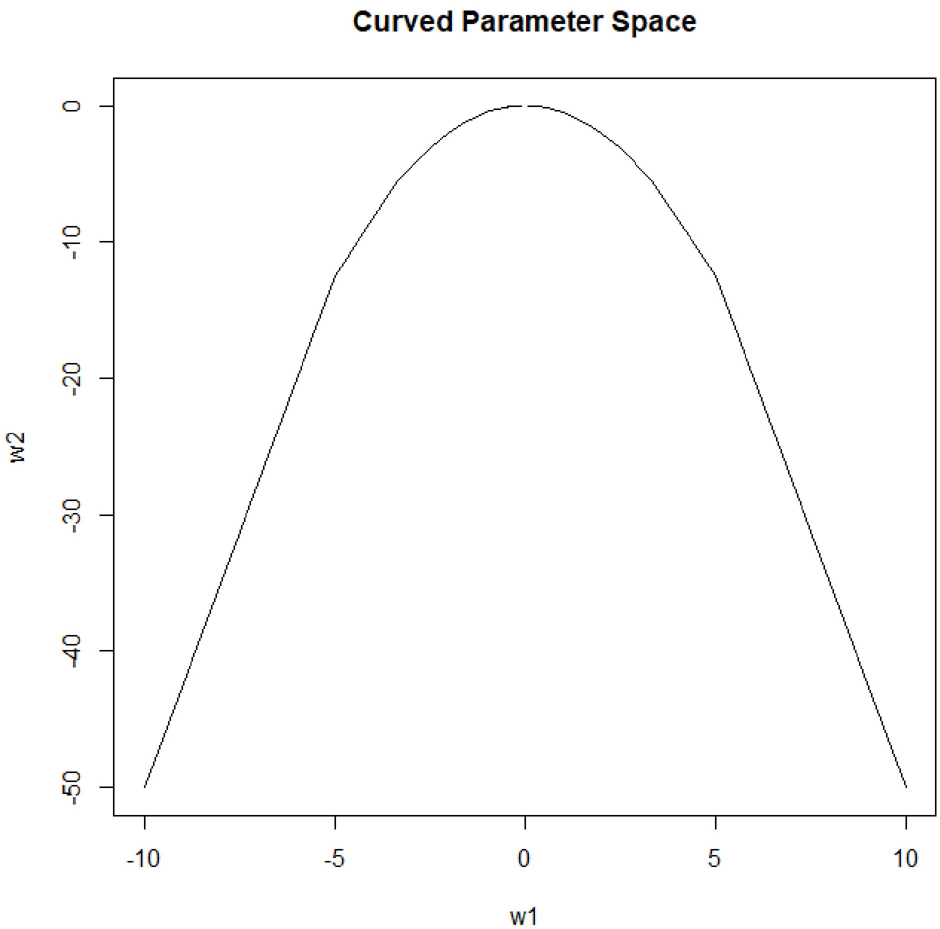
$$\begin{aligned}
f(x) &= \frac{1}{\sqrt{2\pi a\theta^2}} \exp\left(-\frac{(x-\theta)^2}{2a\theta^2}\right) \\
&= \exp\left(-\frac{1}{2}\ln(2\pi a\theta^2) - \frac{x^2 - 2\theta x + \theta^2}{2a\theta^2}\right) \\
&= \exp\left(-\frac{1}{2}\ln(2\pi a) - \ln(\theta) - \frac{x^2}{2a\theta^2} + \frac{x}{a\theta} - \frac{1}{2a}\right) \\
&\text{Let } c(\theta) = \exp\left(-\frac{1}{2}\ln(2\pi a) - \ln(\theta) - \frac{1}{2a}\right), h(x) = 1, \\
&w_1(\theta) = \frac{1}{a\theta}, w_2(\theta) = -\frac{1}{2a\theta^2}, \\
&t_1(x) = x, t_2(x) = x^2 \\
&\Rightarrow f(x) = c(\theta)h(x)e^{w_1(\theta)t_1(x)+w_2(\theta)t_2(x)}.
\end{aligned}$$

Thus, the family is an exponential family.

(ii) The parameter vector is $\theta = (\theta_1, \theta_2)$, where $\theta_1 = \frac{1}{a\theta}$ and $\theta_2 = -\frac{1}{2a\theta^2}$. The curve on which the θ parameter vector lies is

$$w_2(\theta) = -\frac{1}{2a}w_1(\theta)^2.$$

(iii) The graph of the curved parameter space with $a = 1$ is



Problem 8. Show that the following are not exponential families.

(a) The uniform(a, b) distributions.

(b) The (location-scale) logistic(μ, β) distributions. (See Example 3.10 in the notes.)

(a) The pdf of uniform(a, b) distributions is

$$f(x) = \frac{1}{b-a}, \quad a < x < b.$$

The support of f is (a, b), which depends on the parameters a and b. Thus, the uniform(a, b) distributions are not an exponential family.

(b) The pdf of logistic(μ, β) distributions is

$$f(x) = \frac{e^{-\frac{x-\mu}{\beta}}}{\beta(1 + e^{-\frac{x-\mu}{\beta}})^2}, \quad -\infty < x < \infty.$$

$$
\begin{aligned}
f(x) &= \frac{e^{-\frac{x-\mu}{\beta}}}{\beta(1+e^{-\frac{x-\mu}{\beta}})^2} \\
&= \exp\left(-\frac{x-\mu}{\beta} - \ln(\beta) - 2\ln(1+e^{-\frac{x-\mu}{\beta}})\right) \\
&\text{Let } c(\mu,\beta) = \exp(-\ln(\beta)), h(x) = 1, \\
&w_1(\mu,\beta) = \frac{1}{\beta}, w_2(\mu,\beta) = -\frac{1}{\beta}, \\
&t_1(x) = x, t_2(x) = \ln(1+e^{-\frac{x-\mu}{\beta}}) \\
&\Rightarrow f(x) = c(\mu,\beta)h(x)e^{w_1(\mu,\beta)t_1(x)+w_2(\mu,\beta)t_2(x)}.
\end{aligned}
$$

However, $t_2(x)$ depends on the parameters μ and β. Thus, the logistic(μ,β) distributions are not an exponential family.