Homework 8

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Stat 610 Distribution Theory

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Problem 1. Let X and Y be independent random variables with the *same* cdf $F(x)$. Define $U = \min(X, Y)$ and $V = \max(X, Y)$.

(a) Show that $\mathsf{P}(u < U \leq V \leq v) = [F(v) - F(u)]^2$, if $u < v$. Hint: think about the event in terms of the values of X and Y.

(b) Explain why $\mathsf{P}(U \leq u, V \leq v) = \mathsf{P}(V \leq v) - \mathsf{P}(u < U \leq V \leq v)$, and then deduce the joint cdf for (U, V).

(c) What are the marginal cdfs?

(d) Now assume $F(x)$ has pdf $f(x)$. What are the joint and marginal pdfs for (U, V)?

(a) We have

$$\begin{aligned}
\mathsf{P}(u < U \leq V \leq v) &= \mathsf{P}(u < \min(X, Y) \leq \max(X, Y) \leq v) \\
&= \mathsf{P}(u < X \leq v, u < Y \leq v) \\
&= \mathsf{P}(u < X \leq v) \cdot \mathsf{P}(u < Y \leq v) \\
&= [F(v) - F(u)] \cdot [F(v) - F(u)] \\
&= [F(v) - F(u)]^2.
\end{aligned}$$

(b) We have

$$\begin{aligned}
\mathsf{P}(U \leq u, V \leq v) &= \mathsf{P}(\min(X, Y) \leq u, \max(X, Y) \leq v) \\
&= \mathsf{P}(X \leq v, Y \leq v) - \mathsf{P}(u < \min(X, Y) \leq \max(X, Y) \leq v) \\
&= F(v)^2 - [F(v) - F(u)]^2.
\end{aligned}$$

Thus the joint cdf of (U, V) is

$$F_{U,V}(u, v) = \mathsf{P}(U \leq u, V \leq v) = F(v)^2 - (F(v) - F(u))^2.$$

(c) The marginal cdf of U is

$$\begin{aligned} F_U(u) = \mathsf{P}(U \le u) &= \lim_{v\to\infty} F_{U,V}(u,v) \\ &= \lim_{v\to\infty} [F(v)^2 - (F(v)-F(u))^2] \\ &= 1-(1-F(u))^2 = 2F(u)-F(u)^2. \end{aligned}$$

The marginal cdf of V is

$$\begin{aligned} F_V(v) = \mathsf{P}(V \le v) &= \lim_{u\to-\infty} F_{U,V}(u,v) \\ &= \lim_{u\to-\infty} [F(v)^2 - (F(v)-F(u))^2] \\ &= F(v)^2-(F(v)-0)^2 = F(v)^2. \end{aligned}$$

(d) The joint pdf of (U,V) is

$$f_{U,V}(u,v) = \frac{\partial^2}{\partial u \partial v}[F(v)^2-(F(v)-F(u))^2] = 2f(u)f(v).$$

The marginal pdf of U is

$$f_U(u) = \frac{d}{du}[2F(u)-F(u)^2] = 2(1-F(u))f(u).$$

The marginal pdf of V is

$$f_V(v) = \frac{d}{dv}[F(v)^2] = 2F(v)f(v).$$

Problem 2. Suppose $R \sim$ exponential(2) (same as chi-square(2)) and $\Theta \sim$ uniform$(-\pi,\pi)$, independent.

(a) Find the joint pdf for $(X_1,X_2) = (R\sin(\Theta), R\cos(\Theta))$. (This is a 1-1 transformation.)

(b) Are X_1 and X_2 dependent or independent? What are their marginal distributions?

