Homework 2

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Stat 610 Distribution Theory

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Problem 1. *Statistical Inference* by Casella and Berger, 2nd Edition, Chapter 1, Exercise 34.

34. Two litters of a particular rodent species have been born, one with two brown-haired and one gray-haired (litter 1), and the other with three brown-haired and two gray-haired (litter 2). We select a litter at random and then select an offspring at random from the selected litter.

   (a) What is the probability that the animal chosen is brown-haired?

   (b) Given that a brown-haired offspring was selected, what is the probability that the sampling was from litter 1?

(a) Let A be the event that the animal chosen is brown-haired, Let B_i be the event that the sampling was from litter i, $i = 1, 2$. Then we have

$$\mathrm{P}(A) = \mathrm{P}(A|B_1)\mathrm{P}(B_1) + \mathrm{P}(A|B_2)\mathrm{P}(B_2) = \frac{2}{3} \cdot \frac{1}{2} + \frac{3}{5} \cdot \frac{1}{2} = \frac{19}{30}.$$

(b) By Bayes' theorem, we have

$$\mathrm{P}(B_1|A) = \frac{\mathrm{P}(A|B_1)\mathrm{P}(B_1)}{\mathrm{P}(A)} = \frac{\frac{2}{3} \cdot \frac{1}{2}}{\frac{19}{30}} = \frac{5}{19}.$$

Problem 2. (a) Prove this alternative to Bayes' rule:

$$\log\left(\frac{\mathrm{P}(A|B)}{\mathrm{P}(A^c|B)}\right) = \log\left(\frac{\mathrm{P}(A)}{\mathrm{P}(A^c)}\right) + \log\left(\frac{\mathrm{P}(B|A)}{\mathrm{P}(B|A^c)}\right).$$

This expression is useful in genetics: conditional log odds of disease (A) given gene (B) = unconditional log odds of disease + log ratio of gene prevalence.

(b) Suppose $B_1, \ldots, B_n$ are disjoint. Show that

$$\mathrm{P}\left(B_j | \bigcup_{i=1}^{n} B_i\right) = \frac{\mathrm{P}(B_j)}{\sum_{i=1}^{n} \mathrm{P}(B_i)}. \quad \text{for each } j \in \{1, \ldots, n\}.$$

(a) By definition, we have

$$\log\left(\frac{\mathrm{P}(A|B)}{\mathrm{P}(A^c|B)}\right) = \log\left(\frac{\frac{\mathrm{P}(A\cap B)}{\mathrm{P}(B)}}{\frac{\mathrm{P}(A^c\cap B)}{\mathrm{P}(B)}}\right) = \log\left(\frac{\mathrm{P}(A\cap B)}{\mathrm{P}(A^c\cap B)}\right).$$

By the definition of conditional probability again and $\log(AB) = \log(A) + \log(B)$, we have

$$\log\left(\frac{\mathrm{P}(A\cap B)}{\mathrm{P}(A^c\cap B)}\right) = \log\left(\frac{\mathrm{P}(A)\mathrm{P}(B|A)}{\mathrm{P}(A^c)\mathrm{P}(B|A^c)}\right) = \log\left(\frac{\mathrm{P}(A)}{\mathrm{P}(A^c)}\right) + \log\left(\frac{\mathrm{P}(B|A)}{\mathrm{P}(B|A^c)}\right).$$

(b) By definition, we have

$$\mathrm{P}\left(B_j|\bigcup_{i=1}^n B_i\right) = \frac{\mathrm{P}\left(B_j\cap\bigcup_{i=1}^n B_i\right)}{\mathrm{P}\left(\bigcup_{i=1}^n B_i\right)}.$$

Since $B_1,\ldots,B_n$ are disjoint, we have

$$\mathrm{P}\left(B_j\cap\bigcup_{i=1}^n B_i\right) = \mathrm{P}(B_j) \quad\text{and}\quad \mathrm{P}\left(\bigcup_{i=1}^n B_i\right) = \sum_{i=1}^n \mathrm{P}(B_i).$$

Thus we have

$$\mathrm{P}\left(B_j|\bigcup_{i=1}^n B_i\right) = \frac{\mathrm{P}(B_j)}{\sum_{i=1}^n \mathrm{P}(B_i)}.$$

Problem 3. Yule's Paradox. A new drug is introduced to combat a disease. Let R be the event that the drug is applied and I be the event the patient improves.

