one dimensional problem.

$$m\frac{d^2x}{dt^2} = F(x)$$

$$\int_{x_1}^{x_2} m\frac{dv}{dt}\,dx = \int_{x_1}^{x_2} F(x)\,dx$$

$$\int_{x_a}^{x_b} \frac{dv}{dt}\cdot dx$$

make change of variables

$$dx = \left(\frac{dx}{dt}\right)\cdot dt = v\,dt$$

$$= \int_{t_a}^{t_b} \frac{dv}{dt}\cdot v\,dt$$

$$= \int_{t_a}^{t_b} dt\,\frac{d}{dt}\left(\frac{1}{2}mv^2\right)$$

$$= \frac{1}{2}mv(t_b)^2 - \frac{1}{2}mv(t_a)^2$$

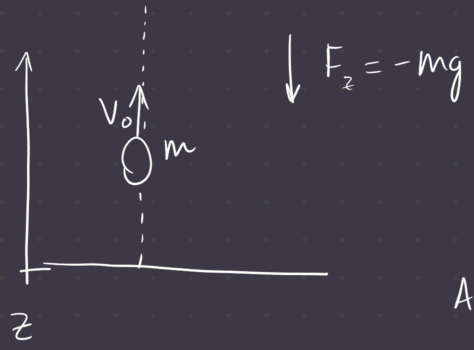
I can still hear you say

that you'd never break the chain.

$\partial_x\partial_x\partial_x 6$

$$\int_{x_a}^{x_b} F(x)\,dx$$

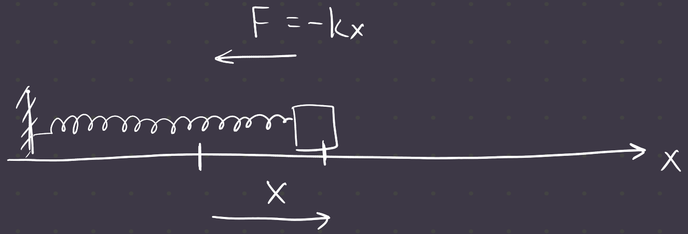
$$\frac{1}{2}mv^2 - \frac{1}{2}mv_0^2 = \int_{z_0}^{z} dx\,(-mg)$$



$$\frac{1}{2}mv^2 - \frac{1}{2}mv_0^2 = mg(z_0 - z)$$

At $z = h_{max}$, $V = 0 \Rightarrow \quad \frac{1}{2}mv_0^2 = mg(h_{max} - z_0)$

$$h_{max} = \frac{v_0^2}{2g} + z_0$$



$$\frac{1}{2}mv^2 - \frac{1}{2}mv_0^2 = -k\int_{x_0}^{x} dx\cdot x$$

$$\frac{1}{2}mv^2 - \frac{1}{2}mv_0^2 = -\frac{1}{2}kx^2 + \frac{1}{2}kx_0^2$$

$$\frac{1}{2}mv^2 + \frac{1}{2}kx^2 = \frac{1}{2}mv_0^2 + \frac{1}{2}kx_0^2$$

Say at $t = t_0$, $\quad v(t_0) = v_0 = 0 \quad$ and $\quad x(t_0) = x_0 \neq 0$

$$\frac{1}{\cancel{2}}mv^2 + \frac{1}{\cancel{2}}kx^2 = \frac{1}{\cancel{2}}kx_0^2 \qquad v^2 = \frac{k}{m}(x_0^2 - x^2)$$

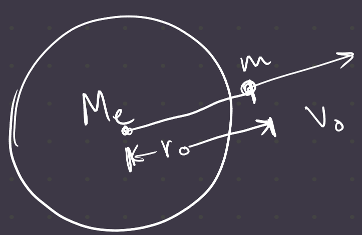
$$v = \sqrt{\frac{k}{m}}\,(x_0^2 - x^2)^{1/2} \equiv v(x)$$

