0.1 Special Relativity

In Special Relativity, we use a lot of 4-vector notation to represent spacetime coordinates. Starting with the concept of tensors,

$$
\begin{aligned}
x^\mu &\equiv (x^0, x^1, x^2, x^3) && \text{Contravariant Tensor} \\
x_\mu &\equiv (x_0, x_1, x_2, x_3) && \text{Covariant Tensor}
\end{aligned}
$$

For the Cartesian Coordinate system, $x^0 = ct, x^1 = x, x^2 = y, x^3 = z$, and in general, $x^\mu \neq x_\mu$.

$$\boxed{x^\mu = (ct, x, y, z)}$$

The Minkowski Metric $(-, +, +, +)$ signature is given as-

$$\eta_{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad \text{if we use the } (ct, x, y, z) \text{ coordinate system.}$$

The rank of the tensors is given by the number of upper and lower indices it is written with. The typical notation to write the type of tensor is (upper indices, lower indices).

For example-

- $X^\mu \rightarrow$ Rank (1,0) contravariant tensor (4-vector)
- $X_\nu \rightarrow$ Rank(0,2) covariant tensor (4-vector)
- $\eta_{\mu\nu} \rightarrow$ Rank (0,2) covariant tensor (metric)

By definition, the contravariant metric $\underline{\eta^{\mu\nu} \text{ is the inverse of } \eta_{\mu\nu} \text{ i.e. } \sum_\alpha \eta^{\mu\alpha}\eta_{\alpha\nu} = \delta^\mu_\nu}$.

$$\eta_{\mu\nu} = \eta^{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

0.2 Dot Product

The 4-vector dot prouct is defined using the Minkowski metric-

$$a \cdot b = \sum_{\mu,\nu} \underbrace{\eta_{\mu\nu}}_{(0,2)} \underbrace{a^\mu b^\nu}_{(2,0)} = \eta_{00}a^0b^0 + \eta_{11}a^1b^1 + \eta_{22}a^2b^2 + \eta_{33}a^3b^3 = -a^0b^0 + a^1b^1 + a^2b^2 + a^3b^3$$

The (0,2) tensor cancels with (2,0) tensor to give a (0,0) tensor.

Einstein Summation Notation: We will use this summation convention throughout the notes wherever writing Σ's becomes cumbersome . We sum over repeated lower and upper indices.

$$\sum_{\mu,\nu} a^\mu b^\nu \equiv \eta_{\mu\nu} a^\mu b^\nu$$

Apart from Dot Products, $\eta^{\mu\nu}$ and $\eta_{\mu\nu}$ can be used for raising and lowering indices respectively.

If a_μ is given, then its raised version can be written as-

$$\boxed{a^\mu = \sum_\nu \eta^{\mu\nu} a_\nu \equiv \eta^{\mu\nu} a_\nu}$$

Similarly, if a^ν is given, then its lowered version can be written as-

$$\boxed{a_\mu = \sum_\nu \eta_{\mu\nu} a^\nu \equiv \eta_{\mu\nu} a^\nu}$$

Using these, we can define a dot product $a \cdot b$ as-

$$a \cdot b = \eta_{\mu\nu} a^\mu b^\nu = \begin{cases} \underbrace{\eta_{\mu\nu} a^\mu}\, b^\nu & = a_\nu b^\nu \\ \underbrace{\eta_{\mu\nu} b^\nu}\, a^\mu & = a^\mu b_\mu \end{cases}$$

Exercises:

1. $x^\mu = (ct, x, y, z)$. Find $x \cdot x$ and x_μ.
2. $p^\mu = (E/c, p_x, p_y, p_z)$. Find $p \cdot p$ and p_μ.

Solutions:

1. (a) The dot product of x with itself can be calculated as follows-

$$x \cdot x = \sum_{\mu,\nu} \eta_{\mu\nu} x^\mu x^\nu = x^\mu x_\mu = x_\mu x^\mu = -(ct)^2 + x^2 + y^2 + z^2$$

(b) To lower the index of x^μ, we use-

$$x_\mu = \sum_{\mu,\nu} \eta_{\mu\nu} x^\nu = \eta_{\mu 0} x^0 + \eta_{\mu 1} x^1 + \eta_{\mu 2} x^2 + \eta_{\mu 3} x^3$$

$$\Rightarrow x_0 = -ct, x_1 = x, x_2 = y, x_3 = z \Rightarrow \boxed{x_\mu = (-ct, x, y, z)}$$

2. (a) Again, the dot product of p with itself can be calculated as

$$p \cdot p = \sum_{\mu,\nu} \eta_{\mu\nu} p^\mu p^\nu = p^\mu p_\mu = p_\mu p^\mu = -(E/c)^2 + p_x^2 + p_y^2 + p_z^2$$

