Project 3: Block Walking on Pascal's Triangle

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1 Introduction

Today, I want to explore a topic I learned in high school from a math competition book called *The Art of Problem Solving Volume 2* by Sandor Lehoczky and Richard Rusczyk [1]. A familiar fact about Pascal's triangle is that its entries correspond to the binomial coefficients, namely $\binom{n}{k}$ is the entry in the n-th row and k-th column (starting from 0). A related topic concerning Pascal's triangle is called block walking. Imagine you are starting at the top of the triangle (see

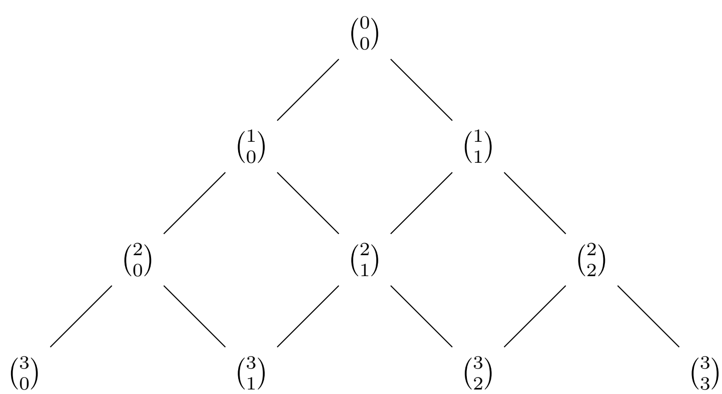

Figure 1: The first few rows of Pascal's triangle

Figure 1). At each step, you can either move down to the left or down to the right. For example, to get to the entry $\binom{3}{1}$, you can take the following paths:

- Down left, down right, down left
- Down right, down left, down left
- Down left, down left, down right

In total, there are 3 different paths to get to $\binom{3}{1}$. More generally, to get to the entry $\binom{n}{k}$, you need to take k steps down to the right and $n-k$ steps down to the left. Thus, the total number of steps is n, and you just need to choose which k of those n steps are down to the right. Therefore this important equation holds:

$$\text{Number of paths to } \binom{n}{k} = \binom{n}{k}. \tag{1}$$

We will now use this block walking idea to prove a nice combinatorial theorem.

2 Application of Block Walking

Theorem (Sum of Squares of Binomial Coefficients Identity)

$$\sum_{k=0}^{n} \binom{n}{k}^2 = \binom{2n}{n}.$$

Proof. Consider a block walking scenario where you start at the top of Pascal's triangle and want to get to the entry $\binom{2n}{n}$. To get there, we simply apply equation 1 to get that the total number of different paths to reach $\binom{2n}{n}$ is $\binom{2n}{n}$.

Now, let's consider an alternative way to count the number of paths to get to $\binom{2n}{n}$. We can break the journey into two parts: first, we walk down to the n-th row of Pascal's triangle, and then we walk from there down to $\binom{2n}{n}$. When we reach the n-th row, we could be at any entry $\binom{n}{k}$ for $k = 0, 1, 2, \ldots, n$. The number of different paths to get to $\binom{n}{k}$ is given by $\binom{n}{k}$ using equation 1. From $\binom{n}{k}$, we need to take another n steps to reach $\binom{2n}{n}$. In this second part of the journey, we need to take $(n-k)$ steps down to the right and k steps down to the left. The number of different paths from $\binom{n}{k}$ to $\binom{2n}{n}$ is given by $\binom{n}{n-k}$. However, this is also equal to $\binom{n}{k}$ as shown below:

$$\binom{n}{n-k} = \frac{n!}{(n-k)!(n-(n-k))!} = \frac{n!}{(n-k)!k!} = \frac{n!}{k!(n-k)!} = \binom{n}{k}.$$

Therefore, for each entry $\binom{n}{k}$ in the n-th row, the total number of paths from the top of the triangle to $\binom{2n}{n}$ that pass through $\binom{n}{k}$ is given by $\binom{n}{k} \cdot \binom{n}{k} = \binom{n}{k}^2$. To get the total number of paths to $\binom{2n}{n}$, we need to sum this quantity over all possible values of k:

$$\sum_{k=0}^{n} \binom{n}{k}^2.$$

Since both counting methods count the same number of paths, we have

$$\sum_{k=0}^{n} \binom{n}{k}^2 = \binom{2n}{n},$$

which completes the proof. □

3 Conclusion

The theorem we proved isn't particularly useful on its own although it is aesthetically pleasing in its symmetry. However, a much more general version of this theorem is called Vandermonde's identity.

Theorem (Vandermonde's Identity)

$$\sum_{k=0}^{r} \binom{m}{k}\binom{n}{r-k} = \binom{m+n}{r}.$$

This identity has many applications in a huge variety of fields, and it can also be proved using the block walking idea we discussed today, which I encourage you to try on your own!

4 Bibliography

References

[1] Sandor Lehoczky and Richard Rusczyk. *The Art of Problem Solving Volume 2: and Beyond*. AoPS, 2006.