Collective Synchronization in Kuramoto Model

IDC 621 Modelling Complex Systems

Kunal Verma
MS18148

24th March 2021

1 Introduction

The Kuramoto Model is a model used to describe a large set of coupled oscillators which exhibit synchronization. The model makes several assumptions such as weak coupling, that the oscillators are identical, and that interactions depend sinusoidally on the phase difference between each pair of objects. The most general expression of oscillators coupling to the collective rhythm generated by the whole population is given by-

$$\dot{\theta}_i = \omega_i + \left[\sum_{j=1}^{N} X(\theta_j)\right] Z(\theta_i) \qquad i = 1, 2, 3...N.$$

where θ_i and ω_i are the phase and the natural frequency of the oscillator i, $X(\theta_j)$ is the phase dependent influence and $Z(\theta_i)$ is the sensitivity function. Then long-term dynamics of a system of weakly coupled and identical limit cycle oscillators can be simplified as-

$$\dot{\theta}_i = \omega_i + \sum_{j=1}^{N} \Gamma_{ij}(\theta_j - \theta_i) \qquad i = 1, 2, 3...N.$$

where Γ_{ij} are the interaction functions. This reduction to a phase model is a tremendous simplification, but still too difficult to analyze in a general manner. So, Kuramoto used the simplest possible case of equally weighted, all-to-all, purely sinusoidal coupling with the coupling constant K such that the governing equations become-

$$\boxed{\dot{\theta}_i = \omega_i + \frac{K}{N}\sum_{j=1}^{N} \sin(\theta_j - \theta_i)} \qquad i = 1, 2, 3...N.$$

For simplicity, some further assumptions were also made, namely-

- $g(\omega)$ is unimodal.
- $g(\omega)$ is symmetric about its mean frequency Ω such that $g(\Omega - \omega) = g(\Omega + \omega)$.
- Due to rotational symmetry, we can set $\Omega = \langle g(\omega) \rangle = 0$, $\Rightarrow g(-\omega) = g(\omega)$.

1.1 The Order Parameter

To visualize the dynamics better, a complex order parameter can be defined as follows:

$$re^{i\psi} = \frac{1}{N}\sum_{j=1}^{N} e^{i\theta_j}$$

which describes the macroscopic collective rhythm produced by the whole population. The radius $r(t)$ measures the phase coherence and $\psi(t)$ measures the average phase. Using the order parameter, the Kuramoto dynamics can be written ina a simpler form as-

$$\boxed{\dot{\theta}_i = \omega_i + Kr\sin(\psi - \theta_i), \qquad i = 1, 2, 3...N} \tag{1}$$

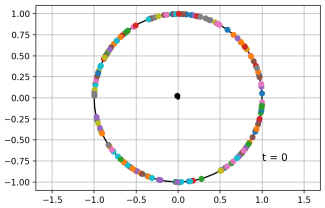
This shows that the oscillators interact only through mean field quantities r and ψ. Kuramoto's analysis showed that as we increase the coupling strength K, and the value exceeds K_c, the incoherent state becomes unstable and clusters of oscillators mutually synchronize, and this phase locking transition is precisely what we aim to study using our simulations!

2 Simulations

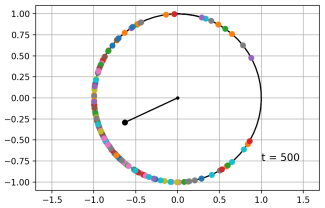
The oscillators start with randomly distributed phases θ_i and the natural frequencies ω_i are distributed according to the distribution function $g(\omega)$ which can be generated using rejection sampling. The N differential equations (1) can then be integrated numerically using the Euler method to obtain the time evolution $\theta_i(t)$ and $r_(t)$.

