Homework 3

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Stat 610 Distribution Theory

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Problem 1. *Statistical Inference* by Casella and Berger, 2nd Edition, Chapter 1, Exercise 47 parts b and c.

47. Prove the following functions are cdfs.

  (b) $(1+e^{-x})^{-1}$, $x \in (-\infty, \infty)$
  (c) $e^{-e^{-x}}$, $x \in (-\infty, \infty)$

Theorem 2.6. $F(x)$ is a cdf for some rv X if and only if

  i. $x \le y \Rightarrow F(x) \le F(y)$ (nondecreasing)
  ii. $F(y) \to F(x)$ as $y \downarrow x$ (right-continuous)
  iii. $F(x) \to 0$ as $x \to -\infty$ and $F(x) \to 1$ as $x \to \infty$ (total measure 1)

(b) i. $F'(x) = \frac{e^{-x}}{(1+e^{-x})^2}$, which is nonnegative for all x, so F is nondecreasing.
  ii. Since F is differentiable everywhere, it is continuous everywhere, so it is also right continuous.
  iii.
$$\lim_{x\to-\infty} \frac{1}{1+e^{-x}} = \frac{1}{\infty} = 0.$$
$$\lim_{x\to\infty} \frac{1}{1+e^{-x}} = \frac{1}{1} = 1.$$

(c) i. $F'(x) = e^{-x}e^{-e^{-x}}$, which is nonnegative for all x, so F is nondecreasing.
  ii. Since F is differentiable everywhere, it is continuous everywhere, so it is also right continuous.
  iii.
$$\lim_{x\to-\infty} e^{-e^{-x}} = e^{-e^{\infty}} = e^{-\infty} = 0.$$
$$\lim_{x\to\infty} e^{-e^{-x}} = e^{-e^{-\infty}} = e^{0} = 1.$$

Problem 2. Suppose we sample 10 individuals at random and with replacement from a population that has three categories C_1, C_2, C_3 with respective proportions 10%, 30% and 60%. Let X_i be the number in the sample from C_i, $i = 1, 2, 3$. Then (recall Example 1.11 on Slide 31 in the notes)

$$\mathrm{P}(X_1 = i, X_2 = j \text{ and } X_3 = k) = \frac{10!}{i!j!k!} 0.1^i 0.3^j 0.6^k, \quad \text{if } i + j + k = 10,\ i \geq 0,\ j \geq 0,\ k \geq 0.$$

(a) Determine the probability that X_1 and X_2 are both equal 1 (keeping in mind what X_3 must be).

(b) Find the probability that $X_1 = X_2$ by splitting the event into disjoint cases according to their common value, and then get the conditional probability that $X_1 = X_2 = 1$, given $X_1 = X_2$. (Recall Problem 2(b) of Assignment 2.)

(c) Call C_1 "Success" and C_1^c "Failure" to argue that X_1 has binomial(10, 0.1) distribution. Draw a similar conclusion about X_2.

(d) Are the events $\{X1 = 1\}$ and $\{X_2 = 1\}$ independent? Explain.

(a) Since $i + j + k = 10$, $X_3 = k = 10 - i - j = 8$.

$$\mathrm{P}(X_1 = 1, X_2 = 1, X_3 = 8) = \frac{10!}{1!1!8!}(0.1)(0.3)(0.6^8) = 90(0.1)(0.3)(0.6^8) \approx 0.0453.$$

(b) Let the common value be $t = X_1 = X_2$, then $X_3 = 10 - 2t$.
Since $X_3 \geq 0$, thus $t = 0, 1, 2, 3, 4, 5$.

$$\mathrm{P}(X_1 = X_2 = t, X_3 = 10 - 2t) = \sum_{t=0}^{5} \frac{10!}{t!t!(10-2t)!} 0.1^t 0.3^t 0.6^{10-2t} \approx 0.120.$$

$$\mathrm{P}(X_1 = X_2 = 1 \mid X_1 = X_2) = \frac{\mathrm{P}(X_1 = 1, X_2 = 1)}{\mathrm{P}(X_1 = X_2)} \approx \frac{0.0453}{0.120} \approx 0.378.$$

(c) Declare "Success" = draw is in C_1 (prob 0.1), "Failure" = not in C_1 (prob 0.9). Across 10 independent draws

$$\mathrm{P}(X_1 = k) = \binom{10}{k} 0.1^k 0.9^{10-k}, \quad k = 0, 1, \ldots, 10.$$

So $X_1 \sim$ binomial(10, 0.1).
Similarly, declare "Success" = draw is in C_2 (prob 0.3), "Failure" = not in C_2 (prob 0.7). Across 10 independent draws

$$\mathrm{P}(X_2 = k) = \binom{10}{k} 0.3^k 0.7^{10-k}, \quad k = 0, 1, \ldots, 10.$$

So $X_2 \sim$ binomial(10, 0.3).

