Lecture 19:

Non-relativistic limit of Dirac eqn:

a) Dirac hamiltonian in $A_0(\bar{r}) = 0$, $\bar{A} \neq 0$

$$H_D^0 = \alpha \cdot \bar{p} + \beta m$$

$$\boxed{\begin{aligned} \bar{\alpha} &= \gamma^0 \bar{\gamma} \\ &= \begin{pmatrix} 0 & \bar{\sigma} \\ \bar{\sigma} & 0 \end{pmatrix} \\ \bar{\beta} &= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \end{aligned}}$$

$\Rightarrow$ TISE : $H_D \psi = E \psi$

Minimal prescription:

$$\hat{p} = -i\bar{\nabla} \longrightarrow (\bar{p} - Q\bar{A})$$

$$= -i(\bar{\nabla} - i\underset{\overset{\shortparallel}{qe}}{Q}\bar{A})$$

$$= -i\bar{D}$$

$$\begin{pmatrix} m - E + V & -i\bar{\sigma}\cdot D \\ -i\bar{\sigma}\cdot\bar{D} & -m - E + V \end{pmatrix} \begin{pmatrix} u \\ L \end{pmatrix} = 0 \quad \Bigg\} \begin{matrix} H_D\psi = E\psi \\ \psi = \begin{pmatrix} u \\ L \end{pmatrix} \end{matrix}$$

$$\Rightarrow u = \left(\frac{1}{m - (E - V)}\right) i\bar{\sigma}\cdot\bar{D}\, L \quad ; \quad L = \left(\frac{-1}{m + (E - V)}\right) i\bar{\sigma}\cdot\bar{D}\, u$$

Free particle:

$$\begin{pmatrix} \chi \\ \dfrac{\bar{\sigma}\cdot\bar{p}}{E + m}\chi \end{pmatrix} \xrightarrow[\frac{\bar{p}}{m} \to 0]{N.R} \begin{pmatrix} \chi \\ 0 \end{pmatrix} \qquad \chi = \begin{pmatrix} a \\ b \end{pmatrix}$$

$$\mathcal{U} = \frac{1}{m-E}(\cancel{i}\bar{\sigma}\cdot\bar{D})\frac{(\cancel{-1})}{E+m}(\cancel{i}\bar{\sigma}\cdot\bar{D})\mathcal{U}$$

$$= \frac{1}{m^2-E^2}(\bar{\sigma}\cdot\bar{D})^2 \; \mathcal{U} \qquad E \approx E_{NR} + m$$

$$(m + E_{NR} + m)(\cancel{m} - (E_{NR} + \cancel{m}))$$

$$\approx -2m E_{NR} + \text{small} \quad (E_{NR} \ll m)$$

- $\sigma_i \sigma_j D_i D_j = (\delta_{ij}\mathbb{1} + i\epsilon_{ijk}\sigma_k)D_i D_j$

$$= \bar{D}^2 + i\bar{\sigma}_k B_k(-iqe)$$

If we use $E^2 - m^2 = 2mE_{NR}$;

$$E_{NR}\,\mathcal{U}(\bar{r}) = \Big(-\frac{1}{2m}\underbrace{\bar{D}^2}_{(\bar{\nabla}-iqe\bar{A})} - \underbrace{qe\frac{\bar{\sigma}\cdot\bar{B}}{2M}}_{\text{magnetic moment}}\Big)\mathcal{U}(\bar{r})$$

$\Rightarrow \mu \sim \bar{S} = \bar{\sigma}/2$

For $\bar{A} \neq 0$, $A_0 = 0$:

$$\hookrightarrow H_D^{NR} = \frac{\bar{\pi}^2}{2m} - \bar{\mu}_e\cdot\bar{B} \; ; \quad \bar{\pi} = \bar{p} - Q\bar{A}$$

$$\bar{\mu} = \frac{qe\bar{S}}{m} = \left(\frac{qe}{m}\right)\frac{\bar{\sigma}}{2} \; ; \quad e > 0$$

$$= \left(\frac{geq\hbar}{2mc}\right)\frac{\bar{\sigma}}{2} \Rightarrow \text{Dirac eqn}: \; \underbrace{g}_{\text{lande g-factor}} = 2.$$

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$$\underline{i\epsilon_{ijk}\sigma_k D_i D_j} \qquad \text{test function}$$

$$= \epsilon_{ijk}(\partial_i - iqeA_i)(\partial_j - iqeA_j)\chi$$

$$= \underset{\text{asym}}{\epsilon_{ijk}}\Big[\underbrace{\cancel{\partial_i\partial_j\chi}}_{\text{sym}} - iqe\partial_i(A_j\chi) - iqeA_i\partial_j\chi - \underbrace{\cancel{q^2e^2A_iA_j\chi}}_{\text{sym}}\Big]$$

$$= \underset{\text{asym}}{\epsilon_{ijk}}\Big[-iqe\partial_iA_j\chi - \cancel{iqeA_j\partial_i\chi} - \cancel{iqeA_i\partial_j\chi}\Big] \leftarrow \text{sym}$$

$$= \frac{1}{2}\epsilon_{ijk}(-iqe)\underbrace{\partial_{[i}A_{j]}}_{\epsilon_{ijl}B_l}\chi$$

$$= \delta_{kl}(-iqe)B_l$$

b) $A_0 \neq 0 \Rightarrow V = qeA_0$; $E \to E - V$

Dirac equation : $L(\bar{r}) = \dfrac{1}{E+m-V(\bar{r})} (-i\bar{\sigma}\cdot\bar{D})\, U(\bar{r})$

$$U(\bar{r}) = \frac{1}{E-m-V(\bar{r})} (-i\bar{\sigma}\cdot\bar{D})\, L(\bar{r})$$

$$\Rightarrow \frac{1}{E-m-V} (-i\bar{\sigma}\cdot\bar{D}) \frac{1}{E+m-V} (-i\bar{\sigma}\cdot\bar{D})\, U = U$$

matrix   differential operator

Let $\bar{G} = G_- = E - m - V(\bar{r})$

$G = G_+ = E + m - V(\bar{r})$

$F = -i\bar{\sigma}\cdot\bar{D}$

$$\Rightarrow FG^{-1} = G^{-1}F - G^{-1}[F, G]G^{-1}$$

$$[F,G] = G^{-1}(\cancel{FG} - GF)\, G^{-1}$$

So, we require $[F, G] = [-i\bar{\sigma}\cdot(\bar{\nabla} - iq\cancel{A}), (\cancel{E} + \cancel{m} - V(\bar{r}))]$

