

COLLISION THEORY



If any external force $\vec{F}_{ext} = 0, \Rightarrow \vec{F}_{ext} = \frac{d\vec{P}}{dt} = 0$

$\Rightarrow \vec{P} = \text{constant of motion}$

$\vec{P} = \sum_{i \in \text{system}} \vec{p}_i$

3 eqⁿs $\rightarrow m_1 \vec{v}_1 + m_2 \vec{v}_2 = m_1 \vec{v}_1' + m_2 \vec{v}_2'$ Conservation of momⁿ

Annotations: "known" points to \vec{v}_1, \vec{v}_2 ; "want to find" points to \vec{v}_1', \vec{v}_2' ; "6 unknowns" points to the final velocities.

If on top of this, the interactions occur w.r.t. conservative forces

\Rightarrow total energy is conserved.

$$\frac{1}{2} m_1 |\vec{v}_1|^2 + \frac{1}{2} m_2 |\vec{v}_2|^2 = \frac{1}{2} m_1 |\vec{v}_1'|^2 + \frac{1}{2} m_2 |\vec{v}_2'|^2$$

Annotations: "1" points to the equation; "conservation of energy" points to the right side.

4 eqⁿs & 6 unknowns.

Collisions in One dimension.

Diagram showing 1D collision: $m_1 \rightarrow v_1$ and $m_2 \rightarrow v_2$ before collision, and $m_1 \rightarrow v_1'$ and $m_2 \rightarrow v_2'$ after collision.

$$m_1 v_1 + m_2 v_2 = m_1 v_1' + m_2 v_2'$$

$$\frac{1}{2} m_1 v_1^2 + \frac{1}{2} m_2 v_2^2 = \frac{1}{2} m_1 v_1'^2 + \frac{1}{2} m_2 v_2'^2$$

Cons of momⁿ + Cons of energy \rightarrow 2 eqⁿs
2 unknowns

Example

Elastic collision exactly solvable.

Diagram: $m_1 \rightarrow v$ and $3m_1 \leftarrow v$ before collision, and $m_1 \rightarrow v_1'$ and $3m_1 \rightarrow v_2'$ after collision.

$$p_i: m_1 v - 3m_1 v = p_f: m_1 v_1' + 3m_1 v_2'$$

$$-2v = v_1' + 3v_2' \quad \leftarrow$$

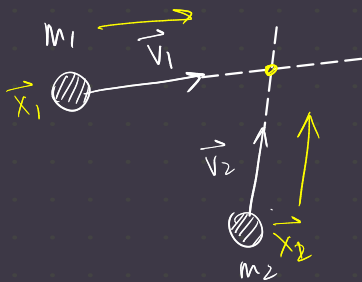
$$\hookrightarrow v^2 + 3(-v)^2 = v_1'^2 + 3v_2'^2 \rightarrow \text{Cons. of Energy}$$

$$4v^2 = v_1'^2 + 3v_2'^2 \leftarrow$$

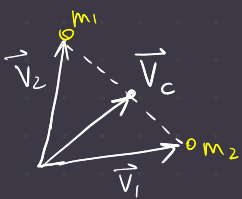
$$v_1' \& v_2' \rightarrow 2 \text{ eqns}$$

Collisions in 3D (or 2D)

In lab frame \rightarrow COM frame.

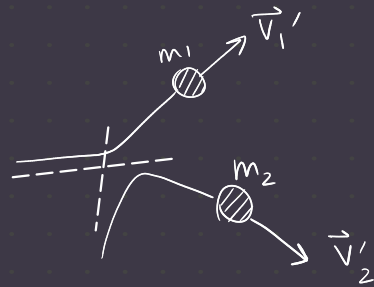


Before

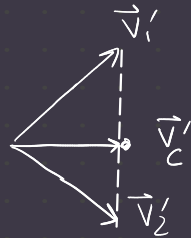


$$\vec{v}_c = \frac{m_1 \vec{v}_1 + m_2 \vec{v}_2}{m_1 + m_2}$$

$$\vec{x}_c = \frac{m_1 \vec{x}_1 + m_2 \vec{x}_2}{m_1 + m_2}$$



After



$$\vec{v}_c' = \frac{m_1 \vec{v}_1' + m_2 \vec{v}_2'}{m_1 + m_2}$$

Why COM frame?

