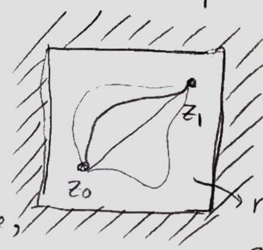
REVISION SHEET

- **Line / Contour integrals**

$$\int_\gamma f(z)\,dz = \int_a^b f(\gamma(t)).\gamma'(t)\,dt \qquad \gamma(t) = x(t) + iy(t), \quad dz = \gamma'(t)\,dt$$

- **Fundamental th^m of $\mathbb{C}$ line integrals** - If $f(z)$ is an analytic function on open region A and γ is a curve on A from z_0 to z_1, then

$$\int_\gamma f'(z)\,dz = f(z_1) - f(z_0)$$



region of analyticity A

Restating, if the integrand has a well defined antiderivative, then integral depends on endpoints z_0 & z_1

$$\int_\gamma g(z)\,dz = G(z_1) - G(z_0)$$

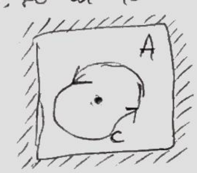
and integrals can just be calculated ordinarily

if antiderative is single valued!

Also, $\int_\gamma f(z)\,dz$ has an antiderative $\Leftrightarrow$ $\int_\gamma f(z)\,dz$ is path independent $\Leftrightarrow$ $\oint_C f(z)\,dz = 0$

Corollary → If $\oint_\gamma f(z)\,dz \neq 0$ when the curve encloses some region inside γ, then antiderivative of $f(z)$ does NOT exist and $f(z)$ isn't analytic!

Note: The fundamental th^m doesn't demand simply connected regions, so it is even applicable in punctured open regions as well!



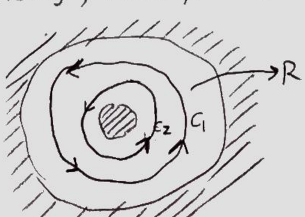
- **Cauchy's Theorem** If A is a simply connected region, $f(z)$ is analytic on A, and C is a simply closed / or even intersecting curve in A, then -

① $\int_C f(z)\,dz = 0$ ② Integrals of f on paths in A are independent

③ $f(z)$ definitely has an antiderivative in A.

- **Extended Cauchy's Theorem** If $f(z)$ is analytic on R where R is a region surrounded by an island of non-analyticity, then for C_1 & C_2 in R,

$$\oint_{C_1} f(z)\,dz = \oint_{C_2} f(z)\,dz$$



• <u>Cauchy's Integral Formula</u> C is a simple closed curve and $f(z)$ is analytic on a region containing C and its interior

$$f(z) = \frac{1}{2\pi i} \oint_C \frac{f(w)}{(w-z)}\, dw$$

z is INSIDE C.

• <u>Cauchy's Integral formula for derivatives</u>

If $f(z)$ & C satisfy the conditions of integral formula, then $\forall z$ inside C

$$\frac{d^n f}{dz^n}(z) = \frac{n!}{2\pi i} \oint_C \frac{f(w)\, dw}{(w-z)^{n+1}} \qquad n = 0, 1, 2, 3 \ldots$$

$\Rightarrow$ An analytic f^n is infinitily diff- hence smooth.

• <u>Triangle inequality for Integrals</u>

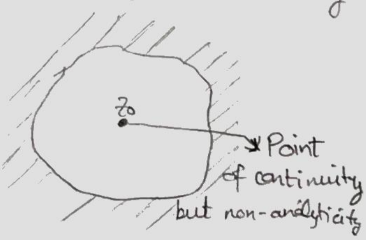
$$\left| \int_\gamma f(z)\, dz \right| \le \int_\gamma |f(z)|\, |dz| \qquad \text{where } |dz| = |\gamma'(t)|\, dt$$

<u>Corrolary</u>: If $|f(z)| < M$ inside C $\Rightarrow \left| \int_\gamma f(z)\, dz \right| < M \cdot (\text{length of } C)$

• <u>Second Extended Cauchy's Theorem</u>: If A is a simply connected region containing z_0 and g is a f^n such that it is

① Continuous on A ② Analytic on $A - \{z_0\}$

then $\oint_C g(z)\, dz = 0$



Also, as a consequence of derivative th^m, if $f(z)$ is analytic on a region containing C (closed loop) and its interior, then derivatives of all orders exist in that region.

REVISION NOTES

- Sum of infinite geometric series. If $|x|<1$, then the sum

$$1+x+x^2+\cdots\cdots = \frac{1}{1-x} \quad (\text{again } |x|<1)$$

- Convergence of power series - $f(z)=\sum_n a_n(z-z_0)^n$.