$$= +i\bar{\sigma}\cdot(\bar{\nabla}V)$$

$$U = \bar{G}^{-1} F G^{-1} F U$$

$$= \bar{G}^{-1}(G^{-1}F - G^{-1}(i\bar{\sigma}\cdot\bar{\nabla}V)G^{-1}) F U$$

$$= \bar{G}^{-1}(G^{-1}F^2 - G^{-2}(i\bar{\sigma}\cdot\bar{\nabla}V)F)\, U$$

Approx: $E - V + m = E_{NR} + m + m - V \approx 2m + \text{small}$

$$\Rightarrow (E-V)^2 - m^2 \simeq (E_{NR} - V)\, 2m$$

$\bar{F} = -i\bar{\sigma}\cdot\bar{D} \Rightarrow F^2 = -(\bar{\sigma}\cdot\bar{D})^2 = -\bar{D}^2 +$ Mag. field

$$\Rightarrow \boxed{\begin{aligned}&\left(-\frac{1}{2}\frac{D^2}{m} - \frac{qe\,\bar{\sigma}\cdot\bar{B}}{2m} + V(r)\right)u + \\ &-\frac{1}{2m}\Big(\underbrace{\bar{\nabla}V\cdot\bar{D}u}_{\mathrm{I}} + i\,\bar{\sigma}\cdot(\underbrace{\bar{\nabla}V\times\bar{D}u}_{\mathrm{II}})\Big) = E_{NR}\,u\end{aligned}}$$

NR limit of Dirac eqn.
with $A_0 \neq 0$, $\bar{A} \neq 0$.

1st term: $(\bar{\nabla}V)\cdot(\bar{\nabla} - iqe\bar{A})u\left(-\frac{1}{4m^2}\right)$ $\quad \hookrightarrow O(A^2)$

$\simeq (\bar{\nabla}V)\cdot(\bar{\nabla}u)$

If V is radial fn: $\Rightarrow \bar{\nabla}V = \hat{r}\,V'(r) = \hat{r}\frac{\partial V}{\partial r}$

$\Rightarrow$ I $\simeq -\frac{1}{4m^2}V'(r)\frac{\partial u}{\partial r}$ $\Big\}$ Darwin term

2nd term: $\frac{-i}{4m^2}\,\bar{\sigma}\cdot\big(\bar{\nabla}V\times(\underbrace{\bar{\nabla}}_{a} - \underbrace{iqe\bar{A}}_{b})u\big)$

2-a: $\frac{-i}{4m^2}\,\bar{\sigma}\cdot(\underbrace{\bar{\nabla}V}_{=\hat{r}V'(r)}\times\bar{\nabla}u) = \frac{1}{4m^2}\frac{V'(r)}{r}\,\bar{\sigma}\cdot(\underbrace{\bar{r}\times(\overbrace{-i\bar{\nabla}}^{\bar{p}})}_{\vec{L}})u$

$= \frac{1}{2m^2}\frac{V'(r)}{r}\underbrace{\left(\frac{\bar{\sigma}}{2}\right)}_{\bar{S}}\cdot\bar{L}\,u$ $\Big\}$ Thomas/Spin orbit term
coupling

2-b: Drop $\sim A_\mu^2$

(also in $\bar{D}^2 u = (\bar\nabla - iqe\bar A)^2 u$)

$$= \bar\nabla^2 - \underbrace{\cancel{q^2e^2\bar A^2}}_{O(A^2)} - iqe\,(\cancel{\bar\nabla\cdot(\bar A u)} + \bar A\cdot\bar\nabla u)$$

$$= (\bar\nabla^2 - 2iqe\bar A\cdot\bar\nabla)u + \text{small}$$

$\boxed{\bar\nabla\cdot\bar A = 0 \quad \text{Coulomb gauge}}$

But darwin term is non-hermitian!

$\to \langle\psi|\varphi\rangle = \int d^3x\, \psi^*(x)\,\phi(x)$

$(\psi, \hat O_D \varphi) \neq (\phi, O_D\psi)^* \longrightarrow$ non-hermitian.

$O_D = (\quad)\, V'(r)\,\partial_r$

$\left(\int d^3x\,(\psi^* V'(r)\partial_r\varphi)\right)^* \longrightarrow \int d^3x\,(-\partial_r(V'(r)\psi(r)))\phi^*$

integration by parts

not hermitian.

Ad-hoc correction:

↳ Retain only hermitian part.

$$O_D \longrightarrow \frac{O_D + O_D^\dagger}{2}$$

- $X = V'(r)\partial_r \quad ; \quad (\psi, X^\dagger\phi) \equiv (X\psi, \phi)$

$$\int \psi^* X^\dagger \phi = \int d^3r\, (V'\partial_r\psi)^*\phi$$

$$= -\int d^3r\, \psi^* \frac{1}{r^2}\partial_r(r^2 V'(r)\phi)$$

$$\Rightarrow \int d\Omega\, r^2 dr \left(-\psi^* \frac{1}{r^2} \partial_r \left(r^2 V'(r) \varphi\right)\right)$$

$$\Rightarrow \int \psi^* X^\dagger \varphi \, d^3r = -\int d^3r\, \psi^* \left(\nabla^2 V(r) + V' \partial_r \varphi\right)$$

$$\Rightarrow \frac{X + X^\dagger}{2} = -\frac{1}{2} \left(\nabla^2 V(r)\right)$$

$$\Rightarrow O_{Darwin}^{(hermitian)} = \frac{1}{8m^2} \left(\nabla^2 V\right)$$

$$V = q e A_0 = -\frac{e^2}{4\pi r} \Rightarrow \nabla^2\left(\frac{1}{r}\right) = -4\pi \delta(\bar{r})$$

Necessity of adding $X^\dagger$;

$$\int \psi^\dagger \psi \, d^3\bar{r} = \int (u^\dagger u + L^\dagger L)\, d^3 r$$

$$L \to \frac{\bar{\sigma} \cdot \bar{p}}{2m} u$$

$$= \int u^\dagger \left(1 + \left(\frac{\bar{\sigma} \cdot \bar{p}}{2m}\right)^2\right) u \, d^3 r$$

$$= \int u^\dagger \left(1 + \frac{p^2}{4m^2}\right) u \, d^3 r$$

$$\Rightarrow \boxed{\int u^\dagger u \, d^3\bar{r} \neq 1} \quad !!$$

So we need to work with $\hat{u} = \Omega u = \left(1 + \frac{\bar{p}^2}{8m^2}\right) u$

$$\hat{u}^\dagger \hat{u} = u^\dagger \left(1 + \frac{\bar{p}^2}{8m^2}\right)^2 u$$

$$\simeq u^\dagger \left(1 + \frac{\bar{p}^2}{4m^2}\right) u + \text{small.} = 1$$

