Relativistic Quantum Mechanics

PHY424 - QUANTUM FIELD THEORY - 1

Grading Structure:

2 Assignments : 20%
2 Mid Terms : 30% → after 1month → after 2months.
1 Final : 50%

Textbooks:

1. Bjorken & Drell, RQM V1, QFT V2
2. Sakurai, Quantum Mechanics.
3. Lahiri and Pal, A First book of QFT
4. Peskin & Schroeder, An introduction to QFT
5. Srednicki, Quantum Field Theory
6. J.D. Jackson, Classical Electrodynamics.

modern replacement.

No separate tutorials. Can contact on whatsapp.

Office hours:

Resources for SR: (1) Martin Martinez GR lectures
(2) klimas.paginas.ufsc.br/files/2020/07/chapter-1.pdf

Relativistic Quantum Mechanics.

For a non-relativistic quantum particle.

$$E = \underbrace{\frac{\vec{p}^2}{2m}}_{KE} + \underbrace{V(\vec{r})}_{PE}$$

The relativistic energy momentum relation is given as

$$E^2 = p^2c^2 + m^2c^4 \quad \text{(for free particle)}$$

$$E = \pm\sqrt{p^2c^2 + m^2c^4} + \underbrace{V(\vec{r})}_{?} \quad \text{(is this even correct?)}$$

and if interactions are described by $V(\vec{r})$ it corresponds to instantaneous forces and action-at-a-distance scenarios in a relativistic theory.

A proper formulation should have causal interactions.

Also, NRQM is actually a single particle / Ensemble of stable particle theory.

The relativistic theory however allows creation & destruction of particles, and a scattering process may give rise to arbitrary no. of particles. The number of particles isn't conserved. ∴ The Hilbert space of the theory intrinsically has states with arbitrary no. of particles

If one is in a non-rel. theory, the energy $<< mc^2$ and the Hilbert space can be considered to have the same dim^n always b/c particle nos. are conserved.

Creation and Destruction of particles $\longrightarrow$ QFT : {HO}

$\uparrow$

$a_i \,,\, a_i^\dagger$

$i = 1, 2, 3 \ldots \infty$

(usually has a cont. index $a_{\vec{p}} \,,\, a_{\vec{p}}^\dagger$)

Then we can start with a ground state and apply $a_{\vec{p}}$ and $a_{\vec{p}}^\dagger$

$|0\rangle \quad : \quad a_{\vec{p}}^\dagger |0\rangle = |\vec{p}\rangle$

Vacuum

one particle

$a_{\vec{p}_1}^\dagger a_{\vec{p}_2}^\dagger |0\rangle = |\vec{p}_1, \vec{p}_2\rangle$ two particles.

$$\prod_i \left(a_{\vec{p}_i}^\dagger\right)^{n_i} |0\rangle = |\overset{\vec{p}_1}{n_1} \; \overset{\vec{p}_2}{n_2} \ldots\ldots\rangle$$

Fock-space state

n_1 particles have $\vec{p}_1$

n_2 particles have $\vec{p}_2$

$\vdots$

(for fermions, n can only be 0, 1)

So, what is QFT?

$\phi(\vec{x}, t) \longrightarrow$ infinite collection of dynamical variables indexed by $\vec{x}$

this field is quantized.

In QM, (Heisenberg formalism) $\quad [\hat{q}(t), \hat{p}(t)] = i\hbar$

EQUAL TIME COMMUTATION RELATIONS

In QFT, $\quad [\phi(\underbrace{\vec{x}, t}_{x^\mu}), \pi(\underbrace{\vec{y}, t}_{y^\mu})] = i\hbar\, \delta^{(d)}(\vec{x} - \vec{y})$

For each $\vec{x}$ & $\vec{y}$, this eq^n defined. (infinite d.o.f. in this eq^n)

Units

Dimensional quantities can be reduced to units of mass, length & time.

$\exists$ different unit systems: cgs, mks, SI, Heaviside-Lorentz $\}$ Conventional units.

Natural units:

In SI, $c = 2.99792\ldots \times 10^{10}$ cm/s. $\quad [c] = LT^{-1}$

$$\beta = \frac{v}{c} \longrightarrow \text{a number} \in [0,1]$$

In natural units, $\underline{\underline{c=1}}$.

$$[c] = LT^{-1} = 1 \Rightarrow L \sim T \text{ (length is equivalent to time)}$$

$$1 \text{ sec} = 3 \times 10^{10} \text{ cm.}$$

(Reduced) Planck's constant $\hbar = \frac{h}{2\pi} = 1.054 \times 10^{-34}$ J s

$$[\hbar] = [p][q] = [E][t] = ML^2T^{-1}$$

For reference, $1\,\text{eV} = 1.6 \times 10^{-19}$ J (unit of energy)

in these units, $\hbar = 6.58 \times 10^{-16}$ eV-sec.

$= 6.58 \times 10^{-22}$ MeV-sec

$= 6.58 \times 10^{-28}$ GeV sec

The next step is to put $\underline{\underline{\hbar = 1}}$

$$[\hbar] = ML^2T^{-1} = 1 = ML\underbrace{LT^{-1}}_{\substack{\shortparallel \\ 1\ (c=1)}} \Rightarrow ML = 1$$

$$\Rightarrow M \sim L^{-1} \sim T^{-1}$$

Also, $E \sim T^{-1} \sim M$

Thus, we define <u>ENERGY NATURAL UNITS</u> as

$$\underline{\hbar = 1, \quad c = 1}$$

Some good rules of thumb.

$\hbar c = 197.326$ MeV-fm

$\text{fm} = 10^{-15}\,\text{m} = 10^{-13}\,\text{m}$ ~ size of a neutron

So writing $\hbar c \approx 200\,\text{MeV-fm} = 1$ (since $\hbar = c = 1$)

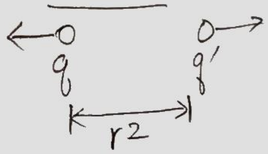
∴ in this unit system

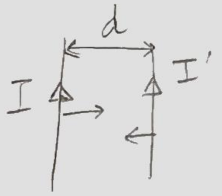
$$1\,\text{fm} = (200\,\text{MeV})^{-1} = 5\,\text{GeV}^{-1}$$

(gives an energy scale that needs to probe that distance)

<u>Electromagnetic units.</u>

$$F_{coul} = \frac{k_1 q q'}{r^2}$$

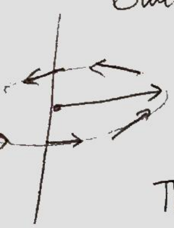




$$F_{amp} = 2 k_2 \frac{I I'}{d}$$

$$\vec{\nabla} \times \vec{E} = -k_3 \frac{\partial \vec{B}}{\partial t}$$

$$B_{wire} = \frac{2 k_2 \tilde{\alpha} I}{d}$$



The Maxwell's eq^{n}s are – (for free space)

$$\vec{\nabla} \cdot \vec{E} = 4\pi k_1 \rho \qquad \vec{\nabla} \times \vec{E} = -k_3 \frac{\partial \vec{B}}{\partial t}$$

$$\vec{\nabla} \times \vec{B} = 4\pi k_2 \tilde{\alpha} \vec{J} + \frac{k_2 \tilde{\alpha}}{k_1} \frac{\partial \vec{E}}{\partial t} \qquad \vec{\nabla} \cdot \vec{B} = 0$$

Combining these we get

$$\nabla^2 \vec{B} - \frac{k_3 k_2 \alpha}{k_1} \frac{\partial^2 \vec{B}}{\partial t^2} = 0 \quad (\text{wave - eq}^n)$$

$$\Rightarrow \frac{k_1}{\tilde{\alpha} k_2 k_3} = c^2$$

Using the Heaviside - Lorentz units (cgs)

$$k_1 = \frac{1}{4\pi} \qquad k_2 = \frac{1}{4\pi c^2} \qquad k_3 = \frac{1}{c} \qquad \tilde{\alpha} = c$$

(J. D. Jackson Appendix)

Natural Units (diff ways to think about it)

In cgs, the quantities have a dimension $[A] = M^a L^b T^c$

If we redefine the fundamental dimensions as S, V, M, then -

- Mandl & Shaw
- q/102527
- q/ 332838
- MIT units notes.
- The nature of natural units. (Rick Van Remortel)

$$[S]_{cgs} = [E][t] = ML^2T^{-2}T = ML^2T^{-1}$$
$$[V]_{cgs} = LT^{-1}$$

$$\Rightarrow \quad S^{a'}V^{b'}M^{c'} = M^{a'}L^{2a'}T^{-a'}L^{b'}T^{-b'}M^{c'}$$
$$= M^{a'+c'}L^{2a'+b'}T^{-a'-b'} = M^{\alpha'}L^{\beta'}T^{\gamma'}$$

$$\alpha' = a'+c'$$
$$\beta' = 2a'+b'$$
$$\gamma' = -(a'+b')$$

$$-(\beta'+2\gamma') = b'$$
$$2a' = \beta'+\beta'+2\gamma' \Rightarrow \underline{a' = \beta'+\gamma'}$$
$$\underline{c' = \alpha'-\beta'-\gamma'}$$

$$\Rightarrow \underline{M^a L^b T^c = S^{b+c}V^{-(b+2c)}M^{a-b-c}}$$

So, there are two schools of thought for using natural units as far as I understand -

(a) Call c the new unit of velocity, & $\hbar$ the new unit of action. We redefine our units of action & velocity s.t. 1 einstein $= c$, and 1 planck $= \hbar$.
So putting $c=1$ doesn't mean it is literally just 1, we are actually setting $c = 1$ velocity unit $=$ 1 einstein. (same for $\hbar$).

So, one can "suppress" the c & $\hbar$ factors in all the eq^{n}s for the natural units & then bring them back using dimensional analysis whenever we need the numbers in conventional unit systems.

So "$\hbar = c = 1$" isn't to be taken literally. It says rescale units s.t. numerical values of c & $\hbar$ in this system is 1.

One can also view it as follows. It's much easier to state the mass of Higgs Boson is 125 GeV, but what it's really telling is the rest mass energy of Higgs Boson with c^{-2} "suppressed" because

$$m_H = 125 \times 10^9 \text{ GeV}/c^2.$$

Therefore, to restore the correct dimensionality, one needs to remember implicitly multiplying by appropriate factors of c & $\hbar$, but meanwhile everything can be expressed in powers of eV.

(also, $\hbar + c = 2$ is utter nonsense in this discussion)

(b) I don't properly understand the second school of thought, but afaik, it says that non-dimensionalize S and V after changing to S, V, M (or E) system. i.e. literally set $c = \hbar = 1$.

That'd mean redefining velocity to be $V' = \frac{V}{c}$ and $S' = \frac{S}{\hbar}$ in this new system and the only fundamental dimⁿ left is Energy (or mass).

However, some textbooks don't even do the first step & directly do

$C = 1 \Rightarrow \quad 3\times10^8 \text{ m/s} = 1 \Rightarrow 3\times10^8 \text{ m} = 1 \text{ s}$ which is confusing as heck

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However, doing the above thing is fine as long as the units aren't literally identical, but instead $L \sim T \sim M^{-1}$ under multiplication factors of $\hbar$ & c. ($\sim$ means the dimensions aren't equal, but equivalent under some transformation)

Lecture-2

07-Sept

Natural Units (contd.)

EM units: Heaviside-Lorentz units are convenient for EM because Maxwell's eqⁿs get a simple form

$$\partial.F = j \quad \text{Inhomogeneous}$$
$$\epsilon.F = 0 \quad \text{homogeneous}$$
(in H-L units)

The natural units $\hbar = 1$, $c = 1$ imply a certain "equivalence" relation of the dimⁿs of M, L, T (upto dim's of $\hbar$ & c)

So, we won't say that $c=1$ & $\hbar=1$ implies that c & $\hbar$ are dimensionless numbers now, or say that $3\times10^8\,m = 1s$. We will however use the symbol '$\sim$' to describe the similarity b/w these dimⁿs upto factors/dimⁿs of c & $\hbar$ and we restore them back whenever needed.

$$\text{So } M \sim L \sim T^{-1} \quad (\text{not } M = L = T^{-1})$$

So, from now on, we'll write every quantity in the units of powers of the energy unit eV. The factors of c & $\hbar$ are set to 1 and are "suppressed" for notational convenience.

So, $\hbar c = 197.3\,\text{MeV-fm} \sim 1$ (since $c=1, \hbar=1$)

$$\Rightarrow 197.3\,\text{MeV} \sim 1\,\text{fm}^{-1} \qquad \text{where } 1\,\text{fm} = 10^{-15}\,m \sim (200\,\text{MeV})^{-1}$$

Similarly, $3\times10^8\,m \sim 1s$.

Let us now express different physical quantities in terms of units of energy on setting $c=1$, $\hbar=1$.

$E=\hbar\omega$ but $\hbar=1 \Rightarrow E=\omega$ ($\hbar$ is suppressed but understood)

$E=mc^2$ but $c=1 \Rightarrow E=m$

$E=pc$ but $c=1 \Rightarrow E=p$ (c is suppresed but understood)

So, let's make a table of different physical quantities. (given $[E]=1$)

A		Mass dim^{n}s.
$\omega=\frac{E}{\hbar}$ but $\hbar=1 \Rightarrow E=\omega$	E & ω have same	$[\omega]=1$
$m=\frac{E}{c^2}$ but $c=1 \Rightarrow E=m$	E & m have same dim^n	$[m]=1$
$p=\frac{E}{c}$ but $c=1 \Rightarrow E=p$	E & p have same	$[p]=1$
$\oint p\,dq=\hbar$ but $\hbar=1$		$[q]=-1$
$t=\frac{l}{c}$ but $c=1$		$[t]=-1$

X

In the above table $[\;] \equiv$ the mass dim^{n}s of the physical quantity.

* In the standard MLT system if a physical quantity A has

$[A]_{conv.} = M^{\alpha}L^{\beta}T^{\gamma}$

$\Rightarrow [A_{conv.}] = M^{\alpha}L^{\beta}T^{\gamma} = \underset{\text{mass}}{M^{\alpha-\beta-\gamma}}\ \underset{\text{action}}{S^{\beta+\gamma}}\ \underset{\text{velocity}}{V^{-\beta-2\gamma}}$

☆ If we suppress the action (S) & velocity dim^{n}s by setting $c=1$, $\hbar=1$, then the $[A_{n.u.}]$ has mass dim^{n}s of $(\alpha-\beta-\gamma)$.

$$\therefore\ M^\alpha L^\beta T^\gamma \sim M^{\alpha-\beta-\gamma} \text{ (mass natural units)}$$

The $\sim$ suppresses the remaining dim^n's of action ($S^{\beta+\gamma}$) and velocity ($V^{-\beta-2\gamma}$).

* Doing the conversion into energy natural units instead, we get

$$M^\alpha L^\beta T^\gamma = E^{\alpha-\beta-\gamma}\, S^{\beta+\gamma}\, V^{\beta-2\alpha}$$

and similarly, when writing quantities in energy n.u., the dim^n's of $\hbar$ & c are dropped out

$$\therefore\ M^\alpha L^\beta T^\gamma \sim E^{\alpha-\beta-\gamma}$$

The $\sim$ suppresses the remaining dim^n's of action ($S^{\beta+\gamma}$) and velocity ($V^{\beta-2\alpha}$)

$\delta = \beta+\gamma$
$\epsilon = \beta - 2\alpha$

for energy natural units

Quantity	α	β	γ	δ (dim^n of S)	ϵ (dim^n of v)	Conversion factor
c (LT^{-1})	0	1	-1	0	1	
$\hbar$ (ML^2T^{-1})	1	2	-1	1	0	
mass (M)	1	0	0	0	-2	$m = E/c^2$
Length (L)	0	1	0	1	1	$E = \hbar \frac{2\pi c}{\lambda} \Rightarrow \lambda = \frac{2\pi}{E}\hbar c$
time (T)	0	0	1	1	0	$E = \hbar\left(\frac{2\pi}{T}\right) \Rightarrow T = \frac{2\pi}{E}\hbar$
Energy (ML^2T^{-2})	1	+2	-2	0	0	$E = E\hbar^0 c^0$
Momentum (MLT^{-1})	1	1	-1	0	-1	$p = E/c$
Force (MLT^{-2})	1	1	-1	-1	-1	$F = dp/dt$ $1/\hbar c$
Action (ML^2T^{-1})	1	2	-1	1	0	$S = \hbar(\,)$

Now let's see what happens with charge. In H-L units,

$$q\phi = \text{Energy} = \frac{q^2}{r} \Rightarrow \text{In H-L system } [q] = \left[[E][r]\right]^{1/2}$$

$$\Rightarrow [q] = M^{1/2} L^{3/2} T^{-1/2}$$

If we impose a similar system for Energy natural units, then

Quantity	α	β	γ	δ	ε	Conversion factor
q	$\frac{1}{2}$	$\frac{3}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\sqrt{\hbar c}$

This means that if we divide q by the reqd. factors of $\hbar$ and c i.e.

$\frac{e^2}{\hbar c}$ will only have energy dim^n^s. But the energy dim^n^s of

q is $(\alpha-\beta-\gamma) = \left(1 - \frac{3}{2} + \frac{1}{2}\right) = 0 \Rightarrow \frac{e^2}{\hbar c} = \text{number}$ (dim^n^-less)

In fact, $\frac{e^2}{\hbar c} = \alpha$ (fine structure constant in c.g.s units.)

Natural Units for temperature.

$$k_B = 1.38\ldots \times 10^{-23} \text{ J/K} = 8.617 \times 10^{-5} \text{ eV/K}$$

$$P_{Boltz} \sim e^{-E/k_B T}$$

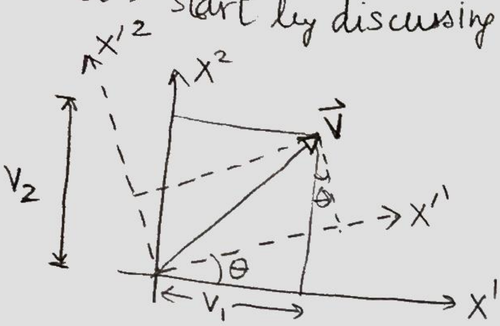
If we now use units where $k = 1$, then $[T] \sim [E]$

In this n.u., we use "$\hbar = c = k_B = 1$". to indicate their suppression

Although I'm still not very comfortable with writing $\hbar = c = k_B = 1$ because it implies that all of $\hbar$, c, k_B have same dim^n^ (din-less to be precise) and can be added to do stupid things like $\hbar + c + k_B = 3$. So I'll try to use $\sim$ as much as possible.

Special Theory of Relativity & Electromagnetism.

Let's start by discussing rotations (how do I derive the rotation matrix?)



$$V_1' = V_1 \cos\theta + V_2 \sin\theta$$
$$V_2' = V_2 \cos\theta - V_1 \sin\theta$$

$$\Rightarrow \begin{pmatrix} V_1 \\ V_2 \end{pmatrix}' = \underbrace{\begin{pmatrix} C_\theta & S_\theta \\ -S_\theta & C_\theta \end{pmatrix}}_{O} \begin{pmatrix} V_1 \\ V_2 \end{pmatrix}$$

Since rotations preserve inner products,

$$\vec{V}' \cdot \vec{W}' = \vec{V} \cdot \vec{W} \quad \text{since} \quad \vec{V}' \cdot \vec{W}' = V'^T W' = V^T \underbrace{O^T O}_{\mathbb{1}} W = V^T W = \vec{V} \cdot \vec{W}$$

This is true in d-spatial dimensions. $(v' = Ov)$

O: dxd orthogonal matrix.

STR

Speed of light $= c$ in all inertial frames (in absence of gravity).
(set $c = 1$)

For light-like events, the distance travelled / time taken $= c$

$$\Rightarrow \left(\frac{d\vec{x}}{dt}\right)^2 = c^2 = 1 \quad \Rightarrow \quad -d\vec{x}^2 + dt^2 = 0$$

the spacetime interval is invariant in all inertial frames

$$\therefore \quad ds^2 = dt^2 - d\vec{x}^2 = dt'^2 - d\vec{x}'^2 = 0$$

The dx^μ and dx'^μ are related by a similar matrix to rot^{n}s.

$$\begin{pmatrix} dx' \\ dt' \end{pmatrix} = \Lambda \begin{pmatrix} dx \\ dt \end{pmatrix}$$

We can write the spacetime interval for the frame S in a certain way

$$(dt \;\; d\vec{x})\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} dt \\ d\vec{x} \end{pmatrix} = dt^2 - d\vec{x}^{\,2}$$

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \eta = g$$

Now if $dt^2 - dx^2 = dt'^2 - dx'^2$

$$\Rightarrow (dt \;\; dx)\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} dt \\ dx \end{pmatrix} = \begin{pmatrix} dt' \\ dx' \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} dt' \\ dx' \end{pmatrix}$$

But we have related $\begin{pmatrix} dt' \\ dx' \end{pmatrix} = \Lambda \begin{pmatrix} dt \\ dx \end{pmatrix}$

$$\Rightarrow (dt \;\; dx)\, \eta \begin{pmatrix} dt \\ dx \end{pmatrix} = (dt \;\; dx)\, \Lambda^T \eta \Lambda \begin{pmatrix} dt \\ dx \end{pmatrix}$$

$\therefore \Lambda^T \eta \Lambda = \eta \longrightarrow$ if this is true, then spacetime interval ds^2 remains invariant.

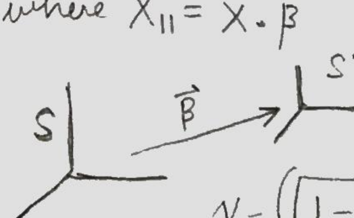
The conventional way in which this is usually set up is

Lorentz transformations

$$x_\perp' = x_\perp$$

$$x_\parallel' = \gamma(x_\parallel - \beta t) \qquad \text{where } x_\parallel = \vec{x}\cdot\hat{\beta}$$

$$t' = \gamma(t - \beta x_\parallel)$$



$$\gamma = \left(\sqrt{1-\beta^2}\right)^{-1}$$

If we take the differentials then

$$dx_\parallel' = \gamma(dx_\parallel - \beta\, dt)$$

$$dt' = \gamma(dt - \beta\, dx_\parallel)$$

So Λ here has the form $\Lambda = \begin{pmatrix} \gamma & -\beta\gamma \\ -\beta\gamma & \gamma \end{pmatrix} = \begin{pmatrix} \cosh\omega & -\sinh\omega \\ -\sinh\omega & \cosh\omega \end{pmatrix}$

where

$\tanh\omega = \beta$

And we see that $$\begin{pmatrix} c & -s \\ -s & c \end{pmatrix}\underbrace{\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} c & -s \\ -s & c \end{pmatrix}}_{\begin{pmatrix} c & -s \\ s & c \end{pmatrix}}$$

$$= \begin{pmatrix} c & -s \\ -s & c \end{pmatrix}\begin{pmatrix} c & -s \\ s & c \end{pmatrix} = \begin{pmatrix} c^2-s^2 & 0 \\ 0 & s^2-c^2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\Rightarrow \Lambda^T \eta \Lambda = \eta$$

So in exactly the same way that

$$O^T \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} O = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

So just like O (orthogonal matrices) preserve the Euclidean metric ($\mathbb{1}$), the Lorentz transformation Λ preserves the Minkowski metric (η).

Infact the Lorentz transf. can be thought of as a generalised rotation -

$$\begin{pmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \end{pmatrix}' = \Lambda \begin{pmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & & & \\ 0 & & \text{Rot.} & \\ 0 & & & \end{pmatrix}$$

Together the Lorentz trafo. & rotations make up the <u>Lorentz group</u>. which is a generalisation of rot^n group.

So we introduce four-vectors $\{x^\mu\} = \{x^0, x^1, x^2, x^3\}$ $\mu = 0, 1, 2, 3$

We can now write Lorentz transf. in tensor notation

$$x'^\mu \doteq \Lambda^\mu{}_\nu \, x^\nu$$

We now define the proper time τ defined as

$$d\tau^2 = dt^2 - d\vec{x}^2$$ which can be written for any 2 events x^μ and $x^\mu + dx^\mu$ and for any frame S.

$$d\tau^2 = dx^\mu \eta_{\mu\nu} dx^\nu \qquad \eta_{\mu\nu} := \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

$$= (dx^0)^2 - (d\vec{x})^2$$

$d\tau^2$ is <u>left invariant</u> under $\Lambda^\mu{}_\nu$ (Lorentz transf.)

$$d\tau'^2 = dx'^\mu \eta_{\mu\nu} dx'^\nu \quad \text{where } dx'^\mu = \Lambda^\mu{}_\nu \, dx^\nu$$

So, $$d\tau'^2 = dx^\alpha \Lambda^\mu{}_\alpha \eta_{\mu\nu} \Lambda^\nu{}_\beta dx^\beta$$
$$= dx^\alpha \left[(\Lambda^T)_\alpha{}^\mu \eta_{\mu\nu} \Lambda^\nu{}_\beta \right] dx^\beta = dx^\alpha dx^\beta (\Lambda^T \eta \Lambda)_{\alpha\beta}$$

But since $d\tau^2$ is left invariant under L.T.

$$d\tau^2 = \eta_{\alpha\beta} dx^\alpha dx^\beta = d\tau'^2 = dx^\alpha dx^\beta (\Lambda^T \eta \Lambda)_{\alpha\beta}$$

$$\Rightarrow \quad \underline{\Lambda^T \eta \Lambda = \eta} \quad (\text{or } \Lambda^\mu{}_\alpha \eta_{\mu\nu} \Lambda^\nu{}_\beta = \eta_{\alpha\beta})$$

We can also see that in Minkowski space, $\eta^2 = \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix}^2 = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}$

$\Rightarrow \eta^{-1} = \eta$

Thus, we define the inverse metric $\eta^{\mu\nu}$ with the same components as $\eta_{\mu\nu}$

$$\eta^{\mu\nu} := \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix} \qquad \Rightarrow \underline{\eta_{\mu\alpha} \eta^{\alpha\nu} = \delta_\mu^\nu}$$

(why define the inverse η i.e. η^{-1} with upper indices?)

Now let's see how we can define $\Lambda_\nu{}^\mu$ from $\tilde{\Lambda}^\mu{}_\nu$ i.e. reverse the order of indices

$$\Lambda^\mu{}_\alpha \underbrace{\eta_{\mu\nu} \Lambda^\nu{}_\beta}_{\tilde{\Lambda}_\mu{}^\lambda} \cdot (\eta^{\lambda\beta}) = \eta_{\alpha\beta} \cdot (\eta^{\lambda\beta})$$

Can we even contract repeated indices both in the second position?

$$\Rightarrow \Lambda^\mu{}_\alpha \tilde{\Lambda}_\mu{}^\lambda = \delta^\lambda_\alpha$$

Lowering and raising indices

From the contravariant form V^μ, we define the covariant form

$V_\mu = \eta_{\mu\nu} V^\nu$ (lowering of index)

We can similarly define raising of indices. We know that

$$V_\mu = \eta_{\mu\nu} V^\nu \xrightarrow[\text{sides by } \eta^{\alpha\mu}]{\text{multiply both}} \eta^{\alpha\mu} V_\mu = \underbrace{\eta_{\mu\nu}\eta^{\alpha\mu}}_{\delta^\alpha_\nu} V^\nu$$

$\Rightarrow V^\alpha = \eta^{\alpha\mu} V_\mu$ (raising of index)

- Need to understand Einstein summation notⁿ & contraction rule properly from somewhere.

Einstein summation convention - repeated indices sum over when one of the index is upstairs (contravariant) and one is downstairs (covariant). Repeated indices which are both or both down don't sum (and more often than not, points that you have definitely made some mistake).

So $A^i{}_j V^j$ is allowed. $A_i{}^j V^j$ isn't.

Contraction rule:

Consider rank-2 tensors (mixed or pure). Einstein summation allows summation over repeated covariant & contravariant indices. However, if the second index of first & the first index of second tensor are repeated, one can interpret this as a matrix multiplication

$$A^i{}_j B^j{}_k = (A \cdot B)^i{}_k \qquad \eta_{\mu\nu} \tilde{\eta}^{\nu\alpha} = (\eta \cdot \tilde{\eta})_\mu{}^\alpha$$

Summation with both repeated indices in the same position still gives rise to another tensor, just that it can't be interpreted as a matrix multiplication

$$T_{\mu\nu} T^{\sigma\nu} = S_\mu{}^\sigma \quad (\text{not the same as } T_{\mu\nu} T^{\nu\sigma} = (T \cdot T)_\mu{}^\sigma)$$

Therefore, in a nutshell, <u>repeated covariant & contravariant indices sum.</u>

however, one can deform this into matrix not^n as well by taking the transpose to switch indices

$$T^{\sigma\nu} = (T^T)^{\nu\sigma} \Rightarrow T_{\mu\nu} T^{\sigma\nu} = T_{\mu\nu} T^{T\nu\sigma} = (T T^T)_\mu{}^\sigma$$

Lecture-3
(09/09)

Special Theory of Relativity (contd): Lorentz transformations.

In the rotation group, we describe rotation of a vector as follows-

$$\vec{X}'_{(d)} = O\,\vec{X}_{(d)}$$

where O is an orthogonal matrix. $O^T O = \mathbb{1}_d = O^T \mathbb{1}_d O$

$$\vec{X}\cdot\vec{X} = X^T \underbrace{\mathbb{1}_d}_{\text{Euclidean metric}} X$$

Therefore, the eqⁿ $O^T \mathbb{1}_d O = \mathbb{1}_d$ can be interpreted as-

"$O_{(d)}$ preserves the Euclidean metric."

We now want to develop a similar structure for Lorentz transf. $\to$ Boosts, Rotⁿs.

Let's define something called a 4-vector as follows-
(in 3+1 dimⁿs)

$$(\underset{t}{\underset{\parallel}{x^0}}, \vec{x}) = [x^\mu] \qquad x^\mu : \mu = 0,1,2,3$$

Then we can write a transformed 4 vector as-

$$x' = \Lambda x \quad \text{(in matrix notⁿ)}$$

$$\begin{pmatrix} x'^0 \\ x'^1 \\ x'^2 \\ x'^3 \end{pmatrix} = \underbrace{\begin{pmatrix} & & & \\ & \Lambda & & \\ & & & \end{pmatrix}}_{4\times 4} \begin{pmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \end{pmatrix}$$

Now just like we define a Euclidean metric which is preserved under rotations of $O(d)$, we similarly define –

Minkowski Metric: $\eta = \text{diag}(1, -\mathbf{1}) = \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix}$

$$\eta_{\mu\nu} = 1 \quad \text{if } \mu = \nu = 0$$
$$= -1 \quad \text{if } \mu = \nu \in \{1,2,3\}$$
$$= 0 \quad \text{if } \mu \neq \nu$$

then the <u>defining property</u> of Lorentz transf. is that it preserves the Minkowski metric

$$\Lambda^T \eta \Lambda = \eta$$

<u>Contravariant</u>: We call an object that transforms like the co-ords. x^μ as contravariant. It transforms with $\Lambda^\mu{}_\nu$

$$V'^\mu = \Lambda^\mu{}_\nu V^\nu$$

We similarly define a <u>co-variant</u> vector as one which is manufactured from V^μ by "lowering its index"

$$V_\mu = \eta_{\mu\nu} V^\nu$$

The metric defines the "lowered" counterpart (the dual vector) of the four vector V^μ.

Now given that <u>V^μ transforms with $\Lambda^\mu{}_\nu$</u>, let's find out how V_μ transforms?

We first prove that the dot product remains invariant under a Lorentz transformation

Say we perform a Lorentz transf. $\Rightarrow V'^{\mu} = \Lambda^{\mu}{}_{\nu} V^{\nu}$

$$\Rightarrow V'^{\mu} \cdot W'_{\mu} = \underset{\text{(def}^n\text{ of lowering index)}}{\eta_{\mu\nu} V'^{\mu} W'^{\nu}} = \eta_{\mu\nu} \Lambda^{\mu}{}_{\alpha} V^{\alpha} \Lambda^{\nu}{}_{\beta} W^{\beta}$$

$$= \left(\Lambda^{\mu}{}_{\alpha} \eta_{\mu\nu} \Lambda^{\nu}{}_{\beta} \right) V^{\alpha} W^{\beta} = \left(\Lambda^{T}{}_{\alpha}{}^{\mu} \eta_{\mu\nu} \Lambda^{\nu}{}_{\beta} \right) V^{\alpha} W^{\beta}$$

But we know from the def^n of L.T. that $\Lambda^{T}{}_{\alpha}{}^{\mu} \eta_{\mu\nu} \Lambda^{\nu}{}_{\beta} = (\Lambda^{T} \eta \Lambda)_{\alpha\beta} = \eta_{\alpha\beta}$

$$\Rightarrow \underline{V'^{\mu} W'_{\mu}} = \eta_{\alpha\beta} V^{\alpha} W^{\beta} = \underline{V^{\mu} W_{\mu}}$$

$\therefore$ Dot products are Lorentz scalars.
(Lorentz inner product)

Using this we can define transf. properties of V_{μ}

$$V'_{\mu} \equiv \tilde{\Lambda}_{\mu}{}^{\nu} V_{\nu} \quad \text{where } \tilde{\Lambda}_{\mu}{}^{\nu} \text{ is the unknown transf. tensor}$$

$$V'_{\mu} W'^{\mu} = \tilde{\Lambda}_{\mu}{}^{\alpha} V_{\alpha} \Lambda^{\mu}{}_{\beta} W^{\beta} = \tilde{\Lambda}_{\mu}{}^{\alpha} \Lambda^{\mu}{}_{\beta} V_{\alpha} W^{\beta}$$

Now since $V'_{\mu} W'^{\mu}$ is supposed to be a Lorentz scalar i.e. $V'_{\mu} W'^{\mu} = V_{\mu} W^{\mu}$

$$\tilde{\Lambda}_{\mu}{}^{\alpha} \Lambda^{\mu}{}_{\beta} V_{\alpha} W^{\beta} \overset{!}{=} V_{\beta} W^{\beta}$$

$$\Rightarrow \tilde{\Lambda}_{\mu}{}^{\alpha} \Lambda^{\mu}{}_{\beta} = \delta^{\alpha}_{\beta} \Rightarrow (\Lambda^{T})_{\beta}{}^{\mu} \tilde{\Lambda}_{\mu}{}^{\alpha} = \delta^{\alpha}_{\beta}$$

$$\Rightarrow \Lambda^{T} \tilde{\Lambda} = \mathbb{1}$$

and hence $\boxed{\tilde{\Lambda} = (\Lambda^{T})^{-1}}$

Therefore, covariant vectors transform with a $\tilde{\Lambda}_\mu{}^\nu$ s.t.

$$V'_\mu = \left[(\Lambda^{-1})^T\right]_\mu{}^\nu V_\nu = V_\nu (\Lambda^{-1})^\nu{}_\mu \qquad \text{(transpose shifts index order)}$$

We now define the inverse metric s.t. $\eta^{\mu\nu}\eta_{\nu\theta} \equiv \delta^\mu_\theta$

$\bar{\eta}\,\eta = \mathbb{1}$ (in matrix not^n)

$\bar{\eta} = \eta^{-1}$

Since $\eta_{\mu\nu} = \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix}$ in flat spacetime, the inverse metric also ends up having the same elements $\eta^{\mu\nu} := \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix}$

Using the property of inverse metric, we can multiply it in the foll. expression

$$\left[\Lambda^\mu{}_\gamma\, \eta_{\mu\nu}\, \Lambda^\nu{}_\theta\right] \cdot \eta^{\theta\lambda} = \left[\eta_{\gamma\theta}\right] \eta^{\theta\lambda}$$

$$\Rightarrow \Lambda^\mu{}_\gamma \underbrace{\eta_{\mu\nu}\, \Lambda^\nu{}_\theta\, \eta^{\theta\lambda}}_{\equiv\, Q_\mu{}^\lambda} = \delta_\gamma^\lambda$$

So, $(\Lambda^T)_\gamma{}^\mu\, Q_\mu{}^\lambda = \delta_\gamma^\lambda \Rightarrow \underline{\underline{Q = (\Lambda^{-1})^T}}$

which is exactly what $\tilde{\Lambda}$ was. So $Q = \tilde{\Lambda}$!

$$\therefore \boxed{\tilde{\Lambda}_\mu{}^\nu = \eta_{\mu\alpha}\, \Lambda^\alpha{}_\beta\, \eta^{\beta\nu}} \quad \text{or} \quad \tilde{\Lambda} = \eta \Lambda \bar{\eta}$$

($\bar{\eta}$ happens to be $\bar{\eta} = \eta$ here)

If we take the determinant on $\Lambda^T \eta \Lambda = \eta$

$$\underbrace{\det(\Lambda^T)}_{\det(\Lambda)} \cancel{\det(\eta)} \det(\Lambda) = \cancel{\det(\eta)}$$

$\Rightarrow \det(\Lambda) = \pm 1$

If $\det(\Lambda) = +1$, then $\Lambda \in \underline{SO}(1,3) \longrightarrow$ "Proper" Lorentz transf.

Let us now talk about $\underline{\text{Tensors}}$

$$V'^{\mu} = \Lambda^{\mu}{}_{\nu} V^{\nu}$$ this is a rank-1 contravariant tensor.

If we instead have 2 indicies, the transformation law is as follows -

$$V'^{\mu_1 \mu_2} = \Lambda^{\mu_1}{}_{\theta_1} \Lambda^{\mu_2}{}_{\theta_2} V^{\theta_1 \theta_2}$$

Why though?

Let's say we define $V^{\mu_1} W^{\mu_2} \equiv T^{\mu_1 \mu_2}$

Then $T^{\mu\nu}$ would transform as a product of transf. of V & W

$$\underbrace{T'^{\mu_1 \mu_2}}_{\text{2 indices}} = V'^{\mu_1} W'^{\mu_2} = \Lambda^{\mu_1}{}_{\nu_1} \Lambda^{\mu_2}{}_{\nu_2} V^{\mu_1} W^{\mu_2} = \underbrace{\Lambda^{\mu_1}{}_{\nu_1} \Lambda^{\mu_2}{}_{\nu_2}}_{2\ \Lambda\ \text{tensors}} T^{\nu_1 \nu_2}$$

Similarly, for a $(0,2)$ tensor

$$T'_{\mu_1 \mu_2} = \tilde{\Lambda}_{\mu_1}{}^{\nu_1} \tilde{\Lambda}_{\mu_2}{}^{\nu_2} T_{\nu_1 \nu_2} = (\Lambda^{-1})^{\nu_1}{}_{\mu_1} (\Lambda^{-1})^{\nu_2}{}_{\mu_2} T_{\nu_1 \nu_2}$$

For a mixed $(1,1)$ tensor

$$T'^{\mu_1}{}_{\mu_2} = \Lambda^{\mu_1}{}_{\theta_1} \tilde{\Lambda}_{\mu_2}{}^{\theta_2} T^{\theta_1}{}_{\theta_2} = \Lambda^{\mu_1}{}_{\theta_1} (\Lambda^{-1})^{\theta_2}{}_{\mu_2} T^{\theta_1}{}_{\theta_2}$$

So, for a general (n,m)-tensor

$$T'^{\mu_1 \dots \mu_n}{}_{\nu_1 \dots \nu_m} = \left(\Lambda^{\mu_1}{}_{\theta_1} \Lambda^{\mu_2}{}_{\theta_2} \dots \Lambda^{\mu_n}{}_{\theta_n}\right)\left(\tilde{\Lambda}_{\nu_1}{}^{\lambda_1} \dots \tilde{\Lambda}_{\nu_m}{}^{\lambda_m}\right) T^{\theta_1 \dots \theta_n}{}_{\lambda_1 \dots \lambda_m}$$

X

Irreducible representations.

Say we have a tensor A s.t. $A_{\mu\nu} = -A_{\nu\mu}$

We know that $A'_{\mu\nu} = \tilde{\Lambda}_\mu{}^\alpha \tilde{\Lambda}_\nu{}^\beta A_{\alpha\beta}$
the transformed A

However for the interchanged indices $A'_{\nu\mu} = \tilde{\Lambda}_\nu{}^\beta \tilde{\Lambda}_\mu{}^\alpha A_{\beta\alpha}$ but $A_{\beta\alpha} = -A_{\alpha\beta}$

$\Rightarrow A'_{\nu\mu} = -A'_{\mu\nu}$ $\therefore$ Anti-symmetrization is preserved within L.T.s!

The same is true for symmetric tensors. By separating a tensor into S & AS parts, one could find an irreducable representation which stays invariant in symmetry of indices

$$S_{\mu\nu} = \underset{\substack{\| \\ -A_{\nu\mu}}}{A_{\mu\nu}} + \underset{\substack{\| \\ B_{\nu\mu}}}{B_{\mu\nu}}$$

X

Another important tensor is the metric which is an invariant tensor of $SO(1,3)$

$$\eta'_{\mu\nu} = \tilde{\Lambda}_\mu{}^\alpha \tilde{\Lambda}_\nu{}^\beta \eta_{\alpha\beta} = (\Lambda^{-1})^T{}_\mu{}^\alpha \, \eta_{\alpha\beta} \, (\Lambda^{-1})^T{}_\nu{}^\beta$$

$$= (\Lambda^{-1})^T{}_\mu{}^\alpha \, \eta_{\alpha\beta} \, (\Lambda^{-1})^\beta{}_\nu = \left((\Lambda^{-1})^T \eta \Lambda^{-1}\right)_{\mu\nu} \overset{\text{by defn.}}{=} (\eta)_{\mu\nu}$$

(ϵ is another one)

Let us now consider <u>derivatives</u>

if $x'^{\mu} = \Lambda^{\mu}{}_{\nu} x^{\nu}$, then by def^n. $\Lambda^{\mu}{}_{\nu} \equiv \dfrac{\partial x'^{\mu}}{\partial x^{\nu}}$ ☆

Can we always use this?

if $x'_{\mu} = \tilde{\Lambda}_{\mu}{}^{\nu} x_{\nu}$, then by def^n. $\tilde{\Lambda}_{\mu}{}^{\nu} = \dfrac{\partial x'_{\mu}}{\partial x_{\nu}}$

One can also write $x^{\mu} = (\Lambda^{-1})^{\mu}{}_{\nu} x'^{\nu} = (\tilde{\Lambda}^{T})^{\mu}{}_{\nu} x'^{\nu}$

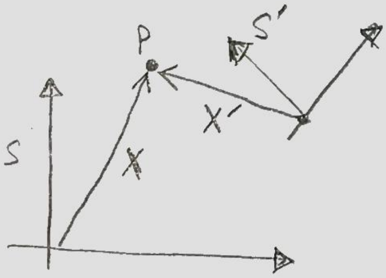
$$\therefore (\tilde{\Lambda}^{T})^{\mu}{}_{\nu} = \frac{\partial x^{\mu}}{\partial x'^{\nu}} = \tilde{\Lambda}_{\nu}{}^{\mu}$$

———— x ————

Let's move onto <u>fields</u> now. A field is defined for every point in spacetime x^{μ}. However, for diff. frames S & S', the field at that pt. should be the same.

$$\phi'(x') = \phi(x) \quad \text{SCALAR FIELD}$$

where $x' = \Lambda x$



but we can consider infinitesimal transformations s.t. $\Lambda = \mathbb{1} + \omega$ → small

$$\Rightarrow \Lambda^{\mu}{}_{\nu} = \delta^{\mu}_{\nu} + \omega^{\mu}{}_{\nu}$$

then $x'^{\mu} \equiv x^{\mu} + \delta x^{\mu} = \delta_{\nu}{}^{\mu} x^{\nu} + \omega^{\mu}{}_{\nu} x^{\nu}$

$$\Rightarrow \delta x^{\mu} = \omega^{\mu}{}_{\nu} x^{\nu}$$

$$\therefore \phi'(x + \delta x) = \phi'(x) + \delta x^{\mu} \frac{\partial \phi'(x^{\mu})}{\partial x^{\mu}} + \dots = \phi(x)$$

Now $\delta\phi \equiv \phi'(x) - \phi(x) = -\delta x^\mu \frac{\partial \phi'(x)}{\partial x^\mu} + \cdots$

$$\approx -\delta x^\mu \frac{\partial \phi(x)}{\partial x^\mu} + \cdots O(\delta^2)$$

$$= -\omega^\mu{}_\nu x_\nu \partial_\mu \phi$$

(We write derivatives as contravariant b/c

$$\frac{\partial}{\partial x'^\mu} = \frac{\partial x^\nu}{\partial x'^\mu} \frac{\partial}{\partial x^\nu} = (\tilde{\Lambda})_\mu{}^\nu \frac{\partial}{\partial x^\nu} \equiv \tilde{\Lambda}_\mu{}^\nu \partial_\nu$$

Hence, covariant)

For differentials,

$$dx'^\mu = \frac{\partial x'^\mu}{\partial x^\nu} dx^\nu = \Lambda^\mu{}_\nu dx^\nu$$

However, if we instead choose VECTOR FIELDS $V^\mu(x)$

we can use the trick of Lorentz scalar to find its transformation i.e.

$$V^\mu(x)\partial_\mu = V'^\mu(x')\partial'_\mu = V'^\mu(x')\frac{\partial}{\partial x'^\mu}$$

$$\Rightarrow V'^\mu(x') = \frac{\partial x'^\mu}{\partial x^\nu} V^\nu(x)$$

Now $V'^\mu(x+\delta x) = V'^\mu(x) + \delta x^\nu \partial_\nu V'^\mu(x) + O(\delta^2)$

$$\Rightarrow \delta V^\mu = V'^\mu(x) - V^\mu(x) = V'^\mu(x) - V'^\mu(x+\delta x)$$

$$= -\delta x^\nu \partial_\nu V'^\mu(x)$$

$$= -\delta x^\nu \partial_\nu V^\mu + \omega^\mu{}_\nu V^\nu + O(\omega^2)$$

didn't understand this at all

Therefore to find the transformation properties of any geometrical object, we can use the "Lorentz scalar trick"

$$"T" = T^{\mu_1 \cdots \mu_n}{}_{\nu_1 \cdots \nu_m} dx^{\nu_1} \ldots dx^{\nu_m} \partial_{\mu_1} \cdots \partial_{\mu_n}$$

$$= T'^{\gamma_1 \cdots \gamma_n}{}_{\lambda_1 \cdots \lambda_m} dx'^{\lambda_1} \ldots dx'^{\lambda_m} \partial'_{\gamma_1} \cdots \partial'_{\gamma_n}$$

Relativistic mechanics of EM

Pair up the energy with momentum $(E, \vec{p}) \longrightarrow$ 4-momentum

In QM, $E \longrightarrow i\dfrac{\partial}{\partial t}$ and $\vec{p} \longrightarrow -i\vec{\nabla}$

So, the 4-momentum operator would be given by

$$(\hat{E}, \hat{\vec{p}}) = \hat{p}^{\mu} = (i\partial_t, -i\vec{\nabla})$$

and $$\hat{p}_{\mu} = (i\partial_t, i\vec{\nabla}) = i\partial_{\mu}$$

THREE GOLDEN RULES FOR TENSORS:

1. Repeated indices sum.
2. adjacent indices can be "thought of as matrix multiplication". Use transpose to switch order of indices. (anything other than (1,1) tensor isn't LITERALLY a matrix so be careful.)
3. The Lorentz transform $\tilde{\Lambda}$ of covariant tensors is inverse transpose of Lorentz transform Λ of contravariant tensors.

$$\tilde{\Lambda} = (\Lambda^{-1})^T$$

Some nice insight by Rubs:

DANGERS OF MATRIX REPRESENTATION OF TENSORS

It might seem harmless to use matrices to denote rank-2 tensors in general. However, that's not a very wise thing to think.

Out of (1,1), (2,0) and (0,2), only a <u>(1,1) tensor is an actual matrix.</u>

A (1,1) tensor takes a co/contravar. vector and gives back the same type of object.

$$A^{\mu}{}_{\nu} V^{\nu} = V^{\mu} \qquad A_{\mu}{}^{\nu} V_{\nu} = V_{\mu}$$

same object   same object.

In other words, a <u>(1,1) tensor</u> does the foll. transf. -

covariant $\longrightarrow$ covariant
Contravariant $\longrightarrow$ contravariant

Hence it <u>works like an "operator" or a matrix</u> in the Linear Algebra sense.

If we relate contra $\rightarrow$ column & co $\rightarrow$ rows

$$\left(A^{\mu}{}_{\nu}\right)\underset{V^{\nu}}{\left(\;\right)} \rightarrow \underset{V^{\mu}}{\left(\;\right)} \qquad \left(\; V_{\nu} \;\right)\left(A^{\nu}{}_{\mu}\right) \longrightarrow \left(\; V_{\mu} \;\right)$$

(Another way to think is - <u>only (1,1) tensors take a vector & covector to give you a number.</u>)

If we however do the same thing with metric $\eta_{\mu\nu}$ (rank 0,2), even though it claims to "raise / lower" indices & take co $\rightarrow$ contra / contra $\rightarrow$ co, the matrix representation suggests that η will leave it the same form of tensor. Hence, only use matrices for other than (1,1) only for ease of matrix multiplication not".

