Homework 2

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Stat 610 Distribution Theory

September 11, 2025

Problem 1. *Statistical Inference* by Casella and Berger, 2nd Edition, Chapter 1, Exercise 34.

34. Two litters of a particular rodent species have been born, one with two brown-haired and one gray-haired (litter 1), and the other with three brown-haired and two gray-haired (litter 2). We select a litter at random and then select an offspring at random from the selected litter.

   (a) What is the probability that the animal chosen is brown-haired?

   (b) Given that a brown-haired offspring was selected, what is the probability that the sampling was from litter 1?

(a) Let A be the event that the animal chosen is brown-haired, Let B_i be the event that the sampling was from litter i, $i = 1, 2$. Then we have

$$\mathrm{P}(A) = \mathrm{P}(A|B_1)\mathrm{P}(B_1) + \mathrm{P}(A|B_2)\mathrm{P}(B_2) = \frac{2}{3} \cdot \frac{1}{2} + \frac{3}{5} \cdot \frac{1}{2} = \frac{19}{30}.$$

(b) By Bayes' theorem, we have

$$\mathrm{P}(B_1|A) = \frac{\mathrm{P}(A|B_1)\mathrm{P}(B_1)}{\mathrm{P}(A)} = \frac{\frac{2}{3} \cdot \frac{1}{2}}{\frac{19}{30}} = \frac{5}{19}.$$

Problem 2. (a) Prove this alternative to Bayes' rule:

$$\log\left(\frac{\mathrm{P}(A|B)}{\mathrm{P}(A^c|B)}\right) = \log\left(\frac{\mathrm{P}(A)}{\mathrm{P}(A^c)}\right) + \log\left(\frac{\mathrm{P}(B|A)}{\mathrm{P}(B|A^c)}\right).$$

This expression is useful in genetics: conditional log odds of disease (A) given gene (B) = unconditional log odds of disease + log ratio of gene prevalence.

(b) Suppose $B_1, \dots, B_n$ are disjoint. Show that

$$\mathrm{P}\left(B_j | \bigcup_{i=1}^{n} B_i\right) = \frac{\mathrm{P}(B_j)}{\sum_{i=1}^{n} \mathrm{P}(B_i)}. \quad \text{for each } j \in \{1, \dots, n\}.$$

(a) By definition, we have

$$\log\left(\frac{\mathrm{P}(A|B)}{\mathrm{P}(A^c|B)}\right) = \log\left(\frac{\frac{\mathrm{P}(A\cap B)}{\mathrm{P}(B)}}{\frac{\mathrm{P}(A^c\cap B)}{\mathrm{P}(B)}}\right) = \log\left(\frac{\mathrm{P}(A\cap B)}{\mathrm{P}(A^c\cap B)}\right).$$

By the definition of conditional probability again and $\log(AB) = \log(A) + \log(B)$, we have

$$\log\left(\frac{\mathrm{P}(A\cap B)}{\mathrm{P}(A^c\cap B)}\right) = \log\left(\frac{\mathrm{P}(A)\mathrm{P}(B|A)}{\mathrm{P}(A^c)\mathrm{P}(B|A^c)}\right) = \log\left(\frac{\mathrm{P}(A)}{\mathrm{P}(A^c)}\right) + \log\left(\frac{\mathrm{P}(B|A)}{\mathrm{P}(B|A^c)}\right).$$