If we implement this, we get:

$$\hat{H}_D \simeq \frac{\bar{p}^2}{2m} + V(r) + \underbrace{\frac{\nabla^2 V}{8m^2}}_{\text{Darwin}} + \underbrace{\frac{V'(r)}{2m^2 r}}_{\text{Spin-orbit}} \bar{S} \cdot \bar{L} - \underbrace{\frac{(\bar{p}^2)^2}{8m^3}}_{\text{1st relativistic correction}}$$

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Lecture 20:

$$\psi \sim \begin{pmatrix} u \\ L \end{pmatrix} e^{i\vec{p}\cdot\vec{r} - iEt}$$

Dirac eqn

$\parallel$

$$(E - V(\vec{r}) - m)\, u = \left(-\vec{\sigma}\cdot\vec{D}\,(E - V + m)^{-1}\,\vec{\sigma}\cdot\vec{D}\right) u$$

$$= \left(E_{NR} - V(\vec{r})\right) u$$

<u>Leading terms in NR limit</u>:

$$\left(\frac{1}{2m}(\vec{\nabla} - iq e\vec{A})^2 - g q \mu_B \vec{S}\cdot\vec{B} + V(\vec{r})\right) u = E_{NR}\, u$$

<u>Problems with NR limit to next order</u>:

(1) $$(E - V + m)^{-1} = (E_{NR} - V + 2m)^{-1}$$

$$= \frac{1}{2m}\left[1 - \frac{(E_{NR} - V)}{2m} + O(m^{-2})\right]$$

So, $H_{NR}\, u = E_{NR}\, u$ is <u>not</u> an eigenvalue eqn in the next order of expansion!

$\because$ $H_{NR} \equiv H_{NR}(E_{NR})$

(2) $$\int \psi^{+}\psi \, d^3\vec{r} = 1 = \int (u^{+}u + L^{+}L)\, d^3\vec{r}$$

$$L = \frac{\vec{\sigma}\cdot\vec{p}}{2m} u + O(1/m^2) + \dots$$

So, we need $$\int d^3\vec{r}\; u^{+}\left(1 + \frac{\vec{p}^2}{4m^2} + \text{small}\right) u = 1$$

$\because$ $\int d^3\vec{r}\; u^{+}u \neq 1 \Rightarrow u$ cannot be ψ_{NR}!!

③ Naively, we obtained $H_{NR} \sim (\bar{\nabla} V) \cdot \bar{\nabla}$

$$\underbrace{O_D}_{\text{Darwin term}} \sim \underbrace{i\,\bar{E}\cdot\bar{p}}_{\text{non-hermitian}}$$

Adhoc fix: Take $\frac{O_D + O_D^\dagger}{2} \to$ hermitian part.

A systematic approach to fix these problems:

$$\int d^3\bar{r}\; \underbrace{u^\dagger \left(1 + \frac{\bar{p}^2}{4m^2}\right)}_{\Omega^\dagger \Omega} u \simeq 1 \qquad : \Omega = 1 + \frac{\bar{p}^2}{8m^2} + \text{small}$$

could be $\Psi_{NR} \longrightarrow \boxed{\hat{u} = \Omega u}$

- $H_{NR} u = \left[\frac{\bar{p}^2}{2m} + V + \frac{1}{2m}\bar{\sigma}\cdot\bar{p}\left(\frac{V}{2m}\right)\bar{\sigma}\cdot\bar{p} - \frac{E_{NR}}{2m}\frac{\bar{p}^2}{2m}\right] u$

$$= E_{NR}\, u$$

$$\Rightarrow \underbrace{\Omega^{-1} H_{NR} \Omega^{-1}}_{\hat{H}'} \hat{u} = E_{NR}\, \Omega^{-2} \hat{u} \quad ; \quad \Omega^{-1} = 1 - \frac{\bar{p}^2}{8m^2} + \dots$$

$$\Rightarrow \hat{H} = H_{NR} - \left\{ \frac{-\bar{p}^2}{2m}, \frac{\bar{p}^2}{2m} + V \right\} + \text{small.}$$

$$= \frac{\bar{p}^2}{2m} + V - \frac{E_{NR}}{2m}\frac{p^2}{2m} + \frac{1}{4m}\bar{\sigma}\cdot p\, V\, \bar{\sigma}\cdot\bar{p}$$

$$+ \frac{V}{4m^2}\bar{p}^2 - \underbrace{\frac{\bar{p}^4}{8m^3}}_{\text{first relativistic correction}} - \frac{1}{8m^2}\left\{\bar{p}^2, V\right\}$$

Now we need $\bar{\sigma}\cdot(\bar{p}V)\bar{\sigma}\cdot\bar{p}$

$$= (\bar{p}V)\cdot\bar{p} + i\bar{\sigma}\cdot((\bar{p}V)\times\bar{p})$$

- $\{\bar{p}^2, V(r)\}\psi = \bar{p}^2(V\psi) + V(\bar{r})\bar{p}^2\psi$

$$= 2(\bar{p}V)\cdot\bar{p}\psi + 2V\bar{p}^2\psi + (\bar{p}^2 V)\psi$$

$$\Rightarrow \frac{1}{4m^2}\left\{\bar{\sigma}\cdot(\bar{p}V)\bar{\sigma}\cdot\bar{p} - \frac{1}{2}\{\bar{p}, V\}\right\}$$

$$= \frac{1}{4m^2}\left\{i\bar{\sigma}\cdot((\bar{p}V)\times\bar{p}) - V\bar{p}^2\psi - \frac{\bar{p}^2 V}{2}\psi\right\}$$