Lecture-4
(10/09)

Relativistic Kinematics & Classical Electrodynamics in 4-vector not^n

We abbreviate the derivative as $\frac{\partial}{\partial x^\mu} \equiv \partial_\mu$

The proper time in the particle's own reference frame is τ. For two infinitesimally separated events

$$ds^2 = d\tau^2 - \cancel{d\vec{\xi}}^{\,2} = dt^2 - d\vec{x}^{\,2} \Rightarrow d\tau^2 = dt^2 - d\vec{x}^{\,2}$$
$$= dx^\mu \eta_{\mu\nu} dx^\nu$$

The 4-velocity $\quad v^\mu \equiv \frac{dx^\mu}{d\tau}$

From the def^n of 4-velocity, we see that the inner product of v^μ with itself is

$$v^\mu v_\mu = v^\mu \eta_{\mu\nu} v^\nu = \frac{dx^\mu}{d\tau}\eta_{\mu\nu}\frac{dx^\nu}{d\tau} = \frac{d\tau^2}{d\tau^2} = 1$$

$$\Rightarrow \underline{v_\mu v^\mu = 1}$$

The 4-momentum of a particle is defined as $\quad p^\mu = m v^\mu$

and from the properties of 4-velocity, it is straightforward to see that-

$$p^\mu p_\mu = m^2 \quad \Rightarrow E^2 - p^2 = m^2 \quad \text{or} \quad \underline{E^2 = p^2 + m^2}$$

$$E = \sqrt{p^2 + m^2}$$

For $|\vec{p}| \ll m \Rightarrow E = \sqrt{\vec{p}^2 + m^2} = m\left(1 + \frac{\vec{p}^2}{m^2}\right)^{1/2}$

and thus $E = m\left(1 + \frac{\vec{p}^2}{2m^2} - \frac{\vec{p}^4}{8m^4} + \cdots\right) = \underbrace{m}_{\text{rest mass}} + \underbrace{\frac{\vec{p}^2}{2m}}_{\text{NR KE}} - \underbrace{\frac{\vec{p}^4}{8m^3}}_{\text{first relativistic correction}} + O(\vec{p}^6)$

For $|\vec{p}| \gg m \Rightarrow E \simeq |\vec{p}|$ (like radiation)

Let's now reformulate Newton's laws for STR. Newton's law says that

$$F^\mu = \frac{dp^\mu}{d\tau} \quad \text{(proper rate of change of 4-momentum)}$$

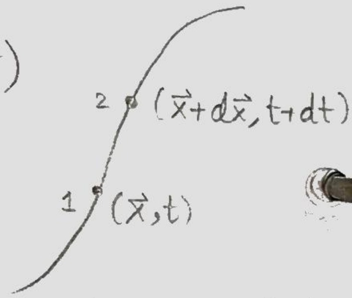
F^μ: 4-force ("fource")

One might remember that for two close events, the $d\tau^2$ is defined as

$$d\tau^2 = dt^2 - d\vec{x}^2 = dt^2\left(1 - \frac{d\vec{x}^2}{dt^2}\right) = dt^2(1 - \vec{\beta}^2)$$

$$\Rightarrow d\tau = dt\sqrt{1 - \vec{\beta}^2}$$

$$\Rightarrow d\tau = \frac{dt}{\gamma} \quad \text{or} \quad dt = \gamma\, d\tau$$



Particle's trajectory in lab's frame

$\frac{d\vec{x}}{dt} = \vec{\beta}$ (particle's velocity from lab frame)

Let us now reformulate EM in this not^n

(Maxwell's eq^n in H-L units)

$$\vec{\nabla}\cdot\vec{E} = \rho \qquad \vec{\nabla}\times\vec{E} = -\partial_t \vec{B}$$

$$\vec{\nabla}\cdot\vec{B} = 0 \qquad \vec{\nabla}\times\vec{B} = \vec{j} + \partial_t \vec{E}$$

We can define a 4-current $j^\mu = (\rho, \vec{j})$. We also aim to write M.E. compactly which we'll do by including $\vec{E}$ & $\vec{B}$ in an anti-symmetric tensor $F^{\mu\nu}$ (field strength tensor)

As we'll see, the Maxwell's eq^{n}s become:

$$\underline{\partial_\mu F^{\mu\nu} = j^\nu} \qquad \underline{\epsilon^{\mu\nu\lambda\sigma} \partial_\nu F_{\lambda\sigma} = 0}$$

Expanding these out,

for $\underline{\nu = 0}$ $\quad F^{00} = -F^{00} \Rightarrow \underline{F^{00} = 0}$

and $\partial_i F^{i0} = j^0 = \rho$ therefore by inspection $\Rightarrow$ $\underline{\underline{F^{i0} = (\vec{E})^i}}$

For $\nu = j$

$$\partial_0 F^{0j} + \partial_i F^{ij} = j^j \Rightarrow -\partial_0 F^{j0} + \partial_i F^{ij} = j^j$$

but we just identified $F^{j0} \equiv (\vec{E})^j$

$$\Rightarrow \underbrace{\partial_i F^{ij}}_{\substack{!!! \\ (\vec{\nabla} \times \vec{B})^j}} = j^j + \partial_0 E^j$$

$\rightarrow$ which looks exactly like Maxwell-Ampere eq^n

So we can say $F^{ij} = \epsilon^{ijk} B_k$ $\quad$ ($B^k = B_k$ here essentially since there is no time comp. involved)

How?

$$(\vec{\nabla} \times \vec{B})^i = \epsilon^{ijk} \partial_j B_k \xrightarrow{\text{switch } i \leftrightarrow j} (\vec{\nabla} \times \vec{B})^j = \epsilon^{jik} \partial_i B_k = \partial_i (\epsilon^{jik} B_k)$$

Now $(\vec{\nabla} \times B)^j = j^j + \partial_0 E_j \Rightarrow \partial_i (\epsilon^{jik} B_k) = j^j + \partial_0 E^j$

but $\partial_i F^{ij} = j^j + \partial_0 E^j \Rightarrow$ $\boxed{F^{ij} = -\epsilon^{ijk} B_k}$

Since $F^{ij} = -\epsilon^{ijk} B_k \Rightarrow F^{12} = -B_3 = -B^3$, $F^{23} = -B_1 = -B^1$, $F^{13} = +B_2 = +B^2$

So, $\boxed{\partial_\mu F^{\mu\nu} = j^\nu}$ reproduces the two inhomog. eq^{n}s.

where $\underline{F^{i0} = E^i}$ and $\underline{F^{ij} = -\epsilon^{ijk} B_k = -\epsilon^{ijk} B^k}$

The homogeneous eq^{n}s are given by "Bianchi identity" $\epsilon^{\mu\nu\lambda\sigma} \partial_\nu F_{\lambda\sigma} = 0$

where $F_{\lambda\sigma} = \eta_{\lambda\alpha} \eta_{\sigma\beta} F^{\alpha\beta} \Rightarrow F_{0i} = -F^{0i} (= F^{i0}), F_{i0} = -F^{i0}$

$F_{ij} = F^{ij} \quad (= -F^{ji})$

Now for $\mu = 0$, ν, λ, σ have to be some permutation of $(1,2,3)$, so -

$$\epsilon^{0ijk} \partial_i F_{jk} = \epsilon^{0ijk} \partial_i \underbrace{F^{jk}}_{= -\epsilon^{jkl} B^l} \equiv -\epsilon^{ijk} \underbrace{\epsilon_{jkl}}_{2\delta^i_l} \partial_i B^l = 0$$

So, $\mu = 0 \Rightarrow \underline{\underline{\vec{\nabla} \cdot \vec{B} = 0}}$

Similarly for $\mu = i \in \{1,2,3\}$, we need to see what $\epsilon^{i\nu\lambda\sigma} \partial_\nu F_{\lambda\sigma}$ gives

$\epsilon^{i\nu\lambda\sigma} \partial_\nu F_{\lambda\sigma} = 0$ (using $F_{00} = 0$)

$$\epsilon^{i\nu 0k} \partial_\nu \underbrace{F_{0k}}_{F_{0k} = -F^{0k} = F^{k0} = E^k} + \epsilon^{i\nu k0} \partial_\nu \underbrace{F_{k0}}_{F_{k0} = -F^{k0} = -E^k} + \epsilon^{i\nu jk} \partial_\nu \underbrace{F_{jk}}_{F^{jk}} = 0$$

$$\epsilon^{i\nu 0k} \partial_\nu E_k - \epsilon^{i\nu k0} \partial_\nu E + \epsilon^{i\nu jk} \partial_\nu F^{jk}$$

$\nu \neq 0$ otherwise $\epsilon = 0$

$$\epsilon^{ij0k} \partial_j E_k - \epsilon^{ijk0} \partial_j E_k + \epsilon^{i\nu jk} \partial_\nu F^{jk} = 0$$

$$\Rightarrow \epsilon^{i\nu jk} \partial_\nu F^{jk} = -2\epsilon^{ij0k} \partial_j E_k = -2\epsilon^{0ijk} \partial_j E_k$$

So, $\epsilon^{i\nu jk}\partial_\nu F^{jk} = -2\epsilon^{ijk}\partial_j E_k$

$$\Rightarrow \epsilon^{i0jk}\partial_0 F^{jk} + \underbrace{\epsilon^{iljk}\partial_l F^{jk}}_{\substack{\text{since } i,j,k,l \in \{1,2,3\}, \\ \text{indices will always repeat} \\ \& \ \epsilon \to 0}} = -2\epsilon^{ijk}\partial_j E_k$$

$$\therefore \ \cancel{-}\epsilon^{ijk}\partial_0 \underset{-\epsilon^{jkl}B_l}{\underset{\parallel}{F^{jk}}} = \cancel{-}2\epsilon^{ijk}\partial_j E_k$$

$$-\underbrace{\epsilon^{ijk}\epsilon_{jkl}}_{\cancel{2}\delta^i_l}\partial_0 B^l = \cancel{2}(\vec{\nabla}\times\vec{E})^i \Rightarrow (\vec{\nabla}\times\vec{E})^i = -\partial_0 B^i$$

Therefore, we do indeed get Faraday's law-

$$(\vec{\nabla}\times\vec{E})^i = -\partial_0 B^i \Rightarrow \vec{\nabla}\times\vec{E} = -\partial_t\vec{B}$$

Therefore we have now written Maxwell's eq^{n}s as

$$\boxed{\partial_\mu F^{\mu\nu} = j^\nu} \qquad \boxed{\epsilon^{\mu\nu\lambda\sigma}\partial_\nu F_{\lambda\sigma} = 0}$$

where $F^{\mu\nu}$ (and $F_{\mu\nu}$) is an anti-symmetric tensor.

$F^{ij} = -\epsilon^{ijk}B_k \qquad F^{i0} = E_i \qquad$ and $\qquad F_{0i} = -F^{0i}$, $\ F_{ij} = F^{ij}$

———— × ———— × ————

We now define a 4-vector pot^n A_μ

$$\boxed{F_{\lambda\sigma} = \partial_\lambda A_\sigma - \partial_\sigma A_\lambda} = \partial_{[\lambda}A_{\sigma]}$$

This ansatz automatically satisfies $\epsilon^{\mu\nu\lambda\sigma}\partial_\nu F_{\lambda\sigma} = 0$

$$\epsilon^{\mu\nu\lambda\sigma}\partial_\nu\partial_{[\lambda}A_{\sigma]} = 2\epsilon^{\mu\nu\lambda\sigma}\partial_\nu\partial_\lambda A_\sigma \equiv 0$$ (since derivatives commute & all terms cancel)

both are anti-symm.

$\therefore$ the vector pot^n automatically solves Bianchi identity.

Let's check it for inhomog. Maxwell's eq^{n}s

$$F^{\mu\nu} = \partial^{[\mu} A^{\nu]}$$

$$\Rightarrow F^{i0} = \partial^i A^0 - \partial^0 A^i = -(\vec{\nabla} A^0)^i - \left(\frac{\partial \vec{A}}{\partial t}\right)^i = (\vec{E})^i \qquad \left(\partial^i = \left(\frac{\partial}{\partial t}, -\vec{\nabla}\right)\right)$$

$$\therefore A^\mu = (A^0 \equiv \phi, \vec{A})$$

$$\Rightarrow F^{ij} = \partial^i A^j - \partial^j A^i = \epsilon^{ijk} B_k$$

Multiplying both sides by ϵ_{ijl}

$$\epsilon_{ijl}(\partial^i A^j - \partial^j A^i) = \underbrace{\epsilon^{kij} \epsilon_{ijl}}_{2\delta^k_l} B_k$$

$$\epsilon_{lij} \partial^i A^j + \epsilon_{lji} \partial^j A^i = 2\delta^k_l B_k$$

$$2(\vec{\nabla} \times \vec{A})^l = 2B^l \Rightarrow \vec{B} = \vec{\nabla} \times \vec{A}$$

The derivation of wave eq^n is trivial if we use A^μ

$$\partial_\mu (\partial^\mu A^\nu - \partial^\nu A^\mu) = \partial_\mu \partial^\mu A^\nu - \partial_\nu \partial_\mu A^\mu$$

$$\partial_\mu \partial^\mu = \partial_0^2 - \vec{\nabla}^2 = \Box \qquad \text{(D'Alembertian)}$$

$$\Box A^\nu - \partial_\nu(\partial_\mu A^\mu) = j^\nu$$

What purpose this?

We also see a gauge invariance here i.e. $A'_\mu \to A_\mu + \partial_\mu \Lambda(x)$

then $\partial_{[\mu} A'_{\nu]} = \partial_{[\mu} A_{\nu]} + \underbrace{\cancel{\partial_{[\mu} \partial_{\nu]}}}_{\text{derivatives commute}} \overset{0}{} \Lambda(x) \Rightarrow \underline{F_{\mu\nu} = F'_{\mu\nu}}$

$(\vec{E}, \vec{B})$ and hence $F_{\mu\nu}$ is gauge invariant, hence we can choose a gauge to simplify eq^{n}s.

Example – $\partial_\mu A^\mu(x) = 0$ (Lorentz Gauge) ★

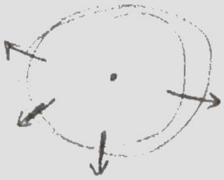
$\vec{\nabla} \cdot \vec{A} = 0$ (Coulomb Gauge)

$A^3 = 0$ (Axial Gauge)

[illegible] $A_\mu \to A_\mu + \partial_\mu \Lambda$ [illegible] the Lorentz & Coulomb gauge conditions

<u>Charge/current conservation</u>: Eq^n of continuity is given as

$$\vec{\nabla} \cdot \vec{j} + \frac{\partial \rho}{\partial t} = 0$$



$$\Rightarrow \underbrace{\frac{\partial}{\partial t} \int_V d^3\vec{x}\, \rho}_{\text{rate of charge flow outwards}} = \underbrace{- \int_{\partial V} \vec{j} \cdot d\vec{s}}_{\text{current flow}}$$

<u>Lorentz force eq^n</u>: We want the Lorentz force eq^n s.t. $(dt = \gamma d\tau)$

$$\frac{d}{dt} p^\mu = F^\mu_{(\text{Lorentz})} \equiv q F^\mu{}_\nu \frac{dx^\nu}{d\tau}$$ ★ (didn't understand how / why this was derived?)

$$\frac{dp^0}{d\tau} = q F^0{}_i \frac{dx^i}{d\tau}$$

$F^0{}_0 = 0$ anyway

$$\frac{dp^i}{d\tau} = q\left(F^i{}_0 \frac{dx^0}{d\tau} + F^i{}_j \frac{dx^j}{d\tau}\right)$$

$$\frac{dp^i}{d\tau} = q\left(F^{i0} \overbrace{\frac{dt}{d\tau}}^{\gamma} - F^{ij} \frac{dx^j}{d\tau}\right)$$

$$\frac{dp^0}{d\tau} = \frac{dE}{d\tau} = q F^0{}_i \frac{dx^i}{d\tau}$$

$$= q F^{i0} \frac{dx^i}{d\tau}$$

(divide by γ) $\Rightarrow$ $$\frac{dp^i}{dt} = q\left(E^i + \underbrace{\epsilon^{ijk} B_k \frac{dx^j}{dt}}_{(\vec{v} \times \vec{B})^i}\right)$$

using $d\tau = \gamma^{-1} dt$ & cancelling on both sides

$$\frac{dp^i}{dt} = q(\vec{E} + \vec{v} \times \vec{B})^i$$

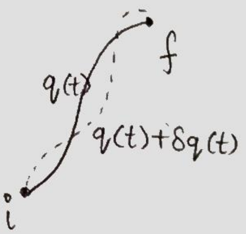
$$\frac{dE}{dt} = q \vec{E} \cdot \vec{v}$$

Classical Mechanics

Point particles : generalised co odinates $q_j(t)$
generalised velocities $\dot{q}_j(t)$

The action $S = \int_i^f dt\, \mathcal{L}(q, \dot{q})$

The EOMs can be derived via the variational principle



Hence, we expect that $\delta S\big|_{\delta q_{i,f}=0} = 0$ for small deviations $\delta q(t)$

$$\delta S = \int_i^f dt\ \frac{\partial L}{\partial q_j}\,\delta q_j + \frac{\partial L}{\partial \dot{q}_j}\,\underbrace{\delta \dot{q}_j}_{d/dt\ \delta q_j(t)}$$

Using IBP

$$\delta S = \int_i^f \left(\frac{\partial L}{\partial q_j} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_j}\right)\delta q(t)\,dt + \underbrace{\cancel{(\partial \text{ terms})}}_{0 \text{ since } \delta q_{i,f} = 0}$$

Putting $\delta S = 0 \Rightarrow$ $\left.\frac{\partial L}{\partial q}\right|_{q(t)} = \frac{d}{dt}\left(\left.\frac{\partial L}{\partial \dot{q}}\right|_{q(t)}\right)$ Euler Lagrange eq^{n}s.

For conservative classical systems, $\mathcal{L}(q, \dot{q}) = T - V$

Noether's theorem.

For every continuous invariance of the action, there is a conserved quantity.

Lecture-5 (PHY424)
(13/09)

Hamiltonian formalism.

In Lagrangian formulation, we use $L(q,\dot{q})$ where q & $\dot{q}$ are co-ords & velocities.

The Hamiltonian formulation is based on the object called $H(p,q)$ Hamiltonian which is a f^n of generalized coords q and generalized momenta p.

It is defined by doing a Legendre transform to go from $\dot{q} \longrightarrow \underbrace{p = \frac{\partial L}{\partial \dot{q}}}_{\text{momentum conjugate to } q}$

$$H(p,q) \equiv p\dot{q} - L(q,\dot{q}) \quad \text{where } p = \frac{\partial L}{\partial \dot{q}}$$

$$\Rightarrow dH = \cancel{p\,d\dot{q}} + \dot{q}\,dp - \frac{\partial L}{\partial q}dq - \cancel{\overset{\parallel p}{\frac{\partial L}{\partial \dot{q}}}\,d\dot{q}}$$

From E-L eq^{n}s, $\frac{\partial L}{\partial q} = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) = \frac{d}{dt}(p) = \dot{p}$

$$\Rightarrow dH = \dot{q}\,dp - \dot{p}\,dq$$

$$\therefore \underline{\dot{q} = \frac{\partial H}{\partial p}} \quad \text{and} \quad \underline{\dot{p} = -\frac{\partial H}{\partial q}}$$

These give the Hamilton's eq^{n}s of motion.

Poisson Bracket Given functions of dynamical variables $f(q_r,p_r)$ & $g(q_r,p_r)$, the "equal time" Poisson brackets are defined as.

$$\{f,g\} = \sum_r \left(\frac{\partial f}{\partial q_r}\frac{\partial g}{\partial p_r} - \frac{\partial f}{\partial p_r}\frac{\partial g}{\partial q_r}\right) = -\{g,f\}$$

At equal times, the Poisson bracket of p & q gives

$$\{q(t),p(t)\} = \overset{1}{\frac{\partial q}{\partial q}}\overset{1}{\frac{\partial p}{\partial p}} - \overset{0}{\frac{\partial q}{\partial p}}\overset{0}{\frac{\partial p}{\partial q}} \Rightarrow \underline{\underline{\{q(t),p(t)\} = 1}}$$

For more than one generalised q, p, we have $\{q_r(p), p_s(t)\} = \delta_{rs}$

Now poisson bracket of q & p with H is -

$$\{q(t), H(q(t), p(t))\} = \frac{\cancel{\partial q}^{\nearrow 1}}{\cancel{\partial q}} \frac{\partial H}{\partial p} - \cancel{\frac{\partial q}{\partial p}}^{\nearrow 0} \frac{\partial H}{\partial q} = \frac{\partial H}{\partial p} = \dot{q}(t)$$

Similarly

$$\{p_{(t)}, H_{(q(t),p(t))}\} = \frac{\partial p}{\partial q} \frac{\partial H}{\partial p} - \frac{\partial p}{\partial p} \frac{\partial H}{\partial q} = -\frac{\partial H}{\partial q} = \dot{p}(t)$$

$\therefore$ For any function $O(p_r, q_r, t)$, we can write the time derivative as -

$$\dot{O} = \frac{dO}{dt} = \frac{\partial O}{\partial t} + \sum_r \frac{\partial O}{\partial p_r} \dot{p}_r + \frac{\partial O}{\partial q_r} \dot{q}_r = \frac{\partial O}{\partial t} + \sum_r \left(\frac{\partial O}{\partial q_r} \frac{\partial H}{\partial p_r} - \frac{\partial O}{\partial p_r} \frac{\partial H}{\partial q_r} \right)$$

$$\Rightarrow \dot{O} = \frac{dO}{dt} = \frac{\partial O}{\partial t} + \{O, H\}$$

For conservative system, O generally doesn't have explicit time dependence so

$\partial O / \partial t \to 0 \quad \Rightarrow \quad \dot{O} = \{O(q(t), p(t)), H\}$

<u>Quantization</u>. (classical $\longrightarrow$ quantum)

observables become operators. PB become commutator $\cdot \left(\frac{1}{i\hbar}\right)$

$$q \longrightarrow \hat{q} \qquad \{\ ,\ \}_{PB} \longrightarrow \frac{[\ ,\ ]}{i\hbar}$$
$$p \longrightarrow \hat{p}$$

(setting $\hbar = 1$)

$$\text{So, } [\hat{q}(t), \hat{p}(t)] = \hat{q}(t)\hat{p}(t) - \hat{p}(t)\hat{q}(t)$$
$$\equiv i\{q, p\}$$
$$= i$$

For >1 position & momentum coordinates, the commutator becomes

$$[\hat{q}_r(t), \hat{p}_s(t)] = i\,\delta_{rs}$$

Quantum dynamics.

The time evolution of the quantum state is unitary & is governed by the Schrödinger eqⁿ

$$i\frac{d}{dt}|\psi(t)\rangle = \hat{H}|\psi(t)\rangle$$

(the most general eqⁿ in abstract vector notⁿ)

One-particle Schrödinger eqⁿ in position representation becomes

$$i\frac{\partial}{\partial t}\psi(\vec{x},t) = \left(-\frac{\nabla^2}{2m} + V(\vec{x})\right)\psi(\vec{x},t)$$

$\langle\vec{x}|\psi(t)\rangle$ position space representation

Non-Relativistic QM. $\longrightarrow$ RQM??

Let's begin by how we got to Schrödinger eqⁿ

for a free particle $E = \underbrace{\frac{\vec{p}^{\,2}}{2m}}_{}$ quantized & project to position rep.

$\vec{p} \longrightarrow \hat{p} := -i\vec{\nabla}$

$E \longrightarrow \hat{H} := i\frac{\partial}{\partial t}$

(operator form in position representation)

$$\hat{H}\psi = \frac{\hat{\vec{p}}^{\,2}}{2m}\psi$$

$$\Rightarrow \quad -\frac{1}{2m}\nabla^2\psi(\vec{x},t) = i\frac{\partial}{\partial t}\psi(\vec{x},t)$$

where $\psi(\vec{x},t) = \langle\vec{x}|\psi(t)\rangle$

One could equivalently work in momentum space s.t.

$$\langle\vec{p}|\psi(t)\rangle = \tilde{\psi}(\vec{p},t)$$

$\vec{p} \longrightarrow \hat{\vec{p}} = \vec{p}$

$\vec{x} \longrightarrow \hat{\vec{x}} := i\,\partial_p$

Essentially, the free particle gives a plane wave soln from the Schrödinger eqn

$$\psi \sim \exp(\pm i\vec{p}\cdot\vec{x} - iEt)$$

Since $\hat{E} = \frac{\hat{\vec{p}}^2}{2m} \Rightarrow$ the eigenvalues of $E > 0$

Now if we go to <u>relativistic quantum mechanics</u>, then the energy momentum relation is quadratic.

$$E^2 = \vec{p}^2 + m^2 \Rightarrow E = \pm\sqrt{p^2+m^2}$$

If we now do the same thing that we did in NRQM, then we now have to make sense of square root of an operator, which gives an infinite power series $\Rightarrow \infty$ order PDE.

So this is not going to work. So we avoid taking the square root, and quantize the quadratic relation

$$\underset{(i\partial_t)^2}{\overset{\downarrow}{E^2}} = \underset{(-i\nabla)^2}{\overset{\downarrow}{\vec{p}^2}} + m^2 \Rightarrow (i\partial_t)^2\phi(\vec{x},t) = ((-i\nabla)^2 + m^2)\phi(\vec{x},t)$$

$$\Rightarrow (\partial_t^2 - \nabla^2 + m^2)\phi(\vec{x},t) = 0$$

Klein-Gordon equation $\rightarrow$ $\underline{(\Box + m^2)\phi(\vec{x},t) = 0}$

$$\left(\frac{mc}{\hbar}\right)^2 = K^2$$ inverse compton wavelength

However, can we interpret ϕ as a single particle wavefunction?

In NRQM, $j^\mu = (\psi^*\psi, \frac{1}{2mi}\psi^*\overleftrightarrow{\nabla}\psi)$

" $\psi^*\vec{\nabla}\psi - \psi\vec{\nabla}\psi^*$ "

This comes from the structure of the Schrödinger eq^n

$$\psi^* \times \left[\left(-\frac{1}{2m}\nabla^2 + V - i\partial_t\right)\psi = 0\right]$$

$$-\psi \times \left[\left(-\frac{1}{2m}\nabla^2 + V + i\partial_t\right)\psi^* = 0\right]$$

$$\Rightarrow \quad \vec{\nabla}\cdot\vec{j}_s + \frac{\partial \rho}{\partial t} = \partial_\mu j_s^\mu = 0$$

Here $j_s^0 = \rho_s = \psi^*\psi = |\psi|^2 \geq 0$ and can thus be interpreted as a probability density.

Also, $\int dV \rho_s = \int dV |\psi|^2 = 1$ (we fix the normalisation)

Now the eq^n of continuity assures that normalization is preserved.

$$\underbrace{\frac{\partial}{\partial t}\int_{V_\infty} \rho_s dV}_{} = -\int_{V_\infty} \vec{\nabla}\cdot\vec{j}_s \, dV = -\cancelto{0}{\int_{S_\infty} d\vec{s}\cdot\vec{j}_s} \quad \text{since } \psi|_\infty = 0$$

this integral is preserved in time since $\partial_t\left[\int dV \rho_s\right] = 0$

Doing the same thing for Klein-Gordon equation -

$$\phi^* \times \left[(\partial_\mu\partial^\mu + m^2)\phi = 0\right] \quad \Rightarrow \quad \phi^*\Box\phi - \phi\Box\phi^* = 0$$

$$-\phi \times \left[(\partial_\mu\partial^\mu + m^2)\phi^* = 0\right] \qquad \partial_\mu(\phi^*\partial^\mu\phi - \phi\partial^\mu\phi^*) = 0$$

So, we can interpret $j^\mu = \underline{\phi^*\partial^\mu\phi - \phi\partial^\mu\phi^*}$ as a conserved current.

$$j^\mu_{KG} = (\phi^*\dot{\phi} - \phi\dot{\phi}^*, \ \phi^*\overleftrightarrow{\nabla}\phi)$$

However, $\phi^* \overleftrightarrow{\partial_t} \phi$ is not ≥ 0 in general.

As an example, if $\phi \sim f(x) e^{-iEt}$, then $\phi^* \sim f(x) e^{iEt}$

$$\Rightarrow \phi^* \overleftrightarrow{\partial_t} \phi \sim f(x) e^{iEt} e^{-iEt}(-iE) - f(x) e^{-iEt} e^{iEt}(iE)$$

$$\sim (-2iE) f(x) \not\geq 0$$

since $E = \pm\sqrt{}$

Therefore, the Klein-Gordon equation doesn't provide a good interpretation of a single particle probability density since j^0 is not positive definite.

Lecture-6 QFT
(14/09)

Dirac Equation – spin ½ RQM.

We need an equation which is first order in time and space. $\exists$ it should be first order in $\vec{\nabla} \sim \vec{p}$ and ∂_t.

Therefore, Dirac's ansatz was argued by saying that if we can find the form of the Hamiltonian s.t. it is linear in $\vec{p}$ and m.

$$\hat{H} = \vec{\alpha} \cdot \vec{p} + \vec{\beta} \cdot m \longleftrightarrow E^2 = p^2 + m^2$$

$$E = \pm\sqrt{p^2+m^2} \rightarrow \text{causes a problem.}$$

then
$$i\partial_t \psi = \hat{H}\psi$$

So can we somehow "factor" $\hat{H}$ into a linear expression of $\vec{p}$ & m?

Dirac's insight was that it's possible if $\vec{\alpha}$ & β are matrices.

If we square the Hamiltonian operator

$$(\alpha_i p^i + \beta m)(\alpha_j p^j + \beta m) = \alpha_i \alpha_j p^i p^j + (\alpha_i \beta + \beta \alpha_i) m p^i + \beta^2 m^2$$
$$\equiv \vec{p}^{\,2} + m^2$$

How can that be?
$$\alpha_i \alpha_j p^i p^j = \vec{p}^{\,2}$$
$$\alpha_i \beta + \beta \alpha_i = 0$$
$$\beta^2 = 1$$

Now $\alpha_i \alpha_j p^i p^j = \frac{1}{2}(\alpha_i \alpha_j p^i p^j + \underbrace{\alpha_j \alpha_i p^j p^i}_{\text{(switching } i \leftrightarrow j)}) = \underbrace{\frac{1}{2}(\alpha_i \alpha_j + \alpha_j \alpha_i) p^i p^j}_{(p^i \text{ \& } p^j \text{ commute})}$

$$= \frac{1}{2}\{\alpha_i, \alpha_j\} p^i p^j$$

What is n? We don't know.

$$\Rightarrow \boxed{\begin{aligned} \{\alpha_i, \alpha_j\} &= 2\delta_{ij}\mathbb{1}_n \qquad \beta^2 = \mathbb{1}_n \\ \{\alpha_i, \beta\} &= 0 \end{aligned}}$$

$\therefore$ We now have the following equation

$$(\alpha_i p^i + \beta m)\psi = i\partial_t \psi$$

$$\beta \times \left[(-i\vec{\alpha}\cdot\vec{\partial} - i\mathbb{1}_n\partial_t + \beta m)\psi = 0 \right]$$

$$\Rightarrow \left(-i\beta\vec{\alpha}\cdot\vec{\partial} - i\beta\,\partial_t + m\mathbb{1}_n\right)\psi = 0$$

$$\left[i(\beta\vec{\alpha}\cdot\vec{\partial} + \beta\partial_t) - m \right]\psi = 0$$

if we identify $\beta\vec{\alpha} = \vec{\gamma}$ and $\beta = \gamma^0$

$$\Rightarrow \gamma^\mu = (\gamma^0 = \beta, \vec{\gamma} = \beta\vec{\alpha})$$

Then we can write the eq^n in a covariant form

$$\boxed{(i\gamma^\mu\partial_\mu - m\mathbb{1}_n)\begin{pmatrix}\psi_1\\ \psi_2\\ \vdots\\ \psi_n\end{pmatrix} = 0}$$

DIRAC EQUATION!

where $\gamma^0 = \beta$ and $\gamma^i = \gamma^0\alpha_i$

Now let's look at the anticommutator of γ^i & γ^j

$$\{\gamma^i, \gamma^j\} = \gamma^0\alpha_i\gamma^0\alpha_j + \gamma^0\alpha_j\gamma^0\alpha_i = \beta\alpha_i\beta\alpha_j + \beta\alpha_j\beta\alpha_i$$

but $\beta\alpha_i = -\alpha_i\beta$

$$\Rightarrow \{\gamma^i, \gamma^j\} = -\alpha_i\beta^2\alpha_j - \alpha_j\beta^2\alpha_i = -\{\alpha_i, \alpha_j\} = -2\delta_{ij}\mathbb{1}_n$$

$$\{\gamma^0, \gamma^0\} = 2\beta^2 = 2\mathbb{1}_n$$

$$\{\gamma^0, \gamma^i\} = \gamma^0\gamma^i + \gamma^i\gamma^0 = \gamma^0\gamma^0\alpha_i + \gamma^0\alpha_i\gamma^0 = \alpha_i - \alpha_i = 0$$

$$\Rightarrow \boxed{\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}\mathbb{1}_n}$$

Clifford Algebra.

The basic question then is, can we find such a set of matrices?

If we can find one such set $\gamma^0 \dots \gamma^3$, then another set

$$\gamma'^{\mu} = S \gamma^{\mu} S^{-1} \qquad \det S \neq 0$$

connected via a similarity transform will also satisfy the same algebra.

$$\Rightarrow \{\gamma'^{\mu}, \gamma'^{\nu}\} = 2\eta^{\mu\nu} \mathbb{1}_n$$

- Now, if $\underline{\underline{m=0}}$, then we need to find a set of matrices s.t.

$$\hat{E}^2 = \hat{p}^2$$

We can, by inspection, do that by choosing $\hat{H} = \hat{\vec{\sigma}} \cdot \hat{\vec{p}}$

So, $\underline{\vec{\alpha} = \vec{\sigma}}$ for $m=0$

This is because $\sigma_i \sigma_j = \mathbb{1}_2 \delta_{ij} + i \epsilon_{ijk} \sigma^k$

$$\text{So, } \hat{H}^2 = \sigma_i p^i \sigma_j p^j = \sigma_i \sigma_j p^i p^j = (\mathbb{1}_2 \delta_{ij} + \epsilon_{ijk} \sigma_k) p^i p^j$$
$$= p^i p_i + \underbrace{\epsilon_{ijk} p^i p^j}_{=0} \sigma_K = \vec{p}^2$$

which was the required relation.

Hence $(\vec{\sigma} \cdot \vec{p}) \psi = -i \vec{\sigma} \cdot \vec{\nabla} \psi = i \partial_t \psi$

What are the gamma matrices here?

- Now for $\underline{\underline{m \neq 0}}$

For d dim's: we can find d-gamma matrices s.t. $\{\gamma^{\mu}, \gamma^{\nu}\} = 2g^{\mu\nu}\mathbb{1}$

$2^n \times 2^n$ matrices

where $[n] = \left[\frac{d}{2}\right]$

→ For $d=2$ (1 space & 1 time dim^n)

$$n = \left[\frac{2}{2}\right] = 1 \Rightarrow \gamma^\mu \text{ are } d=2 \quad 2\times 2 \text{ matrices}$$

For Euclidean metric, we can take $\underline{\gamma^\mu = \{\sigma_1, \sigma_2\}}$ since they anti-commute
$(g^{\mu\nu} = \delta^{\mu\nu})$

For Minkowski metric, then we can take $\gamma^0 = \sigma_1, \gamma^1 = i\sigma_2 \Rightarrow \underline{\gamma^\mu = \{\sigma_1, i\sigma_2\}}$
$(g^{\mu\nu} = \eta^{\mu\nu})$

because $\{\gamma^1, \gamma^2\} = i\{\sigma_1, \sigma_2\} = 0$ and $\gamma^{0\,2} = \mathbb{1}, \gamma^{1\,2} = -\mathbb{1}$

$$\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}\mathbb{1}_2$$

→ For $d=3$ (2 space & 1 time dim^n) $\quad n = \left[\frac{3}{2}\right] = 1$

∴ We'll still have 2x2 matrices but 3 of them.

for Minkowski metric $\eta^{\mu\nu}$, we can similarly take

$$\gamma^0 = \sigma_1 \qquad \gamma^1 = i\sigma_2 \qquad \gamma^2 = i\sigma_3$$

to get $\gamma^\mu = \{\sigma_1, i\sigma_2, i\sigma_3\} \Rightarrow \{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}\mathbb{1}_2$

★ The case we are primarily interested in is $\underline{\underline{d=4}}$ (3 space & 1 time)

$$d=4 \longrightarrow n = \left[\frac{4}{2}\right] = 2 \longrightarrow 2^2 \times 2^2 \text{ matrices.}$$

So we'll have four 4x4 matrices. s.t. $\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}\mathbb{1}_4$

One commonly used set is called <u>Dirac-Pauli</u>.

$$\underline{\gamma^0 = \begin{pmatrix} \mathbb{1}_2 & 0_2 \\ 0_2 & -\mathbb{1}_2 \end{pmatrix}} \qquad \underline{\gamma^i = \begin{pmatrix} 0_2 & \sigma_i \\ -\sigma_i & 0_2 \end{pmatrix}}$$

For the given set, it's easy to see that

$$\gamma^i\gamma^j = \begin{pmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{pmatrix}\begin{pmatrix} 0 & \sigma_j \\ -\sigma_j & 0 \end{pmatrix} = \begin{pmatrix} -\sigma_i\sigma_j & 0 \\ 0 & -\sigma_i\sigma_j \end{pmatrix}$$

$$\Rightarrow \{\gamma^i,\gamma^j\} = \begin{pmatrix} -\{\sigma_i,\sigma_j\} & 0 \\ 0 & -\{\sigma_i,\sigma_j\} \end{pmatrix} = -2\delta_{ij}\mathbb{1}_4$$

$$(\gamma^0)^2 = \mathbb{1}_4$$

Also,
$$\gamma^0\gamma^i = \begin{pmatrix} \mathbb{1}_2 & 0 \\ 0 & -\mathbb{1}_2 \end{pmatrix}\begin{pmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{pmatrix} = \begin{pmatrix} 0 & \sigma_i \\ \sigma_i & 0 \end{pmatrix}$$

$$\gamma^i\gamma^0 = \begin{pmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{pmatrix}\begin{pmatrix} \mathbb{1}_2 & 0 \\ 0 & -\mathbb{1}_2 \end{pmatrix} = \begin{pmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{pmatrix}$$

$$\Rightarrow \{\gamma^0,\gamma^i\} = 0$$

Hence $\Rightarrow \{\gamma^\mu,\gamma^\nu\} = 2\eta^{\mu\nu}\mathbb{1}_4$

$$\gamma^0\gamma^1\gamma^2\gamma^3 = \begin{pmatrix} \mathbb{1}_2 & 0 \\ 0 & -\mathbb{1}_2 \end{pmatrix}\begin{pmatrix} 0 & \sigma_1 \\ -\sigma_1 & 0 \end{pmatrix}\begin{pmatrix} 0 & \sigma_2 \\ -\sigma_2 & 0 \end{pmatrix}\begin{pmatrix} 0 & \sigma_3 \\ -\sigma_3 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & \sigma_1 \\ \sigma_1 & 0 \end{pmatrix}\begin{pmatrix} -\sigma_2\sigma_3 & 0 \\ 0 & -\sigma_2\sigma_3 \end{pmatrix} = \begin{pmatrix} 0 & -\sigma_1\sigma_2\sigma_3 \\ -\sigma_1\sigma_2\sigma_3 & 0 \end{pmatrix}$$

Now $\sigma_2\sigma_3 = i\epsilon_{231}\sigma_1 = i\sigma_1 \Rightarrow \underline{-\sigma_1\sigma_2\sigma_3} = -i\sigma_1^2 = \underline{-i\mathbb{1}_2}$

$$\therefore \boxed{i\gamma^0\gamma^1\gamma^2\gamma^3 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \equiv \gamma^5} \quad \text{and } \underline{\underline{(\gamma^5)^2 = \mathbb{1}_4}}$$

Also, as can be seen for any set of gamma matrices, the product of all the gamma matrices $\prod_\alpha \gamma^\alpha$ anti-commutes with all the individual gamma matrices & hence

$$\underline{\underline{\{\gamma^5,\gamma^\mu\} = 0}}$$

Therefore, the product of all the γ-matrices gives a new γ-matrix which anti-commutes with all of the previous ones & naturally extends to being the next γ-matrix in higher dim^{n}s

i.e. γ^5 is the fifth gamma matrix for $d=5$. (since the dim^{n}s of γ's for $d=4$ is same as $d=5$ $\left(\left[\frac{d}{2}\right]\right)$)

γ^5 is called the Chirality matrix.

Another common representation of gamma matrices called the Chiral representation. Here you take γ^0 as γ^5 of Pauli Dirac rep.

So, $\gamma^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and $\gamma^i = \begin{pmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{pmatrix}$

$$\gamma^0\gamma^i = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{pmatrix} = \begin{pmatrix} -\sigma_i & 0 \\ 0 & \sigma_i \end{pmatrix}$$

$$\gamma^i\gamma^0 = \begin{pmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} \sigma_i & 0 \\ 0 & -\sigma_i \end{pmatrix}$$

$\Rightarrow \{\gamma^0, \gamma^i\} = 0$ and $(\gamma^0)^2 = \mathbb{1}_4$

and of course the γ^i remain the same so $\{\gamma^i, \gamma^j\} = 2\delta_{ij}\mathbb{1}_4$

So, $\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}\mathbb{1}_4$

What does change however is that

$$\gamma^0\gamma^1\gamma^2\gamma^3 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & -\sigma_1\sigma_2\sigma_3 \\ \sigma_1\sigma_2\sigma_3 & 0 \end{pmatrix} = \begin{pmatrix} \sigma_1\sigma_2\sigma_3 & 0 \\ 0 & -\sigma_1\sigma_2\sigma_3 \end{pmatrix}$$

$$= i\begin{pmatrix} \mathbb{1}_2 & 0 \\ 0 & -\mathbb{1}_2 \end{pmatrix} = -i\begin{pmatrix} -\mathbb{1}_2 & 0 \\ 0 & \mathbb{1}_2 \end{pmatrix}$$

$$\Rightarrow \boxed{i\gamma^0_c\gamma^1\gamma^2\gamma^3 = \begin{pmatrix} -\mathbb{1}_2 & 0 \\ 0 & \mathbb{1}_2 \end{pmatrix} \equiv \gamma^5_c}$$

So the role of γ^0 & γ^5 interchange with each other $\gamma^0 \longleftrightarrow \gamma^5$ (upto a proportionality sign) in Dirac-Pauli and Chiral representation.

Some general properties of γ^μ, γ^5

We would like the Hamiltonian to be hermitian since it's an observable, so $H \equiv H^\dagger$

$$H = \vec{\alpha}\cdot\vec{p} + \beta m \Rightarrow H^\dagger = \vec{p}\cdot\vec{\alpha}^\dagger + \beta^\dagger m$$

$\therefore$ for $H = H^\dagger$

$\rightarrow \beta = \beta^\dagger \Rightarrow \underline{\gamma^{0\dagger} = \gamma^0}$

$\rightarrow (\vec{\alpha}\cdot\vec{p})^\dagger = \vec{\alpha}^\dagger\cdot\vec{p} \Rightarrow \hat{\vec{\alpha}}^\dagger = \hat{\vec{\alpha}}$

(since they are obj on diff $\mathcal{H}$ spaces)

Now $\alpha^i = \gamma^0\gamma^i \Rightarrow (\alpha^i)^\dagger = \gamma^{i\dagger}\gamma^{0\dagger} = \gamma^{i\dagger}\gamma^0 \equiv \gamma^0\gamma^i$

Multiply both sides by $\gamma^0 \Rightarrow (\gamma^i)^\dagger = \underbrace{\gamma^i\gamma^0}_{\text{anti-commute}}\gamma^i = -\gamma^i$

So, $\underline{(\gamma^i)^\dagger = -\gamma^i}$

(a) So, the general relation is $\boxed{(\gamma^\mu)^\dagger = \gamma^0\gamma^\mu\gamma^0}$

(b) $\boxed{\mathrm{Tr}\,\gamma^\mu = 0}$ $\quad\forall\mu$

$\mathrm{Tr}(\gamma^0\gamma^i\gamma^0) = \mathrm{Tr}((\gamma^0)^2\gamma^i)$ but since $\gamma^i\gamma^0 = -\gamma^0\gamma^i$

$$\Rightarrow \mathrm{Tr}(\gamma^0\gamma^i\gamma^0) = -\mathrm{Tr}(\underbrace{(\gamma^0)^2}_{=\mathbb{1}}\gamma^i) \Rightarrow \underline{\mathrm{Tr}(\gamma^i) = -\mathrm{Tr}(\gamma^i) = 0}$$

Similarly

$$\mathrm{Tr}(\gamma^i\gamma^0\gamma^i) = \underbrace{\mathrm{Tr}(\overbrace{(\gamma^i)^2}^{=-\mathbb{1}}\gamma^0)}_{\text{(cyclicity)}} = \underbrace{-\mathrm{Tr}(\overbrace{(\gamma^i)^2}^{=-\mathbb{1}}\gamma^0)}_{(\gamma^i\gamma^0 = -\gamma^0\gamma^i)}$$

$\Rightarrow \underline{\mathrm{Tr}(\gamma^0) = 0}$

(c) $\frac{i}{2}[\gamma^\mu, \gamma^\nu] \equiv \sigma^{\mu\nu} = \frac{i}{2}(\gamma^\mu\gamma^\nu - \gamma^\nu\gamma^\mu)$

For $\mu \neq \nu$, $\gamma^\mu\gamma^\nu = -\gamma^\nu\gamma^\mu \Rightarrow \sigma^{\mu\nu} = i\gamma^\mu\gamma^\nu$

$\Rightarrow \sigma^{\nu\mu} = -\sigma^{\mu\nu}$

$\exists$ 6 $\sigma^{\mu\nu}$ uniquevalues

$\sigma^{ij} \longleftrightarrow$ Rotation generators $\quad \hat{S}_K = \frac{1}{4}\epsilon_{Kij}\sigma^{ij}$

$\sigma^{0i} \longleftrightarrow$ Boosts.

$$(\sigma^{\mu\nu})^\dagger = -i(\gamma^\mu\gamma^\nu)^\dagger = -i(\gamma^0\gamma^\nu\underbrace{\gamma^0\gamma^0}_{1}\gamma^\mu\gamma^0)$$

$$= -i(\gamma^0\gamma^\nu\gamma^\mu\gamma^0) = i(\gamma^0\gamma^\mu\gamma^\nu\gamma^0)$$

$$(\sigma^{\mu\nu})^\dagger = i(\gamma^0\gamma^\mu\gamma^\nu\gamma^0)$$

$$\therefore (\sigma^{ij})^\dagger = i(\gamma^0\gamma^i\gamma^j\gamma^0) = i[(-\gamma^i\gamma^0)(-\gamma^0\gamma^j)] = i\gamma^i\gamma^j = \sigma^{ij}$$

$$1$$

$$(\sigma^{0i})^\dagger = i(\gamma^0\gamma^0\gamma^i\gamma^0) = i\gamma^i\gamma^0 = -i\gamma^0\gamma^i = -\sigma^{0i}$$

(d) $\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3 = -\frac{i}{4!}\epsilon_{\mu\nu\sigma\lambda}\gamma^\mu\gamma^\nu\gamma^\lambda\gamma^\sigma$

We define $\epsilon^{0123} = +1 \Rightarrow \epsilon_{0123} = -1$

since $\epsilon_{\mu\nu\lambda\sigma} = \eta_{\mu\mu'}\eta_{\nu\nu'}\eta_{\lambda\lambda'}\eta_{\sigma\sigma'}\epsilon^{\mu'\nu'\lambda'\sigma'}$

So, $\epsilon_{0123} = (1)(-1)(-1)(-1)\epsilon^{0123} = -1$

Lecture - 7 (16/09) Spinor Representation.

Clifford Algebra $\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}\mathbb{1}_4$

Chiral representation of γ-matrices $\gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{pmatrix}$ $\sigma^\mu = (\mathbb{1}_2, \sigma^i)$
$\bar{\sigma}^\mu = (\mathbb{1}, -\sigma^i)$

$\Rightarrow \gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3 = \begin{pmatrix} -\mathbb{1}_2 & 0 \\ 0 & \mathbb{1}_2 \end{pmatrix}$

Let's briefly review some properties of γ matrices discussed in the last lecture.

- $(\gamma^\mu)^\dagger = \gamma^0\gamma^\mu\gamma^0$
- $\sigma^{\mu\nu} = \frac{i}{2}[\gamma^\mu, \gamma^\nu]$
  $= i\gamma^\mu\gamma^\nu$ for $\mu \neq \nu$
  $\downarrow$ generators of Lorentz group.
- $\mathrm{tr}(\gamma^\mu) = 0$

Linear representations of matrix groups.

U(N): leaves $\psi^\dagger\chi$ invariant i.e. $\psi'^\dagger\chi' = \psi^\dagger\chi$
when $\chi' = U\chi$ and $\psi' = U\psi$

$\psi = \begin{pmatrix} \psi_1 \\ \vdots \\ \psi_N \end{pmatrix}$ $\chi = \begin{pmatrix} \chi_1 \\ \vdots \\ \chi_N \end{pmatrix}$ U: N×N unitary matrix s.t. $U^\dagger = U^{-1}$

We can write any such $U = e^{iH}$ where $H = H^\dagger$.

If $U = e^{iH}$, $\Rightarrow U^\dagger = e^{-iH}$

Thus, $\psi'^{\dagger} \chi' = \psi^{\dagger} \underbrace{U^{\dagger} U}_{\mathbb{1}} \chi = \psi^{\dagger} \chi$

The group properties enter the picture by the following observation that

$$(U_1 U_2)^{\dagger} = U_2^{\dagger} U_1^{\dagger} = U_2^{-1} U_1^{-1} = (U_1 U_2)^{-1} \Rightarrow \text{if } U_1, U_2 \in U(N)$$
$$\Rightarrow U_1 \cdot U_2 \in U(N)$$

- $\underline{SU(2)}$: 2x2 unitary matrices with $\det U = +1$ (Rotation group)

$U = e^{i\theta^a \sigma^a}$ ← $\theta^1, \theta^2, \theta^3$ (rot^n angles)

$T^a = \frac{\sigma^a}{2}$ (rotation generators.)

did not understand where we got these eq^{n}s from

$[T^a, T^b] = i\epsilon^{abc} T^c$ Rotation Algebra.

- $\underline{O(N)}$: For $N=2$, we have rot^n matrices

$$\begin{pmatrix} x_1' \\ x_2' \end{pmatrix} = \underbrace{\begin{pmatrix} C_\theta & -S_\theta \\ S_\theta & C_\theta \end{pmatrix}}_{O} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \Rightarrow \vec{x}'^2 = \vec{x}^2$$

And O has the property that $O^T = O^{-1}$

We can generalize orthogonal matrices to $N \times N$ orthogonal matrices.

$$\begin{pmatrix} x_1' \\ x_2' \\ \vdots \\ x_N' \end{pmatrix} = \begin{pmatrix} O_{N\times N} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_N \end{pmatrix}$$

If we expand the rot^n matrix of $N=2$ in powers of θ

$$\begin{pmatrix} x_1' \\ x_2' \end{pmatrix} = \begin{pmatrix} 1 - \theta^2/2! + \dots & -\theta + \theta^3/3! + \dots \\ \theta - \theta^3/3! + \dots & 1 - \theta^2/2! + \dots \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$
$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \theta \underbrace{\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}}_{\text{GENERATOR of } O(2)} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \dots\dots$$

We denote the generator by the symbol T.

For $O(2)$, we identify the generator as $T = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$

To find the generator, we do a Taylor expansion for the parameter (here θ) very small

i.e. $g(\varepsilon) = 1 + \underset{\uparrow \text{ generator.}}{\varepsilon T} + O(\varepsilon^2))$ where $\varepsilon \ll 1$ (very small)

if we take $\varepsilon = \theta/N$ where θ is some finite value of parameter, then applying the transformation enough times

$$\lim_{N\to\infty}\left(1 + \frac{\theta}{N}T\right)^N = \underbrace{e^{\theta T}}$$ gives rise to a finite transformation.

