Assignment 7

Partial solutions for selected problems

1. (b) For integer x and y with $0 \le x \le y \le n$,

$$\mathsf{P}(X_A = x, Y = y) = \mathsf{P}(X_A = x, X_B = y - x, X_C = n - y) = \frac{\binom{M_A}{x}\binom{M_B}{y-x}\binom{M_C}{n-y}}{\binom{N}{n}}.$$

(c,d) $Y \sim$ hypergeometric$(N, M_A + M_B, n)$ and thus the conditional pmf for X, given $Y = y$, is

$$\mathsf{P}(X_A = x | Y = y) = \cdots = \frac{\binom{M_A}{x}\binom{M_B}{y-x}}{\binom{M_A+M_B}{y}}.$$

This is the hypergeometric$(M_A + M_B, M_A, y)$ pmf. Note that it depends on y.

3. (a) $f_{S,T}(s,t) = p^2(1-p)^t$ for integer s, $0 \le s \le t$. [This *does* depend on both s and t.]

(b) $f_S(s) = p(1-p)^s$ for $s \ge 0$ (negative binomial$(1,p)$). $f_{T|S}(t|s) = p(1-p)^{t-s}$ for $t \ge s$.

(c) Since $T = R + S$, $f_{S,R}(s,r) = f_{S,T}(s, r+s) = p^2(1-p)^{r+s}$ for $s \ge 0$ and $r \ge 0$, which is the product of negative binomial$(1,p)$ pmfs.

4. Let $X = S + T$. Then

$$\begin{aligned} f_X(x) &= \sum_{s=0}^{x} f_S(s) f_T(t) = \sum_{s=0}^{x} \binom{m}{s} p^s (1-p)^{m-s} \binom{n}{x-s} p^{x-s}(1-p)^{n-(x-s)} \\ &= \binom{n+m}{x} p^x (1-p)^{m+n-x} \sum_{s=0}^{x} \frac{\binom{m}{s}\binom{n}{x-s}}{\binom{n+m}{x}} \\ &= \binom{n+m}{x} p^x (1-p)^{m+n-x}, \end{aligned}$$

since the sum in the next-to-last line is that of a hypergeomtric pmf. [Or, think about counting the number of ways to select x members from two groups of size m and n.]

5. Exer. 4.4.

(a) $C = \frac{1}{4}$.

(c) Provided completely, the joint cdf is

$$F_{X,Y}(x,y) = \begin{cases} 0 & \text{if } x < 0 \text{ or } y < 0, \\ \frac{x^2y + 2xy^2}{8} & \text{if } 0 \le x < 2,\ 0 \le y < 1, \\ \frac{x^2 + 2x}{8} & \text{if } 0 \le x < 2,\ y \ge 1, \\ \frac{1+y^2}{2} & \text{if } x \ge 2,\ 0 \le y < 1, \\ 1 & \text{if } x \ge 2 \text{ and } y \ge 1. \end{cases}$$

Note that $f_{X,Y}(x,y) = \frac{\partial^2 F_{X,Y}(x,y)}{\partial x \partial y} > 0$ only for $0 \le x < 2$, $0 \le y < 1$.

(d) $\mathsf{P}(Y^2 < X < \sqrt{Y}) = \frac{9}{80}$.

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(e) There are two cases to consider: $t \le 2$ and $t > 2$. For the second case, it may be easier to first find $\mathsf{P}(X + 2Y > t)$. The cdf for T is

$$F_T(t) = \begin{cases} 0 & \text{if } t \le 0, \\ \frac{t^3}{24} & \text{if } 0 \le t \le 2, \\ \frac{-8+6t^2-t^3}{24} & \text{if } 2 \le t \le 4, \\ 1 & \text{if } t \ge 4. \end{cases}$$

[Check that this is continuous at 0, 2 and 4.] The pdf (which also happens to be continuous) is then the derivative of $F_T(t)$.

6. (a) $S \sim \text{gamma}(4, 1/2)$ and $R|\{S = s\} \sim \text{uniform}(0, s)$.

   (b) $f_R(r) = \frac{2(1+2r+2r^2)}{3}\mathrm{e}^{-2r}$ for $r \ge 0$, and $f_{S|R}(s|r) = \frac{4s^2}{1+2r+2r^2}\mathrm{e}^{-2(s-r)}$ for $s \ge r$.

7. (a) We have $x = \frac{dw-bz}{ad-bc}$ and $y = \frac{az-cw}{ad-bc}$, which get substituted into the joint pdf for (X, Y), and that gets multiplied by $\left|\frac{\mathrm{d}x\mathrm{d}y}{\mathrm{d}w\mathrm{d}z}\right|$. In this case, it is a bit easier to find $\frac{\mathrm{d}w\mathrm{d}z}{\mathrm{d}x\mathrm{d}y} = ad - bc$ and take the reciprocal.

   (b) We get $f_{W,Z}(w, z) = \frac{\mathrm{e}^{-w}}{2}$ for $-w \le z \le w$.

   (c) From the joint pdf, we can integrate out z to determine that the marginal pdf of W is the gamma(2,1) pdf, which agrees with other results we have derived. Integrating out w properly ($w > |z|$) shows that Z has the Laplace distribution of Assignment 5, Problem 3 (with $\lambda = 1$).

8. (a) $X \ge 0$ and $0 \le Y \le 1$, with no constraints between them. $T = XY$ and $U = X(1 - Y)$, giving $\frac{\mathrm{d}t\mathrm{d}u}{\mathrm{d}x\mathrm{d}y} = x$. Applying Theorem 4.17 in the notes, these lead to

$$f_{X,Y}(x, y) = x f_{T,U}(xy, x(1-y)) = \frac{x^{\alpha+\beta-1}\mathrm{e}^{-x/\gamma}}{\gamma^{\alpha+\beta}\Gamma(\alpha+\beta)} \times \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} y^{\alpha-1}(1-y)^{\beta-1}.$$

   From this we conclude that X and Y are independent gamma$(\alpha + \beta, \gamma)$ and beta(α, β), respectively.