Now if we know $v(x)$, then $v = \frac{dx}{dt}$

So, $$\frac{dx(t)}{dt} = \sqrt{\frac{k}{m}}\,(x_0^2 - x^2)^{1/2}$$

$$\int_{x_0}^{x} \frac{dx}{\sqrt{x_0^2 - x^2}} = \int_{t_0}^{t} \sqrt{\frac{k}{m}}\, dt$$

$$\hookrightarrow \sin^{-1}\left(\frac{x}{x_0}\right)\Bigg|_{x_0}^{x} = \sqrt{\frac{k}{m}}\, t \Rightarrow \boxed{x = x_0 \sin\left(\sqrt{\frac{k}{m}}\, t + \frac{\pi}{2}\right)}$$



$$F = -\frac{GM_e m}{r^2}$$

$$\frac{1}{2}mv^2 - \frac{1}{2}mv_0^2 = \int_{r_0}^{r} dr\, F(r) = -GM_e m \int_{r_0}^{r} \frac{dr}{r^2}$$

$$\frac{1}{2}mv(r)^2 - \frac{1}{2}mv_0^2 = \cancel{-}GM_e m\left(\cancel{-}\frac{1}{r}\right)\Bigg|_{r_0}^{r} = GM_e m\left(\frac{1}{r} - \frac{1}{r_0}\right)$$

If the mass is to escape the earth, it barely reaches $v(r) = 0$ at $r = r_{max}$.

$$\frac{1}{2}mv_0^2 = GM_e m\left(\frac{1}{r_0} - \frac{1}{r_{max}}\right) \qquad g = \frac{GM_e}{R_e^2}$$

$$\frac{\cancel{GM_e m}}{r_{max}} = \frac{\cancel{GM_e m}^{\,1}}{r_0} - \frac{\frac{1}{2}\cancel{m}v_0^2}{GM_e\cancel{m}} \qquad \frac{1}{r_{max}} = \frac{1}{r_0} - \frac{v_0^2}{2GM_e} = \frac{2GM_e - v_0^2 r_0}{r_0 2GM_e}$$

$$r_{max} = \frac{2GM_e}{\frac{2GM_e}{r_0} - v_0^2}$$

For $\frac{2GM_e}{r_0} > v_0^2$ or $v_0 < \sqrt{\frac{2GM_e}{r_0}}$, $r_{max} > 0 \Rightarrow \exists$ a max. altitude after which mass m returns to Earth.

For $v_0 \approx \sqrt{\frac{2GM_e}{r_0}}$, $r_{max} \longrightarrow \infty$. This is the escape velocity.

In higher dim^ns $(d \neq 1)$

$$\int_{\vec{r}_a}^{\vec{r}_b} m \frac{d\vec{v}(t)}{dt} \cdot d\vec{r} = \int_{\vec{r}_a}^{\vec{r}_b} \vec{F}(\vec{r}) \cdot d\vec{r}$$

$$d\vec{r} = \frac{d\vec{r}}{dt} dt = \vec{v}\, dt.$$

$$\int_{t_a}^{t_b} m \underbrace{\frac{d\vec{v}}{dt} \cdot \vec{v}}_{\frac{1}{2}\frac{d}{dt}(\vec{v}\cdot\vec{v}) = \frac{1}{2}\frac{d}{dt}(v^2)}\, dt = \int_{\vec{r}_A}^{\vec{r}_B} \vec{F}(\vec{r}) \cdot d\vec{r}$$

$$\boxed{\frac{1}{2}mv_b^2 - \frac{1}{2}mv_a^2 = \int_{\vec{r}_a}^{\vec{r}_b} d\vec{r} \cdot \vec{F}(\vec{r})}$$

So,

$$\int_{t_a}^{t_b} \frac{d}{dt}\left(\frac{1}{2} m v^2\right) dt = \int_{\vec{r}_A}^{\vec{r}_B} d\vec{r} \cdot \vec{F}(\vec{r})$$

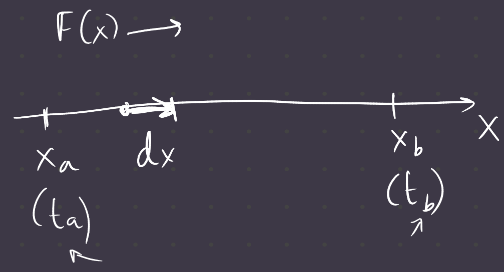
PHY101 Hello Session.

