Assignment 5

Partial solutions for selected problems

1. (a) $\mathsf{E}(Z^3) = \frac{\mathrm{d}^3 e^{t^2/2}}{\mathrm{d}t^3}\Big|_{t=0} = 0$ and $\mathsf{E}(Z^4) = \frac{\mathrm{d}^4 e^{t^2/2}}{\mathrm{d}t^4}\Big|_{t=0} = 3$.

   (b) Expand and use linearity: $\mathsf{E}(X^3) = \mu^3 + 3\mu^2\sigma\mathsf{E}(Z) + 3\mu\sigma^2\mathsf{E}(Z^2) + \sigma^3\mathsf{E}(Z^3) = \mu^3 + 3\mu\sigma^2$. Similarly, $\mathsf{E}(X^4) = \mu^4 + 6\mu^2\sigma^2 + 3\sigma^4$. [Note that the "units", whatever they may be, agree on both sides of each equation.]

2. (c) From the hint, $\mathsf{E}(X) = \mathsf{E}(\int_0^\infty 1_{X>x}\,\mathrm{d}x) = \int_0^\infty \mathsf{E}(1_{X>x})\,\mathrm{d}x = \int_0^\infty \mathsf{P}(X > x)\,\mathrm{d}x$.

3. (a) Split $\mathsf{E}(\mathrm{e}^{tX})$ into an integral below 0 plus an integral above 0 to get

$$M_X(t) = \frac{\lambda}{2(\lambda+t)} + \frac{\lambda}{2(\lambda-t)} = \frac{\lambda^2}{\lambda^2 - t^2} \quad \text{for } |t| < \lambda.$$

   (b) $Q_X(p) = \frac{\log(2p)}{\lambda}$ for $p \le \frac{1}{2}$ and $Q_X(p) = \frac{-\log(2(1-p))}{\lambda}$ for $p \ge \frac{1}{2}$.

4. (a) $W \sim$ negative binomial$(5, .032)$ with mean $\mathsf{E}(W) = \frac{5(1-.32)}{.32}$ and variance $\mathsf{Var}(W) = \frac{5(1-.32)}{.32^2}$.

   (b,c) These are the same value (see Thm. 3.3 in the notes): 55.23%.

5. Exer. 3.24. (Thm. 3.9(i,iii) in the notes can help with some of the expectations.)

   (b) Same as Weibull$(2, 2)$. For moments, either compute $\mathsf{E}(Y^k)$ using the pdf for Y or compute $\mathsf{E}((2X/\beta)^{k/2})$ using the pdf for X: $\mathsf{E}(Y) = \sqrt{\pi/2}$, $\mathsf{Var}(Y) = 2 - \pi/2$.
   For quantiles, it could be easiest to first get the quantiles for $X/\beta \sim$ exponential(1) and then convert: $y_{.5} = \sqrt{2\log(2)}$, $y_{.9} = \sqrt{2\log(10)}$.
   [Make sure all four of these values are *positive*!]

   (e) $F_Y(y) = \mathrm{e}^{-\mathrm{e}^{-(y-\alpha)/\gamma}}$ and $f_Y(y) = \frac{1}{\gamma}\mathrm{e}^{-(y-\alpha)/\gamma}\mathrm{e}^{-\mathrm{e}^{-(y-\alpha)/\gamma}}$, for $y > 0$.
   $Q_Y(p) = \alpha - \gamma\log(Q_X(1-p)) = \alpha - \gamma\log(-\log(p))$ (since Y is a decreasing function of X).

6. (a) Be careful with the $j = 0$ term and consider what happens as $t \to 0$.

   (b) Both are 0.86794.

7. $\mathsf{P}(Z^2 \le x) = \mathsf{P}(-\sqrt{x} \le Z \le \sqrt{x}) = \Phi(\sqrt{x}) - \Phi(-\sqrt{x})$, where $\Phi(z)$ is the standard normal cdf. Taking a derivative and using symmetry,

$$f_{Z^2}(x) = \frac{\mathrm{d}(\Phi(\sqrt{x}) - \Phi(-\sqrt{x}))}{\mathrm{d}x} = \frac{2}{\sqrt{x}}\Phi'(\sqrt{x}) = \frac{1}{\sqrt{x}}\Big(\frac{\mathrm{e}^{-z^2/2}}{\sqrt{2\pi}}\Big)\Big|_{z=\sqrt{x}} = \frac{x^{-1/2}\mathrm{e}^{-x/2}}{\sqrt{2\pi}},$$

   which must be the gamma$(\frac{1}{2}, 2)$ pdf. This is the same as chi-square(1).