So, for $O(2)$, $T = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and $e^{\theta T} = \begin{pmatrix} c_\theta & -s_\theta \\ s_\theta & c_\theta \end{pmatrix}$

since $e^{\theta T} = \sum_{n=0}^{\infty} \frac{1}{n!}\theta^n T^n = c_\theta \mathbb{1} + s_\theta T$

So, in orthogonal group, the orthogonal matrix can be written as

$O = e^A$ where A is an anti-symmetric matrix

$$O^T = O^{-1} = e^{-A} \stackrel{!}{=} (e^A)^T = e^{(A^T)}$$

Now since A is a $N\times N$ anti-symm. matrix, we have 0s along the diagonal and entries in upper triangle are the only independent parameters since the lower triangle is just those values negated.

So, we can write $A = \sum_{\substack{i<j \\ i=1}}^{N} \theta_{ij} A^{[i,j]}$ where $A^{[i,j]}$ represents anti-symmetric matrices.

how can a transformation have > 1 generators?

$$A^{[12]} = \begin{pmatrix} 0 & 1 & \cdots & 0 \\ -1 & 0 & \ddots & \\ \vdots & & \ddots & \\ 0 & \cdots & & 0 \end{pmatrix} \qquad A^{[1,3]} =: \begin{pmatrix} 0 & 0 & 1 & \cdots & 0 \\ 0 & 0 & \ddots & & \\ -1 & & & \ddots & \\ \vdots & & & & 0 \\ 0 & \cdots & & & \end{pmatrix}$$ and so on

$$\text{So } A = \sum_{\substack{i<j \\ i=1}}^{N} \theta_{ij} A^{[ij]} = \theta_{12} A^{[1,2]} + \theta_{23} A^{(2,3)} + \dots$$
$$+ \theta_{13} A^{[1,3]} + \dots$$
$$+ \dots$$

θ_{ij} are called the parameters.

$O(N)$ has $\underline{\underline{\dfrac{N \cdot (N-1)}{2}}}$ independent parameters.

(i can be chosen in N ways, then j can be chosen in $N-1$ ways. But since the order doesn't matter, divide by 2.)

$$\left.\begin{array}{l} N=2 \longrightarrow 1 \\ N=3 \longrightarrow 3 \\ N=4 \longrightarrow 6 \end{array}\right] \rightarrow \text{no. of parameters.}$$

This is all about orthogonal groups.

Unitary groups.

very similar to $O(N)$, except for the fact that we now want to see matrices s.t.

if $\psi' = U\psi$, $\psi \in \mathbb{C}^N$

then $\psi'^{\dagger}\psi' = \psi^{\dagger} U^{\dagger} U \psi \overset{!}{=} \psi^{\dagger}\psi$

then $\underline{U^{\dagger} \overset{!}{=} U^{-1}}$ for $U \in U(N)$

This is ensured if $U = e^{iH}$ and $H = H^{\dagger}$

SU(N) is the same as $U(N)$ i.e. $U^{\dagger}U = \mathbb{1}$ with the additional $\det(U) \overset{!}{=} +1$

This can be done by demanding $tr(H) = 0$.

So, in a nutshell

$$\left[\begin{array}{lll} \psi' = U\psi & H = \theta_A T^A \rightarrow \text{generators.} & T^{A\dagger} = T^A \\ U = e^{iH} & \uparrow & \\ & \text{real parameters} & \rightarrow \text{how this?} \end{array}\right]$$

Lorentz Group: $SO(1,3) \longleftrightarrow SO(4)$

$\downarrow$ 3 rot^{n}s + 3 boosts.

no. of parameters = 6
||
no. of generators = 6

One of the representation of Lorentz group comes up when we study tensors

$$V'^{\mu} = \Lambda^{\mu}{}_{\nu} V^{\nu} \qquad \text{(Vector representation)}$$

$SO(1,3)$: 4 dimensional

BUT, there is another independent 4 component representation which is called the spinor representation.

Generators of the Lorentz group in the spinor representation are called $J^{\mu\nu}$. $(\mu<\nu)$

$J^{\mu\nu}(\mu<\nu) \Rightarrow$

Boosts	Rot^n.
J^{01}	J^{12}
J^{02}	J^{13}
J^{03}	J^{23}

6 GENERATORS

It so happens that $J^{\mu\nu}$ can be defined as -

$$J^{\mu\nu} = \frac{i}{2}\sigma^{\mu\nu} \qquad \text{where } \sigma^{\mu\nu} = i\gamma^{\mu}\gamma^{\nu} \quad \mu\neq\nu$$

(see last lecture)

Recall:

$$(\sigma^{\mu\nu})^\dagger = \gamma^0 \sigma^{\mu\nu} \gamma^0 \longrightarrow (\sigma^{0i})^\dagger = -\sigma^{0i}$$
$$(\sigma^{ij})^\dagger = +\sigma^{ij}$$

Now we write spin rot^n matrix in the same way we decomposed orthogonal matrices in terms of generators & parameters.

$$S = e^{i\sum_{\mu<\nu} \omega_{\mu\nu} J^{\mu\nu}}$$

but we know that $\sigma^{\mu\nu} = -\sigma^{\nu\mu} \Rightarrow J^{\mu\nu} = -J^{\nu\mu}$

So, $\Rightarrow$ $S = e^{i\sum_{\mu<\nu} \omega_{\mu\nu} J^{\mu\nu}} = e^{\frac{i}{2}\omega_{\mu\nu} J^{\mu\nu}}$

6 independent parameters

6 independent generators

$= e^{i\omega\cdot J/2}$ ★

Is $S = e^{-\omega\sigma/4}$ then?

SPINOR ROT^N MATRIX

if we define $\omega_{\mu\nu} = -\omega_{\nu\mu}$

$\therefore$ $\psi'(x') = S\psi(x) \longrightarrow$ transf. of spinors

transf of spinors is similar to transf. of vectors $V'^\mu(x') = \Lambda^\mu{}_\nu V^\mu(x)$

Infact we can write $\Lambda = e^{\frac{i}{2}\omega_{\mu\nu} L^{\mu\nu}}$ where $L^{\mu\nu}$ is the analog of $J^{\mu\nu}$ for vectors.

The reason is that the commutators of J & L are same in form.

$$[L_{\mu\nu}, L_{\lambda\sigma}] = C_{\mu\nu\lambda\sigma\delta\gamma} L_{\delta\gamma}$$

same coeffs.

$$[J_{\mu\nu}, J_{\lambda\sigma}] = C_{\mu\nu\lambda\sigma\delta\gamma} J_{\delta\gamma}$$

$\Rightarrow$ Multiplication rules of S & Λ are same!

i.e.
$$\left.\begin{aligned} L(\omega)\, L(\omega') &= L(\omega'') \\ S(\omega)\, S(\omega') &= S(\omega'') \end{aligned}\right]$$ What does this even mean?

Let's now go back to gamma matrices.

(a) $\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3$ which has the properties $\boxed{\begin{aligned}(\gamma^5)^\dagger &= \gamma^5 \\ (\gamma^5)^2 &= \mathbb{1}_4\end{aligned}}$

(in chiral rep.)

We can check $(\gamma^5)^2 = \mathbb{1}_4$ easily

$$(\gamma^5)^2 = \gamma^5(\gamma^5)^\dagger = \cancel{i}\gamma^0\gamma^1\gamma^2\gamma^3(\cancel{-i}\gamma^{3\dagger}\gamma^{2\dagger}\gamma^{1\dagger}\gamma^{0\dagger})$$

But $(\gamma^i)^\dagger = -\gamma^i$ & $(\gamma^0)^\dagger = \gamma^0$

$$\Rightarrow (\gamma^5)^2 = \gamma^0\gamma^1\gamma^2\gamma^3\gamma^3\gamma^2\gamma^1\gamma^0(-1)^3$$

From $\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}\mathbb{1}_4$, we see that $(\gamma^i)^2 = -\mathbb{1}$ & $(\gamma^0)^2 = \mathbb{1}_4$

$$\text{So, } (\gamma^5)^2 = \gamma^0\gamma^1\gamma^2\gamma^3\gamma^3\gamma^2\gamma^1\gamma^0(-1)^3 = (-1)^6\,\mathbb{1}_4 = \underline{\underline{\mathbb{1}_4}}$$

(b) We also see that $\boxed{\{\gamma^\mu, \gamma^5\} = 0}$

$$\gamma^\mu\gamma^5 = \gamma^\mu(i\gamma^0\gamma^1\gamma^2\gamma^3)$$

Now $\mu \in \{0,1,2,3\}$. So when we push it through till the last position, γ^μ anti-commutes everytime $\nu \neq \mu$. So it will pick up 3 (-) signs no matter what μ

$$\text{So } \gamma^\mu\gamma^5 = \gamma^\mu(i\gamma^0\gamma^1\gamma^2\gamma^3) = -i\gamma^0\gamma^1\gamma^2\gamma^3\gamma^\mu = -\gamma^5\gamma^\mu$$

hence, they anti-commute.

# of L.I. independent matrices → Why are we talking about this?

$$\{ \mathbb{1}_4, \gamma^\mu, \gamma^5, \gamma^5\gamma^\mu, \sigma^{\mu\nu} \} \equiv \Gamma^A$$
$$1 \quad 4 \quad 1 \quad 4 \quad 6$$

Basis for expanding any matrix which transforms spinors.

Why? Don't we only have 6 independent generators? Are they even related?

$\psi \in \mathbb{C}^4$

Matrices $\underset{4\times4}{M} = M_A \Gamma^A$

So, we now state that if we have a $\tilde{\gamma}^\mu$ s.t.

$$\underline{\tilde{\gamma}^\mu = U \gamma^\mu U^\dagger}$$

then it has all the properties of γ^μ, and also satisfies same hermiticity props.
$\tilde{\gamma}^{\mu\dagger} = \tilde{\gamma}^0 \tilde{\gamma}^\mu \tilde{\gamma}^0$ and satisfies Clifford Algebra.

Also, if we take $\gamma^{\mu\prime} = S \gamma^\mu S^{-1}$
it satisfies Clifford Algebra, but hermiticity properties might not be same! However if S is unitary then $S^{-1} = S^\dagger$ and then hermiticity properties would be satisfied.

Also, the converse is true i.e. if γ^μ & $\hat{\gamma}^\mu$ satisfy Clifford Alg., & same hermiticity props, then $\exists$ M s.t.

$$\hat{\gamma}^\mu = M \gamma^\mu M^{-1} \quad \text{where } M^{-1} = M^\dagger$$

Going back to Dirac Eq^n, we have

$$(i\gamma^\mu \partial_\mu - m)\psi = 0$$

$$\Big(\qquad\Big)_{4\times4}\begin{pmatrix}\psi_1\\ \vdots\\ \psi_4\end{pmatrix} = \begin{pmatrix}0\\0\\0\\0\end{pmatrix}$$

We introduce Feynman slash notation –

$$\boxed{a_\mu \gamma^\mu \equiv \not{a}}$$

It enjoys various properties, some of them as follows-

$$\not{a}^2 = a_\mu \gamma^\mu a_\nu \gamma^\nu = a_\nu \gamma^\nu a_\mu \gamma^\mu$$

So, $\not{a}^2 = a_\mu a_\nu \gamma^\mu \gamma^\nu = a_\mu a_\nu \gamma^\nu \gamma^\mu$ (since a_μ, a_ν commute)

$$2\not{a}^2 = a_\mu a_\nu (\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu) \Rightarrow \not{a}^2 = a_\mu a_\nu \frac{\{\gamma^\mu, \gamma^\nu\}}{2}$$

but $\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}\mathbb{1}_4$

So, $\underline{\underline{\not{a}^2}} = a_\mu a_\nu \eta^{\mu\nu} = \underline{\underline{a_\mu a^\mu}} = a\cdot a$ $\quad\boxed{\not{a}^2 = a_\mu a^\mu}$

Therefore, Dirac Eq^n in slash notation is

$$(i\not{\partial} - m)\psi = 0$$

If we multiply by $(i\not{\partial} + m) \Rightarrow (i\not{\partial}+m)(i\not{\partial}-m)\psi = 0$

$\Rightarrow (-(\not{\partial})^2 - m^2)\psi = 0$ $\quad$ but $(\not{\partial})^2 = \partial_\mu \partial_\mu = \Box$

So this gives $\underline{(\Box + m^2)\psi = 0}$

and Klein Gordon eq^n is trivially obtained.

Lorentz Covariance of Dirac Eqn.

Recall Maxwell's eqns defined Lorentz covariance!

$$\partial_\mu F^{\mu\nu} = j^\nu \qquad \epsilon^{\mu\nu\lambda\sigma}\partial_\nu F_{\lambda\sigma} = 0$$

Doing L.T., we get the same form of eqn

$$\partial'_\mu F'^{\mu\nu} = j'^\nu \qquad \epsilon^{\mu\nu\lambda\sigma}\partial'_\nu F'_{\lambda\sigma} = 0$$

$\therefore$ Maxwell's eqns are form invariant under L.T. where

$$j'^\mu(x') = \Lambda^\mu{}_\nu \, j^\nu(x)$$
$$F'^{\mu\nu}(x') = \Lambda^\mu{}_{\mu'} \Lambda^\nu{}_{\nu'} F^{\mu'\nu'}(x)$$
$$\partial'_\mu = \tilde{\Lambda}_\mu{}^\nu \partial_\nu$$
$$\epsilon'^{\mu\nu\lambda\sigma} = \epsilon^{\mu\nu\lambda\sigma} \rightarrow \text{invariant tensor of Lorentz group)}$$

<u>Form invariance ≡ "Covariance"</u>

So, $(i\gamma^\mu \partial_\mu - m)\psi(x) = 0$

$\Rightarrow$ Under L.T.

$$(i\gamma^\mu \partial'_\mu - m)\psi'(x') = 0$$

However ψ is not a wavefn but a spinor.

So, $\psi'_\alpha(x') = \underbrace{S_{\alpha\beta}\,\psi_\beta}_{\text{summed.}}(x)$

$\gamma^\mu \rightarrow$ invariant object.

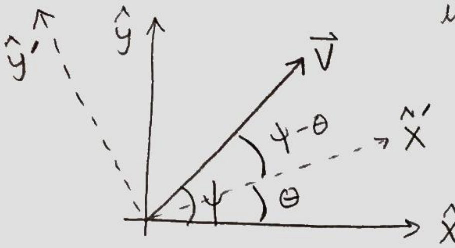
Is it really a fourvector then?

Lecture - 8

(17/09)

Crash course of Lie Groups (continuous symmetries)

Suppose we have a vector $\vec{v}$ in $\hat{x}$-$\hat{y}$ plane. If we rotate the axes s.t. we now have $\hat{x}'$ and $\hat{y}'$ as new axes.



$$V_x = |\vec{v}| \cos\phi \qquad V_y = |\vec{v}| \sin\phi$$

$$\downarrow \qquad\qquad \downarrow$$

$$V'_x = |\vec{v}| \cos(\phi-\theta) \qquad V'_y = |\vec{v}| \sin(\phi-\theta)$$

$$= |\vec{v}| \left(C_\phi C_\theta + S_\phi S_\theta\right) \qquad = |\vec{v}| \left(S_\phi C_\theta - C_\phi S_\theta\right)$$

$$= V_x C_\theta + V_y S_\theta \qquad = -V_x C_\theta + V_y S_\theta$$

So, we get the transformation as -

$$\begin{pmatrix} V_x' \\ V_y' \end{pmatrix} = \begin{pmatrix} C_\theta & S_\theta \\ -S_\theta & C_\theta \end{pmatrix} \begin{pmatrix} V_x \\ V_y \end{pmatrix} \Rightarrow V' = O(\theta)\, V$$

Let's now see what this matrix looks like when expanded in powers of θ (about $\theta = 0$)

$$O = \begin{pmatrix} 1-\theta^2/2! + \cdots & \theta - \theta^3/3! + \cdots \\ -\theta + \theta^3/3! + \cdots & 1 - \theta^2/2! + \cdots \end{pmatrix} = \underbrace{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}}_{= \mathbb{1}} + \theta \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} + O(\theta^2)$$

For small enough θ, we get a rot^n close enough to $\mathbb{1}$. The infinitesimal difference b/w $O(\theta)$ where $\theta \ll 1$ and $\mathbb{1}$ is -

$$\delta O = \underbrace{\theta}_{\substack{\text{parameter} \\ SO(2)}} \underbrace{\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}}_{\substack{\text{Matrix generator} \\ \text{of } SO(2)}} \equiv T$$

We usually write an i in front of θT so here we will write $T \equiv -i\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$

then $\delta O = i\theta T + O(\theta^2)$

Repeating this for $d=3$

$$\begin{pmatrix} V_1' \\ V_2' \\ V_3' \end{pmatrix} = \Big(\underbrace{O(\theta_a)}_{3\times 3} \Big) \begin{pmatrix} V_1 \\ V_2 \\ V_3 \end{pmatrix}$$

a bunch of f^n's of parameters

There's not much use in writing down the explicit form of O. So we'll analyze a general orthogonal matrix in 3×3.

The primary property of orthogonal matrices is that $V'^T V' = V^T V$
$\Rightarrow O^T O = \mathbb{1}$

Now suppose O is a f^n of some parameters θ_a i.e. $O : \theta_a \to O_{3\times 3}$ and then we do a Taylor expansion about $\theta_a = 0$

$$O(\theta_a) = \mathbb{1}_3 + \theta_a A_a + \cdots$$

then $O^T(\theta_a) = \mathbb{1}_3 + \theta_a A^T_a + \cdots$

$$OO^T = \mathbb{1}_3 + \theta_a A_a + \theta_a A^T_a + \ldots O(\theta_a^2) \ldots \overset{!}{=} \mathbb{1}_3$$

$\Rightarrow A_a^T \overset{!}{=} -A_a$ $\therefore$ Generators of $SO(d)$ are anti-symmetric!

So, $A_{ij} = -A_{ji}$ (components of generators)

$$\begin{pmatrix} 0 & A_{12} & A_{13} \\ -A_{12} & 0 & A_{23} \\ -A_{13} & -A_{23} & 0 \end{pmatrix} \longrightarrow \text{only three independent parameters.}$$

Now we don't have 3 independent parameters for every A_a. We only have 3 independent parameters in total. Why did I then reach the conclusion that each A_a has 3 independent parameters?

The simplest way of writing O in terms of generators is -

$$O = \mathbb{1} + \Theta_1 A_1 + \Theta_2 A_2 + \Theta_3 A_3 + \cdots$$

I still don't understand the way we argue that A has 3 independent params, because we showed that it's A_a which has 3 independent params, not A. However, let's assume that means $\exists$ 3 independ. param of $O_{3\times3}$.

Here we can identify $A_3 = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, $A_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}$, $A_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}$

We can also decompose A differently with new parameters φ_{ij} where we define $\varphi_{ij} = -\varphi_{ji}$

$$O = \mathbb{1} + \varphi_{23} A_{23} + \varphi_{31} A_{31} + \varphi_{12} A_{12} + \cdots$$

where $A_{23} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}$, $A_{31} = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$, $A_{12} = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

If we also define $A_{ij} = -A_{ji}$ (new matrices with opp. sign)

Then $O = \mathbb{1} + \frac{1}{2}\sum_{i,j} \varphi_{ij} A_{ij}$

$$O_{kl} = \mathbb{1}_{kl} + \frac{1}{2}\sum_{ij} \varphi_{ij} (A_{ij})_{kl}$$

need to think about this again.

$$(A_{ij})_{kl} = (\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}) \equiv \delta_{i[k}\delta_{l]j}$$

$$\boxed{(A_{(ij)})_{kl} = \delta_{i[k}\delta_{l]j}} \longleftrightarrow \varphi_{ij} = -\varphi_{ji}$$

For a general $d > 3$, we have $\frac{d(d-1)}{2}$ parameters.

The generators of $SO(d)$ are thus given by -

$$\boxed{(A_{(ij)})_{kl} = \delta_{i[k}\delta_{l]j} \qquad \varphi_{ij} = -\varphi_{ji}}$$

GENERATORS (Anti-hermitian) $\quad i,j = 1,2\ldots d \quad$ PARAMETERS

# of independent parameters = # of generators $= \frac{d(d-1)}{2}$

Doing this same thing for $\underline{\underline{SU(N)}}$

$$\begin{pmatrix}\psi_1\\ \psi_2\\ \vdots\\ \psi_N\end{pmatrix}' = U\begin{pmatrix}\psi_1\\ \psi_2\\ \vdots\\ \psi_n\end{pmatrix}$$

Defining property $\psi^\dagger\psi = \psi'^\dagger\psi'$

$\Rightarrow U^\dagger = U^{-1}$

Writing U in terms of its parameters Θ_a

$$U(\Theta_a) = \mathbb{1} + i\Theta_A T^A + O(\theta^2)$$
$$U^\dagger(\Theta_a) = \mathbb{1} - i\Theta_A T^{A\dagger} + O(\theta^2)$$

Since $U^\dagger U \stackrel{!}{=} \mathbb{1} \Rightarrow U^\dagger U = \mathbb{1} + i\Theta_A(T^A - T^{A\dagger}) + O(\theta^2)$

$\Rightarrow \underline{T^A = T^{A\dagger}}$ Generators of $SU(N)$ must be Hermitian.

So, right now, my main confusion is, how does one find the generators of SO(N) & SU(N)? How do we find the generators & # of ind-parameters? I want a proper derivation of this!

<u>Note</u>: For $SO(d)$, $O = \mathbb{1} + \frac{1}{2}\Theta_{ij}A_{(ij)} + O(\theta^2)$ where $A_{(ij)}$ was anti-Hermitian

If we define $\underline{T_{(ij)} = +iA_{(ij)}}$

$$O = \mathbb{1} + \frac{i}{2}\Theta_{ij}T_{(ij)} + O(\Theta^2)$$

Lorentz Group. $SO(1,3)$

The Lorentz transformation on the co-ordinates is

$$X'^{\mu} = \Lambda^{\mu}{}_{\nu} X^{\nu} \quad \longrightarrow 4\times 4 \text{ matrices}$$

The defining properties of $\Lambda^{\mu}{}_{\nu}$ is that -

$$\Lambda^{\mu}{}_{\theta} \Lambda^{\nu}{}_{\sigma} \eta_{\mu\nu} = \eta_{\theta\sigma}$$

If we consider an infinitesimal Lorentz transformation -

$$\delta\Lambda^{\mu}{}_{\theta} = (\delta^{\mu}{}_{\theta} - \omega^{\mu}{}_{\theta})$$

$$\Rightarrow \delta V^{\mu} \simeq V'^{\mu} - V^{\mu} = -\omega^{\mu}{}_{\theta} V^{\theta}$$

We can also see that -

$$\eta_{\mu\nu} \underbrace{(\delta^{\mu}{}_{\theta} - \omega^{\mu}{}_{\theta})}_{\delta\Lambda^{\mu}{}_{\theta}} \underbrace{(\delta^{\nu}{}_{\sigma} - \omega^{\nu}{}_{\sigma})}_{\delta\Lambda^{\nu}{}_{\sigma}} = \eta_{\theta\sigma} - \omega_{\theta\sigma} - \omega_{\sigma\theta} + O(\omega^2)$$

but $\eta_{\mu\nu} \delta\Lambda^{\mu}{}_{\theta} \delta\Lambda^{\nu}{}_{\sigma} \overset{!}{=} \eta_{\theta\sigma} \Rightarrow \boxed{\omega_{\sigma\theta} = -\omega_{\theta\sigma}}$

property of Lorentz group params.
(ANTI-SYMM. !!)

For $d=4$, $SO(1,3)$, we have 6 independent Lorentz group parameters.

Now, we know that $SO(3)$: 3D spatial rot'^s $\longrightarrow$ 3 rotation parameters.

This leaves $6-3=3$ boost parameters.

So, Parameters of $SO(1,3)$ Lorentz group can be labelled by

$$\left.\begin{array}{llll} \omega_{ij} = -\omega_{ji} & (\text{3-D rot'}^{\text{s}}) & i,j=1,2,3 & \longrightarrow 3 \\ \omega_{0i} = -\omega_{i0} & (\text{Boosts}) & & \longrightarrow 3 \end{array}\right\} 6$$

Lorentz Covariance of Dirac Eqⁿ.

For Maxwell's equations

$$\partial_\mu F^{\mu\nu} = j^\nu \xrightarrow{S\to S'} \partial'_\mu F'^{\mu\nu} = j'^\nu$$

where $j'^\nu(x') = \Lambda^\nu{}_\sigma \, j^\sigma(x)$ ⟶ This is how 4-vectors transforms.

Now, for Dirac eqⁿ -

$$(i\gamma^\mu \partial_\mu - m)\psi_{(x)} = 0 \xrightarrow{S\to S'} (i\gamma^\mu \partial'_\mu - m)\psi'(x') = 0$$

and just like we have a linear relation b/w 4-vector transformations, we would like a similar situation to prevail for spinors.

So, we'd expect $\psi'_\alpha(x') = S^{(\omega)}_{\alpha\beta}\,\psi_\beta(x)$ ⟶ This is how spinors transform.

↑ fⁿ of parameters of Lorentz group $\omega_{\mu\nu}$

Objects which tranform like the wavefⁿ of Dirac eqⁿ are defined to be spinors.

$S_{\alpha\beta}$ ⟶ Lorentz transform in spinor representation.

What is a representation anyway?

Say we have a matrix M of some group with parameter θ i.e. $M(\theta)$ & another matrix of same group $M(\theta')$. The idea of groups is that

$$M(\theta)M(\theta') = M(\theta''(\theta,\theta')) \text{ since } a\circ b \in \text{group}.$$

The new parameter θ'' is a fⁿ of θ & θ' i.e. $\theta''(\theta,\theta')$. This defines group multiplication.

The idea of generators is that we write $M_{(\theta)} = e^{i\theta\cdot T}$ & $M(\theta') = e^{i\theta'\cdot T}$

BCH $e^{i\theta\cdot T} e^{i\theta'\cdot T} = \exp i\left[(\theta+\theta')\cdot T + \theta_a\theta'_b\,[T_a, T_b] + \text{all commutators}\right]$

$= \exp(i\theta''\cdot T)$

The only way for this thing to work is that

$[T^a, T^b] \sim T^c$ & only then we'd be able to match coeffs on both sides.

$$\Rightarrow \quad [T^a, T^b] = i\, \underset{\substack{\uparrow \\ \text{structure const.}}}{C^{ab}{}_c}\, T^c \qquad \text{COMMUTATOR ALGEBRA}$$

$\therefore$ The commutator algebra describes the group multiplication rule.

Now suppose instead of T^as, we had some other set of matrices $\mathcal{J}^a$, then if we define objects

$\mathcal{M} = e^{i\theta^a \mathcal{J}^a}$ which satisfy the same commutation algebra

$$[\mathcal{J}^a, \mathcal{J}^b] = i\, C^{ab}{}_c\, \mathcal{J}^c$$

$$\Rightarrow \mathcal{M}(\theta)\,\mathcal{M}(\theta') = \mathcal{M}(\underset{\substack{\uparrow \\ \text{same rule as for M}}}{\theta''(\theta, \theta')})$$

$\therefore$ If two objects satisfy the same commutation algebra, then the group multiplication rule for both would be same, and they would be different representations of the same group.

VERY SIMILARLY, even though spinors and 4-vectors (both 4-component objects) live in different spaces, their multiplication rule is the same $\omega''(\omega, \omega')$

i.e. $\Lambda(\omega)\,\Lambda(\omega') = \Lambda(\omega''(\omega,\omega'))$ ← 4-vector rep.

$S(\omega)\,S(\omega') = S(\omega''(\omega,\omega'))$ ← spinor rep.

and hence they are diff. representations of the same $SO(1,3)$ group.

Lecture - 9 (QFT)

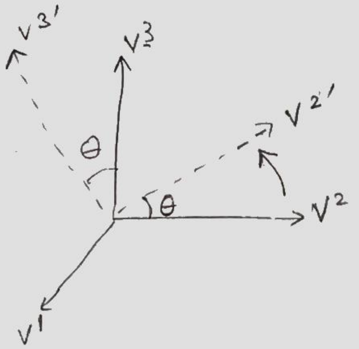
(21/09)

SO(3) Rotation group $\longrightarrow$ special orthogonal group with $n=3$ (3x3 matrices)

The Lorentz group SO(1,3) consists of Boosts $\oplus$ Rotations which are called the Lorentz transformations.

Recalling rot^n's in 2-3 plane -

$$\begin{pmatrix} V^{2'} \\ V^{3'} \end{pmatrix} = \begin{pmatrix} C_\theta & S_\theta \\ -S_\theta & C_\theta \end{pmatrix} \begin{pmatrix} V^2 \\ V^3 \end{pmatrix}$$



In first order

$$\delta V^2 = \theta V^3$$
$$\delta V^3 = -\theta V^2$$

We call θ as the parameter ω_{23}, for rot^n around 2-3 plane.

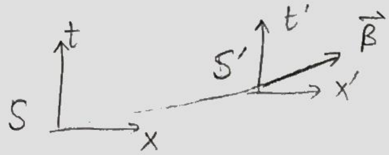
Similar rot^n's in 1-3 and 1-2 plane are determined by parameters ω_{13} and ω_{12} respectively.

Then come the boosts, say in the x-direction (or "1"-direction), -

$$\begin{pmatrix} V^{0'} \\ V^{1'} \end{pmatrix} = \gamma \begin{pmatrix} 1 & -\beta \\ -\beta & 1 \end{pmatrix} \begin{pmatrix} V^0 \\ V^1 \end{pmatrix}$$

$$= \begin{pmatrix} \cosh\omega & -\sinh\omega \\ -\sinh\omega & \cosh\omega \end{pmatrix} \begin{pmatrix} V^0 \\ V^1 \end{pmatrix}$$

where $\tanh\omega = \beta$, $\gamma = \sqrt{1-\beta^2}$



for small β (or ω), in first order.

$$\delta V^0 = -\omega V^1$$
$$\delta V^1 = -\omega V^0$$

Now, knowing the infinitesimal form of boosts & rotations, we can write the transformation ~~of~~ a (parameter)·(generator) acting on vectors.

Let us define these generators as –

for O(3): $\delta V_\ell = \frac{i}{2}\,\omega_{mn}\left(T^{mn}\right)_{\ell j} V^j$

parameters (ω_{mn}), generator (T^{mn})

we are just guessing a form of the infinitesimal transformations from knowledge of O(3) transf. covered in Lec-8.

And this has to coincide with the transform. rules we just got from L.T. matrix

i.e. $\delta V^\ell = \omega_{\ell j} V^j$

$\left(\begin{array}{l}\text{since } \delta V^2 = \omega_{23} V^3 \\ \delta V^3 = \omega_{32} V^2 \\ \text{where } \omega_{32} = -\omega_{23}\end{array}\right)$

$T^{mn} = -T^{nm}$

Hence, $\frac{i}{2}\omega_{mn}\left(T^{mn}\right)_{\ell j} \overset{!}{=} \omega_{\ell j}$

$\left(\begin{array}{l}\omega_{mn} T^{mn} \to \text{matrix} \\ (\text{matrix})_{\ell j} V^j \to \text{transf. vector}\end{array}\right)$

$\Rightarrow \left(T^{mn}\right)_{\ell j} = -i\left(\delta_{\ell m}\delta_{nj} - \delta_{\ell n}\delta_{mj}\right)$

since $\left[\frac{i}{2}\omega_{mn}\right]\cdot\left[-i\left(\delta_{\ell m}\delta_{nj} - \delta_{\ell n}\delta_{mj}\right)\right] = \frac{1}{2}\left(\omega_{\ell j} - \omega_{j\ell}\right) = \omega_{\ell j}$

$\therefore \left(T^{mn}\right)_{\ell j} = -i\,\delta_{\ell[m}\delta_{n]j}$

The generators satisfy the following commutation algebra –

$\left(\left[T_{mn}, T_{pq}\right]\right)_{rs} = i\left(\delta_{[p[m} T_{n]q]}\right)_{rs}$ → how?

<u>Note:</u>

The object (T^{mn}) here is actually a matrix corresponding to rot^n about m–n plane and ω_{mn} is a parameter associated to that rot^n. The mn aren't separate indices, but can be thought of as independent when defining $T^{mn} = -T^{nm}$ and $\omega_{nm} = -\omega_{mn}$ & then putting a $\frac{1}{2}$ factor.

Let's take an example –

$$[\underset{mn}{T_{12}}, \underset{pq}{T_{23}}] = i\left(\delta_{[2[1}T_{2]3]}\right)$$

$$\delta_{[2[1}T_{2]3]} = \delta_{[21}T_{23]} - \delta_{[22}T_{13]}$$
$$= \delta_{21}T_{23} - \delta_{31}T_{22} - \delta_{22}T_{13} + \delta_{32}T_{12}$$
$$= -T_{23}$$

$$\Rightarrow [T_{12}, T_{23}] = -i\,T_{13} = +i\,T_{31}$$

- <u>Note about anti-symm. not^n:</u>

$$A_{[i[j}B_{k]l]} = A_{i[j}B_{k]l} - A_{l[j}B_{k]i}$$
$$= A_{ij}B_{kl} - A_{ik}B_{jl} - A_{lj}B_{ki} + A_{lk}B_{ji}$$

where $X_{[a}Y_{b]} = X_aY_b - X_bY_a$

One can notice a pattern there. Since 12 $\longrightarrow$ rot^n about 1–2 plane, and so on, in 3D, one can think of rot^n about axes instead. So, 12 is a rot^n about z i.e. 3, 23 is a rotation about x i.e. 1, and 31 is a rot^n about y i.e. 2.

Thus, we define $\underline{L_i = \frac{1}{2}\epsilon_{ijk}T_{jk}}$

$$\Rightarrow [L_i, L_j] = i\,\epsilon_{ijk}L_k \longrightarrow \begin{aligned}[L_3, L_1] &= iL_2\\ [L_1, L_2] &= iL_3\\ [L_2, L_3] &= iL_1\end{aligned}$$

<u>This is the general structure of SO(3) (or even SO(N)) rotations!</u>

We can now copy this same structure to form Lorentz transformations which include rot^n in the following manner -

$$\delta V^\mu = -\omega^\mu{}_\nu V^\nu \qquad \mu = 0, i = 0,1,2,3,4$$

Rot^n $\Rightarrow$ $\delta V^i = -\omega^i{}_j V^j$ where $\omega^i{}_j = \eta^{ik}\omega_{kj} = \eta^{ii}\omega_{ij} + 0$

$= +\omega_{ij} V^j \qquad\qquad = (-1)\omega_{ij}$

(just as we found on 1)

Boost in x $\Rightarrow$ $\delta V^0 = -\omega^0{}_1 V^1 = -\omega_{01} V^1$

and $\delta V^1 = -\omega^1{}_0 V^0 = +\omega_{10} V^0 = -\omega_{01} V^0$

$\omega_{01} = -\omega_{10} = -\omega = -\tanh^{-1}\beta$

$\omega_{23} = -\omega_{32} = -\theta$

Therefore, for counting of parameters -

$$\omega^\mu{}_\nu = \eta^{\mu\lambda}\underset{\substack{\uparrow \\ \text{anti-symm. parameter} = -\omega_{\nu\lambda}}}{\omega_{\lambda\nu}}$$

$\therefore$ the no. of independent params are given by $\frac{4(4-1)}{2} = \frac{4 \cdot 3}{2} = \underline{\underline{6}}$

So, we have a total of 6 ind. params $\rightarrow$ $\omega_{ij} = -\omega_{ji}$ $\boxed{3}$; $\omega_{0i} = -\omega_{i0}$ $\boxed{3}$ $\Big\}$ 6 params. of Lorentz group.

For Lorentz transformations, we write

$$\delta V^\mu = \frac{i}{2}\left(\underset{\substack{\uparrow \\ 0,1,2,3}}{\omega_{MN}} L^{MN}\right)^\mu{}_\nu V^\nu$$

where L^{MN} is now the Lorentz generator (the Lorentz version of T^{mn}).

$$(L^{MN})^\mu{}_\nu = -i\,\delta_\nu^{[M}\eta^{N]\mu}$$

Interlude on Canonical Parameterization.

There an infinite no. of ways in which the infinitesimal form reduces to-

$$\delta V_l = \frac{i}{2}\omega_{mn}(T^{mn})_{lj} V^j \equiv \delta O_{lj} V^j$$

$$O_{lj} = \mathbb{1} + \frac{i}{2}\omega_{mn}(T^{mn})_{lj} + \mathcal{O}(\omega^2)$$

The canonical form of parameterization is using the exponential f^n

$$O \equiv \exp\left(\frac{i}{2}\omega_{mn}(T^{mn})\right) = \sum_{r=0}^{\infty}\frac{1}{r!}\left(\frac{i}{2}\omega_{mn}(T^{mn})\right)^r$$

Let's check if this indeed works

$$\delta V^\mu = \frac{1}{2}\left(\omega_{MN}\,\delta_\nu^{[M}\eta^{N]\mu}\right)V^\nu = \frac{1}{2}\omega_{MN}\left(\delta_\nu^M\eta^{N\mu} - \delta_\nu^N\eta^{M\mu}\right)V^\nu$$

$$= \frac{1}{2}\left(\omega_{\nu N}\eta^{N\mu} - \omega_{M\nu}\eta^{M\mu}\right)V^\nu = \frac{1}{2}\left(\eta^{\mu N}\omega_{N\nu} - \eta^{\mu M}\omega_{M\nu}\right)V^\nu$$

$$= \underline{-\omega^\mu{}_\nu V^\nu} \checkmark$$

Let's compare the commutation algebra of T & L.

Starting with T

$$[T_{mn}, T_{rs}]_{de} = (T_{mn})_{df}(T_{rs})_{fe} - (T_{rs})_{df}(T_{mn})_{fe}$$

$$= (-i)^2\left(\delta_{d[m}\delta_{n]f}\,\delta_{f[r}\delta_{s]e} - \delta_{d[r}\delta_{s]f}\,\delta_{f[m}\delta_{n]e}\right)$$

$$= \frac{+1}{i}\left(i\delta_{d[m}\delta_{n][r}\delta_{s]e} - i\delta_{d[r}\delta_{s][m}\delta_{n]e}\right)$$

$$= \frac{+1}{i}\begin{pmatrix} i\,\delta_{d[m}\delta_{n]r}\,\delta_{se} - i\,\delta_{d[m}\delta_{n]s}\,\delta_{re} \\ -i\,\delta_{d[r}\delta_{s]m}\,\delta_{ne} + i\,\delta_{d[r}\delta_{s]e}\,\delta_{nm} \end{pmatrix}$$

$$= \frac{1}{i}\begin{pmatrix} -(T^{mn})_{dr}\,\delta_{se} + (T^{mn})_{ds}\,\delta_{re} \\ +(T^{rs})_{dm}\,\delta_{ne} - (T^{rs})_{de}\,\delta_{nm} \end{pmatrix}$$

$$= \frac{1}{i}\left((T^{mn})_{d[s}\,\delta_{r]e} - (T^{rs})_{d[m}\,\delta_{e]n} \right)$$

$$\overset{\text{somehow}}{=} i\left(\delta_{[r[m}\,T_{n]s]}\right)_{de}$$

We now want to do the exact same thing with L^{MN}'s.-

$$[L^{MN}, L^{PQ}] = -iL^{[N[Q}\eta^{P]M]}$$

Hence, in summary, the Lorentz group $\xrightarrow{} SO(1,3)$ generators are given by

$$\boxed{(L^{MN})^{\mu}{}_{\nu} = -i\,\delta_{\nu}{}^{[M}\,\eta^{N]\mu}}$$

Satisfying the commutation relations -

$$\boxed{[L^{MN}, L^{PQ}] = -i\,L^{\{N[Q}\eta^{P]M\}}} \qquad \{\} \equiv [\,]$$

which is very similar to generators of orthogonal group $O(3)$.

Lorentz Covariance of Dirac eqⁿ.

$$\left(i\gamma^{\mu}\partial_{\mu} - m\right)\psi(x) = 0$$

4x4 constant matrices

$$\begin{pmatrix}\psi_1\\ \psi_2\\ \psi_3\\ \psi_4\end{pmatrix} \equiv [\psi_\alpha]$$

We would expect that under Lorentz transform -

$$\left(i\gamma^{\mu}\partial'_{\mu} - m\right)\psi'(x') = 0$$

where $\psi'(x')$ is related to $\psi(x)$ via a L.T.

$$\underline{\psi'(x') = S(\omega)\psi(x)}$$

4x4 matrix

(just like $V'^{\mu}_{(x')} = \Lambda^{\mu}{}_{\nu} V^{\nu}(x)$ or in matrix notⁿ $V'(x') = \Lambda V(x)$)

Now, we are in the search of S. Let's first discuss the transf. rule of ∂_μ

$$\partial'_{\mu} = \tilde{\Lambda}_{\mu}{}^{\nu}\partial_{\nu} \Rightarrow \frac{\partial}{\partial x'^{\mu}} = \underset{\displaystyle \frac{\partial x^{\nu}}{\partial x'^{\mu}}}{\overset{\displaystyle\tilde{\Lambda}_{\mu}{}^{\nu}}{\equiv}} \frac{\partial}{\partial x^{\nu}}$$

(because chain rule)

where $\tilde{\Lambda}$ is related to Λ via -

$$\tilde{\Lambda}_{\mu}{}^{\nu} = \eta_{\mu\theta}\,\eta^{\nu\lambda}\,\Lambda^{\theta}{}_{\lambda}$$

So, coming back to the Dirac eqⁿ in frame F′

$$\left(i\gamma^{\mu}\underset{\displaystyle \tilde{\Lambda}_{\mu}{}^{\nu}\partial_{\nu}}{\underset{\shortparallel}{\partial'_{\mu}}} - m\right)S\psi(x) = 0$$

$\therefore$ We need a $S(\omega)$ s.t.

$$(i\gamma^\mu \tilde{\Lambda}_\mu{}^\nu \partial_\nu - m\mathbb{1}) S\psi(x) = 0 \iff (i\gamma^\nu \partial_\nu - m\mathbb{1})\psi(x) = 0$$

Now multiplying by S^{-1} from LHS

$$(i \underline{S^{-1}\gamma^\mu S \tilde{\Lambda}_\mu{}^\nu} \partial_\nu - m\mathbb{1})\psi(x) = 0 \iff (i\underline{\gamma^\nu}\partial_\nu - m\mathbb{1})\psi(x) = 0$$

$$\Rightarrow \left[S^{-1}\gamma^\mu S (\Lambda^{-1})^\nu{}_\mu = \gamma^\nu\right] \times \Lambda^\theta{}_\nu \Rightarrow S^{-1}\gamma^\theta S = \Lambda^\theta{}_\nu \gamma^\nu$$

★ Does $\Lambda^\mu{}_\nu$ commute with S & γ^μ matrices?
* Yes. because $\Lambda^\mu{}_\nu$ is a no. whereas S & γ^μ are matrices.

So, we want to find S matrix s.t.

$$\boxed{S\gamma^\mu S^{-1} = (\Lambda)^\mu{}_\nu \gamma^\nu}$$

We said sometime back that $\Lambda^\mu{}_\nu = \delta^\mu_\nu - \omega^\mu{}_\nu + O(\omega^2)$

and is enough for infinitesimal level. $(\Lambda^{-1})^\mu{}_\nu = \delta^\mu_\nu + \omega^\mu{}_\nu + O(\omega^2)$

So, if we write S as -

why 4?

$$S = \mathbb{1}_4 + \frac{i}{4}\omega_{\mu\nu}\beta^{\mu\nu} + O(\omega^2)$$

$$S^{-1} = \mathbb{1}_4 - \frac{i}{4}\omega_{\mu\nu}\beta^{\mu\nu} + O(\omega^2)$$ check by multiplying SS^{-1}

parameters ($\omega_{\mu\nu}$), 4x4 matrices ($\beta^{\mu\nu}$)

$$SS^{-1} = \mathbb{1}_4 + O(\omega^2)$$

So, writing the condition on S in form of params & generators

$$\left(\frac{i}{4}\omega_{\mu\theta}\beta^{\mu\theta} + \mathbb{1}\right)\gamma^\nu\left(-\frac{i}{4}\omega_{\mu\theta}\beta^{\mu\theta} + \mathbb{1}\right)$$

$$\Rightarrow \frac{i}{4}\omega_{\mu\theta}[\beta^{\mu\theta}, \gamma^\nu] + \cancel{\gamma^\nu} = \cancel{\gamma^\nu} + \omega^\nu{}_\mu \gamma^\mu$$

$$\frac{i}{4}\cancel{\omega_{\mu\theta}}[\beta^{\mu\theta},\gamma^\nu] = \omega^\nu{}_\mu\,\gamma^\mu$$
$$= \eta^{\nu\theta}\omega_{\theta\mu}\gamma^\mu = -\eta^{\nu\theta}\omega_{\mu\theta}\gamma^\mu$$
$$\overset{?}{=} -\frac{\omega_{\mu\theta}}{2}(\eta^{\nu\theta}\gamma^\mu - \eta^{\nu\mu}\gamma^\theta)$$
$$= -\frac{\cancel{\omega_{\mu\theta}}}{2}\eta^{\nu[\theta}\gamma^{\mu]}$$

→ why anti-symm. this?

$$\Rightarrow \boxed{[\beta^{\mu\theta},\gamma^\nu] = +2i\,\eta^{\nu[\theta}\gamma^{\mu]}}$$

Let's now see if we can guess any sol^n s

$$\sigma^{\mu\nu} \equiv \frac{i}{2}[\gamma^\mu,\gamma^\nu] \longrightarrow \sigma^{ij} = i\gamma^i\gamma^j \quad i\neq j$$

So let's see if $\beta^{\mu\theta} \overset{?}{=} (\text{const.})\,\sigma^{\mu\theta} \quad \mu\neq\theta$

$\overset{?}{=} (\text{const.})\; i\gamma^\mu\gamma^\theta \quad \mu\neq\theta$

Let's then check -

$[A,BC] = \{A,B\}C - B\{C,A\}$

$$[\gamma^\mu\gamma^\theta,\gamma^\nu] = -[\gamma^\nu,\gamma^\mu\gamma^\theta]$$
$$= -(\{\gamma^\nu,\gamma^\mu\}\gamma^\theta - \gamma^\mu\{\gamma^\theta,\gamma^\nu\})$$
$$= -2(\eta^{\nu\mu}\gamma^\theta - \gamma^\mu\eta^{\theta\nu}) = 2(\eta^{\nu\theta}\gamma^\mu - \eta^{\nu\mu}\gamma^\theta)$$
$$= 2\eta^{\nu[\theta}\gamma^{\mu]}$$

$$\Rightarrow [\sigma^{\mu\nu},\gamma^\nu] = 2i\,\eta^{\nu[\theta}\gamma^{\mu]} \Rightarrow \boxed{\begin{aligned}\beta^{\mu\theta} &= \sigma^{\mu\theta}\\ &= i\gamma^\mu\gamma^\theta\end{aligned}} \quad \underline{\underline{\mu\neq\nu}}$$

Therefore,

$$\boxed{S = \exp\left(\frac{i}{4}\omega_{\mu\nu}\sigma^{\mu\nu}\right)}$$

where $\sigma^{\mu\nu} = \frac{i}{2}[\gamma^\mu,\gamma^\nu]$

Very similarly

$$S^{-1} = \exp\left(-\frac{i}{4}\omega_{\mu\nu}\sigma^{\mu\nu}\right) = \exp\left(-\frac{i}{4}\omega.\sigma\right)$$

If we define $\mathcal{S}^{\mu\nu} = \frac{1}{2}\sigma^{\mu\nu}$

$$\Rightarrow \boxed{\begin{aligned} S &= e^{\frac{i}{2}\omega.\mathcal{S}} \\ \Lambda &= e^{\frac{i}{2}\omega.L} \end{aligned}}$$

The commutation relation are given by

$$[L^{\mu\nu}, L^{\lambda\theta}] = -i\,\eta^{\{\lambda[\mu}L^{\nu]\theta\}} \qquad \text{vector rep.}$$

$$[\mathcal{S}^{\mu\nu}, \mathcal{S}^{\lambda\theta}] = -i\,\eta^{\{\lambda[\mu}\mathcal{S}^{\nu]\theta\}} \qquad \text{spinor rep.}$$

Check by putting $\mathcal{S}^{\mu\nu} = \frac{1}{2}\sigma^{\mu\nu} = \frac{i}{4}[\gamma^\mu, \gamma^\nu]$

$$\delta x^\mu = (\underbrace{\delta^\mu{}_\nu - \omega^\mu{}_\nu + \dots}_{\text{infinitesimal } \Lambda^\mu{}_\nu})\, x^\nu$$

Therefore $\psi'(x') = \psi'(x+\delta x) = \psi'(x^\mu - \omega^\mu{}_\nu x^\nu)$

$$= \psi'(x^\mu) - \omega^\mu{}_\nu x^\nu \partial_\mu \psi + O(\omega^2)$$

$$\equiv S\psi(x) = \left(\mathbb{1} + \frac{i}{4}\omega_{\mu\nu}\sigma^{\mu\nu}\right)\psi$$

$$\Rightarrow \quad \delta\psi \equiv \psi'(x) - \psi(x) = -\frac{i}{2}(\omega_{\mu\nu}J^{\mu\nu})\psi(x) + O(\omega^2)$$

where $J^{\mu\nu} = \underbrace{i\,x^{[\mu}\partial^{\nu]}}_{\text{ORBITAL GENERATOR}} + \underbrace{\frac{1}{2}\sigma^{\mu\nu}}_{\mathcal{S}^{\mu\nu}\ \text{SPIN GENERATOR}}$

since $\psi'(x') \simeq \psi'(x) + \delta x^\mu \partial_\mu \psi + \dots = \psi'(x) - \omega^\mu{}_\nu x^\nu \partial_\mu \psi$

$$\stackrel{!}{=} \psi(x) + \frac{i}{4}\omega_{\mu\nu}\sigma^{\mu\nu}\psi$$

So while discussing unitary rotations, we remember that for

$\psi' = U\psi$ we have $U = \exp(i\theta \cdot T)$ $\quad T_A^{\dagger} = T_A$

$U^{\dagger} = \exp(-i\theta \cdot T) = U^{-1}$

$\Rightarrow \psi'^{\dagger} = \psi^{\dagger} U^{\dagger} = \psi^{\dagger} U^{-1}$

which also ensured $\psi^{\dagger} \chi = \psi'^{\dagger} \chi^{\dagger}$

However, S doesn't share this property.

$$S = \exp\left(\frac{i}{4} \omega \cdot \sigma\right) \Rightarrow S^{\dagger} = \exp\left(-\frac{i}{4} \omega \cdot \sigma^{\dagger}\right)$$

$$S^{\dagger} = \exp\left(-\frac{i}{4}\left(\omega_{ij} \sigma^{\dagger ij} + 2\omega_{0i} \sigma^{\dagger 0i}\right)\right)$$

If we recall $\sigma^{0i} = i\gamma^0\gamma^i$ where $\gamma^{\mu\dagger} = \gamma^0 \gamma^{\mu} \gamma^0$

$\Rightarrow \gamma^{0\dagger} = \gamma^0$ and $\gamma^{i\dagger} = -\gamma^i$

$$\Rightarrow (\sigma^{0i})^{\dagger} = -i(-\gamma^i)(\gamma^0) = i\gamma^i\gamma^0 = -i\gamma^0\gamma^i$$
$$= -\sigma^{0i}$$

Similarly $\sigma^{\dagger ij} = \sigma^{ij}$

$$\text{So, } S^{\dagger} = \exp -\frac{i}{2}\left(\omega_{ij}\sigma^{ij} - 2\omega_{0i}\sigma^{0i}\right)$$

However, notice that

$$\underline{\sigma^{ij}\gamma^0 = \gamma^0\sigma^{ij}}$$

because $\gamma^i\gamma^j\gamma^0 = \gamma^0\gamma^i\gamma^j = (-1)^2\gamma^i\gamma^j\gamma^0$

whereas $\underline{\sigma^{0i}\gamma^0 = -\gamma^0\sigma^{0i}}$

Therefore,

$$S^\dagger \gamma^0 = \exp\left[\frac{-i}{4}\left(\underset{\substack{\uparrow\\ \text{Commutes}\\ \text{with } \gamma^0}}{\omega_{ij}\sigma^{ij}} - 2\omega_{0i}\underset{\substack{\uparrow\\ \text{Anti}\\ \text{commutes with } \gamma^0}}{\sigma^{0i}}\right)\right]\gamma^0$$

$$\Rightarrow S^\dagger \gamma^0 = \gamma^0 \exp\left[-\frac{i}{4}\left(\omega_{ij}\sigma^{ij} + 2\omega_{0i}\sigma^{0i}\right)\right] = \underline{\underline{\gamma^0 S^{-1}}}$$

So; if we define an object $\bar{\psi}$ where

$$\bar{\psi} = \psi^\dagger \gamma^0$$

then $\bar{\psi}'$ can be calculated as-

$$\bar{\psi}' = (S\psi)^\dagger \gamma^0 = \psi^\dagger S^\dagger \gamma^0 = \psi^\dagger \gamma^0 S^{-1} = \bar{\psi} S^{-1}$$

where we used $\underline{\underline{S^\dagger \gamma^0 = \gamma^0 S^{-1}}}$

Therefore, we have the nice property of transf.

$$\underline{\bar{\psi}'\chi' = \bar{\psi}\chi}$$

$$\bar{\psi}' = \bar{\psi} S^{-1}$$
$$\chi' = S\chi$$

$\bar{\psi}$ now transforms as $\boxed{\bar{\psi}' = \bar{\psi} S^{-1}}$

Lecture-10

(23-09)

Under a L.T., the Dirac wave f^n transforms as -

$$\psi'(x') = S\,\psi(x)$$

where $S = \exp\left(\frac{i}{4}\omega_{\mu\nu}\sigma^{\mu\nu}\right)$

$$\sigma^{\mu\nu} = \frac{i}{2}[\gamma^\mu, \gamma^\nu] \begin{cases} \to \sigma^{ij} & \text{Rotations} \\ \to \sigma^{0i} & \text{boosts.} \end{cases}$$

The total generator $J^{\mu\nu}$ has an orbital & a spin part

$$\underset{\text{(spinor)}}{J^{\mu\nu}} = \underset{\substack{\uparrow \\ i x^{[\mu}\partial^{\nu]}}}{L^{\mu\nu}} + \frac{\sigma^{\mu\nu}}{2}$$

Also, we saw that the spinor transformation S is s.t. $S^\dagger \neq S^{-1}$ but

$$S^\dagger \gamma^0 = \gamma^0 S^{-1}$$

$\Rightarrow$ for a Dirac wave f^n $\psi' = S\psi \rightarrow \psi'^\dagger = \psi^\dagger S^\dagger$

Then defining $\bar{\psi} = \psi^\dagger \gamma^0 = (\psi_1^* \; \psi_2^* \; \psi_3^* \; \psi_4^*)\underset{\searrow 4\times4 \text{ matrix}}{\gamma^0}$ $\longrightarrow$ row object.