For such series, $\exists R \geq 0$ s.t. if $R>0$ then the series converges for $|z-z_0|<R$ to an analytic function. R is called the radius of convergence. ($R=0$ series diverges)

- Ratio test. $L=\lim_{n\to\infty}\left|\frac{c_{n+1}}{c_n}\right|$ for the series $\sum_n c_n$ $\quad L<1$ series converges, $L>1$ series diverges

- <u>Taylor's Theorem</u>: Suppose $f(z)$ is an analytic function in region A and the point $z_0 \in A$, then

$$f(z)=\sum_{n=0}^{\infty} a_n(z-z_0)^n \qquad a_n=\frac{1}{n!}\left.\frac{d^nf}{dz^n}\right|_{z_0}=\frac{1}{2\pi i}\int_\gamma \frac{f(z)}{(z-z_0)^{n+1}}\,dz$$

converges for $|z-z_0|<R$ contained in A

γ: simple curve surrounding z_0 in A

Each series expansion about z_0 has a unique expansion!

The radius of convergence is the distance from z_0 to closest singularity ξ.

Taylor's theorem is used for expansion about analytical points.

- <u>Laurent series</u>: Suppose $f(z)$ is analytic on the annulus $A: r_1<|z-z_0|<r_2$

then $$f(z)=\sum_{n=1}^{\infty}\frac{b_n}{(z-z_0)^n}+\sum_{n=0}^{\infty}a_n(z-z_0)^n$$

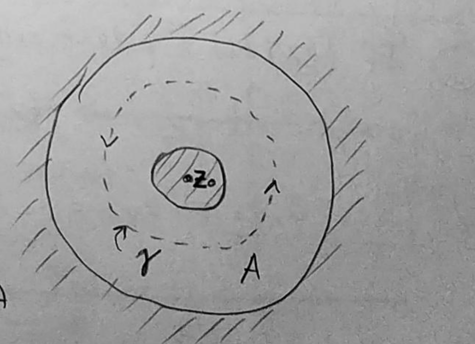
$$a_n=\frac{1}{2\pi i}\int_\gamma \frac{f(w)}{(w-z_0)^{n+1}}\,dw$$

$$b_n=\frac{1}{2\pi i}\int_\gamma f(w)\,(w-z)^{n-1}\,dw$$

Depending on the region of interest, expanding $f(z)$ as a Laurent series might give different expansions. Let our expansion be about $z=0$ & $z=0$ & $z=1$ are poles, then we can expand $f(z)$ on two annuli

$A_1: 0<|z|<1$

$A_2: 1<|z|<\infty$



γ is inside A

- Poles and Zeroes

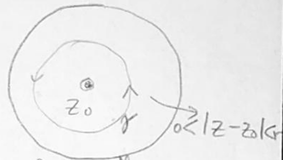
$f(z)$ is analytic at $z_0 \rightarrow f(z) = a_n(z-z_0)^n + a_{n+1}(z-z_0)^{n+1} + \cdots$ — nth order zero at $z = z_0$

$f(z)$ has an isolated singularity at $z_0 \rightarrow f(z) = \frac{b_n}{(z-z_0)^n} + \frac{b_{n-1}}{(z-z_0)^{n-1}} + \cdots \frac{b_1}{z-z_0} + a_0 + a_1(z-z_0) + \cdots$ — nth order pole at z_0

- **Residues** Let $f(z)$ be some function with an isolated singularity but analytic on $0<|z-z_0|<r$ and Laurent series in the same region (neighbourhood of singul.) be -

$$f(z) = \sum_{n=1}^{\infty} \frac{b_n}{(z-z_0)^n} + \sum_{n=0}^{\infty} a_n (z-z_0)^n$$

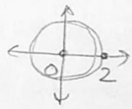
Then $\text{Res}(f, z_0) = b_1$



$$\int_\gamma f(z)\,dz = 2\pi i\, b_1$$

γ: Some small circle surrounding z_0 & no other singularities

- While calculating residue, the annular region across which you expand the function should NOT contain any other singularity. Both Taylor & Laurent Exp - demand an analytic region. Ex - $f(z) = \frac{1}{z(z-2)}$, I can only do Laurent exp. in the region ($0<|z|<2$, not more than this, but small is allowed.) that's why



- Residue Calculation Properties

$\text{Res}(f, z_0) = g(z_0)$

① If $g(z) = (z-z_0) f(z)$ is analytic, then z_0 is either a simple pole or f is analytic. and ↑

② If $f(z)$ has a simple zero at z_0, then $1/f(z)$ has a simple pole at $z = z_0$, then

$$\text{Res}(1/f(z), z_0) = 1/f'(z_0)$$

[expand $f(z) = a_1(z-z_0) + a_2(z-z_0)^2 + \cdots \Rightarrow \frac{1}{f(z)} = \frac{1}{a_1(z-z_0)} \left[\underbrace{\frac{1}{1 + \frac{a_2}{a_1}(z-z_0) + \cdots}}_{\text{analytic with 0th order zero}} \right]$

