Angular Momentum and Angular velocity as vectors.

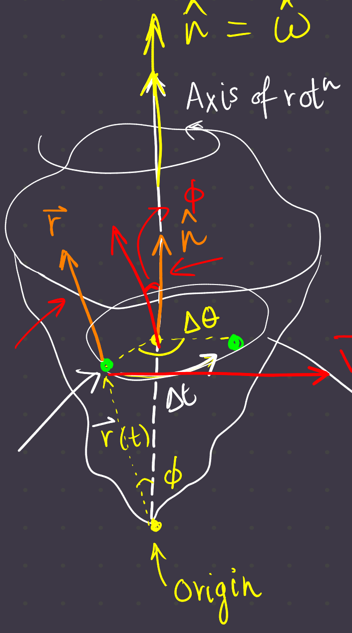
Rotations do not commute, but infinitesimal rotations do!

$$\frac{d\vec{\theta}}{dt} = \frac{d\theta_x}{dt}\hat{i} + \frac{d\theta_y}{dt}\hat{j} + \frac{d\theta_z}{dt}\hat{k}$$

$$= \omega_x \hat{i} + \omega_y \hat{j} + \omega_z \hat{k}$$

$\vec{\omega} \rightarrow$ vector

Relation b/w $\vec{v}$ and $\vec{\omega}$

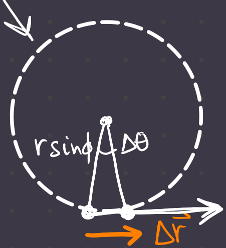


Axis of rotⁿ along $\hat{n}$

angle b/w $\hat{n}$ and $\vec{r}(t)$ is constant $= \phi$

In time Δt, the particle moves through an angle $\Delta\theta$

$$\Rightarrow \underbrace{|\Delta\vec{r}|}_{} = \underbrace{r\sin\phi}_{\text{radius}} \; \underbrace{\Delta\theta}_{\text{small arc angle}}$$



$|\hat{n} \times \vec{r}| = r\sin\phi$

$$\lim_{\Delta t \to 0} \frac{|\Delta\vec{r}|}{\Delta t} = \lim_{\Delta t \to 0} r\sin\phi \frac{\Delta\theta}{\Delta t}$$

$$\left|\frac{d\vec{r}}{dt}\right| = r\sin\phi \frac{d\theta}{dt}$$

$$\vec{v} = \frac{d\vec{r}}{dt}$$

$$|\vec{v}| = r\sin\phi\,\omega$$

$$|\vec{v}| = r\sin\phi\,\omega$$

$$= |\hat{n} \times \vec{r}|\,\omega$$

$$= |\underbrace{\hat{n}\omega}_{} \times \underbrace{\vec{r}}_{}|$$

$$\boxed{\vec{v} = \vec{\omega} \times \vec{r}}$$

For fixed axis rot^n, $\vec{L} \parallel \vec{\omega}$ and $\vec{L} = I\vec{\omega}$

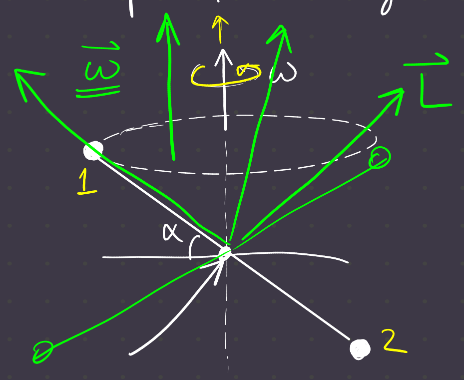
But this isn't true in general.