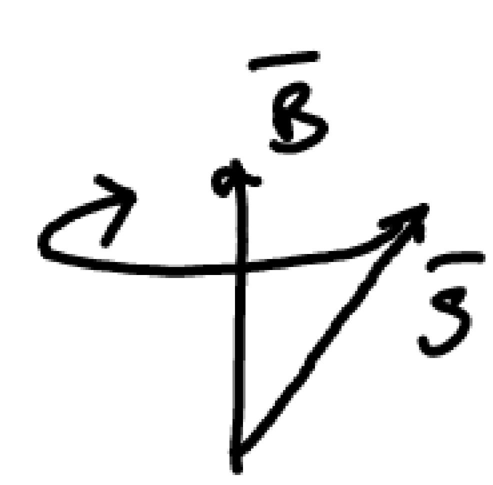
$$\Rightarrow \Bigg(\frac{\bar{p}^2}{2m} + V - \underbrace{\frac{\bar{p}^4}{8m^3}}_{\text{first relativistic correction}} - \underbrace{\frac{Q\bar{\sigma}\cdot(\bar{E}\times\bar{p})}{4m^2}}_{\text{thomas spin orbit term}} - \underbrace{Q\frac{\bar{\nabla}\cdot\bar{E}}{8m^2}}_{\text{darwin}}\Bigg)\hat{u} = E_{NR}\hat{u}$$

$$= \underbrace{\hat{H}_{NR}}_{\text{hamitian}}\hat{u}$$

Thomas spin orbit term:

→ Magnetic moment of e^- sees nuclear electric field as a magnetic field. → $-\mu_e\cdot\bar{B}$

→ Moreover, the e^- Spin $\bar{S}$ precesses in $\vec{B}$.



$$\bar{B} \sim -\bar{v}_e\times\bar{E} \sim -\frac{\bar{p}\times\bar{E}}{m}$$

$$(\Delta H) \simeq -\bar{\mu}_e\cdot\left(\frac{\bar{E}\times\bar{p}}{m}\right) \quad ; \quad \mu_e = \frac{Q}{m}\bar{S} = \frac{Q}{2m}\bar{\sigma}$$

$= 2\times$ Spin orbit/thomas.

→ Inclusion of spin precession reduces this

Spin orbit term : $V(r) \to \bar{\nabla} V = V'(r)\hat{r}$

$\hookrightarrow \frac{1}{2m^2} \frac{V'(r)}{r} \bar{S}\cdot\bar{L}$

Darwin term : $-\frac{q_e}{8m^2} \bar{\nabla}\cdot\bar{E}_{nucleus} = +\frac{q_e}{8m^2} \nabla^2 A_0 = \frac{\nabla^2 V}{8m^2}$.

$\nabla\cdot\bar{E} = \rho(\bar{r}) = -\nabla^2 A_0$

$\hookrightarrow$ for point nucleus, $\rho(\bar{r}) = q_n \delta^3(\bar{r})$

$\hookrightarrow$ leads to $Z_N |e|^2 \delta^3(\bar{r})$ in H_{NR}.

$\Rightarrow \underset{S}{(\Delta E)_{Darwin}} \sim Z_N |e|^2 |\psi(0)|^2$

$\langle\psi|\Delta H_{Darwin}|\psi\rangle = \int d^3r\, \psi^* \Delta H \psi$

} Only s-wave parts contribute to this.

Physically, darwin term is due to "Jittering motion" with frequency $\sim m_e$.

$\psi(t) \sim e^{iEt}$; $E = m +$ b.e. + relativistic

$(m >> E_{NR})$

$\langle V(\underbrace{x+\delta x}_{\text{high freq.}})\rangle = \langle V(\bar{x}) + \underbrace{\cancel{\delta\bar{x}\cdot\bar{\nabla}V}}_{\text{average}=0} + \frac{1}{2}\delta x_i \delta x_j \partial_i\partial_j V + \ldots\rangle$

$= \langle(\delta\bar{x})^2\rangle \nabla^2 V \simeq \frac{1}{6m^2}\nabla^2 V$.

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Zitterbewegung :

- Heisenberg eqn. of motion for a free particle :

[$O(t)$; $|\Psi\rangle$ - independent of t]

$$\rightarrow \frac{d}{dt}\Omega_H(t) = -i\left[\Omega_H(t), H\right] + \underbrace{\frac{\partial \Omega_H}{\partial t}}_{=0}$$

0 (Autonomous system)

Constants of motion :

- $H_D = \bar{\alpha}\cdot\bar{p} + \beta m$

$$\frac{d}{dt}(\bar{p}_i) = -i\left[p_i, \bar{\alpha}\cdot\bar{p} + \cancel{\beta m}\right] = 0$$

$\therefore$ $\bar{p}$ is a const. of motion.

$$\frac{d(L_H)_i}{dt} = -i\left[\epsilon_{ijk}\, x^j p^k , \bar{\alpha}\cdot\bar{p} + \cancel{\beta m}\right]$$

$$= (\bar{\alpha}\times\bar{p})_i$$

$$\frac{d}{dt}\left(\frac{\Sigma_H}{2}\right)_i = -(\bar{\alpha}\times\bar{p})_i \quad \Rightarrow \quad \frac{d}{dt}\bar{J} = 0 \; ; \; \bar{J} = L + \frac{\Sigma}{2}$$

Trajectory of a free Dirac particle of momentum $\bar{p}$:

$$\dot{x}_i = -i[\bar{x}_i, \bar{\alpha}\cdot\bar{p} + \beta m] = +\alpha_i(t) \neq 0$$

$\alpha_{\text{Heisenberg}}$

$$\bar{\alpha}_S = \gamma^0 \bar{\gamma} \Rightarrow \bar{\alpha}_S^2 = \gamma^0 \gamma^i \gamma^0 \gamma^i = \mathbb{1}_4$$

$$\Rightarrow \bar{\alpha}_H^2 = \mathbb{1}_4$$

$\therefore$ eigenvalues of $\bar{\alpha}$ are ± 1.

Velocity $= \bar{\alpha}(t)$; and eigenvalues are $\pm 1 \; (= \pm c)$

So, measurement of velocity of massive dirac particle should always give c. How ??