$$\vec{F}_{\text{ext}} = M \ddot{\vec{R}} \rightarrow \text{COM}$$

total mass

$$\vec{F}_{\text{ext}} = (m_1 + m_2) \frac{d\vec{v}_c}{dt}$$

||
0

$$\frac{d\vec{v}_c}{dt} = \frac{d^2 \vec{R}_{\text{COM}}}{dt^2} = \ddot{\vec{R}}$$

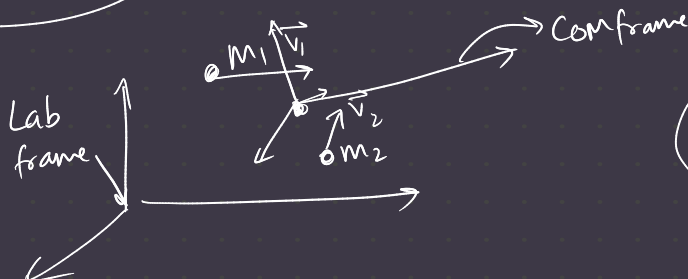
$\Rightarrow \vec{v}_c = \text{constant of motion}$



if you shift to the COM frame

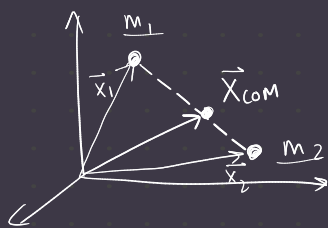
\Rightarrow COM frame is an inertial frame.

$\vec{v}_c = 0$ in the COM.



$$\vec{p}_{\text{COM}} = 0$$

$$\vec{V} \equiv \vec{V}_1 - \vec{V}_2$$

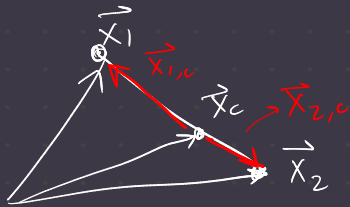


$$\frac{d}{dt} \vec{X}_c$$

$$= \frac{d}{dt} \frac{m_1 \vec{X}_1 + m_2 \vec{X}_2}{m_1 + m_2}$$

$$\Rightarrow$$

$$\vec{V}_c = \frac{m_1 \vec{V}_1 + m_2 \vec{V}_2}{m_1 + m_2}$$



$$\vec{X}_{1,c} = \vec{X}_1 - \vec{X}_c$$

$$\vec{X}_{2,c} = \vec{X}_2 - \vec{X}_c$$

$$\vec{V}_{1,c} = \vec{V}_1 - \vec{V}_c$$

$$\vec{V}_{2,c} = \vec{V}_2 - \vec{V}_c$$



velocity of $m_{1,2}$ w.r.t. the COM frame.



$$\mu = \frac{m_1 m_2}{m_1 + m_2}$$

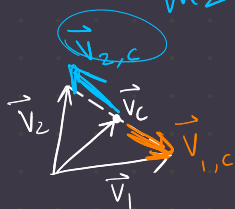
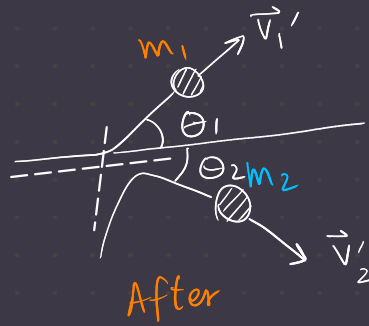
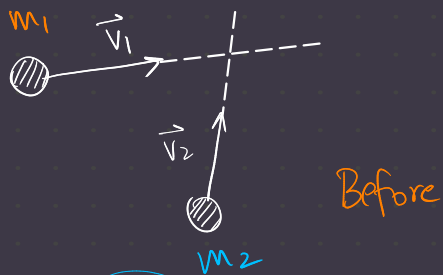
$$\vec{V} \equiv \vec{V}_1 - \vec{V}_2$$

