Exam 2 2023

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Stat 610 Distribution Theory

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Problem 1. Suppose T has pdf $f_T(t) = \frac{1}{t\log(2)}$ for $1 \leq t \leq 2$ and the conditional distribution of X, given T, is exponential with scale parameter T. That is, $f_{X|T}(x|t) = \frac{1}{t}e^{-x/t}$ for $x > 0$.

(a) Express the marginal pdf for X as a simple integral and then attempt the integration. It will help to note that the anti-derivative of $\frac{a}{t^2}e^{-a/t}$ is $e^{-a/t}$.

(b) Find $\mathsf{E}(X)$ by using iterated expectation.

(a) The marginal pdf for X is

$$\begin{aligned}
f_X(x) &= \int_1^2 f_{X|T}(x|t) f_T(t) dt \\
&= \int_1^2 \frac{1}{t}e^{-x/t} \cdot \frac{1}{t\log(2)} dt \\
&= \frac{1}{\log(2)} \int_1^2 \frac{1}{t^2} e^{-x/t} dt \\
&= \frac{1}{\log(2)} \left[\frac{1}{x}e^{-x/t}\right]_{t=1}^{t=2} \\
&= \frac{1}{x\log(2)} \left(e^{-x/2} - e^{-x}\right), \quad x > 0.
\end{aligned}$$

(b) By iterated expectation, we have

$$\begin{aligned}
\mathsf{E}(X) &= \mathsf{E}\left(\mathsf{E}(X|T)\right) \\
&= \mathsf{E}(T) \\
&= \int_1^2 t \cdot \frac{1}{t\log(2)} dt \\
&= \frac{1}{\log(2)} \int_1^2 1 dt \\
&= \frac{1}{\log(2)}.
\end{aligned}$$

Problem 2. Let $T \sim \text{gamma}(\alpha, 1)$ and $R \sim \text{uniform}(0, 1)$, independent. Let $(X, Y) = (RT, (1-R)T)$.

(a) Compute $\mathsf{E}(XY)$. Hint: it may help to think about this in terms of expectations for (R, T).

(b) Find the joint pdf for (X, Y). (Note that $T = X + Y$ and $R = \frac{X}{X+Y}$.)

(a)

$$
\begin{aligned}
\mathsf{E}(XY) &= \mathsf{E}\left((RT)((1-R)T)\right) \\
&= \mathsf{E}\left(T^2 R(1-R)\right) \\
&= \mathsf{E}\left(T^2\right) \mathsf{E}(R(1-R)) \\
&= (\mathrm{Var}(T) + [\mathsf{E}(T)]^2)\left(\mathsf{E}(R) - \mathsf{E}(R^2)\right) \\
&= (\alpha + \alpha^2)\left(\frac{1}{2} - \frac{1}{3}\right) \\
&= (\alpha + \alpha^2) \cdot \frac{1}{6} \\
&= \frac{\alpha(\alpha+1)}{6}.
\end{aligned}
$$

(b) The inverse transformation is

$$r = \frac{x}{x+y}, \quad t = x + y.$$

The Jacobian determinant is

$$J = \begin{vmatrix} \frac{\partial r}{\partial x} & \frac{\partial r}{\partial y} \\ \frac{\partial t}{\partial x} & \frac{\partial t}{\partial y} \end{vmatrix} = \begin{vmatrix} \frac{y}{(x+y)^2} & -\frac{x}{(x+y)^2} \\ 1 & 1 \end{vmatrix} = \frac{1}{x+y}.$$

The joint pdf for (X, Y) is

$$
\begin{aligned}
f_{X,Y}(x, y) &= f_{R,T}\left(\frac{x}{x+y}, x+y\right)|J| \\
&= f_R\left(\frac{x}{x+y}\right) f_T(x+y) \cdot \frac{1}{x+y} \\
&= 1 \cdot \frac{1}{\Gamma(\alpha)}(x+y)^{\alpha-1} e^{-(x+y)} \cdot \frac{1}{x+y} \\
&= \frac{1}{\Gamma(\alpha)}(x+y)^{\alpha-2} e^{-(x+y)}, \quad x > 0, y > 0.
\end{aligned}
$$

Problem 3. Recall that $\mathsf{Cov}(X, Y) = \mathsf{E}((X - \mu_X)(Y - \mu_Y))$. Prove, for any real-valued a, that

$$\mathsf{E}((X - \mu_X)(Y - \mu_Y)) = \mathsf{E}((X - a)(Y - \mu_Y)).$$

Do not assume or use properties about covariances; just use expectations.

$$\begin{aligned}
\mathsf{E}((X-a)(Y-\mu_Y)) &= \mathsf{E}((X-\mu_X+\mu_X-a)(Y-\mu_Y)) \\
&= \mathsf{E}((X-\mu_X)(Y-\mu_Y)) + \mathsf{E}((\mu_X-a)(Y-\mu_Y)) \\
&= \mathsf{E}((X-\mu_X)(Y-\mu_Y)) + (\mu_X-a)\mathsf{E}(Y-\mu_Y) \\
&= \mathsf{E}((X-\mu_X)(Y-\mu_Y)) + (\mu_X-a)(\mu_Y-\mu_Y) \\
&= \mathsf{E}((X-\mu_X)(Y-\mu_Y)) + 0 \\
&= \mathsf{E}((X-\mu_X)(Y-\mu_Y)).
\end{aligned}$$

Problem 4. Suppose $\beta > 0$ and W has probability mass function

$$f_W(w) = c(\beta)w^2 e^{-(w/\beta)^2}$$

for $w = 1, 2, 3, \ldots$, and $c(\beta)$ is defined to be the constant so that $f_W(w)$ sums to 1. Does this model define an exponential family with parameter β? Why or why not?

An exponential family can be written in the form

$$f(w) = c(\beta)h(w)e^{g(\beta)t(w)},$$

for functions c, h, g, t. We can rewrite this pmf as

$$\begin{aligned}
f_W(w) &= c(\beta)w^2 e^{-(w/\beta)^2} \\
&= c(\beta)w^2 e^{-w^2/\beta^2} \\
&= c(\beta)h(w)e^{g(\beta)t(w)}, \quad \text{where } h(w) = w^2, \quad g(\beta) = -\frac{1}{\beta^2}, \quad t(w) = w^2.
\end{aligned}$$

Thus, this model defines an exponential family with parameter β.