Then $\bar{\psi}' = (\psi^\dagger\gamma^0)' = \psi^\dagger S^\dagger \gamma^0 = \psi^\dagger \gamma^0 S^{-1} = \bar{\psi} S^{-1}$

So, given $\bar{\psi} = \psi^\dagger\gamma^0 \Rightarrow \boxed{\bar{\psi}' = \bar{\psi} S^{-1}}$

So, $\underline{\bar{\psi}'(x')\,\psi'(x') = \bar{\psi}\psi}$ (scalar density)

(but $\psi^{\dagger\prime}\psi' \neq \psi^\dagger\psi$)

In fact, we can form an object

$$\bar{\psi}\gamma^\mu\psi = \psi^\dagger\gamma^0\gamma^\mu\psi = \begin{cases} \psi^\dagger\psi & \text{for } \mu=0 \\ \psi^\dagger\gamma^0\gamma^i\psi & \text{for } \mu\neq 0 \end{cases}$$

Defining, $j^\mu = \bar{\psi}\gamma^\mu\psi$, then it actually transforms like a Lorentz vector. $\psi^\dagger\psi$ actually comes out to be the $\mu=0$ comp. of this current.

Coming back to Dirac eq^n

$$(i\gamma^\mu\overrightarrow{\partial_\mu} - m)\psi = 0 \xrightarrow[\text{Conjugate}]{\text{Herm.}} \psi^\dagger(-i\gamma^{\mu\dagger}\overleftarrow{\partial_\mu} - m) = 0$$

$\gamma^{\mu\dagger} = \gamma^0\gamma^\mu\gamma^0$

Do γ^0 & $\overleftarrow{\partial_\mu}$ commute?

$$\left[\psi^\dagger(i\gamma^0\gamma^\mu\gamma^0\overleftarrow{\partial_\mu} - m) = 0\right] \times \gamma^0$$

$$\underbrace{\psi^\dagger\gamma^0}_{\bar{\psi}}(-i\gamma^\mu\overleftarrow{\partial_\mu} - m\gamma^0) = 0 \Rightarrow \bar{\psi}(-i\gamma^\mu\overleftarrow{\partial_\mu} - m) = 0$$

So, we have $\bar{\psi}(i\overrightarrow{\not\partial} - m)\psi = 0$ and $\bar{\psi}(i\overleftarrow{\not\partial} + m)\psi = 0$

$$\Rightarrow i\bar{\psi}\gamma^\mu\partial_\mu\psi + i\partial_\mu\bar{\psi}\gamma^\mu\psi = 0$$

$$\Rightarrow i\partial_\mu(\bar{\psi}\gamma^\mu\psi) = 0 \Rightarrow \underline{j^\mu = \bar{\psi}\gamma^\mu\psi \text{ is a conserved current.}}$$

$$\boxed{\partial_\mu j^\mu = 0} = \partial_0 j^0 + \vec{\nabla}\cdot\vec{j}$$

$$j^0 = \psi^\dagger\psi \qquad j^i = \bar{\psi}\gamma^i\psi$$

Let's now <u>prove</u> that j^μ is a Lorentz vector.

$j^\mu \equiv \bar{\psi}\gamma^\mu\psi$, then the transformed version is –

$$j^{\mu\prime}(x') = \bar{\psi}'(x')\gamma^\mu\psi'(x') = \bar{\psi}_{(x)}S^{-1}\gamma^\mu S\psi_{(x)}$$

And we remember that $S^{-1}\gamma^\mu S = \Lambda^\mu{}_\nu\, \gamma^\nu$

$$\Rightarrow \quad j^{\mu\prime}(x') = \bar{\psi}(x)\, \Lambda^\mu{}_\nu\, \gamma^\nu \psi(x)$$
$$= \underbrace{\Lambda^\mu{}_\nu\, \bar{\psi}(x)}\gamma^\nu \psi(x) = \Lambda^\mu{}_\nu\, j^\nu(x)$$

how did Λ & $\bar{\psi}$ commute over here? because $\Lambda^\mu{}_\nu$ is a number.

$$\therefore \boxed{j^{\mu\prime}_{\text{Dirac}}(x') = \Lambda^\mu{}_\nu\, j^\mu_{\text{Dirac}}(x)}$$

- Free Dirac particle solution

Consider the trial sol^n $\psi(x) \sim e^{-iEt}\, e^{i\vec{p}\cdot\vec{x}} = e^{-ip_\mu x^\mu} \equiv e^{-i\underline{p}\cdot\underline{x}}$

$$p^\mu x_\mu = Et - \vec{p}\cdot\vec{x}$$

$$H_D = \vec{\alpha}\cdot\vec{p} + \beta m \qquad \alpha^i = \gamma^0\gamma^i, \quad \beta = \gamma^0$$

So, $(H_D)^2 \stackrel{!}{=} \vec{p}^{\,2} + m^2 = E^2$

$$E = \pm\sqrt{p^2 + m^2}$$

Consider the situation in the rest frame. $(\vec{p}=0)$

$$i\partial_t \psi = H_D \psi = \gamma^0 m \psi$$

If we consider the sol^n $\psi \sim e^{-iEt} \Rightarrow (E - \gamma^0 m)\psi = 0$

Since $(\gamma^0)^2 = \mathbb{1}_4$, the eigenvalues of γ^0 are supposed to be ± 1.

$\Rightarrow E = \pm m$ are the energy eigenvalues in the rest frame.

for $\vec{p} \neq 0$ $\quad E = \pm\sqrt{m^2 + p^2}$ (both -ve & +ve energy sol^{n}s.)

ψ here is a 4-component wave f^n $\begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \end{pmatrix}$

Then if we use an ansatz $\psi \sim e^{-ip\cdot x}\, W$ ← a 4-component object.

$$(\underbrace{\gamma^\mu p_\mu - m\mathbb{1}_4}_{4\times 4 \text{ matrix}})\, e^{-ip_\mu x^\mu} \underbrace{W}_{4\text{-component}} = 0$$

$$\Rightarrow \Big(\gamma^0 \underbrace{E}_{=\pm\sqrt{p^2+m^2}} + \gamma^i p_i - m\mathbb{1}_4\Big)\begin{pmatrix} W_1 \\ W_2 \\ W_3 \\ W_4 \end{pmatrix} = 0 \longrightarrow$$ 4 linearly independent sol^{n}s. (2 +ve E and 2 -ve E)

We expect 4 solutions : 2 +ve energy and 2 -ve energy solutions for a given $\vec{p}$, which will span the 4-D space.

+ve energy $\psi_{(\vec{p},\, s,\, +)}(x) \equiv N\, \underbrace{u_s(p)}_{\text{momentum dependent}}\, e^{-ip\cdot x} \longrightarrow \sim e^{-iE_p t}$, $s = 1,2$

N: normalization

$$p_0 \equiv +\sqrt{p^2 + M^2} = E_p > 0$$

-ve energy $\psi_{(\vec{p},\, s,\, -)}(x) \equiv \tilde{N}\, V_s(p)\, e^{+ip\cdot x} \longrightarrow \sim e^{-i(-E_p)t}$, $s = 1,2$

Now note that $\psi_+ \sim e^{+i\vec{p}\cdot\vec{x}}$ and $\hat{\vec{p}} = -i\vec{\nabla} \Rightarrow$ momentum > 0

$$\hat{\vec{p}}\,\psi_+(\vec{p}) = \vec{p}\,\psi_+(\vec{p})$$

but for $\psi_- \sim e^{-i\vec{p}\cdot\vec{x}} \Rightarrow \hat{\vec{p}}\,\psi_-(\vec{p}) = -\vec{p}\,\psi_-(\vec{p})$

So, $U_s(\vec{p})$ and $V_s(-\vec{p})$ will span the spinor space at constant $\vec{p}$ (i.e. the space of Dirac wavef^{n}s for particles of a given 3-momentum $\vec{p}$)

$$(\gamma^\mu \hat{p}_\mu - m)\begin{pmatrix} e^{-ip\cdot x}\, U_s(\vec{p}) \\ e^{+ip\cdot x}\, V_s(\vec{p}) \end{pmatrix} = 0.$$

$$\therefore \left.\begin{aligned} (\not{p} - m)\, U_s(\vec{p}) &= 0 \\ (\not{p} + m)\, V_s(\vec{p}) &= 0 \end{aligned}\right\}\ s = 1,2$$

We can also find the corresponding eqⁿs for $\overline{U_s}$ and $\overline{V_s}$

$$U_s^\dagger(\not{p}^\dagger - m) = 0 \qquad \text{where } \not{p}^\dagger = \gamma^0 \not{p} \gamma^0$$

$$U_s^\dagger(\gamma^0 \not{p} \gamma^0 - \gamma^0 m \gamma^0) = 0$$

$$U_s^\dagger \gamma^0 (\not{p} - m)\gamma^0 = 0 \Rightarrow U_s^\dagger \gamma^0 (\not{p} - m) = 0$$

$\therefore$ $\underline{\overline{U_s}(\not{p} - m) = 0}$ and similarly, $\underline{\overline{V_s}(\not{p} + m) = 0}$

Normalization of wavefunctions.

$$U_r^\dagger(\vec{p})\, U_s(\vec{p}) = 2E_p\, \delta_{rs} = V_r^\dagger(\vec{p})\, V_s(\vec{p})$$

$$V_r^\dagger(\vec{p})\, U_s(-\vec{p}) = 0 = U_r^\dagger(\vec{p})\, V_s(-\vec{p})$$

$U_r(\vec{p})$ and $V_r(-\vec{p})$ correspond to the same eigenvalue of $\hat{\vec{p}}$, but different energies ($+E_p$ and $-E_p$) $\therefore$ they must be orthogonal b/c eigenvectors with diff. eigenvalues are orthogonal, always.

Explicit form of the Dirac Pauli Representation.

$$\gamma^0 \equiv \begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix} \qquad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix} \qquad \gamma^5 = \begin{pmatrix} 0 & \mathbb{1}_2 \\ \mathbb{1}_2 & 0 \end{pmatrix}$$

$$\sigma^{0i} = i\begin{pmatrix} 0 & \sigma^i \\ \sigma^i & 0 \end{pmatrix}$$

$$(\not{p} - m) = p_0 \gamma^0 + p_i \gamma^i - m \qquad \sigma^{ij} = \epsilon^{ijk}\begin{pmatrix} \sigma^k & 0 \\ 0 & \sigma^k \end{pmatrix}$$

$$= E_p \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} - \begin{pmatrix} 0 & \vec{\sigma}\cdot\vec{p} \\ -\vec{\sigma}\cdot\vec{p} & 0 \end{pmatrix} - m\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

We don't know what E_p is right now.

$$= \begin{pmatrix} E_p - m & -\vec{p}\cdot\vec{\sigma} \\ +\vec{p}\cdot\vec{\sigma} & -(E_p + m) \end{pmatrix}$$

Writing $U = \begin{pmatrix} \phi_t \\ \phi_b \end{pmatrix} \rightarrow$ 2 components.

$$\begin{pmatrix} E_p - m & -\vec{\sigma}\cdot\vec{p} \\ \vec{\sigma}\cdot\vec{p} & -(E_p+m) \end{pmatrix} \begin{pmatrix} \phi_t \\ \phi_b \end{pmatrix} = 0$$ since $(\not{p} - m)\, u_s = 0$

$$\left.\begin{aligned} (E_p - m)\phi_t &= \vec{\sigma}\cdot\vec{p}\, \phi_b \\ \vec{\sigma}\cdot\vec{p}\, \phi_t &= (E_p+m)\phi_b \end{aligned}\right\} \Rightarrow (E_p - m)\phi_t = \frac{(\vec{\sigma}\cdot\vec{p})^2}{(E_p+m)}\phi_t$$

$\left[E_p^2 - m^2 - (\vec{\sigma}\cdot\vec{p})^2\right]\phi_t = 0$ & similarly $\left[E_p^2 - m^2 - (\vec{\sigma}\cdot\vec{p})^2\right]\phi_b = 0$

$(\vec{\sigma}\cdot\vec{p})^2 = \vec{p}^{\,2}$ since $\sigma_i \sigma_j = \delta_{ij} + i\epsilon_{ijk}\sigma_k$

> isn't this = 0?

So, $(E_p^2 - m^2 - \vec{p}^{\,2})\begin{pmatrix} \phi_t \\ \phi_b \end{pmatrix} = 0$

Let's say $\phi_t = \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \chi^+$ or we can take $\chi^- = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$$\Rightarrow u_\pm = N_\pm \begin{pmatrix} \phi_t \\ \phi_b \end{pmatrix} = N_\pm \begin{pmatrix} \chi^\pm \\ \frac{(\vec{\sigma}\cdot\vec{p})}{E_p+m}\chi^\pm \end{pmatrix}$$

So, $$u_\pm^\dagger u_\pm = |N_\pm|^2 \left(\chi_\pm^\dagger,\ \chi_\pm^\dagger \frac{(\vec{\sigma}\cdot\vec{p})}{(E_p+m)}\right) \begin{pmatrix} \chi_\pm \\ \frac{(\vec{\sigma}\cdot\vec{p})}{(E_p+m)}\chi_\pm \end{pmatrix}$$

$$= |N_\pm|^2 \cdot \left(\chi_\pm^\dagger \chi_\pm + \chi_\pm^\dagger \frac{\vec{p}^{\,2}}{(E_p+m)^2}\chi_\pm\right)$$

$$= |N_\pm|^2 \left(1 + \frac{\vec{p}^{\,2}}{(E_p+m)^2}\right)$$ (on subbing in $\chi_\pm$)

$$= |N_\pm|^2 \frac{E_p^2 + m^2 + 2mE_p + \vec{p}^{\,2}}{(E_p+m)^2} = \frac{2E_p\,\cancel{(E_p+m)}}{(E_p+m)^{\cancel{2}}}|N_\pm|^2$$

$$\Rightarrow \boxed{\text{If } |N|^2 = (E_p+m) \Rightarrow u_\pm^\dagger u_\pm = 2E_p}$$

So, we write the soln as

$$u_\pm(\vec{p}) = \sqrt{E_p+m}\begin{pmatrix} \chi_\pm \\ \frac{\vec{\sigma}\cdot\vec{p}}{(E_p+m)}\chi_\pm \end{pmatrix}$$

Similarly, doing the same exercise for $v_\pm(\vec{p})$, you get -

$$v_\pm(\vec{p}) = \sqrt{E_p+m}\begin{pmatrix} \frac{\vec{\sigma}\cdot\vec{p}}{E_p+m}\chi_\pm \\ \chi_\pm \end{pmatrix}$$

We can now check explicitly if $u_s^\dagger(\vec{p})\, v_{s'}(-\vec{p}) \overset{?}{=} 0$

$$u_s(\vec{p})\, v_{s'}(-\vec{p}) = (E_p+m)\left(\chi_s^\dagger, \chi_s^\dagger \frac{\vec{\sigma}\cdot\vec{p}}{E_p+m}\right)\begin{pmatrix} -\frac{\vec{\sigma}\cdot\vec{p}}{E_p+m}\chi_{s'} \\ \chi_{s'} \end{pmatrix}$$

$$= (E_p+m)\,\chi_s^\dagger \underbrace{\left(-\frac{\vec{\sigma}\cdot\vec{p}}{E_p+m} + \frac{\vec{\sigma}\cdot\vec{p}}{E_p+m}\right)}_{\overset{||}{0}}\chi_s \equiv 0$$

If we check for the case of $\vec{p}=0$ solutions

$$u_\pm(\vec{p}=0) = \sqrt{2m}\begin{pmatrix} \chi_\pm \\ 0 \end{pmatrix} \quad \text{and} \quad v_\pm(\vec{p}=0) = \sqrt{2m}\begin{pmatrix} 0 \\ \chi_\pm \end{pmatrix}$$

then $u_\pm(\vec{p})$ and $v_\pm(\vec{p})$ can be obtained by Lorentz boosts applied on u & v in rest frame.

$$S_{\text{boost}} \equiv \exp\left(\frac{i}{4}\omega_{\mu\nu}\sigma^{\mu\nu}\right) = \exp\left(\frac{i}{2}\omega_{0i}\,\sigma^{0i}\right)$$

$i\gamma^0\gamma^i$

$\vec{p}$ Lorentz boost

$$= \exp\left(-\frac{1}{2}\gamma^0\,\vec{\omega}\cdot\vec{\gamma}\right)$$

Exercise:

$$\left.\begin{aligned} S(\vec{\omega})\, u_\pm(\vec{p}=0) &= ?? \\ S(\vec{\omega})\, v_\pm(\vec{p}=0) &= ?? \end{aligned}\right\}$$ free particle solns with $\vec{p}\neq 0$

Non-relativistic limit $\quad |\vec{p}| \ll m \quad \left(E_p \simeq m + \frac{p^2}{2m} + \cdots\right)$

then $\dfrac{\vec{\sigma}\cdot\vec{p}}{E_p+m} = \dfrac{\vec{\sigma}\cdot\vec{p}}{2m + \frac{p^2}{2m}} \approx \dfrac{\vec{\sigma}\cdot\vec{p}}{2m}$

velocity

$$\left|\frac{\vec{\sigma}\cdot\vec{p}}{E+m}\right| = \left|\frac{\vec{\sigma}\cdot\vec{V}}{2}\right| = \frac{v}{2} \ll 1$$ in the non-relativistic regime.

For $E>0$ & $|\vec{p}| \ll m$, $u_\pm \approx \begin{pmatrix} \chi_\pm \\ 0_2 \end{pmatrix}$ and $v_\pm \approx \begin{pmatrix} 0_2 \\ \chi_\pm \end{pmatrix}$

Exercises: Do the same thing Chiral representation.

$$\gamma^0 = \begin{pmatrix} 0 & \mathbb{1}_2 \\ \mathbb{1}_2 & 0 \end{pmatrix} \qquad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}$$

$$\gamma^5 = \begin{pmatrix} -\mathbb{1}_2 & 0 \\ 0 & \mathbb{1}_2 \end{pmatrix}$$ and $\sigma^{0i} = i\begin{pmatrix} \sigma^i & 0 \\ 0 & \sigma^i \end{pmatrix}$

$$\sigma^{ij} = \epsilon_{ijk}\begin{pmatrix} \sigma^k & 0 \\ 0 & \sigma^k \end{pmatrix}$$

$$P_\pm = \frac{1 \mp \gamma^5}{2} = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & 0 \end{pmatrix} \text{ for } -$$

$$= \begin{pmatrix} 0 & 0 \\ 0 & \mathbb{1}_2 \end{pmatrix} \text{ for } +$$

Find u_s & v_s in Chiral representation.

Another standard representation is the Majorana rep^n.

Lecture - 11
(24/09)

Free particle solutions to Dirac Eq^n: $\quad e^{-iE_p t}\, e^{i\vec{p}\cdot\vec{x}}$

$\psi_+ \quad E>0 \qquad u_s(\vec{p})\, e^{-ip\cdot x}$

$\psi_- \quad E<0 \qquad v_s(\vec{p})\, e^{ip\cdot x}$

$$p_\mu = (E_p, \vec{p}), \qquad E_p = \sqrt{\vec{p}^2+m^2}$$

$$\uparrow e^{-i(-E_p t)}\, e^{-i\vec{p}\cdot\vec{x}}$$

$$u_s(\vec{p}) = \sqrt{E_p+m}\begin{pmatrix} \chi_\pm \\ \dfrac{\vec{\sigma}\cdot\vec{p}}{E_p+m}\chi_\pm \end{pmatrix} \qquad E>0 \text{ solution.}$$

$(\hat{p}\psi_+ = \vec{p}\psi_+)$

$$v_s(\vec{p}) = \sqrt{E_p+m}\begin{pmatrix} \dfrac{\vec{\sigma}\cdot\vec{p}}{E_p+m}\chi_\pm \\ \chi_\pm \end{pmatrix} \qquad E<0 \text{ solution.}$$

$(\hat{p}\psi_- = -\vec{p}\psi_-)$

In NRQM, we have the total angular momentum $\vec{J}$ as

$$\vec{J} = \vec{L} + \underbrace{\vec{S}}_{\frac{\vec{\sigma}}{2}}$$

and we find a very similar expression for generators in RQM.

$$J_k = \frac{1}{2}\epsilon_{kij}J^{ij} = L_k + S_k \qquad \text{where } J^{\mu\nu} = L^{\mu\nu} + S^{\mu\nu}, \quad S^{\mu\nu} = \frac{\sigma^{\mu\nu}}{2}$$

$$S_k \rightarrow \frac{1}{4}\epsilon_{kij}\sigma^{ij} = \frac{1}{2}\begin{pmatrix} \sigma_k & 0 \\ 0 & \sigma_k \end{pmatrix}$$

In the non-relativistic limit, the $u_s(\vec{p})$ wavef^{n}s become the typical spin $\frac{1}{2}$ wavefns ($|\uparrow\rangle$ and $|\downarrow\rangle$) and S_k becomes the spin operator (?)

Somehow. How even?

Where did spin even enter the picture?

What about negative energy sol^{n}s though?
We'll come to this after a while.

Bilinear Covariants under $O(1,3) \to SO(1,3)$, Parity.

$$\psi'(x') = S\psi(x)$$

where $S = \exp\left(\frac{i}{4}\omega_{\mu\nu}\sigma^{\mu\nu}\right)$ where $\sigma^{\mu\nu} = \frac{i}{2}[\gamma^\mu, \gamma^\nu]$

Now $S^\dagger \neq S^{-1}$, however $S^\dagger\gamma^0 = \gamma^0 S^{-1}$

$$\Rightarrow \bar{\psi}'(x') = (S\psi)^\dagger\gamma^0 = \psi^\dagger S^\dagger \gamma^0 = \psi^\dagger\gamma^0 S^{-1} \qquad \bar{\psi} \equiv \psi^\dagger\gamma^0$$
$$= \bar{\psi} S^{-1}$$

Another property from which we derived S was $S^{-1}\gamma^\mu S = \Lambda^\mu{}_\nu \gamma^\nu$

We'll now discuss the formalism for improper Lorentz transformation, and in particular the parity transf

Parity transform: $(x^0, \vec{x}) \longrightarrow (x^0, -\vec{x})$

This can be written as $x'^\mu = (\Lambda_P)^\mu{}_\nu x^\nu$

$$\Rightarrow (\Lambda_P)^\mu{}_\nu = \mathrm{diag}(1, -\mathbb{1}_3) = \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix}$$

$\det \Lambda_P = -1$ (improper L.T.) $\in O(1,3)$ but $\notin SO(1,3)$

Now we also want to find the parity transformation for spinors.

$$\psi'(x') = S_P\psi(x) \qquad S_P = ??$$

Since there is no reason for Dirac eq^n to not transform covariantly under parity, we continue down that route -

$$\left(i(\gamma^0\partial_0' + \vec{\gamma}\cdot\vec{\partial}') - m\right) S_P\psi(x) = 0$$

But the Dirac eq^n in original frame is

$$\left[i(\gamma^0\partial_0 + \vec{\gamma}\cdot\vec{\partial}) - m\right]\psi = 0$$

Now $\partial_0' = \partial_0$ & $\vec{\partial}' = -\vec{\partial}$ since $x^{0\prime} = x^0$ but $\vec{x}' = -\vec{x}$

$\Rightarrow$ In the parity transformed frame

$$S_P^{-1} \times \left[\left(i(\gamma^0\partial_0 - \vec{\gamma}\cdot\vec{\partial}) - m\right) S_P \psi = 0\right]$$

$$\Rightarrow \left[i(S_P^{-1}\gamma^0 S_P \partial_0 - S_P^{-1}\vec{\gamma} S_P \cdot\vec{\partial}) - m\right]\psi = 0$$

$$\Rightarrow \boxed{S_P^{-1}\gamma^0 S_P = \gamma^0} \quad \text{and} \quad \boxed{S_P^{-1}\vec{\gamma} S_P = -\vec{\gamma}}$$

So, we can see that if-

$\boxed{S_P = \eta\gamma^0}$ and $\boxed{S_P^{-1} = \frac{1}{\eta}\gamma^0}$ where $|\eta| = 1$, with $\frac{1}{\eta} = \eta^*$

phase (to preserve norm)

it satisfies the above conditions.

Let's now look at the properties of other objects under the (proper L.T. + parity) transformations.

$$\bar{\psi}'\psi' = \bar{\psi} S_L^{-1} S_L \psi = \bar{\psi}\psi \qquad \text{(scalar)}$$

For parity $\leftrightarrow$, let's check this explicitly-

$$\psi' = S_P\psi = \eta\gamma^0\psi$$

$$\Rightarrow \psi'^\dagger = \psi^\dagger\gamma^{0\dagger}\eta^* \Rightarrow \psi'^\dagger\gamma^0 \equiv \bar{\psi}' = \underbrace{\psi^\dagger\gamma^0}_{\bar{\psi}}\eta^*\gamma^0 = \bar{\psi}\gamma^0\eta^*$$

$\Rightarrow$ Under parity

$$\bar{\psi}'\psi' = \bar{\psi}\gamma^0\underbrace{\eta^*\eta}_{1}\gamma^0\psi = \bar{\psi}\psi$$

(with $\gamma^0\gamma^0 = \mathbb{1}$)

$\therefore \bar{\psi}\psi$ is a proper scalar.

In the last lecture, we also said that $j^\mu = \bar{\psi}\gamma^\mu\psi$ is a Lorentz vector

because $j^{\mu\prime}(x') = \bar{\psi}'\gamma^\mu\psi' = \bar{\psi}(x)\underbrace{S^{-1}\gamma^\mu S}_{\Lambda^\mu{}_\nu\gamma^\nu}\psi = \Lambda^\mu{}_\nu\, S^{-1}\gamma^\nu S$

$= \Lambda^\mu{}_\nu\, j^\nu(x)$

Under parity,

$$\bar{\psi}'\gamma^\mu\psi' = \bar{\psi}\, S_P^{-1}\gamma^\mu S_P\psi = \bar{\psi}\gamma^0\overbrace{\eta^*\gamma^\mu\eta}^{1}\gamma^0\psi = \bar{\psi}\gamma^0\gamma^\mu\gamma^0\psi$$

now $\gamma^0\gamma^\mu\gamma^0 = \begin{Bmatrix} \gamma^0 & \mu=0 \\ -\gamma^i & \mu=i \end{Bmatrix} \Rightarrow \bar{\psi}'\gamma^\mu\psi' = \begin{Bmatrix} \bar{\psi}\gamma^0\psi & \mu=0 \\ -\bar{\psi}\gamma^i\psi & \mu=i \end{Bmatrix}$

$$\Rightarrow \boxed{(j^0, \vec{j}) \xrightarrow{\text{Parity}} (j^0, -\vec{j})}$$

$\bar{\psi}\gamma^\mu\psi$: Vector bilinear covariant.

Now out of the 16 independent γ-matrices, we had γ^5

$\gamma^5 \equiv i\gamma^0\gamma^1\gamma^2\gamma^3$ where $\{\gamma^5, \gamma^\mu\} = 0$

$= \frac{i}{4!}\epsilon_{\mu\nu\lambda\sigma}\gamma^\mu\gamma^\nu\gamma^\lambda\gamma^\sigma$

$\epsilon_{0123} = +1$ (T.A.S.)

$$\Rightarrow S^{-1}\gamma^5 S = \frac{i}{4!}\epsilon_{\mu\nu\lambda\sigma}\, S^{-1}\gamma^\mu \underbrace{}_{SS^{-1}}\gamma^\nu\underbrace{}_{SS^{-1}}\gamma^\lambda\underbrace{}_{SS^{-1}}\gamma^\sigma S$$

$$= \frac{i}{4!}\underbrace{\left(\epsilon_{\mu\nu\lambda\sigma}\Lambda^\mu{}_{\mu'}\Lambda^\nu{}_{\nu'}\Lambda^\lambda{}_{\lambda'}\Lambda^\sigma{}_{\sigma'}\right)}_{\equiv\ \epsilon_{\mu'\nu'\lambda'\sigma'}\det(\Lambda)}\gamma^{\mu'}\gamma^{\nu'}\gamma^{\lambda'}\gamma^{\sigma'}$$

$$= \frac{i}{4!}\epsilon_{\mu'\nu'\lambda'\sigma'}\gamma^{\mu'}\gamma^{\nu'}\gamma^{\lambda'}\gamma^{\sigma'}\det(\Lambda) = \det(\Lambda)\,\gamma^5$$

$$\Rightarrow \boxed{S^{-1}\gamma^5 S = \begin{Bmatrix} +\gamma^5 & \text{for } S_L \\ -\gamma^5 & \text{for } S_P \end{Bmatrix}}$$

$\therefore \quad \bar{\Psi}'(x')\gamma^5\psi'(x') = \begin{cases} +\bar{\Psi}\gamma^5\psi & \text{for } S_L \rightarrow \text{proper Lorentz} \\ -\bar{\Psi}\gamma^5\psi & \text{for } S_p \rightarrow \text{parity.} \end{cases}$

$\Rightarrow \bar{\Psi}\gamma^5\psi$ is a PSEUDO-SCALAR.

• Similarly, if we consider the transformation of the quantity $\bar{\Psi}\gamma^5\gamma^\mu\psi$

$$\bar{\Psi}'\gamma^5\gamma^\mu\psi' = \bar{\Psi}S^{-1}\gamma^5\underbrace{}_{SS^{-1}}\gamma^\mu S\psi = \bar{\Psi}(S^{-1}\gamma^5 S)\underbrace{(S^{-1}\gamma^\mu S)}_{\Lambda^\mu{}_\nu\gamma^\nu}\psi$$

$$= \Lambda^\mu{}_\nu\,\bar{\Psi}(S^{-1}\gamma^5 S)\gamma^\nu\psi$$

$$\bar{\Psi}(S^{-1}\gamma^5 S)\gamma^\nu\psi = \begin{Bmatrix} \bar{\Psi}\gamma^5\gamma^\nu\psi \\ -\bar{\Psi}\gamma^5\gamma^\nu\psi \end{Bmatrix} = \bar{\Psi}\gamma^5\gamma^\nu\psi\,\det(\Lambda)$$

If $\det\Lambda = +1 \longrightarrow$ vector law

If $\det\Lambda = -1 \longrightarrow$ extra $\ominus$ sign.

Axial vector.

Also called PSEUDO-VECTOR.

• Now, consider the object $\bar{\Psi}\sigma^{\mu\nu}\psi = \frac{i}{2}\bar{\Psi}[\gamma^\mu,\gamma^\nu]\psi = i\bar{\Psi}\gamma^\mu\gamma^\nu\psi$ for $\mu\neq\nu$

Therefore the transf. of this -

$$\bar{\Psi}'\sigma^{\mu\nu}\psi' = \bar{\Psi}S^{-1}\sigma^{\mu\nu}S\psi = i\bar{\Psi}\underbrace{S^{-1}\gamma^\mu S}_{\Lambda^\mu{}_{\mu'}\gamma^{\mu'}}\underbrace{S^{-1}\gamma^\nu S}_{\Lambda^\nu{}_{\nu'}\gamma^{\nu'}}\psi$$

$$= \Lambda^\mu{}_{\mu'}\Lambda^\nu{}_{\nu'}\,i\bar{\Psi}\gamma^\mu\gamma^\nu\psi = \Lambda^\mu{}_{\mu'}\Lambda^\nu{}_{\nu'}\,\bar{\Psi}\sigma^{\mu\nu}\psi$$

Let's look at the effect by parity on $\mu=0, \nu=i$

$$\bar{\Psi}'\gamma^0\gamma^i\psi' = (\bar{\Psi}\gamma^0\eta^*)(\gamma^0\gamma^i)(\eta\gamma^0\psi) = \bar{\Psi}\overbrace{\gamma^0\gamma^0}^{1}\gamma^i\gamma^0\psi$$

$$= -\bar{\Psi}\gamma^0\gamma^i\psi$$

But $\bar{\Psi}'\gamma^i\gamma^j\psi' = \bar{\Psi}\gamma^0\gamma^i\gamma^j\gamma^0\psi = \bar{\Psi}\gamma^i\gamma^j\psi$

So, $\bar{\Psi}'\gamma^0\gamma^i\psi' = -\bar{\Psi}\gamma^0\gamma^i\psi$

$\bar{\Psi}'\gamma^i\gamma^j\psi' = \bar{\Psi}\gamma^i\gamma^j\psi$

But we proved for a general case that $\bar{\Psi}'\sigma^{\mu\nu}\psi' = \bar{\Psi}\sigma^{\mu\nu}\psi$. Why this then? & What is this mixed $\pm$ signs? Is this a pseudo-tensor or what?

This gives us a total of 16 bilinear covariants -

$$\bar{\Psi}(x)\begin{Bmatrix} \mathbb{1} \\ \gamma^5 \\ \gamma^\mu \\ \gamma^5\gamma^\mu \\ \sigma^{\mu\nu} \end{Bmatrix}\psi(x)$$

→ scalar
→ vector
→ pseudoscalar
→ pseudovector
→ tensor

The totally anti-symm. tensor is a pseudotensor as well.

$$\epsilon'^{\mu\nu\lambda\sigma} = \Lambda^\mu{}_{\mu'}\Lambda^\nu{}_{\nu'}\Lambda^\lambda{}_{\lambda'}\Lambda^\sigma{}_{\sigma'}\,\epsilon^{\mu'\nu'\lambda'\sigma'} = \det(\Lambda)\,\epsilon^{\mu\nu\lambda\sigma}$$

invariant density of weight 1.

DIRAC SEA.

$$\begin{cases} \psi_{(s,\vec{p},+)}(x) = u_s(\vec{p})\,e^{-ip\cdot x} & E>0 \\ \psi_{(s,\vec{p},-)}(x) = v_s(\vec{p}')\,e^{+ip'\cdot x}\Big|_{p'=(E_p,-\vec{p})} & E<0 \end{cases}$$

These sol^{n}s span the Hilbert space at momentum $\vec{p}$

General sol^n $$\psi(\vec{x}) = \sum_s \int C_{\vec{p},s}\,u_{\vec{p},s}\,e^{-ip\cdot x}\,d^3\vec{p} + \sum_s \int \tilde{C}_{\vec{p},s}\,v_{\vec{p},s}\,e^{ip\cdot x}\,d^3\vec{p}$$

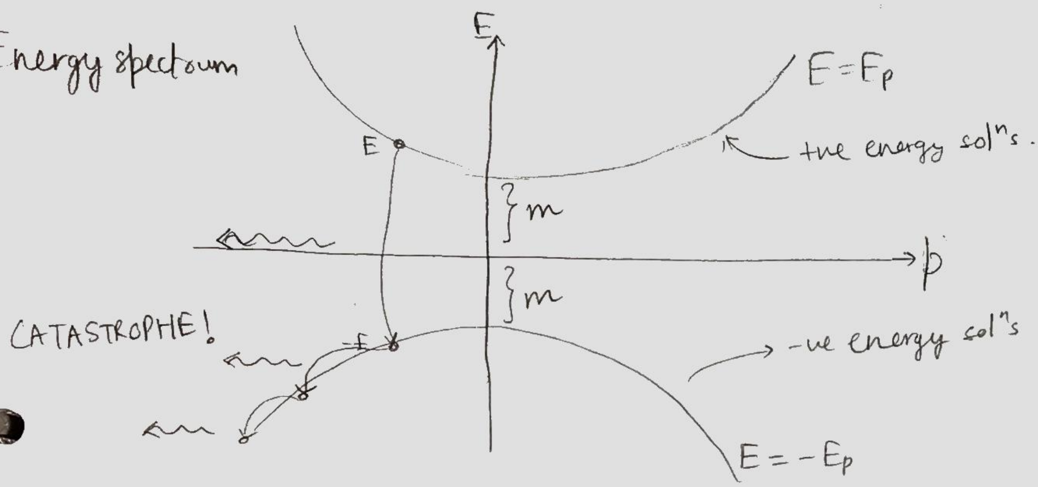
spin index

-ve energy states

how to interpret these?

Spin $\frac{1}{2}$ particles obey Fermi-Dirac statistics, and consequently the Pauli exclusion principle.

Energy spectrum



In case a particle falls down from some $+E \rightarrow -E$ & emits a photon, due to availability of lower & lower states it will keep going down & emit energy, which is not what's physically observed.

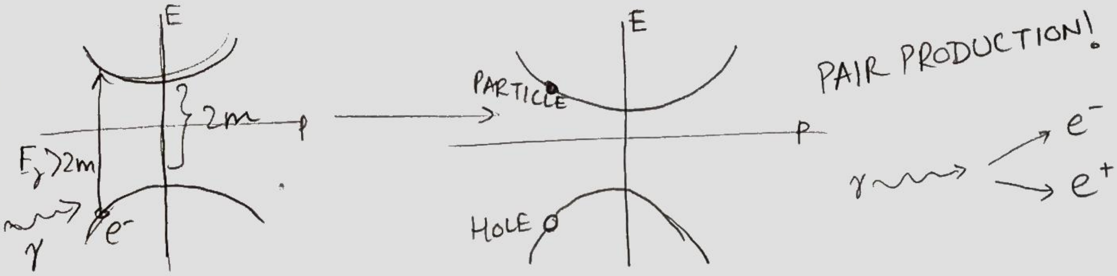
Therefore, Dirac postulated that all the -ve energy states are filled (i.e. have occupation # = 1) when there is no observable particle is present. (DIRAC VACUUM / DIRAC SEA OF FILLED FERMION STATES)

$\therefore$ particles can't fall into lower energy states.

Dirac sea $\Longrightarrow$ ∞ -ve energy sol^{n}s.

However, experiments detect differences / changes in energy. Therefore, the common ∞ background energy is unobservable.

However, we can observe the effects of the presence of the Dirac sea by interacting the vacuum with a photon of energy $>2m$.



$Q^{(e^-)}_{vacuum} \longrightarrow -\infty\,!$ also unobservable

So, the absence / emptying Dirac sea states behave as particles of positive energy and positive charge.
(creation of holes)

$(\vec{p}, E, Q = +e)$ where $E > 0$

Relativistic particle of mass m, $Q = +e$

$e^- \longleftrightarrow e^+$ (holes of Dirac sea)

ELECTRON POSITRON

$\Rightarrow$ Every particle has an anti-particle (which is a hole of Dirac sea) with opposite charge but all the other physical properties are same.

$$-E_p \xrightarrow{\gamma} E_p = \sqrt{E_p^2 + m^2}$$

Lecture-12

(28/09)

Projection operators.

Basic property: $P_{\hat{e}}\vec{V} = \vec{V}\cdot\hat{e}\,\hat{e}$

$P_{\hat{e}}^2 = P_{\hat{e}}$

and $P_{\hat{e}_1} P_{\hat{e}_2} = 0$ if $\hat{e}_1 \perp \hat{e}_2$

(1) ENERGY PROJECTION OPERATORS

Dirac Pauli Rep^n of Clifford Algebra γ^μ

$$\gamma^0 = \begin{pmatrix} \mathbb{1}_2 & 0 \\ 0 & -\mathbb{1}_2 \end{pmatrix} \qquad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}$$

In the last lecture, we derived the free particle spinors.

for $E>0$, $\quad u_s(\vec{p}) = \sqrt{E_p+m}\begin{pmatrix} \chi_s \\ \dfrac{\vec{\sigma}\cdot\vec{p}}{E_p+m}\chi_s \end{pmatrix}$

$s = \pm 1$

$\chi_s^\dagger \chi_{s'} = \delta_{ss'}$ (orthonormal basis of 2-comp. spinor)

$$\begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} = \Psi = \sum_{s=\pm1} C_s \chi_s \quad \text{where } C_s = \chi_s^\dagger \Psi$$

$$\Rightarrow \Psi = \sum_{s=\pm1} \chi_s \chi_s^\dagger \Psi \Rightarrow \sum_s \chi_s \chi_s^\dagger = \mathbb{1}_2$$

so χ_s is a complete basis.

for $E<0$ $\quad v_s(\vec{p}) = \sqrt{E_p+m}\begin{pmatrix} \dfrac{\vec{\sigma}\cdot\vec{p}}{E_p+m}\chi_s \\ \chi_s \end{pmatrix}$

We can now define

$$U_{\vec{p},s} = N e^{-iE_pt + i\vec{p}\cdot\vec{x}}\, u_s(\vec{p})$$

$$V_{\vec{p},s} = N' e^{+iE_pt - i\vec{p}\cdot\vec{x}}\, v_s(\vec{p})$$

eigenstates of $\hat{p} = -i\vec{\nabla}$ and $\hat{E} = i\partial_t$ with eigenvalues $+\vec{p}$ & E_p and $-\vec{p}$ & $-E_p$

(same as $\psi_\pm$)

Since $u_s(\vec{p}) = \sqrt{E_p+m} \begin{pmatrix} \chi_s \\ \frac{\vec{\sigma}\cdot\vec{p}}{E_p+m}\chi_s \end{pmatrix}$

$$\Rightarrow \bar{u}_s(\vec{p}) = \sqrt{E_p+m}\left(\chi_s^\dagger, \ \chi_s^\dagger \frac{(\vec{\sigma}\cdot\vec{p})}{E_p+m}\right)\underbrace{\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}}_{\gamma^0}$$

$$= \sqrt{E_p+m}\left(\chi_s^\dagger, \ -\chi_s^\dagger \frac{(\vec{\sigma}\cdot\vec{p})}{E_p+m}\right)$$

If we now compute the following outer product -

$$\sum_{s=\pm1} u_s \bar{u}_s = \sum_s (E_p+m)\begin{pmatrix} \chi_s \\ \frac{\vec{\sigma}\cdot\vec{p}}{E_p+m}\chi_s \end{pmatrix}\left(\chi_s^\dagger, -\chi_s^\dagger \frac{\vec{\sigma}\cdot\vec{p}}{E_p+m}\right)$$

$$= \sum_s \begin{pmatrix} \chi_s\chi_s^\dagger & -\chi_s\chi_s^\dagger \frac{\vec{\sigma}\cdot\vec{p}}{E_p+m} \\ \frac{\vec{\sigma}\cdot\vec{p}}{E_p+m}\chi_s\chi_s^\dagger & -\frac{\vec{\sigma}\cdot\vec{p}}{E_p+m}\chi_s\chi_s^\dagger\frac{\vec{\sigma}\cdot\vec{p}}{E_p+m} \end{pmatrix} \cdot (E_p+m)$$

but using $\sum_s \chi_s\chi_s^\dagger = \mathbb{1}_2$, we get

$$\Rightarrow \sum_s u_s\bar{u}_s = \begin{pmatrix} E_p+m & -\vec{\sigma}\cdot\vec{p} \\ +\vec{\sigma}\cdot\vec{p} & -\frac{\vec{p}^2}{E_p+m} \end{pmatrix} \overset{\vec{p}^2 = E_p^2 - m^2}{=\!=} \begin{pmatrix} m+E_p & -\vec{\sigma}\cdot\vec{p} \\ \vec{\sigma}\cdot\vec{p} & m-E_p \end{pmatrix}$$

$$= m\mathbb{1}_4 + E_p\begin{pmatrix} \mathbb{1}_2 & 0 \\ 0 & -\mathbb{1}_2 \end{pmatrix} - \begin{pmatrix} 0 & -\vec{\sigma} \\ \vec{\sigma} & 0 \end{pmatrix}\vec{p} = \underline{\underline{m\mathbb{1}_4 + \not{p}}}$$

since $\not{p} = \gamma^\mu p_\mu = \gamma^0 p_0 + \gamma^i p_i = \gamma^0 p^0 - \vec{\gamma}\cdot\vec{p}$ since $p_\mu = (p^0, -\vec{p})$

So, what we have found is

$$\boxed{\sum_{s=\pm1} \underset{\substack{\uparrow \\ 4\times1}}{u_s} \underset{\substack{\uparrow \\ 1\times4}}{\bar{u}_s} = \not{p} + \underset{\substack{\uparrow \\ 4\times4}}{m\mathbb{1}}}$$

Now, we know that

$$(\not{p}-m)\, u_{s'}(\vec{p}) = 0$$

$$\Rightarrow (\not{p}+m)\, u_{s'}(\vec{p}) = 2m\, u_{s'}(\vec{p})$$

We now define an operator $\underline{\Lambda_+(\vec{p}) \equiv \frac{\not{p}+m}{2m}}$ which has the property that

$$\Lambda_+ u_s(\vec{p}) = u_s(\vec{p})$$

But what is the effect of Λ_+ on $v_s(\vec{p})$? We remember the eq^n that $(\not{p}+m)\, v_s(\vec{p}) = 0$

$$\Rightarrow \Lambda_+ v_s(\vec{p}) = 0$$

Therefore, if we we have an arbitrary linear combination like a spinor of mom^n $\vec{p}$, $[c_s u_s(\vec{p}) + d_s v_s(-\vec{p})]$, then applying Λ_+ on this gives

$$[c_s u_s(\vec{p}) + \cancel{d_s v_s(\vec{p})}] = c_s u_s(\vec{p})$$

Therefore, $\underline{\Lambda_+ \text{ is a projection onto the +ve energy subspace.}}$!

Similarly, we can construct the object $\boxed{\sum_s v_s(\vec{p})\, \bar{v}_s(\vec{p}) = \not{p} - m\mathbb{1}_2}$

and then define $\underline{\Lambda_-(\vec{p}) \equiv \frac{m-\not{p}}{2m}}$

$$\Rightarrow \Lambda_-(\vec{p})\, v_s(\vec{p}) = v_s(\vec{p}) \quad \text{and} \quad \Lambda_- u_s(\vec{p}) = 0$$

$\underline{\Lambda_- \text{ here is the projector onto the -ve energy space.}}$

(2) HELICITY PROJECTION OPERATORS.