(a) The joint pdf of (R,Θ) is

$$f_{R,\Theta}(r,\theta) = f_R(r)f_\Theta(\theta) = \frac{1}{2}e^{-\frac{1}{2}r}\cdot\frac{1}{2\pi} = \frac{1}{4\pi}e^{-\frac{1}{2}r},$$

for $r>0$ and $-\pi<\theta<\pi$. The transformation is

$$g(r,\theta) = (r\sin(\theta), r\cos(\theta)) = (x_1,x_2).$$

The inverse transformation is

$$g^{-1}(x_1,x_2) = (r,\theta) = \left(\sqrt{x_1^2+x_2^2}, \tan^{-1}\left(\frac{x_1}{x_2}\right)\right).$$

The Jacobian determinant is

$$J = \begin{vmatrix} \sin(\theta) & r\cos(\theta) \\ \cos(\theta) & -r\sin(\theta) \end{vmatrix} = -r(\sin^2(\theta) + \cos^2(\theta)) = -r.$$

Thus the joint pdf of (X_1, X_2) is

$$\begin{aligned} f_{X_1,X_2}(x_1, x_2) &= f_{R,\Theta}(g^{-1}(x_1, x_2)) \cdot |J| \\ &= \frac{1}{4\pi} e^{-\frac{1}{2}\sqrt{x_1^2+x_2^2}} \cdot \sqrt{x_1^2 + x_2^2} \\ &= \frac{\sqrt{x_1^2 + x_2^2}}{4\pi} e^{-\frac{1}{2}\sqrt{x_1^2+x_2^2}}, \end{aligned}$$

for $-\infty < x_1, x_2 < \infty$.

(b) The joint pdf $f_{X_1,X_2}(x_1, x_2)$ cannot be factored into the product of two functions of x_1 and x_2 separately, so X_1 and X_2 are dependent. The marginal pdf of X_1 is

$$\begin{aligned} f_{X_1}(x_1) &= \int_{-\infty}^{\infty} f_{X_1,X_2}(x_1, x_2)\, dx_2 \\ &= \int_{-\infty}^{\infty} \frac{\sqrt{x_1^2 + x_2^2}}{4\pi} e^{-\frac{1}{2}\sqrt{x_1^2+x_2^2}}\, dx_2 \end{aligned}$$

Apparently this can be simplified by using Bessel functions, but we did not cover that in class. Similarly, the marginal pdf of X_2 is

$$\begin{aligned} f_{X_2}(x_2) &= \int_{-\infty}^{\infty} f_{X_1,X_2}(x_1, x_2)\, dx_1 \\ &= \int_{-\infty}^{\infty} \frac{\sqrt{x_1^2 + x_2^2}}{4\pi} e^{-\frac{1}{2}\sqrt{x_1^2+x_2^2}}\, dx_1. \end{aligned}$$

Problem 3. Recall Problem 6 of Assignment 7, where (R, S) has joint pdf $f_{R,S}(r, s) = \frac{8}{3}s^2 e^{-2s}$ for $0 \le r \le s$. You may use the solutions from that problem.

(a) Find the mean and variance of S.

(b) Find $\mathsf{E}(R|S = s)$ and $\mathsf{E}(R^2|S = s)$. Then use iterated expectation and the variance partition formula to compute the mean and variance of R.

(c) Use iterated expectation to find $\mathsf{E}(RS) = \mathsf{E}(\mathsf{E}(RS|S))$.

(d) Now use the above to determine $\mathsf{Cov}(R, S)$.

(a) From the solution of Problem 6 of Assignment 7, we know that S follows a gamma distribution with parameters $\alpha = 4$ and $\beta = \frac{1}{2}$. Since the mean and variance of a gamma distribution are $\alpha\beta$ and $\alpha\beta^2$, respectively, we have

$$\mathsf{E}(S) = 4 \cdot \frac{1}{2} = 2, \quad \mathsf{Var}(S) = 4 \cdot \left(\frac{1}{2}\right)^2 = 1.$$

