The N-Queens Problem

-a constrained optimization

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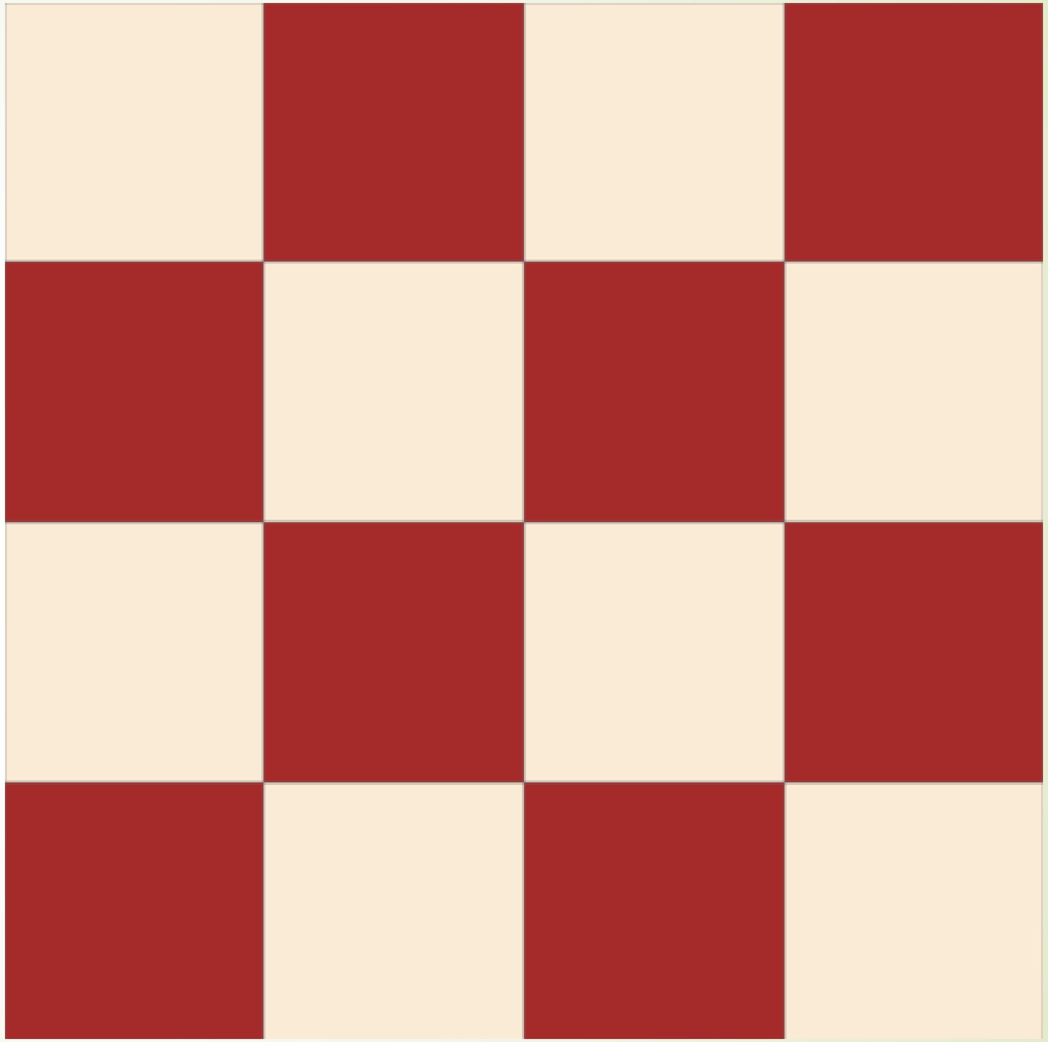
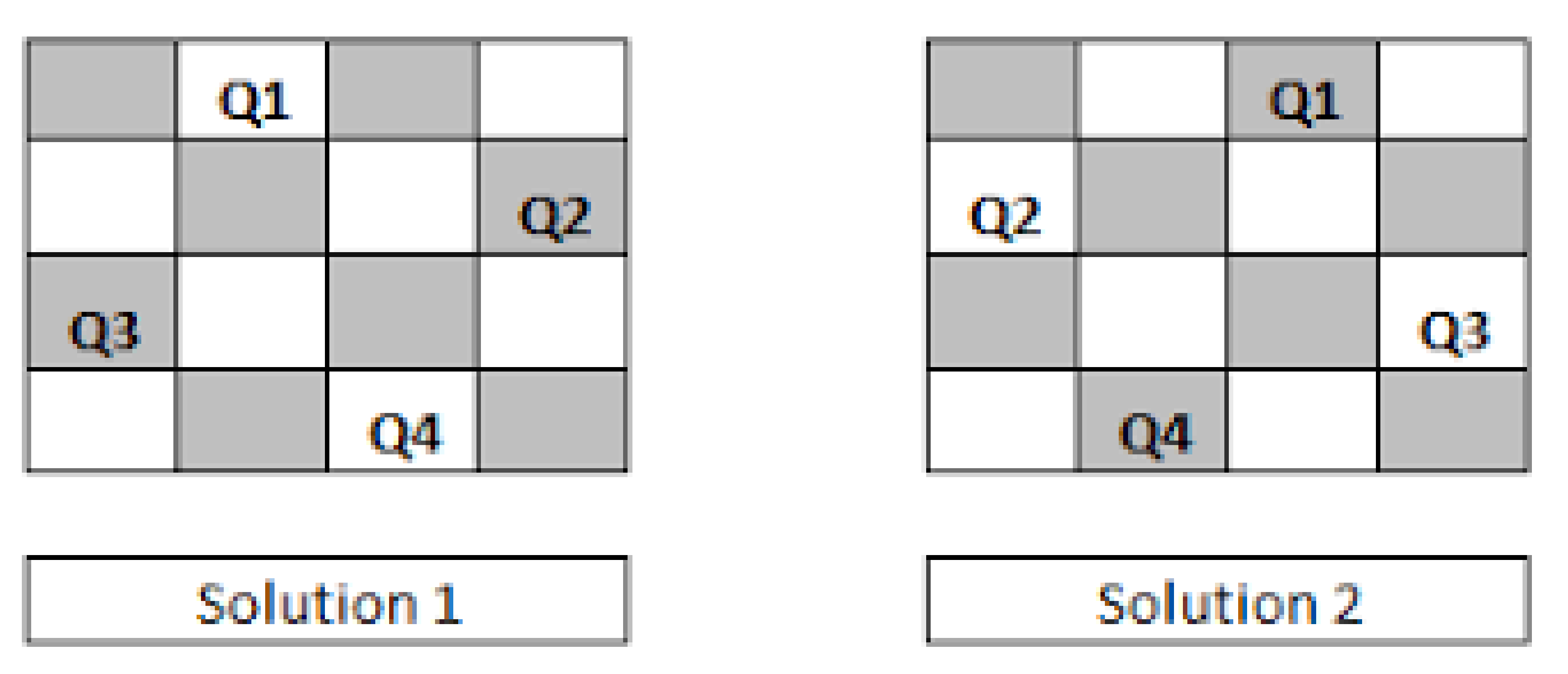
N-queens problem