Helicity ≡ spin along the momentum direction $\vec{p}$

$$H_p = \frac{\vec{S}\cdot\vec{p}}{|\vec{p}|} = \vec{S}\cdot\hat{p} \longrightarrow \text{eigenvalues } \pm 1/2$$

$$\Sigma_p = 2H_p = 2\vec{S}\cdot\hat{p} \longrightarrow \text{eigenvalues } \pm 1$$

For NR spin $1/2$: $\vec{S} = \frac{\vec{\sigma}}{2}$

However for Dirac particles $(\vec{\Sigma})_i = \frac{1}{2}\epsilon_{ijk}\sigma^k = \begin{pmatrix} \sigma_i & 0 \\ 0 & \sigma_i \end{pmatrix}$

Therefore, we define $\Sigma_p = \begin{pmatrix} \vec{\sigma}\cdot\hat{p} & 0 \\ 0 & \vec{\sigma}\cdot\hat{p} \end{pmatrix}$

And since $(\vec{\sigma}\cdot\hat{p})^2 = (\hat{p}\cdot\hat{p}) = \mathbb{1}_2 \Rightarrow (\Sigma_p)^2 = \mathbb{1}_4 \longrightarrow$ eigenvalues ± 1

So, we define the "Helicity projection operator" as-

$$\boxed{\Pi_\pm(\vec{p}) \equiv \left(\frac{1 \pm \Sigma_p}{2}\right)}$$

$$\Rightarrow (\Pi_\pm(\vec{p}))^2 = \frac{\mathbb{1}_4 + (\Sigma_p)^2 \pm 2\Sigma_p}{4} = \frac{\cancel{2}(\mathbb{1}_4 \pm 2\Sigma_p)}{\cancel{4}\,2} = \Pi_\pm(\vec{p})$$

and $\Pi_+\Pi_- = \frac{\mathbb{1}_4 - (\Sigma_p)^2}{4} = \frac{\mathbb{1}_4 - \mathbb{1}_4}{4} = 0$

and $\Pi_+(\vec{p}) + \Pi_-(\vec{p}) = \mathbb{1}_4$

Let's also calculate behaviour of helicity projection ops with energy projection ops.

$$[\Lambda_s(\vec{p}), \Pi_{s'}(\vec{p})] = \left[\frac{s\not{p} + m\mathbb{1}_4}{2m}, \frac{1 \pm s'\Sigma_p}{2}\right]$$

$$[\Lambda_s, \Pi_{s'}] = \frac{ss'}{4m}[\not{p}, \Sigma_p] = \frac{ss'}{4m}[\gamma^0 p_0 - \vec{\gamma}\cdot\vec{p}, \Sigma_p]$$

Now since $\gamma^0 p_0 = p_0 \begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix}$ (D.P.) and $\Sigma_p = \vec{\sigma}\cdot\hat{p}\begin{pmatrix} \mathbb{1}_2 & 0 \\ 0 & \mathbb{1}_2 \end{pmatrix}$ and both of these are diagonal matrices, they'll obviously commute. So, now we are left to evaluate -

$$[\vec{\gamma}\cdot\vec{p}, \Sigma_p] = ? \qquad \vec{\gamma}\cdot\vec{p} = \begin{pmatrix} 0 & -\vec{\sigma}\cdot\vec{p} \\ \vec{\sigma}\cdot\vec{p} & 0 \end{pmatrix}$$

$$\frac{1}{|\vec{p}|}\begin{pmatrix} 0 & -\vec{\sigma}\cdot\vec{p} \\ +\vec{\sigma}\cdot\vec{p} & 0 \end{pmatrix}\begin{pmatrix} \vec{\sigma}\cdot\vec{p} & 0 \\ 0 & \vec{\sigma}\cdot\vec{p} \end{pmatrix} - \frac{1}{|\vec{p}|}\begin{pmatrix} \vec{\sigma}\cdot\vec{p} & 0 \\ 0 & \vec{\sigma}\cdot\vec{p} \end{pmatrix}\begin{pmatrix} 0 & -\vec{\sigma}\cdot\vec{p} \\ +\vec{\sigma}\cdot\vec{p} & 0 \end{pmatrix}$$

$$= \frac{1}{|\vec{p}|}\begin{pmatrix} 0 & -(\vec{\sigma}\cdot\vec{p})^2 \\ (\vec{\sigma}\cdot\vec{p})^2 & 0 \end{pmatrix} - \frac{1}{|\vec{p}|}\begin{pmatrix} 0 & -(\vec{\sigma}\cdot\vec{p})^2 \\ (\vec{\sigma}\cdot\vec{p})^2 & 0 \end{pmatrix} = \underline{\underline{0}}$$

$$\Rightarrow \quad \underline{[\Lambda_s(\vec{p}), \Pi_{s'}(\vec{p}')] = 0}$$

$\therefore$ Helicity and energy projections are simultaneously diagonalizable.

e.g.

Say $\vec{p} = p\hat{z}$

$$u_\pm(p\hat{z}) = \sqrt{E_p+m}\begin{pmatrix} \chi_\pm \\ \frac{p}{E_p+m}\underbrace{\sigma_z\chi_\pm}_{\pm\chi_\pm} \end{pmatrix}$$

then $u_\pm(p\hat{z})$ are eigenstates of

$\Sigma_p = \begin{pmatrix} \sigma_3 & 0 \\ 0 & \sigma_3 \end{pmatrix}$ with eigenvalues ± 1.

For arbitrary directions of $\vec{p}$, we can take a combination

$$\alpha u_+ + \beta u_- = \begin{pmatrix} \alpha\chi_+ + \beta\chi_- \\ \frac{\vec{\sigma}\cdot\vec{p}}{E_p+m}(\alpha\chi_+ + \beta\chi_-) \end{pmatrix}\sqrt{E_p+m}$$

6

and then demand that

we would like these to be be eigenstates of Σ_p as well

$$\vec{\sigma}\cdot\hat{p}\,(\alpha_s \chi_+ + \beta_s \chi_-) \overset{!}{=} s\,(\alpha_s \chi_+ + \beta \chi_-)$$

$$\Rightarrow \begin{pmatrix} p_3 - s & p_- \\ p_+ & -(p_3+s) \end{pmatrix} \begin{pmatrix} \alpha_s \\ \beta_s \end{pmatrix} = 0 \quad \text{where } p_\pm = p_1 \pm i p_2$$

[Note: here $\hat{p}$ is the unit vector.]

$$\Rightarrow \det(\ \downarrow\ ) = 0$$

$$-(p_3^2 - s^2) - (\hat{p}_1^2 + \hat{p}_2^2) = -(p_1^2 + p_2^2 + p_3^2) + 1 = 0$$

$$\Rightarrow -1+1 = 0 = 0 \ \checkmark$$

Hence, the solⁿ of the homogeneous eqⁿ exists.

$$\text{Infact, } \frac{\alpha_s}{\beta_s} = \frac{-p_-}{p_3 - s} = \frac{p_3 + s}{p_+} \quad (???)$$

what does dividing unit vectors even mean?

We can also normalize it to get

$$|\alpha_s|^2 + |\beta_s|^2 = 1$$

(Normalization + phase convention)

p_1, p_2, p_3 are components of the unit vector $\hat{p}$.

Lecture-14.

Projection operators:

→ Energy projection operators.

$$\Lambda_\pm(\vec{p}) = \frac{\pm \not{p} + m}{2m} = \frac{\sum_s u_s \bar{u}_s(\vec{p})}{2m}$$

$\Rightarrow \quad \Lambda_+ v_s = 0 \quad$ and $\quad \Lambda_+ u_s = u_s$

For Λ_- : $+ \to -$, $u \to v$

let's calculate the quantity

$$\sum_{ss'} |\bar{u}_s O u_{s'}|^2 = \sum_{ss'} (\bar{u}_s O u_{s'})^\dagger (\bar{u}_s O u_{s'})$$

$$(u_s^\dagger \gamma^0 O u_{s'})^\dagger = \underbrace{u'^\dagger_{s'}}_{\gamma^0\gamma^0} O^\dagger \gamma^0 u_s = \bar{u}'_{s'} \gamma^0 O^\dagger \gamma^0 u_s$$

$$\Rightarrow \sum_{ss'} (\bar{u}_s O u_{s'})^\dagger (\bar{u}_s O u_{s'}) = \sum_{ss'} \bar{u}'_{s'c} [\gamma^0 O^\dagger \gamma^0]_{cd} \underbrace{u_{s_d} \bar{u}_{s_a}}_{(\not{p}+m)_{da}} O_{ab} \underbrace{u'_{s'_b}}_{(\not{p}'+m)_{bc}}$$

$$= (\gamma^0 O^\dagger \gamma^0)_{cd} (\not{p}+m)_{da} O_{ab} (\not{p}'+m)_{bc}$$

$$= 4m^2 \mathrm{tr}\,(\gamma^0 O^\dagger \gamma^0 \Lambda_+(\vec{p}) O \Lambda_+(\vec{p}'))$$

Λ_+ projectors prove to be crucial to simplify such trace calculations!

(This part can be ignored w.l.o.c.)

→ Helicity: $\quad \Sigma_p = \vec{\Sigma}\cdot\hat{p} = \dfrac{\vec{\Sigma}\cdot\vec{p}}{|\vec{p}|}$

Helicity projection → $\quad \Pi_\pm(\vec{p}) = \dfrac{1 \pm \Sigma_p}{2}$

$$\Pi_\pm^2 = \Pi_\pm$$
$$\Pi_\pm \Pi_\mp = 0$$

$[\Lambda_s, \Pi_{s'}] = 0 \qquad s, s' \in \{\pm 1\}$

here ± refers to helicity sign.

here ± refers to energy sign

The projection operators are simultaneously digonalizable. Also, they are Hermitian. $\Lambda^\dagger = \Lambda$, $\Pi^\dagger = \Pi$

For $\vec{p} = p\hat{z}$

$$u_\pm(\vec{p}) = \sqrt{E_p+m}\begin{pmatrix} \chi_\pm \\ \frac{p\sigma_3 \chi_\pm}{E_p+m}\end{pmatrix} = \sqrt{E_p+m}\begin{pmatrix} \chi_\pm \\ \frac{\pm p \chi_\pm}{E_p+m}\end{pmatrix}$$

and we can see that

$$\Sigma_p u_s(p\hat{z}) = \begin{pmatrix} \sigma_3 & 0 \\ 0 & \sigma_3 \end{pmatrix}\begin{pmatrix} \chi_\pm \\ \frac{\pm p \chi_\pm}{E_p+m}\end{pmatrix} = \begin{pmatrix} \pm\chi_\pm \\ \frac{p \chi_\pm}{E_p+m}\end{pmatrix} = \pm\begin{pmatrix} \chi_\pm \\ \frac{\pm p \chi_\pm}{E_p+m}\end{pmatrix}$$

$\Rightarrow$ Σ_p has eignvalues ± 1.

★ Therefore, the free particle spinors that we constructed (u_s & v_s), are eigenstates of helicity, even for arbitrary directions of momentum. Hence, we can label the states by the helicity.

- For arbitrary direction $\vec{p}$, we take $\alpha_s u_+ + \beta_s u_- = \begin{pmatrix} \alpha\chi_+ + \beta\chi_- \\ \frac{\vec{\sigma}\cdot\vec{p}}{E_p+m}(\alpha\chi_+ + \beta\chi_-)\end{pmatrix}$

We demand that $\vec{\sigma}\cdot\hat{p}\,(\alpha_s\chi_+ + \beta_s\chi_-) \stackrel{!}{=} s(\alpha_s\chi_+ + \beta_s\chi_-)$ —(1)

$$\Rightarrow \begin{pmatrix} p_3 - s & p_- \\ p_+ & -(p_3+s)\end{pmatrix}\begin{pmatrix}\alpha_s \\ \beta_s\end{pmatrix} = 0, \quad p_\pm = p_1 \pm i p_2$$

$\det(\,) = -(p_1^2 + p_2^2 + p_3^2) + s^2 = 0$ as should have been.

$\Rightarrow$ Demand (1) is always satisfied.

$$\text{Also, } \left.\begin{aligned}(p_3 - s)\alpha_s + p_-\beta_s &= 0 \\ (p_+)\alpha_s + (p_3+s)\beta_s &= 0\end{aligned}\right\} \Rightarrow \frac{\alpha_s}{\beta_s} = -\frac{p_-}{p_3-s} = \frac{p_3+s}{p_+}$$

So, we define eigenspinors of Σ_p as follows. The new $\chi_\pm$ are-

$$\xi_+ = \frac{1}{\sqrt{2|\vec{p}|(p_3+|\vec{p}|)}} \begin{pmatrix} p_3 + |\vec{p}| \\ p_1 + ip_2 \end{pmatrix}$$

$$\xi_- = \frac{1}{\sqrt{2|\vec{p}|(p_3+|\vec{p}|)}} \begin{pmatrix} -p_1 + ip_2 \\ p_3 + |\vec{p}| \end{pmatrix}$$

Work these out! And how did we get two $\xi_\pm$ by subbing into $(\alpha_\pm \chi_+ + \beta_\pm \chi_-)$?

$$\Rightarrow \boxed{u_+(\vec{p}) = \sqrt{E_p+m} \begin{pmatrix} \xi_\pm(\vec{p}) \\ \frac{\vec{\sigma}\cdot\vec{p}}{E_p+m}\, \xi_\pm(\vec{p}) \end{pmatrix}}$$

are the new helicity Σ_p eigenspinors!

If $\vec{p} = p\hat{z}$, $\xi_\pm \xrightarrow{\text{reduces to}} \chi_\pm$.

Another way to get these is to consider the projection operator of helicity

$$\Pi_\pm = \frac{(1 \pm \vec{\Sigma}\cdot\hat{p})}{2} = \frac{1}{2}\begin{pmatrix} 1 \pm \frac{\vec{\sigma}\cdot\vec{p}}{|\vec{p}|} & 0 \\ 0 & 1 \pm \frac{\vec{\sigma}\cdot\vec{p}}{|\vec{p}|} \end{pmatrix}$$

Here p_1, p_2, p_3 are components of $\vec{p}$.

Now act with Π_+ on $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$$\vec{\sigma}\cdot\hat{p} = \begin{pmatrix} p_3 & p_1 - ip_2 \\ p_1 + ip_2 & -p_3 \end{pmatrix} \Rightarrow \Pi_+ = \frac{1}{2|\vec{p}|}\begin{pmatrix} \begin{pmatrix} p_3 + |\vec{p}| & p_1 - ip_2 \\ p_1 + ip_2 & |\vec{p}| - p_3 \end{pmatrix} & 0_{2\times2} \\ 0_{2\times2} & \vec{\sigma}\cdot\vec{p} + 1 \end{pmatrix}$$

$$\Pi_+ \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \frac{1}{2|\vec{p}|}\begin{pmatrix} p_3 + |\vec{p}| \\ p_1 + ip_2 \\ 0 \\ 0 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} \frac{p_3 + p}{|\vec{p}|} \\ \frac{p_1 + ip_2}{|\vec{p}|} \\ 0 \\ 0 \end{pmatrix} \longrightarrow \xi_+ \text{ upto a factor of normaliz}$$

Projection of $\begin{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ \begin{pmatrix} 0 \\ 0 \end{pmatrix} \end{pmatrix}$ onto Π_+ gives $\begin{pmatrix} \xi_+ \\ \begin{pmatrix} 0 \\ 0 \end{pmatrix} \end{pmatrix}$.

(Why? Why does it give the eigenspinor?)

(3) CHIRALITY PROJECTION OPERATOR:

$$\gamma^5 = \gamma_5 = i\gamma^0\gamma^1\gamma^2\gamma^3$$

$(\gamma^5)^2 = \mathbb{1}_4 \Rightarrow$ eigenvalues of γ^5 have to be ± 1.

So, the chirality eigenspinor would be of the form $\psi_\pm$ s.t.

$$\gamma^5 \psi_\pm = \pm\psi_\pm$$

So, we define the chirality projection operator as -

$$P_\pm = \frac{\mathbb{1} \pm \gamma^5}{2}$$

$$(P_\pm)^2 = \frac{1+1\pm 2\gamma^5}{2} = P_\pm$$

It's interesting to notice that in the limit $m \to 0$

$$\Pi_\pm = \frac{1}{2}\left(1 \pm \frac{\vec{\Sigma}\cdot\vec{p}}{|\vec{p}|}\right) \xrightarrow[m\to 0]{} P_\pm$$

Proof:

$$u_\pm = \sqrt{E_p+m}\begin{pmatrix} \chi_\pm \\ \frac{\vec{\sigma}\cdot\vec{p}}{E_p+m}\chi_\pm \end{pmatrix}$$

As $m \to 0$, $E_p \to |\vec{p}| \Rightarrow \quad u_\pm = \sqrt{|\vec{p}|}\begin{pmatrix} \chi_\pm \\ \vec{\sigma}\cdot\hat{p}\,\chi_\pm \end{pmatrix}$

In Dirac Pauli Rep^n, (for $\vec{p} = p\hat{z}$)

$$\gamma^5 = \begin{pmatrix} 0 & \mathbb{1}_2 \\ \mathbb{1}_2 & 0 \end{pmatrix} \Rightarrow P_\pm = \frac{\mathbb{1}\pm\gamma_5}{2} = \frac{1}{2}\begin{pmatrix} \mathbb{1}_2 & \pm\mathbb{1}_2 \\ \pm\mathbb{1}_2 & \mathbb{1}_2 \end{pmatrix}$$

2x1 objects

$$\Rightarrow P_+\begin{pmatrix} \psi_u \\ \psi_e \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} \psi_u \\ \psi_e \end{pmatrix} = \frac{1}{2}\begin{pmatrix} \psi_u+\psi_e \\ \psi_u+\psi_e \end{pmatrix}$$

$$P_-\begin{pmatrix}\psi_u\\ \psi_e\end{pmatrix} = \frac{1}{2}\begin{pmatrix}1 & -1\\ -1 & 1\end{pmatrix} = \frac{1}{2}\begin{pmatrix}\psi_u-\psi_e\\ -(\psi_u-\psi_e)\end{pmatrix}$$

The above two projections along $P_\pm$ give the eigenspinors of the chirality operator γ^5.

So, when $m \to 0$

$$u_\pm = \sqrt{|\vec{p}|}\begin{pmatrix}\chi_\pm\\ \underbrace{\vec{\sigma}\cdot\hat{p}}_{\text{helicity op. along } \hat{z}}\ \chi_\pm\end{pmatrix} = \sqrt{|\vec{p}|}\begin{pmatrix}\chi_\pm\\ \pm\chi_\pm\end{pmatrix}$$

which we recognize exactly as the eigenspinors of γ^5.

We call $\psi_L = \psi_u + \psi_e$ and $\psi_R = \psi_u - \psi_e$

$\Rightarrow$ the eigenspinors are then $\begin{pmatrix}\psi_L\\ \psi_L\end{pmatrix}$ (for +) & $\begin{pmatrix}\psi_R\\ -\psi_R\end{pmatrix}$ (for −)

However, a much simpler rep^n of γ^5 is in the Chiral rep^n.

$$\gamma^5 = \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$$

In this rep^n, $\dfrac{1+\gamma^5}{2} = \begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}$ and $\dfrac{1-\gamma^5}{2} = \begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix}$

So, $\dfrac{1+\gamma^5}{2}\begin{pmatrix}\psi_u\\ \psi_e\end{pmatrix} = \begin{pmatrix}\psi_u\\ 0\end{pmatrix} = \Psi_L$

$\dfrac{1-\gamma^5}{2}\begin{pmatrix}\psi_u\\ \psi_e\end{pmatrix} = \begin{pmatrix}0\\ \psi_e\end{pmatrix} = \Psi_R$

This is all for the case when $\vec{p} = p\hat{z}$.

(Though actually on that note, when did we even talk about $\vec{p}$ here? okay, I guess we are talking about $\vec{p}$ of <u>massless particles</u>.)

So now let's find eigenspinors of γ^5 for arbitrary $\vec{p}$. So we demand

$$\gamma^5(\alpha_s u_+ + \beta_s u_-) = \underset{\substack{\downarrow \\ \pm 1}}{s}(\alpha_s u_+ + \beta_s u_-)$$

Let's work in D. P. Rep^n.

$$\gamma^5 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \alpha_s u_+ + \beta_s u_- = \sqrt{E_p+m}\begin{pmatrix} \alpha_s \chi_+ + \beta_s \chi_- \\ \frac{\vec{\sigma}\cdot\vec{p}}{E_p+m}(\alpha\chi_+ + \beta\chi_-) \end{pmatrix}$$

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} \begin{pmatrix}\alpha \\ \beta\end{pmatrix} \\ \frac{\vec{\sigma}\cdot\vec{p}}{E_p+m}\begin{pmatrix}\alpha \\ \beta\end{pmatrix} \end{pmatrix} = s\begin{pmatrix} \begin{pmatrix}\alpha \\ \beta\end{pmatrix} \\ \frac{\vec{\sigma}\cdot\vec{p}}{E_p+m}\begin{pmatrix}\alpha \\ \beta\end{pmatrix} \end{pmatrix}$$

$$\begin{pmatrix} \frac{\vec{\sigma}\cdot\vec{p}}{E_p+m}\begin{pmatrix}\alpha \\ \beta\end{pmatrix} \\ \begin{pmatrix}\alpha \\ \beta\end{pmatrix} \end{pmatrix} = s\begin{pmatrix} \begin{pmatrix}\alpha \\ \beta\end{pmatrix} \\ \frac{\vec{\sigma}\cdot\vec{p}}{E_p+m}\begin{pmatrix}\alpha \\ \beta\end{pmatrix} \end{pmatrix}$$

$$\Big\downarrow \text{ God knows how}$$

$$\begin{pmatrix}\alpha \\ \beta\end{pmatrix} = \frac{s}{E_p+m}\begin{pmatrix} \alpha p_3 - \beta p_- \\ \alpha p_+ - \beta p_3 \end{pmatrix}$$

$$\begin{pmatrix} \frac{s p_3}{E_p+m} - 1 & \frac{s p_-}{E_p+m} \\ \frac{s p_+}{E_p+m} & -1 - \frac{p_3 s}{E_p+m} \end{pmatrix}\begin{pmatrix}\alpha \\ \beta\end{pmatrix} = 0$$

$$\det(M) = 1 - \frac{p^2}{(E_p+m)^2} = \frac{2m}{E_p+m} \overset{?}{=} 0$$

$\therefore$ A linear combination of u_+ & u_- can be an eigenspinor of chirality γ^5 iff $m = 0$!

When $m=0$, then $u_\pm \Rightarrow \begin{pmatrix} \xi_\pm(\vec{p}) \\ \pm\, \xi_\pm(\vec{p}) \end{pmatrix} \longrightarrow$ Chirality eigenspinors.

$$\gamma^5 \begin{pmatrix} \xi_\pm(\vec{p}) \\ \pm\, \xi_\pm(\vec{p}) \end{pmatrix} = \pm \begin{pmatrix} \xi_\pm(\vec{p}) \\ \pm\, \xi_\pm(\vec{p}) \end{pmatrix}$$

$\therefore$ In the lim $m \to 0$, we get the eigenspinors of chirality from eigenspinors of helicity.

(4) SPIN PROJECTION OPERATOR:

In the rest frame $(\vec{p}=0)$, $\Sigma_{\vec{p}}$ is ill-defined.

Spin 4-vector for massive particles -

$$S^\mu = \left(\frac{\vec{p}\cdot\hat{s}}{m} \,;\; \hat{s} + \frac{(\vec{p}\cdot\hat{s})\vec{p}}{m(E_p+m)} \right)$$

$$\xrightarrow[\vec{p}\to 0]{} (0, \hat{s})$$

$\therefore$ If $\hat{s} = \hat{p}$, $\quad S^\mu = \frac{1}{m}\left(|\vec{p}|, m\hat{p} + \frac{p^2}{E_p+m}\hat{p} \right) = \frac{1}{m}\left(|\vec{p}|, E\hat{p}\right)$ $\quad$ ($p^2 = E_p^2 - m^2$)

$$S_\mu S^\mu = \left(\frac{\vec{p}\cdot\hat{s}}{m}\right)^2 - \left(\underset{1}{\underset{\shortparallel}{\hat{s}^2}} + \frac{(\vec{p}\cdot\hat{s})^2\vec{p}^{\,2}}{m^2(E_p+m)^2} + \frac{2(\vec{p}\cdot\hat{s})^2}{m(E_p+m)} \right)$$

$$= \underline{\underline{-1}} \quad \text{(after simplifying)}$$

$$p_\mu S^\mu = \frac{\vec{p}\cdot\hat{s}}{m} E_p - \hat{s}\cdot\vec{p} - \frac{(\vec{p}\cdot\hat{s})\vec{p}^{\,2}}{m(E_p+m)}$$

$$= \underline{\underline{0}}$$

Now let's look at $\gamma^5 \not{s}$

$$(\gamma^5 \not{s})^2 = \gamma^5 \not{s}\, \gamma^5 \not{s} = -(\gamma^5)^2 \not{s}^2 = -\not{s}^2 = -s_\mu s^\mu = +1$$

So, we introduce operators -

$$\boxed{P_\uparrow = \frac{1+\gamma_5 \not{s}}{2}} \qquad \boxed{P_\downarrow = \frac{1-\gamma_5 \not{s}}{2}}$$

spin projection operators

$$P_\uparrow^2 = P_\uparrow \,, \quad P_\downarrow^2 = P_\downarrow \,, \quad P_\uparrow P_\downarrow = 0$$

Say, now we take $\hat{s} = (1,0,0)$

then $S^\mu_{\text{rest}\,(\vec{p}=0)} = (0, \hat{s})$

$$\Rightarrow \quad \not{s} = \gamma^\mu s_\mu = -\hat{s}\cdot\vec{\gamma} = -\gamma^1$$

$$\gamma^5 \not{s} = i\gamma^0\gamma^1\gamma^2\gamma^3(-\gamma^1) = -i\gamma^0(-1)\gamma^2\gamma^3 = i\gamma^0\gamma^2\gamma^3$$

$$= i\gamma^0 \Sigma^{23}$$ what is this new Σ?

$$[\gamma^0(\Sigma)_1, (\Sigma_1)] = 0$$ god knows what this is.

$\therefore$ eigenstates of $\gamma^5 \not{s}$ are also eigenstates of Σ^1

i.e $P_\uparrow$ and $P_\downarrow$ project out components of spin up or spin down.

Lecture-15.

(05/10/21)

Coupling a Dirac particle (e^-)

For a charged particle, in CM, we replace the momentum $\vec{p}$ with

$$\vec{p} \longrightarrow \vec{\pi} = \vec{p} - q\vec{A} \quad \text{in non-relativistic case.}$$

In relativistic mechanics, $p_\mu \longrightarrow \pi_\mu = p_\mu - QA_\mu$ $\quad$ $A_\mu \rightarrow$ four vector potential.

Now in RQM, $\hat{p}_\mu = i\partial_\mu$

Minimal substitution: $i\partial_\mu \longrightarrow i\partial_\mu - QA_\mu = i(\partial_\mu + iQA_\mu)$

i.e. $\partial_\mu \longrightarrow \underline{D_\mu = \partial_\mu + iQA_\mu}.$ Covariant derivative.

Jargon Alert:

The reason for introducing is deriving Dirac eq^n from Dirac Lagrangian via Hamilton's principle $\delta S = 0$.

$$S = \int d^4x\, \mathcal{L}(\psi, \partial_\mu\psi), \text{ then } \delta S = 0 \longrightarrow \text{eq}^n\text{s of motion}$$

$$\mathcal{L}_{Dirac} = \bar{\psi}(i\gamma^\mu\partial_\mu - m)\psi$$

ψ & $\bar{\psi}$ are treated as independent objects.

The Dirac Lagrangian $\mathcal{L}_{Dirac}$ has what's called a U(1) invariance.

$$\left.\begin{aligned}\psi &\longrightarrow e^{iq\theta}\psi \\ \bar{\psi} &\longrightarrow e^{-iq\theta}\bar{\psi}\end{aligned}\right\} \Rightarrow \mathcal{L} \text{ remains invariant!}$$

$$\left(\text{since } \bar{\psi}' = \psi'^{\dagger}\gamma^0 = (e^{iq\theta}\psi)^\dagger\gamma_0 = e^{-iq\theta}\bar{\psi}\right)$$

"U(1) invariance implies that $\exists$ a conserved current, $(\partial_\mu j^\mu = 0)$ which happens to be $j^\mu = \bar{\psi}\gamma^\mu\psi$.

Some more jargon: If we have a conserved j^μ, we can introduce a "connection" A_μ, and promote $\theta \to \theta(x)$ to a local parameter. Then the theory has a gauge invariance.

The way to make $\mathcal{L}_D$ gauge invariant is $\mathcal{L}_D \to \mathcal{L}_{D+EM}$. That will guarantee gauge invariance $U(1)$ of $\mathcal{L}_D$. $(\partial_\mu \to D_\mu)$

So, back to actual content, to introduce EM coupling, replace $\partial_\mu \to D_\mu$.

$$Q = q\underset{\uparrow}{e} \qquad \Rightarrow \qquad Q_{e^-} = -e \qquad e>0$$

elementary charge of proton

$$Q_{e^+} = +e$$

So, $D_\mu \psi_{e^-} = (\partial_\mu + iQ_{e^-}A_\mu)\psi_{e^-} = (\partial_\mu - ieA_\mu)\psi_{e^-}$

$\therefore$ Dirac eq^n coupled with EM

$$\boxed{i\gamma^\mu(\partial_\mu + iqeA_\mu)\psi - m\psi = 0}$$

The adjoint of above eq^n is

$$\gamma^{\mu\dagger} = \gamma^0\gamma^\mu\gamma^0$$

$$-i\psi^\dagger(\overleftarrow{\partial_\mu} - iqeA_\mu)\gamma^{\mu\dagger} - m\psi^\dagger = 0$$

$$[i\psi^\dagger\gamma^0(\overleftarrow{\not\partial} - iqe\not A)\gamma^0 + m\psi^\dagger = 0] \times \gamma^0$$

$$\boxed{i\bar\psi(\overleftarrow{\not\partial} - iqe\not A) + m\bar\psi = 0}$$

$$\left.\begin{array}{l} [(\partial_\mu\bar\psi)i\gamma^\mu + qe\bar\psi\not A + m\bar\psi = 0] \times \psi \\ \bar\psi \times [i\gamma^\mu\partial_\mu\psi - qe\not A\psi - m\psi = 0] \end{array}\right\} \Rightarrow \partial_\mu(\bar\psi\gamma^\mu\psi) = 0$$

same old current j^μ but this can now actually be thought of as electric current.

Charge conjugate wavefn.

$$\text{If} \quad \psi \to e^{iq\theta}\psi, \text{ then } \psi^* \to e^{-iq\theta}\psi^* = e^{i(-q)\theta}\psi^*$$

So, the conjugate ψ^* actually describes a particle with opposite electric charge. We are now interested in Lorentz transformation of ψ^*.

Now ψ transforms as $\psi' = e^{\frac{i\sigma\cdot\omega}{4}}\psi$

$$\Rightarrow \quad \psi^{*\prime} = e^{-\frac{i\sigma^*\cdot\omega}{4}}\psi^*$$

But $\sigma^* \neq \pm\sigma \Rightarrow \psi^*$ doesn't transform in a nice way.

So, we say let's construct a new object ψ^c called "charge conjugate"

$$\psi^c \equiv \underbrace{\tilde{C}}_{4\times 4 \text{ matrix}}\psi^*$$

$$\Rightarrow (\psi^c)' = \tilde{C}(S_L\psi)^* = \tilde{C}S_L^*\psi^* = \tilde{C}S_L^*\tilde{C}^{-1}\underbrace{C\psi^*}_{\psi^c}$$

So, we now demand that

$$\boxed{\tilde{C}S_L^*\tilde{C}^{-1} = S_L} \quad \leftarrow \text{defn of } C$$

If $\tilde{C}$ is defined like this, then $\psi_c' = S_L\psi_c$ (just like $\psi' = S_L\psi$ but $Q(\psi^*) = -Q(\psi)$)

However by convention, we define

$$\psi^c = C\bar{\psi}^T = C(\psi^\dagger\gamma^0)^T = (C\gamma^{0T})\psi^*$$

This is what we defined as $\tilde{C} \equiv C\gamma^{0T}$

We can get the transformation props. of ψ^c from here as well

we know $\bar{\psi}' = \bar{\psi}S_L^{-1} \Rightarrow \psi^{c\prime} = C(\bar{\psi}S_L^{-1})^T = C(S_L^{-1})^T\bar{\psi}^T$

$$= C(S_L^{-1})^T C^{-1}\underbrace{C\bar{\psi}^T}_{\psi^c}$$

$$\therefore \quad \psi^{c\prime} = C(S_L^{-1})^T C^{-1}\psi^c$$

So, if we demand $\boxed{C(S_L^{-1})^T C^{-1} = S_L}$, then the transf. property is

$$\boxed{\psi^{c\prime} = S_L \psi^c}$$

So, $S_L = \exp\left(\frac{i}{4}\omega\cdot\sigma\right) \Rightarrow S_L^{-1} = \exp\left(-\frac{i}{4}\omega\cdot\sigma\right)$

So, $(S_L^{-1})^T = \exp\left(-\frac{i}{4}\omega\cdot\sigma^T\right)$

Now, for we use the following property of exponentials -

$$Ce^A C^{-1} = C\sum_{n=0}^{\infty}\frac{1}{n!}A^n C^{-1} = \sum_{n=0}^{\infty}\frac{1}{n!}(CAC^{-1})^n = e^{CAC^{-1}}$$

Since $(CAC^{-1})^n = CA\cancel{C^{-1}}\cancel{C}A\cancel{C^{-1}}\cancel{C}A\cancel{C^{-1}} \ldots \cancel{C}AC^{-1}$
$= CA^nC^{-1}$

$\Rightarrow$ The demand on C becomes

$$C\, e^{-\frac{i\omega\cdot\sigma^T}{4}} C^{-1} = e^{-\frac{i\omega}{4}\cdot C\sigma^T C^{-1}} \stackrel{!}{=} e^{\frac{i\omega}{4}\cdot\sigma}$$

$$\Rightarrow \boxed{C(\sigma^{\mu\nu})^T C^{-1} = -\sigma^{\mu\nu}}$$

Can we find such a C?

In Dirac Pauli, $\sigma^{ij} = i\begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0\end{pmatrix}\begin{pmatrix} 0 & \sigma^j \\ -\sigma^j & 0\end{pmatrix} = \epsilon_{ijk}\begin{pmatrix}\sigma^k & 0 \\ 0 & \sigma^k\end{pmatrix}$.

So, $\left.\begin{aligned} \sigma^{12} &= \begin{pmatrix}\sigma^3 & 0 \\ 0 & \sigma^3\end{pmatrix} \\ \sigma^{23} &= \begin{pmatrix}\sigma^1 & 0 \\ 0 & \sigma^1\end{pmatrix}\end{aligned}\right\}$ symmetric

$\sigma^{31} = \begin{pmatrix}\sigma^1 & 0 \\ 0 & \sigma^1\end{pmatrix}\Big\}$ Anti-symmetric

$$\sigma^{0i} = i\gamma^0\gamma^i = i\begin{pmatrix} 0 & \sigma^i \\ \sigma^i & 0 \end{pmatrix}$$

So, $\sigma^{01^T} = \begin{pmatrix} 0 & i\sigma^{1T} \\ i\sigma^{1T} & 0 \end{pmatrix} = \sigma^{01}$

$$\sigma^{02^T} = \begin{pmatrix} 0 & i\sigma^{2T} \\ i\sigma^{2T} & 0 \end{pmatrix} = -\sigma^{02}$$

$$\sigma^{03^T} = \begin{pmatrix} 0 & i\sigma^{03^T} \\ i\sigma^{03^T} & 0 \end{pmatrix} = \sigma^{03}$$

What now? idk.

Lecture - 16
(07/10/2021)

Coupling of Dirac particle to EM

Prescription: $\partial_\mu \longrightarrow D_\mu = \partial_\mu + iqeA_\mu$

U(1) invariance is associated to the charge.

$$\psi_{(q)} \rightarrow e^{iq\theta}\psi(q)$$
$$(\partial_\mu\theta = 0;\ \text{global})$$

If we get an invariance under $\psi_{(q)} \rightarrow \psi_{(q)}\, e^{iq\theta(x)}$ s.t. $\partial_\mu\theta(x) \neq 0$ (local)

$$\partial_\mu\psi \rightarrow \left(\partial_\mu\psi + iq(\partial_\mu\theta)\psi\right)$$

then $\partial_\mu\psi$ doesn't transform homogeneously.

So, the reason for introducing D_μ is that $D_\mu\psi$ transforms homogeneously

$$\underline{D_\mu\psi \longrightarrow e^{iq\theta(x)}D_\mu\psi} \qquad \left(\text{if } A_\mu \rightarrow A_\mu - \frac{1}{e}\partial_\mu\theta(x)\right)$$

under a U(1) gauge invariance.

$\therefore$ We write Dirac eqn for a particle of charge $Q = qe$

$$(i\gamma^\mu D_\mu - m)\psi(x) = 0 \quad \Rightarrow \quad \underline{(i\overrightarrow{\not{D}} - m)\psi = 0}$$

Taking conjugate of Dirac eqn & multiplying by γ^0

$$\psi^\dagger(-i\gamma^0\gamma^\mu\gamma^0 D_\mu^\dagger - m)\cdot\gamma^0 = 0$$
$$\bar{\psi}(-i\gamma^\mu \overleftarrow{D}_\mu^{\dagger} - m) = 0$$

Now $D_\mu^\dagger = \partial_\mu - iqeA_\mu \equiv \bar{D}_\mu$
$(q \longrightarrow -q)$

$$\Rightarrow \underline{\bar{\psi}(i\overleftarrow{\bar{\not{D}}} + m) = 0}$$

If we take the transpose of the conjugate Dirac eq^n

$\xrightarrow{\text{transpose}}$ $(i\bar{\not{D}}^T + m)\bar{\psi}^T = 0 \Rightarrow (i\gamma^{\mu T}\bar{D}_\mu + m)\bar{\psi}^T = 0$

↑ covariant derivative for particle with charge $-eq$

If we can find matrices C s.t. $C(\gamma^\mu)^T C^{-1} = -\gamma^\mu$, then-

$$C(i\gamma^\mu \bar{D}_\mu + m)C^{-1}\underbrace{C\bar{\psi}^T}_{\psi^c} = 0$$

using our demand,

$$(i\gamma^\mu \bar{D}_\mu - m)\psi^c = 0$$

→ which looks just like the original Dirac eq^n, with $\bar{D}_\mu = D_\mu(-q)$

The only difference is ψ obeys Dirac eq^n of charge $+qe$ and ψ^c obeys Dirac eq^n of charge $-qe$.

Now, we know ψ transforms as $\psi' = S_L\psi$. How does ψ^c transform?

$$\psi_c' = C\bar{\psi}'^T = C(\bar{\psi}S_L^{-1})^T = C(S_L^{-1})^T\bar{\psi}^T$$
$$= C(S_L^{-1})^T C^{-1} C\bar{\psi}^T = C S_L^{-1\,T} C^{-1}\psi^c$$

But $C(S_L^{-1})^T C^{-1} = C\exp\left(-\frac{i}{4}\omega.\sigma^T\right)C^{-1} = \exp\left(-\frac{i}{4}\omega.C\sigma^T C^{-1}\right)$

Now earlier, we demanded $C(\gamma^\mu)^T C^{-1} = -\gamma^\mu$

$$\Rightarrow C(\sigma^{\mu\nu})^T C^{-1} = iC(\gamma^\mu\gamma^\nu)^T C^{-1} \quad (\text{for } \mu \neq \nu)$$

$$= iC\gamma^{\nu T}.C^{-1}C.\gamma^{\mu T}C^{-1} = i(-1)^2\gamma^\nu\gamma^\mu = -i\gamma^\mu\gamma^\nu = -i\sigma^{\mu\nu}$$

$$\therefore \quad C\gamma^{\mu T}C^{-1} = -\gamma^\mu \Rightarrow C\sigma^{\mu\nu T}C^{-1} = -\sigma^{\mu\nu}$$

$$\therefore \quad C\,(S_L^{-1})^T C^{-1} = \exp\left(-\tfrac{i}{4}\,\omega\cdot C\sigma C^{-1}\right) = \exp\left(\tfrac{i}{4}\,\omega\cdot\sigma\right) = S_L$$

$$\Rightarrow \boxed{\psi_c' = S_L \psi_c} \qquad \text{if} \quad C\gamma^{\mu T} C^{-1} = -\gamma^\mu$$

(same as ψ)

So the problem now is to find these matrices C.

$$\gamma^{\mu T} = \begin{pmatrix} \gamma^0 \\ \gamma^1 \\ \gamma^2 \\ \gamma^3 \end{pmatrix}^T$$

Pauli Dirac representation

$\gamma^{0T} = \gamma^0$

Now $\gamma^{1T} = \begin{pmatrix} 0 & \sigma^1 \\ -\sigma^1 & 0 \end{pmatrix}^T = \begin{pmatrix} 0 & -\sigma^1 \\ \sigma^1 & 0 \end{pmatrix} = -\gamma^1$

$$\gamma^{2T} = \begin{pmatrix} 0 & \sigma^2 \\ -\sigma^2 & 0 \end{pmatrix}^T = \begin{pmatrix} 0 & -\sigma^{2T} \\ +\sigma^{2T} & 0 \end{pmatrix} = \begin{pmatrix} 0 & +\sigma^2 \\ -\sigma^2 & 0 \end{pmatrix} = \gamma^2$$

$$\gamma^{3T} = \begin{pmatrix} 0 & \sigma^3 \\ -\sigma^3 & 0 \end{pmatrix}^T = \begin{pmatrix} 0 & -\sigma^3 \\ \sigma^3 & 0 \end{pmatrix} = -\gamma^3$$

$$\Rightarrow \gamma^{\mu T} = \begin{pmatrix} \gamma^0 \\ -\gamma^1 \\ \gamma^2 \\ -\gamma^3 \end{pmatrix}$$

We can show that if we choose $C = \eta\,\gamma^2\gamma^0$ with $|\eta| = 1$, that will do the job of satisfying our demand.

$\underline{C = \eta\,\gamma^2\gamma^0} \Rightarrow \underline{C^{-1} = \eta^*\,\gamma^2\gamma^0}$ since

$$\begin{aligned} CC^{-1} &= |\eta|^2 \gamma^2\gamma^0\gamma^2\gamma^0 \\ &= -|\eta|^2\gamma^2\gamma^{0\,2}\gamma^2 \\ &= -|\eta|^2(\gamma^2)^2 = (-1)^2|\eta|^2 = \mathbb{1} \end{aligned}$$

So, let's check if we get $C\gamma^{\mu T}C^{-1} = -\gamma^\mu$

$$C\gamma^{0T}C^{-1} = \gamma^2\underbrace{\gamma^0\cdot\gamma^0}_{\mathbb{1}}\cdot\gamma^2\gamma^0 = (\gamma^2)^2\gamma^0 = -\gamma^0$$

$$C\gamma^{1T}C^{-1} = \gamma^2\gamma^0(-\gamma^1)\gamma^2\gamma^0 = -\gamma^2\gamma^{0\,2}\gamma^1\gamma^2 = -\gamma^2\gamma^1\gamma^2 = (\gamma^2)^2\gamma^1 = -\gamma^1$$

$$C\gamma^{2T}C^{-1} = \gamma^2\gamma^0\cdot\gamma^2\gamma^2\gamma^0 = -\gamma^2\gamma^0\gamma^0 = -\gamma^2$$

$$C\gamma^{3T}C^{-1} = \gamma^2\gamma^0(-\gamma^3)\gamma^2\gamma^0 = \text{same as } \gamma^1 \text{ calculation} = -\gamma^3$$

✓ Yes. it does satisfy the demand.

Conventionally, we take $\eta = i \Rightarrow C = i\gamma^2\gamma^0$

Now $i\gamma^2 = \begin{pmatrix} 0 & i\sigma^2 \\ -i\sigma^2 & 0 \end{pmatrix} = \epsilon$

$i\sigma^2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \equiv \epsilon$

anti-symmetric unit matrix

$$\therefore\ C = \begin{pmatrix} 0 & \epsilon \\ -\epsilon & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 0 & -\epsilon \\ -\epsilon & 0 \end{pmatrix}$$

Aside: $SU(2)$ $\quad \underset{2\times 2}{U^\dagger} = U^{-1} \quad \det U = 1$

Under $SU(2)$, a 2-spinor $\psi' = U\psi \Rightarrow \psi'^\dagger\psi' = \psi^\dagger\psi$

The canonical parameterization of $SU(2)$ transformation is

$$U = \exp\left(\frac{i}{2}\,\vec{\theta}\cdot\vec{\sigma}\right) = \exp\left(i\vec{\theta}\cdot\vec{T}\right) \qquad \text{Defining } \vec{T} = \frac{\vec{\sigma}}{2}$$

$(\theta^1, \theta^2, \theta^3)$ Pauli matrices.

Now, let's see how ψ^* transforms in $SU(2)$

Since $\sigma^{2*} = -\sigma^2$

$$\psi^{*\prime} = U^*\psi^* = \exp\left(-\frac{i}{2}\left(\theta^1\sigma^1 - \theta^2\sigma^2 + \theta^3\sigma^3\right)\right)\psi^*$$

However if define $\tilde{\psi} = \epsilon\psi^* = (i\sigma^2)\psi^*$

$$\Rightarrow \tilde{\psi}' = (i\sigma^2)\psi'^* = (i\sigma^2)U^*\psi^*$$
$$= i\sigma^2 \exp\left(-\frac{i}{2}\left(\theta^1\sigma^1 - \theta^2\sigma^2 + \theta^3\sigma^3\right)\right)$$

But σ^2 anti-commutes with $\sigma^{1,3}$ but commutes with itself.

$$\text{So, } \sigma^2\left(\theta^1\sigma^1 - \theta^2\sigma^2 + \theta^3\sigma^3\right) = -\left(\vec{\theta}\cdot\vec{\sigma}\right)\sigma^2$$

$$\Rightarrow \underline{\tilde{\psi}'} = i\sigma^2 U^*\psi^* = iU\sigma^2\psi^* = \underline{U\tilde{\psi}}$$

Thus, $\epsilon\psi^*$ transforms just like ψ!

We are doing a very analogous thing for $SO(4)$ be defining

ψ^c which transforms just like ψ.

$$\psi^c = C\bar{\Psi}^T = C(\psi^\dagger\gamma^0)^T = i\gamma^2 \cancel{\gamma^0} \cancel{\gamma^{0*}} \psi^*$$
$$= \begin{pmatrix} 0 & \epsilon \\ -\epsilon & 0 \end{pmatrix} \psi^* = \begin{pmatrix} 0 & \epsilon \\ -\epsilon & 0 \end{pmatrix} \begin{pmatrix} \psi_t^* \\ \psi_b^* \end{pmatrix} = \begin{pmatrix} \epsilon\,\psi_b^* \\ -\epsilon\,\psi_t^* \end{pmatrix}$$

Summary: CHARGED DIRAC PARTICLE

$$(i\cancel{D} - m)\psi = 0 \qquad D = \partial + iqeA$$
$$(i\cancel{\bar{D}} - m)\psi^c = 0 \qquad \bar{D} = \partial - iqeA$$
$$(A_\mu \xrightarrow{\text{conj.}} -A_\mu)$$
$$\psi^c = C\bar{\Psi}^T, \quad C = i\gamma^2\gamma^0$$

ELECTRON IN AN EM FIELD

$$\vec{B} = \vec{\nabla}\times\vec{A} \qquad \vec{E} = -\vec{\nabla}A_0 - \partial_0\vec{A}$$

So, $A_\mu = (0, -\vec{A})$

The Dirac eq^n for this particle is

$$(i\gamma^\mu\partial_\mu - m)\psi = qe\cancel{A}\psi = qe\gamma^\mu A_\mu\psi$$
$$(i\vec{\gamma}\cdot\vec{\nabla} + i\gamma^0\partial_t - m)\psi = -qe\vec{\gamma}\cdot\vec{A}\psi + qe\gamma^0A^0\psi$$

Now multiplying both sides by γ^0 from left.

$$(i\gamma^0\vec{\gamma}\cdot\vec{\nabla} + i\gamma^0\partial_t - m)\psi = -qe\gamma^0\vec{\gamma}\cdot\vec{A}\psi + qeA^0\psi$$
$$(i\vec{\alpha}\cdot\vec{\nabla} + i\partial_t - m\beta)\psi = -qe\vec{\alpha}\cdot\vec{A}\psi + qeA^0\psi$$
$$i\partial_t\psi = (\vec{\alpha}\cdot\hat{\vec{p}} + \beta m - qe\vec{\alpha}\cdot\vec{A} + qeA^0)\psi$$
$$\boxed{i\partial_t\psi = \vec{\alpha}\cdot(\hat{\vec{p}} - qe\vec{A})\psi + (\beta m + qeA^0)\psi}$$

electric energy.

(a) e^- in a static $\vec{E}$ field

$\vec{A} = 0 \qquad A_0(\vec{x},t) = A_0(\vec{x})$

$\Rightarrow \quad i\partial_t \psi = (\vec{\alpha}\cdot\hat{\vec{p}} + \beta m + V(\vec{x}))\,\psi$ where $V = qeA_0 = -eA_0 \quad (q=-1)$

(b) e^- in a $\vec{B}$ field

$\vec{A} \neq 0, \quad A_0 = 0$

$\Rightarrow \quad i\partial_t \psi = \left[\vec{\alpha}\cdot(\hat{\vec{p}} + e\vec{A}) + \beta m\right]\psi$

Let's now take the Dirac eq^n and analyse it for a central pot^n problem.

Central Pot^n $\quad V = V(|\vec{r}|) = V(r)$

Angular momentum operators are given by

$\hat{L}_i = \epsilon_{ijk}\,\hat{x}^j\,\hat{p}^k$ and total ang. mom^n. $\vec{J} = \vec{L} + \vec{S}$

Let's first calculate some commutators.

- $[\hat{L}_i, \vec{\alpha}\cdot\hat{\vec{p}}] = \alpha_\ell[\epsilon_{ijk}\hat{x}^j\hat{p}^k, \hat{p}^\ell] = \epsilon_{ijk}\alpha_\ell[\hat{x}^j\hat{p}^k, \hat{p}^\ell]$

$= \epsilon_{ijk}\alpha_\ell\,\hat{x}^j\underbrace{[\hat{p}^k,\hat{p}^\ell]}_{=0} + \epsilon_{ijk}\alpha_\ell\underbrace{[\hat{x}^j,\hat{p}^\ell]}_{=(i\delta^{j\ell})}\hat{p}^k$

$= \epsilon_{ijk}\alpha_\ell\, i\delta^{j\ell}\hat{p}^k = i\epsilon_{ijk}\alpha_j\hat{p}_k = i(\vec{\alpha}\times\hat{\vec{p}})_i$

- $[\hat{L}^i, \beta m] = 0$

- $[\hat{L}_i, V(r)] = 0$ (rotational invariance)

$\Rightarrow \quad [\hat{L}_i, H_D^{(\vec{A}=0)}] = i(\vec{\alpha}\times\vec{p})_i \overset{\text{(in Heisenberg picture)}}{=} i\dfrac{d\hat{L}_i}{dt}$

We also need the commutator with spin part.

- $[\Sigma_j, H_D] = ?$

$$\Sigma_j = \begin{pmatrix} \sigma^j & 0 \\ 0 & \sigma^j \end{pmatrix} \qquad H_D = \vec{\alpha}\cdot\hat{\vec{p}} + \underbrace{\beta}_{=\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}} m + V\mathbb{1}$$

trivially commute.