(b) We have

$$f_{R|S}(r|s) = \frac{f_{R,S}(r,s)}{f_S(s)} = \frac{\frac{8}{3}s^2 e^{-2s}}{\frac{8}{3}s^3 e^{-2s}} = \frac{1}{s},$$

for $0 \le r \le s$. Thus

$$\mathsf{E}(R|S=s) = \int_0^s r \cdot \frac{1}{s}\, dr = \frac{s}{2},$$

and

$$\mathsf{E}(R^2|S=s) = \int_0^s r^2 \cdot \frac{1}{s}\, dr = \frac{s^2}{3}.$$

By iterated expectation, we have

$$\mathsf{E}(R) = \mathsf{E}(\mathsf{E}(R|S)) = \mathsf{E}\left(\frac{S}{2}\right) = \frac{1}{2}\mathsf{E}(S) = 1.$$

By the variance partition formula, we have

$$\begin{aligned}
\mathsf{Var}(R) &= \mathsf{E}(\mathsf{Var}(R|S)) + \mathsf{Var}(\mathsf{E}(R|S)) \\
&= \mathsf{E}\left(\mathsf{E}(R^2|S) - (\mathsf{E}(R|S))^2\right) + \mathsf{Var}\left(\frac{S}{2}\right) \\
&= \mathsf{E}\left(\frac{S^2}{3} - \left(\frac{S}{2}\right)^2\right) + \frac{1}{4}\mathsf{Var}(S) \\
&= \mathsf{E}\left(\frac{S^2}{3} - \frac{S^2}{4}\right) + \frac{1}{4} \\
&= \mathsf{E}\left(\frac{S^2}{12}\right) + \frac{1}{4} \\
&= \frac{1}{12}\mathsf{E}(S^2) + \frac{1}{4}.
\end{aligned}$$

Since $\mathsf{Var}(S) = \mathsf{E}(S^2) - (\mathsf{E}(S))^2$, we have

$$\mathsf{E}(S^2) = \mathsf{Var}(S) + (\mathsf{E}(S))^2 = 1 + 2^2 = 5.$$

Thus

$$\mathsf{Var}(R) = \frac{1}{12} \cdot 5 + \frac{1}{4} = \frac{5}{12} + \frac{3}{12} = \frac{8}{12} = \frac{2}{3}.$$

(c) By iterated expectation, we have

$$\mathsf{E}(RS) = \mathsf{E}(\mathsf{E}(RS|S)) = \mathsf{E}\left(S \cdot \mathsf{E}(R|S)\right) = \mathsf{E}\left(S \cdot \frac{S}{2}\right) = \frac{1}{2}\mathsf{E}(S^2) = \frac{1}{2} \cdot 5 = \frac{5}{2}.$$

(d) We have

$$\mathsf{Cov}(R,S) = \mathsf{E}(RS) - \mathsf{E}(R)\mathsf{E}(S) = \frac{5}{2} - 1 \cdot 2 = \frac{5}{2} - 2 = \frac{1}{2}.$$

Problem 4. Suppose $W \sim \text{Poisson}(\lambda)$ and the conditional distribution of X, given $W = w$, is exponential(w^2) if $w > 0$ and $X = 0$ if $w = 0$. So, in particular, $\mathsf{E}(X|W) = W^2$. Find the mean and variance of X. You may use (without proof) the following fact:

$$\mathsf{E}(W(W-1) \times \cdots \times (W-k)) = \lambda^{k+1}, \quad k = 1, 2, \ldots$$

Since W is Poisson, its mean and variance are both λ. Since the conditional distribution of X given $W = w$ is exponential(w^2) for $w > 0$, we have

$$\mathsf{E}(X|W = w) = w^2, \quad \mathsf{Var}(X|W = w) = w^4.$$

Thus,

$$\begin{aligned} \mathsf{E}(X) &= \mathsf{E}(\mathsf{E}(X|W)) = \mathsf{E}(W^2) \\ &= \mathsf{Var}(W) + (\mathsf{E}(W))^2 = \lambda + \lambda^2. \end{aligned}$$