(a) Suppose the following proportions are reported, based on an experimental sample of patients.

	R	R^c	
I	0.412	0.276	0.688
I^c	0.155	0.157	0.312
	0.567	0.433	1.000

Find the chance the patient improves given that he takes the drug. Find the chance of improvement when he does not get the drug. What does this say about the efficacy of the drug?

(b) What the researchers *did not* report is that patients can easily be categorized into two groups: those in the early stages of the disease (E) and those not (E^c). The proportions then become as follows.

	$E\cap R$	$E\cap R^c$	$E^c\cap R$	$E^c\cap R^c$
I	0.392	0.216	0.020	0.060
I^c	0.077	0.039	0.078	0.118
	0.469	0.255	0.098	0.178

Now compare the chance of improvement with and without the drug just for the patients in the early stages. Do the same for the patients in the later stages. What do you think of the efficacy of the drug now? How should this experiment have been designed differently? ("Design" here means "number of patients chosen for each group in the study.")

(a) Let A be the event that the patient improves. Then we have

$$\mathrm{P}(A|R) = \frac{\mathrm{P}(A \cap R)}{\mathrm{P}(R)} = \frac{0.412}{0.567} \approx 0.7267, \quad \mathrm{P}(A|R^c) = \frac{\mathrm{P}(A \cap R^c)}{\mathrm{P}(R^c)} = \frac{0.276}{0.433} \approx 0.6374.$$

Thus the chance of improvement is higher when the patient takes the drug.

(b) Let E be the event that the patient is in the early stages of the disease. Then we have

$$\mathrm{P}(A|E \cap R) = \frac{\mathrm{P}(A \cap E \cap R)}{\mathrm{P}(E \cap R)} = \frac{0.392}{0.469} \approx 0.8358$$

$$\mathrm{P}(A|E \cap R^c) = \frac{\mathrm{P}(A \cap E \cap R^c)}{\mathrm{P}(E \cap R^c)} = \frac{0.216}{0.255} \approx 0.8471.$$

Thus for patients in the early stages, the chance of improvement is slightly higher when the patient does not take the drug.
Similarly, we have

$$\mathrm{P}(A|E^c \cap R) = \frac{\mathrm{P}(A \cap E^c \cap R)}{\mathrm{P}(E^c \cap R)} = \frac{0.020}{0.098} \approx 0.2041$$

$$\mathrm{P}(A|E^c \cap R^c) = \frac{\mathrm{P}(A \cap E^c \cap R^c)}{\mathrm{P}(E^c \cap R^c)} = \frac{0.060}{0.178} \approx 0.3371.$$

Thus for patients in the later stages, the chance of improvement is higher when the patient does not take the drug.

Based on the above results, it seems that the drug is not effective.

The experiment should have been designed to have the same proportion of patients in the early stages and in the later stages for both groups (taking the drug and not taking the drug).

Problem 4. Consider sampling with replacement from a population of size N in which M individuals would respond Yes (and $N - M$ would respond No).

(a) Show that the first and second responses are independent. As the responses are binary, it suffices to show that events $A_i = \{$"i-th response is Yes"$\}$, are independent, $i = 1, 2$.

(b) Show that the n responses for a sample of size n are *mutually* independent. As in part (a), you can simply show that $A_1, \ldots, A_n$ are mutually independent.

(a) We have

$$\mathrm{P}(A_1) = \frac{M}{N}, \quad \mathrm{P}(A_2) = \frac{M}{N}, \quad \mathrm{P}(A_1 \cap A_2) = \frac{M}{N} \cdot \frac{M}{N} = \mathrm{P}(A_1)\mathrm{P}(A_2).$$

Thus A_1 and A_2 are independent.

(b) For any k events $A_{i_1}, A_{i_2}, \ldots, A_{i_k}$ where $1 \leq i_1 < i_2 < \cdots < i_k \leq n$, we have

$$\mathrm{P}(A_{i_1} \cap A_{i_2} \cap \cdots \cap A_{i_k}) = \left(\frac{M}{N}\right)^k = \mathrm{P}(A_{i_1})\mathrm{P}(A_{i_2}) \cdots \mathrm{P}(A_{i_k}).$$

Thus $A_1, A_2, \ldots, A_n$ are mutually independent.