(d)

$$\mathrm{P}(X_1 = 1) = \binom{10}{1} 0.1^1 0.9^9 \approx 0.387.$$

$$\mathrm{P}(X_2 = 1) = \binom{10}{1} 0.3^1 0.7^9 \approx 0.121.$$

$$\mathrm{P}(X_1 = 1)\mathrm{P}(X_2 = 1) \approx 0.387 \times 0.121 \approx 0.047.$$

$$\mathrm{P}(X_1 = 1, X_2 = 1) \approx 0.0453 \neq \mathrm{P}(X_1 = 1)\mathrm{P}(X_2 = 1) \approx 0.047.$$

So the events $\{X_1 = 1\}$ and $\{X_2 = 1\}$ are not independent.

Problem 3. Determine whether the following functions are cdfs or not, explaining your reasoning. If it is a cdf, is it discrete, continuous or a mixture of the two? You may assume the interval given for the variable is the ostensible support (no probability outside that interval). It may help to plot the functions.

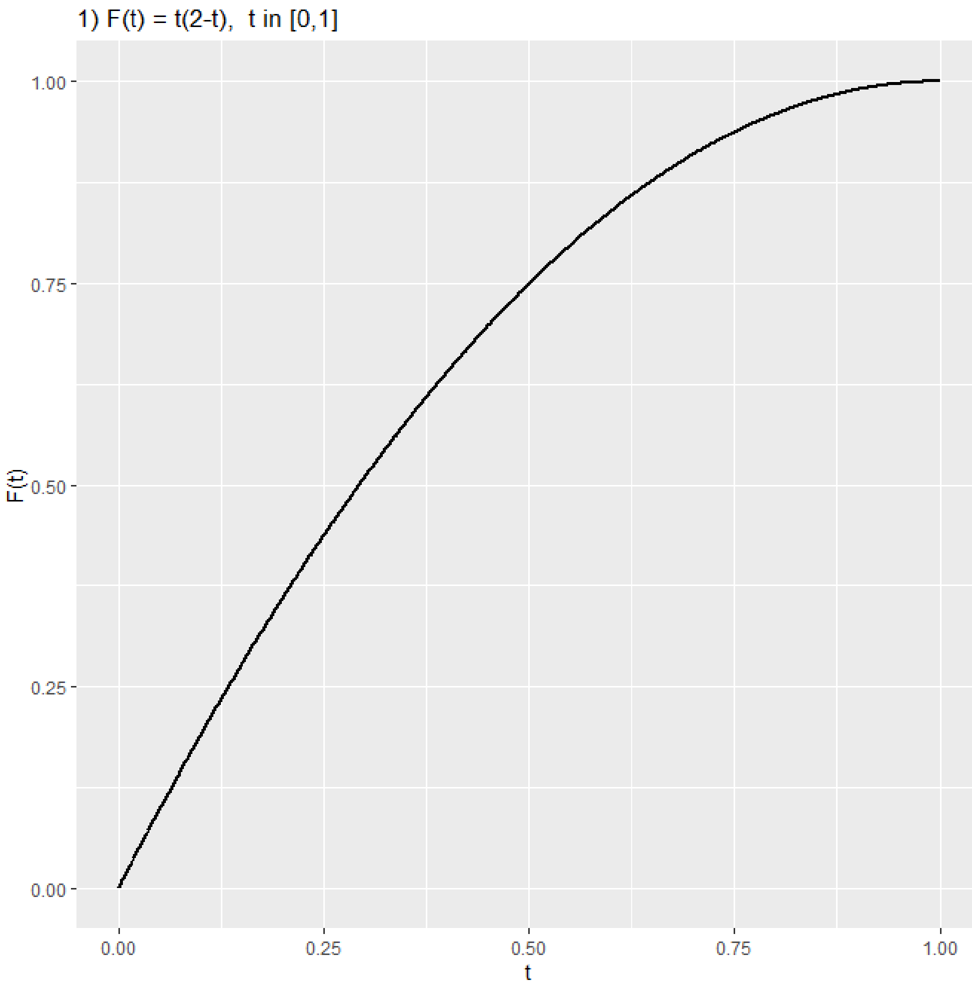
(a) $F(t) = t(2 - t)$ for $t \in [0, 1]$.

(b) $F(t) = t(2 - t)$ for $t \in [0, 2]$.

(c) $F(t) = t(2 - t)1_{[0,1/2)}(t) + \frac{t+7}{8}1_{[1/2,1]}(t)$ for $t \in [0, 1]$.

