Exam 2 2023

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Stat 610 Distribution Theory

November 4, 2025

Problem 1. X and Y are independent random variables with moment generating functions $M_X(t) = \frac{e^t}{1-t^2}$ and $M_Y(t) = e^{e^t-t-1}$, respectively. Find $\mathsf{Var}(X+Y)$.

$$\begin{aligned}
\mathsf{Var}(X+Y) &= \mathsf{Var}(X) + \mathsf{Var}(Y) \\
&= \frac{d^2 \ln(M_X(t))}{dt^2}\bigg|_{t=0} + \frac{d^2 \ln(M_Y(t))}{dt^2}\bigg|_{t=0} \\
&= \frac{d^2}{dt^2}\left(t - \ln(1-t^2)\right)\Big|_{t=0} + \frac{d^2}{dt^2}\left(e^t - t - 1\right)\Big|_{t=0} \\
&= \frac{d}{dt}\left(1 + \frac{2t}{1-t^2}\right)\Big|_{t=0} + \frac{d}{dt}\left(e^t - 1\right)\Big|_{t=0} \\
&= \left(2(1-t^2)^{-2} + 2t(-2)(-2t)(1-t^2)^{-3}\right)\Big|_{t=0} + e^t\Big|_{t=0} \\
&= 2 + 1 = 3
\end{aligned}$$

Problem 2. (S,T) has joint pdf $f_{S,T}(s,t) = \frac{(s+t)^2}{6}e^{-s-t}$ for $s > 0, t > 0$.

(a) Let $W = S + T$ and $Z = \frac{S}{S+T}$. Find the joint pdf for (W, Z) and identify.

(b) Accept as given that $E(S) = E(T) = 2$. Find $\mathsf{Cov}(S,T)$.

(a) We have the transformation

$$\begin{cases} W = S + T \\ Z = \frac{S}{S+T} \end{cases}$$

The inverse transformation is

$$\begin{cases} S = WZ \\ T = W(1-Z) \end{cases}$$

The Jacobian determinant is

$$J = \begin{vmatrix} \frac{\partial S}{\partial W} & \frac{\partial S}{\partial Z} \\ \frac{\partial T}{\partial W} & \frac{\partial T}{\partial Z} \end{vmatrix} = \begin{vmatrix} Z & W \\ 1-Z & -W \end{vmatrix} = -WZ - W(1-Z) = -W$$

Thus, the joint pdf for (W, Z) is

$$\begin{aligned}
f_{W,Z}(w,z) &= f_{S,T}(wz, w(1-z)) \cdot |J| \\
&= \frac{(wz + w(1-z))^2}{6} e^{-wz-w(1-z)} \cdot w \\
&= \frac{w^2}{6} e^{-w} \cdot w = \frac{w^3}{6} e^{-w}
\end{aligned}$$

for $w > 0$ and $0 < z < 1$. This shows that W and Z are independent, with $W \sim$ gamma$(4,1)$ and $Z \sim$ uniform$(0,1)$.

(b) We have

$$\begin{aligned}
\mathsf{Cov}(S,T) &= \mathsf{E}(ST) - \mathsf{E}(S)\mathsf{E}(T) \\
&= \mathsf{E}(W^2 Z(1-Z)) - 4 \\
&= \mathsf{E}(W^2)\mathsf{E}(Z(1-Z)) - 4 \\
&= \left(\mathsf{Var}(W) + (\mathsf{E}(W))^2\right)\left(\mathsf{E}(Z) - \mathsf{E}(Z^2)\right) - 4 \\
&= (4+16)\left(\frac{1}{2} - \frac{1}{3}\right) - 4 \\
&= 20 \cdot \frac{1}{6} - 4 = \frac{10}{3} - 4 = -\frac{2}{3}
\end{aligned}$$

Problem 3. Suppose X and Y are positive integer-valued random variables with joint pmf

$$f_{X,Y}(x,y) = C 1_{\{1,\dots,10\}}(x) 1_{\{1,\dots,x\}}(y)$$

for some constant C.
What is the best predictor of Y as a function of X? Explain.

Normalize the pmf:

$$\begin{aligned}
1 &= \sum_{x=1}^{10} \sum_{y=1}^{x} C \\
&= C \sum_{x=1}^{10} x = C \cdot \frac{10 \cdot 11}{2} = 55C \\
\Rightarrow C &= \frac{1}{55}
\end{aligned}$$

Then the conditional pmf of Y given X is

$$f_{Y|X}(y|x) = \frac{f_{X,Y}(x,y)}{f_X(x)} = \frac{\frac{1}{55}}{\sum_{y=1}^{x} \frac{1}{55}} = \frac{1}{x}$$

for $y = 1, \dots, x$. Thus,

$$\mathsf{E}(Y|X=x) = \sum_{y=1}^{x} y \cdot \frac{1}{x} = \frac{x(x+1)}{2x} = \frac{x+1}{2}$$

This is the best predictor of Y as a function of X since it minimizes the mean squared error by the orthogonality principle.

Problem 4. Show that the beta distributions form a two-parameter exponential family.

The pdf of a beta distribution is

$$f_X(x) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} x^{\alpha-1}(1-x)^{\beta-1}$$

for $0 < x < 1$, $\alpha > 0$, and $\beta > 0$. We can rewrite it as

$$\begin{aligned}
f_X(x) &= \exp\left(\ln\left(\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} x^{\alpha-1}(1-x)^{\beta-1}\right)\right) \\
&= \exp\left(\ln\left(\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}\right) + (\alpha-1)\ln(x) + (\beta-1)\ln(1-x)\right) \\
&= \exp\left((\alpha-1)\ln(x) + (\beta-1)\ln(1-x) + \ln\left(\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}\right)\right)
\end{aligned}$$

This is in the form of a two-parameter exponential family with

$$f(x) = c(\theta)h(x)e^{w_1(\theta)t_1(x)+w_2(\theta)t_2(x)}$$

where

$$c(\theta) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}, \quad h(x) = 1, \quad w_1(\theta) = \alpha-1, \quad w_2(\theta) = \beta-1, \quad t_1(x) = \ln(x), \quad t_2(x) = \ln(1-x)$$