So, we only need to calculate $[\Sigma_j, \vec{\alpha}\cdot\vec{p}] = ?$

In Pauli-Dirac, $\vec{\alpha} = \gamma^0\vec{\gamma} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 0 & \vec{\sigma} \\ -\vec{\sigma} & 0 \end{pmatrix} = \begin{pmatrix} 0 & \vec{\sigma} \\ \vec{\sigma} & 0 \end{pmatrix}$

$$[\Sigma_j, \vec{\alpha}\cdot\hat{\vec{p}}] = [\Sigma_j, \alpha_k]\,\hat{p}_k$$

So, we need to calculate $\left[\begin{pmatrix} \sigma_j & 0 \\ 0 & \sigma_j \end{pmatrix}, \begin{pmatrix} 0 & \sigma^k \\ \sigma^k & 0 \end{pmatrix}\right]$

$$= \begin{pmatrix} 0 & \sigma^j\sigma^k \\ \sigma^j\sigma^k & 0 \end{pmatrix} - \begin{pmatrix} 0 & \sigma^k\sigma^j \\ \sigma^k\sigma^j & 0 \end{pmatrix} = 2i\,\epsilon_{jkl}\begin{pmatrix} 0 & \sigma^l \\ \sigma^l & 0 \end{pmatrix}$$

$$= 2i\,\epsilon_{jkl}\,\alpha^l$$

$$\Rightarrow [\Sigma_j, \vec{\alpha}\cdot\vec{p}] = 2i\,\epsilon_{jkl}\,\alpha^l\hat{p}^k = -2i(\vec{\alpha}\times\vec{p})_j = [\Sigma_j, H_D]$$

$$\Rightarrow i\frac{d(\Sigma_j/2)}{dt} = -i(\vec{\alpha}\times\vec{p})_j$$

$\frac{\Sigma_j}{2} \equiv S_j$ in RQM $\Rightarrow$ $i\frac{dS_j}{dt} = -i(\vec{\alpha}\times\vec{p})_j$ → opposite of $[\hat{L}_i, H^D]$

$$\Rightarrow [\vec{J}, H_D] = [\vec{L}, H_D^{(\vec{A}=0)}] + [\vec{S}, H_D^{(\vec{A}=0)}]$$

$$= i(\vec{\alpha}\times\vec{p}) - i(\vec{\alpha}\times\vec{p}) = 0$$

$\therefore \boxed{[\vec{J}, H_D^{(\vec{A}=0)}] = 0}$ $\vec{J}$ commutes with H_D for a spherically symmetric potential $V(r)$

and in Heisenberg picture $\frac{d\vec{J}}{dt} = 0 \Rightarrow$ $\vec{J}$ (total angular mom^n) is conserved in a central potential!

So, $\underline{(J^3, \vec{J}^2, H_0 + V(r))}$ can be simultaneously diagonalised and form a complete set of commuting observables (CSCO).

So, we have quantum nos. – j, j_3, n

Question – Are there anymore COMs?

There is another. If we define $\hat{K} \equiv \gamma^0 (\vec{\Sigma} \cdot \vec{J} - \frac{1}{2})$

writing $\vec{J} = \frac{\vec{\Sigma}}{2} + \vec{L} \Rightarrow \vec{\Sigma} \cdot \vec{L} + \frac{\vec{\Sigma}^2}{2} = \vec{\Sigma} \cdot \vec{L} + \frac{3}{2} \mathbb{1}$

So, $\underline{\hat{K} \equiv \gamma^0 (\vec{\Sigma} \cdot \vec{L} + \mathbb{1})}$

Then eigenvalues of $\hat{K}$ labelled by κ discriminate $\vec{S}$ (parallel/anti-parallel) to $\vec{L}$.

$$\hat{K}^2 = \gamma^0 (\vec{\Sigma} \cdot \vec{L} + \mathbb{1}) \gamma^0 (\vec{\Sigma} \cdot \vec{L} + \mathbb{1})$$

γ^0 commutes with $\vec{\Sigma}$ (check), so we can shift it around

$$\hat{K}^2 = (\vec{\Sigma} \cdot \vec{L} + \mathbb{1})^2 = (\vec{\Sigma} \cdot \vec{L})^2 + \mathbb{1} + 2 \vec{\Sigma} \cdot \vec{L}$$
$$= \Sigma_i \Sigma_j L_i L_j + \mathbb{1} + 2 \Sigma_i L_i$$

$$\Sigma_i \Sigma_j L_i L_j = \begin{pmatrix} \sigma_i \sigma_j & 0 \\ 0 & \sigma^i \sigma^j \end{pmatrix} L_i L_j \qquad \text{but } \sigma^i \sigma^j = \delta_{ij} \mathbb{1} + i \epsilon_{ijk} \sigma_k$$

$$= \mathbb{1} (\vec{L})^2 + i \epsilon_{ijk} \Sigma_k L_i L_j = \mathbb{1} (\vec{L})^2 + \frac{i}{2} \epsilon_{ijk} \Sigma_k \underbrace{[L_i, L_j]}_{= i \epsilon_{ijl} L_l}$$

$$= (\vec{L}^2) \mathbb{1} + \frac{i}{2} (i) \underbrace{\epsilon_{ijk} \epsilon_{ijl}}_{2 \delta^k_l} \Sigma_k L_l = (\vec{L})^2 \mathbb{1} - \vec{\Sigma} \cdot \vec{L}$$

$$\Rightarrow \hat{K}^2 = (\vec{L})^2 \mathbb{1} - \vec{\Sigma} \cdot \vec{L} + \mathbb{1} + 2 \vec{\Sigma} \cdot \vec{L}$$
$$= \underbrace{\left(\vec{L} + \frac{\vec{\Sigma}}{2}\right)^2}_{\vec{J}^2} - \underbrace{\left(\frac{\vec{\Sigma}}{2}\right)^2}_{= \frac{3}{4} \mathbb{1}} + \mathbb{1} = \vec{J}^2 + \frac{1}{4}$$

$$\boxed{\hat{K}^2 = \vec{J}^2 + \frac{1}{4}}$$

But now J is a commuting observable, and we can use its eigenvalue

$$K^2 = j(j+1) + \frac{1}{4} = \left(j+\frac{1}{2}\right)^2$$

$\Rightarrow$ eigenvalues of $\hat{K}$ are $\underline{K = \pm\left(j+\frac{1}{2}\right)}$

And since j is a half-integer $\Rightarrow$ K is an integer.

Lecture - 17

(08/10/2021)

Constants of motion in a spherically symmetric potⁿ (Dirac particle).

Operator:	$\hat{H}_D$	J_3	$\vec{J}^2$	$K =$
Eigenvalue:	E	j_3	$j(j+1)$	
Q. no:	n	$j_3 = j_z$	j	

For $\hat{K}$ to also be included in the CSCO, we need to check if it commutes with the Hamiltonian-

$$[\hat{K}, H_D] = \left[\gamma^0 \vec{\Sigma}\cdot\vec{J} - \frac{\gamma^0}{2}, H_D\right]$$

$$= \gamma^0\vec{\Sigma}\cdot[\vec{J}, H_D] + \gamma^0[\vec{\Sigma}, H_D]\cdot\vec{J} + [\gamma^0, H_D]\vec{\Sigma}\cdot\vec{J} - \frac{1}{2}[\gamma^0, H_D]$$

- $[\gamma^0, \vec{\alpha}\cdot\vec{p} + \gamma^0 m + V(r)] = [\gamma^0, \vec{\alpha}\cdot\vec{p}] + \underset{0}{\underset{\parallel}{[\gamma^0, \gamma^0 m]}} + \underset{0}{\underset{\parallel}{[\gamma^0, V(r)\mathbb{1}]}}$

$$= [\gamma^0, \vec{\alpha}\cdot\hat{\vec{p}}] = [\gamma^0, \gamma^0\vec{\gamma}\cdot\vec{p}]$$

$$= \gamma^0\gamma^0\vec{\gamma}\cdot\vec{p} - \gamma^0\vec{\gamma}\cdot\vec{p}\,\gamma^0 = -2\gamma^0\vec{\gamma}\cdot\vec{p}\,\gamma^0 = \underline{\underline{-2\vec{\alpha}\cdot\hat{\vec{p}}\,\gamma^0}}$$

- $[\Sigma_i, H_D] = [\Sigma_i, \alpha_j]\,\hat{p}_j = -2i\,(\vec{\alpha}\times\vec{p})_i$ (calculation done in prev. lecture)

- $\vec{\alpha}\cdot\vec{p}\;\vec{\Sigma}\cdot\vec{J} = \begin{pmatrix} 0 & \sigma_i \\ \sigma_i & 0 \end{pmatrix}\begin{pmatrix} \sigma_j & 0 \\ 0 & \sigma_j \end{pmatrix} J_j\, p_i$

$= \begin{pmatrix} 0 & \sigma_i\sigma_j \\ \sigma_i\sigma_j & 0 \end{pmatrix} p_i J_j = \begin{pmatrix} 0 & \delta_{ij} + i\epsilon_{ijk}\sigma_k \\ \delta_{ij} + i\epsilon_{ijk}\sigma_k & 0 \end{pmatrix} p_i J_j$

since $\gamma^5_{DP} \equiv \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

$\vec{\alpha}\cdot\vec{p}\;\vec{\Sigma}\cdot\vec{J} = \gamma^5\,\vec{p}\cdot\vec{J} + i\,\epsilon_{ijk}\,\alpha_k\, p_i\, J_j$

$= \gamma^5\,\vec{p}\cdot\vec{J} + i\,\vec{J}\cdot(\vec{\alpha}\times\vec{p})$

Now using the above results, we can find $[\hat{K}, H_D]$

$[\hat{K}, \hat{H}_D] = \gamma^0\vec{\Sigma}\cdot\cancelto{0}{[\vec{J}, H_D]} + \gamma^0[\vec{\Sigma}, H_D]\cdot\vec{J} + [\gamma^0, H_D]\,\vec{\Sigma}\cdot\vec{J} - \frac{1}{2}[\gamma^0, H_D]$

$= \gamma^0[\Sigma_i, H_D]\,J_i + [\gamma^0, H_D]\left(\vec{\Sigma}\cdot\vec{J} - \frac{1}{2}\right)$

$= \gamma^0(-2i)\,\vec{J}\cdot(\vec{\alpha}\times\vec{p}) + (-2\,\vec{\alpha}\cdot\vec{p}\,\gamma^0)\left(\vec{\Sigma}\cdot\vec{J} - \frac{1}{2}\right)$

$= \gamma^0\left[-2i\,\vec{J}\cdot\vec{\alpha}\times\vec{p} + 2\,\vec{\alpha}\cdot\vec{p}\left(\vec{\Sigma}\cdot\vec{J} - \frac{1}{2}\right)\right]$

$= \gamma^0\left[2\left(\gamma^5\,\vec{p}\cdot\vec{J} + \cancel{i\,\vec{J}\cdot(\vec{\alpha}\times\vec{p})}\right) - \vec{\alpha}\cdot\vec{p} - \cancel{2i\,\vec{J}\cdot(\vec{\alpha}\times\vec{p})}\right]$

$= 2\gamma^0\gamma^5\,\vec{p}\cdot\left(\vec{L} + \frac{\vec{\Sigma}}{2}\right) - \gamma^0\vec{\alpha}\cdot\vec{p}$

$\vec{\alpha}\cdot\vec{p} \rightarrow \gamma^0\vec{\gamma}$

Now $\gamma^0\gamma^5\vec{\Sigma} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} \vec{\sigma} & 0 \\ 0 & \vec{\sigma} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} \vec{\sigma} & 0 \\ 0 & \vec{\sigma} \end{pmatrix} = \begin{pmatrix} 0 & \vec{\sigma} \\ -\vec{\sigma} & 0 \end{pmatrix} \equiv \vec{\gamma}$

and $\vec{p}\cdot\vec{L} = \vec{p}\cdot(\vec{r}\times\vec{p}) = 0$

$\Rightarrow\; = \cancel{2}\cdot\vec{\gamma}\cdot\frac{\vec{p}}{\cancel{2}} - \gamma^{02}\,\vec{\gamma}\cdot\vec{p} = \vec{\gamma}\cdot\vec{p} - \vec{\gamma}\cdot\vec{p} = \underline{\underline{0}}$

Hence, $\boxed{[\hat{K}, \hat{H}_D] = 0}$

One can also check $[\hat{K}, J_i] = [\gamma^0 \Sigma_j J_j, J_i] - \frac{1}{2}[\gamma^0, L_i + \frac{\Sigma_i}{2}]$

Check $[\hat{K}, J_i] = 0$ and $[\hat{K}, \vec{J}^2] = 0$

$\therefore$ The complete commuting set for $V(r) + H_0^{(D)}$ is given by
$\uparrow$ free

$\underline{\{H_D, \vec{J}^2, J_3, \hat{K}\}}$ $\qquad K = \pm(j + \frac{1}{2})$

Eigenvalue eq^n of $\hat{K}$

eigenfunction $\psi_K = \begin{pmatrix} \psi_K^u \\ \psi_K^D \end{pmatrix} \Rightarrow \hat{K}\psi_K = K\begin{pmatrix} \psi_K^u \\ \psi_K^D \end{pmatrix}$

$$\hat{K} \equiv \underset{\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}}{\underset{\parallel}{\beta}} (\vec{\Sigma}\cdot\vec{L} + \mathbb{1}) = \begin{pmatrix} \vec{\sigma}\cdot\vec{L} + \mathbb{1}_2 & 0 \\ 0 & -(\vec{\sigma}\cdot\vec{L} + \mathbb{1}_2) \end{pmatrix}$$

So, $(\vec{\sigma}\cdot\vec{L} + \mathbb{1}_2)\begin{pmatrix} \psi_K^u \\ \psi_K^D \end{pmatrix} = K\begin{pmatrix} \psi_K^u \\ -\psi_K^L \end{pmatrix}$ $\leftarrow$ HOW?

$$\vec{L}^2 = (\vec{J} - \tfrac{\vec{\Sigma}}{2})^2 = \vec{J}^2 - \underset{(\vec{L} + \frac{\vec{\Sigma}}{2})\cdot\vec{\Sigma}}{\underset{\parallel}{\vec{J}\cdot\vec{\Sigma}}} + \frac{3}{4}\mathbb{1} = \vec{J}^2 - \vec{L}\cdot\vec{\Sigma} - \frac{3}{4}\mathbb{1}$$

$$= \vec{J}^2 - \begin{pmatrix} \vec{\sigma}\cdot\vec{L} & 0 \\ 0 & \vec{\sigma}\cdot\vec{L} \end{pmatrix} - \frac{3}{4}\mathbb{1} = \begin{pmatrix} -\vec{\sigma}\cdot\vec{L} + \vec{J}^2 - 3/4 & 0 \\ 0 & -\vec{\sigma}\cdot\vec{L} + \vec{J}^2 - 3/4 \end{pmatrix}$$

$\Rightarrow$ Two component wavef^{n}s ψ_K^u and ψ_K^D are eigenf^{n}s of $\vec{J}^2, \hat{K}$. They are also eigenf^{n}s of $\vec{L}^2$ but with different values of l.

From the expression for $\vec{L}^2$;

$$\vec{\sigma}\cdot\vec{L} + \mathbb{1} = \vec{J}^2 - \vec{L}^2 + \frac{1}{4}$$

Now since $(\vec{\sigma}\cdot\vec{L} + \mathbb{1})\begin{pmatrix}\psi^u \\ \psi_D\end{pmatrix} = K\begin{pmatrix}\psi^u \\ -\psi^D\end{pmatrix}$

$$\Rightarrow \left(\vec{J}^2 - \vec{L} + \frac{1}{4}\right)\begin{pmatrix}\psi^u \\ \psi^D\end{pmatrix} = K\begin{pmatrix}\psi^u \\ -\psi^D\end{pmatrix}$$

$$\Rightarrow \quad K = j(j+1) - l_u(l_u+1) + \frac{1}{4} \quad -(a)$$ why is $j_u = j_L$

$$-K = j\cdot(j+1) - l_D(l_D+1) + \frac{1}{4} \quad -(b)$$

Putting $K = \pm\left(j+\frac{1}{2}\right)$ in (a) and (b)

			$\lvert l_u - l_D \rvert$
$\Rightarrow$ • $K = j+\frac{1}{2} \longrightarrow$	$l_u = j - \frac{1}{2}$	$l_D = j + \frac{1}{2}$	1
• $K = -\left(j+\frac{1}{2}\right) \longrightarrow$	$l_D = j + \frac{1}{2}$	$l_D = j - \frac{1}{2}$	1

For a given j, $l_u = j \mp \frac{1}{2}$ as $K = \pm\left(j+\frac{1}{2}\right)$

$j = \frac{1}{2}$	$l = 0$	$s = \frac{1}{2}$	$\uparrow\uparrow$	parallel $\vec{L}$ & $\vec{S}$
$j = \frac{1}{2}$	$l = 1$	$s = \frac{1}{2}$	$\uparrow\downarrow$	anti-parallel $\vec{L}$ & $\vec{S}$

PARITY:

$$\psi'(x') = P\psi(x) = \gamma^0\psi(x)$$

$$\psi'_{\pm}(\vec{x}) = P\psi_{\pm}(-\vec{x}) = \gamma^0\,\psi_{\pm}(-\vec{x})$$

$$\gamma^0 = \begin{pmatrix}1 & 0 \\ 0 & -1\end{pmatrix}, \quad \psi_{\pm}(-\vec{x}) = \begin{pmatrix}\psi^u_{\pm}(-\vec{x}) \\ \psi^L_{\pm}(-\vec{x})\end{pmatrix}$$

So, $P\,\psi_\pm(\vec{x}) = \pm\,\psi_\pm(\vec{x}) \Rightarrow \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} \psi_\pm^u(-\vec{x}) \\ \psi_\pm^L(-\vec{x}) \end{pmatrix} = \pm \begin{pmatrix} \psi_u(+\vec{x}) \\ \psi_L(\vec{x}) \end{pmatrix}$

$$\therefore\quad \psi_\pm^u(-\vec{x}) = \pm\,\psi_\pm^u(\vec{x})$$
$$\psi_\pm^L(-\vec{x}) = \mp\,\psi_\pm^L(\vec{x})$$

Eigenf^{n}s of $\vec{J}^2$, J_3, will be built from $\vec{L}^2, L_3$, $\vec{S}^2 = \frac{3}{4}$, S_3. These eigenf^{n}s are known as spherical harmonics $\otimes$ spin eigenf^{n}s.
$\left(\left|\frac{1}{2}, \pm\frac{1}{2}\right\rangle\right)$

Parity of $Y_{\ell m} \to (-1)^\ell$

However as discussed earlier, $|\ell^u - \ell^L| = 1 \Rightarrow$ the parity of ψ^u differs by ψ^L by a (-1). So their parity will be opposite.

$$\begin{pmatrix} \psi^u \\ \psi^L \end{pmatrix} = \begin{pmatrix} g(r)\,\mathcal{Y}^{j_3}_{j\,l_u} \\ i f(r)\,\mathcal{Y}^{j_3}_{j\,l_L} \end{pmatrix}$$

$\mathcal{Y}^{j_3}_{j\,l} \to$ generalised spherical harmonics.

$\langle \theta, \phi | j, j_3, l \rangle$

$\uparrow$

obtained from $|\ell, m\rangle \left|\frac{1}{2}, s_z\right\rangle$

Calculation of Clebsch Gordon coeffs is like this -

$$\left|j, j-\tfrac{1}{2}, j_3\right\rangle = \alpha\left|j-\tfrac{1}{2}, j_3-\tfrac{1}{2}\right\rangle \cdot \left|\tfrac{1}{2}, \tfrac{1}{2}\right\rangle + \beta\left|j-\tfrac{1}{2}, j_3+\tfrac{1}{2}\right\rangle\left|\tfrac{1}{2}, -\tfrac{1}{2}\right\rangle$$

Projecting along $\langle\theta, \phi|$

$$\mathcal{Y}^{j_3}_{j,\,j-\frac{1}{2}} = \alpha\, Y^{j_3-\frac{1}{2}}_{j-\frac{1}{2}}(\theta,\phi)\begin{pmatrix} 1 \\ 0 \end{pmatrix} + \beta\, Y^{j_3+\frac{1}{2}}_{j-\frac{1}{2}}(\theta,\phi)\begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$\mathcal{Y}^{j_3}_{j,\,j+\frac{1}{2}} = \delta\, Y^{j_3-\frac{1}{2}}_{j+\frac{1}{2}}(\theta,\phi)\begin{pmatrix} 1 \\ 0 \end{pmatrix} + \gamma\, Y^{j_3+\frac{1}{2}}_{j+\frac{1}{2}}(\theta,\phi)\begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$\psi_+ = \begin{pmatrix} g_+(r)\, \mathcal{Y}^{j_3}_{j,j-\frac{1}{2}} \\ i f_+(r)\, \mathcal{Y}^{j_3}_{j,j+\frac{1}{2}} \end{pmatrix} \qquad \psi_- = \begin{pmatrix} g_-(r)\, \mathcal{Y}^{j_3}_{j,j+\frac{1}{2}} \\ i f_-(r)\, \mathcal{Y}^{j_3}_{j,j-\frac{1}{2}} \end{pmatrix}$$

$\uparrow$ +ve Parity $\qquad$ $\uparrow$ -ve Parity.

Dirac Equation.

$$\vec{\sigma}\cdot\vec{p} = \underset{\overset{\shortparallel}{1}}{(\vec{\sigma}\cdot\hat{x})^2}\ \vec{\sigma}\cdot\vec{p} = \frac{\vec{\sigma}\cdot\vec{x}}{r}\ \underbrace{\vec{\sigma}\cdot\vec{x}\ \vec{\sigma}\cdot\vec{p}}_{\underbrace{\sigma_i\sigma_j}_{(\delta_{ij}+i\epsilon_{ijk}\sigma_k)} x_i p_j}$$

$$= \frac{\vec{\sigma}\cdot\vec{x}}{r}\left(\vec{x}\cdot(-i\vec{\nabla}) + i\,\vec{\sigma}\cdot\vec{L}\right)$$

$$= \frac{\vec{\sigma}\cdot\vec{x}}{r}\left(-ir\frac{\partial}{\partial r} + i\,\vec{\sigma}\cdot\vec{L}\right)$$

↳ $\vec{\sigma}\cdot\hat{x}$ is a pseudo-scalar operator. $[\vec{J}, \vec{\sigma}\cdot\hat{x}] = 0$

under parity $\vec{\sigma}\cdot\hat{x} \longrightarrow -\vec{\sigma}\cdot\hat{x}$

$\therefore$ $\vec{\sigma}\cdot\hat{x}\,\mathcal{Y}^{j_3}_{j\ell}$ has <u>same</u> j, j_3 but opposite parity to $\mathcal{Y}^{j_3}_{j\ell}$

$\frac{\vec{\sigma}\cdot\vec{x}}{r} \rightarrow$ spin-1 object $\longleftarrow \ell = 1$

Application of $\vec{\sigma}\cdot\hat{x}$ can change ℓ by 0 or 1.

$$\vec{\sigma}\cdot\hat{x}\ \mathcal{Y}^{j_3}_{j\,\ell_u} = \pm\ \mathcal{Y}^{j_3}_{j\,\ell_D}$$

$$\vec{\sigma}\cdot\hat{x} = C_\theta \sigma_3 + S_\theta C_\phi \sigma_1 + S_\theta S_\phi \sigma_2$$

$$= \begin{pmatrix} C_\theta & S_\theta e^{-i\phi} \\ S_\theta e^{i\phi} & -C_\theta \end{pmatrix}$$

So, $$\vec{\sigma}\cdot\hat{x}\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix} C_\theta \\ S_\theta e^{i\phi}\end{pmatrix}$$

$$\vec{\sigma}\cdot\hat{x}\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix} S_\theta e^{-i\phi} \\ -C_\theta\end{pmatrix}$$

$$\vec{\sigma}\cdot\hat{x}\, y^{j_3}_{j,j-\frac{1}{2}} = \vec{\sigma}\cdot\hat{x}\left(\alpha y^{j_3-\frac{1}{2}}_{j,j-\frac{1}{2}}\begin{pmatrix}1\\0\end{pmatrix} + \beta y^{j_3+\frac{1}{2}}_{j,j-\frac{1}{2}}\begin{pmatrix}0\\1\end{pmatrix}\right)$$

$$= \alpha y^{j_3-\frac{1}{2}}_{j,j-\frac{1}{2}}\begin{pmatrix} C_\theta \\ S_\theta e^{i\phi}\end{pmatrix} + \beta y^{j_3+\frac{1}{2}}_{j,j-\frac{1}{2}}\begin{pmatrix} S_\theta e^{-i\phi} \\ -C_\theta\end{pmatrix}$$

$$= \begin{pmatrix} \alpha y^{j_3-\frac{1}{2}}_{j,j-\frac{1}{2}}\cos\theta + \beta y^{j_3+\frac{1}{2}}_{j,j-\frac{1}{2}}\sin\theta\, e^{-i\phi} \\ \alpha y^{j_3-\frac{1}{2}}_{j,j-\frac{1}{2}}\sin\theta\, e^{i\phi} - \beta y^{j_3+\frac{1}{2}}_{j,j-\frac{1}{2}}\cos\theta \end{pmatrix}$$

$$\alpha = \sqrt{\frac{j+j_3}{2j}} \qquad \beta = \sqrt{\frac{j-j_3}{2j}} \qquad \delta = -\sqrt{\frac{j-j_3}{2(j+1)}} \qquad \gamma = \sqrt{\frac{j+1+j_3}{2(j+1)}}$$

CB coeffs

Using these, we see that $\boxed{\vec{\sigma}\cdot\hat{x}\, y^{j_3}_{j,j-\frac{1}{2}} = -y^{j_3}_{j,j+\frac{1}{2}}}$

Lecture-18.
(11-10-21)

Dirac particle in V(r) (spherically symmetric)

In, NRQM, when we analyze problems of central potⁿ like 3D SHO or the classic Hydrogen atom, we begin by identifying constants of motion

conserved ops. $H_D \to E_n, \vec{L}^2, L_3$ $\Big\} \to \psi_{nlm} \sim R_{nl}(r)\, Y_{lm}(\theta,\phi)$

q. nos. n, l, m

$\Rightarrow$ Diff. eqⁿ in r for R_{nl}, in θ, ϕ for Y_{lm}

$R_{nl} \to$ Laguerre polynomials , $Y_{lm} \to$ spherical harmonics.

$$E_n = -13.6 \frac{Z^2}{a_0} \qquad a_0 = \frac{1}{\alpha m_e} = \frac{4\pi}{e^2 m_e}$$

Let's try out this problem in Dirac's relativistic QM.

$$H_D = \vec{\alpha}\cdot\hat{\vec{p}} + \beta m + V(r)\,\mathbb{1} \qquad \vec{\alpha}_{DP} = \begin{pmatrix} 0 & \vec{\sigma} \\ \vec{\sigma} & 0 \end{pmatrix} \quad \beta_{DP} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

and we aim to solve the eigenvalue eqⁿ

$$\hat{H}_D \psi = E\psi$$

$$\Rightarrow \begin{pmatrix} m - E + V & \vec{\sigma}\cdot\vec{p} \\ \vec{\sigma}\cdot\vec{p} & -m-E+V \end{pmatrix} \begin{pmatrix} u \\ D \end{pmatrix} = 0$$

Apart from q. nos. n, l, m, we found other constants of motion. The complete set of commuting operators is –

$$\hat{K} = \left(\beta\, \vec{\Sigma}\cdot\vec{J} - \tfrac{1}{2}\right) = \gamma^0(\vec{\Sigma}\cdot\vec{L} + 1)$$

H_D	$\vec{J}^2$	J_3	$\hat{K}$
E_n	$j(j+1)$	j_3	k

↑ half integers. $(j = l \oplus \frac{1}{2})$

$$\hat{K}\psi_k = k\psi_k \Rightarrow \begin{pmatrix} \vec{\sigma}\cdot\vec{L}+\mathbb{1} & 0 \\ 0 & -(\vec{\sigma}\cdot\vec{L}+\mathbb{1}) \end{pmatrix}\begin{pmatrix} U_k \\ D_k \end{pmatrix} = k\begin{pmatrix} U_k \\ D_k \end{pmatrix}$$

So, $\vec{\sigma}\cdot\vec{L}\; U_k = (k-1)\,U_k$

$\vec{\sigma}\cdot\vec{L}\; D_k = -(k+1)\,D_k$.

Now recalling the def^n of $\hat{K}$, $\hat{K}^2 = \vec{J}^2 + \frac{1}{4} = (j+\frac{1}{2})^2 = k^2$ for eigenf^{n}s.

Hence $k = \pm(j+\frac{1}{2})$

Now, if U_k, D_k are eigenf^{n}s of $\vec{L}^2$ (since they are simult. diag.) with eigenvalue $l_U(l_U+1)$, $l_D(l_D+1)$, then we can say that

$$k = \pm(j+\tfrac{1}{2}) \longrightarrow \begin{array}{l} l_U = j \mp \frac{1}{2} \\ l_D = j \pm \frac{1}{2} \end{array}$$

(which can be obtained by writing $\vec{J}^2 - \vec{L} + \frac{1}{4} = \vec{\sigma}\cdot\vec{L} + \mathbb{1}$)

We also concluded that Parity is also a good symmetry here

$|l_U - l_D| = 1 \Rightarrow U$ & D have opposite parities. Therefore, the form of the eigenf^{n}s has generalized spherical harmonics.

$$\begin{pmatrix} g_+(r)\, \mathcal{Y}^{j_3}_{j,\,j-\frac{1}{2}} \\ i f_+(r)\, \mathcal{Y}^{j_3}_{j,\,j+\frac{1}{2}} \end{pmatrix} \begin{array}{l} \to l_U \\ \to l_D \end{array}$$

$\to$ +ve parity state $\longrightarrow$ how did you conclude that?

$$\psi_\pm(-\vec{x}) = \pm\psi_\pm(\vec{x}) \longrightarrow \text{Parity eigenstates}$$

We then showed that $\vec{\sigma}\cdot\hat{x}\; \mathcal{Y}^{j_3}_{j,\,l_U} = -\,\mathcal{Y}^{j_3}_{j,\,l_D}$

Using this relation for $\vec{\sigma}\cdot\hat{x}$, we can write the following -

$$\vec{\sigma}\cdot\vec{p} = -i\vec{\sigma}\cdot\vec{\nabla} = (\vec{\sigma}\cdot\hat{x})^2\,\vec{\sigma}\cdot\vec{p} = \frac{\vec{\sigma}\cdot\hat{x}}{r}\left(-ir\partial_r + i\vec{\sigma}\cdot\vec{L}\right)$$

We can now use these operators to be applied on the Dirac eq^n.

$$\vec{\sigma}\cdot\vec{p}\,D = \frac{\vec{\sigma}\cdot\vec{x}}{r}\Big(-ir\,\partial_r + i\underbrace{\vec{\sigma}\cdot\vec{L}}_{-(k+1)}\Big)\left(if\,\mathcal{Y}^{j_3}_{j\,l_D}\right)D$$

$U, D \equiv \psi_{u,D}$

Dirac eq^n in 2-components

$$\vec{\sigma}\cdot\vec{p}\;\psi_D = (m+E-V)\,\psi_u$$
$$\vec{\sigma}\cdot\vec{p}\;\psi_u = (m+E-V)\,\psi_D$$

Also, $\vec{\sigma}\cdot\hat{x}\,\psi_u = -\psi_D$

Plugging in and using $\psi \sim \begin{pmatrix} g\,\mathcal{Y}^{j_3}_{j\,l_u} \\ if\,\mathcal{Y}^{j_3}_{j,\,l_D}\end{pmatrix}\begin{matrix}\to \psi_u \\ \to \psi_D\end{matrix}$

$$-f'(r) - \frac{(k+1)}{r}f_{(r)} = (E-V-m)\,g(r)$$

$$g'(r) - \frac{(k-1)}{r}g = (E-V+m)\,f(r)$$

how did I write these? This makes no sense.

Defining $R = rf$ and $G = rg$ gives

$$\begin{cases} F'(r) + \frac{k}{r}F = -(E-V(r)+m)\,G \\ G'(r) - \frac{k}{r}G = (E-V+m)\,F \end{cases}$$

These DEs can be solved for any spherically symm. $V(r)$ to get the Dirac wavef^{n}s & H_D eigenvalues!

- For Hydrogenic atom, $V(r) = -\frac{Ze^2}{4\pi r}$

B.C. $\int \psi^\dagger\psi\, d^3x < \infty$

If we define $\alpha_1 = m+E$, $\alpha_2 = E-m$, $\gamma = \frac{Ze^2}{4\pi}$

$\rho = \sqrt{\alpha_1\alpha_2}\, r$

$$\Rightarrow \left(\frac{\partial}{\partial\rho} + \frac{k}{\rho}\right) F(\rho) - \left(\sqrt{\frac{\alpha_2}{\alpha_1}} - \frac{\gamma}{\rho}\right) G(\rho) = 0$$

$$\left(\frac{\partial}{\partial\rho} - \frac{k}{\rho}\right) G(\rho) - \left(\sqrt{\frac{\alpha_1}{\alpha_2}} + \frac{\gamma}{\rho}\right) F(\rho) = 0$$

$$\left.\begin{aligned} F(\rho) &= e^{-\rho} \rho^s \sum_{m=0}^{\infty} a_m \rho^m \\ G(\rho) &= e^{-\rho} \rho^s \sum_{m=0}^{\infty} b_m \rho^m \end{aligned}\right\} \text{Frobenius ansatz}$$

B.C.s $\Rightarrow$ series should truncate.

Truncation $\Rightarrow$ Eigenvalues E are discrete, which gives

$$E_{n'k} = \frac{m_e}{\sqrt{1 + \gamma^2/(s+n')^2}}$$

$s = \sqrt{k^2 - \gamma^2} = \sqrt{\left(j+\frac{1}{2}\right)^2 - Z^2\alpha_{EM}^2}$

$n' = 0, 1, 2 \ldots\ldots$

$$E = m\left[1 + \frac{Z^2\alpha_{EM}^2}{\left(n' + \sqrt{(j+\frac{1}{2})^2 - Z^2\alpha_{EM}^2}\right)}\right]^{1/2}$$

Define $n = n' + \underbrace{\left(j+\frac{1}{2}\right)}_{=|K|}$ → half integer. min. value $\frac{1}{2}$ → integer principle q. no.

$\therefore n \geq 1$

Expanding in powers of $\alpha_{EM} \approx 1/137$

$$E_{n,j} = m_e\left(1 - \underbrace{\frac{Z^2\alpha_{EM}^2}{2n^2}}_{\text{Rydberg term}} - \underbrace{\frac{1}{2}\frac{Z^4\alpha^4}{n^3}\left(\frac{1}{|K|} - \frac{3}{4n}\right)}_{\text{relativistic corrections.}} + \mathcal{O}(\alpha^6)\right) + \ldots$$

$$\frac{\alpha_{em}^2 m_e}{2} = \frac{\alpha_{EM}}{2a_0} \qquad a_0 = \text{Bohr radius} = \frac{1}{\alpha m_e}$$

If we take $\Delta H^{fs} = \left(-\frac{\vec{p}^{\,4}}{8m^3} - e\frac{\vec{\sigma}\cdot\vec{E}\times\vec{p}}{4m^2} - \frac{e}{8m^2}\vec{\nabla}\cdot\vec{E} \right)$

spin-orbit coupling — Darwin term.

$\vec{E}$: electric field at location of the electron $\vec{r}$.

$$= -\frac{p^4}{8m^3c^2} + \frac{1}{2m}\frac{V'(r)}{r}\vec{S}\cdot\vec{L} + \frac{1}{8m^2}\nabla^2 V$$

and then we solve for $\langle \Delta H \rangle$ (first order perturbation correction)

then we get the same relativistic correction term as obtained by expanding in powers of α_{em}.

Spectroscopic not^n: n, l_u, j

$$k = \pm\left(j+\frac{1}{2}\right) \longleftrightarrow l_u = j \mp \frac{1}{2} \qquad n \begin{pmatrix} s \\ p \\ d \\ f \\ \vdots \end{pmatrix} \begin{matrix} j \\ \downarrow \\ \frac{1}{2}, \frac{3}{2} \cdots \end{matrix}$$

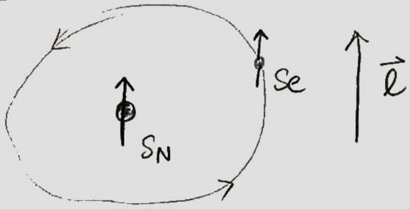
$j = \frac{1}{2} \longrightarrow l = 0, 1 \longrightarrow ns_{1/2}$ or $np_{1/2}$ $\Rightarrow$ Degenerate states since $E \equiv E(n, j)$

So, $2p_{1/2}$ and $2s_{1/2}$ are degenerate from Dirac theory. But actually,

$E_{s_{1/2}} - E_{p_{1/2}} \neq 0$, and was actually measured in 1947, and is known as Lamb shift ~ 1060 MHz.

This splitting exists due to self energy + vacuum polarization corrections, which is properly done in the framework of QFT.

Hyperfine interactions.



Total ang. mom^n $\vec{F} = \vec{S_e} + \vec{S_N}$ $\left(\frac{1}{2} \oplus \frac{1}{2}\right)$

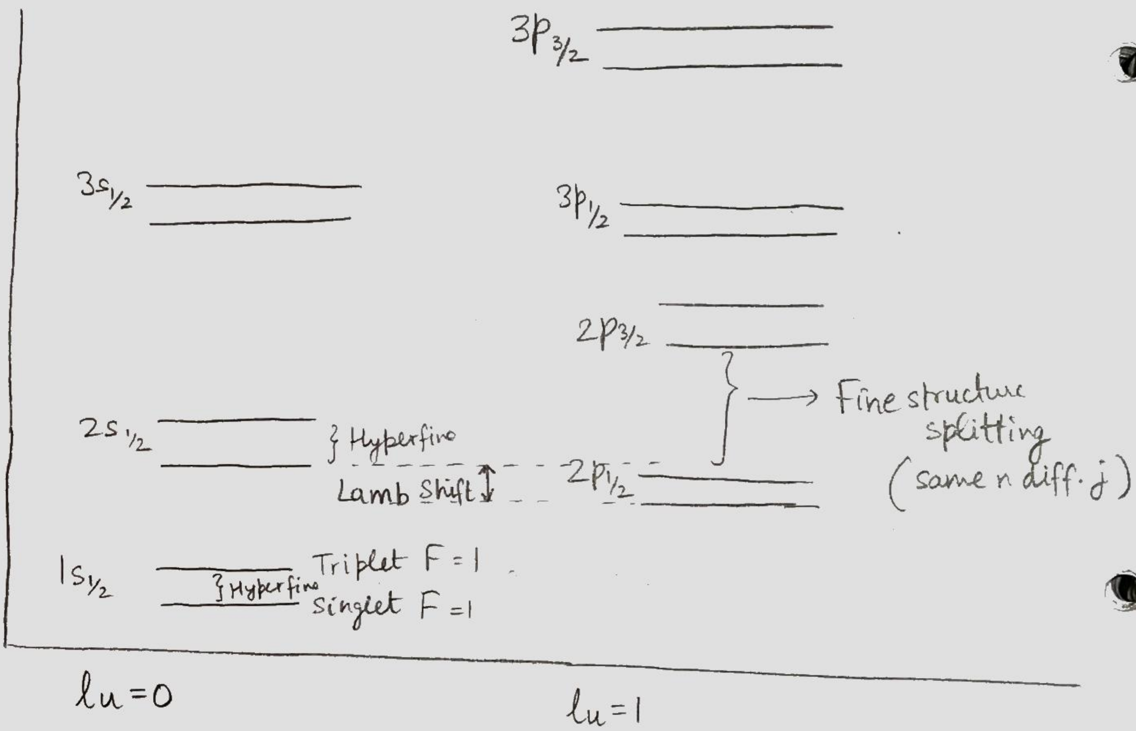
$\Rightarrow$ Total electronic + nuclear spin is 0 or 1.

$\Delta H_{hyperfine} \sim \vec{\mu}_N \cdot \vec{\mu}_e \sim \vec{S}_N \cdot \vec{S}_e$

$\rightarrow$ H atom orbitals split $\Delta E_n^{hyperfine} \approx \alpha^4 \left(\frac{m_e}{m_p}\right) \frac{m_e}{n^3}$

whereas $\Delta E_n^{fine\text{-}structure} \sim \frac{\alpha^4 m}{2n^3} \sim 5\times10^{-5}\times10\,eV \sim 10^{11}\,Hz$

$\frac{m_e}{m_p} = \frac{0.5}{938} \sim \frac{1}{1876} \sim 0.5\times10^{-3} \Rightarrow \Delta E_n^{hyperfine} \sim 10^9\,Hz$



Fine-structure > Hyperfine ≈ Lamb Shift.

Propagators
&
Classical Field Theory

Lecture - 22

(25-10-2021)

Green's functions for Partial Differential Equations:

Consider Poisson eq^n. $\nabla^2 \phi(\vec{x}) = -\rho(\vec{x})$

The central idea behind the Green's function is that if we know how to solve for $G(\vec{x},\vec{x}')$ given

$$\nabla^2 G(\vec{x},\vec{x}') = +\delta^{(3)}(\vec{x}-\vec{x}')$$

then we can write the general sol^n of $\phi(\vec{x})$ for Poisson's eq^n as

$$\phi(\vec{x}) = \phi_0(\vec{x}) - \int d^3\vec{x}' \; G(\vec{x},\vec{x}')\,\rho(\vec{x}')$$

$\uparrow$ Sol^n of homogeneous Laplace eq^n $\nabla^2\phi_0(\vec{x})=0$

because action of ∇^2 on ϕ now gives $-\rho(\vec{x})$.

Infact, whenever we have an operator M acting on vector V equal to source J i.e.

$$\underset{\text{operator}}{M} \; \underset{\text{vectors}}{V} = J$$

then we can write the general sol^n. $V = V_0 + M^{-1}J$. where $MV_0 = 0$ (null vector)

Now, the object we are interested in finding is the Green's function -

$$\nabla^2 G(\vec{x},\vec{x}') \equiv \nabla^2 G(\vec{x}-\vec{x}') = \delta^{(3)}(\vec{x}-\vec{x}')$$

$\uparrow$ in ∞ space (Changing origin doesn't change anything)

Now $\delta^3(\vec{x}-\vec{x}') = \frac{1}{(2\pi)^3}\int d^3\vec{q}\; e^{i\vec{q}\cdot(\vec{x}-\vec{x}')}$

If we Fourier tranform $G(\vec{x},\vec{x}')$ we get -

$$G(\vec{x}-\vec{x}') = \frac{1}{(2\pi)^3}\int d^3\vec{q}\; \tilde{G}(\vec{q})\, e^{i\vec{q}\cdot(\vec{x}-\vec{x}')}$$

So, $\nabla^2 G = \frac{1}{(2\pi)^3}\int d^3\vec{q}\; \tilde{G}(\vec{q})\, \underbrace{\nabla^2 e^{i\vec{q}\cdot(\vec{x}-\vec{x}')}}_{-\vec{q}^{\,2} e^{i\vec{q}\cdot(\vec{x}-\vec{x}')}} = \int \frac{d^3\vec{q}}{(2\pi)^3}\left[-\tilde{G}(q)\,\vec{q}^{\,2}\right] e^{i\vec{q}\cdot(\vec{x}-\vec{x}')}$

Equating $\nabla^2 G = +\delta^3(\vec{x}-\vec{x}')$

$$\Rightarrow \quad \tilde{G}(\vec{q}) = \frac{-1}{|\vec{q}|^2}$$

Now since $G(\vec{x}-\vec{x}') = G(\vec{r}) = \int \frac{d^3\vec{q}}{(2\pi)^3}\cdot\left(\frac{-1}{|\vec{q}|^2}\right) e^{i\vec{q}\cdot(\vec{x}-\vec{x}')}$

$$= \frac{-1}{(2\pi)^3}\int d^3\vec{q}\cdot\frac{1}{|\vec{q}|^2}\cdot e^{i\vec{q}\cdot(\vec{x}-\vec{x}')}$$

$$\int d^3\vec{q} = \int_0^\infty dq\, q^2 \int_0^{2\pi} d\phi_q \int_0^\pi d\theta_q \sin\theta_q$$

take $\xi = \cos\theta$

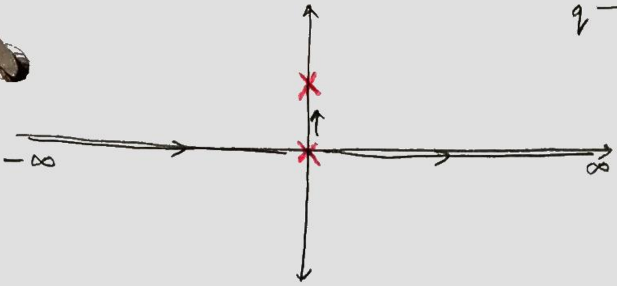
$d\xi = -\sin\theta\, d\theta$

$$\Rightarrow G(\vec{r}) = \frac{-1}{(2\pi)^3}\cdot(2\pi)\int_0^\infty dq\, q^2 \int_0^\pi d\theta \sin\theta\, \frac{e^{i|\vec{q}||\vec{x}-\vec{x}'|\cos\theta}}{|\vec{q}|^2}$$

$$= \frac{-1}{(2\pi)^2}\int_0^\infty dq\, \cancel{q^2}\underbrace{\int_{-1}^{+1} d\xi\, \frac{e^{iqr\xi}}{\cancel{q^2}}}_{\frac{e^{iqr}-e^{-iqr}}{iqr}}$$

$$= \frac{-1}{4\pi^2}\cdot\int_0^\infty \frac{2\sin(qr)}{qr}\,dq = \frac{-1}{2\pi^2 r}\int_0^\infty \frac{\sin qr}{q}\,dq$$

$$= \frac{-1}{4\pi^2 r}\int_{-\infty}^{\infty} dq\, \frac{\sin qr}{q}$$



$q \to q - i\epsilon$

Shift the pole & now do the contour integral over the real axis.

However, in this case, we can get away without doing the calculation since we know that $\nabla^2 \cdot \frac{1}{r} = -\delta^{(3)}(\vec{r}) \Rightarrow \underbrace{G(\vec{r}) = -\frac{1}{4\pi|\vec{x}-\vec{x}'|}}_{\text{potential at } \vec{x} \text{ of a unit charge at } \vec{x}'}$

$$\Rightarrow \phi(\vec{x}) = \phi_0(\vec{x}) + \int d^3\vec{x}' \frac{\rho(\vec{x}')}{4\pi|\vec{x}-\vec{x}'|} \quad \to dQ(\vec{x}')$$

This was the solⁿ of Green function for Poisson's eqⁿ. We can consider other eqⁿs as well -

Wave eqⁿ: $\Box \vec{A} \equiv (\nabla^2 - \partial_t^2)\vec{A}(\vec{x},t) = \vec{j}(\vec{x},t)$

Klein-Gordon: $(\Box + m^2)\phi(\vec{x},t) = 0$

Dirac eqⁿ: $(i\gamma^\mu \partial_\mu - m)_{\alpha\beta} \psi_\beta(x) = 0$

Schrödinger type eqⁿ: $(i\partial_t - H)|\psi\rangle = 0$

Let's start by now finding the <u>Green fⁿ for K-G eqⁿ</u>

$$(\Box + m^2)\, G(\underline{x} - \underline{x}') = -\delta^4(\underline{x} - \underline{x}')$$

$$-\delta^4(\underline{x} - \underline{x}') = \frac{-1}{(2\pi)^4} \int d^4\underline{p}\; e^{-i\underline{p}\cdot(\underline{x}-\underline{x}')} \qquad \text{—— Eqn. (a)}$$

$\uparrow\; p_0(t-t') - \vec{p}\cdot(\vec{x}-\vec{x}')$

Again writing down the F.T. of the Green function $G(\underline{x}-\underline{x}')$

$$G(\underline{x}-\underline{x}') = \frac{1}{(2\pi)^4}\int d^4p\ \tilde{G}(p)\ e^{-ip\cdot(\underline{x}-\underline{x}')}$$

$$\Box = \partial_\mu \partial^\mu = \partial_t^2 - \partial_x^2 - \partial_y^2 - \partial_z^2$$

then $\Box\, e^{-ip\cdot(\underline{x}-\underline{x}')} = \partial_\mu \partial^\mu (e^{-ip\cdot\underline{x}}) = -\underbrace{p_\mu p^\mu}_{p^2} e^{-ip\cdot\underline{x}}$

So, $\Box\, G(\underline{x}-\underline{x}') = \frac{-1}{(2\pi)^4}\int d^4p\ \tilde{G}(p)\ p^2\ e^{-ip\cdot(\underline{x}-\underline{x}')}$ —— Eqn.(b)

Equating $(\Box + m^2) G(\underline{x}-\underline{x}') = -\delta^4(\underline{x}-\underline{x}')$ from (a) & (b)

$$\Rightarrow \underbrace{(-p^2+m^2)}_{= (-p_0^2 + E_p^2)} \tilde{G}(p) = -1 \quad \Rightarrow \quad \tilde{G}(p) = \frac{1}{p_0^2 - E_p^2}$$

Remember that p is just a dummy. No one said it's momentum yet.

$$\text{So, } G(\underline{x}-\underline{x}') = \int \frac{d^3\vec{p}}{(2\pi)^3}\int_{-\infty}^{\infty}\frac{dp_0}{2\pi}\ \frac{e^{-i(p_0(t-t') - \vec{p}\cdot(\vec{x}-\vec{x}'))}}{p_0^2 - E_p^2}$$

$p_0^2 - E_p^2 \longrightarrow (p_0 - E_p)(p_0 + E_p)$

Integrand of p_0 integral has poles at $p_0 = \pm E_p$!

We need to specify a rule to make p_0 integral well defined. Different rules yield Green's f^{n}s obeying different boundary conditions appropriate to different situations.

Rule: prescription to move poles off the $\mathbb{R}$ axis

Some of the BCs that arise are -

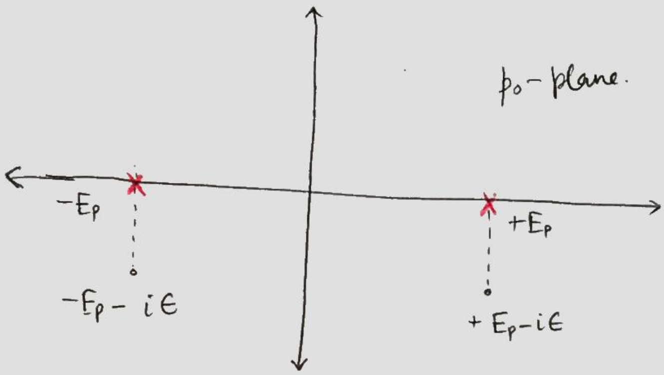
- retarded boundary conditions.
- Feynman boundary conditions.

Continue from 36:04.