By the variance partition formula, we have

$$\begin{aligned} \mathsf{Var}(X) &= \mathsf{E}(\mathsf{Var}(X|W)) + \mathsf{Var}(\mathsf{E}(X|W)) \\ &= \mathsf{E}(W^4) + \mathsf{Var}(W^2). \end{aligned}$$

We can express W^4 in terms of falling factorials as

$$W^4 = W(W-1)(W-2)(W-3) + 6W(W-1)(W-2) + 7W(W-1) + W.$$

Thus,

$$\begin{aligned} \mathsf{E}(W^4) &= \mathsf{E}(W(W-1)(W-2)(W-3)) + 6\mathsf{E}(W(W-1)(W-2)) \\ &\quad + 7\mathsf{E}(W(W-1)) + \mathsf{E}(W) \\ &= \lambda^4 + 6\lambda^3 + 7\lambda^2 + \lambda. \end{aligned}$$

Next, we compute $\mathsf{Var}(W^2)$ as

$$\begin{aligned} \mathsf{Var}(W^2) &= \mathsf{E}(W^4) - (\mathsf{E}(W^2))^2 \\ &= (\lambda^4 + 6\lambda^3 + 7\lambda^2 + \lambda) - (\lambda + \lambda^2)^2 \\ &= (\lambda^4 + 6\lambda^3 + 7\lambda^2 + \lambda) - (\lambda^4 + 2\lambda^3 + \lambda^2) \\ &= 4\lambda^3 + 6\lambda^2 + \lambda. \end{aligned}$$

Therefore, we have

$$\begin{aligned} \mathsf{Var}(X) &= \mathsf{E}(W^4) + \mathsf{Var}(W^2) \\ &= (\lambda^4 + 6\lambda^3 + 7\lambda^2 + \lambda) + (4\lambda^3 + 6\lambda^2 + \lambda) \\ &= \lambda^4 + 10\lambda^3 + 13\lambda^2 + 2\lambda. \end{aligned}$$

Problem 5. Recall Exercise 4.4 (Problem 5 of Assignment 7):
A pdf is defined by

$$f(x,y) = \begin{cases} C(x+2y) & \text{if } 0 < y < 1 \text{ and } 0 < x < 2 \\ 0 & \text{otherwise.} \end{cases}$$

where $C = \frac{1}{4}$. Now do the following.

(a) Determine the conditional pdf for Y, given $X = x$, and the conditional pdf for X, given $Y = y$.

(b) Find $g(x) = \mathsf{E}(Y|X = x)$ and $h(y) = \mathsf{E}(X|Y = y)$. Are either of these linear? Are they inverses of each other?

(c) Plot $y = g(x)$ versus x and y versus $x = h(y)$ on the same graph. Comment on what they indicate about predicting one random variable from the other.

(d) Compute $\mathsf{E}(X)$, $\mathsf{E}(Y)$ and $\mathsf{E}(XY)$ by whatever method(s) you prefer.

(e) Compute $\mathsf{Cov}(X, Y)$.

(a) The marginal pdf of X is

$$\begin{aligned} f_X(x) &= \int_0^1 f(x,y)\,dy \\ &= \int_0^1 C(x+2y)\,dy = C\,(x+1)\,, \end{aligned}$$

for $0 < x < 2$. The marginal pdf of Y is

$$\begin{aligned} f_Y(y) &= \int_0^2 f(x,y)\,dx \\ &= \int_0^2 C(x+2y)\,dx = C\,(1+4y)\,, \end{aligned}$$

for $0 < y < 1$. Thus the conditional pdf of Y given $X = x$ is

$$f_{Y|X}(y|x) = \frac{f(x,y)}{f_X(x)} = \frac{C(x+2y)}{C(x+1)} = \frac{x+2y}{x+1}, \quad 0 < y < 1.$$

The conditional pdf of X given $Y = y$ is

$$\begin{aligned} f_{X|Y}(x|y) &= \frac{f(x,y)}{f_Y(y)} = \frac{C(x+2y)}{C(1+4y)} \\ &= \frac{x+2y}{1+4y}, \quad 0 < x < 2. \end{aligned}$$