$$\vec{V}_{1,c} = \frac{\mu}{m_1} \vec{V}$$

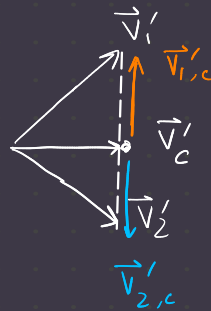
$$\vec{V}_{2,c} = -\frac{\mu}{m_2} \vec{V}$$

$$\vec{p}_c = m_1 \vec{V}_{1,c} + m_2 \vec{V}_{2,c} = \mu \vec{V} - \mu \vec{V} = \underline{\underline{0}}$$

In lab frame

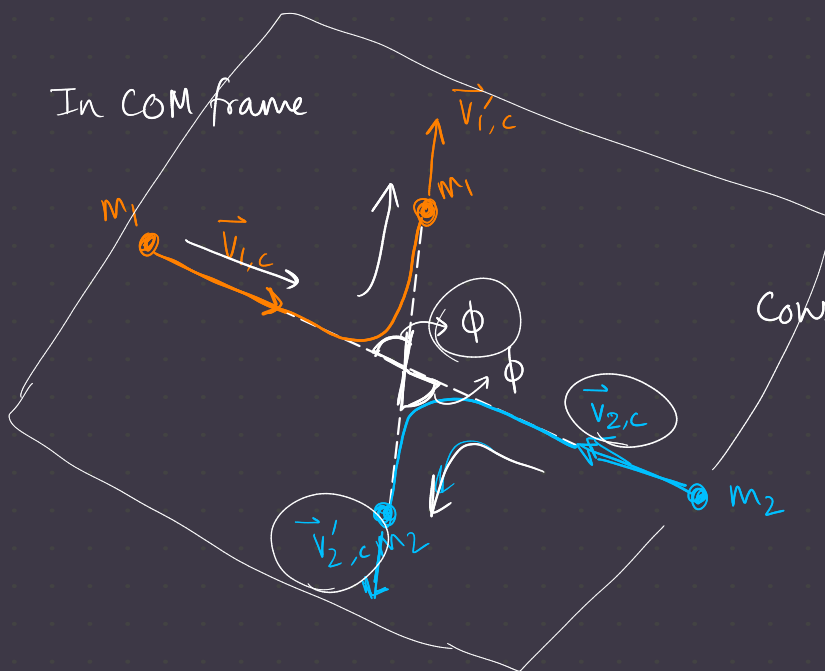


$$\vec{V} = \frac{m_1 \vec{v}_1 + m_2 \vec{v}_2}{m_1 + m_2}$$



$$\vec{V}' = \frac{m_1 \vec{v}_1' + m_2 \vec{v}_2'}{m_1 + m_2}$$

In COM frame



Conserve momentum in the two directions

$$\begin{aligned} m_1 \vec{v}_{1,c} - m_2 \vec{v}_{2,c} &= 0 \\ m_1 \vec{v}_{1,c}' - m_2 \vec{v}_{2,c}' &= 0 \end{aligned}$$

If your collision was elastic

$$\frac{1}{2} m_1 v_{1,c}^2 + \frac{1}{2} m_2 v_{2,c}^2 = \frac{1}{2} m_1 v_{1,c}'^2 + \frac{1}{2} m_2 v_{2,c}'^2$$

⇓

$$v_{1,c} = v_{1,c}'$$

$$v_{2,c} = v_{2,c}'$$

6 unknowns + 4 eqns

1 free parameter