# Path Integrals in Quantum Mechanics

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# Motivation

- Stemmed from the desire to obtain QM using the Lagrangian (rather than the Hamiltonian) as the starting point.
- Completely equivalent to the standard formulations of QM by Schrödinger, Heisenberg.
- Intuitively similar to the double slit interference.

Assume knowledge of QM and deduce path integral formalism from it.

- Time evolution in Schrödinger picture

$$|\psi(t_f)\rangle = \hat{U}(t_f, t_i)\,|\psi(t_i)\rangle$$

where $\hat{U}(t_f, t_i) = e^{-i\hat{H}(t_f - t_i)}$

# From Schrödinger to Feynman (contd.)

$\hat{\mathbb{1}} = \int dx_i \, |x_i\rangle\langle x_i|$

- In position-space representation,

$$\begin{aligned}
\langle x_f|\psi(t_f)\rangle &= \langle x_f|\,\hat{U}(t_f,t_i)\mathbb{1}\,|\psi(t_i)\rangle \\
&= \int \mathrm{d}x_i \,\langle x_f|\,\hat{U}(t_f,t_i)|x_i\rangle\,\langle x_i|\psi(t_i)\rangle \\
\implies \psi(x_f,t_f) &= \int \mathrm{d}x_i \,\underbrace{K(x_f,t_f;x_i,t_i)}_{\text{propagator}}\,\psi(x_i,t_i)
\end{aligned}$$

- $K(x_f,t_f;x_i,t_i) = \langle x_f|\hat{U}(t_f,t_i)|x_i\rangle$ is the probability amplitude of a particle to go from $x_i$ to $x_f$ in time $\Delta t = t_f - t_i$.

# Derivation of the Path Integral

$t_i = 0 \quad \epsilon \quad 2\epsilon \quad 3\epsilon \quad \cdots \quad N\epsilon = t_f$

Splice the time interval into $N$ intervals of size $\epsilon$ s.t. $\Delta t = N\epsilon$.

$$K = \langle x_f | \left(e^{-i\hat{H}\epsilon}\right)^N |x_i\rangle = \langle x_f | \underbrace{e^{-i\hat{H}\epsilon} e^{-i\hat{H}\epsilon} \ldots e^{-i\hat{H}\epsilon}}_{N \text{ times}} |x_i\rangle .$$

Insert $\mathbb{1}$ between each exponential ($N-1$ in total)

$$K = \langle x_f | e^{-i\hat{H}\epsilon}\, \mathbb{1}\, e^{-i\hat{H}\epsilon}\, \mathbb{1} \ldots \mathbb{1}\, e^{-i\hat{H}\epsilon} |x_i\rangle$$

$$= \int \mathrm{d}x_{N-1} \ldots \mathrm{d}x_1 \, \langle x_f |\, e^{-i\hat{H}\epsilon} | x_{N-1}\rangle \, \langle x_{N-1} | e^{-i\hat{H}\epsilon} | x_{N-2}\rangle \ldots \langle x_1 | e^{-i\hat{H}\epsilon}\, |x_i\rangle$$

$x_n \equiv x(t = n\epsilon)$

Relabel $x_f = x_N$ and $x_i = x_0$.

$$K = \int \mathrm{d}x_{N-1} \ldots \mathrm{d}x_1 \prod_{n=0}^{N-1} \underbrace{\langle x_{n+1} | e^{-i\hat{H}\epsilon} | x_n \rangle}_{M_n}$$

**Next task:** Simplify the matrix element $\mathcal{M}_n = \langle x_{n+1}|e^{-i\hat{H}\epsilon}|x_n\rangle$.

$$\mathcal{M}_n = \langle x_{n+1}|\mathbb{1}\, e^{-i\hat{H}\epsilon}|x_n\rangle = \int \mathrm{d}p\ \underbrace{\langle x_{n+1}|p\rangle}_{I_1}\underbrace{\langle p|e^{-i\hat{H}\epsilon}|x_n\rangle}_{I_2}$$

$\hat{T}+\hat{V}$

In general, $e^{\hat{A}+\hat{B}} \neq e^{\hat{A}} e^{\hat{B}}$

In the limit $N \to \infty$ or $\epsilon \to 0$, we use the approximation

$$e^{\epsilon(-i\hat{H})} = e^{\epsilon(-i(\hat{T}+\hat{V}))} = e^{-i\epsilon\hat{T}(\hat{p})}\, e^{-i\epsilon\hat{V}(\hat{x})} + \cancel{\mathcal{O}(\epsilon^2)}^{\,0}\ \text{in the limit } \epsilon\to 0$$

$$\implies \lim_{\epsilon\to 0} I_2 = \langle p|e^{-i\epsilon\frac{\hat{p}^2}{2m}}\, e^{-i\epsilon\hat{V}(\hat{x})}|x_n\rangle = e^{-i\epsilon\frac{p^2}{2m}}\, e^{-i\epsilon V(x_n)}\,\langle p|x_n\rangle$$