- Find a configuration of n queens not attacking each other
- What is the maximum number of queens that can be placed on an n x n chessboard such that no two attack one another?
- ANSWER : N

4-Queen Problem





GOAL CONFIGURATION

Objective function

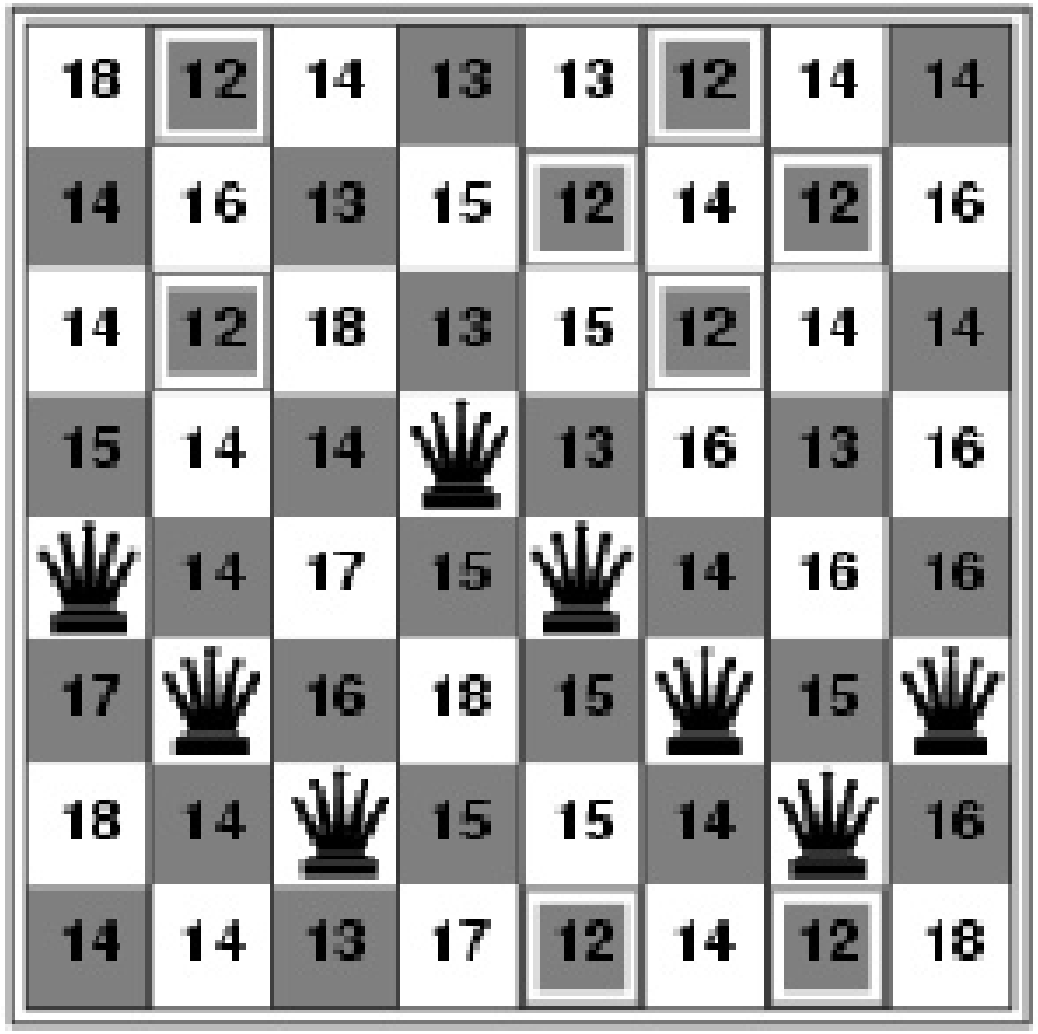
Let $x_{i,j}$ = 1, if a queen occupies the square
= 0, if the square is empty

$$\max f = \sum_{i=1}^{N} \sum_{j=1}^{N} x_{i,j} = N$$

Heuristics

- Is any approach to problem solving, learning, or discovery that employs a practical method not guaranteed to be optimal or perfect, but sufficient for the immediate goals
- Where finding an optimal solution is impossible or impractical, heuristic methods can be used to speed up the process of finding a satisfactory solution

- Heuristics in our problem???