27-02-2022

Work & Energy

One-dimensional motion.

$$m\frac{d^2x}{dt^2} = F(x) \qquad \frac{dx}{dt} \equiv v(t)$$



$$\int_{x_a}^{x_b} m\frac{dv}{dt}\,\underbrace{dx}_{} = \int_{x_a}^{x_b} F(x)\,dx$$

$$dx = \left(\frac{dx}{dt}\right)dt = v\,dt$$

$\frac{dx}{dt} = v$

$x \longrightarrow t$

$$\int_{v_a \leftarrow t_a}^{\cancel{v_b} \leftarrow t_b} \cancel{dt}\, mv\,\underbrace{\frac{dv}{\cancel{dt}}}_{} = \int_{x_a}^{x_b} dx\,F(x)$$

$$\frac{1}{2}\left(v\frac{dv}{dt} + \frac{dv}{dt}\cdot v\right) = \frac{dv}{dt}\cdot v$$

$$\frac{d}{dt}\left(\frac{1}{2}v^2\right)$$

$$\int_a^b dx\,\frac{d}{dx}F(x) = F(b) - F(a)$$

$$\int_{t_a}^{t_b} dt\,\frac{d}{dt}\left(\frac{1}{2}mv^2\right) = \int_{x_a}^{x_b} dx\,F(x)$$

$$\frac{1}{2}mv(t_b)^2 - \frac{1}{2}mv(t_a)^2 = \int_{x_a}^{x_b} dx\,F(x) \Rightarrow \boxed{\frac{1}{2}mv_b^2 - \frac{1}{2}mv_a^2 = \int_{x_a}^{x_b} dx\,F(x) = W_{ba}}$$

$v(t_b) = v_b$, $v(t_a) = v_a$

$K \equiv \frac{1}{2}mv^2$

Kinetic energy

$$K_b - K_a = W_{ba}$$

$$m\frac{d^2x}{dt^2} = F(x)$$

☆ Start with $v_0 \longrightarrow v$

$x_0 \longrightarrow x$

$$\frac{1}{2}mv^2 = \underbrace{\int_{x_0}^{x} dx\,F(x)}_{f(x)} + \frac{1}{2}mv_0^2$$

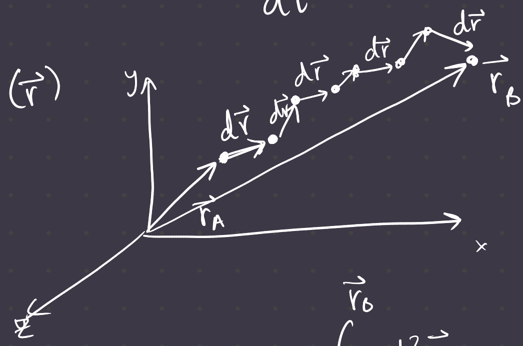
$$v \equiv v(x) = \underbrace{\frac{dx}{dt}} \qquad \int_{x_0}^{x} \frac{dx}{v(x)} = \int_{t_0}^{t_1} dt$$

In higher dimensional

$$\vec{x} \equiv (x, y, z) \equiv \vec{r}$$

$$m\frac{d^2\vec{r}}{dt^2} = \vec{F}(\vec{r})$$

$\vec{F}(\vec{r})$



$$\underbrace{\int_{\vec{r}_A}^{\vec{r}_B} \vec{F}(\vec{r}) \cdot d\vec{r}} = m \int_{\vec{r}_A}^{\vec{r}_B} d\vec{r} \cdot \frac{d^2\vec{r}}{dt^2}$$

$\vec{r}(t_a) = \vec{r}_A$

$\vec{r}(t_B) = \vec{r}_B$

$$m\int_{\vec{r}_A}^{\vec{r}_B} \frac{d^2\vec{r}}{dt^2} \cdot d\vec{r} = m\int_{\vec{r}_A}^{\vec{r}_B} \frac{d\vec{v}}{dt} \cdot d\vec{r}$$

$$d\vec{r} = \left(\frac{d\vec{r}}{dt}\right) dt = \vec{v}\, dt$$

$$= m\int_{t_a}^{t_b} dt\, \underbrace{\vec{v} \cdot \frac{d\vec{v}}{dt}}_{= \frac{d}{dt}\left(\frac{1}{2}\vec{v}\cdot\vec{v}\right)} = \int_{t_a}^{t_b} dt\, \frac{d}{dt}\left(\frac{1}{2}mv^2\right)$$