(b) We have

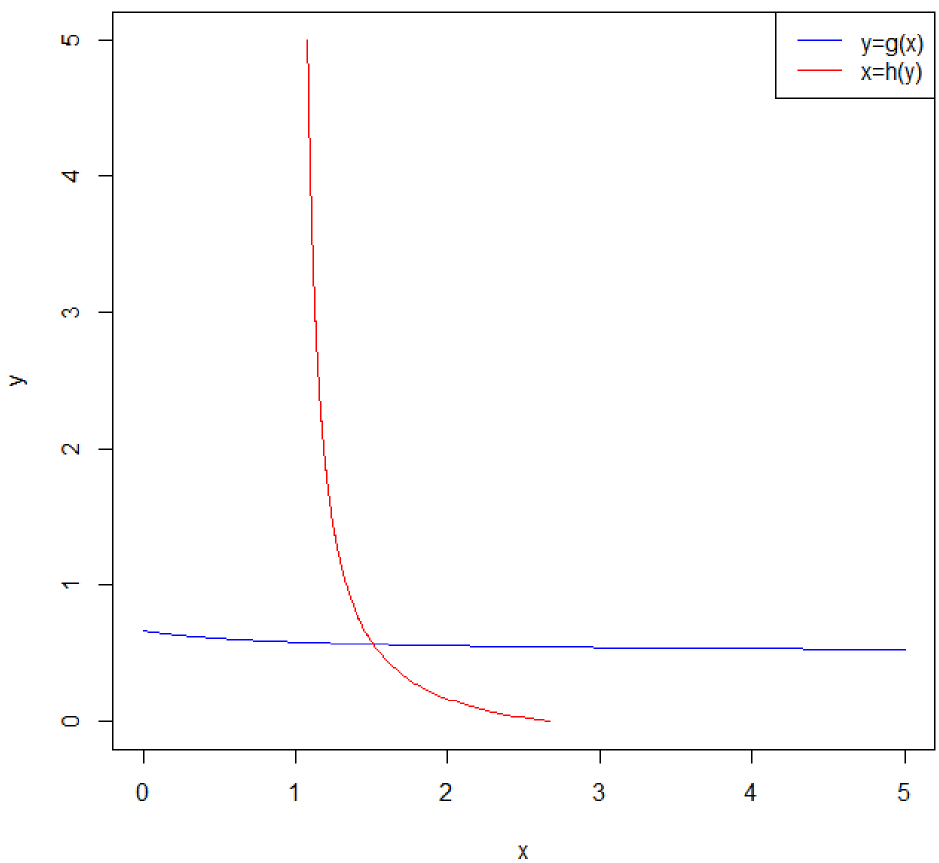
$$\begin{aligned} g(x) = \mathsf{E}(Y|X=x) &= \int_0^1 y \cdot \frac{x+2y}{x+1}\,dy \\ &= \frac{1}{x+1}\int_0^1 (xy+2y^2)\,dy \\ &= \frac{1}{x+1}\left(\frac{x}{2}+\frac{2}{3}\right) = \frac{3x+4}{6(x+1)}. \end{aligned}$$

Similarly,

$$
\begin{aligned}
h(y) = \mathsf{E}(X|Y=y) &= \int_0^2 x \cdot \frac{x+2y}{1+4y}\,dx \\
&= \frac{1}{1+4y}\int_0^2 (x^2+2yx)\,dx \\
&= \frac{1}{1+4y}\left(\frac{8}{3}+4y\right) = \frac{8+12y}{3(1+4y)}.
\end{aligned}
$$

Neither $g(x)$ nor $h(y)$ is linear. They are not inverses of each other since $g(h(y)) \neq y$ and $h(g(x)) \neq x$.

(c) The plot is shown below.