Trajectory :

$$-i\left[\bar{x}, \frac{\bar{p}^2}{2m}\right] = \frac{\bar{p}}{m} = \bar{v} \quad \text{in NRQM}$$

- $$\frac{d^2\bar{x}}{dt^2} = \frac{d}{dt}\bar{\alpha}_H(t) = -i\left[\bar{\alpha}_H(t), \bar{\alpha}\cdot\bar{p} + \beta m\right]$$

$$[A, B] = AB + BA - 2BA = \{A, B\} - 2BA$$

$$= -i\left(\{\alpha_j, H_D\} - 2H_D\alpha_j\right)$$

$$\Rightarrow \frac{d\bar{\alpha}_j}{dt} = -i2\bar{p}_j + 2iH_D\alpha^j$$

$$\{\alpha_i, \alpha_j\} = 2\delta_{ij}$$

$$\{\alpha_i, \beta\} = 0$$

$$\Rightarrow \{\alpha_j, H_D\} = 2p_j$$

Consider:

$$\frac{d}{dt}\Big(\underbrace{e^{-2iH_D t}}_{\text{integrating factor}}\bar{\alpha}_j\Big) = e^{-2iH_D t}\frac{d\bar{\alpha}_j}{dt} - 2i e^{-2iH_D t} H_D \alpha_j$$

$$= -i2p^j e^{-2iH_D t}$$

Integrate this:

$$\Rightarrow \quad e^{-2iH_D t}\bar{\alpha}_j(t) = H_D^{-1} p^j e^{-2iH_D t} + \underset{\hookrightarrow \text{const.}}{\bar{a}}$$

- $\bar{v}(0) = H_D^{-1}\bar{p} + \bar{a}$

$$\Rightarrow \bar{v}(t) = \bar{\alpha}(t) = H_D^{-1}\bar{p} + \underbrace{e^{2iH_D t}}_{\substack{\text{fast fluctuations} \\ (\text{freq} \sim 2m + \text{small})}}\left(\bar{v}(0) - H_D^{-1}\bar{p}\right)$$

$$\langle \bar{v}(t)\rangle_{\text{avg. over } T \gg \frac{1}{2m}} = \frac{\bar{p}}{H_D} = \frac{\bar{p}}{E}$$

$$E = \gamma m \; ; \; \bar{p} = \gamma m \bar{v} \Rightarrow \frac{\bar{p}}{E} = \bar{v} \; !$$

So, on long time scales, the usual relativistic $\bar{v}$ momentum relation $\bar{v} = \frac{\bar{p}}{E}$ holds true. The fast fluctuation does, however, have observable effects, e.g. in Darwin term where the $\bar{e}$ in an atom instead of seeing a smooth nuclear potential, actually samples a range of $\bar{r}$ value due to its relativistic jitter at freq. $\nu \sim 2m$.

$$\Rightarrow \langle \delta x_i \, \delta x_j \, \partial_i \partial_j V \rangle \simeq \frac{1}{3} \underbrace{\langle \delta \bar{x}^2 \rangle}_{\text{non-zero for dirac particle.}} \cdot \nabla^2 V$$

There are additional counter-intuitive effects for relativistic particles which further invalidates the single particle interpretation.

→ Klein paradox

→ Interference of +ve and -ve freq. components

These will motivate the foundations of field theory.

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Lecture 21:

Zitter bewegung: "Jittering motion"

Heisenberg picture: $\frac{d\bar{x}}{dt} = \bar{\alpha}_H(t)$

$\bar{\alpha}_S = \begin{pmatrix} 0 & \bar{\sigma} \\ \bar{\sigma} & 0 \end{pmatrix}$; $\bar{\alpha}_H(t) = \mathcal{U}^\dagger(t)\,\bar{\alpha}_S\,\mathcal{U}(t)$; $\mathcal{U} = e^{-iH_D t}$

$$\underset{\substack{\parallel \\ -i[\bar{x}_H(t),\, H_D]}}{\bar{\alpha}_H(t)} = \bar{p}\,H_D^{-1} + \underset{\substack{\uparrow \\ \text{Fast fluctuation at freq. } \nu \sim 2m}}{e^{i2H_D t}} \left(\bar{\alpha}_H(0) - \bar{p}\,H_D^{-1}\right)$$

(w/ amplitude $\sim$ compton $\lambda = \frac{\hbar}{mc}$)

$1\text{ eV} \sim 10^{15}\text{ Hz}$

$1\text{ MeV} \sim 10^{21}\text{ Hz}$

Schrodinger picture:

$\frac{dO_S}{dt} = 0$; $i\frac{d}{dt}|\psi_S(t)\rangle \neq 0$.

$\langle \psi_S(t)|\,\bar{\alpha}_S\,|\psi_S(t)\rangle$

$$\psi(\bar{x},t) = \int \frac{d^3\bar{p}}{\sqrt{(2\pi)^3 2E_p}} \sum_s \Big(\underbrace{f_s(\bar{p})\, u_s(\bar{p})\, e^{\overset{-iE_pt + i\bar{p}\cdot\bar{x}}{-i\bar{p}\cdot\bar{x}}}}_{\psi_+(\bar{x})} + \underbrace{\hat{f}_s(\bar{p})^*\, v_s(\bar{p})\, e^{\underset{iE_pt - i\bar{p}\cdot\bar{x}}{i\bar{p}\cdot\bar{x}}}}_{\psi_-(\bar{x})} \Big)$$

$$\parallel$$

$$\hat{f}_s(-\bar{p})^*\, v_s(-\bar{p})\, e^{iE_pt + i\bar{p}\cdot\bar{x}}.$$

$$\Rightarrow \Psi(\bar{x}, t) = \underbrace{\mathop{S}_{\bar{p}, s}}_{\text{generalized sum.}} \left(f_s(\bar{p})\, \mathcal{U}_{\bar{p},s} + f_s^{+}(\bar{p}) V_{\bar{p},s} \right)$$

$$\mathcal{U}_{\bar{p},s} = \frac{1}{\sqrt{(2\pi)^3 2E_p}}\, u_s(\bar{p})\, e^{-ip\cdot\bar{x}}$$

$$= \sum c_k |e_k\rangle \qquad \boxed{E_p = \sqrt{p^2 + m^2}}$$

$$\Rightarrow \overline{\Psi}(\bar{x}, t) = \psi^{+}(\bar{x}, t)\gamma^0$$

$$= \mathop{S}_{\bar{p}, s} \left(f_s^{*}(\bar{p})\, \overline{u}_s(\bar{p})\, e^{ip\cdot x} + \hat{f}_s(\bar{p})\, \overline{v}_s(\bar{p}) e^{-ip\cdot x} \right)$$

- $f_r(\bar{q}) = (\mathcal{U}_r(\bar{q}), \Psi)$

$$= \int \frac{d^3\bar{x}}{\sqrt{2E_q (2\pi)^3}}\, u_r^{+}(q, x)\, e^{iq\cdot x}\, \psi(x)$$