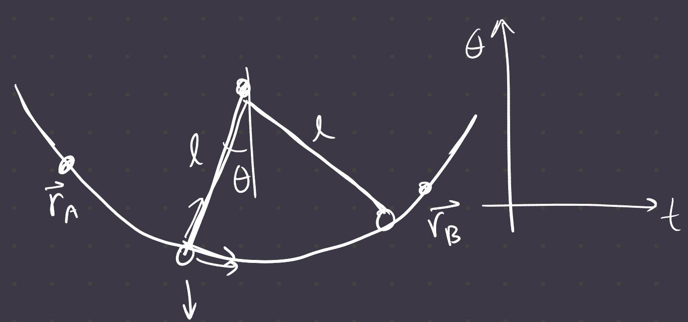
$$v^2 = v_x^2 + v_y^2 + v_z^2$$

$$\boxed{\underset{\text{Energy}}{\frac{1}{2}mv(t_b)^2 - \frac{1}{2}mv(t_a)^2} = \underset{\text{Work}}{\int_{C,\,\vec{r}_A}^{\vec{r}_B} d\vec{r} \cdot \vec{F}(\vec{r})}}$$

$\vec{v} \cdot \vec{v}$

$\vec{v}(t) \cdot \vec{v}(t)$

Line integral



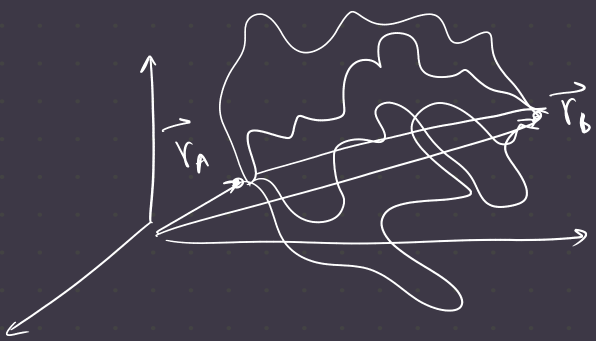
Useful in two cases –

- if motion is constrained.
- $\vec{F}$ is a "conservative force".

$\vec{r}(t)$

$$\oint_{\vec r_A}^{\vec r_B} d\vec r \cdot \vec F(\vec r) = f(\vec r_B) - f(\vec r_A)$$

A conservative force is defined as the one where $\oint \vec F \cdot d\vec r$ is INDEPENDENT of the path & only dependent upon the endpoints.



For conservative forces

$$\boxed{\oint_{\vec r_A}^{\vec r_B} d\vec r \cdot \vec F(\vec r) = -\left(U(\vec r_B) - U(\vec r_A)\right)}$$

$U \equiv$ pot^n energy

For a <u>conservative system</u> $\quad K_b - K_a = \oint_{\vec r_A}^{\vec r_B} d\vec r \cdot \vec F(\vec r) = -\left[(U_b + C) - (U_a + C)\right]$

$$K_b - K_a = -U_b + U_a \Rightarrow \underbrace{K_a + U_a}_{(\vec x_a, t_a)} = \underbrace{K_b + U_b}_{(\vec x_b, t_b)} = E$$

total mechanical energy

$$\oint_{\vec r_A}^{\vec r_B} d\vec r \cdot F(\vec r) = -\left(U(\vec r_B) - U(\vec r_A)\right)$$
$$= -\left[(\underbrace{U(\vec r_B) + C}) - (U(\vec r_A) + C)\right]$$

$$E = K + (U + C)$$

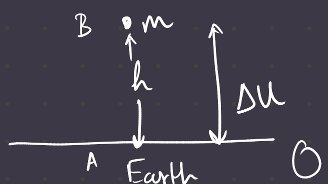
arbitrary const.

$\underline{\underline{\Delta E}}$

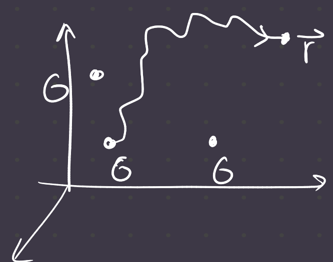
$$U(\vec r) \equiv -\oint_{G}^{\vec r} d\vec r \cdot \vec F(\vec r)$$

reference pt.