(b) To lower the index of p^μ, we use-

$$p_\mu = \sum_{\mu,\nu} \eta_{\mu\nu} p^\nu = \eta_{\mu 0} p^0 + \eta_{\mu 1} p^1 + \eta_{\mu 2} p^2 + \eta_{\mu 3} p^3$$

$$\Rightarrow x_0 = -E/c, x_1 = p_x, x_2 = p_y, x_3 = p_z \Rightarrow \boxed{p_\mu = (-E/c, p_x, p_y, p_z)}$$

0.3 Lorentz Transformations

A Lorentz Transformation is a transformation of frame of references related by a Lorentz boost or rotations. It is denoted by Λ^μ_ν (1,1) tensor.

$$a^\mu \to a'^\mu = \sum_\nu \Lambda^\mu_\nu a^\nu$$

$$b_\mu \to b'_\mu = \sum_\nu \Lambda^\mu_\nu b_\nu$$

As a rule of thumb, for each upper index $^\mu$, a Λ^μ_ν appears. For each lower index $_\mu$, a $(\Lambda^\mu_\nu)^{-1}$ appears.

Thus, a $X_{\mu\nu}$ would transform with two Λ^{-1} transforms, and a $Y^{\mu\nu}$ would transform with two Λ transforms.

$$X'_{\mu\nu} = \sum_{\alpha,\beta} (\Lambda^\alpha_\mu)^{-1} (\Lambda^\beta_\nu)^{-1} X_{\alpha\beta}$$

$$Y'^{\mu\nu} = \sum_{\alpha,\beta} (\Lambda^\mu_\alpha)\,(\Lambda^\nu_\beta) Y^{\alpha\beta}$$

The Minkowski metric tensor is invariant under Lorentz Transformation.

$$\eta_{\mu\nu} \to \eta'_{\mu\nu} = \sum_{\alpha,\beta} (\Lambda^\alpha_\mu)^{-1} (\Lambda^\beta_\nu)^{-1} \eta_{\alpha\beta} = \eta_{\mu\nu}$$

$$\eta^{\mu\nu} \to \eta'^{\mu\nu} = \sum_{\alpha,\beta} (\Lambda^\mu_\alpha)(\Lambda^\nu_\beta) \eta^{\alpha\beta} = \eta^{\mu\nu}$$

Another interesting point is that the dot products are invariant under a Lorentz Transformation. The dot product is defined as $a \cdot b = \eta_{\mu\nu} a^\mu b^\nu$. Under a Lorentz Transformation, $a \cdot b \to a' \cdot b'$.

$$a' \cdot b' = \eta_{\mu\nu}\, \Lambda^\mu_\alpha\, a^\alpha\, \Lambda^\nu_\beta\, b^\beta = \underbrace{\Lambda^\mu_\alpha\, \Lambda^\nu_\beta\, \eta_{\mu\nu}}\, a^\alpha b^\beta = \eta_{\alpha\beta}\, a^\alpha b^\beta = a \cdot b$$

As a corrolary to this, the dot product $p \cdot p = -E^2/c^2 + |\vec{p}|^2$ is also invariant under L.T. Thus-

$$p \cdot p = -(\frac{E^2}{c^2} - (p_x^2 + p_y^2 + p_z^2)) = -m_0^2 c^2 \qquad \text{is an invariant quantity under L.T.}$$

We call m_0 as *Invariant Mass*.

Exercise: Prove that speed of light c is also invariant under Lorentz Transformations.

0.4 Quantizing the 4-vectors

Quantizing the Lorentz invariant dot product of p^μ -

$$\sum_\mu p_\mu\, p^\mu = -m_0^2 c^2 \qquad \longrightarrow \qquad \sum_\mu \hat{p}_\mu\, \hat{p}^\mu\, \psi(\vec{x},t) = -m_0^2 c^2 \psi(\vec{x},t)$$

Extrapolating the momentum 4-vector operator by the usual method (even though there doesn't exist any time operator in QM)-

$$\hat{p}_\mu = \frac{\hbar}{i}\partial_\mu = \frac{\hbar}{i}\frac{\partial}{\partial x^\mu} = \frac{\hbar}{i}\left(\frac{1}{c}\frac{\partial}{\partial t}, \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)$$

$$\Rightarrow \hat{p}^\mu = \frac{\hbar}{i}\partial^\mu = \frac{\hbar}{i}\frac{\partial}{\partial x_\mu} = \sum_\nu \eta^{\mu\nu}\hat{p}_\nu = \frac{\hbar}{i}\left(-\frac{1}{c}\frac{\partial}{\partial t}, \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)$$