Problem 5. Consider rolling two fair dice.

(a) Define events $A =$ "the first die is 6", $B =$ "the second die is 2", $C =$ "total of the dice is 7". Are the events pairwise independent? Are they mutually independent?

(b) Now consider $A =$ "the first die is even", $B =$ "the two dice are the same", $C =$ "total is 8, 9, or 10". Are the events pairwise independent? Are they mutually independent?

(c) Suppose A, B, C are *pairwise* independent events, not necessarily mutually independent. Must A be independent of $B \cap C$ or of $B \cup C$? Prove or show counterexamples.

(a) We have

$$\mathrm{P}(A) = \frac{1}{6}, \quad \mathrm{P}(B) = \frac{1}{6}, \quad \mathrm{P}(C) = \frac{1}{6}.$$

$$\mathrm{P}(A \cap B) = \frac{1}{36} = \mathrm{P}(A)\mathrm{P}(B), \quad \mathrm{P}(A \cap C) = \frac{1}{36} = \mathrm{P}(A)\mathrm{P}(C), \quad \mathrm{P}(B \cap C) = \frac{1}{36} = \mathrm{P}(B)\mathrm{P}(C).$$

Thus A, B, C are pairwise independent. However, we have

$$\mathrm{P}(A \cap B \cap C) = 0 \neq \mathrm{P}(A)\mathrm{P}(B)\mathrm{P}(C).$$

Thus A, B, C are not mutually independent.

(b) We have

$$\mathrm{P}(A) = \frac{1}{2}, \quad \mathrm{P}(B) = \frac{1}{6}, \quad \mathrm{P}(C) = \frac{1}{3}.$$

$$\mathrm{P}(A \cap B) = \frac{1}{12} = \mathrm{P}(A)\mathrm{P}(B), \quad \mathrm{P}(A \cap C) = \frac{1}{6} = \mathrm{P}(A)\mathrm{P}(C), \quad \mathrm{P}(B \cap C) = \frac{1}{18} = \mathrm{P}(B)\mathrm{P}(C).$$

Thus A, B, C are pairwise independent. However, we have

$$\mathrm{P}(A \cap B \cap C) = \frac{1}{36} \neq \mathrm{P}(A)\mathrm{P}(B)\mathrm{P}(C).$$

Thus A, B, C are not mutually independent.

(c) No. Consider the following counterexample. Let $A = \{1,2\}, B = \{1,3\}, C = \{2,3\}$. Then we have

$$\mathrm{P}(A) = \mathrm{P}(B) = \mathrm{P}(C) = \frac{1}{3}, \quad \mathrm{P}(A \cap B) = \mathrm{P}(A \cap C) = \mathrm{P}(B \cap C) = \frac{1}{6}.$$

Thus A, B, C are pairwise independent. However, we have

$$\mathrm{P}(A \cap B \cap C) = 0 \neq \mathrm{P}(A)\mathrm{P}(B)\mathrm{P}(C).$$

Thus A, B, C are not mutually independent.
Also, we have

$$\begin{aligned}\mathrm{P}(A \cap (B \cap C)) &= 0 \neq \mathrm{P}(A)\mathrm{P}(B \cap C), \\ \mathrm{P}(A \cap (B \cup C)) &= \frac{1}{3} \neq \mathrm{P}(A)\mathrm{P}(B \cup C).\end{aligned}$$

Thus A is neither independent of $B \cap C$ nor of $B \cup C$.

Problem 6. Two shaved dice are such that the chance of a 1 or 6 is 1/5 each, and the chance of a 2, 3, 4 or 5 is 3/20 each.

(a) Find the chance of a total of 7 is obtained when the dice are rolled.

(b) Now suppose the dice are repeatedly rolled independently for a total of 6 times. Let Y be the number of times that a total of 7 is obtained. Find the pmf and cdf of Y and plot both. (A line graph works best for a pmf: draw a line or narrow bar of height $f_Y(x)$ at value x. Also, be sure to explicitly indicate the cdf's value at each of the jump points with, say, a small darkened circle.)