- $\hat{f}_r^{*}(\bar{q}) = (V_r(\bar{q}), \Psi(\bar{x}))$

Also, orthonormality $(\mathcal{U}_r(\bar{q}), \mathcal{U}_s(\bar{p})) = \delta_{rs}\,\delta^3(\bar{p} - \bar{q})$

$$(V_r(\bar{q}), V_s(\bar{p})) = \delta_{rs}\,\delta^3(\bar{p} - \bar{q})$$

$$(\mathcal{U}_r(\bar{q}), V_s(\bar{p})) = 0.$$

We require $\langle \bar{\alpha}_s \rangle = (\Psi, \bar{\alpha}_s \Psi)$

$$\langle \bar{\alpha}_s \rangle = \underset{\bar{p},r}{S} \; \underset{\bar{q},s}{S} \Big[f^*_{\bar{p},r} f_{\bar{q},s} \int d^3\bar{x}\, u^+_{\bar{p},r}(x)\, \bar{\alpha}\, u_{\bar{q},s}(x)$$

$$+ \hat{f}_{\bar{p},r} \hat{f}^*_{\bar{q},s} \int d^3\bar{x}\, V^+_{\bar{p},r}(x)\, \bar{\alpha}\, V_{\bar{q},s}(x)$$

$$+ f^*_{\bar{p},r} \hat{f}^*_{\bar{q},s} \int d^3\bar{x}\, u^+_{\bar{p},r}(x)\, \bar{\alpha}\, V_{\bar{q},s}(x)$$

$$+ \text{hermitian conj.} \Big]$$

- $$\int d^3x\, e^{ip\cdot x} e^{-iq\cdot x} = \int d^3x\, e^{i(E_p - E_q)t + i(\bar{q}-\bar{p})\cdot\bar{x}}$$

$$= (2\pi)^3 \underset{E_p = E_q}{\delta^3(\bar{p}-\bar{q})} e^{i(E_p - E_q)t} = (2\pi)^3$$

- $$\int d^3x\, e^{i(E_p t - \bar{p}\cdot\bar{x})} e^{i(E_q t - \bar{q}\cdot\bar{x})}$$

$$= (2\pi)^3 \underset{E_p = E_q}{\delta^3(\bar{p}+\bar{q})}\, e^{i(E_p + E_q)t} = \underbrace{e^{2iE_p t}}_{\text{fast fluctuations!}}$$

$$\Rightarrow u^+_{\bar{p},r}\, \bar{\alpha}\, v_{-\bar{p},s} = \left(\sqrt{E_p + m}\right)^2 \left(\chi_r^+ , \frac{\bar{\sigma}\cdot\bar{p}}{E_p+m}\chi_r^+ \right) \begin{pmatrix} \bar{\sigma}\chi_r \\ -\dfrac{\bar{p}\,(\bar{\sigma}\cdot\bar{p})\,\chi_r}{E_p + m} \end{pmatrix}$$

$$= 2E_p \chi_r^+ \bar{\sigma} \chi_r - \frac{2\bar{p}}{E_p + m} \chi_r^+ \bar{\sigma}\cdot\bar{p}\, \chi_r$$

$$\Rightarrow \langle \bar{\alpha} \rangle = \underset{\bar{p},r}{S} \left(|f_{\bar{p},r}|^2 + |f_{\bar{q},s}|^2 \right) \frac{\bar{p}}{E_p}$$

$$+ \underset{\bar{p},r}{S} \underbrace{f^*_{\bar{p},r} \hat{f}^*_{-\bar{p},r} e^{-2iE_p t}}_{\text{Zwitterbewegung}} \chi_r^+ \left(\bar{\sigma} - \frac{\bar{p}\,\bar{\sigma}\cdot\bar{p}}{(E_p + m)E_p} \right) \chi_r + \text{h.c.}$$

$\Rightarrow$ Interference of +ve and −ve frequency oscillations.

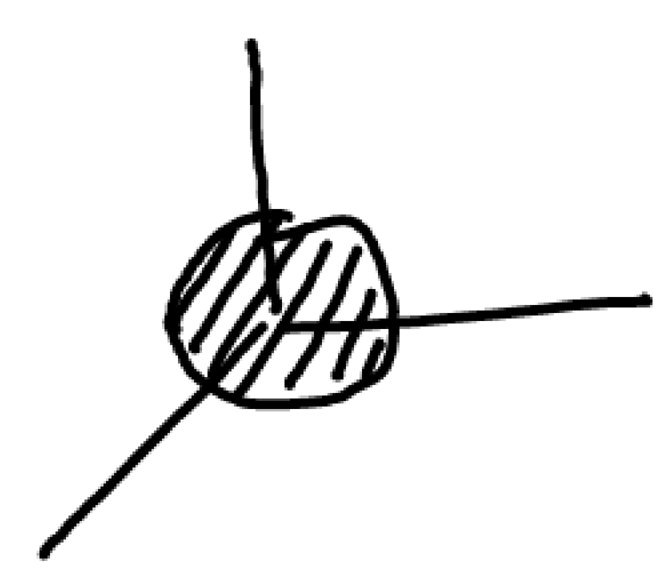
But −ve frequency components are always present to a significant extent for a Dirac particle wave-packet which is localized to a width $\Delta x \lesssim 1/m$.

$\therefore \left| \dfrac{\hat{f}_{\bar{p}}}{f_{\bar{p}}} \right| \sim \dfrac{\bar{p}}{2m} \Rightarrow$ If $\bar{p} \sim m$, −ve and +ve freq. are comparable.

So, −ve freq. cannot be ignored!

e.g. $\psi(\bar{x},0) = \begin{pmatrix} \phi(\bar{x}) \\ 0 \\ 0 \\ 0 \end{pmatrix}$ $\quad \phi(\bar{x})$ has support $\sim (\Delta\bar{x})$

$\hookrightarrow \phi(\bar{x}) \neq 0$ in a region of size $|\Delta\bar{x}|$



$$\psi(\bar{x},t) = \sum_{\bar{p},r} \underbrace{f_{\bar{p},r}}_{(u_{\bar{p},r}(\bar{x},0),\, \psi(\bar{x},0))} u_{\bar{p},r}(\bar{x},t) + \underbrace{\hat{f}^{*}_{p,s}}_{(V_{\bar{p},s}(\bar{x},0),\, \psi(\bar{x},0))} V_{\bar{p},s}(\bar{x},t)$$

$$\Rightarrow f_{\bar{p},r} \simeq \tilde{\phi}(\bar{p}) \sqrt{E_p+m} \left(u_r^{+} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \right)$$

$$\hat{f}_{\bar{p},r} \simeq \tilde{\phi}(\bar{p}) \sqrt{E_p+m}\; u_r^{+} \frac{\bar{\sigma}\cdot\bar{p}}{E_p+m} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\boxed{\begin{array}{c} \tilde{\phi}(\bar{p}) \\ \downarrow \\ \text{fourier transform of } \phi(x) \end{array}}$$

$$\Rightarrow \quad \frac{\hat{f}_{-\bar{p},1}}{f_{\bar{p},1}} \sim \frac{-p_3}{E_p+m} \quad ; \quad \frac{\hat{f}_{-\bar{p},2}}{f_{\bar{p},1}} \sim \frac{-p_+}{E_p+m}$$

If $\tilde{\phi}(\bar{p}) \neq 0$ for $\bar{p}$ upto $|\bar{p}| \sim m$

i.e. $\phi(\bar{x})$ is localized to $\Delta x \sim \frac{1}{m}$.