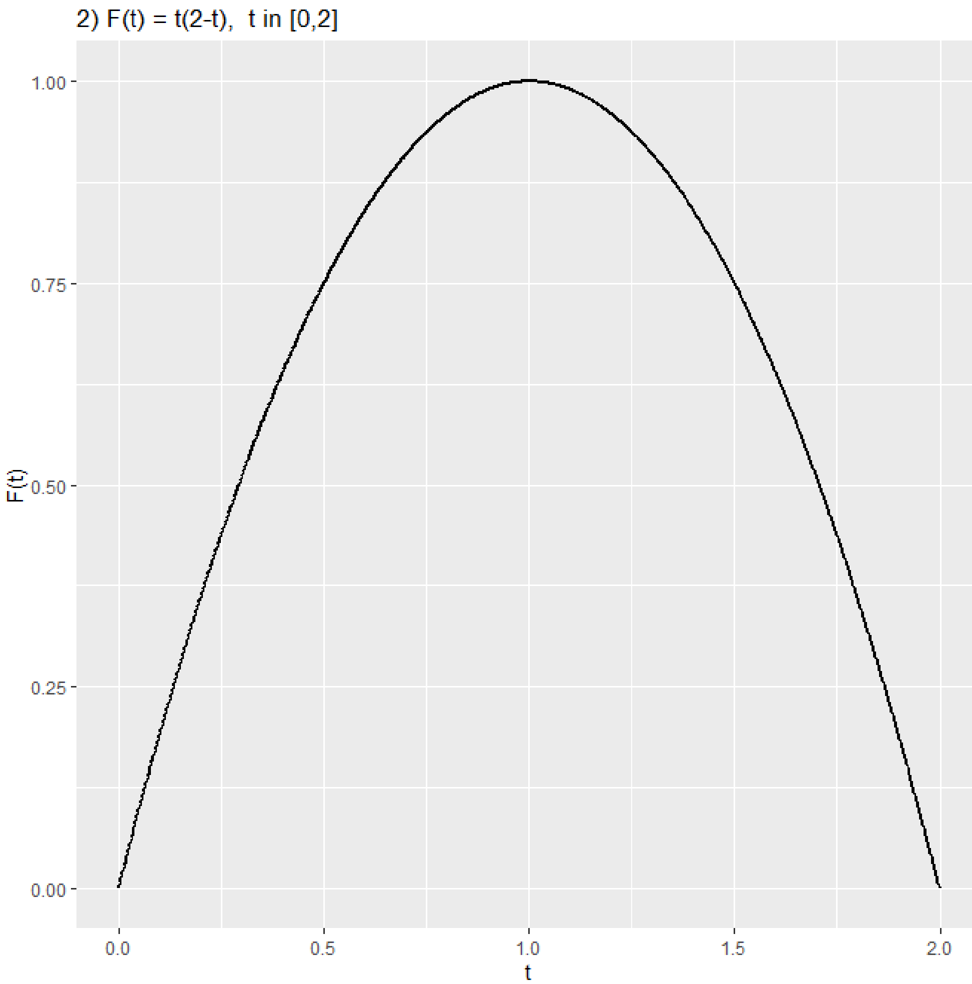
(d) $F(t) = \frac{1}{5}1_{[0,\infty)}(t) + \frac{1}{4}1_{[1/2,\infty)}(t) + \frac{1}{2}1_{[3/4,\infty)}(t) + \frac{1}{20}1_{[1,\infty)}(t)$ for $t \in [0, 1]$.

(a) graph of F:



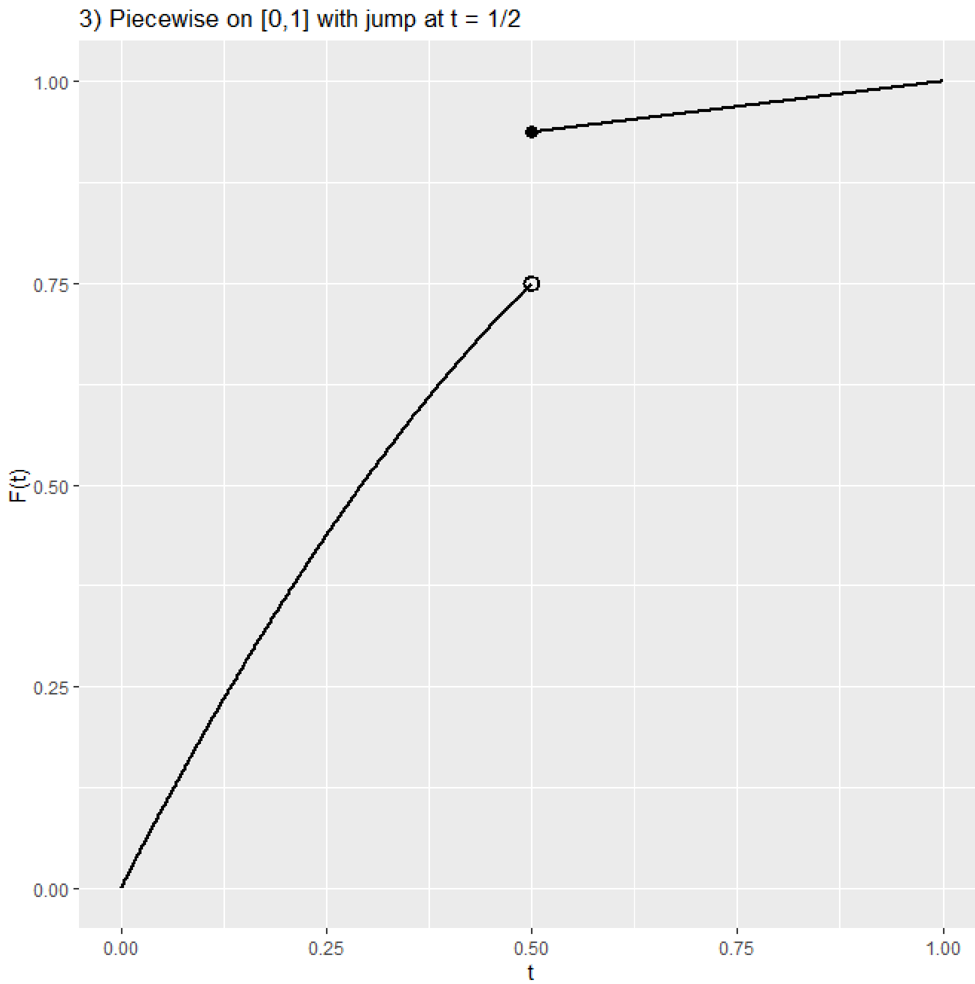
Assuming $F(t) = 0$ for $t < 0$ and $t > 1$, then F fails both right-continuity and total measure 1. So F is not a cdf. However, if we assume $F(t) = 0$ for $t < 0$ and $F(t) = 1$ for $t > 1$, then F is a cdf, which is continuous.