So, $\text{Res}(\frac{1}{f}, z_0) = \frac{1}{a_1} = \frac{1}{f'(z_0)}$]

③ If $f_{(z)}$ has a simple pole at z_0 and $g(z)$ is analytic at z_0, then-

$$\text{Res}(fg, z_0) = g(z_0)\,\text{Res}(f, z_0)$$ & if $g(z_0) \neq 0$ $$\text{Res}(f/g, z_0) = \frac{1}{g(z_0)} \text{Res}(f, z_0)$$

④ If $f(z)$ has a pole of order k at z_0, then $g(z) = (z-z_0)^k f(z)$ is analytic at $z_0 \Rightarrow g(z) = a_0 + a_1(z-z_0) + \cdots$

$$\text{Res}(f, z_0) = a_{k-1} = \frac{1}{(k-1)!} \frac{d^{k-1}}{dz^{k-1}} g(z_0)$$

Cauchy Residue Theorem – Suppose $f(z)$ is analytic in the region A except for a set of isolated singularities. Also suppose C is a simple closed curve in A that doesn't go through any of the singularities, then –

$$\oint_C f(z)\,dz = 2\pi i \sum (\text{residues of } f \text{ inside } C)$$

If the curve C is big enough to contain all the singularities, then

$$\text{Res}(f,\infty) = \frac{-1}{2\pi i}\oint_C f(z)\,dz = -\sum(\text{sum of residues of } f)$$

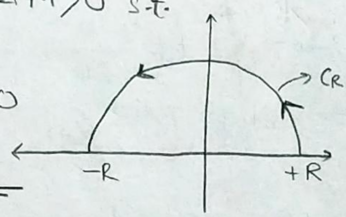
→ So, if f is analytic in C except for some finite no. of singularities (with C big enough to contain all of them)

$$\oint_C f(z)\,dz = 2\pi i \,\text{Res}\left(\frac{1}{\omega^2} f\left(\frac{1}{\omega}, 0\right)\right)$$

CALCULATING DEFINITE INTEGRALS

- Theorem: (a) If $f(z)$ is defined in upper plane and $\exists\, a>1$ & $M>0$ s.t.

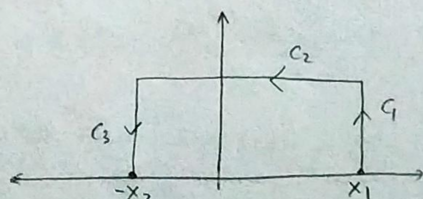
$$|f(z)| < \frac{M}{|z|^a} \left(\text{decays faster than } \frac{1}{z}\right) \Rightarrow \lim_{R\to\infty} \int_{C_R} f(z)\,dz = 0$$



(b) If $f(z)$ is defined in the lower plane, use lower semicircle. Same result.

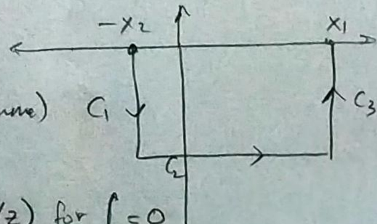
Theorem: (a) $f(z)$ is defined in upper half plane and $a>0$ & it decays like $1/z$ i.e.

$$|f(z)| < \frac{M}{|z|} \text{ for } M>0 \Rightarrow \lim_{\substack{X_1\to\infty \\ X_2\to\infty}} \int_{C_1+C_2+C_3} f(z)\,e^{iaz}\,dz = 0$$



(b) for $a<0$ and $f(z)$ defined in lower plane

$$\lim_{X_1, X_2\to\infty} \int_{C_1+C_2+C_3} f(z)\,e^{iaz}\,dz = 0 \quad \text{(basically Jordan's Lemma)}$$



→ For integrals with e^{iax} $\boxed{\lim_{|z|\to\infty} |f(z)| = 0}$ is enough. (decay like $1/z$) for $\int_{C_R} = 0$

For integrals of form $\int f(z)\,dz$, $\boxed{\lim_{|z|\to\infty} |z f(z)| = 0}$ is needed for $\int_{C_R} = 0$ (decay faster than $\frac{1}{z}$)

- For integrals of the type $\int_{-\infty}^{\infty}$ and $\int_{0}^{\infty}$ the f^n should decay faster than $1/z$

i.e. $\lim_{|z|\to\infty} |zf(z)| = 0$ or $|f(z)| < \frac{|M|}{|z|^a}$ for $a > 1$, then choose a contour,

and calculate $\oint f(z)dz$ at $\lim_{R\to\infty}$ (with one of contour lines as x-axis)