$$K = \frac{1}{2}mv^2$$



$U = mgh$



In 1D

$$\int_a^b dx \underbrace{\frac{df}{dx}} = f(b) - f(a) \qquad K_b - K_a = \int_{x_a}^{x_b} dx\, F(x)$$

$$\int_{x_a}^{x_b} dx\, F(x) = -U(x_b) + U(x_a)$$

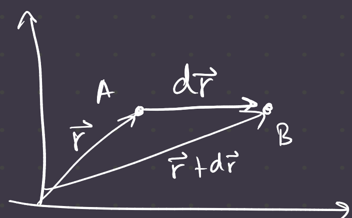
$$\int dx\, F(x) = -\int dx \frac{dU}{dx}$$

$$\int_{x_a}^{x_b} dx \frac{dU}{dx} = U(x_b) - U(x_a)$$

$$\boxed{F(x) = -\frac{dU(x)}{dx}}$$

- 3D

$$\int_{\vec r_A}^{\vec r_B} d\vec r \cdot \vec F(\vec r) = -\left(U(\vec r_B) - U(\vec r_A)\right)$$



$$\int_{\vec r}^{\vec r + d\vec r} d\vec r\,' \cdot \underbrace{\vec F(\vec r\,')} = -\left(U(\vec r + d\vec r) - U(\vec r)\right)$$

$$\|$$

$$\vec F(\vec r) \cdot d\vec r = -\left(\underbrace{U(\vec r + d\vec r) - U(\vec r)}\right)$$

$$\underset{dU}{U(\vec r + \Delta \vec r) - U(\vec r)} = \underbrace{\frac{\partial U}{\partial x}} dx + \underbrace{\frac{\partial U}{\partial y}} dy + \underbrace{\frac{\partial U}{\partial z}} dz$$

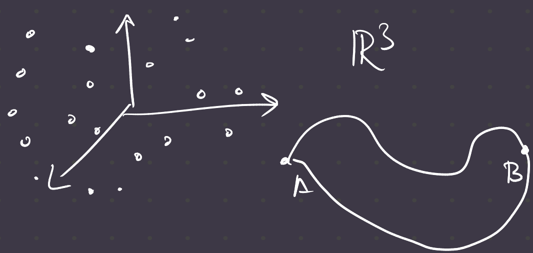
$$\vec F(\vec r) \cdot d\vec r = -\left(\frac{\partial U}{\partial x} dx + \frac{\partial U}{\partial y} + \frac{\partial U}{\partial z} dz\right)$$

$$U : \mathbb{R}^3 \longrightarrow \mathbb{R}$$

$$= -\left(\frac{\partial U}{\partial x}, \frac{\partial U}{\partial y}, \frac{\partial U}{\partial z}\right) \cdot (dx, dy, dx)$$

scalar

$$\vec F(\vec r) \cdot d\vec r = -\vec\nabla U \cdot d\vec r \quad \Rightarrow \quad \boxed{\vec F(\vec r) = \vec\nabla U(\vec r)}$$



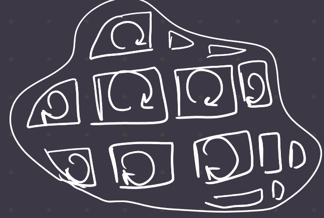
How to find if F is conservative?

$$\oint_{\vec r_A}^{\vec r_B} d\vec r \cdot \vec F(\vec r) = -\left(U(\vec r_B) - U(\vec r_A)\right)$$

$$\boxed{\oint d\vec r \cdot \vec F(\vec r) = 0}$$

C

=

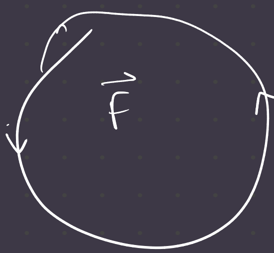


$\underline{\vec{\nabla} \times \vec{F}}$

$C \to$ closed loop

$$\oint d\vec{r} \cdot \vec{F}(\vec{r}) = \int d\vec{a} \cdot (\vec{\nabla} \times \vec{F})$$

summing of the $\vec{F}$ along the boundary = Summing up all the rot^n curls in all the infinitesimal loop



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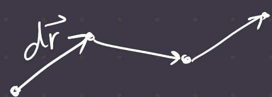
Griffiths Electrodynamics Ch-1

$$\boxed{\oint d\vec{r} \cdot \vec{F}(\vec{r}) = \int d\vec{a} \cdot (\vec{\nabla} \times \vec{F})}$$

Stokes' Th^m

for conservative fields

$$\oint d\vec{r} \cdot \vec{F}(\vec{r}) = 0 = \int d\vec{a} \cdot (\vec{\nabla} \times \vec{F})$$



$\boxed{\vec{\nabla} \times \vec{F} = 0} \to$ if a force field is conservative