(b) graph of F:



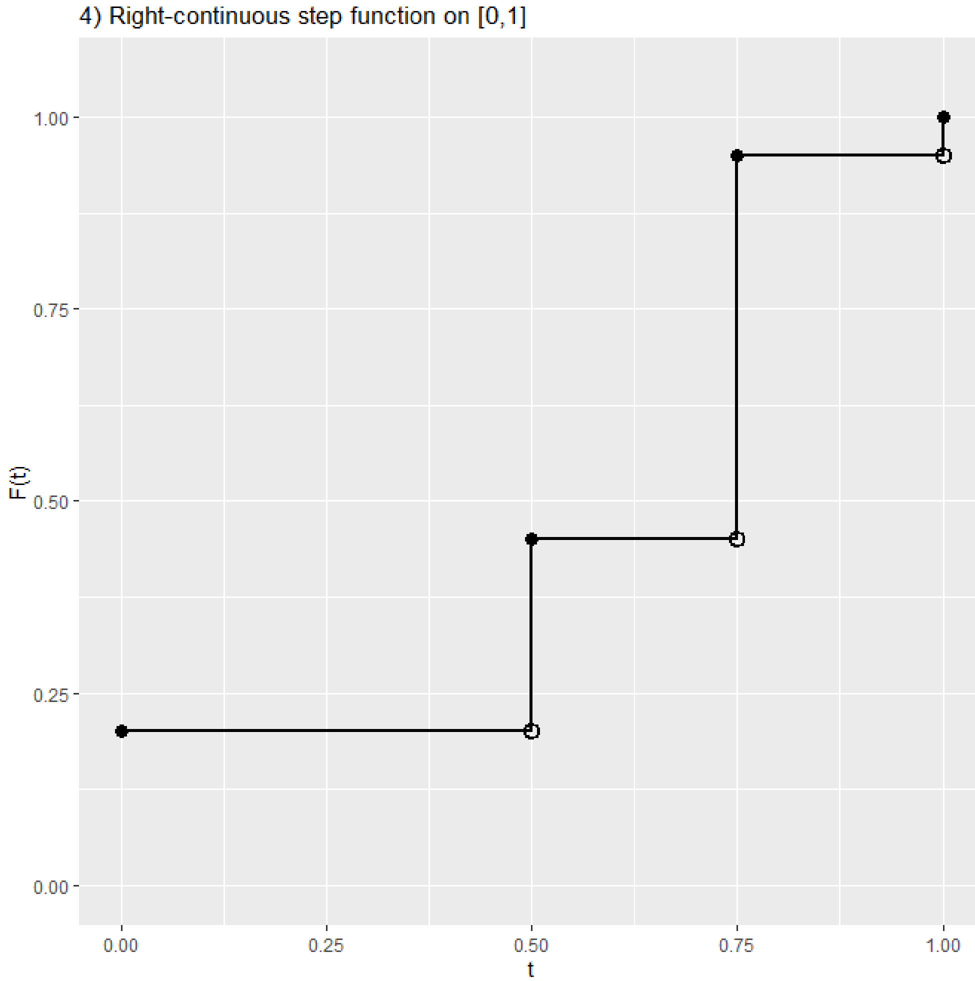
F fails nondecreasing, since $F'(t) = 2 - 2t < 0$ for $t > 1$, so F is not a cdf.

(c) graph of F:



Assuming $F(t) = 0$ for $t < 0$ and $t > 1$, then F fails both right-continuity and total measure 1. So F is not a cdf. However, if we assume $F(t) = 0$ for $t < 0$ and $F(t) = 1$ for $t > 1$, then F is a cdf, which is a mixture of continuous and discrete.

(d) graph of F:



F satisfies all three properties of a cdf, so F is a cdf, which is discrete.

Problem 4. Let U have uniform(0,1) distribution (see Example 2.6 in the notes). Let $V = \sqrt{U}$. Show that V is a continuous random variable and find its cdf. Find the pdf and plot both the cdf and the pdf.