(a) Let A be the event that a total of 7 is obtained when the dice are rolled. Then we have

$$\begin{aligned}\mathrm{P}(A) &= \mathrm{P}((1,6)) + \mathrm{P}((2,5)) + \mathrm{P}((3,4)) + \mathrm{P}((4,3)) + \mathrm{P}((5,2)) + \mathrm{P}((6,1)) \\ &= 2 \cdot \frac{1}{5} \cdot \frac{1}{5} + 2 \cdot \frac{3}{20} \cdot \frac{3}{20} + 2 \cdot \frac{3}{20} \cdot \frac{3}{20} \\ &= \frac{17}{100}.\end{aligned}$$

(b) Since each roll is independent and the chance of a total of 7 is $\frac{17}{100}$, we have

$$Y \sim \text{Binomial}\left(n = 6, p = \frac{17}{100}\right).$$

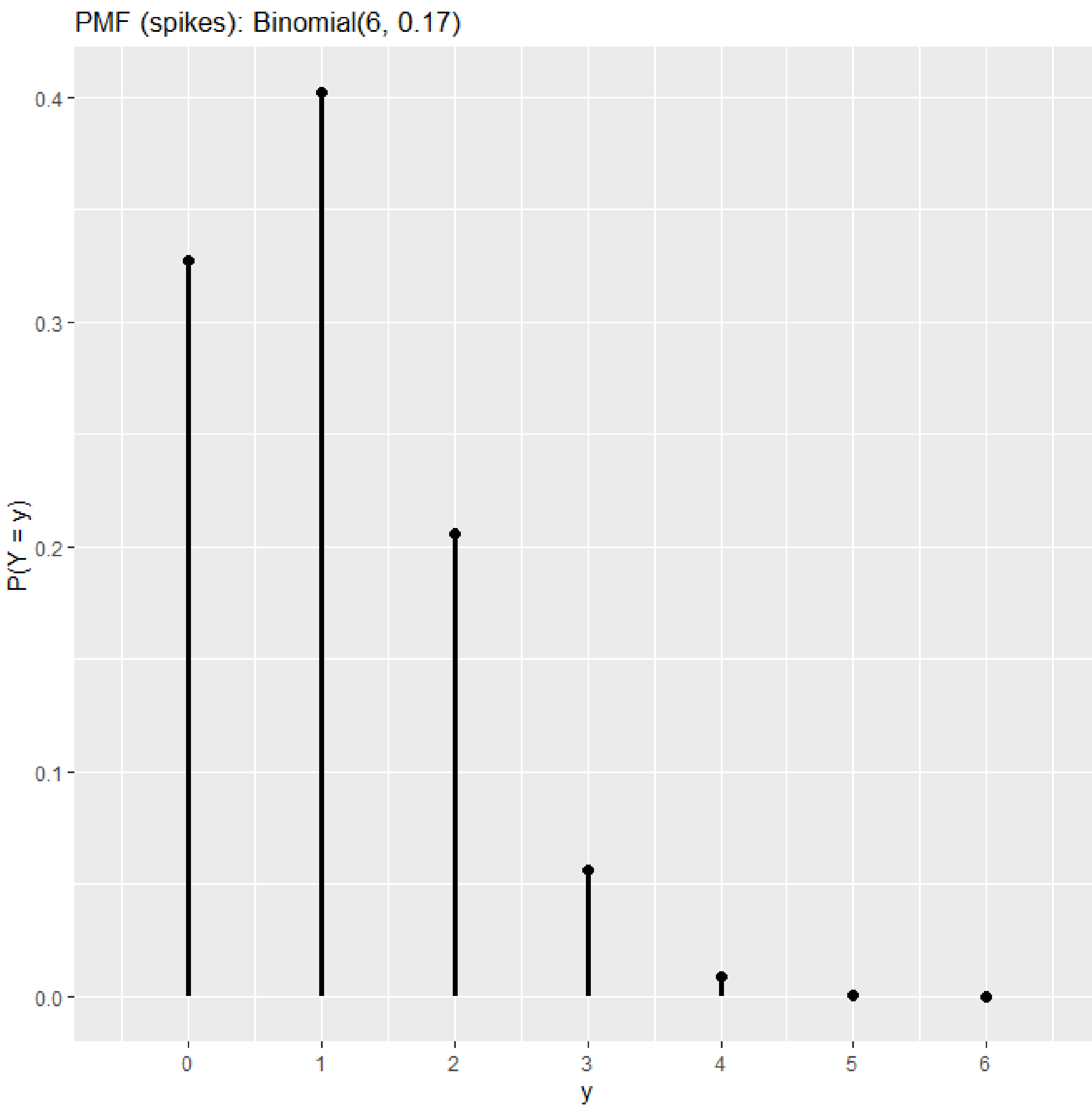
Thus the pmf of Y is given by

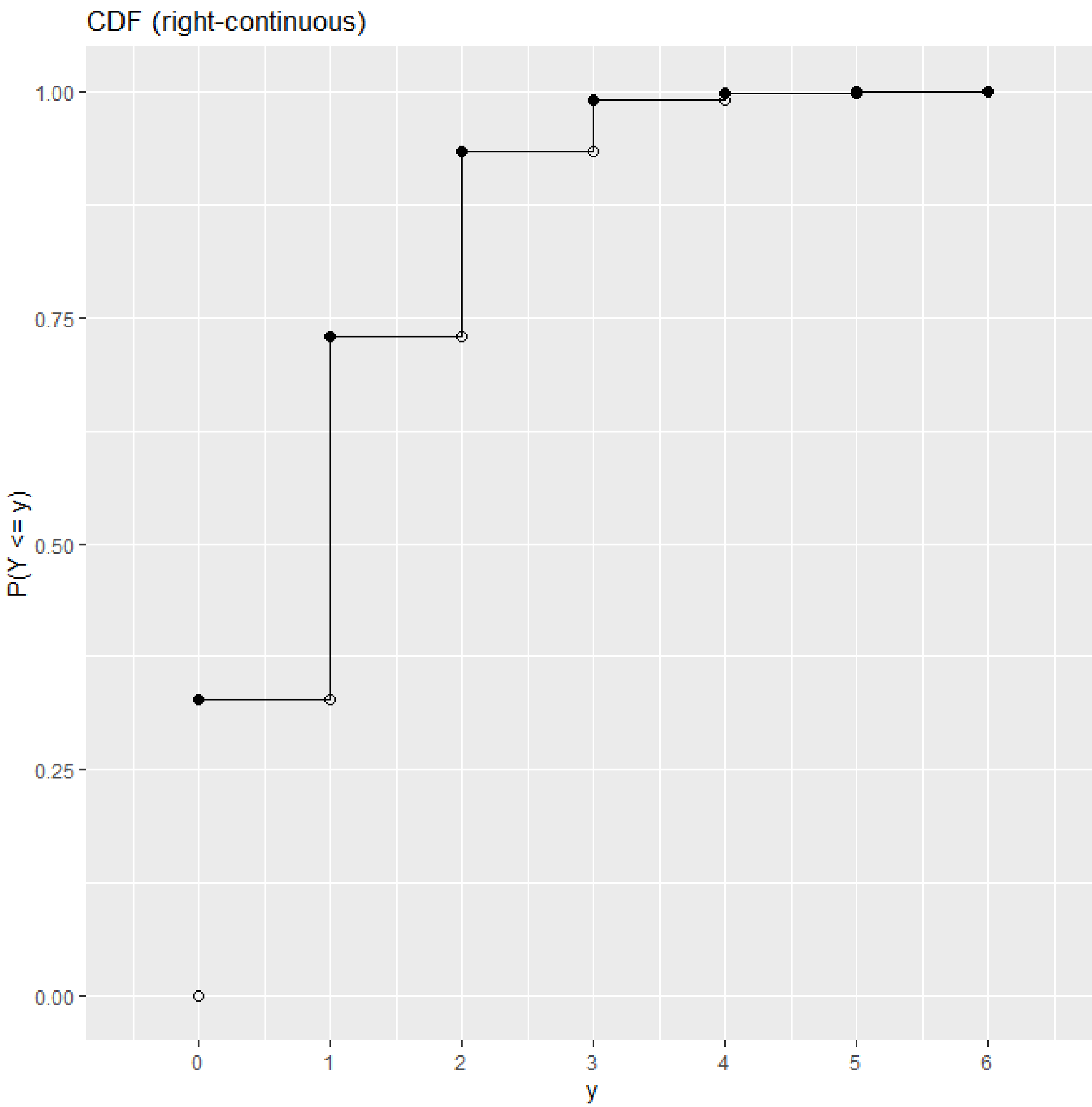
$$f_Y(y) = \binom{6}{y} \left(\frac{17}{100}\right)^y \left(1 - \frac{17}{100}\right)^{6-y}, \quad y = 0, 1, 2, 3, 4, 5, 6.$$

