Printed Name: ______________________________

STAT610 Exam II
Thursday, 31 October 2024

Instructions.

- You have 75 minutes.
- You may use a stand-alone calculator, but no other resources.
- Please write your solutions on separate sheets of paper, and *return them in order* with this sheet on top.
- You may use any results from class or homework as long as you explain clearly the result you are using (by name or description).
- To ensure credit, explain or show all of your steps.
- There are 60 points total.
- Attempt all parts of a question; some may not rely on successful completion of earlier parts.

1. (10 points) Suppose T has pdf $f_T(t) = \frac{1}{t\log(2)}$ for $1 \le t \le 2$ and the conditional distribution of X, given T, is exponential with *scale* parameter T. That is, $f_{X|T}(x|t) = \frac{1}{t}\mathrm{e}^{-x/t}$ for $x > 0$.
   (a) (10 points) Express the marginal pdf for X as a simple integral and then attempt the integration. It will help to note that the anti-derivative of $\frac{a}{t^2}\mathrm{e}^{-a/t}$ is $\mathrm{e}^{-a/t}$.
   (b) (10 points) Find $\mathsf{E}(X)$ by using iterated expectation.
2. Let $T \sim \text{gamma}(\alpha, 1)$ and $R \sim \text{uniform}(0, 1)$, independent. Let $(X, Y) = (RT, (1-R)T)$.
   (a) (10 points) Compute $\mathsf{E}(XY)$. Hint: it may help to think about this in terms of expectations for (R, T).
   (b) (10 points) Find the joint pdf for (X, Y). (Note that $T = X + Y$ and $R = \frac{X}{X+Y}$.)
3. (10 points) Recall that $\mathsf{Cov}(X, Y) = \mathsf{E}((X - \mu_X)(Y - \mu_Y))$. Prove, for any real-valued a, that
$$\mathsf{E}((X - \mu_X)(Y - \mu_Y)) = \mathsf{E}((X - a)(Y - \mu_Y)).$$
   Do not assume or use properties about covariances; just use expectations.
4. (10 points) Suppose $\beta > 0$ and W has probability mass function
$$f_W(w) = c(\beta)w^2\mathrm{e}^{-(w/\beta)^2} \quad \text{for } w = 1, 2, 3 \ldots,$$
   and $c(\beta)$ is defined to be the constant so that $f_W(w)$ sums to 1. Does this model define an exponential family with parameter β? Why or why not?

(distribution formulas on next page)

Some common probability density functions and probability mass functions are provided below.

beta(α, β) $f(x) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} x^{\alpha-1}(1-x)^{\beta-1}$, for $0 < x < 1$, $\alpha > 0$, $\beta > 0$.
$\mathsf{E}(X) = \frac{\alpha}{\alpha+\beta}$, $\mathsf{Var}(X) = \frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}$.

binomial(n, p) $f(x) = \binom{n}{x} p^x (1-p)^{n-x}$, for $x = 0, 1, \ldots, n$; $0 < p < 1$.
$\mathsf{E}(X) = np$, $\mathsf{Var}(X) = np(1-p)$.

exponential(β) $f(x) = \frac{1}{\beta} \mathrm{e}^{-x/\beta}$, for $x > 0$; $\beta > 0$.
$\mathsf{E}(X) = \beta$, $\mathsf{Var}(X) = \beta^2$.

gamma(α, β) $f(x) = \frac{1}{\beta^\alpha \Gamma(\alpha)} x^{\alpha-1} \mathrm{e}^{-x/\beta}$, for $x > 0$; $\alpha > 0$, $\beta > 0$.
$\mathsf{E}(X) = \alpha\beta$,$\mathsf{Var}(X) = \alpha\beta^2$.

geometric(p) $f(x) = p(1-p)^{x-1}$, for $x = 1, 2, \ldots$; $0 < p < 1$.
$\mathsf{E}(X) = \frac{1}{p}$, $\mathsf{Var}(X) = \frac{1-p}{p^2}$.

hypergeometric(N, M, n) $f(x) = \binom{M}{x}\binom{N-M}{n-x} / \binom{N}{n}$, for $x = 0, 1, \ldots, n$; $n > 0$, $M > 0$, $N > 0$.
$\mathsf{E}(X) = \frac{nM}{N}$, $\mathsf{Var}(X) = \frac{N-n}{N-1} \frac{nM}{N} (1 - \frac{M}{N})$.

negative binomial(k, p) $f(x) = \binom{k+x-1}{k-1} p^k (1-p)^x$, for $x = 0, 1, 2, \ldots$; $0 < p < 1$, $k > 0$.
$\mathsf{E}(X) = \frac{k(1-p)}{p}$, $\mathsf{Var}(X) = \frac{k(1-p)}{p^2}$.

normal(μ, σ^2) $f(x) = \frac{1}{\sqrt{2\pi}\sigma} \mathrm{e}^{-(x-\mu)^2/2\sigma^2}$, for $-\infty < x < \infty$; $-\infty < \mu < \infty$, $\sigma^2 > 0$.
$\mathsf{E}(X) = \mu$, $\mathsf{Var}(X) = \sigma^2$.

Poisson(λ) $f(x) = \frac{\lambda^x}{x!} \mathrm{e}^{-\lambda}$, for $x = 0, 1, 2, \ldots$; $\lambda > 0$.
$\mathsf{E}(X) = \lambda$, $\mathsf{Var}(X) = \lambda$.

uniform(a, b) $f(x) = \frac{1}{b-a}$, for $a < x < b$; $a < b$.
$\mathsf{E}(X) = \frac{a+b}{2}$, $\mathsf{Var}(X) = \frac{(b-a)^2}{12}$.

Weibull(γ, β) $f(x) = \frac{\gamma}{\beta} x^{\gamma-1} \mathrm{e}^{-x^\gamma/\beta}$, for $x > 0$; $\gamma > 0$, $\beta > 0$.
$\mathsf{E}(X) = \beta^{1/\gamma}\Gamma(1 + \frac{1}{\gamma})$, $\mathsf{Var}(X) = \beta^{2/\gamma}(\Gamma(1 + \frac{2}{\gamma}) - \Gamma(1 + \frac{1}{\gamma})^2)$.