Retarded Boundary conditions (causal boundary conditions)

Start by complexifying the variable p_0. $p_0 \in \mathbb{C}$

$$G(\underline{x}-\underline{x}') = \int \frac{d^3p}{(2\pi)^3} \int_{-\infty}^{\infty} \frac{dp_0}{2\pi} \frac{e^{-i(p_0(t-t') - \vec{p}\cdot(\vec{x}-\vec{x}'))}}{p_0^2 - E_p^2}$$



poles go from

$$\boxed{\pm E_p \longrightarrow \pm E_p - i\epsilon}$$

$$\int_{-\infty}^{\infty} \frac{dp_0}{2\pi} \frac{e^{-ip_0(t-t')}}{p_0^2 - E_p^2} \longrightarrow \text{poles at } p_0 = \pm E_p$$

If we shift poles from $\pm E_p \longrightarrow \pm E_p - i\epsilon$

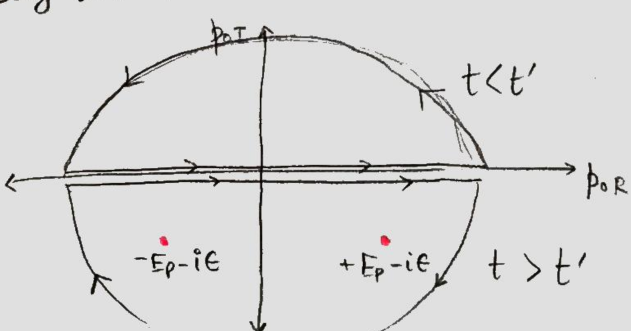
$$\int_{-\infty}^{\infty} \frac{dp_0}{2\pi} \frac{e^{-ip_0(t-t')}}{(p_0 - (E_p - i\epsilon))(p_0 - (-E_p - i\epsilon))}$$

Now since $p_0 \in \mathbb{C}$, $p_0 = p_{0R} + i p_{0I} \Rightarrow e^{-ip_0(t-t')} = e^{-ip_{0R}(t-t')} e^{p_{0I}(t-t')}$

For $t>t'$, this function damps out at infinities in the lower half.

For $t<t'$, this function damps out at infinities in the upper half.

Corresponding to both these cases, we can choose closed contours



As a result, for $t<t'$ $\oint_{\text{upper half}} \frac{dp_0}{2\pi} \frac{e^{-ip_0(t-t')}}{p_0^2 - E_p^2} = 0 \rightarrow$ no part dependent on $t<t'$

and for the case $t>t'$, $\oint_{\text{lower half}} \frac{dp_0}{2\pi} \frac{e^{-ip_0(t-t')}}{p_0^2 - E_p^2} \underset{\uparrow \text{ clockwise}}{=} -2\pi i \left[\text{Res}\left(f(E_p - i\epsilon)\right) + \text{Res}\left(f(-E_p - i\epsilon)\right)\right] \cdot \Theta(t-t')$

★ For a pole at $p_0 = P$, the $\text{Res}(f(P))$ is the coefficient factor of $\frac{1}{p_0 - P}$ evaluated at $p_0 = P$.

$$\frac{e^{-ip_0(t-t')}}{(p_0 - (E_p - i\epsilon))(p_0 + (E_p + i\epsilon))} = \frac{1}{2E_p}\left[\frac{1}{p_0 - (E_p - i\epsilon)} - \frac{1}{p_0 - (-E_p - i\epsilon)}\right] \frac{e^{-ip_0(t-t')}}{2\pi}$$

$$\text{Res}(E_p - i\epsilon) = \frac{1}{2E_p} \cdot \frac{e^{-ip_0(t-t')}}{2\pi}\Bigg|_{p_0 = E_p - i\epsilon} = \frac{1}{2E_p} \frac{e^{-iE_p(t-t')}}{2\pi}$$

$$\text{Res}(-E_p - i\epsilon) = \frac{1}{2E_p} \frac{(-1) \cdot e^{-ip_0(t-t')}}{2\pi}\Bigg|_{p_0 = -E_p - i\epsilon} = \frac{-1}{2E_p} \frac{e^{iE_p(t-t')}}{2\pi}$$

$$\Rightarrow \int_{-\infty}^{\infty} \frac{dp_0}{2\pi} \frac{e^{-ip_0(t-t')}}{(p_0 - (E_p - i\epsilon))(p_0 + (E_p + i\epsilon))} = \frac{i}{2E_p}\left(e^{iE_p(t-t')} - e^{-iE_p(t-t')}\right)\Theta(t-t')$$

$$\Rightarrow \boxed{G_R(\underline{x} - \underline{x}') = \frac{-i}{(2\pi)^3} \int \frac{d^3\vec{p}}{2E_p} e^{i\vec{p}\cdot(\vec{x} - \vec{x}')}\left(e^{-iE_p(t-t')} - e^{iE_p(t-t')}\right)\Theta(t-t')}$$

Feynman Boundary conditions.

$$\tilde{G}_F(p) = \tilde{\Delta}_F(p) = \frac{1}{p^2 - m^2} \xrightarrow[\text{prescription}]{\text{Feynman}} \frac{1}{p^2 - m^2 + i\epsilon'}$$

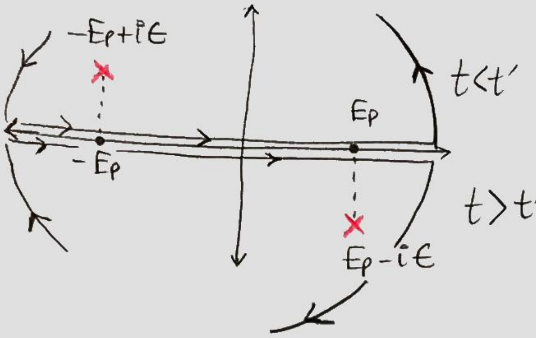
$\epsilon' = 2E_p\epsilon$

small $\rightarrow 0$ at end of calc.

$$p^2 - m^2 + i\epsilon = p_0^2 - E_p^2 + i\epsilon'$$

$$= p_0^2 - (E_p^2 - 2iE_p\epsilon) = p_0^2 - (E_p - i\epsilon)^2 + \cancel{\epsilon^2}^{\,0}$$

So, $p_0^2 - E_p^2 \xrightarrow{\text{Feynman}} p_0^2 - (E_p - i\epsilon)^2 \Rightarrow$ Poles at $\boxed{p_0 = \pm (E_p - i\epsilon)}$



So, both semi-circles now enclose a pole.

$$\frac{1}{(p_0-(E_p-i\epsilon))(p_0-(-E_p+i\epsilon))} = \frac{1}{2E_p}\left(\frac{1}{p_0-(E_p-i\epsilon)} - \frac{1}{p_0-(-E_p+i\epsilon)}\right)$$

So, $$G_F(\underline{x}-\underline{x}') = \int \frac{d^3\vec{p}}{(2\pi)^3} \int_{-\infty}^{\infty} \frac{dp_0}{2\pi} \frac{1}{2E_p}\left(\frac{1}{p_0-(E_p-i\epsilon)} - \frac{1}{p_0+(E_p-i\epsilon)}\right) e^{-ip_0(t-t')} e^{i\vec{p}\cdot(\vec{x}-\vec{x}')}$$

$$\int_{-\infty}^{\infty} \frac{dp_0}{2\pi} \cdot \frac{1}{2E_p}\left(\frac{1}{p_0-(E_p-i\epsilon)} - \frac{1}{p_0+(E_p-i\epsilon)}\right) e^{-ip_0(t-t')} = \underset{\substack{\text{lower half} \\ (t>t')}}{\oint dp_0} + \underset{\substack{\text{upper half} \\ (t<t')}}{\oint dp_0}$$

$$= 2\pi i\left[-\theta(t-t')\,\mathrm{Res}(f(E_p-i\epsilon)) + \theta(t'-t)\,\mathrm{Res}(f(-E_p+i\epsilon))\right]$$

$$= \frac{2\pi i}{2\pi}\left[\theta(t'-t)\frac{e^{+iE_p(t-t')}}{-2E_p} - \theta(t-t')\frac{e^{-iE_p(t-t')}}{2E_p}\right]$$

$$= \frac{-i}{2E_p}\left[e^{iE_p(t-t')}\theta(t'-t) + e^{-iE_p(t-t')}\theta(t-t')\right]$$

$$\Rightarrow \boxed{\Delta_F(\underline{x}-\underline{x}') = \frac{-i}{(2\pi)^3}\int \frac{d^3\vec{p}}{2E_p} e^{+i\vec{p}\cdot(\vec{x}-\vec{x}')}\left[e^{-iE_p(t-t')}\theta(t-t') + e^{iE_p(t-t')}\theta(t'-t)\right]}$$

Doing $\vec{p} \to -\vec{p}$ on the 2nd term

$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} d^3\vec{p} \to (-1)^6 \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} d^3p$

$$\Rightarrow \boxed{\Delta_F = \frac{-i}{(2\pi)^3}\int \frac{d^3\vec{p}}{2E_p}\left[e^{-ip\cdot(\underline{x}-\underline{x}')}\theta(t-t') + e^{ip\cdot(\underline{x}-\underline{x}')}\theta(t'-t)\right]}$$

Lecture-23.

(26-10-2021)

Propagator for the Dirac eqn:

$$(i\gamma^\mu \partial_\mu - m)_{\alpha\beta}\, \psi_\beta(\underline{x}) = j_\alpha(\underline{x})$$

The Green's function $S(\underline{x}-\underline{x}')$ for this eqn should satisfy:

$$(i\gamma^\mu \partial_\mu - m)_{\alpha\beta}\, S_{\beta\gamma}(\underline{x}-\underline{x}') = \delta_{\alpha\gamma}\, \delta^{(4)}(\underline{x}-\underline{x}')$$

If we can find such a $S_{\beta\gamma}(\underline{x}-\underline{x}')$, then given

$$\psi_\alpha = \psi^{(0)}_\alpha(\underline{x}) + \int d^4\underline{x}'\, S_{\alpha\beta}(\underline{x}-\underline{x}')\, j_\beta(\underline{x}')$$

$$\Rightarrow (i\gamma^\mu \partial_\mu - m)_{\alpha\beta}\, \psi_\beta(\underline{x}) = \underbrace{(i\gamma^\mu \partial_\mu - m)_{\alpha\beta}\, \psi^{(0)}_\beta(\underline{x})}_{=0} + \int d^4\underline{x}'\, \underbrace{(i\not{\partial} - m)_{\alpha\beta}\, S_{\beta\gamma}(\underline{x}-\underline{x}')}_{\delta_{\alpha\gamma}\, \delta^4(\underline{x}-\underline{x}')}\, j_\gamma(\underline{x}')$$

$$= j_\alpha(\underline{x})$$

Let's now find $S(\underline{x}-\underline{x}')$ satisfying

$$(i\gamma^\mu \partial_\mu - m)_{\alpha\beta}\, S_{\beta\gamma}(\underline{x}-\underline{x}') = \delta_{\alpha\gamma}\, \delta^{(4)}(\underline{x}-\underline{x}')$$

$$S(\underline{x}-\underline{x}') = \int \frac{d^4\underline{p}\; e^{-i\underline{p}\cdot(\underline{x}-\underline{x}')}}{(2\pi)^4}\, \underbrace{\tilde{S}(\underline{p})}_{\text{matrix}}$$

$$(i\gamma^\mu \partial_\mu - m)\, S(\underline{x}-\underline{x}') = \int \frac{d^4\underline{p}}{(2\pi)^4}\, (i\gamma^\mu(-ip_\mu) - m)\, \tilde{S}(\underline{p})\, e^{-i\underline{p}\cdot(\underline{x}-\underline{x}')}$$

$$= \int \frac{d^4\underline{p}}{(2\pi)^4}\, (\gamma\cdot\underline{p} - m)\, \tilde{S}(\underline{p})\, e^{-i\underline{p}\cdot(\underline{x}-\underline{x}')} \qquad \text{—— Eq}^n\text{(a)}$$

On the RHS $\quad \delta^4(\underline{x}-\underline{x}')\cdot \underset{\substack{\uparrow \\ \text{matrix}}}{\delta} = \int \frac{d^4\underline{p}}{(2\pi)^4}\, \mathbb{1}\, e^{-i\underline{p}\cdot(\underline{x}-\underline{x}')} \qquad \text{—— Eq}^n\text{(b)}$

$(\delta \equiv \mathbb{1})$

Equating (a) and (b), we get

$$(\underline{\gamma}\cdot\underline{p} - m)\,\tilde{S}(p) = \mathbb{1}$$

Multiplying both sides by $(\not{p}+m)$, we get

$$(\not{p}+m)(\not{p}-m)\,\tilde{S}(p) = (\not{p}+m)$$

$$(\underline{p}^2 - m^2)\,\tilde{S}(p) = (\not{p}+m) \Rightarrow \tilde{S}_{\alpha\beta}(p) = \underbrace{\frac{(\not{p}+m)_{\alpha\beta}}{\underline{p}^2-m^2}}_{\text{exact form.}} \equiv \underbrace{\left(\frac{1}{\not{p}-m}\right)_{\alpha\beta}}_{\text{shorthand.}}$$

$$S^{(F)}_{\alpha\beta}(\underline{x}-\underline{x}') = \int \frac{d^4\underline{p}}{(2\pi)^4}\, \frac{(\not{p}+m)_{\alpha\beta}\, e^{-i\underline{p}\cdot(\underline{x}-\underline{x}')}}{\underbrace{\underline{p}^2 - m^2}_{\text{poles at } \pm E_p} + i\epsilon}$$

Feynman prescription

$$\boxed{S_{\alpha\beta} = (i\not{\partial}_x + m)_{\alpha\beta}\,\Delta_F(\underline{x}-\underline{x}')} \longrightarrow$$ Δ_F here is the Klein-Gordon Green's function for Feynman B.C.s.

$$= (i\not{\partial}+m)\left(\frac{-i}{(2\pi)^3}\right)\int\frac{d^3\vec{p}}{2E_p}\left[e^{-i\underline{p}\cdot(\underline{x}-\underline{x}')}\,\Theta(t-t') + e^{i\underline{p}\cdot(\underline{x}-\underline{x}')}\,\Theta(t'-t)\right]$$

$$= \frac{-i}{(2\pi)^3}\left[\int\frac{d^3\vec{p}}{2E_p}\,(\not{p}+m)\,e^{-i\underline{p}\cdot(\underline{x}-\underline{x}')}\,\Theta(t-t') - \int\frac{d^3\vec{p}}{2E_p}\,(\not{p}-m)\,e^{i\underline{p}\cdot(\underline{x}-\underline{x}')}\,\Theta(t'-t)\right] = S_F(\underline{x}-\underline{x}')$$

Propagator for Non-relativistic Schrödinger eqn. (NR Scattering theory)

$$i\partial_t\,\psi(\vec{x},t) = \left(-\frac{\nabla^2}{2m} + V(\vec{x})\right)\psi(\vec{x},t)$$

Introduce a causal / retarded Green's function

$$\psi(\vec{x},t) = \int d^3\vec{x}' \; i\, G(\vec{x},\vec{x}',t,t')\, \psi(\vec{x}',t') \quad \text{for } t>t'$$

$$\Rightarrow \theta(t-t')\,\psi(\underline{x}) = \int d^3\vec{x}'\, i\, G(\underline{x}-\underline{x}')\, \psi(\underline{x}')$$

where $G(\underline{x}-\underline{x}')=0$ for $t<t'$

$$(i\partial_t - \underset{\downarrow \; \left(\frac{-\nabla^2}{2m}+V\right)}{H})(\theta(t-t')\psi(\underline{x}))$$

$$= i\partial_t\theta(t-t')\psi(\vec{x},t) + \theta(t-t')\, i\partial_t\psi(\vec{x},t) - \theta(t-t')\, H\psi(\vec{x},t)$$

$$= i\partial_t\theta(t-t')\psi(\vec{x},t) + \theta(t-t')\underbrace{(i\partial_t - H)\psi(\vec{x},t)}_{\overset{!}{=}0}$$

$$= i\,\delta(t-t')\,\psi(\vec{x},t)$$

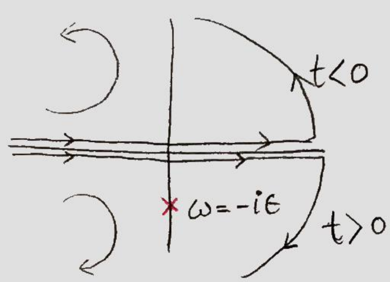
$$\therefore (i\partial_t - H)\int d^3\vec{x}'\, i\, G(\underline{x}-\underline{x}')\,\psi(\vec{x}',t) = i\,\delta(t-t')\psi(\vec{x},t)$$

$$\Rightarrow \underline{(i\partial_t - H)\, G(\underline{x}-\underline{x}') \overset{!}{=} \delta^{(4)}(\underline{x}-\underline{x}')}$$

Math aside:

Heavyside step f^n $\boxed{\Theta(t) = \frac{-1}{2\pi i}\int_{-\infty}^{\infty} \frac{e^{-i\omega t}}{\omega + i\epsilon}\, d\omega}$ goes to 0 at end

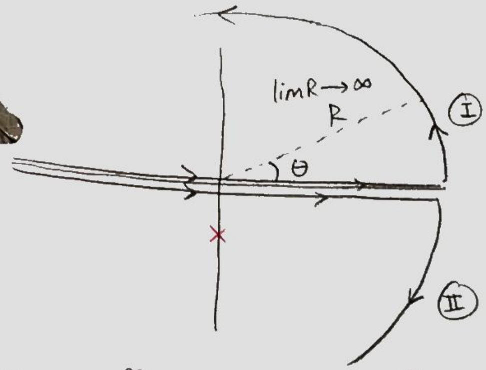
pole at $\omega = -i\epsilon$



$$= \frac{-1}{2\pi i}(-2\pi i)\,\mathrm{Res}(f(-i\epsilon))$$

$$= \lim_{\epsilon\to 0} e^{-\epsilon t} \xrightarrow{\text{how?}} = \begin{cases} 1 & \text{for } t>0 \\ 0 & \text{for } t<0 \end{cases}$$

To evaluate $\Theta(t=0)$, $\Theta(0) = \frac{-1}{2\pi i}\int_{-\infty}^{\infty} \frac{d\omega}{\omega + i\epsilon}$



Now the curve (I) doesn't enclose any poles so -

$$-\frac{1}{2\pi i}\oint_{I} \frac{d\omega}{\omega+i\epsilon} = 0$$

$$\omega = Re^{i\theta}$$
$$d\omega = iRe^{i\theta}d\theta$$
$$\frac{d\omega}{\omega} = \frac{iRe^{i\theta}d\theta}{Re^{i\theta}} \Rightarrow \frac{d\omega}{\omega} = i\,d\theta$$

$$\oint_{I} = -\frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{d\omega}{\omega+i\epsilon} \quad -\frac{1}{2\pi \cancel{i}}\int_{0}^{\pi}\cancel{i}\,d\theta$$

$$= -\frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{d\omega}{\omega+i\epsilon} \quad - \frac{1}{2} \equiv 0$$

$$\Rightarrow \quad -\frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{d\omega}{\omega+i\epsilon} = \frac{1}{2}$$

This can also be done using the contour (II)

Using this defn of $\Theta(t)$

$$\frac{d}{dt}\Theta(t) = -\frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{d\omega\,\partial_t e^{-i\omega t}}{\omega+i\epsilon} = \frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega\, e^{-i\omega t} = \delta(t)$$

As before we define $G_0(\underline{x}-\underline{x}') = \int \frac{d^4p}{(2\pi)^4}\,\tilde{G}_0(p)\,e^{-ip\cdot(\underline{x}-\underline{x}')}$

free case $V(x)=0$

$$\left(i\partial_t - \frac{\nabla^2}{2m}\right)G_0(\underline{x}-\underline{x}') = \int\frac{d^4p}{(2\pi)^4}\left(p_0 - \frac{\vec{p}^{\,2}}{2m}\right)\tilde{G}_0(p)\,e^{-ip\cdot(\underline{x}-\underline{x}')}$$

But $\left(i\partial_t - \frac{\nabla^2}{2m}\right)G_0(\underline{x}-\underline{x}') \overset{!}{=} \delta^4(\underline{x}-\underline{x}') = \int\frac{d^4p}{(2\pi)^4}\,e^{-ip\cdot(\underline{x}-\underline{x}')}$

$$\Rightarrow \quad \tilde{G}_0(p) = \frac{1}{p_0 - \vec{p}^{\,2}/2m}$$

pole at $p_0 = \vec{p}^{\,2}/2m$.

We want $G(\underline{x}-\underline{x}') \sim \Theta(t-t')$ for causal boundaries. So, by adding $+i\epsilon$ to denominator of $\tilde{G}_0(\underline{p})$ allow integration like we did for the Heaviside fn.

We have $$G(\underline{x}-\underline{x}') = \int \frac{d^3\vec{p}}{(2\pi)^3} e^{+i\vec{p}\cdot(\vec{x}-\vec{x}')} \int_{-\infty}^{\infty} dp_0 \frac{e^{-ip_0(t-t')}}{p_0 - \frac{\vec{p}^2}{2m} + i\epsilon} \cdot \frac{1}{(2\pi)}$$

define $p_0' = p_0 - \frac{\vec{p}^2}{2m} \Rightarrow dp_0' = dp_0$ and $e^{-ip_0(t-t')} = e^{-ip_0'(t-t')} e^{-i\vec{p}^2/2m\,(t-t')}$

$$\Rightarrow \int_{-\infty}^{\infty} dp_0 (\quad) = e^{-\frac{i\vec{p}^2}{2m}(t-t')} \int_{-\infty}^{\infty} \underbrace{\frac{dp_0'}{2\pi} \frac{e^{-ip_0'(t-t')}}{p_0' + i\epsilon}}_{-i\Theta(t-t')}$$

$$\therefore G_0(\underline{x}-\underline{x}') = \frac{-i}{(2\pi)^3} \Theta(t-t') \int \frac{d^3\vec{p}}{(2\pi)^3} e^{i\vec{p}\cdot(\vec{x}-\vec{x}')} e^{-i\vec{p}^2/2m\,(t-t')}$$

$$\int \frac{d^3\vec{p}}{(2\pi)^3} e^{-i[\vec{p}^2/2m(t-t') - \vec{p}\cdot(\vec{x}-\vec{x}')]} = \left(\frac{m}{2i\pi(t-t')}\right)^{3/2} \exp\left(\frac{i(\vec{x}-\vec{x}')^2 m}{2(t-t')}\right)$$

Procedure to do this integral?

Lecture-24 (PROPAGATOR THEORY)

(28-10-2021)

Schrödinger propagator.

$$\left(i\partial_t + \frac{\nabla^2}{2m}\right) G_0(\underline{x}-\underline{x}') = \delta^4(\underline{x}-\underline{x}')$$

$$\Rightarrow \quad G_0(\underline{x}-\underline{x}') = -i\theta(t-t')\int \frac{d^3\vec{p}}{(2\pi)^3}\, e^{-i\underline{p}\cdot(\underline{x}-\underline{x}')}\Big|_{p^0=\frac{\vec{p}^2}{2m}-i\epsilon}$$

$$= -i\theta(t-t')\left[\left(\frac{m}{2\pi i(t-t')}\right)^{3/2}\exp\left(\frac{i(\vec{x}-\vec{x}')^2 m}{2(t-t')}\right)\right]$$

It is important to note that the Green's f^ns also have a form where they can be written in terms of the normalized eigenf^ns of the operator for which the Green's f^n is evaluated. Here the operator is $\left(i\partial_t + \frac{\nabla^2}{2m}\right)$

$$G_0(\underline{x}-\underline{x}') = -i\theta(t-t')\int d^3\vec{p}\;\phi_{\vec{p}}(\vec{x},t)\,\phi^*_{\vec{p}}(\vec{x}',t')$$

$$\phi_{\vec{p}}(\underline{x}) = \frac{e^{-i\underline{p}\cdot\underline{x}}}{(2\pi)^{3/2}}$$ ⟶ eigenf^ns of the Hamiltonian with $p_0 = \frac{\vec{p}^2}{2m}$

$$\left(i\partial_t + \frac{\nabla^2}{2m}\right)\phi_{\vec{p}}(\underline{x}) = 0$$

Completeness relation:

$$\int d^3\vec{p}\;\varphi_{\vec{p}}(\vec{x},t)\,\varphi_{\vec{p}}^*(\vec{x}',t) = \delta^{(3)}(\vec{x}-\vec{x}')$$

In general, $G_0(\underline{x}-\underline{x}') = -\theta(t-t')\sum_n \psi_n(\underline{x})\,\psi_n^*(\underline{x}')$

Introducing interactions.

$H_0 \longrightarrow H = H_0 + V(\vec{x})$

$$(i\partial_t - H)\,G(\underline{x}-\underline{x}') = \delta^{(4)}(\underline{x}-\underline{x}')$$ ⟶ full Green f^n.

$$\Rightarrow (i\partial_t - H_0)\,G(\underline{x}-\underline{x}') = \delta^{(4)}(\underline{x}-\underline{x}') + V(\vec{x})\,G(\underline{x}-\underline{x}')$$

Now in the eq^n $(i\partial_t - H_0)\, G(\underline{x}-\underline{x}') = \underbrace{\delta^4(\underline{x}-\underline{x}') + V(\underline{x})\, G(\underline{x},\underline{x}')}_{\rho(\underline{x},\underline{x}')}$

Now, for the above eq^n, we need to solve for the $G(\underline{x}-\underline{x}')$. The Green's function of the operator $(i\partial_t - H_0)$ is $\tilde{G}(\underline{x},\underline{x}') = G_0(\underline{x},\underline{x}')$, so -

$$G(\underline{x}-\underline{x}') = \phi_0(\underline{x}) + \int d^4\underline{x}''\, \underset{\substack{\parallel \\ G_0}}{\tilde{G}(\underline{x}-\underline{x}'')}\, \rho(\underline{x}'',\underline{x}')$$

$$= \phi_0(\underline{x}) + \int d^4\underline{x}'' \left[G_0(\underline{x}-\underline{x}'')\, \delta^4(\underline{x}''-\underline{x}') + G_0(\underline{x}-\underline{x}'')\, V(\underline{x}'')\, G(\underline{x}''-\underline{x}') \right]$$

$$= \phi_0(\underline{x}) + G_0(\underline{x}-\underline{x}') + \int d^4\underline{x}''\, G_0(\underline{x}-\underline{x}'')\, V(\underline{x}'')\, G(\underline{x}''-\underline{x}')$$

homogenous sol^n.
but we can set $=0$

$$\Rightarrow \boxed{G(\underline{x}-\underline{x}') = G_0(\underline{x}-\underline{x}') + \int d^4\underline{x}''\, G_0(\underline{x}-\underline{x}'')\, V(\underline{x}'')\, G(\underline{x}''-\underline{x}')}$$

Through iteration, we can write the above equation as-

$$G(\underline{x}-\underline{x}') = G_0(\underline{x}-\underline{x}') + \int d^4\underline{x}_1\, G_0(\underline{x}-\underline{x}_1)\, V(\underline{x}_1) \left[G_0(\underline{x}_1-\underline{x}') + \int d^4\underline{x}_2\, G_0(\underline{x}_1-\underline{x}_2)\, V(\underline{x}_2)\, G(\underline{x}_2-\underline{x}') \right]$$

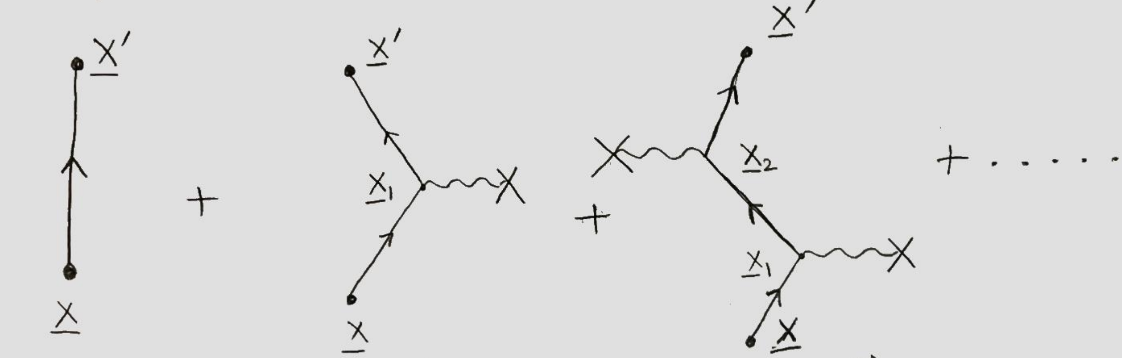
$$= G_0(\underline{x}-\underline{x}') + \int d^4\underline{x}_1\, G_0(\underline{x}-\underline{x}_1)\, V(\underline{x}_1)\, G_0(\underline{x}_1-\underline{x}')$$
$$+ \iint d^4\underline{x}_1\, d^4\underline{x}_2\, G_0(\underline{x}-\underline{x}_1)\, V(\underline{x}_1)\, G_0(\underline{x}_1-\underline{x}_2)\, V(\underline{x}_2)\, G(\underline{x}_2-\underline{x}')$$

put G into this again.
Keep iterating!

$G_0(\underline{x}-\underline{x}')$ =  , $V(\underline{x}_i)$ = 

free propogation

So, this perturbation series expansion (done via iteration) can be represented as



(G_0) $\qquad$ $(G_0 V G)$ $\qquad$ $(G_0 V G_0 V G)$

The earlier iteration process can be written clearly as follows -

$$G = G_0 + G_0 V G$$

$$G = G_0 + G_0 V (G_0 + G_0 V G)$$

$$= G_0 + G_0 V G_0 + G_0 V G_0 V (G_0 + G_0 V G)$$

$$= G_0 + G_0 V G_0 + G_0 V G_0 V G_0 + G_0 V G_0 V G_0 V G_0 + \cdots$$

<u>Non-relativistic Scattering.</u>

Let's say we start with a wave-packet $\phi_i(\vec{x}', t')$ at $t' \to -\infty$.

Then we evaluate the wave at a later time using the propagator -

$$\psi(\vec{x}, t) = \lim_{t' \to -\infty} i \int d^3\vec{x}'\, G(\underline{x} - \underline{x}')\, \phi_i(\vec{x}', t')$$

Using $G(\underline{x} - \underline{x}') = G_0(\underline{x} - \underline{x}') + \int d^4\underline{x}_1\, G_0(\underline{x} - \underline{x}_1)\, V(\underline{x}_1)\, G(\underline{x}_1 - \underline{x}')$

$$\Rightarrow \psi(\vec{x}, t) = \lim_{t' \to -\infty} i \int d^3\vec{x}' \left[G_0(\underline{x} - \underline{x}')\, \phi_i(\underline{x}') + \int d^4\underline{x}_1\, G_0(\underline{x} - \underline{x}_1)\, V(\underline{x}_1)\, G(\underline{x}_1 - \underline{x}')\, \phi_i(\underline{x}') \right.$$

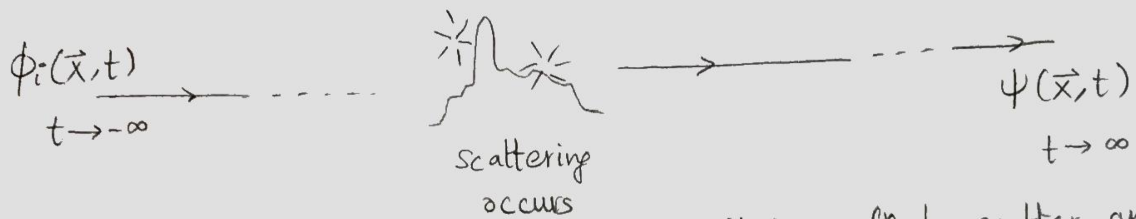
$$= \phi_i(\vec{x}, t) + \lim_{t' \to -\infty} \int d^4\underline{x}_1\, G_0(\underline{x} - \underline{x}_1)\, V(\underline{x}_1) \int d^3\vec{x}'\, G(\underline{x}_1 - \underline{x}')\, \phi_i(\underline{x}')$$

$$\psi(\vec{x}, t) = \phi_i(\vec{x}, t) + \int d^4\underline{x}_1\, G_0(\underline{x} - \underline{x}_1)\, V(\underline{x}_1)\, \psi(\vec{x}_1, t_1)$$

So,

$$\underbrace{\psi(\vec{x},t) = \phi_i(\vec{x},t)}_{\text{free particle propagation}} + \underbrace{\int d^4\underline{x}_1 \; G_0(\underline{x}-\underline{x}_1)\, V(\underline{x}_1)\, \psi(\vec{x}_1,t_1)}_{\text{scattered propagation.}}$$

Notice that we actually haven't solved anything. $\psi(\vec{x}_1,t_1)$ is as good an unknown as $G(\underline{x}_1-\underline{x}')$ was.



We are interested in finding the amplitude of initial wavef^n to scatter and reach the final wavef^n. To do that, we need to consider an overlap of time developed free (initial) wavef^n with the final state that we expect it to be in.

S-matrix

S-matrix theory provides us with the formalism to go from $\phi_i(\vec{x}',t')\big|_{t'\to-\infty}$ to $\phi_f(\vec{x},t)\big|_{t\to+\infty}$.

The S-matrix element is defined as follows -

$$S_{fi} = \langle f|i\rangle = \lim_{t\to\infty} \int d^3\vec{x}\; \phi_f^*(\vec{x},t)\, \psi_i^{(+)}(\vec{x},t)$$

$\phi_f(\vec{x},t)$ is the final free particle state attained after scattering.

$\psi_i^{(+)}(\vec{x},t)$ is the initial time developed free wavef^n which scatters with the potential. $(+)$ denotes forward moving direction.

Now $\psi_i^{(+)}(\vec{x},t) = \phi_i(\vec{x},t) + \int d^4\underline{x}_1\, G_0(\underline{x}-\underline{x}_1)\, V(\underline{x}_1)\, \psi_i^{(+)}(\vec{x}_1,t_1)$

$$S_{fi} = \lim_{t\to\infty} \int d^3\vec{x}\, \underbrace{\phi_f^*(\vec{x},t)\Big[\phi_i(\vec{x},t)}_{\langle\phi_f|\phi_i\rangle} + \int d^4\underline{x}_1\, G_0(\underline{x}-\underline{x}_1)\, V(\underline{x}_1)\, \psi_i^{(+)}(\vec{x}_1,t_1)\Big]$$

$$= \lim_{t\to\infty} \underbrace{\delta^{(3)}(\vec{k}_f - \vec{k}_i)}_{\substack{|||\\ \delta_{fi}}} + \int d^4\underline{x}_1 \int d^3\vec{x}\, \phi_f^*(\vec{x},t)\, G_0(\underline{x}-\underline{x}_1)\, V(\underline{x}_1)\, \psi_i^{(+)}(\vec{x}_1,t_1)$$

Now $\quad i\int d^3\vec{x}\, G_0(\underline{x}-\underline{x}_1)\, \phi_f^*(\vec{x},t) \equiv \phi_f^*(\vec{x}_1,t_1)$ ← since G_0 is the propagator of free states.

$$\Rightarrow \int d^3\vec{x}\, \phi_f^*(\vec{x},t)\, G_0(\underline{x}-\underline{x}_1) = -i\phi_f^*(\vec{x}_1,t_1)$$

So, $S_{fi} = \delta_{fi} - i\int d^4\underline{x}_1\, \phi_f^*(\vec{x}_1,t_1)\, V(\underline{x}_1)\, \psi_i^{(+)}(\vec{x}_1,t_1)$

↑ what about this?

The expansion of $\psi_i^{(+)}$ also mimics the expansion of $G(\underline{x}-\underline{x}')$, and is as follows -

$$\psi_i^{(+)}(\vec{x},t) = \phi_i(\vec{x},t) + \int d^4\underline{x}_1\, G_0(\underline{x}-\underline{x}_1)\, V(\underline{x}_1)\Big[\phi_i(\vec{x}_1,t_1) + \int d^4\underline{x}_2\, G_0(\underline{x}_1-\underline{x}_2)\, V(\underline{x}_2)\, \psi_i^{(+)}(\vec{x}_2,t_2)\Big]$$

$$= \phi_i(\vec{x},t) + \int d^4\underline{x}_1\, G_0(\underline{x}-\underline{x}_1)\, V(\underline{x}_1)\, \phi_i(\vec{x}_1,t_1)$$

$$+ \int d^4\underline{x}_1\, d^4\underline{x}_2\, G_0(\underline{x}-\underline{x}_1)\, V(\underline{x}_1)\, G_0(\underline{x}_1-\underline{x}_2)\, V(\underline{x}_2)\, \phi_i(\vec{x}_2,t_2)$$

$$+ \cdots\cdots$$

Dirac Theory case

- Free Dirac propagator : $(i\cancel{\partial} - m)\, S_F(\underline{x}-\underline{x}') = \delta^4(\underline{x}-\underline{x}')$

Green's f^n for Dirac particle (free)

$$\Rightarrow S_F(\underline{x}-\underline{x}') = (i\cancel{\partial}+m)\, \Delta_F(\underline{x}-\underline{x}')$$

$$= \int \frac{d^3\vec{p}}{(2\pi)^3} \cdot \frac{2m}{2E_p} \left[\theta(t-t')\, \Lambda_+(p)\, e^{-ip\cdot(\underline{x}-\underline{x}')} + \theta(t'-t)\, \Lambda_-(p)\, e^{+ip\cdot(\underline{x}-\underline{x}')} \right]$$

$$= S_F^{(+)} + S_F^{(-)}$$

$S_F^{(+)}$: propagates +ve energy sol^{n}s forward in time.

$S_F^{(-)}$: propagates −ve energy sol^{n}s backward in time.

- Interacting case (QED): $\partial_\mu \longrightarrow \partial_\mu + iqeA_\mu$

$$(i\gamma^\mu \partial_\mu - m + \underbrace{eA_\mu \gamma^\mu}_{\text{potential term}})_{\alpha\gamma}\, \underbrace{S_{\gamma\beta}}_{\text{full Dirac propagator}}(\underline{x}-\underline{x}') = \delta_{\alpha\beta}\, \delta^{(4)}(\underline{x}-\underline{x}')$$

We know that $S_F^{\circ}(\underline{x}-\underline{x}') = (i\cancel{\partial}+m)\, \Delta_F(\underline{x}-\underline{x}')$

So, repeating the same steps -

potential term like in NR-scattering

$$S_F(\underline{x}-\underline{x}') = S_F^{\circ}(\underline{x}-\underline{x}') + e\int d^4y\; S_F^{\circ}(\underline{x}-\underline{y})\, \cancel{A}(\underline{y})\, S_F(\underline{y}-\underline{x}')$$

$$= S_F^{(0)}(\underline{x}-\underline{x}') + e\int d^4y\; S_F^{(0)}(\underline{x}-\underline{y})\, \cancel{A}(\underline{y})\, S_F^{(0)}(\underline{y}-\underline{x}')$$

$$+ e^2 \int d^4y'\; S_F^{(0)}(\underline{y}-\underline{y}')\, \cancel{A}(\underline{y}')\, S_F^{(0)}(\underline{y}'-\underline{x}')$$

$$+ \ldots\ldots$$

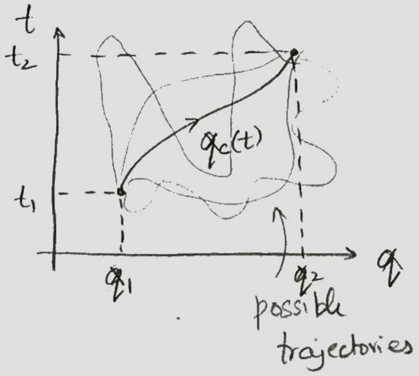
So, $S_F = S_F^{(0)} + e\, S_F^{(0)} \cancel{A}\, S_F^{(0)} + e^2\, S_F^{(0)} \cancel{A}\, S_F^{(0)} \cancel{A}\, S_F^{(0)} + \ldots\ldots$

Lecture-25. (Classical Field Theory)

(29-10-2021)

Lagrangian formalism.

2nd order eq^{n}s of motion have a $L \equiv L(q_i(t), \dot{q}_i(t))$. Since we consider autonomous systems, we don't have explicit t dependence.



$$S = \int_{①}^{②} dt \, L(q(t), \dot{q}(t))$$

We fix $q_i(t_1) = q_{1_i}$ and $q_i(t_2) = q_{2_i}$

The principle of extremal action:

The trajectory b/w ① & ② is the path $q_c(t)$ such for small variations of $\delta q(t)$, $\delta S[q_c(t)] = 0$.

So, $q(t) = q_c(t) + \delta q(t) \longrightarrow$ arbitrary

$$\delta S[q(t)] = \int_{①}^{②} dt \left(\frac{\partial L}{\partial q} \delta q + \frac{\partial L}{\partial \dot{q}} \underbrace{\delta \dot{q}}_{\frac{d}{dt}\delta q(t)} \right)$$

$$= \int_{①}^{②} dt \left(\frac{\partial L}{\partial q} - \frac{d}{dt} \frac{\partial L}{\partial \dot{q}} \right) \delta q + \int_{①}^{②} dt \cdot \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}} \delta q \right) \longrightarrow \left. \frac{\partial L}{\partial \dot{q}} \delta q \right|_{①}^{②} = 0$$

since by def^n $\delta q(t_1) = \delta q(t_2) = 0$

So, for $\delta S[q(t)] = 0$, then q must satisfy -

$$\boxed{\frac{\partial L}{\partial q} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}} \right) = 0}$$

Euler-Lagrange eq^{n}s.

Noether's Theorem:

For every continuous invariance of the Lagrangian, there exists an associated constant of motion.

(1) Time translation invariance

$t \longrightarrow t' = t + a \rightarrow$ This is a symmetry of autonomous Lagrangians.

$q'(t') = q(t) = q'(t+\delta t)$ because any 'a' works.

$$q'(t+\delta t) = q'(t) + \delta t \cdot \underbrace{\dot{q}'(t)}_{\approx \dot{q}(t)} + \mathcal{O}(\delta t^2)$$

$$\Rightarrow \delta q(t) = q'(t) - q(t) = -\delta t\, \dot{q}(t)$$

Now, if the Lagrangian is invariant under this transformation, then

$$L'(t+\delta t) = L'(t) + \delta t \underbrace{\frac{dL'}{dt}}_{\approx \frac{dL}{dt}} + \mathcal{O}(\delta t^2) = L(t)$$

$$\Rightarrow \delta L \equiv L'(t) - L(t) = -\delta t \frac{dL'}{dt}$$

since $L'(t) = L(t)$ for a continuous time symmetry, $\delta L = 0$

$$\delta L = -\delta t \frac{dL}{dt} = -\frac{d}{dt}(L\,\delta t)$$

but we also know that $\delta L = \frac{\partial L}{\partial q}\delta q + \frac{\partial L}{\partial \dot{q}}\delta \dot{q} + \cancel{\frac{\partial L}{\partial t}\delta t}$

autonomous assumption.

$$= \underbrace{\left(\frac{\partial L}{\partial q} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}}\right)}_{=0}\delta q + \frac{d}{dt}\left[\left(\frac{\partial L}{\partial \dot{q}}\right)\underbrace{\delta q}_{= -\delta t\, \dot{q}(t)}\right]$$

$$= -\frac{d}{dt}\left(\delta t\, \dot{q}(t) \frac{\partial L}{\partial \dot{q}}\right)$$

So, $-\frac{d}{dt}\left(\delta t\, \dot{q}(t)\, \frac{\partial L}{\partial \dot{q}}\right) = -\frac{d}{dt}(L\,\delta t)$

$$\Rightarrow \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\dot{q}(t) - L\right) \underset{\substack{\uparrow \\ \text{arbitrary}}}{\delta t} = 0$$

$$\Rightarrow \left(\frac{\partial L}{\partial \dot{q}}\dot{q}(t) - L\right) = \underset{\substack{\uparrow \\ \text{CONSERVATION OF ENERGY FOR AUTONOMOUS SYSTEMS.}}}{p\dot{q} - L} \text{ is a conserved quantity!}$$

(2) let's now work and see what happens with translation invariance

$$q_{(t)} \longrightarrow q'_{(t)} = q_{(t)} + b \qquad\qquad L = L(q, \dot{q})$$

If $L(q(t)) = L(q'(t))$ i.e. $q \to q'$ is a symmetry of the Lagrangian, then we can say $\delta L = 0$

$$\delta L = \left[\frac{\partial L}{\partial q} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right)\right]\delta q + \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\delta q\right)$$

In this case $\delta q(t) = q'(t) - q(t) = b \longrightarrow b$ is a continuous paramater

So, on a trajectory that satisfies E-L, $\partial_q L - D_t(\partial_{\dot{q}} L) = 0$

$$\Rightarrow \delta L = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}} \cdot a\right) = a\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) = 0$$

$\frac{\partial L}{\partial \dot{q}} = p$ is defined as the canonical momentum, so

$\frac{d}{dt}p = 0 \Rightarrow$ momentum conservation

$\therefore$ Translation invariance $\Rightarrow$ momentum conservation.

(3) Rotation symmetry

Say we have several q_i & $\quad q_i \longrightarrow q_i' = O_{ij}\, q_j$ where

$$O^T = O^{-1} \Rightarrow \sum_i q_i^2 = \sum_i q_i'^2 = 1$$

So, we can write $\quad O = e^{\omega}$ where $\omega^T = -\omega$

$$\Rightarrow O_{ij} = \delta_{ij} + \underset{\substack{\uparrow \\ \text{A.S.}}}{\omega_{ij}} + \mathcal{O}(\omega^2)$$

So, $\delta q_i = \omega_{ij}\, q_j$

Rotational invariance $\Rightarrow L(q'(t)) = L(q(t))$

Now again $\quad \delta L = \underbrace{\left[\frac{\partial L}{\partial q_i} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot q_i}\right)\right]}_{=0} \delta q_i + \frac{d}{dt}\left[\frac{\partial L}{\partial \dot q_i}\, \delta q_i\right]$

If E-L eq^n's are satisfied by $q_i(t)$, then

So, $\delta L = \dfrac{d}{dt}\left(\dfrac{\partial L}{\partial \dot q^i}\, \omega^i{}_j\, q^j\right) = 0$

arbitrarily small A-S.

So, $\quad \omega^i{}_j \dfrac{\partial L}{\partial \dot q^i} q^j = \dfrac{1}{2}\left[\omega^i{}_j \dfrac{\partial L}{\partial \dot q^i} q^j + \omega^j{}_i \dfrac{\partial L}{\partial \dot q^j} q^i\right]$

$$= \frac{1}{2}\left[\omega^i{}_j \frac{\partial L}{\partial \dot q^i} q^j - \omega^i{}_j \frac{\partial L}{\partial \dot q^j} q^i\right] = \frac{\omega_{ij}}{2}\left(\frac{\partial L}{\partial \dot q^{[i}}\, q^{j]}\right)$$

$$\Rightarrow \delta L = \frac{\omega^i{}_j}{2} \frac{d}{dt}\left(\frac{\partial L}{\partial \dot q^{[i}}\, q^{j]}\right) = 0$$

$\therefore \quad \dfrac{\partial L}{\partial \dot q^{[i}}\, q^{j]} \equiv L_{ji}$ is CONSTANT.

But what is L_{ji}?

In 3D -

$L_{ji} = q_{[j} p_{i]}$

If we define $L_k = \frac{1}{2} \epsilon_{kij} L_{ij} = \frac{1}{2} \epsilon_{kij} (q_j p_i - q_i p_j)$
$= (\vec{q} \times \vec{p})_k$

↑ ANGULAR MOMENTUM

Hence, rotational symmetry ⟹ ANG. MOM^N. conservation.

Classical Field Theory. (relativistic)

require the action to be Lorentz invariant.

$S = \int dt \, L$ in classical mechanics.

However, we see an issue b/c integral is only over dt and not $d^3\vec{x}$, so this discriminates b/w space & time. So, we should integrate over $d^4\underline{x}$.

In going from particles → fields, we replace -

$q_i(t) \longrightarrow \phi_i(\vec{x}, t)$

label of d.o.f.

both space & time derivatives. So, it treats them on same footing

$L(q, \dot{q}) \longrightarrow \mathcal{L} = \mathcal{L}(\phi(\vec{x},t), \dot{\phi}(\vec{x},t), \vec{\nabla}\phi(\vec{x},t))$

$S = \int dt \, L(q, \dot{q})$

$S = \int d^4\underline{x} \, \mathcal{L}$ → Lagrangian density

where
$d^4\underline{x}' = d^4\underline{x}$

$x'^\mu = \Lambda^\mu{}_\nu x^\nu + a^\mu$

So, we get the following for fields -

$$S = \int d^4\underline{x} \, \mathcal{L}(\phi(\underline{x}), \partial_\mu \phi(\underline{x}))$$

So, to get the eq^{n}s of motion for fields, we look at the extremal action principle here.

$$\delta S = 0 \qquad \text{with } \phi_i(\vec{x}) \text{ and } \phi_f(\vec{x}) \text{ held fixed.}$$

$$\delta S = \int d^4\underline{x} \left(\frac{\partial \mathcal{L}}{\partial \phi} \delta\phi + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \delta(\partial_\mu \phi) \right)$$

$$= \int d^4\underline{x} \left(\frac{\partial \mathcal{L}}{\partial \phi} - \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \right) \right) \delta\phi + \underbrace{\int d^4\underline{x}\, \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \delta\phi \right)}$$

$$\underbrace{\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}}_{\overset{\shortparallel}{0}} \delta\phi \Big|_{S_{i,f}} \to = 0$$

For $\delta S = 0$

$\Rightarrow$ We get the E-L equations

$$\boxed{\frac{\partial \mathcal{L}}{\partial \phi_r} - \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_r)} \right) = 0}$$

<u>Example</u>-

$$\mathcal{L} = \frac{1}{2} \partial_\mu \phi \partial^\mu \phi - \frac{m^2}{2} \phi^2 = \frac{1}{2} \left(\dot{\phi}^2 - (\vec{\nabla}\phi)^2 \right) - \frac{m^2}{2} \phi^2$$

$$\frac{\partial \mathcal{L}}{\partial \phi} = -m^2 \phi \qquad \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} = \partial^\mu \phi$$

$$\Rightarrow -m^2\phi - \partial_\mu(\partial^\mu \phi) = 0 \Rightarrow (\partial_\mu \partial^\mu + m^2)\phi = 0 \to \text{K.G. equation.}$$

If we add a pot^n to $\mathcal{L}$ i.e. $\mathcal{L} = \frac{1}{2} \partial_\mu \partial^\mu \phi - \frac{m^2}{2} \phi^2 - V(\phi)$

$$\Rightarrow \boxed{(\partial_\mu \partial^\mu + m^2)\phi_i = -\frac{\partial V}{\partial \phi_i}} \to \text{K.G. eq}^n \text{ for a potential.}$$

Lecture-26
(01-11)

Math aside:

$$\delta F[\phi, \vec{\nabla}\phi, \varphi, \dots] = \int d^3\vec{x}\ \frac{\delta F}{\delta \phi}\delta\phi + \frac{\delta F}{\delta \vec{\nabla}\phi}\delta(\vec{\nabla}\phi) + \frac{\delta F}{\delta \varphi}\delta(\varphi) + \frac{\delta F}{\delta \dots}\delta(\dots) + \dots$$

$$\delta f(\phi, \vec{\nabla}\phi, \varphi, \dots) = \int d^3\vec{x}\ \frac{\partial f}{\partial \phi}\delta\phi + \frac{\partial f}{\partial \vec{\nabla}\phi}\delta(\vec{\nabla}\phi) + \frac{\partial f}{\partial \varphi}\delta(\varphi) + \frac{\partial f}{\partial(\dots)}\delta(\dots) + \dots$$

$$\frac{\delta F}{\delta \rho(\vec{r})} = \frac{\partial f}{\partial \rho} - \vec{\nabla}\cdot\frac{\partial f}{\partial \vec{\nabla}\rho}$$

Classical Field Theory:

$$S = \int d^4\underline{x}\ \mathcal{L}(\phi, \partial_\mu\phi) \xrightarrow[\text{least action}]{\text{Principle of}} \frac{\partial \mathcal{L}}{\partial \phi} = \partial_\mu\left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu\phi)}\right)$$

Action S has dimensions of $\hbar$. ($\hbar = 1$)

$[\ ] \equiv$ mass dim^{n}s.

$$S = \int d^4\underline{x}\ \mathcal{L}$$

$$[S] = 0, \quad [d^4\underline{x}] = -4 \Rightarrow [\mathcal{L}] = +4$$

$$\mathcal{L} = (\partial\phi)^2 - \frac{m^2\phi^2}{2!} - \frac{g\phi^3}{3!} - \frac{\lambda\phi^4}{4!} - \sum_n \lambda^{(n)}\frac{\phi^n}{n!}$$

	Length dim^n	Mass dim^n
$[\partial/\partial x^\mu]$	-1	$+1$
$[\mathcal{L}]$	-4	$+4$
$[\phi]$	-1	$+1$
m, g	-1	$+1$
λ	0	0
$\lambda^{(n)}$	$n-4$	$4-n$

In general, in d-dim^{n}s,

$$[d^d \underline{x}] = -d \qquad [\mathcal{L}] = +d$$

$$[\mathcal{L}] = [(\partial\phi)^2] = +d \Rightarrow [\partial\phi] = \frac{d}{2} \Rightarrow [\phi]_d = \frac{d}{2} - 1$$

since $[\partial] = +1$

$$[\phi]_3 = \frac{1}{2} \qquad [\phi]_2 = 0$$

Functional derivatives

If we have independent variables z^A, then $\frac{\partial z^A}{\partial z^B} \equiv \delta^A_B$.