# Derivation of the Path Integral (contd.)

Matrix element $\mathcal{M}_n$

$$\lim_{\epsilon\to 0} \mathcal{M}_n = \int \mathrm{d}p\, e^{-i\epsilon \frac{p^2}{2m}}\, e^{-i\epsilon V(x_n)} \langle x_{n+1}|p\rangle \langle p|x_n\rangle$$

$$\langle x|p\rangle = \frac{1}{\sqrt{2\pi}} e^{ipx}$$

$$\implies \lim_{\epsilon\to 0} \mathcal{M}_n = \int_{-\infty}^{+\infty} \frac{\mathrm{d}p}{2\pi} e^{-i\epsilon\left[\frac{p^2}{2m} + V(x_n) - p\left(\frac{x_{n+1}-x_n}{\epsilon}\right)\right]}$$

upon completing the square

$$= \underbrace{\left(\frac{m}{2\pi i\epsilon}\right)^{1/2}}_{\text{from Gaussian integral}} e^{i\epsilon\left[\frac{m}{2}\left(\frac{x_{n+1}-x_n}{\epsilon}\right)^2 - V(x_n)\right]}$$

# Derivation of the Path Integral (contd.)

$$K = \int \mathrm{d}x_{N-1} \ldots \mathrm{d}x_1 \prod_{n=0}^{N-1} \mathcal{M}_n$$

Substitute $\mathcal{M}_n$ back into the propagator $K$

$$K = \lim_{\epsilon \to 0} \left( \frac{m}{2\pi i \epsilon} \right)^{N/2} \int \mathrm{d}x_{N-1} \ldots \mathrm{d}x_1 \; e^{i \sum_{n=0}^{N-1} \epsilon \left[ \frac{m}{2} \left( \frac{x_{n+1} - x_n}{\epsilon} \right)^2 - V(x_n) \right]}$$

# Derivation of the Path Integral (contd.)

$$K = \lim_{\epsilon \to 0} \left( \frac{m}{2\pi i \epsilon} \right)^{N/2} \int \mathrm{d}x_{N-1} \ldots \mathrm{d}x_1 \, e^{i \sum_{n=0}^{N-1} \epsilon \left[ \frac{m}{2} \left( \frac{x_{n+1} - x_n}{\epsilon} \right)^2 - V(x_n) \right]}$$

analyze the exponent

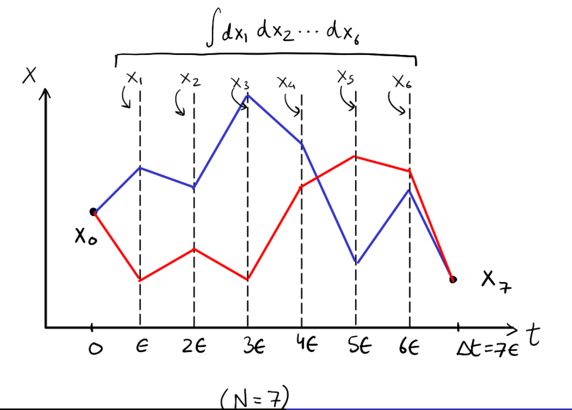
Now, in the $\lim_{\epsilon \to 0}$ (or $\lim_{N \to \infty}$)

$$\sum_{n=0}^{N-1} \epsilon \left[ \frac{m}{2} \left( \frac{x_{n+1} - x_n}{\epsilon} \right)^2 - V(x_n) \right] \to \int_{t_i}^{t_f} \mathrm{d}t \left[ \frac{m}{2} \left( \frac{\mathrm{d}x}{\mathrm{d}t} \right)^2 - V(x(t)) \right]$$

$$= \int_{t_i}^{t_f} \mathrm{d}t \, \mathcal{L}(x, \dot{x}) = S[x(t)]$$

This gives the **propagator** in terms of a **path integral**

$$\boxed{K = \underbrace{\lim_{\substack{N\to\infty \\ \epsilon\to 0}} \left(\frac{m}{2\pi i\epsilon}\right)^{N/2} \int \prod_{n=1}^{N-1} \mathrm{d}x_n}_{\int \mathcal{D}[x(t)]} \; e^{iS[x(t)]} = \int \mathcal{D}[x(t)] \; e^{iS[x(t)]}}$$



# References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