From the plot, we can see that both $g(x)$ and $h(y)$ are decreasing functions. This indicates that when predicting Y from X, a larger value of X tends to correspond to a smaller predicted value of Y, and vice versa.

(d)

$$
\begin{aligned}
\mathsf{E}(X) &= \int_0^2 x f_X(x)\,dx = \int_0^2 xC(x+1)\,dx = C\int_0^2 (x^2+x)\,dx = C\left(\frac{8}{3}+2\right) = \frac{7}{6},\\
\mathsf{E}(Y) &= \int_0^1 y f_Y(y)\,dy = \int_0^1 yC(1+4y)\,dy = C\int_0^1 (y+4y^2)\,dy = C\left(\frac{1}{2}+\frac{4}{3}\right) = \frac{7}{12},\\
\mathsf{E}(XY) &= \int_0^2\int_0^1 xyf(x,y)\,dy\,dx\\
&= \int_0^2\int_0^1 xyC(x+2y)\,dy\,dx\\
&= C\int_0^2 \left(x^2\int_0^1 y\,dy + 2x\int_0^1 y^2\,dy\right)dx\\
&= C\int_0^2 \left(x^2\cdot\frac{1}{2} + 2x\cdot\frac{1}{3}\right)dx\\
&= C\int_0^2 \left(\frac{x^2}{2} + \frac{2x}{3}\right)dx\\
&= C\left(\frac{8}{6}+\frac{4}{3}\right) = C\cdot\frac{16}{6}\\
&= \frac{2}{3}.
\end{aligned}
$$

(e)

$$
\begin{aligned}
\mathsf{Cov}(X,Y) &= \mathsf{E}(XY) - \mathsf{E}(X)\mathsf{E}(Y)\\
&= \frac{2}{3} - \frac{7}{6}\cdot\frac{7}{12}\\
&= \frac{2}{3} - \frac{49}{72} = \frac{48}{72} - \frac{49}{72} = -\frac{1}{72}.
\end{aligned}
$$

Problem 6. Assume the random pair (S,T) has joint distribution

$$
f_{S,T}(s,t) = \frac{1}{2}(s+t)e^{-(s+t)}, \quad s>0, t>0.
$$

(a) Find $\mathsf{E}(S)$, $\mathsf{E}(S^2)$, $\mathsf{E}(T)$, $\mathsf{E}(T^2)$ and $\mathsf{E}(ST)$. Hint: you can express each of these in terms of some gamma function values.

(b) Determine $\mathsf{Corr}(S,T)$.

(a) We have

$$\begin{aligned}
\mathsf{E}(S) &= \int_0^\infty \int_0^\infty s \cdot \frac{1}{2}(s+t)e^{-(s+t)}\,dt\,ds \\
&= \frac{1}{2}\int_0^\infty se^{-s}\left(\int_0^\infty (s+t)e^{-t}\,dt\right)ds \\
&= \frac{1}{2}\int_0^\infty se^{-s}(s+1)\,ds \\
&= \frac{1}{2}(\Gamma(3)+\Gamma(2)) = \frac{1}{2}(2+1) = \frac{3}{2}. \\
\mathsf{E}(S^2) &= \int_0^\infty \int_0^\infty s^2 \cdot \frac{1}{2}(s+t)e^{-(s+t)}\,dt\,ds \\
&= \frac{1}{2}\int_0^\infty s^2e^{-s}(s+1)\,ds \\
&= \frac{1}{2}(\Gamma(4)+\Gamma(3)) = \frac{1}{2}(6+2) = 4,
\end{aligned}$$

By symmetry,

$$\begin{aligned}
\mathsf{E}(T) &= E[S] = \frac{3}{2}, \\
\mathsf{E}(T^2) &= E[S^2] = 4.
\end{aligned}$$

