**Assignment 8**

Partial solutions for selected problems

1. (a) This follows from $\{u < U \le V \le v\} = \{u < X \le v\} \cap \{u < Y \le v\}$.

   (b)

$$F_{U,V}(u,v) = \begin{cases} F(u)(2F(v) - F(u)) & \text{if } u \le v, \\ (F(v))^2 & \text{if } v < u. \end{cases}$$

   It follows that $F_U(u) = F(u)(2 - F(u))$ and $F_V(v) = (F(v))^2$.

   (c) $f_{U,V}(u,v) = 2f(u)f(v)1_{u<v}$. [This *does* integrate to 1.] $f_U(u) = 2f(u)(1 - F(u))$ and $f_V(v) = 2f(v)F(v)$.

2. First, the possible values of $(X_1, X_2)$ are the whole real plane. Next, compute $\frac{\mathrm{d}x\mathrm{d}y}{\mathrm{d}r\mathrm{d}\theta} = r = \sqrt{x_1^2 + x_2^2}$. Thus,

$$f_{X_1,X_2}(x_1,x_2) = \frac{1}{r} f_{R,\theta}(r,\theta) = \cdots = \frac{\mathrm{e}^{-\sqrt{x_1^2+x_2^2}/2}}{4\pi\sqrt{x_1^2 + x_2^2}}.$$

   Clearly, $X_1$ and $X_2$ are not independent.

   [This actually was not the problem I intended. Instead it should have been $(X_1, X_2) = (\sqrt{R}\sin(\Theta), \sqrt{R}\cos(\Theta))$ which are independent standard normal rvs.]

3. (a) Use what is known about gamma random variables: $\mathsf{E}(S) = 2$, $\mathsf{Var}(S) = 1$.

   (b) (Recall that the conditional distribution for $R$, given $S = s$, is uniform$(0, s)$.) $\mathsf{E}(R|S) = \frac{1}{2}S$ and $\mathsf{E}(R^2|S) = \frac{1}{3}S^2$. Thus, $\mathsf{Var}(R|S) = \frac{1}{12}S^2$. Additionally,

$$\mathsf{E}(R) = \mathsf{E}(\mathsf{E}(R|S)) = \mathsf{E}(\tfrac{1}{2}S) = 1, \quad \mathsf{Var}(R) = \mathsf{Var}(\tfrac{1}{2}S) + \mathsf{E}(\tfrac{1}{12}S^2) = \tfrac{2}{3}.$$

   (c,d)

$$\mathsf{E}(RS) = \mathsf{E}(S\mathsf{E}(R|S)) = \mathsf{E}(\tfrac{1}{2}S^2) = \tfrac{5}{2},$$

   and thus $\mathsf{Cov}(R, S) = \frac{1}{2}$.

4. First, using the given property for $W$, find

$$\mathsf{E}(W^2) = \lambda + \lambda^2 \quad \text{and} \quad \mathsf{E}(W^4) = \lambda + 7\lambda^2 + 6\lambda^3 + \lambda^4.$$

   Then $\mathsf{E}(X) = \mathsf{E}(\mathsf{E}(X|W)) = \mathsf{E}(W^2) = \lambda + \lambda^2$. Also,

$$\mathsf{E}(X^2) = \mathsf{E}(\mathsf{E}(X^2|W)) = \mathsf{E}(2(W^2)^2) = 2\lambda + 14\lambda^2 + 12\lambda^3 + 2\lambda^4.$$

   Then $\mathsf{Var}(X) = \mathsf{E}(X^2) - (\mathsf{E}(X))^2 = 2\lambda + 13\lambda^2 + 10\lambda^3 + \lambda^4$. Alternatively, use the variance partition formula.

5. (a) $f_{Y|X}(y|x) = \frac{x+2y}{x+1}$, $g(x) = \mathsf{E}(Y|X = x) = \frac{3x+4}{6(x+1)}$. $f_{X|Y}(x|y) = \frac{x+2y}{1+2y}$, $h(y) = \mathsf{E}(X|Y = y) = \frac{4(2+3y)}{3(1+2y)}$. $g(x)$ and $h(y)$ are not inverses of each other. [Nor could they be – they never are unless $X$ and $Y$ have a perfect 1-1 relationship.]

   (c,d) $\mathsf{E}(X) = \frac{7}{6}$, $\mathsf{E}(Y) = \frac{7}{12}$, $\mathsf{E}(XY) = \frac{2}{3}$, $\mathsf{Cov}(X, Y) = -\frac{1}{72}$.

6. $\mathsf{Var}(S) = \mathsf{Var}(T) = \frac{7}{4}$, $\mathsf{Cov}(S, T) = -\frac{1}{4}$, $\mathsf{Corr}(S, T) = -\frac{1}{7}$.

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7. (a) $X$ and $Y$ are not independent. They have the same pmf $f(x) = \frac{5-2|x|}{13}$ for $x \in \{-2,-1,0,1,2\}$.
   (b) $\mathsf{Cov}(X,Y) = 0$.

8. (a) Apply the theorem to the independent random variables $\mathrm{e}^{tX_1}, \ldots, \mathrm{e}^{tX_1}$.
   (b) Use the binomial$(n,p)$ mgf $(1-p+p\mathrm{e}^t)^n$, the gamma$(\alpha,\gamma)$ mgf $(1-\gamma t)^{-\alpha}$ and the Poisson$(\lambda)$ mgf $\mathrm{e}^{\lambda(\mathrm{e}^t-1)}$.