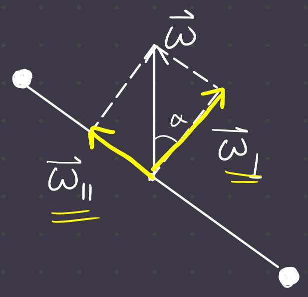
Angular Momentum vector

$$\vec{L} = I\vec{\omega}$$

$\vec{\omega} \rightarrow$ vector

Example of rotating skew rod (Example 8.4)





$\rightarrow \omega_{\perp} = \omega \cos\alpha$

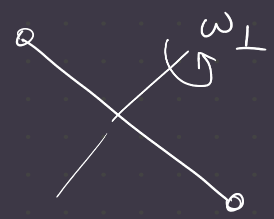
$\rightarrow \omega_{\parallel} = \omega \sin\alpha$

Choose midpoint of the rod as the origin

$$\vec{L} = \text{~~rot}^n\text{ due to } \underline{\omega_{\parallel}}\text{~~}^{\,0} + \underbrace{\text{rot}^n \text{ due to } \underline{\omega_{\perp}}}$$

$$= I\vec{\omega}_{\perp}$$

$$|\vec{L}| = I\omega\cos\alpha$$



HW: try calculating the torque $\vec{\tau}$ by taking the time derivative $\left(\frac{d\vec{L}}{dt}\right)$

Example 8.5

$$\vec{\tau} = \frac{d\vec{L}}{dt}$$

Comment about conservation of angular momentum (section 8.5)

$$\vec{L} = \sum_j m_j \vec{r}_j \times \vec{v}_j$$

[For an isolated system $\vec{L}$ is conserved.]

Inertia Tensor.

$\vec{L} \not\parallel \vec{\omega}$ tensor

$\vec{L} = I\vec{\omega}$ ~~scalar~~ tensor

$$\boxed{\vec{L} = I\,\vec{\omega}}$$

$\vec{L}$, $\vec{\omega}$ vectors

rank (1,1)

$$\begin{pmatrix} L_x \\ L_y \\ L_z \end{pmatrix} = \underbrace{\begin{pmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{yx} & I_{yy} & I_{yz} \\ I_{zx} & I_{zy} & I_{zz} \end{pmatrix}}\begin{pmatrix} \omega_x \\ \omega_y \\ \omega_z \end{pmatrix}$$

a $\mathbb{I}$

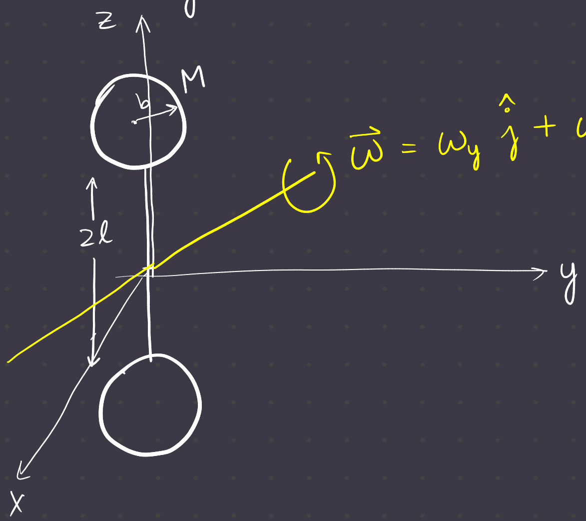
$$\begin{pmatrix} a & o & o \\ o & b & o \\ o & o & a \end{pmatrix} \longrightarrow \text{fixed axis rotn}$$

$\sum_j \to$ sum over all the particles in the system

$$\underbrace{\begin{aligned} I_{xx} &= \sum_j m_j\,(y_j^2 + z_j^2) \\ I_{yy} &= \sum_j m_j\,(x_j^2 + z_j^2) \\ I_{zz} &= \sum_j m_j\,(x_j^2 + y_j^2) \end{aligned}}_{\text{moments of inertia}}$$

$$\underbrace{\begin{aligned} I_{xy} &= -\sum_j m_j\,x_j\,y_j = I_{yx} \\ I_{yz} &= -\sum_j m_j\,y_j\,z_j = I_{zy} \\ I_{xz} &= -\sum_j m_j\,x_j\,z_j = I_{zx} \end{aligned}}_{\text{products of inertia}}$$