Using the change of variables $u = s+t$, we have

$$\begin{aligned}
\mathsf{E}(ST) &= \frac{1}{2}\int_0^\infty ue^{-u}\int_0^u s(u-s)\,ds\,du \\
&= \frac{1}{2}\int_0^\infty ue^{-u}\left(\frac{u^3}{6}\right)du \\
&= \frac{1}{12}\int_0^\infty u^4e^{-u}\,du \\
&= \frac{1}{12}\Gamma(5) = \frac{1}{12}\cdot 24 = 2.
\end{aligned}$$

(b) We have

$$\begin{aligned}
\mathsf{Cov}(S,T) &= \mathsf{E}(ST) - \mathsf{E}(S)\mathsf{E}(T) \\
&= 2 - \frac{3}{2}\cdot\frac{3}{2} = 2 - \frac{9}{4} = -\frac{1}{4}, \\
\mathsf{Var}(S) &= \mathsf{E}(S^2) - (\mathsf{E}(S))^2 = 4 - \left(\frac{3}{2}\right)^2 = 4 - \frac{9}{4} = \frac{7}{4}, \\
\mathsf{Var}(T) &= \mathsf{Var}(S) = \frac{7}{4}.
\end{aligned}$$

Thus, we have

$$\begin{aligned}\mathsf{Corr}(S,T) &= \frac{\mathsf{Cov}(S,T)}{\sqrt{\mathsf{Var}(S)\mathsf{Var}(T)}}\\ &= \frac{-\frac{1}{4}}{\sqrt{\frac{7}{4}\cdot\frac{7}{4}}} = \frac{-\frac{1}{4}}{\frac{7}{4}} = -\frac{1}{7}.\end{aligned}$$

Problem 7. Suppose X and Y are *integer-valued* rvs with joint pmf

$$f_{X,Y}(x,y) = \frac{1}{13}, \quad \text{if } |x+y| \le 2 \text{ and } |x-y| \le 2.$$

(It may help to graph the possible points (x,y).)

(a) Are X and Y independent? What are their marginal pmfs?

(b) What is $\mathsf{Cov}(X,Y)$?

(a) The possible points (x,y) are

$(-2,0),(-1,-1),(-1,0),(-1,1),(0,-2),(0,-1),(0,0),(0,1),(0,2),(1,-1),(1,0),(1,1),(2,0).$

Thus, the marginal pmf of X is

$$f_X(x) = \begin{cases}\frac{1}{13} & x=-2\\ \frac{3}{13} & x=-1\\ \frac{5}{13} & x=0\\ \frac{3}{13} & x=1\\ \frac{1}{13} & x=2\\ 0 & \text{otherwise.}\end{cases}$$

The marginal pmf of Y is $f_Y(y) = f_X(y)$ by symmetry. Since $f_{X,Y}(0,0) = \frac{1}{13} \neq f_X(0)f_Y(0) = \frac{5}{13}\cdot\frac{5}{13} = \frac{25}{169}$, X and Y are not independent.

(b) We have

$$\begin{aligned}\mathsf{E}(X) &= \sum_x x f_X(x) = -2\cdot\frac{1}{13} - 1\cdot\frac{3}{13} + 0\cdot\frac{5}{13} + 1\cdot\frac{3}{13} + 2\cdot\frac{1}{13} = 0,\\ \mathsf{E}(Y) &= \mathsf{E}(X) = 0,\\ \mathsf{E}(XY) &= \sum_x\sum_y xy f_{X,Y}(x,y)\\ &= (0+1+0-1+0+0+0+0+0-1+0+1+0)\cdot\frac{1}{13} = 0.\end{aligned}$$

Thus, we have

$$\begin{aligned}\mathsf{Cov}(X,Y) &= \mathsf{E}(XY) - \mathsf{E}(X)\mathsf{E}(Y)\\ &= 0 - 0\cdot 0 = 0.\end{aligned}$$

Problem 8. (a) Use Theorem 4.22 in the notes to prove that if $X_1, \ldots, X_k$ are independent random variables with respective mgfs $M_1(t), \ldots, M_k(t)$ then the mgf for $S = X_1 + \cdots + X_k$ is $M_S(t) = \prod_{i=1}^k M_i(t)$.