Then $\lim_{R\to\infty} \oint f(z)dz = \sum [\text{Res}(f) \text{ in the upper plane}]$

- For integrals of the type $\int_{-\infty}^{\infty} e^{iax} f(x)\,dx$, the function should should decay like $\frac{1}{z}$ i.e. $\lim_{|z|\to\infty} |f(z)| = 0$ or $|f(z)| < \frac{|M|}{|z|}$ in the upper plane (important)

Then again $\lim_{R\to\infty} \oint f(z) e^{iaz} dz = \sum [\text{Res}(f(z)e^{iaz}) \text{ in upper plane}]$

- Trigonometric Integral

$$\int_0^{2\pi} f(\sin\theta, \cos\theta)\, d\theta = ?$$

$$\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2} = \frac{z + z^{-1}}{2} \qquad d\theta = \frac{dz}{iz}$$

$$\sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i} = \frac{z - z^{-1}}{2i}$$

$z = e^{i\theta}$ unit circle

$$\Rightarrow I = \oint_{|z|=1} \underbrace{f\left(\frac{z-z^{-1}}{2i}, \frac{z+z^{-1}}{2i}\right)\frac{dz}{iz}}_{g(z)} = \oint_{|z|=1} g(z)\,dz = \sum \text{Res}(g(z))$$

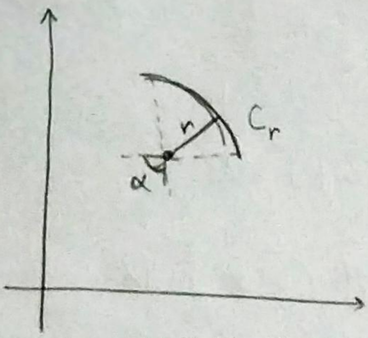
- Cauchy Principle Value

$$\text{p.v.}\left(\int_{-\infty}^{\infty} f(x)dx\right) = \lim_{\substack{R\to\infty \\ r_i\to 0}} \left[\int_{-R}^{x_1-r_1} f(x)dx + \int_{x_1+r_1}^{x_2-r_2} + \cdots \int_{x_n+r_n}^{R} f(x)dx\right]$$

$x_i \to$ discontinuities

If $f(x)$ has finite discontinuities, and $\int_{-\infty}^{\infty} f(x)dx$ converges, then so does $\text{p.v.}\left(\int_{-\infty}^{\infty} f(x)dx\right)$ and $\int_{-\infty}^{\infty} f(x)dx = \text{p.v.}\left(\int_{-\infty}^{\infty} f(x)dx\right)$

• Integrals over portions of circles (Generalised Residue th^m)



Suppose $f(z)$ has a simple pole at z_0. Let C_r be the arc $\gamma(t) = z_0 + re^{it}$ $\quad \theta_0 \leq t \leq \theta_0 + \alpha$

$$\Rightarrow \lim_{r \to 0} \int_{C_r} f(z)\,dz = \alpha i \,\text{Res}(f, z_0)$$

ONLY FOR SIMPLE POLES

• Gamma Functions

$$\Gamma(z) = \int_0^\infty e^{-t}\, t^{z-1} \xrightarrow{u = \ln(1/t)} \Gamma(z) = \int_0^1 \left(\ln \frac{1}{u}\right)^{z-1} du$$

$\downarrow x^2 = t$

$$\Gamma(z) = 2\int_0^\infty e^{-x^2} x^{2z-1}\,dx$$

Another alternate def^n

$$\Gamma(z) = \lim_{n\to\infty} \frac{1\cdot 2\cdot 3 \cdots\cdots n \quad n^z}{z(z+1)(z+2)\cdots\cdots(z+n)}$$

$\text{Re}(z) > 0$ $\qquad z \neq 0, -1, -2, -3 \ldots \Leftarrow$ poles

$$\Gamma\left(\tfrac{1}{2}\right) = \sqrt{\pi}$$

$$\Gamma(z+1) = z\,\Gamma(z) \Rightarrow \Gamma(z-1) = \frac{\Gamma(z)}{z-1} \qquad \Gamma(z) = (z-1)!$$

• Diagamma & Polygamma

$$\ln \Gamma(z+1) = \lim_{n\to\infty} \left[\ln(n!) + z\ln(n) - \sum_{m=1}^{n} \ln(z+m)\right]$$

Digamma $\frac{d}{dz} \ln(\Gamma(z+1)) = \lim_{n\to\infty} \left[\ln(n) - \sum_{m=1}^{n} \frac{1}{(z+m)}\right] = \psi'(z+1) = \frac{[\Gamma(z+1)]'}{\Gamma(z+1)}$