18 12 14 13 13 12 14 14
14 16 13 15 12 14 12 16
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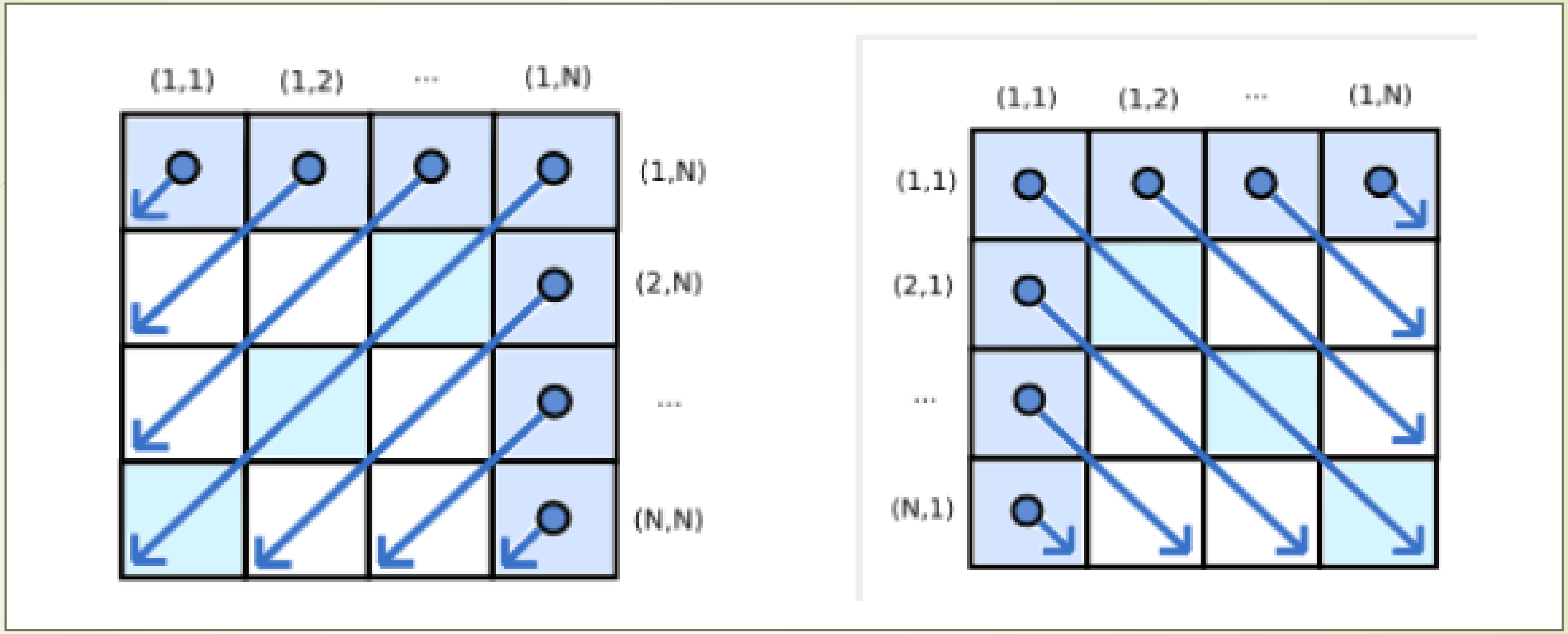


Backward Diagonals and Forward Diagonals

Conditions

$$\max\ f = \sum_{i=1}^{N}\sum_{j=1}^{N} x_{i,j}$$

subject to:

$$\begin{aligned}
\sum_{j=1}^{N} x_{i,j} &= 1 && \forall\ i = 1,2,\ldots,N\\
\sum_{i=1}^{N} x_{i,j} &= 1 && \forall\ j = 1,2,\ldots,N\\
x_{1,j} + \sum_{m=1}^{N-j} x_{1+m,j+m} &\le 1 && \forall\ j = 1,2,\ldots,N\\
x_{i,1} + \sum_{m=1}^{N-i} x_{i+m,1+m} &\le 1 && \forall\ i = 1,2,\ldots,N\\
x_{1,j} + \sum_{m=1}^{j-1} x_{1+m,j-m} &\le 1 && \forall\ j = 1,2,\ldots,N\\
x_{i,N} + \sum_{m=1}^{N-i} x_{i+m,N-m} &\le 1 && \forall\ i = 1,2,\ldots,N
\end{aligned}$$

$$x_{i,j} \in \{0,1\}\ \forall\ i,j = 1,2,\ldots,N.$$

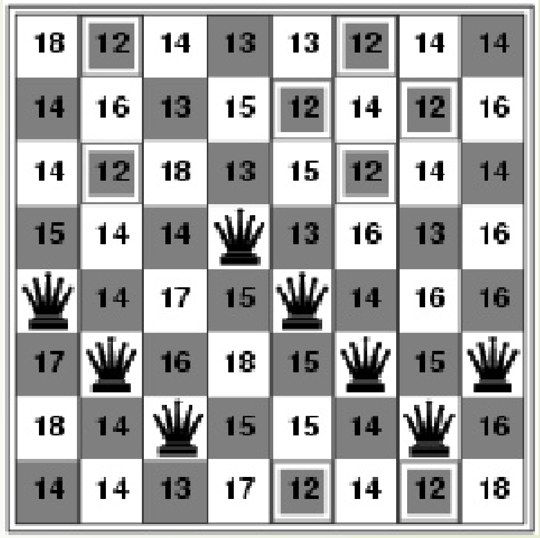
Local Search Optimization Techniques

- Hill Climbing Search
- Local Beam Search
- Genetic Algorithm
- Simulated Annealing