The cdf of Y is given by

$$F_Y(y) = \sum_{k=0}^{\lfloor y \rfloor} f_Y(k) = \sum_{k=0}^{\lfloor y \rfloor} \binom{6}{k} \left(\frac{17}{100}\right)^k \left(1 - \frac{17}{100}\right)^{6-k}, \quad y \geq 0.$$

The following are the plots of the pmf and cdf of Y.


PMF (spikes): Binomial(6, 0.17)
0.4
0.3
0.2
0.1
0.0
P(Y = y)
0
1
2
3
4
5
6
y



Problem 7. *Statistical Inference* by Casella and Berger, 2nd Edition, Chapter 1, Exercise 51.

51. An appliance store receives a shipment of 30 microwave ovens, 5 of which are (unknown to the manager) defective. The store manager selects 4 ovens at random, without replacement, and tests to see if they are defective. Let X = number of defectives found. Calculate the pmf and cdf of X and plot the pmf and cdf.

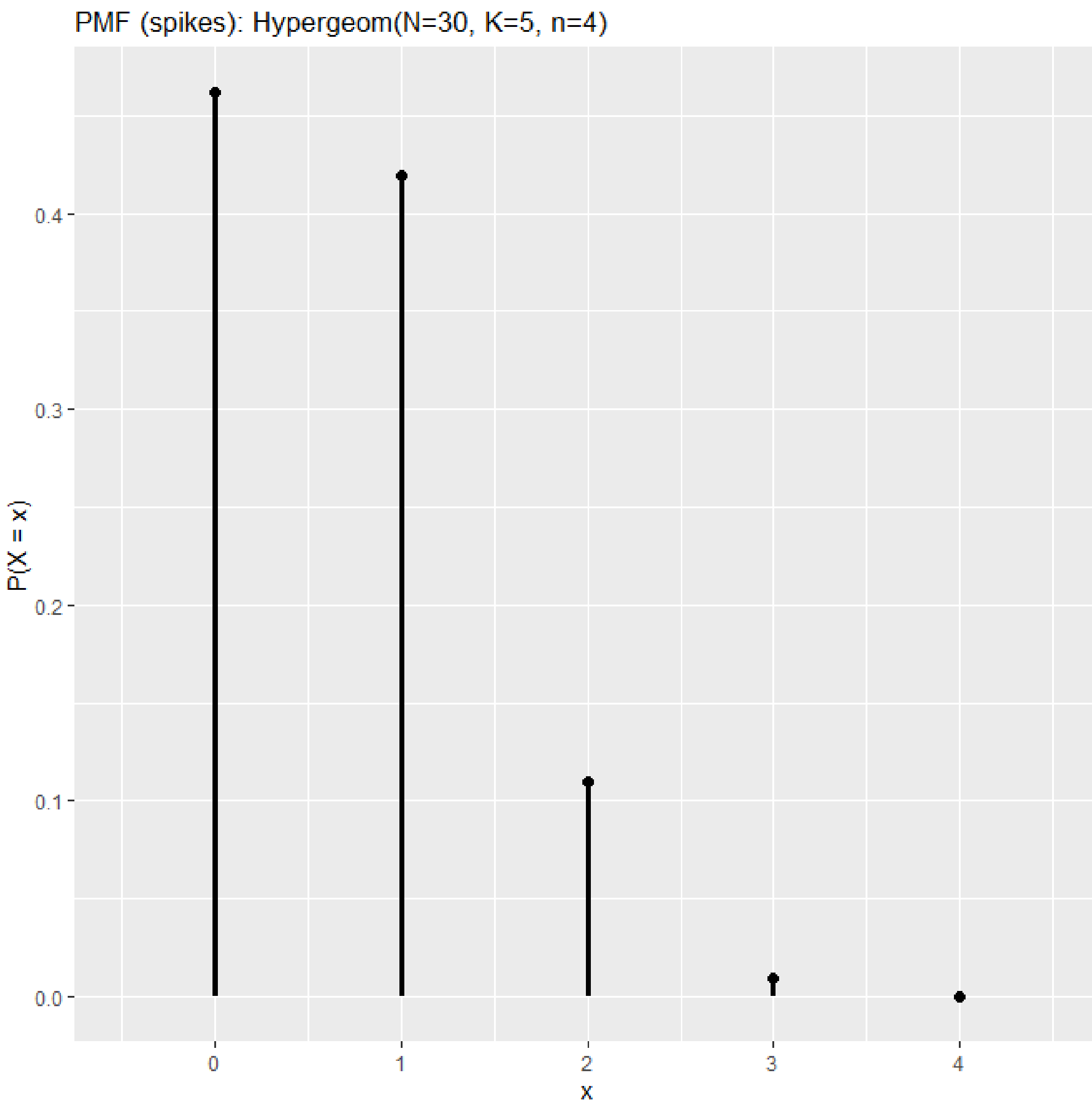
    The pmf of X is given by

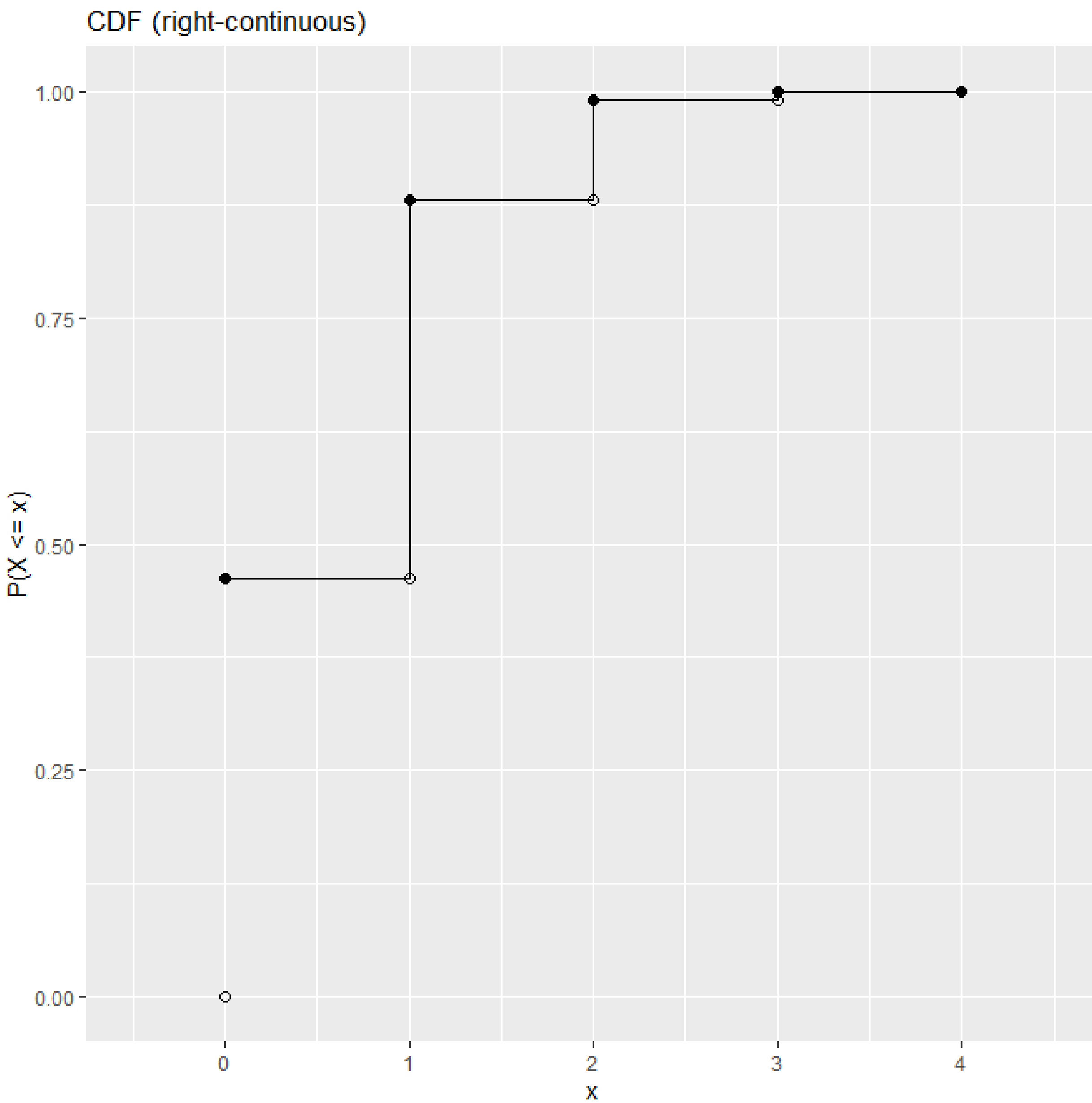
$$f_X(x) = \frac{\binom{5}{x}\binom{25}{4-x}}{\binom{30}{4}}, \quad x = 0, 1, 2, 3, 4.$$