Poly gama $\psi^{(a)}(z+1) = \frac{d^a}{dz^a}(\ln(\Gamma(z+1)) = (-1)^a (a-1)! \sum_{m=1}^{\infty} \frac{1}{(z+m)^a} \quad a \geq 2$

- Beta Function

$$\boxed{\beta(m+1,n+1) = \frac{\Gamma(m+1)\Gamma(n+1)}{\Gamma(m+n+2)} = \frac{m!\,n!}{(m+n+1)!} = 2\int_0^{\pi/2} \cos^{2m+1}\theta \sin^{2n+1}\theta \, d\theta}$$

→ sub in $\cos^2\theta = t \Rightarrow \beta(m+1,n+1) = \int_0^1 t^m (1-t)^n \, dt$

→ sub in $t = x^2 \Rightarrow \beta(m+1,n+1) = 2\int_0^1 x^{2m+1} (1-x^2)^n \, dx$

→ sub in $t = u/1+u \Rightarrow \beta(m+1,n+1) = \int_0^\infty \frac{u^m}{(1+u)^{m+n+2}} \, du$

- Branches

For fractional powers, $z^{1/n} = r^{1/n} \exp\left(\frac{i\phi}{n}\right) \underbrace{\exp\left(\frac{i2\pi m}{n}\right)}$ → n valued
$1 \leq m \leq n$

To find powers of complex nos., $z^a = e^{a\log(z)}$ and then use the principal branch for $\log(z)$

- The branch is defined as the choice of range for a given multivalued function $f(z)$.
The branch cut removes the parts of the domain which make $f(z)$ (such as $\log(z)$ or $z^{1/n}$) discontinuous by defining the branch for $\arg(z) = \theta + 2n\pi$

- Cauchy Reimann for $f(z) = u(x,y) + iv(x,y)$, f is analytic if $\partial_x u = \partial_y v$, $\partial_x v = -\partial_y u$

For a f^n $f(r,\theta)$, if $f(r,\theta+2\pi) \neq f(r,\theta)$, we face an identity crisis since the same pt. on z-plane maps to more than one value of f. So, we introduce a branch cut accordingly to make f single valued in the reqd. domain (usually $0 \leq \theta < 2\pi$)

Revision Notes

Differential Eq^{n}s

- **Exact first order D.E.** $\rightarrow Q(x,y)\frac{dy}{dx} + P(x,y) = 0 \Rightarrow P(x,y)dx + Q(x,y)dy = 0$

if $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x} \Rightarrow$ arising from some $d\varphi = P dx + Q dy \Rightarrow \varphi = \text{constant}$

- **LINEAR first order O.D.E** if D.E. isn't exact, make it exact

$\left(\frac{d}{dx} + p(x)\right) y(x) = q(x)$ multiply by $\alpha(x)$ to make it exact

$$\alpha_{(x)} = e^{\int^{x} p(x')dx'} \Rightarrow \boxed{y(x) = e^{-\int^{x} p(x')dx'}\left[\int q(x)\, e^{\int^{x} p(x')dx'} dx + C\right]}$$

- **LINEAR Homogeneous second order O.D.E** $\left[\frac{d^2}{dx^2} + P(x)\frac{d}{dx} + Q(x)\right] y(x) = 0$

Use $y(x) = \sum_{i=0}^{\infty} a_i x^{i+k}$ find indicial eq^{n} and recursion relation $\quad a_0 \neq 0 \quad$ set $a_1 = 0$ as long as it is arbitrary

★ If we have $ay'' + by' + c = 0$ with a, b, c const, then factorise the D.E. as $\left(\frac{d}{dx} - s_1\right)\left(\frac{d}{dx} - s_2\right) y = 0$

- **Second solution of Linear hom. 2nd order ODE**

$$W = \begin{vmatrix} f_1 & f_2 & f_3 \\ f_1' & f_2' & f_3' \\ f_1'' & f_2'' & f_3'' \end{vmatrix}$$

$\rightarrow = 0$ linearly dependent f^{n}s

$\rightarrow \neq 0$ linearly independent f^{n}s

For y_1 & y_2, sol^{n}s of $y'' + P(x)y' + Q(x)y = 0$

$$\boxed{W_{(x)} = W_0\, e^{-\int^{x} p(x')dx'}}$$

But since $\frac{d}{dx}\left(\frac{y_2}{y_1}\right) = \frac{W}{y_1^2} \Rightarrow \boxed{y_2(x) = W_0\, y_1(x) \int dx \frac{e^{-\int^{x} p(x')dx'}}{y_1(x)^2}}$

$W = y_1 y_2' - y_2 y_1' \Rightarrow W' = y_1 y_2'' - y_2 y_1'' \Rightarrow W' = -2p(x) W$

Sturm Liouville Theory

- **Self adjoint.** Given $\hat{L} = p_0(x)\hat{D}_x^2 + p_1(x)\hat{D}_x + p_2(x)$

$$\boxed{\hat{\bar{L}} = p_0(x)\hat{D}_x^2 + (2p_0' - p_1)\hat{D}_x + (p_0'' - p_1' + p_2)}$$

- **Making an operator self adjoint** If $\mathcal{L}$ is not a self adjoint operator -

i.e $\mathcal{L}y = y'' + P(x)y' + Q(x)y$ Assume $y = M_{(x)} z(x)$ where $M(x)$ is the adjust fn.