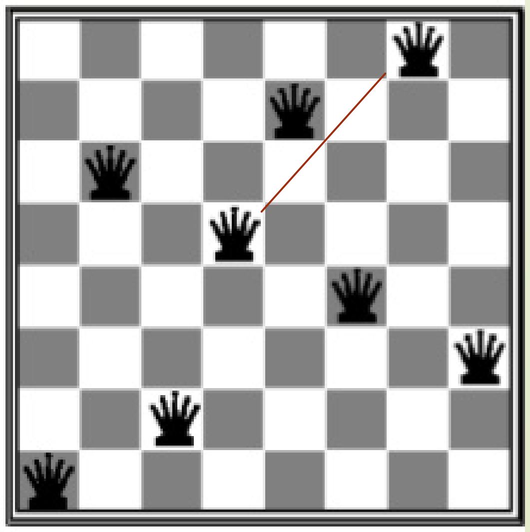
Hill Climbing Search

- a.k.a. greedy local search
- Move in the direction of increasing evaluation function

For H = 17



For H = 1



Genetic Algorithm

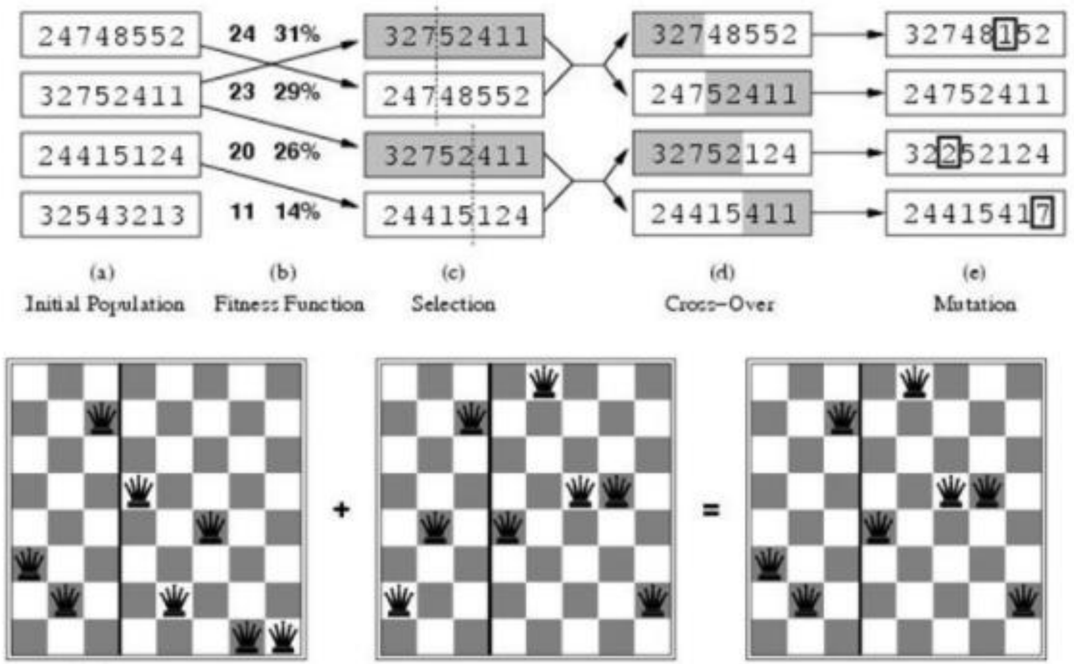
01
Maintain a list of potential solutions

02
Modify potential solutions in parallel

03
Crossover
- Randomly swap X number of Queen positions

04
Mutation
- Randomly change a Queen's position to a new random location

Solution in $O(n^3)$



	numiter (in 10 runs)		
n	min	max	average
8	1	10	4.0
10	16	113	49.1
30	212	1546	917.9
50	491	83641	17592.3
75	1114	16118	5711.7
100	3395	14581	8877.7
200	13345	48013	22879.6
300	20168	38084	27748.2
500	36471	167404	89406.4

