STAT610 Assignment 7
due Wednesday, 22 October 2025

Warning: This assignment involves a lot of algebra and calculus. Plan your time accordingly.

1. A random sample of size n is obtained *without* replacement from a finite population of size N that has 3 separate categories A,B,C with sizes M_A, M_B, M_C, respectively, such that $M_A + M_B + M_C = N$.

   (a) Let X_A, X_B, X_C be the numbers of individuals sampled from the three categories. Find the joint pmf for (X_A, X_B, X_C). Keep in mind that $X_A + X_B + X_C = n$.

   (b) Now let $Y = X_A + X_B$. Find the joint pmf for (X_A, Y).

   (c) What is the marginal pmf for Y? Hint: think about the sampling and what happens if you ignore the distinction between categories A and B.

   (d) Find the conditional pmf for X_A, given $Y = y$. What distribution is this (name and parameters)?

2. Let (X, Y) have joint cdf $F(x, y) = \mathsf{P}(X \le x, Y \le y)$.

   (a) Show that, for $a < b$ and $c < d$,

   $$\mathsf{P}(a < X \le b, c < Y \le d) = F(b, d) - F(b, c) - F(a, d) + F(a, c).$$

   (b) Simplify the expression in (a) for the independence case: $F(x, y) = F_X(x)F_Y(y)$.

3. Suppose T has negative binomial$(2, p)$ distribution and the conditional pmf for S, given $T = t$, is $f_{S|T}(s|t) = \frac{1}{t+1}$, $s \in \{0, \dots, t\}$.

   (a) Find the joint pmf for (S, T). Be sure to indicate the range with any restrictions.

   (b) Find the marginal pmf for S and the conditional pmf for T, given $S = s$.

   (c) Show that S and $R = T - S$ are independent and have the same distribution.

4. Suppose $S \sim$ binomial(m, p) and $T \sim$ binomial(n, p) (same second parameter p), with S and T independent. Use the convolution formula to prove $S + T \sim$ binomial$(m + n, p)$. Hint: factor out the result and observe that a hypergeometric pmf remains.

5. Chapter 4 Exercise 4(a–c). For part (c), be sure to give the joint cdf for <u>all</u> (x, y) in the real plane, and then check yourself by deriving the joint pdf for all (x, y).

   Add the following.

   (d) Find $\mathsf{P}(Y^2 < X < \sqrt{Y})$.

   (e) Find $\mathsf{P}(X + 2Y \le t)$ for $t \in [0, 4]$ and deduce the pdf for $T = X + 2Y$. You will need to consider a couple cases of the double integral separately. Be sure to check that your pdf is valid.

6. Suppose (R, S) has joint pdf $f_{R,S}(r, s) = \frac{8}{3}s^2 \mathrm{e}^{-2s}$ for $0 \le r \le s$.

   (a) Find the marginal pdf for S and the conditional pdf for R, given $S = s$. Try to identify the distributions by name, giving appropriate values to the parameters.

   (b) Find the marginal pdf for R and the conditional pdf for S, given $R = r$.

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7. (a) Prove Corollary 4.19.ii in the notes, by applying Theorem 4.18.
   (b) Suppose X, Y are independent exponential(1) random variables and $W = X + Y$, $Z = X - Y$. Find the joint pdf for (W, Z). Be aware of the range for the random pair: Z can be positive or negative but it also is restricted by W.
   (c) Find the marginal distributions for W and Z. (Note: W is a special case of Example 4.10 in the notes.)

8. (a) Let T and U be independent with $T \sim \text{gamma}(\alpha, \gamma)$ and $U \sim \text{gamma}(\beta, \gamma)$ (with the same scale parameter γ). Let $X = T + U$ and $Y = T/(T + U)$. Determine the joint pdf for (X, Y). Identify the marginal distributions by name and describe, in words, what the joint distribution of (X, Y) is. Hint: review distributions defined in Section 3.3 of the notes.
   (b) Using the result for the distribution of X in part (a), prove by mathematical induction (iteration) that if $T_1, \ldots, T_k$ are independent random variables such that $T_i \sim \text{gamma}(\alpha_i, \gamma)$, then $T_1 + \cdots + T_k \sim \text{gamma}(\alpha_1 + \cdots + \alpha_k, \gamma)$.