$\mathcal{L}y = \mathcal{L}(Mz) = (\mathcal{L}M)z$ Now adjust $M(x)$ s.t. the self adjoint condition is satisfied $(p_1 = p_0')$

then $\mathcal{L}M$ is a S.A. operator.

- **Gram Schmidt Orthogonalization** $\{v_i\} \rightarrow$ linearly independent set

① $\tilde{V}_0 = V_0 \longrightarrow$ normalize to get f_0

② $\tilde{V}_1 = V_1 - \langle f_0 | V_1 \rangle \longrightarrow$ normalize to get f_1

③ $\tilde{V}_2 = V_2 - \langle f_0 | V_2 \rangle - \langle f_1 | V_2 \rangle \rightarrow$ norm. to get f_2

Subtract projections with orthonormalized vectors. And then normalize again!

GREEN FUNCTION

- **Solving inhomogeneous eq^n's by constructing G.F.**

Let's say the D.E. is $\underline{p_0 y''_{(x)} + p_1 y'_{(x)} + p_2 y_{(x)} = f(x)}$ $\mathcal{L} \equiv p_0 \hat{D}_x^2 + p_1 \hat{D}_x + p_2$

So, $\underline{\mathcal{L}y(x) = f(x)}$. Now we convert the diff. problem to an integral problem

Let $\exists$ G s.t. $\underline{\mathcal{L}G(x,x') = \delta(x-x')}$, then $\underline{y(x) = \int_a^b G(x,x') f(x') dx'}$

To find G.F. solve this when $x \neq x'$ $\mathcal{L}G(x,x') = 0$

You'll get sol^ns like $G_1(x)$ & $G_2(x)$, then

$$\boxed{G(x,x') = \begin{cases} \alpha G_1(x) + \beta G_2(x) & a<x<x'<b \\ \gamma G_1(x) + \xi G_2(x) & b>x>x'>a \end{cases}}$$

To find $\alpha, \beta, \gamma, \xi$ -

① use boundary conditions at $y(x=a,b)$

② Continuity of $G(x,x')$ at $x=x'$

③ Appropriate dis cont. jump $\underline{\Delta \partial_x G \big|_{x=x'} = \frac{1}{p_0'}}$

then

$$\boxed{G(x,x') = \int_a^x (\gamma G_1 + \xi G_2) f(x') dx' + \int_x^b (\alpha G_1 + \beta G_2) f(x') dx'}$$

- Eigen f^n expansion of Green f^n

Given a general D.E. $\boxed{\mathcal{L} y(x) - \mu\, \omega(x)\, y(x) = \underbrace{f(x)}_{\text{source}}}$

then if we find the complete set of eigenvalues / f^n's for $\mathcal{L}$ $\boxed{\mathcal{L}\varphi_n(x) = \lambda_n\, \omega(x)\, \varphi_n(x)}$

$\{\varphi_n(x)\}$ → normalize them

then we can write the sol^n $y(x) = \sum_{n=0}^{\infty} c_n \varphi_n(x)$

$$\Rightarrow (\mathcal{L} - \mu\omega(x)) \sum_n c_n \varphi_n(x) = f(x) = \sum_{n=0}^{\infty} \left[(\lambda_n - \mu)\, \omega(x)\, c_n\right] \varphi_n(x)$$

take the integral with $\varphi_m^*(x)$ multiplied on both sides

$$\int_a^b \varphi_m^*(x) f(x)\, dx \quad \sum_{n=0}^{\infty} \int_a^b (\lambda_n - \mu)\, c_n\, \omega(x)\, \varphi_m^*(x)\, \varphi_n(x)\, dx$$

$$\int_a^b \varphi_m^*(x) f(x)\, dx = (\lambda_n - \mu)\, c_m \Rightarrow c_m = \int_a^b \frac{\varphi_m^*(x) f(x)}{(\lambda_n - \mu)}\, dx$$

then $$y(x) = \int_a^b \underbrace{\sum_{n=0}^{\infty} \frac{\varphi_n^*(x')\, \varphi_n(x)}{(\lambda_n - \mu)}}_{G(x,x')} f(x')\, dx' \Rightarrow \boxed{G(x,x') = \sum_{n=0}^{\infty} \frac{\varphi_n^*(x')\, \varphi_n(x)}{(\lambda_n - \mu)}}$$