So, if we use only +ve wave, we cannot localize to scales $< 1/m$.

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Klein paradox:

In a region of constant potential $V(x)$, we have from the Dirac eqn.:

$$H_D = \bar{\alpha}\cdot(\bar{p} - \underbrace{q\bar{A}}_{=0}) + \beta m + \underbrace{V(\bar{x})}_{\text{const.}}$$

Solve for u:

$$\underbrace{(\bar{\sigma}\cdot\bar{p})^2}_{=\bar{p}^2} u_{\bar{p},r} = (E - V + m)(E - V - m)\, u_{\bar{p},r}$$

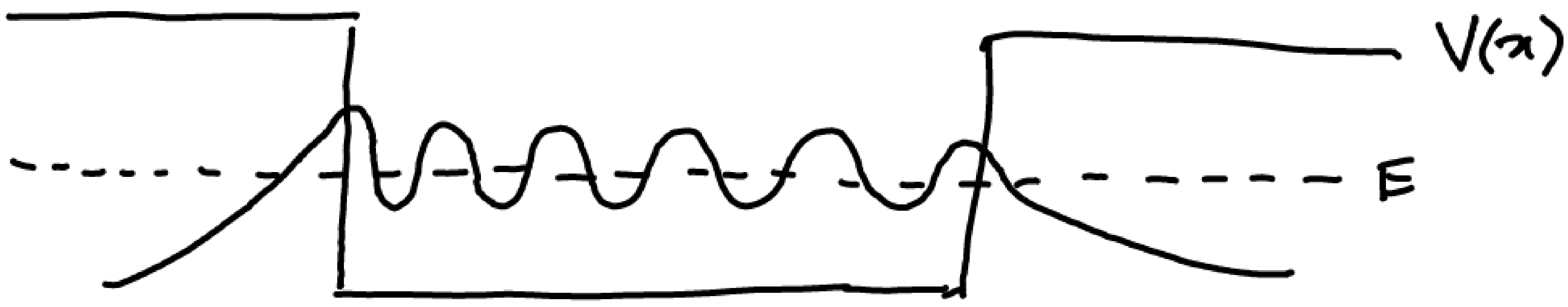
So, $\bar{p}^2 = (E-V+m)(E-V-m) = (2m + E_{NR} - V)(E_{NR} - V)$

$\boxed{E_{NR} = E - m}$ Energy rel. to rest mass.

① $V > 0$: Repulsive potential

If $\bar{p}^2 > 0 \Rightarrow$ Wavefn. is oscillatory

$\bar{p}^2 < 0 \Rightarrow$ Wavefn. is damped



- $\bar{p}^2 > 0$ if

i) $E - V + m > 0$, $E - V - m > 0$

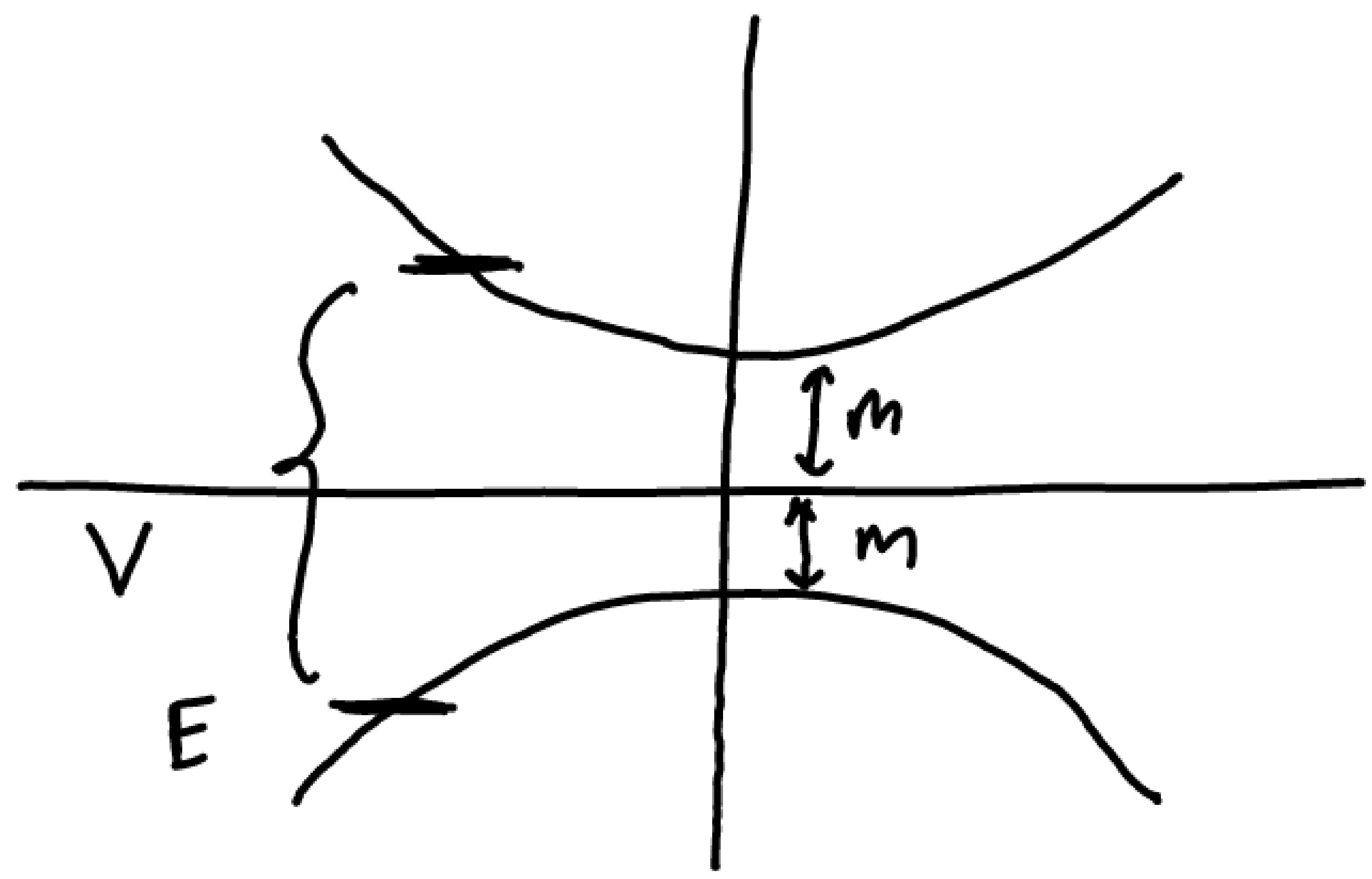
is usual ; $2m + E_{NR} > V$ and $\underline{E_{NR} > V}$