Thus, the $\hat{p}_\mu\, \hat{p}^\mu$ operator equation can be written as-

$$\sum_\mu \hat{p}_\mu\, \hat{p}^\mu \psi(\vec{x},t) = -\hbar^2\left(-\frac{1}{c^2}\partial_t^2 + \nabla^2\right)\psi(\vec{x},t) = \frac{-m_0^2 c^2}{\hbar^2}\psi(\vec{x},t)$$

And in all its glory, we get the Klein-Gordon Equation-

$$\boxed{\left(-\frac{1}{c^2}\frac{\partial^2}{\partial t^2} + \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)\psi(t,\vec{x}) = \frac{m_0^2 c^2}{\hbar^2}\psi(t,\vec{x})}$$

Using the d'Alembertian operator-

$$\Box \equiv -\partial_\mu \cdot \partial^\mu = -\frac{1}{\hbar^2}p_\mu \cdot p^\mu = \frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2} - \frac{\partial^2}{\partial z^2}$$

we can rewrite the equation as-

$$\boxed{\left(\Box + \frac{m_0^2 c^2}{\hbar^2}\right)\psi(t,\vec{x}) = 0}$$

Using $\psi(t,\vec{x}) = \exp\left(\frac{i}{\hbar}p_\mu \cdot x^\mu\right)$ as a trial solution-

$$\Box\psi = \left(\frac{1}{c^2}\partial_t^2 - \nabla^2\right)\exp\left(\frac{-i}{\hbar}(Et - \vec{p}\cdot\vec{x})\right) = -\frac{1}{\hbar^2}\left(\frac{E^2}{c^2} - |\vec{p}|^2\right)\psi = -\frac{1}{\hbar^2}m_0^2 c^2 \psi$$

This relates the parameters $E, \vec{p}, m_0$ as

$$E = \pm c\sqrt{m_0^2 c^2 + |\vec{p}|^2}$$

However, E isn't energy yet. It is just some parameter of the trial wavefunction, for now.

0.4.1 Conserved Current j^μ

We have the Klein-Gordon Equation as-

$$(\partial^\mu \cdot \partial_\mu - m_0^2c^2)\psi = 0 \qquad \xrightarrow{\text{Complex Conjugation}} (\partial^\mu \cdot \partial_\mu - m_0^2c^2)\psi^* = 0$$

Now multiplying first equation by ψ^* and the second one by ψ and subtracting them-

$$\psi^*(\partial^\mu \cdot \partial_\mu - m_0^2c^2)\psi - \psi(\partial^\mu \cdot \partial_\mu - m_0^2c^2)\psi^* = 0 \Rightarrow \underline{\psi^*\,\partial^\mu\partial_\mu\,\psi - \psi\,\partial^\mu\partial_\mu\,\psi^* = 0} \tag{1}$$

Now, a crucial thing in the derivative notation is that-

$$\partial_\mu = \frac{\partial}{\partial x^\mu} = \left(\frac{1}{c}\frac{\partial}{\partial t}, \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) \qquad \partial^\mu = \frac{\partial}{\partial x_\mu} = \left(-\frac{1}{c}\frac{\partial}{\partial t}, \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)$$

Before moving further, let's calculate the quantity $\partial_\mu(\psi \cdot \partial^\mu\psi^*)$ and $\partial_\mu(\psi^* \cdot \partial^\mu\psi)$

$$\partial_\mu(\psi \cdot \partial^\mu\psi^*) = \frac{\partial}{\partial x^\mu}\left(\psi\frac{\partial}{\partial x_\mu}\psi^*\right) = \psi\frac{\partial^2\psi^*}{\partial x^\mu \partial x_\mu} + \frac{\partial\psi}{\partial x^\mu}\frac{\partial\psi^*}{\partial x_\mu} = \psi\partial_\mu\partial^\mu\psi^* + \partial_\mu\psi\partial^\mu\psi^* \tag{2}$$

$$\partial_\mu(\psi^* \cdot \partial^\mu\psi) = \frac{\partial}{\partial x^\mu}\left(\psi^*\frac{\partial}{\partial x_\mu}\psi\right) = \psi^*\frac{\partial^2\psi}{\partial x^\mu \partial x_\mu} + \frac{\partial\psi^*}{\partial x^\mu}\frac{\partial\psi}{\partial x_\mu} = \psi^*\partial_\mu\partial^\mu\psi + \partial_\mu\psi^*\partial^\mu\psi \tag{3}$$