While solving a simpler problem of $\boxed{\mathcal{L} y(x) = f(x)}$ & $\boxed{\mathcal{L}\varphi_n = \lambda_n \varphi_n(x)}$ $\omega(x) = 1$

then $$\boxed{G(x,x') = \sum_{n=0}^{\infty} \frac{\varphi_n^*(x)\, \varphi_n(x)}{\lambda_n}}$$ and thus $y(x) = \int_a^b G(x,x') f(x)\, dx'$

Remember: if weight f^n is introduced in eigenvalue eq^n, it will also be introduced in the inner product. Reason? I still don't know...

- Def^n of Hermitian adjoint (which I always forget) $\langle f | A g \rangle = \langle g | A^\dagger f \rangle^*$

Revision Sheet for Special functions

1. Legendre Polynomials

- $(1-x^2) P_\ell''(x) - 2x P_\ell'(x) + \ell(\ell+1) P_\ell(x) = 0$ $\rightarrow \frac{d}{dx}\left[(1-x^2) P_\ell'\right] + \ell(\ell+1) P_\ell = 0$

  Legendre diff. eqⁿ

- Series expansion

$$P_\ell(x) = \sum_{k=0}^{\lfloor \ell/2 \rfloor} (-1)^k \frac{(2\ell-2k)!}{2^\ell \, k! \, (\ell-k)! \, (\ell-2k)!} x^{\ell-2k}$$

$\Rightarrow$ The Legendre polynomial $P_\ell(x)$ is a ℓ-degree polynomial.

- Generating fⁿ $\quad g(x,t) = (1-2xt+t^2)^{-1/2} = \sum_{n=0}^{\infty} t^n P_n(x)$

- Recurrence relations.

  using $\partial_t g \rightarrow$ $x(2n+1) P_n(x) = (n+1) P_{n+1}(x) + n P_{n-1}(x)$

  using $\partial_x g \rightarrow$ $2x P_n'(x) + P_n(x) = P'_{n+1}(x) + P'_{n-1}(x)$

other recursions

$\rightarrow P'_{n+1}(x) - P'_{n-1}(x) = (2n+1) P_n(x)$, $\rightarrow (1-x^2) P_n'(x) = n P_{n-1}(x) - nx P_n(x)$

$\rightarrow P'_{n-1}(x) = -n P_n(x) + x P_n'(x)$, $\rightarrow (1-x^2) P_n'(x) = (n+1) x P_n(x) - (n+1) P_{n+1}(x)$

- Special values, Parity & orthoganility

  $\rightarrow P_n(-x) = P_n(x)(-1)^n \Rightarrow P_n(-1) = (-1)^n$ $\quad \underline{\underline{x=1}}$

  $\rightarrow P_{2n}(0) = \frac{(2n)!\,(-1)^n}{2^{2n}(n!)^2}$, $\quad P_{2n+1}(0) = 0$ $\quad \underline{\underline{x=0}}$

$\rightarrow \int_{-1}^{+1} P_n P_m \, dx = \frac{2}{2n+1} \delta_{n,m}$ (multiply by P_m in Legendre eqⁿ for orthogonality)

- Rodrigues' formula $\quad P_n(x) = \frac{d^n}{dx^n}\left(\frac{1}{2^n n!} (x^2-1)^n\right)$

2. Associated Legendre Polynomials.

$$P_\ell^m(x) = (1-x^2)^{m/2} \frac{d^m}{dx^m} P_\ell(x)$$ $\rightarrow$ defⁿ in terms of Legendre Polynomial

→ General Legendre ODE $(1-x^2)\, P_\ell^{m\,''}(x) - 2x\, P_\ell^{m\,'}(x) + \left[\ell(\ell+1) - \dfrac{m^2}{1-x^2} \right] P_\ell^m = 0$

For $m=0$, $P_\ell^0(x) = P_\ell(x)$

Extending the defⁿ for $-\ell \le m \le \ell$ $\quad P_\ell^m(x) = \dfrac{(1-x^2)^{m/2}}{2^\ell \ell!} \dfrac{d^{m+\ell}}{dx^{m+\ell}} (x^2-1)^\ell$

• Parity $\quad P_\ell^m(-x) = (-1)^{\ell+m} P_\ell^m(x)$

$P_\ell^{-m}(x) = (-1)^m \dfrac{(\ell-m)!}{(\ell+m)!} P_\ell^m(x)$

③ $\boxed{\text{Hermite functions}}$

• Generating function $\quad g(x,t) = e^{-t^2+2xt} = \sum_{n=0}^{\infty} H_n(x) \dfrac{t^n}{n!} \Rightarrow \left. \dfrac{\partial^n}{\partial t^n} g(x,t) \right|_{t=0} = H_n(x)$

• Recurrence relations

$2x\, H_n(x) - 2n\, H_{n-1}(x) = H_{n+1}(x)$ → using $\partial_t g$

$H_n'(x) = 2n\, H_{n-1}(x)$ → using $\partial_x g$

Recursion relations are immensly useful (paired with orthogonality) to solve integrals.