Since U is continuous, and $V = \sqrt{U}$ is a continuous function, so V is also continuous. Because $g(u) = \sqrt{u}$ is nondecreasing, we can get the cdf of V by inverting g:

$$g^{-1}(v) = v^2, \quad v \in [0, 1].$$

For $v \in [0, 1]$,

$$F_V(v) = \mathrm{P}(V \leq v) = \mathrm{P}(\sqrt{U} \leq v) = \mathrm{P}(U \leq v^2) = F_U(v^2).$$

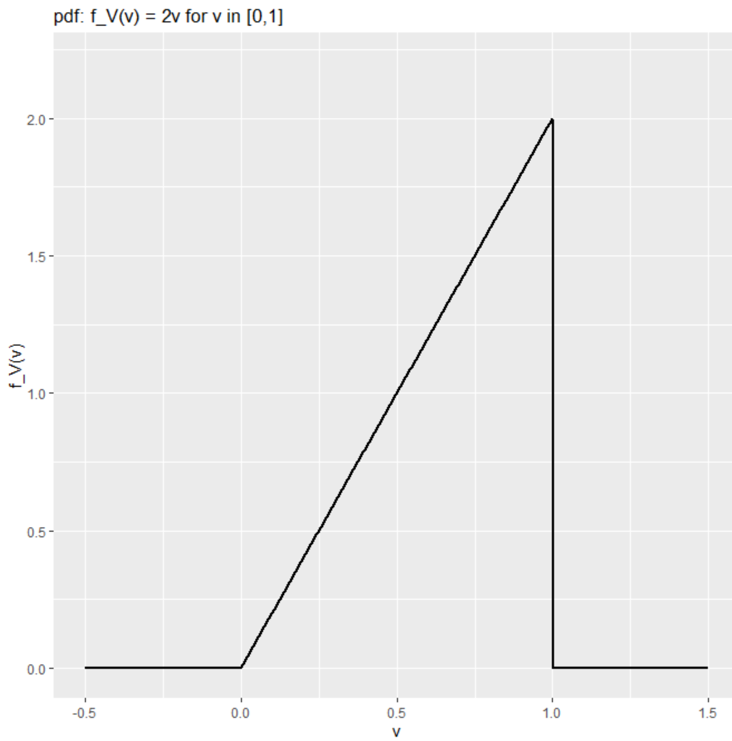
However, it must be noted that $F_U(u) = 0$ for $u < 0$ and $F_U(u) = 1$ for $u > 1$. So the cdf of V is

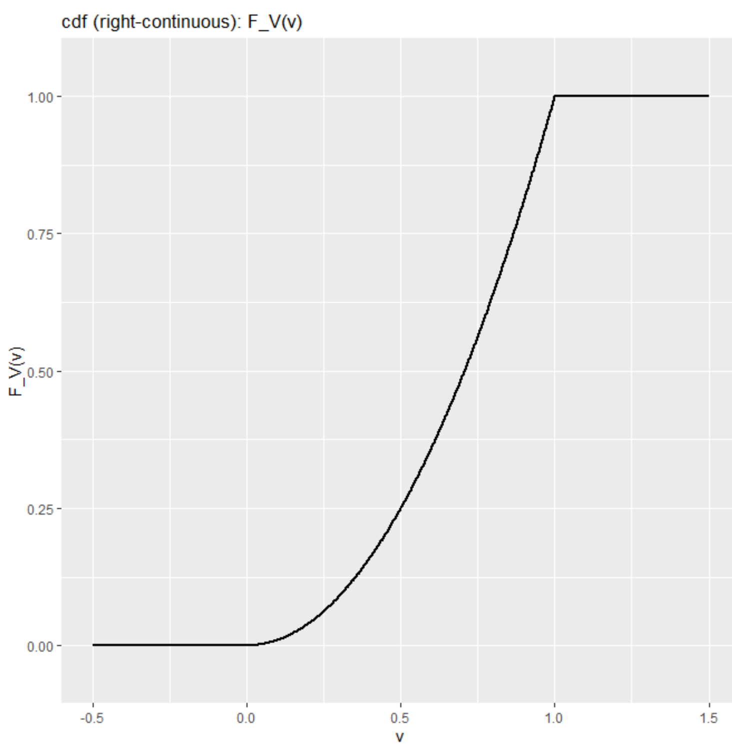
$$F_V(v) = v^2 1_{[0,1]}(v) + 0 \cdot 1_{(-\infty,0)}(v) + 1 \cdot 1_{(1,\infty)}(v).$$

The pdf of V is

$$f_V(v) = F'_V(v) = 2v1_{[0,1]}(v).$$

The graphs of the pdf and cdf of V are shown below.





Problem 5. Suppose T_1 and T_2 are two random variables such that $T_1(s) \le T_2(s)$ for all outcomes s in the sample space. Prove that $F_{T_1}(t) \ge F_{T_2}(t)$ for all real t. Hint: compare appropriate events about the random variables.

For any real t, let $A = \{s : T_1(s) \le t\}$ and $B = \{s : T_2(s) \le t\}$. If $s \in B$, then $T_2(s) \le t$. Since $T_1(s) \le T_2(s)$, so $T_1(s) \le t$ implies $s \in A$. Thus $B \subseteq A$. Thus $\mathrm{P}(A) \ge \mathrm{P}(B)$, i.e.

$$F_{T_1}(t) = \mathrm{P}(T_1 \le t) \ge \mathrm{P}(T_2 \le t) = F_{T_2}(t).$$

Problem 6. *Statistical Inference* by Casella and Berger, 2nd Edition, Chapter 2, Exercise 1 parts a and b.