2.1 Evolution of r with time

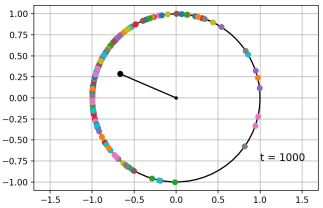
To see how the order parameter r varies with time, we setup a simulation with $g(\omega)$ as a normalised Gaussian of $\sigma = 1.5$ with $K = 4$. The graphs below show how a system of 200 coupled oscillators evolve in time.



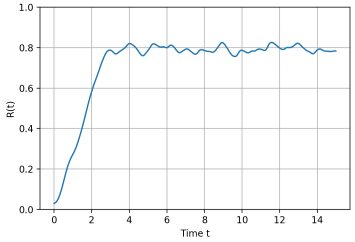
(a) 0 time steps



(b) 500 time steps



(c) 1000 time steps



(d) $r(t)$ plot for $K = 4$

Figure 1: Evolution of a collection of oscillators ($dt = 0.01$)

2.2 Variation of r_∞ with different K

In his earliest work, Kuramoto analyzed the model without the benefit of any simulations. He guessed the long-term behaviour in the limit $N \to \infty$ and by using symmetry and self-consistency arguments. From his analysis, he was able to conclude that the critical coupling K_c at the onset of collective synchronization is

$$K_c = \frac{2}{\pi g(0)}$$

Thus, we can verify Kuramoto's theoretical prediction with our simulations, by plotting the equilibrium values of r, which we'll call r_∞ against different values of K for which it is evaluated.

In the analysis done in the following sections, we use the Euler method for numerical integration for 1500 steps with $dt = 0.01$. The number of oscillators in the system are always 2000, and the variation of r_∞ with different K values is to be observed for various values of the scale parameters of the distributions, like the variance σ for Gaussian and HWHM γ for Lorentzian.
In this term paper, we will particularly explore three different distribution functions $g(\omega)$-

1. Lorentzian
2. Gaussian
3. Exponential Kink

2.2.1 Lorentzian Distribution

The normalised Lorentzian distribution is given by-

$$g(\omega) = \frac{\gamma}{\pi(\gamma^2 + \omega^2)} \implies K_c = 2\gamma$$

where the scale parameter γ specifies the half-width at half-maximum. With the given

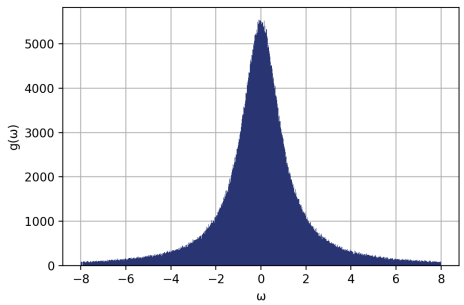
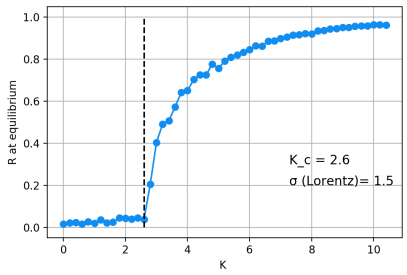
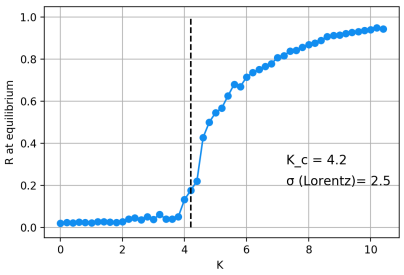


Figure 2: Lorentzian Distribution generated by rejection sampling

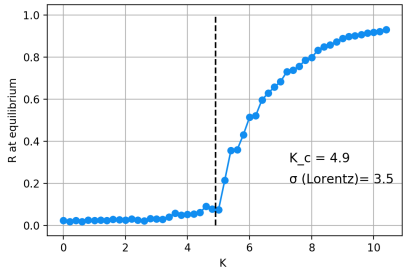
$g(\omega)$, we aim to see the phase locking transition in the r_∞ v/s K plot and see if it matches with Kuramoto's theoretical prediction from his analysis. We'll run the simulations for different values of the scale parameter γ, namely $\gamma = 1.5$, 2.5, 3.5 and 4.5.