→ $H_{2n+1}(0) = 0 \quad H_{2n} = (-1)^n \dfrac{(2n)!}{n!}$ → $H_n(-x) = (-1)^n H_n(x)$

• Rodrigues's formula $\quad H_n(x) = (-1)^n e^{x^2} \dfrac{d^n}{dx^n} e^{-x^2}$ $\quad$ differentiate $g(x,t)$ n times & set $t=0$

Self adjoint Hermite's D.E. $\quad e^{-x^2} \left[H_n''(x) - 2x H_n'(x) + 2n H_n(x) \right] = 0$

→ $\displaystyle\int_{-\infty}^{\infty} H_m(x) H_n(x) \underbrace{e^{-x^2}}_{\text{weight } f^n} dx = \sqrt{\pi}\, 2^n n!\, \delta_{m,n}$ $\quad$ orthogonality condition

4 Bessel functions

- **Generating function** $g(x,t) = e^{x/2(t-1/t)} = \sum_{n=-\infty}^{\infty} J_n(x)\, t^n$

Series expansion $J_n(x) = \sum_{s=0}^{\infty} \frac{(-1)^s \left(\frac{x}{2}\right)^{n+2s}}{(n+s)!\, s!}$

In the above expansion $n \to -n \Rightarrow J_{-n}(x) = (-1)^n J_n(x)$

similarly use $g(x,-t) = g(-x,t)$

$\to J_n(-x) = (-1)^n J_n(x)$

- **Recursion relations**

$\to J_{n-1}(x) + J_{n+1}(x) = \frac{2n}{x} J_n(x)$ → using $\partial_t g$

$\to J_{n-1}(x) - J_{n+1}(x) = 2 J_n'(x)$ → using $\partial_x g$

using these two, we get $\to J_{n-1}(x) = \frac{n}{x} J_n(x) + J_n'(x)$

and $\to J_{n+1}(x) = \frac{n}{x} J_n(x) - J_n'(x)$

and $\frac{d}{dx}\left(x^n J_n(x)\right) = x^n J_{n-1}(x)$ → can prove using induction

- **Bessel eq^n** $x^2 J_\nu'' + x J_\nu' + (x^2 - \nu^2) J_\nu(x) = 0$

- **Integral Representation** use $t = e^{i\varphi}$ or some combination thereof in $g(x,t)$

$\cos(x\sin\theta) = J_0 + 2\sum_{n=1}^{\infty} J_{2n} \cos(2n\theta)$

$\sin(x\sin\theta) = 2\sum_{n=0}^{\infty} J_{2n+1} \sin((2n+1)\theta)$

then using fourier series orthogonality

$J_{2n}(x) = \frac{1}{\pi}\int_0^{\pi} \cos(x\sin\theta)\cos(2n\theta)\, d\theta$, $J_{2n+1}(x) = \frac{1}{\pi}\int_0^{\pi} \sin(x\sin\theta)\sin(2n+1\cdot\theta)\, d\theta$

orthogonality → $\int_0^a J_\nu\left(k_{\nu m}\frac{\rho}{a}\right) J_\nu\left(k_{\nu n}\frac{\rho}{a}\right) \rho\, d\rho = \frac{a^2 J_{\nu+1}^2(k_{\nu n})}{2}\, \delta_{n,m}$

expansion $\Rightarrow f(x) = \sum_{m=1}^{\infty} C_{\nu m} J_\nu\left(\frac{\alpha_{\nu m} x}{a}\right)$

NOTES:

* When doing integrals like $\int H_n(x)\, e^{-x^2/2}$ or $\int J_\nu(x)\, dx$ or something, we can use generating f^n, and then taking the integral of $g(x,t)$ has the coeff. of expansion as the integral we want to evaluate. (by collecting $t^n/n!$)

* All the special f^n's form an orthogonal basis in a given range, so any f^n can be expanded in terms of them.

* Use recursion relation for simplifying integrals. Immensely useful for $|H_n|^2$ or $|P_n|^2$ type.

* Don't even try to integrate generating f^n of $P_n(x)$

* $\frac{d}{dx}(fg')h - \frac{d}{dx}(fh')g = \frac{d}{dx}\left(f\,(g'h - h'g)\right)$