The cdf of X is given by

$$F_X(x) = \sum_{k=0}^{\lfloor x \rfloor} f_X(k) = \sum_{k=0}^{\lfloor x \rfloor} \frac{\binom{5}{k}\binom{25}{4-k}}{\binom{30}{4}}, \quad x \geq 0.$$

The following are the plots of the pmf and cdf of X.





Problem S. uppose $g(x) = 4\left(\frac{1}{3}\right)^x + \left(\frac{2}{3}\right)^x$ for $x = 1, 2, \ldots$, and $g(x) = 0$ otherwise.

(a) Find $c > 0$ such that $f(x) = cg(x)$ is a valid pmf.

(b) Find the corresponding cdf.

(c) Consider this game: A fair coin is flipped. If a Head appears then a fair die is rolled until either a 1 or a six appears. If a Tail is flipped then the die is rolled until a 2, 3, 4, or 5 appears. Let X be the number of rolls and show that f (the function in part (a)) is its pmf.

(a) Since

$$\sum_{x=1}^{\infty} g(x) = 4\sum_{x=1}^{\infty}\left(\frac{1}{3}\right)^x + \sum_{x=1}^{\infty}\left(\frac{2}{3}\right)^x = 4\cdot\frac{\frac{1}{3}}{1-\frac{1}{3}} + \frac{\frac{2}{3}}{1-\frac{2}{3}} = 4\cdot\frac{1}{2} + 2 = 4,$$

we need $c = \frac{1}{4}$ for $f = cg$ to be a pmf, since we need $\sum f = 1$.

(b) The cdf is

$$F(x) = \sum_{k=1}^{\lfloor x \rfloor} f(k) = \frac{1}{2}\left(1-\left(\frac{1}{3}\right)^{\lfloor x \rfloor}\right) + \frac{1}{2}\left(1-\left(\frac{2}{3}\right)^{\lfloor x \rfloor}\right) = 1-\frac{1}{2}\left[\left(\frac{1}{3}\right)^{\lfloor x \rfloor} + \left(\frac{2}{3}\right)^{\lfloor x \rfloor}\right],$$

with $F(x) = 0$ for $x < 1$.

(c) We have

$$\mathrm{P}(X = x|H) = \left(1-\frac{2}{6}\right)^{x-1}\frac{2}{6} = \left(\frac{2}{3}\right)^{x-1}\left(\frac{1}{3}\right) = \frac{1}{2}\left(\frac{2}{3}\right)^x,$$

and

$$\mathrm{P}(X = x|T) = \left(1-\frac{4}{6}\right)^{x-1}\frac{4}{6} = \left(\frac{1}{3}\right)^{x-1}\left(\frac{2}{3}\right) = 2\left(\frac{1}{3}\right)^x.$$

With $\mathrm{P}(H) = \mathrm{P}(T) = \frac{1}{2}$, we get

$$\mathrm{P}(X = x) = \frac{1}{2}\mathrm{P}(X = x|H) + \frac{1}{2}\mathrm{P}(X = x|T) = \frac{1}{4}\left(\frac{2}{3}\right)^x + \left(\frac{1}{3}\right)^x = f(x), \quad x = 1, 2, \ldots$$