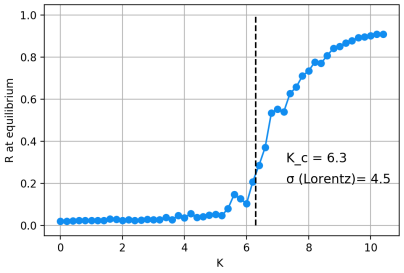
(a) $\gamma = 1.5$



(b) $\gamma = 2.5$



(c) $\gamma = 3.5$



(d) $\gamma = 4.5$

Figure 3: r_∞ v/s K plots for Lorentzian distribution.

-	$\gamma = 1.5$	$\gamma = 2.5$	$\gamma = 3.5$	$\gamma = 4.5$
Expected	3.0	5.0	7.0	9.0
Observed	2.6	4.2	4.9	6.3

From our simulations, we see that expected and observed K_c are quite close to each other for small γ, but the observed K_c is considerabely smaller than the theoretically expected value for $\gamma = 3.5,\ 4.5$.

2.2.2 Gaussian Distribution

The normalised Gaussian distribution is defined as-

$$g(\omega) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{x^2}{2\sigma^2}\right) \implies K_c = \sigma\sqrt{\frac{8}{\pi}}$$

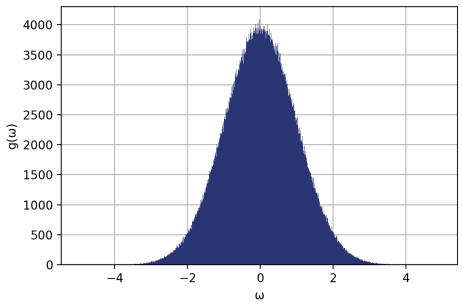
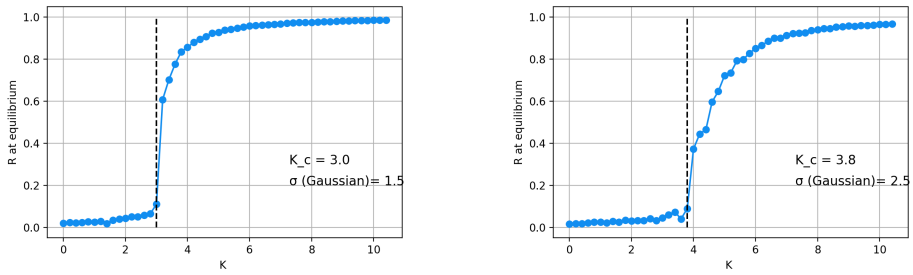
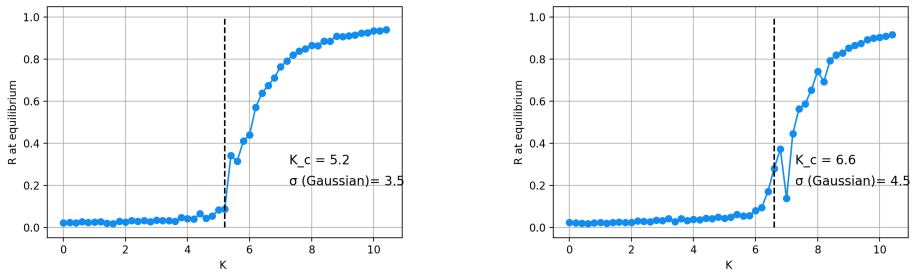


Figure 4: Gaussian Distribution generated by rejection sampling

Again, we will run the simulations for $\sigma = 1.5,\ 2.5,\ 3.5,\ 4.5$.



(a) $\sigma = 1.5$ (b) $\sigma = 2.5$



(c) $\sigma = 3.5$ (d) $\sigma = 4.5$

Figure 5: r_∞ v/s K plots for Gaussian distribution.