1. In each of the following, find the pdf of Y. Show that the pdf integrates to 1.

   (a) $Y = X^3$ and $f_X(x) = 42x^5(1-x),\ 0 < x < 1$.

   (b) $Y = 4X + 3$ and $f_X(x) = 7e^{-7x},\ 0 < x < \infty$.

   (a) $Y = X^3 \Rightarrow X = Y^{1/3}$, and

   $$\frac{dX}{dY} = \frac{1}{3}Y^{-2/3}.$$

   The support of Y is $(0, 1)$.

   $$f_Y(y) = f_X(y^{1/3})\left|\frac{dX}{dY}\right| = 42(y^{1/3})^5(1-y^{1/3}) \cdot \frac{1}{3}y^{-2/3} = 14y(1-y^{1/3}), \quad 0 < y < 1.$$

   $$\begin{aligned}\int_0^1 14y(1-y^{1/3})dy &= \int_0^1 14(y - y^{4/3})dy \\ &= \left[7y^2 - \frac{14}{\frac{4}{3}+1}y^{\frac{4}{3}+1}\right]_0^1 \\ &= 7 - \frac{14}{\frac{4}{3}+1} = 7 - 6 = 1.\end{aligned}$$

   (b) $Y = 4X + 3 \Rightarrow X = \frac{Y-3}{4}$, and

   $$\frac{dX}{dY} = \frac{1}{4}.$$

   The support of Y is $(3, \infty)$.

   $$f_Y(y) = f_X\left(\frac{y-3}{4}\right)\left|\frac{dX}{dY}\right| = 7e^{-7\frac{y-3}{4}} \cdot \frac{1}{4} = \frac{7}{4}e^{-\frac{7}{4}(y-3)}, \quad y > 3.$$

   $$\int_3^\infty \frac{7}{4}e^{-\frac{7}{4}(y-3)}dy = \left[-e^{-\frac{7}{4}(y-3)}\right]_3^\infty = 0 - (-1) = 1.$$

Problem 7. Let X be a continuous random variable, *not necessarily nonnegative,* with cdf F_X and pdf f_X.

(a) Find the cdf for $Y = |X|$ and use it to find the pdf for Y. Be sure that both satisfy the necessary properties.

(b) Apply the above to the standard normal density $f_X(x) = \frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}$. (This gives the so-called half-normal pdf.)

(a) For $y < 0$, $F_Y(y) = \mathrm{P}(Y \le y) = 0$.
For $y \ge 0$, $F_Y(y) = \mathrm{P}(Y \le y) = \mathrm{P}(|X| \le y) = \mathrm{P}(-y \le X \le y) = F_X(y) - F_X(-y)$.
Thus, the pdf of Y is

$$f_Y(y) = F_Y'(y) = f_X(y) + f_X(-y), \quad y \ge 0.$$

$f_Y(y) \ge 0$ for all y since f_X is nonnegative and

$$\begin{aligned}\int_{-\infty}^{\infty} f_Y(y)dy &= \int_0^{\infty} (f_X(y) + f_X(-y))dy \\ &= \int_0^{\infty} f_X(y)dy + \int_{-\infty}^{0} f_X(y)dy \\ &= \int_{-\infty}^{\infty} f_X(y)dy = 1.\end{aligned}$$

(b) Plugging in the pdf of X gives

$$f_Y(y) = F_Y'(y) = f_X(y) + f_X(-y) = \frac{1}{\sqrt{2\pi}}e^{-\frac{y^2}{2}} + \frac{1}{\sqrt{2\pi}}e^{-\frac{(-y)^2}{2}} = \frac{2}{\sqrt{2\pi}}e^{-\frac{y^2}{2}}, \quad y \ge 0.$$

$f_Y(y) \ge 0$ for all y and

$$\begin{aligned}\int_{-\infty}^{\infty} f_Y(y)dy &= \int_0^{\infty} \frac{2}{\sqrt{2\pi}}e^{-\frac{y^2}{2}}dy \\ &= 2\int_0^{\infty} \frac{1}{\sqrt{2\pi}}e^{-\frac{y^2}{2}}dy \\ &= 2 \cdot \frac{1}{2} = 1.\end{aligned}$$

Problem 8. Let $h(x) = xe^{-2x}$ for $x > 0$, and $h(x) = 0$ otherwise.

(a) Find $c > 0$ such that $f(x) = ch(x)$ is a valid pdf.

(b) Find the corresponding cdf. (Check that it satisfies the required properties of a cdf.)

(c) Plot both the pdf and the cdf.

(d) Suppose X has pdf $f(x)$. Find the cdf and pdf for $Y = \log(X)$. (This is the natural logarithm.)