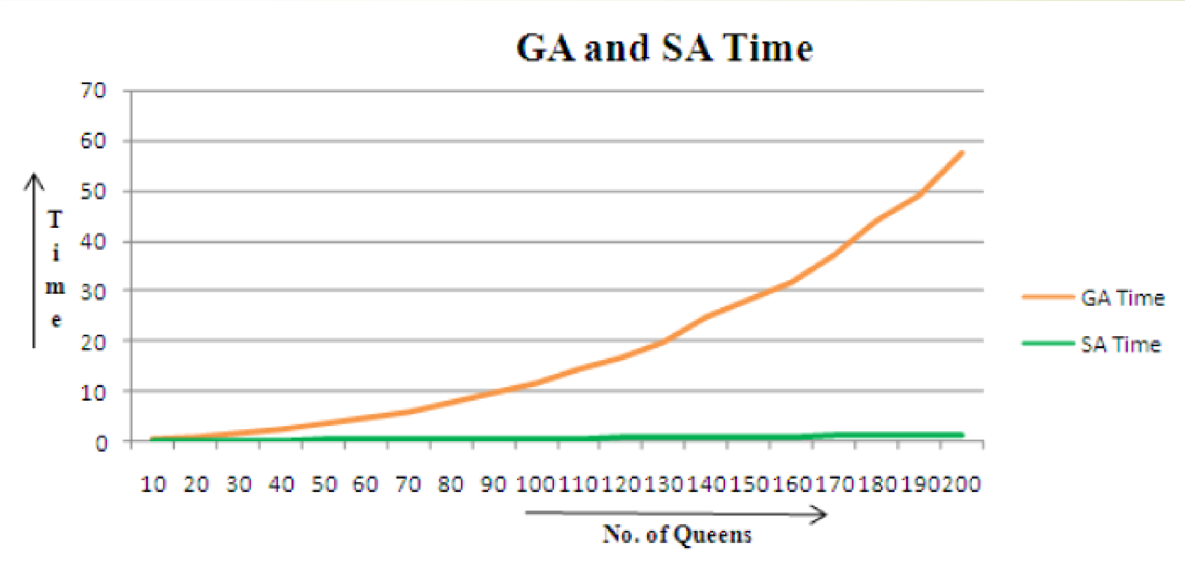
Simulated Annealing

Step 1	Start with a random initial placement. Initialize a high temperature
Step 2	Modify the placement through a defined change
Step 3	Calculate the difference in the energy due to the change made.
Step 4	Depending on the difference in energy, accept or reject the change.
Step 5	Update the temperature value by reducing the temperature very slowly.

Analysis

n	numiter (in 10 runs)		
	min	max	average
8	66	803	492.8
10	202	2006	947.8
30	1403	3777	2159.9
50	1815	5426	2848.6
75	3288	9874	6091.3
100	4482	11769	7872.7
200	9185	38681	21708.2
300	21687	31917	24636.2
400	30103	66846	48435.7
500	36205	83222	56629.7

Comparison of Execution Time of SA vs GA



Backtracking

- Place a queen in the top row, then note the column and diagonals it occupies
- Then place a queen in the next row down, taking care not to place it in the same column or diagonal. Keep track of the occupied columns and diagonals and move on to the next row
- If no position is open in the next row, we back track to the previous row and move the queen over to the next available spot in its row and the process starts over again

Time is an issue

- Difficult for a search algorithm such as backtracking to find a solution for the N-queens problem in an acceptable amount of time.

- It took over 9 months to get the results for N = 25

("N")	Total Solutions	Unique Solutions
1	1	1
2	0	0
3	0	0
4	2	1
5	10	2
6	4	1
7	40	6
8	92	12
9	352	46
10	724	92
11	2,680	341
12	14,200	1,787
13	73,712	9,233
14	365,596	45,752
15	2,279,184	285,053
16	14,772,512	1,846,955
17	95,815,104	11,977,939
18	666,090,624	83,263,591
19	4,968,057,848	621,012,754
20	39,029,188,884	4,878,666,808
21	314,666,222,712	39,333,324,973
22	2,691,008,701,644	336,376,244,042
23	24,233,937,684,440	3,029,242,658,210
24	227,514,171,973,736	?
25	2,207,893,435,808,352	?

Thank You!!