-	$\sigma = 1.5$	$\sigma = 2.5$	$\sigma = 3.5$	$\sigma = 4.5$
Expected	2.4	4.0	5.5	7.1
Observed	3.0	3.8	5.2	6.6

For Gaussian distribution, we see that the expected and observed values of K_c are fairly close compared to those in Lorentzian distribution.

2.2.3 Exponential Kink Distribution

The normalised exponential kink distribution is given by-

$$g(\omega) = \frac{1}{2a} \exp\left(- \frac{|x|}{a} \right) \implies K_c = \frac{4a}{\pi}$$

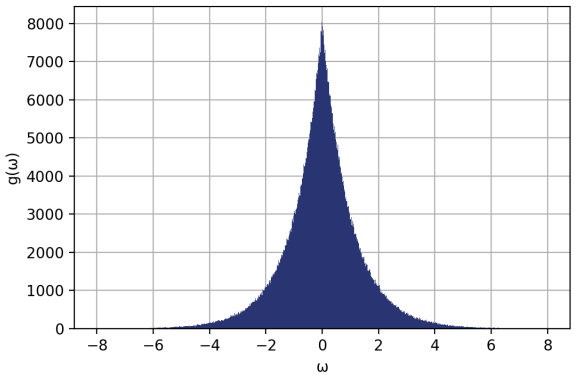
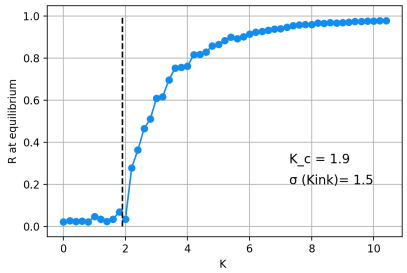
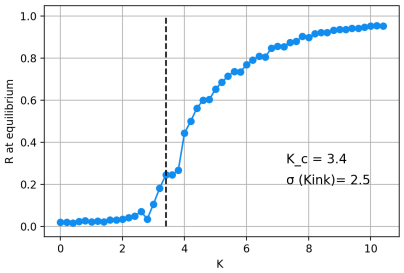


Figure 6: Exp. Kink Distribution generated by rejection sampling

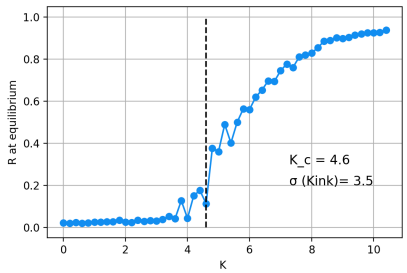
As always, we run the simulations for $a = 1.5,\ 2.5,\ 3.5,\ 4.5$ and we see that the observed values of K_c for the Kink distribution are the closest to the theoretically expected values.



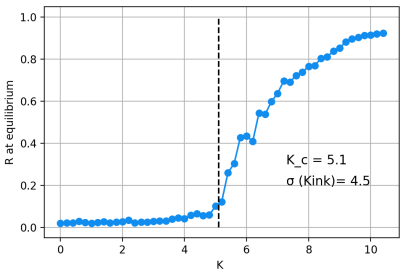
(a) $a = 1.5$



(b) $a = 2.5$



(c) $a = 3.5$



(d) $a = 4.5$

Figure 7: r_∞ v/s K plots for Kink distribution.

-	$a = 1.5$	$a = 2.5$	$a = 3.5$	$a = 4.5$
Expected	1.9	3.2	4.5	5.7
Observed	1.9	3.4	4.6	5.1

3 Conclusion

We were able to observe collective synchronization in a system of coupled oscillators starting with different distributions $g(\omega)$ of their natural frequencies ω_i. By numerically integrating the equations, we saw that as the coupling of the oscillators increases via K, the synchronization improves, and that the observed values for the critical K_c were very close to Kuramoto's theoretical analysis.

Code

The code for this project was written in Python 3.7.6 and can be found on the following GitHub repository: `https://github.com/kaizokugarizoro/ComplexSystems`.