(b) Use the result in part (a) to confirm the following.

i. If $S \sim \text{binomial}(m, p)$ and $T \sim \text{binomial}(n, p)$, independent, then $S + T \sim \text{binomial}(m + n, p)$. (This was done by use of a convolution in Assignment 6 Problem 4.)

ii. If T and U are independent with $T \sim \text{gamma}(\alpha, \gamma)$ and $U \sim \text{gamma}(\beta, \gamma)$ then $T + U \sim \text{gamma}(\alpha + \beta, \gamma)$. (You also showed this in Assignment 7 Problem 8 using a different method.)

iii. If $X \sim \text{Poisson}(\lambda)$ and $Y \sim \text{Poisson}(\mu)$, independent, then $X + Y \sim \text{Poisson}(\lambda + \mu)$. (This was also done by convolution in Theorem 4.10 of the notes.)

Theorem 4.22. Suppose $(X_1, \ldots, X_k)$ has joint distribution. Then $X_1, \ldots, X_k$ are independent if and only if

$$\mathsf{E}(g_1(X_1) \times \cdots \times g_k(X_k)) = \mathsf{E}(g_1(X_1)) \times \cdots \times \mathsf{E}(g_k(X_k))$$

whenever these expectations exist. $(g_1(X_1), \ldots, g_k(X_k))$ are also independent.
In particular, if X and Y are independent and have finite means then $\mathsf{E}(XY) = \mathsf{E}(X)\mathsf{E}(Y)$.

(a) Since $X_1, \ldots, X_k$ are independent, by Theorem 4.22, we have

$$\begin{aligned}
M_S(t) &= \mathsf{E}(e^{tS}) = \mathsf{E}(e^{t(X_1+\cdots+X_k)}) \\
&= \mathsf{E}(e^{tX_1} \times \cdots \times e^{tX_k}) \\
&= \mathsf{E}(e^{tX_1}) \times \cdots \times \mathsf{E}(e^{tX_k}) \\
&= M_1(t) \times \cdots \times M_k(t) = \prod_{i=1}^k M_i(t).
\end{aligned}$$

(b) i. The mgf of $S \sim \text{binomial}(m, p)$ is

$$M_S(t) = (1 - p + pe^t)^m,$$

and the mgf of $T \sim \text{binomial}(n, p)$ is

$$M_T(t) = (1 - p + pe^t)^n.$$

Thus, the mgf of $S + T$ is

$$M_{S+T}(t) = M_S(t)M_T(t) = (1 - p + pe^t)^{m+n},$$

which is the mgf of $\text{binomial}(m + n, p)$.

ii. The mgf of $T \sim \text{gamma}(\alpha, \gamma)$ is

$$M_T(t) = (1 - \gamma t)^{-\alpha}, \quad t < \frac{1}{\gamma},$$

and the mgf of $U \sim \text{gamma}(\beta, \gamma)$ is

$$M_U(t) = (1 - \gamma t)^{-\beta}, \quad t < \frac{1}{\gamma}.$$

Thus, the mgf of $T + U$ is

$$M_{T+U}(t) = M_T(t) M_U(t) = (1 - \gamma t)^{-(\alpha+\beta)},$$

which is the mgf of $\text{gamma}(\alpha + \beta, \gamma)$.

iii. The mgf of $X \sim \text{Poisson}(\lambda)$ is

$$M_X(t) = e^{\lambda(e^t - 1)},$$

and the mgf of $Y \sim \text{Poisson}(\mu)$ is

$$M_Y(t) = e^{\mu(e^t - 1)}.$$

Thus, the mgf of $X + Y$ is

$$M_{X+Y}(t) = M_X(t) M_Y(t) = e^{(\lambda+\mu)(e^t - 1)},$$

which is the mgf of $\text{Poisson}(\lambda + \mu)$.