Now, let's calculate the second term in both of the above expressions-

$$\partial_\mu\psi\partial^\mu\psi^* = \frac{\partial\psi}{\partial x^\mu}\frac{\partial\psi^*}{\partial x_\mu} = \left(\frac{1}{c}\frac{\partial\psi}{\partial t}, \frac{\partial\psi}{\partial x}, \frac{\partial\psi}{\partial y}, \frac{\partial\psi}{\partial z}\right) \cdot \left(\frac{-1}{c}\frac{\partial\psi^*}{\partial t}, \frac{\partial\psi^*}{\partial x}, \frac{\partial\psi^*}{\partial y}, \frac{\partial\psi^*}{\partial z}\right)$$

$$\partial_\mu\psi^*\partial^\mu\psi = \frac{\partial\psi^*}{\partial x^\mu}\frac{\partial\psi}{\partial x_\mu} = \left(\frac{1}{c}\frac{\partial\psi^*}{\partial t}, \frac{\partial\psi^*}{\partial x}, \frac{\partial\psi^*}{\partial y}, \frac{\partial\psi^*}{\partial z}\right) \cdot \left(\frac{-1}{c}\frac{\partial\psi}{\partial t}, \frac{\partial\psi}{\partial x}, \frac{\partial\psi}{\partial y}, \frac{\partial\psi}{\partial z}\right)$$

We see that these two terms in equations (2) and (3) are equal , and if we subtract (2) from (3) , we get-

$$\partial_\mu(\psi^* \cdot \partial^u\psi) - \partial_\mu(\psi \cdot \partial^\mu\psi^*) = \psi^*\partial_\mu\partial^\mu\psi + \cancel{\partial_\mu\psi^*\partial^\mu\psi} - \psi\partial_\mu\partial^\mu\psi^* - \cancel{\partial_\mu\psi\partial^\mu\psi^*} \overset{\text{eqn.(1)}}{=} 0$$

So, we see that-

$$\partial_\mu(\psi^* \cdot \partial^u\psi - \psi \cdot \partial^\mu\psi^*) = 0 \Rightarrow (\psi^* \cdot \partial^u\psi - \psi \cdot \partial^\mu\psi^*) \text{ is a conserved quantity!}$$

$$\boxed{j^\mu = \frac{i\hbar}{2m_0}(\psi^* \cdot \partial^u\psi - \psi \cdot \partial^\mu\psi^*)} \qquad \partial_\mu j^\mu = 0$$

If we expand out $\partial_\mu j^\mu$ as follows-

$$\partial_\mu j^\mu = \frac{1}{c}\frac{\partial}{\partial t}\left(\frac{i\hbar}{2m_0}\left(\psi^* \cdot \frac{-1}{c}\frac{\partial\psi}{\partial t} - \psi \cdot \frac{-1}{c}\frac{\partial\psi^*}{\partial t}\right)\right) + \vec{\nabla} \cdot \left(\frac{i\hbar}{2m_0}(\psi^*\vec{\nabla}\psi - \psi\vec{\nabla}\psi^*)\right)$$

$$= \frac{\partial}{\partial t}\underbrace{\left(\frac{-i\hbar}{2m_0c^2}\left(\psi^* \cdot \frac{\partial\psi}{\partial t} - \psi \cdot \frac{\partial\psi^*}{\partial t}\right)\right)}_{\rho} + \vec{\nabla} \cdot \underbrace{\left(\frac{i\hbar}{2m_0}(\psi^*\vec{\nabla}\psi - \psi\vec{\nabla}\psi^*)\right)}_{\vec{J}}$$

this implies we can write the above in a convenient notation-

$$\boxed{\frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot \vec{J} = 0}$$

where $\rho = j^0/c$. Thus, Klein Gordan equation predicts that the non-relativistic probability current $\vec{J}$ isn't conserved if $\partial\rho/\partial t \neq 0$. Hence, <u>the wavefunction can't be interpreted as a probability amplitude!</u>

0.5 Lagrangian Approach to Klein-Gordon Equation

In non-relativistic mechanics, time is always treated as a special coordinate. The classical Lagrangian L is usually a function of configuration space coordinate q and $\dot{q}$.

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L(q,\dot{q})}{\partial \dot{q}}\right)\bigg|_{q(t)} = \frac{\partial L(q,\dot{q})}{\partial q}\bigg|_{q(t)}$$

But in relativistic mechanics, time and space coordinates are treated on an equal footing, hence L must be a function of ψ and all of its derivatives $\partial_\mu \psi$ with $\mu = 0, 1, 2, 3$. So, the relativistic Lagrangian can be written as-

$$L_{\mathrm{rel}}(\psi, \dot{\psi}, \partial_i \psi) = L_{\mathrm{rel}}(\psi, \partial_\mu \psi)$$