ii) $E - V + m < 0$; $E - V - m < 0$

$$\left.\begin{array}{l} \Rightarrow 2m + E_{NR} < V_{repulsive} > 0 \\ \quad E_{NR} < V_{repulsive} \end{array}\right\} \text{again oscillatory ??}$$

→ So, if the potential is so repulsive s.t. $V > 2m$, then we get oscillatory / undamped particle propagation even in the classically forbidden region. Since $V > 2m + \ldots$ this should be associated with pair creation.

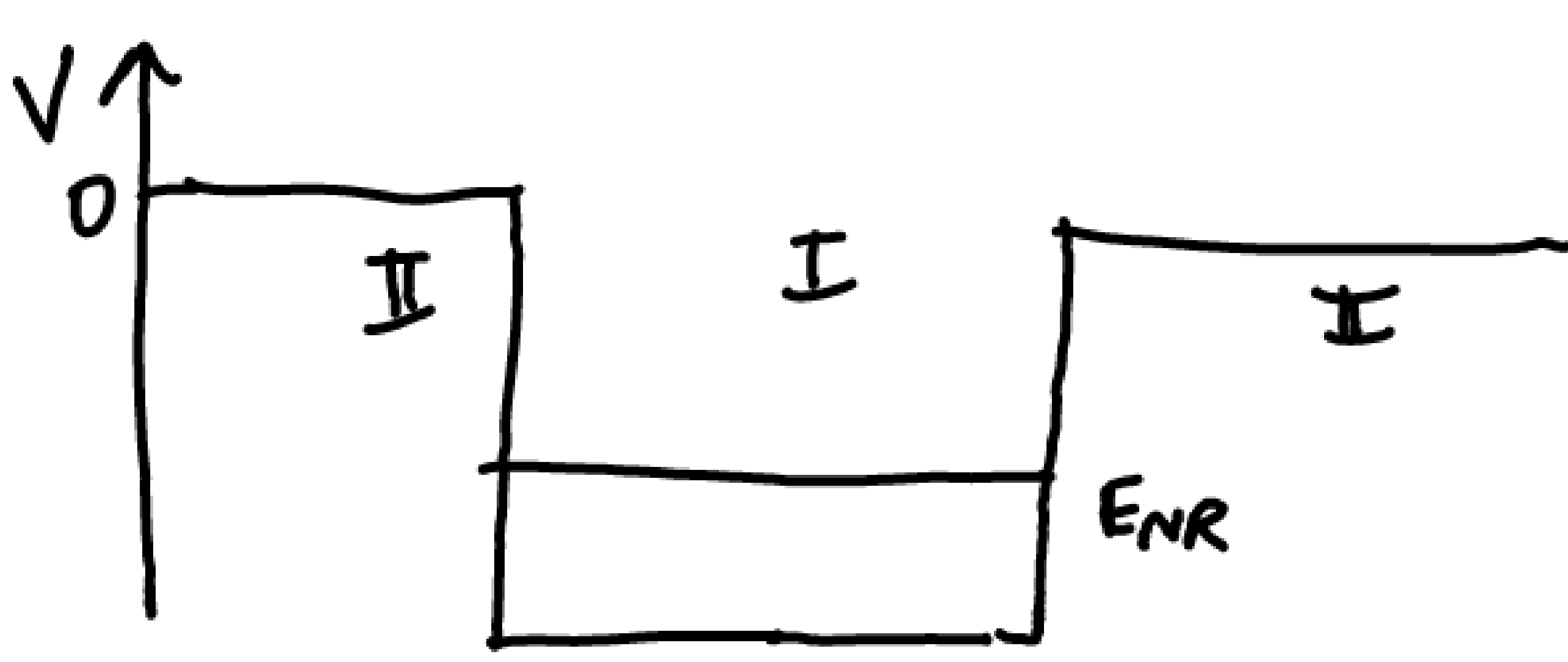
→ These oscillatory solutions are basically −ve energy states raised by $V > 0$ to +ve energy, leaving a hole which itself behaves as a particle of mass m.



$$V + E = V - \sqrt{p^2 + m^2}$$

↳ particle-antiparticle pair which propagates undamped.

② $V < 0$; Attractive potential



$V_I = -|V_0|$

<u>N.R case</u> : $P_I = \sqrt{2m(|V_0| + E_{NR})}$ ~ real

$$E_{NR} = \frac{P^2}{2m} - |V_0|$$

⇒ In region I → it is oscillatory.

In region II, $P_{II} = \sqrt{2mE_{NR}}$ ~ img.

⇒ Classically forbidden, damped propagation.

In relativistic case,

$$P_{I} = \sqrt{p^2} = \sqrt{(E_{NR}-V)(2m+E_{NR}-V)}$$
$$= \sqrt{(|V|-|E_{NR}|)(2m+|V|-|E_{NR}|)}$$

$$P_{II} = \sqrt{|E_{NR}|(|E_{NR}|-2m)}$$

Suppose the well is very deep:

$$|E_{NR}| > 2m$$

$\Rightarrow$ P_{II} is oscillatory.

$\Rightarrow$ Sufficiently strong attractive potential creates an $e^- - e^+$ pair (out of vacuum!) which propagates without damping.

→ Thus, the Klein paradox is intelligible as e^+e^- pair creation due to large $|V|$ but again with the Dirac Sea, i.e. ∞ of particles and states playing an essential role.

→ Thus we require a better theory: QFT!

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