(a) To make $f(x) = ch(x)$ a valid pdf, we need

$$\int_{-\infty}^{\infty} f(x)dx = 1 \implies c\int_{0}^{\infty} h(x)dx = 1.$$

First, we compute the integral:

$$\int_{0}^{\infty} h(x)dx = \int_{0}^{\infty} xe^{-2x}dx.$$

Using integration by parts with $u = x$ and $dv = e^{-2x}dx$, we have $du = dx$ and $v = -\frac{1}{2}e^{-2x}$. Thus,

$$\int_{0}^{\infty} xe^{-2x}dx = \left[-\frac{1}{2}xe^{-2x}\right]_{0}^{\infty} + \frac{1}{2}\int_{0}^{\infty} e^{-2x}dx.$$

The first term evaluates to 0 by L'Hôpital's rule

$$\lim_{x\to\infty} -\frac{x}{2e^{2x}} = \lim_{x\to\infty} -\frac{1}{4e^{2x}} = 0.$$

and the second term is

$$\frac{1}{2}\int_{0}^{\infty} e^{-2x}dx = \frac{1}{2}\left[-\frac{1}{2}e^{-2x}\right]_{0}^{\infty} = \frac{1}{2}\cdot\frac{1}{2} = \frac{1}{4}.$$

Therefore,

$$\int_{0}^{\infty} h(x)dx = \frac{1}{4} \implies c\cdot\frac{1}{4} = 1 \implies c = 4.$$

So the valid pdf is

$$f(x) = 4xe^{-2x}, \quad x > 0.$$

(b) The corresponding cdf is

$$F(x) = \int_{-\infty}^{x} f(t)dt = \int_{0}^{x} 4h(t)dt = 4\int_{0}^{x} te^{-2t}dt.$$

Using integration by parts again with $u = t$ and $dv = e^{-2t}dt$, we find

$$\int_{0}^{x} te^{-2t}dt = \left[-\frac{1}{2}te^{-2t}\right]_{0}^{x} + \frac{1}{2}\int_{0}^{x} e^{-2t}dt.$$

The first term evaluates to $-\frac{1}{2}xe^{-2x}$, and the second term is

$$\frac{1}{2}\int_{0}^{x} e^{-2t}dt = \frac{1}{2}\left[-\frac{1}{2}e^{-2t}\right]_{0}^{x} = \frac{1}{4}(1 - e^{-2x}).$$

Therefore,

$$\int_{0}^{x} te^{-2t}dt = -\frac{1}{2}xe^{-2x} + \frac{1}{4}(1 - e^{-2x}),$$

and

$$F(x) = 4\left(-\frac{1}{2}xe^{-2x} + \frac{1}{4}(1-e^{-2x})\right) = -2xe^{-2x} + (1-e^{-2x}).$$

Thus,

$$F(x) = 1 - e^{-2x} - 2xe^{-2x}, \quad x > 0.$$

For $x \leq 0$, $F(x) = 0$.
We check the properties of a cdf:

i. $F(x)$ is nondecreasing since $F'(x) = f(x) = 4xe^{-2x} \geq 0$ for $x > 0$.

ii. $F(x)$ is right-continuous since it is differentiable everywhere.

iii.

$$\lim_{x\to-\infty} F(x) = 0, \text{ since } F(x) = 0 \text{ for } x \leq 0.$$

$$\lim_{x\to\infty} F(x) = \lim_{x\to\infty}(1 - e^{-2x} - 2xe^{-2x}) = 1 - 0 - \infty \cdot 0(\text{indeterminate form}).$$

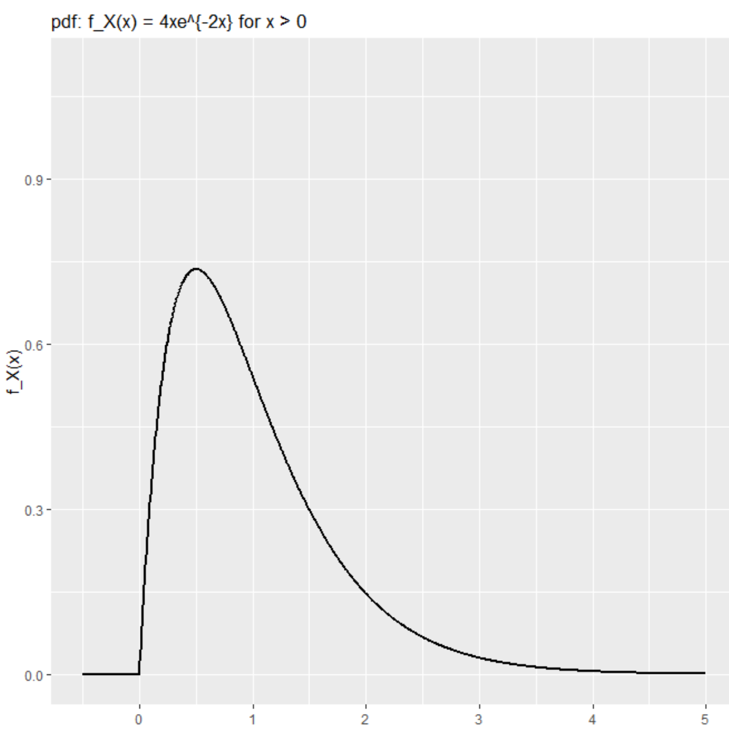
Using L'Hôpital's rule on $2xe^{-2x}$:

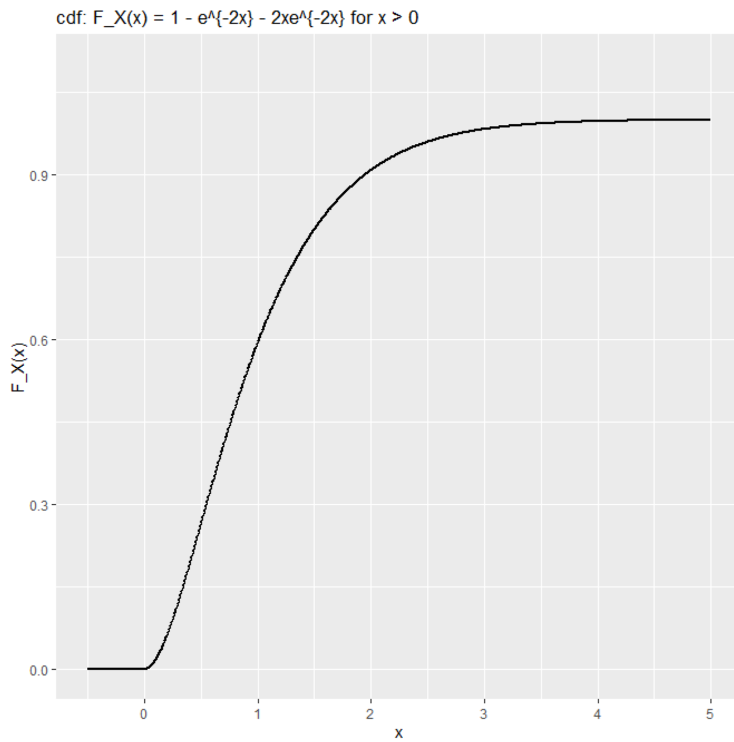
$$\lim_{x\to\infty} 2xe^{-2x} = \lim_{x\to\infty}\frac{2x}{e^{2x}} = \lim_{x\to\infty}\frac{2}{2e^{2x}} = 0.$$

Thus,

$$\lim_{x\to\infty} F(x) = 1 - 0 - 0 = 1.$$

(c) The graphs of the pdf and cdf are shown below.





(d) Since $Y = \log(X) \Rightarrow X = e^Y$, and

$$\frac{dX}{dY} = e^Y.$$

The support of Y is $(-\infty, \infty)$.

$$f_Y(y) = f_X(e^y)\left|\frac{dX}{dY}\right| = 4(e^y)e^{-2(e^y)} \cdot e^y = 4e^{2y}e^{-2e^y}, \quad y \in (-\infty, \infty).$$

This is a valid pdf since $f_Y(y) \geq 0$ for all y and

$$\begin{aligned}\int_{-\infty}^{\infty} f_Y(y)dy &= \int_{-\infty}^{\infty} 4e^{2y}e^{-2e^y}dy \\ &= \int_0^{\infty} 4ue^{-2u}du \text{ (using substitution } u = e^y) \\ &= 1 \text{ (as computed in part (a)).}\end{aligned}$$

The cdf of Y is

$$F_Y(y) = \int_{-\infty}^{y} f_Y(t)dt = \int_{-\infty}^{y} 4e^{2t}e^{-2e^t}dt.$$

Using substitution $u = e^t$, $du = e^t dt$, we have

$$F_Y(y) = \int_0^{e^y} 4ue^{-2u}du.$$

Using integration by parts with $v = u$ and $dw = e^{-2u}du$, we find

$$\begin{aligned}
\int_0^{e^y} 4ue^{-2u}du &= 4\left[-\frac{1}{2}ue^{-2u}\right]_0^{e^y} + 4\cdot\frac{1}{2}\int_0^{e^y} e^{-2u}du \\
&= -2e^y e^{-2e^y} + 2\int_0^{e^y} e^{-2u}du \\
&= -2e^y e^{-2e^y} + 2\left[-\frac{1}{2}e^{-2u}\right]_0^{e^y} \\
&= -2e^y e^{-2e^y} + (1 - e^{-2e^y}).
\end{aligned}$$

We check the properties of a cdf:

i. $F_Y(y)$ is nondecreasing since $F'_Y(y) = f_Y(y) = 4e^{2y}e^{-2e^y} \geq 0$ for all y.

ii. $F_Y(y)$ is right-continuous since it is differentiable everywhere.

iii.
$$\lim_{y\to-\infty} F_Y(y) = 0, \text{ since } e^y \to 0.$$

$$\lim_{y\to\infty} F_Y(y) = \lim_{y\to\infty}(1 - e^{-2e^y} - 2e^y e^{-2e^y}) = 1 - 0 - \infty\cdot 0 (\text{indeterminate form}).$$

Using L'Hôpital's rule on $2e^y e^{-2e^y}$:

$$\lim_{y\to\infty} 2e^y e^{-2e^y} = \lim_{y\to\infty}\frac{2e^y}{e^{2e^y}} = \lim_{y\to\infty}\frac{2e^y}{2e^y e^{2e^y}} = \lim_{y\to\infty} e^{-2e^y} = 0.$$

Thus,
$$\lim_{y\to\infty} F_Y(y) = 1 - 0 - 0 = 1.$$