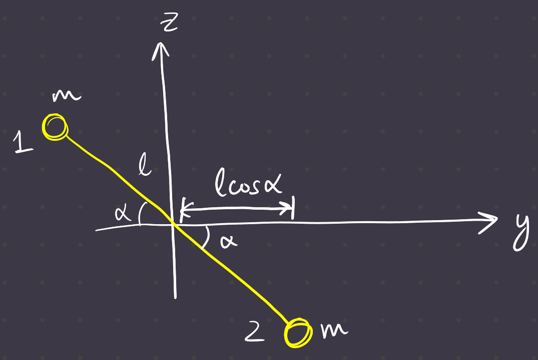
Rotating dumbell



$$\vec{\omega} = \omega_y\,\hat{j} + \omega_z\,\hat{k}$$

Rotating Skew Rod

At $t=0$, start in yz plane



At $t=0$

$x_1 = 0$ $\qquad$ $x_2 = 0$

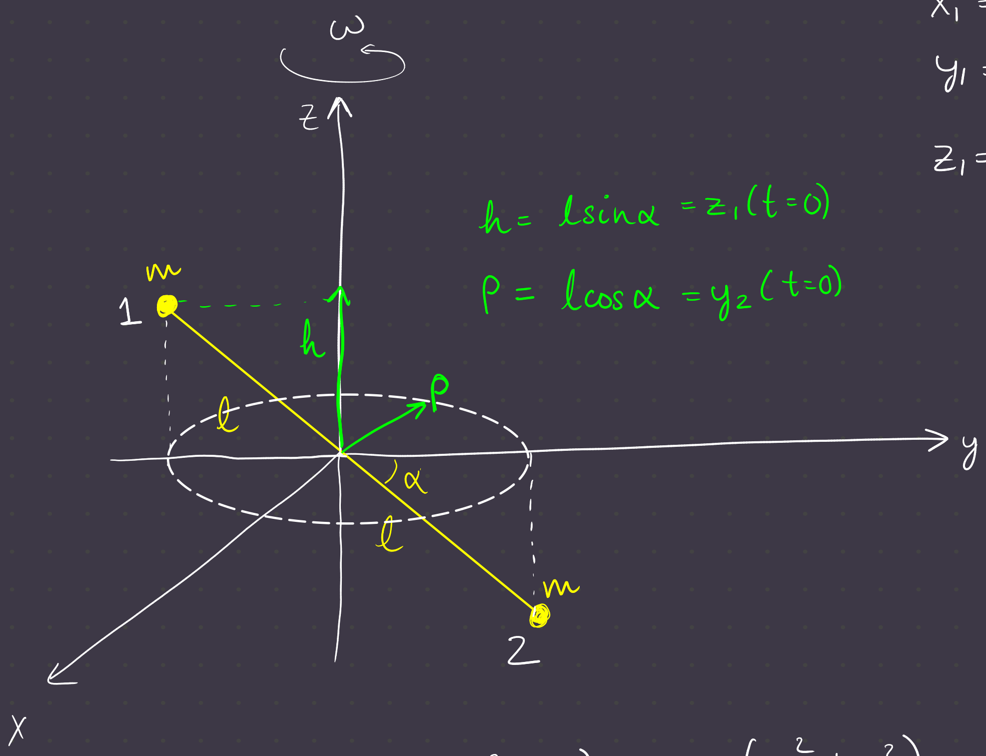
$y_1 = -l\cos\alpha$ $\qquad$ $y_2 = l\cos\alpha$

$z_1 = l\sin\alpha$ $\qquad$ $z_3 = -l\sin\alpha$

$x_1 = +\rho\sin\omega t$ $\qquad$ $x_2 = -\rho\sin\omega t$

$y_1 = -\rho\cos\omega t$ $\qquad$ $y_2 = +\rho\cos\omega t$

$z_1 = +h$ $\qquad$ $z_2 = -h$



$h = l\sin\alpha = z_1(t=0)$

$\rho = l\cos\alpha = y_2(t=0)$

$$I_{xx} = m_1(y_1^2+z_1^2) + m_2(y_2^2+z_2^2)$$

$$I_{yy} = m_1(x_1^2+z_1^2) + m_2(x_2^2+z_2^2)$$

$$\vdots$$

$$I_{xy} = -m_1x_1y_1 - m_2x_2y_2$$

$$I_{yz} = -my_1z_1 - m_2y_2z_2$$

$$\vdots$$

$$I = \begin{pmatrix} & & \\ & & \\ & & \end{pmatrix}_{3\times3}$$