In CFT, $\phi(\vec{x},t) = \phi(\underline{x})$, we view the value of ϕ at each $\vec{x} \in \mathbb{R}^{d-1}$ as an independent degree of freedom.

($\vec{x}$ is like a label analogue, so $\forall \vec{x} \in \mathbb{R}^{d-1}$, $\phi(\vec{x}, t_0)$ is an infinite no. of independent variables.)

$$\frac{\partial q_i(t)}{\partial q^j(t)} = \delta^i_j \quad i,j = 1, \dots N \quad \xrightarrow{\text{as } N \to \infty} \quad \boxed{\frac{\delta\phi(\underline{x})}{\delta\phi(\underline{y})} \equiv \delta^{(d)}(\underline{x} - \underline{y}) = \delta^{(d-1)}(\vec{x} - \vec{y})\,\delta(t_x - t_y)}$$

↑ ↑ labels as $N \to \infty$

$$\pi = \frac{\partial \mathcal{L}}{\partial \dot{\phi}} \longrightarrow \text{momentum density field} = \dot{\phi} \text{ (conservative fields).}$$

In Hamiltonian field theory, we also consider the momentum densities to also be independent variables satisfying -

$$\boxed{\frac{\delta\pi(\underline{x})}{\delta\pi(\underline{x}')} = \delta^{(d)}(\underline{x} - \underline{x}')}$$

$\frac{\delta}{\delta\phi(\underline{x})}$ functional derivative is defined to possess analogs of the standard partial derivatives, such as Liebniz rule, chain rule etc.

So, $$\frac{\delta}{\delta\phi(\underline{x})}\left[O_1(y)\,O_2(\underline{z})\right] = \frac{\delta O_1(y)}{\delta\phi(\underline{x})}\,O_2(\underline{z}) + O_1(y)\,\frac{\delta O_2(\underline{z})}{\delta\phi(\underline{x})}$$

- $$\frac{\delta S}{\delta\phi(\underline{x}')} = \frac{\delta}{\delta\phi(\underline{x}')}\int d^4\underline{x}\left(\frac{1}{2}\partial_\mu\phi\,\partial^\mu\phi - V(\phi)\right)$$

how valid is this step?

$$= \int d^4\underline{x}\; 2\cdot\frac{1}{2}\partial_\mu\phi\;\partial^\mu\left(\frac{\delta\phi(\underline{x})}{\delta\phi(\underline{x}')}\right) - V'(\phi(\underline{x}))\,\frac{\delta\phi(\underline{x})}{\delta\phi(\underline{x}')}\Bigg)$$

$$= \int d^4\underline{x}\left[\partial_\mu\phi\,\partial^\mu(\delta(\underline{x}-\underline{x}')) - V'(\phi(\underline{x}))\,\delta(\underline{x}-\underline{x}')\right]$$

$$= \cancel{\partial_\mu\phi\,\delta(\underline{x}-\underline{x}')}^{\,0}\Big|_{\partial c} - \left(\int d^4\underline{x}\;\delta(\underline{x}-\underline{x}')\,\partial_\mu\partial^\mu\phi + V'(\phi(\underline{x}))\,\delta(\underline{x}-\underline{x}')\right)$$

$$= \int d^4\underline{x}\;\delta^4(\underline{x}-\underline{x}')\left[-\partial_\mu\partial^\mu\phi - V'(\phi(\underline{x}))\right] = 0 \quad \text{(for the classical path)}$$

$$\Rightarrow \Box\phi = -\frac{\partial V}{\partial\phi}$$

so does this mean
$$\frac{\delta(\partial_\mu\phi)}{\delta\phi(\underline{x})} = \partial_\mu\frac{(\delta\phi)}{\delta\phi(\underline{x})}$$
but $\frac{\delta(\phi)}{\delta(\partial_\mu\phi)} = 0$?

- What if we take functional derivative of S w.r.t. $\dot\phi$

$$\frac{\delta S}{\delta\dot\phi(\underline{x})} = \frac{\delta}{\delta\dot\phi}\int d^4\underline{x}'\left(\frac{1}{2}\dot\phi^2 - (\vec\nabla'\phi)^2 - V(\phi(\underline{x}'))\right)$$

$$= \int d^4\underline{x}'\left(2\cdot\frac{1}{2}\dot\phi\;\delta(\underline{x}'-\underline{x})\right) = \int d^4\underline{x}'\;\dot\phi(\underline{x}')\,\delta(\underline{x}-\underline{x}') = \dot\phi(\underline{x})$$
$$= \pi(\underline{x})$$

Poisson Brackets - (functional)

$$\boxed{\begin{aligned}&\{O(\phi(\vec{x},t),\pi(\vec{x},t)),\ O'(\phi(\vec{x}',t),\pi(\vec{x}',t))\}\\ &\equiv \int d^3\vec{y}\left[\frac{\delta O}{\delta\phi(\vec{y},t)}\frac{\delta O'}{\delta\pi(\vec{y},t)} - \frac{\delta O}{\delta\pi(\vec{y},t)}\frac{\delta O'}{\delta\phi(\vec{y},t)}\right]\end{aligned}}$$

Let's calculate $\{\phi(\vec{x},t),\pi(\vec{x}',t)\}$ using this prescription

$$\{\phi(\vec{x},t),\pi(\vec{x}',t)\} = \int d^3\vec{y}\ \frac{\delta\phi(\vec{x},t)}{\delta\phi(\vec{y},t)}\frac{\delta\pi(\vec{x}',t)}{\delta\pi(\vec{y},t)} - \underbrace{\frac{\delta\phi(\vec{x},t)}{\delta\pi(\vec{y},t)}}_{=0}\underbrace{\frac{\delta\pi(\vec{x}',t)}{\delta\phi(\vec{y},t)}}_{=0}$$

$$= \int d^3\vec{y}\ \delta^{(3)}(\vec{x}-\vec{y})\ \delta^{(3)}(\vec{x}'-\vec{y})$$

$$= \delta^{(3)}(\vec{x}-\vec{x}')$$

$$\Rightarrow \underline{\{\phi(\vec{x},t),\pi(\vec{x}',t)\} = \delta^{(3)}(\vec{x}-\vec{x}')}$$

Similarly,

$$\underline{\{\phi(\vec{x},t),\phi(\vec{x}',t)\} = 0 = \{\pi(\vec{x},t),\pi(\vec{x}',t)\}}$$

at equal times.

Hamiltonian Density.

$$H = p\dot{q} - L \longrightarrow \mathcal{H} = \pi\dot{\phi} - \mathcal{L}$$

$$H = \int d^3\vec{x}\ \mathcal{H}(\phi,\pi)$$

$$L = \int d^3\vec{x}\ \mathcal{L}(\phi,\partial_\mu\phi)$$

For $\mathcal{L} = \frac{1}{2}\dot{\phi}^2 - \frac{1}{2}(\vec{\nabla}\phi)^2 - V(\phi) \Rightarrow \frac{\partial\mathcal{L}}{\partial\dot{\phi}} = \dot{\phi} = \pi$

$$\Rightarrow \mathcal{H} = \pi(\vec{x},t)\,\dot{\phi}(\vec{x},t) - \mathcal{L} = \pi^2 - \left(\frac{\pi^2}{2} - \frac{(\vec{\nabla}\phi)^2}{2} - V(\phi)\right)$$

$$= \frac{\pi^2}{2} + \frac{(\vec{\nabla}\phi)^2}{2} + V(\phi)$$

Some other examples of relativistic Lagrangian densities -

(1) Electro magnetism

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - j_\mu A^\mu = -\frac{1}{4} \partial_{[\mu} A_{\nu]} \partial^{[\mu} A^{\nu]} - j \cdot A$$

Now $\partial_{[\mu} A_{\nu]} \partial^{[\mu} A^{\nu]} = \partial_{[\mu} A_{\nu]} \left(\partial^\mu A^\nu - \partial^\nu A^\mu \right)$

$$= \partial_{[\mu} A_{\nu]} \partial^\mu A^\nu - \underbrace{\partial_{[\mu} A_{\nu]} \partial^\nu A^\mu}_{\text{(switch labels } \mu \leftrightarrow \nu)} = \partial_{[\mu} A_{\nu]} \partial^\mu A^\nu - \partial_{[\nu} A_{\mu]} \partial^\mu A^\nu$$

$$= \left(\partial_{[\mu} A_{\nu]} - \underbrace{\partial_{[\nu} A_{\mu]}}_{= -\partial_{[\mu} A_{\nu]}} \right) \partial^\mu A^\nu = 2 \partial_{[\mu} A_{\nu]} \partial^\mu A^\nu$$

$$\Rightarrow \mathcal{L} = -\frac{1}{2} \partial_{[\mu} A_{\nu]} \partial^\mu A^\nu - j_\mu A^\mu$$

another way $= -\frac{1}{4} \partial_{[\mu} A_{\nu]} \partial_{[\lambda} A_{\sigma]} \eta^{\mu\lambda} \eta^{\nu\sigma} - j \cdot A$

So, $\delta S = \int d^4\underline{x}\, \delta \left(-\frac{1}{4} \partial_{[\mu} A_{\nu]} \partial^{[\mu} A^{\nu]} - j^\mu A_\mu \right) = \int d^4\underline{x} \left(-\frac{1}{4} \begin{bmatrix} \partial_{[\mu} A_{\nu]} \partial^{[\mu} \delta A^{\nu]} \\ + \partial_{[\mu} \delta A_{\nu]} \partial^{[\mu} A^{\nu]} \end{bmatrix} - j^\mu \delta A_\mu \right)$

$$= \int d^4\underline{x} \left(-\frac{1}{2} \partial_{[\mu} \delta A_{\nu]} \partial^{[\mu} A^{\nu]} - j^\mu \delta A_\mu \right)$$

$$= \int d^4\underline{x} \left(\underbrace{- \partial_\mu \delta A_\nu \partial^\mu A^\nu + \partial_\nu \delta A_\mu \partial^\mu A^\nu}_{\text{integrate by parts}} - j^\mu \delta A_\mu \right)$$

$$= \int d^4\underline{x} \left(+ \delta A_\nu \partial_\mu \partial^\mu A^\nu - \delta A_{\underset{\nu}{\mu}} \partial_{\underset{\mu}{\nu}} \partial^{\underset{\nu}{\mu}} A^{\underset{\mu}{\nu}} - j^{\underset{\nu}{\mu}} \delta A_\mu \right)$$

$$= \int d^4\underline{x} \quad \delta A_\nu \left[\partial_\mu \partial^\mu A^\nu - \partial_\mu \partial^\nu A^\mu - j^\nu \right]$$

If we set $\delta S = 0$ for arbitrary variations of δA_ν

$$\Rightarrow \boxed{\partial_\mu F^{\mu\nu} = j^\nu} \longrightarrow \text{Inhomogeneous Maxwell eq}^{n}\text{s.}$$

For the EM Lagrangian,

$$\pi^{\mu} = \frac{\partial \mathcal{L}}{\partial \dot{A}_{\mu}} \Rightarrow \pi^{0} = 0 \text{ because } F^{\mu\nu} \text{ doesn't depend on } \dot{A}_0 .$$

We can also do $\frac{\delta S}{\delta A_{\mu}(\underline{x})} = 0$, and try to find the field eq^{n}'s in the same way.

$$\frac{\delta A_{\nu}(\underline{x})}{\delta A_{\mu}(\underline{x}')} = \delta(\underline{x} - \underline{x}')\, \delta^{\mu}{}_{\nu}$$

(2) Dirac Lagrangian density.

$$\mathcal{L} = \bar{\psi}(i\gamma^{\mu}\partial_{\mu} - m)\psi$$

$$\mathcal{L}_{QED} = \bar{\psi}(i\gamma^{\mu} \underset{\substack{\| \\ (\partial_{\mu} + ieqA_{\mu})}}{D_{\mu}} - m)\psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}$$

$$= \bar{\psi}(i\not{\partial} - m)\psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} - qe\underbrace{\bar{\psi}\gamma^{\mu}\psi}_{j^{\mu}}A_{\mu}$$

Free Dirac: $\mathcal{L} = \bar{\psi}(i\overrightarrow{\not{\partial}} - m)\psi$

$$S = \int d^4\underline{x}\; \bar{\psi}(i\overrightarrow{\not{\partial}} - m)\psi$$

$$\delta S = \int d^4\underline{x}\left(\delta\bar{\psi}(i\overrightarrow{\not{\partial}} - m)\psi + \bar{\psi}(i\overrightarrow{\not{\partial}} - m)\delta\psi\right)$$

↑ integrate by parts

$$= \int d^4\underline{x}\,(\delta\bar{\psi}(i\overrightarrow{\not{\partial}} - m)\psi - \int d^4\underline{x}\;\bar{\psi}(i\overleftarrow{\not{\partial}} + m)\delta\psi$$

For arbitrary $\delta\psi$ & $\delta\bar{\psi}$

$$(i\overrightarrow{\not{\partial}} - m)\psi = 0$$

$$\bar{\psi}(i\overleftarrow{\not{\partial}} + m) = 0$$

$$\frac{\delta \psi_\beta(y)}{\delta \psi_\alpha(x)} = \delta^\beta_\alpha \, \delta^4(\underline{y}-\underline{x}) \, , \qquad \pi_\psi = \frac{\partial \mathcal{L}}{\partial \dot{\psi}} = i\psi^\dagger \gamma^0 \gamma^0 = i\psi^\dagger$$

③ Complex Scalar fields. $\phi_i = \phi_i^*$

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi_1 \partial^\mu \phi_1 + \partial_\mu \phi_2 \partial^\mu \phi_2) - \frac{m^2}{2}(\phi_1^2 + \phi_2^2) - V(\phi_1, \phi_2)$$

Define $\frac{\phi_1 + i\phi_2}{\sqrt{2}} = \phi$, $\frac{\phi_1 - i\phi_2}{\sqrt{2}} = \phi^*$

$$\mathcal{L} = (\partial \phi^*)(\partial \phi) - m^2 \phi^* \phi - V(\phi^* \phi)$$ → U(1) invariant potⁿ.

$$\delta S = 0 \Rightarrow \Box \phi = -\frac{\partial V}{\partial \phi^*} - m^2 \phi$$

$$\Box \phi^* = -\frac{\partial V}{\partial \phi} - m^2 \phi^*$$

Lecture - 27

(02-11-2021)

Noether's theorem:

For every continuous symmetry which leaves the action unchanged, there exists an associated conserved quantity and hence, a COM.

$$\partial_\mu j^\mu = 0 = \partial_0 j^0 + \vec{\nabla} \cdot \vec{j}$$

$$\Rightarrow \partial_0 \int d^3\vec{x} \, j = -\int_V \vec{\nabla} \cdot j \, d^3x = -\oint_{S = \partial V} j \cdot ds \longrightarrow 0$$

$\therefore$ a divergenceless current $\partial_\mu j^\mu = 0 \Rightarrow \int d^3\vec{x}\, j^0 = $ COM.

Let's try some examples -

1(a). U(1) Global invariance

$$\mathcal{L} = \partial_\mu \phi^* \partial^\mu \phi - f(\phi^*\phi)$$

Under $\phi(x) \longrightarrow \phi'(\underline{x}) = e^{i\theta}\phi$

$$\phi^*\phi \longrightarrow \phi'^*\phi' = \phi^*\phi$$

Now, let's see what $\delta\phi$ is. $\delta\phi = \phi'(\underline{x}) - \phi(x) = i\theta\phi + O(\theta^2)$

$$\delta\phi^* = -i\theta\phi^* + O(\theta^2)$$

Using the eq^{n}s of motion -

$$\partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\right) = \frac{\partial\mathcal{L}}{\partial\phi} \Rightarrow \partial_\mu(\partial^\mu\phi^*) = -f'(\phi^*\phi)\cdot\phi^*$$

$$\Rightarrow \underline{\Box\phi^* = -f'(\phi^*\phi)\phi^*} \left(= -\frac{\partial V}{\partial\phi}\right) \quad - (1)$$

Generally $f(x) = m^2 x + \lambda x^2 + \dots \Rightarrow f'(x) = m^2 + 2\lambda x + \dots$

similarly, doing derivatives in EL w.r.t. ϕ^*

$$\Rightarrow \underline{\Box\phi = -f'(\phi^*\phi)\phi} \quad - (2)$$

Now, evaluating $\phi^* \cdot (2) - \phi\ (1) \Rightarrow \phi^*\partial_\mu\partial^\mu\phi - \phi\,\partial_\mu\partial^\mu\phi^* = 0$

$$\Rightarrow \partial_\mu(\underbrace{\phi^*\partial^\mu\phi - \phi\,\partial^\mu\phi^*}_{j^\mu}) = 0$$

$$\boxed{j^\mu = i\phi^* \overleftrightarrow{\partial^\mu}\phi} \Rightarrow Q = \int d^3\vec{x}\, j^0 = i(\phi^*\dot{\phi} - \dot{\phi}^*\phi)$$

is a constant of motion.

1(b) $\mathcal{L} = \bar{\psi}(i\not{\partial} - m)\psi$ fermion field

again $\psi \longrightarrow e^{i\theta}\psi$ $\qquad \delta\psi = i\theta\psi$

$\bar{\psi} \longrightarrow e^{-i\theta}\bar{\psi}$ $\qquad \delta\bar{\psi} = -i\theta\bar{\psi}$

$\bar{\psi}\times[\,(i\overrightarrow{\not{\partial}} - m)\psi = 0\,]$

$[\,\bar{\psi}(i\overleftarrow{\not{\partial}} + m) = 0\,]\times\psi$

Subtracting the above eq^{n}s gives

$$\partial_\mu(\bar{\psi}\gamma^\mu\psi) = 0 \Rightarrow \underline{j^\mu = \bar{\psi}\gamma^\mu\psi}$$

1(c) N-Fermi fields

$$\mathcal{L} = \sum_{i=1}^{N} \bar{\psi}_i(i\gamma^\mu\partial_\mu - m)\psi_i = \bar{\Psi}(i\gamma^\mu\partial_\mu - m)\Psi$$

where $\Psi = (\psi_1\ \psi_2 \ldots \psi_N)^T$

$i \rightarrow$ index for field no.
$A \rightarrow$ index for independent $U(N)$ parameters & generators.

$\Psi \longrightarrow \Psi'_{(\underline{x})} = U\Psi_{(\underline{x})}$

$\Psi^\dagger \longrightarrow \Psi^{\dagger\prime}_{(\underline{x})} = \Psi^\dagger U^\dagger_{(\underline{x})}$

these are internal transfos. not co-ord. transfos! so $\underline{x}' = \underline{x}$

For $U \in U(N)$,

$$U = \exp(i\theta_A T^A)$$

real. $\quad A = 1, \ldots N$

$T^A = T^{A\dagger}$

$\Rightarrow \delta\Psi = i\theta_A T^A \Psi$, $\qquad \delta\Psi^\dagger = -i\theta_A \Psi^\dagger T^A$

or $\delta\psi_i = i\theta_A (T^A)_{ij}\psi^j$

$\Rightarrow N^2$ conserved currents $(j^\mu)^A = \bar{\Psi}\gamma^\mu T^A \Psi$

$U(1)$ is a special case of $U(N)$.

$$U(N) \cong SU(N) \times SU(1) \qquad U = \hat{U}e^{i\varphi}$$

$\det \hat{U} = +1$

$\hat{U}^\dagger = \hat{U}^{-1}$

$\hat{U}^{-1} = \hat{u}^{-1} e^{-i\varphi \mathbb{1}_N}$ and $\det(U) = \det(\hat{u}) \cdot e^{iN\varphi}$

$\therefore \varphi = \frac{1}{N} \arg(\det(U))$

$\hat{u} = \exp(i\theta^{\hat{A}} \hat{T}^{\hat{A}})$ where $\hat{A} = 1, \dots, N^2-1$.

where $\hat{T}^{\hat{A}\dagger} = \hat{T}^{\hat{A}}$ and $tr(\hat{T}^{\hat{A}}) = 0$

Let's say if we have the invariance of the Lagrangian.

$$\mathcal{L}' = \mathcal{L} \Rightarrow \delta\mathcal{L} = 0 \qquad \mathcal{L} \equiv \mathcal{L}(\phi_i, \partial_\mu \phi_i)$$

$$\delta\mathcal{L} = \frac{\partial \mathcal{L}}{\partial \phi_i} \delta\phi_i + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_i)} \underbrace{\delta(\partial_\mu \phi_i)}_{\equiv\, \partial_\mu(\delta\phi_i)}$$

$$= \left(\frac{\partial \mathcal{L}}{\partial \phi_i} - \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_i)}\right) \delta\phi_i + \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_i)} \delta\phi_i\right)$$

If $\phi_i(\underline{x})$ is a sol^n to E-L eq^{n}s, then $\partial_{\phi_i}\mathcal{L} - \partial_\mu\left(\partial_{\partial_\mu \phi_i} \mathcal{L}\right) \equiv 0$

$$\Rightarrow \delta\mathcal{L} = 0 = \partial_\mu\left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_i)} \delta\phi_i\right) \longrightarrow \text{true for any internal symmetry}$$

$$\phi_i'(\underline{x}) = M_{ij}(\theta)\, \phi_j$$
$$= (\delta_{ij} + \underbrace{i\theta_A T^A_{ij} + \dots}_{\delta\phi_i})\, \phi_j$$

$$\Rightarrow \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_i)} \underset{\substack{\uparrow \\ \text{arbitoary}}}{i\theta_A} T^A{}_{ij}\, \phi_j\right) = 0$$

$$\Rightarrow \boxed{j^{\mu A} = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_i)} T^A_{ij}\, \phi_j}$$ is the conserved current.

If we have N complex scalar fields $\Phi = (\phi_1 \cdots \phi_N)^T$

$$\mathcal{L} = \partial^\mu \Phi^\dagger \partial_\mu \Phi - f(\Phi^\dagger \Phi)$$

$$\Phi' = \mathcal{U}\Phi \longrightarrow \Phi'^\dagger = \Phi^\dagger \mathcal{U}^\dagger$$

$$\Rightarrow \delta\phi_i = i\theta_A T^A{}_{ij}\, \phi_j \longrightarrow \delta\phi_i^* = \phi_j^*(-i\theta_A T^A_{ji})$$

$$= i\theta_A (-T^A)^T_{ij}\, \phi_j^*$$

Now the conserved current is given by the eqn –

$$\partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_i^*)} \delta\phi_i^* + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_i)} \delta\phi_i \right) = 0 \qquad \forall\, \theta^A$$

$$\partial_\mu \left[i\theta_A \left[\underbrace{\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_i^*)}}_{= (\partial_\mu \phi_i)} (-T^A)^T_{ij}\, \phi_j^* + \underbrace{\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_i)}}_{= (\partial_\mu \phi_i^*)} (T^A)_{ij}\, \phi_j \right]\right] = 0$$

$$\Rightarrow (j^\mu)^A = i\left(\partial^\mu \phi_i (T^A)^T_{ij}\, \phi_j^* - \partial^\mu \phi_i^* (T^A)_{ij}\, \phi_j\right)$$

$$= i\left(\phi_j^* T^A_{ji}\, \partial^\mu \phi_i - \partial^\mu \phi_i^* T^A{}_{ij}\, \phi_j\right)$$

$$= i\left(\phi_i^* T^A_{ij}\, \partial^\mu \phi_j - \partial^\mu \phi_i^* T^A{}_{ij}\, \phi_j\right)$$

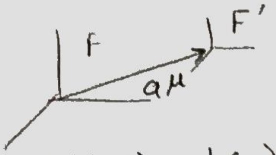
$$= i\, \Phi^\dagger T^A \overleftrightarrow{\partial} \Phi$$

$$\Rightarrow \partial_\mu j^{\mu A} = 0 \quad \text{where} \quad \boxed{j^{\mu A} = i\, \Phi^\dagger T^A \overleftrightarrow{\partial} \Phi}$$

Spacetime symmetries.

Here, the action is invariant, and the Lagrangian can shift by a total derivative.

(1) Poincaré symmetry (co-ord. shift invariance)



$$x'^\mu = x^\mu - a^\mu$$

$$\phi'(x') = \phi(x) \Rightarrow \phi'(x-a) = \phi'(x) - a^\mu \partial_\mu \phi + \cdots \equiv \phi(x)$$

$$\Rightarrow \delta\phi = \phi'(x) - \phi(x) = +a^\mu \partial_\mu \phi$$

Now $\mathcal{L}'(\phi'(x'), \partial_\mu' \phi'(x')) = \mathcal{L}(\phi, \partial_\mu \phi)$

$$\Rightarrow \delta\mathcal{L} = \mathcal{L}'(\phi, \partial\phi) - \mathcal{L}(\phi, \partial\phi) \underset{\uparrow}{=} +a^\mu \partial_\mu \mathcal{L}$$

by a similar logic

Now along the E-L soln, the variation in the Lagrangian is just

$$\delta\mathcal{L} = \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \delta\phi \right) = \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} a_\nu \partial^\nu \phi \right)$$

$$\partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} a^\nu \partial_\nu \phi \right) = \partial_\mu \left(a^\mu \mathcal{L} \right)$$

$$a^\nu \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \partial_\nu \phi - \delta^\mu{}_\nu \mathcal{L} \right) = 0$$

$\Rightarrow$ invariance of action under shifts of origin of spacetime implies that $\underline{\partial_\mu \Theta^\mu{}_\nu = 0}$

where
$$\boxed{\Theta^\mu{}_\nu \equiv \sum_A \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi^A)} \partial_\nu \phi^A - \delta^\mu{}_\nu \mathcal{L}} \longrightarrow$$ CANONICAL STRESS TENSOR

Remark: $\partial_\mu j^\mu = 0 \Rightarrow$ if $j^{\mu\prime} = j^\mu + \partial_\nu A^{\mu\nu}$ where $A^{\mu\nu} = -A^{\nu\mu}$ (antisymm-tensor)

then $\partial_\mu j^{\mu\prime} = \partial_\mu j^\mu + \underbrace{\partial_\mu \partial_\nu}_{\text{symm.}} \underbrace{A^{\mu\nu}}_{\text{antisymm.}}$, $\equiv 0$

$\therefore$ The conserved current is arbitrary upto a divergence of AS tensor.

Let's take the example of $\quad \mathcal{L} = \frac{1}{2}(\partial\phi)^2 - V(\phi)$

$$\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} = \partial^\mu \phi$$

$$\therefore \Theta^\mu{}_\nu = \partial^\mu\phi\, \partial_\nu \phi - \delta^\mu_\nu \Big(\underbrace{\tfrac{1}{2}(\partial\phi)^2}_{=\frac{1}{2}(\dot\phi^2 - \vec\nabla\phi^2)} - V(\phi) \Big)$$

$$\Rightarrow \Theta^0{}_0 = \dot\phi^2 - \left(\tfrac{1}{2}\dot\phi^2 - (\vec\nabla\phi)^2 - V(\phi)\right)$$

$$= \frac{1}{2}\left(\dot\phi^2 + \vec\nabla\phi^2 + V(\phi)\right) = \mathcal{H} \quad \text{(Hamiltonian density)}$$

$$H = \int d^3x\, \mathcal{H} = \int d^3x\, \Theta^0{}_0$$

$\therefore$ As a consequence of shift invariance, we automatically have conservation of energy $\frac{dH}{dt} = 0$.

$$(\hat{\vec{p}})_i = -i(\vec\nabla)_i = -i\partial_i$$

$$\Theta^0{}_i = \partial^0\phi\, \partial_i \phi - \cancel{\delta^0_i}\,\mathcal{L} \quad (\to 0)$$

$$= i\dot\phi(-i\vec\nabla)_i$$

$$= i\pi\,(\hat{\vec{p}})_i$$

first quantized momentum / particle momentum.

For instance,

$$\left\{ \int d^3x \, \Theta^0{}_i(\vec{x},t) \, , \, \phi(\vec{y},t) \right\}_{PB}$$

$$= \int d^3x \, \underbrace{\left\{ \pi(\vec{x},t), \phi(\vec{y},t) \right\}}_{= -\delta^{(3)}(\vec{x}-\vec{y})} \frac{\partial}{\partial x^i} \phi(\vec{x},t)$$

$$= -\partial_i \phi(\vec{x},t) = -\left(\vec{\nabla} \phi(\vec{x},t)\right)_i$$

(b) Lorentz invariance.

$$x'^\mu = \Lambda^\mu{}_\nu x^\nu = x^\mu - \omega^\mu{}_\nu x^\nu + O(\omega^2)$$

where $\omega_{\mu\nu} = -\omega_{\nu\mu}$

$$\phi'_A(x') = S_{AB} \phi_B(x)$$

$$S = e^{i\omega\cdot\sigma/2}$$

↑
spin field

$$\Rightarrow \delta\phi_A = (+\omega^\mu{}_\nu x^\nu) \partial_\mu \phi_A + S_{AB} \phi_B(x)$$

$$= (\omega^\mu{}_\nu x^\nu) \partial_\mu \phi_A + \frac{i}{2} \omega_{\mu\nu} (\sigma^{\mu\nu})_{AB} \phi(x)_B$$

spin generators.

Lecture - 28.

(08 - 11 - 2021)

Noether's theorem (contd.)

If under some transformation (continuous), $\underline{x} \to \underline{x}'$, $\phi(\underline{x}) \longrightarrow \phi'(\underline{x}')$, the action

S remains invariant / Lagrangian density changes by a four-divergence,

$\exists$ a conserved four-current $\partial_\mu j^\mu = 0$

If. $\mathcal{L}(\phi, \partial_\mu \phi, \phi^*, \partial_\mu \phi^*) = \partial\phi^* \partial\phi - f(\phi^*\phi)$

Then $\phi(\underline{x}) \longrightarrow \phi'(\underline{x}) = e^{i\theta}\phi(\underline{x})$ (global U(1) transf.)

$$\Rightarrow \mathcal{L}(\phi', \phi^{*\prime}, \partial_\mu\phi', \partial_\mu\phi^{*\prime}) = \mathcal{L}(\phi, \phi^*, \partial_\mu\phi, \partial_\mu\phi^*)$$

Since $\phi'(\underline{x}) = \left(\mathbb{1} + i\theta + \mathcal{O}(\theta^2)\right)\phi(\underline{x}) \Rightarrow \delta\phi(\underline{x}) = i\theta\,\phi(\underline{x})$

$\phi^{*\prime}(\underline{x}) = (\mathbb{1} - i\theta + \mathcal{O}(\theta^2))\,\phi^*(\underline{x}) \Rightarrow \delta\phi^*(\underline{x}) = -i\theta\,\phi^*(\underline{x})$

Then we know that if eqⁿs of motion are satisfied by ϕ & ϕ^* i.e.

$\partial_\phi \mathcal{L} = \partial_\mu\left(\partial_{\partial_\mu\phi}\mathcal{L}\right)$ and $\partial_{\phi^*}\mathcal{L} = \partial_\mu\left(\partial_{\partial_\mu\phi^*}\mathcal{L}\right)$, then-

$$\delta\mathcal{L} = \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\cdot i\phi + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi^*)}(-i\phi^*)\right) = 0$$

For $\mathcal{L}$ of the type $\mathcal{L} = \partial_\mu\phi\,\partial^\mu\phi^* - f(\phi^*\phi)$

$$\delta\mathcal{L} = \partial_\mu\Big(\underbrace{i\phi\,\partial_\mu\phi^* - i\phi^*\,\partial_\mu\phi}_{-j^\mu}\Big) = 0$$

$\Rightarrow \boxed{j^\mu = i\phi^* \overleftrightarrow{\partial_\mu}\,\phi}$ is a conserved current corresponding to U(1) symmetry.

If we integrate the j^0 component

$$\int d^3\vec{x}\; j^0 = \int d^3\vec{x}\; i(\phi^*\dot{\phi} - \dot{\phi}^*\phi) \equiv Q_V \text{ (conserved charge in vol. V)}$$

If $V \to \infty$, then $\int_{V_\infty} d^3x\, j^0 = Q$, and $\frac{dQ}{dt} = -\oint_{S_\infty} \vec{\nabla}\cdot j\, d^3\vec{x} = 0$

For a general variation of $\delta\phi$ of the form -

$$\delta\phi_i = i\theta_A T^A_{ij}\,\phi_j(\underline{x}) \qquad i = 1, 2 \ldots N$$

$$\Rightarrow j^{\mu A} = \sum_{ij} i\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_i)} T^A_{ij}\,\phi_j \qquad \text{and } \partial_\mu j^{\mu A} = 0$$

• As an example, for $\mathcal{L} = \sum_i \bar{\psi}_i (i\not{\partial} - m)\psi_i$

Now if we have a symmetry transf. $\psi \to U\psi = e^{i\theta_A T^A}\psi$ (generators of U(N))

$$\Rightarrow \delta\psi_i = i\theta_A T^A \psi_i \qquad \delta\bar{\psi}_i = -i\theta_A T^A \bar{\psi}_i$$

$$\partial_\mu\left(\sum_i \frac{\partial\mathcal{L}}{\partial(\partial_\mu\psi_i)}\delta\psi_i + \cancelto{0}{\frac{\partial\mathcal{L}}{\partial(\partial_\mu\bar{\psi}_i)}}\delta\bar{\psi}_i\right) = 0$$

$$\partial_\mu\left(\sum_i \bar{\psi}_i (i\gamma^\mu)\cdot(i\theta_A T^A\psi_i)\right) = 0$$

$$\theta^A\partial_\mu\left(\sum_i \bar{\psi}_i T^A\gamma^\mu\psi_i\right) = 0 \Rightarrow j^{\mu A} = \sum_i \bar{\psi}_i T^A\gamma^\mu\psi_i$$

• For $\boxed{\text{translational symmetry}}$ $\quad x^\mu \to x^\mu + \delta x^\mu$

$$\phi \to \phi_A'(\underline{x}') = \phi_A(\underline{x})$$

$$\Rightarrow \delta\phi_A = -\delta x^\mu\,\partial_\mu\phi$$

3

Since $\mathcal{L}'(\phi') = \mathcal{L}(\phi)$ (since Lagrangian is also like a scalar field).

$$\mathcal{L}'(\phi') = \mathcal{L}'(\phi) + \delta\phi\, \partial_\mu \mathcal{L} \equiv \mathcal{L}(\phi)$$

$\Rightarrow$

How is $\delta\mathcal{L} = -\delta x^\mu \partial_\mu \mathcal{L}$?? Still need to think about it.

(his argument is that $\mathcal{L}$ is a scalar fn. and scalar f's transf. as $-\delta x^\mu \partial_\mu \mathcal{L}$)

$$\delta\mathcal{L}(\phi, \partial\phi) = \frac{\partial\mathcal{L}}{\partial\phi}\delta\phi + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\delta(\partial_\mu\phi) = \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial\,\partial_\mu\phi}\delta\phi\right) \text{ along } \phi_{cl}(\underline{x})$$

$$\text{So, } \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}(\delta x^\nu \partial_\nu\phi) - \delta x^\nu \delta^\mu_\nu \mathcal{L}\right) = 0$$

$$\Rightarrow \partial_\mu \Theta^\mu{}_\nu = 0 \quad \text{where } \Theta^\mu{}_\nu = \sum_A \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_A)}\partial_\nu\phi^A - \delta^\mu_\nu \mathcal{L}$$

Canonical stress tensor.

If we consider a Lagrangian like

$$\mathcal{L} = \frac{1}{2}\partial\phi\,\partial\phi - V(\phi)$$

$$\Theta^0{}_\nu = \dot{\phi}^A \partial_\nu\phi^A - \delta^0_\nu \mathcal{L} \Rightarrow \Theta^0{}_0 = \dot{\phi}^{A2} - \mathcal{L} = \dot{\phi}^A \pi^A - \mathcal{L} = \mathcal{H}.$$

$$\text{Similarly, } \Theta^0{}_i = \dot{\phi}^A \partial_i\phi^A = \pi^A(\partial_i\phi^A)$$

$$P_i = \int d^3x\, \Theta^0{}_i \quad = \text{conserved 3-momentum in the field.}$$

Let's now derive the current for $\boxed{\text{Lorentz symmetries.}}$

$$\delta x^\mu = -\omega^\mu{}_\nu x^\nu \qquad \omega_{\mu\nu} = -\omega_{\nu\mu}$$

$$\phi'_A(x') = \phi'_A(x) + \delta x^\mu \partial_\mu\phi_A = \phi_A(x) \Rightarrow \delta\phi_A = \omega^\mu{}_\nu x^\nu \partial_\mu\phi_A$$

(if spin = 0)

$$\text{So, } \delta\phi_A = -\frac{\omega_{\mu\nu}}{2} x^{[\mu}\partial^{\nu]}\phi_A$$

However, if we consider spinor fields, then

$$\delta\phi_A = \frac{i}{2}\,\omega_{\mu\nu}\left(i x^{[\mu}\partial^{\nu]} + \frac{1}{2}\sigma^{\mu\nu}\right)\phi_A(x)$$

Now, $\delta\mathcal{L} = -\delta x^{\mu}\partial_\mu \mathcal{L}$ since it's a scalar field.

And, $\delta\mathcal{L} = \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_A)}\,\delta\phi_A\right) = -\delta x^{\mu}\partial_\mu\mathcal{L}$

$$\Rightarrow \quad \delta x^{\mu}\partial_\mu\mathcal{L} + \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_A)}\,\delta\phi_A\right) = 0$$

So, $\partial_\mu\left(\delta x^{\nu}\delta^{\mu}_{\nu}\mathcal{L} + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_A)}\,\delta\phi_A\right) = 0$

$$\Rightarrow \partial_\mu\left(-\omega^{\nu}_{\ \lambda}x^{\lambda}\delta^{\mu}_{\nu}\mathcal{L} + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_A)}\cdot\frac{i}{2}\omega_{\nu\lambda}\left(i x^{[\nu}\partial^{\lambda]} + \frac{1}{2}\sigma^{\nu\lambda}\right)\phi\right) = 0$$

$$\partial_\mu\left(\underbrace{-\omega_{\nu\lambda}x^{\lambda}\eta^{\mu\nu}\mathcal{L}}_{-\frac{1}{2}\omega_{\nu\lambda}x^{[\lambda}\eta^{\nu]\mu}\mathcal{L}} + \frac{\omega_{\nu\lambda}\partial\mathcal{L}}{2\,\partial(\partial_\mu\phi_A)}\left(-x^{[\nu}\partial^{\lambda]}\phi_A + \frac{i}{2}\sigma^{\nu\lambda}_{AB}\phi_B\right)\right) = 0$$

$$\Rightarrow \frac{\omega_{\nu\lambda}}{2}\partial_\mu\Bigg(\underbrace{\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_A)}\left(x^{[\lambda}\partial^{\nu]}\phi_A + \frac{i}{2}\sigma^{\nu\lambda}_{AB}\phi_B\right) - x^{[\lambda}\eta^{\nu]\mu}\mathcal{L}}_{\text{conserved current.}}\Bigg) = 0$$

$$\boxed{M^{\mu,\nu\lambda} = \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_A)}\left(x^{[\lambda}\partial^{\nu]}\phi_A + \frac{i}{2}\sigma^{\nu\lambda}_{AB}\phi_B\right) - x^{[\lambda}\eta^{\nu]\mu}\mathcal{L}}$$

μ is the index over which we take the divergence ∂_μ.

Lorentz current density.

$\partial_\mu M^{\mu,\nu\lambda} = 0$ $\quad$ $M^{\mu,\nu\lambda}$ is a conserved current.

The Lorentz current can be written in a simple form as follows -

$$M^{\mu,\nu\lambda} = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_A)} x^{[\lambda}\left(\partial^{\nu]}\phi_A - \eta^{\nu]\mu}\mathcal{L}\right) + \frac{i}{2}\frac{\partial \mathcal{L}}{\partial(\partial_\mu\phi_A)}\sigma^{\nu\lambda}_{AB}\phi_B$$

$$\Rightarrow \boxed{M^{\mu,\nu\lambda} = x^{[\lambda}\Theta^{\mu\nu]} + \frac{i}{2}\frac{\partial \mathcal{L}}{\partial(\partial_\mu\phi_A)}\sigma^{\nu\lambda}_{AB}\phi_B}$$

where $\Theta^{\mu\nu}$ is the canonical stress tensor in the raised form.

$M^{\mu,\lambda\nu}$ is manifestly A.S in ν & λ

$$\underline{M^{\mu,\lambda\nu} = -M^{\mu,\nu\lambda}}$$ (due to the [ ] anti-commutation)

Now, we know that $\partial_\mu M^{\mu,\nu\lambda} = 0$

$$\Rightarrow \partial_\mu\Big(x^{[\lambda}\Theta^{\mu\nu]} + \underbrace{\frac{i}{2}\frac{\partial \mathcal{L}}{\partial(\partial_\mu\phi_A)}\sigma^{\nu\lambda}_{AB}\phi_B}_{\Sigma^{\mu,\nu\lambda}}\Big) = \partial_\mu(x^{[\lambda}\Theta^{\mu\nu]}) + \partial_\mu\Sigma^{\mu,\nu\lambda}$$

$$= \delta^{[\lambda}_\mu \Theta^{\mu\nu]} + x^{[\lambda}\underbrace{\partial_\mu\Theta^{\mu\nu]}}_{=0} + \partial_\mu\Sigma^{\mu,\nu\lambda} = 0$$

because $\Theta^{\mu\nu}$ is itself a conserved current!

$$= \Theta^{[\lambda\nu]} + \partial_\mu\Sigma^{\mu,\nu\lambda}$$

So, $\Theta^{\lambda\nu} - \Theta^{\nu\lambda} = -\partial_\mu \Sigma^{\mu,\nu\lambda}$

↑ SPIN CURRENT!

Therefore, <u>existence of spin makes $\Theta^{\lambda\nu}$ non-symmetric.</u>

Here $\boxed{\Sigma^{\mu,\nu\lambda} = \frac{i}{2}\frac{\partial \mathcal{L}}{\partial(\partial_\mu\phi_A)}\sigma^{\nu\lambda}_{AB}\phi_B}$ spin current.

$\left(\Sigma^{\mu,\lambda\nu} = -\Sigma^{\mu,\nu\lambda}\right)$ ↑ ANTI-SYMM.

$$\Rightarrow \underline{M^{\mu,\nu\lambda} = x^{[\lambda}\Theta^{\mu\nu]} + \Sigma^{\mu,\nu\lambda}}$$

This relation b/w the non-symmetry of $\Theta^{\mu\nu}$ and the divergence of the spin current can be exploited to define

$$T^{\mu\nu}_{(\text{symm.})} = \underset{\substack{\uparrow \\ \text{Canonical} \\ \text{stress tensor.}}}{\Theta^{\mu\nu}} + \underset{\substack{\uparrow \\ \text{Belinfante density.}}}{B^{\mu\nu}}$$

to get $T^{\mu\nu}_{(\text{sym.})} = T^{\nu\mu}_{(\text{sym.})}$. Here, $B^{\mu\nu}$ is an object defined as-

$$B^{\mu\nu} = \frac{1}{2}\partial_\lambda\left(\Sigma^{\mu,\nu\lambda} + \Sigma^{\nu,\mu\lambda} - \Sigma^{\lambda,\nu\mu}\right)$$

and $\partial_\mu B^{\mu\nu} = 0$

<u>Main point</u>: if $\partial_\mu j^\mu = 0$, then if $\tilde{j}^\mu \equiv j^\mu + Y^\mu$:

$$\partial_\mu \tilde{j}^\mu = 0$$

if
- $\partial_\mu Y^\mu = 0$
- $\tilde{Q} = \int d^3x\, \tilde{j}^0(\underline{x}) = Q + \int d^3x\, Y^0$

  So, provided $\int d^3x\, Y^0 = 0$, $\tilde{Q} = Q$.
- $Y^\mu = \partial_\lambda(X^{\mu\lambda})$ for some $X^{\mu\lambda}$ is enough to ensure this.

Lecture-29.

(09-11-2021)

- Canonical Stress tensor $\longleftrightarrow$ invariance under $\delta x^{\mu} = a^{\mu}$.

$$\boxed{\Theta^{\mu}{}_{\nu} = \pi^{\mu}\,\partial_{\nu}\phi - \delta^{\mu}_{\nu}\,\mathcal{L}}$$

where $\pi^{\mu} = \dfrac{\partial \mathcal{L}}{\partial(\partial_{\mu}\phi)}$ $\rightarrow$ field 4-mom^n

- Lorentz currents $\longleftrightarrow$ invariance under $\delta x^{\mu} = -\omega^{\mu}{}_{\nu}\, x^{\nu}$

$$\boxed{M^{\mu}{}_{\lambda\nu} = x_{[\lambda}\,\Theta^{\mu}{}_{\nu]} + \frac{i}{2}\pi^{\mu} S_{\lambda\nu}\,\phi}$$

$$= -M^{\mu}{}_{\nu\lambda}$$

- $\Theta^{0}{}_{i} = \pi\,\partial_i\phi = \pi(\vec{\nabla}\phi)_i$

$$\int d^3\vec{x}\;\Theta^{0}{}_{i} = \int d^3\vec{x}\;\pi(\vec{\nabla}\phi)_i = i\int d^3\vec{x}\,\pi\,\hat{p}_i\,\phi$$

(from the def^n of first quantized mom^n $\hat{\vec{p}} = -i\vec{\nabla}$)

If we define $P_{\mu} \equiv \int d^3\vec{x}\;\Theta^{0}{}_{\mu}$, then -

$$\int d^3\vec{x}\;\Theta^{0}{}_{i} = i\int d^3\vec{x}\;\pi\,\hat{p}_i\,\phi = -\vec{P}_i \quad \text{(const. of motion.)}$$

$$\Rightarrow \underline{\vec{P} = \int d^3\vec{x}\;(-i\pi)\,\hat{\vec{p}}\,\phi}\;, \text{ and } \frac{d\vec{P}}{dt} = 0$$

- $\Theta^{0}{}_{0} = \pi\dot{\phi} - \mathcal{L} = \mathcal{H}$

So, $\displaystyle\int d^3\vec{x}\;\Theta^{0}{}_{0} = \int d^3\vec{x}\,(\pi\dot{\phi} - \mathcal{L}) = H$

So, we generally expect the conserved charges to be of the form -

$$Q = \int d^3\vec{x}\;(-i\pi)\,(\text{First quantized operator})\,\phi$$

For example, if we talk about $U(1)$ invariance for ϕ, ϕ^* fields,

then $\delta\phi = i\theta\phi$ & $\delta\phi^* = -i\theta\phi^*$

$$\Rightarrow j^\mu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \delta\phi + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi^*)} \delta\phi^*$$

If $\mathcal{L} = \partial_\mu \phi^* \partial^\mu \phi - f(\phi^*\phi)$, then the conserved current (ignoring the θ parameter) -

$$j^\mu = i\phi\partial_\mu \phi^* - i\phi^*\partial_\mu \phi$$

Also, for this $\mathcal{L}$, $\pi^\mu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} = \partial_\mu \phi^* \Rightarrow j^\mu = i\phi\,\pi^\mu - i\phi^*\pi^{\mu*}$

Therefore, $j^0 = (-i\pi)(-\mathbb{1})\phi + (-i\pi^*)(\mathbb{1})\phi^*$

$$\Rightarrow Q = \int d^3\vec{x} \left[(-i\pi)(-\mathbb{1})\phi + (-i\pi^*)(\mathbb{1})\phi^* \right]$$

So here, the first quantized operator is almost trivial $(\pm \mathbb{1})$.

If we would have $U(N)$ symmetry, then we would have had -

$$j^{0A} = (-i\pi)_i \underbrace{(-T^A)_{ij}}_{\text{first quantized operator}} \phi_j + (-i\pi^*)_i \underbrace{(T^A)_{ij}}_{\text{first quantized operator}} \phi_j$$

STRESS ENERGY TENSOR FOR A GRAVITY-COUPLED THEORY.

So, we know $M^\mu{}_{\lambda\nu} = x_{[\lambda} \Theta^\mu{}_{\nu]} + \underbrace{\Sigma^\mu{}_{\lambda\nu}}_{\text{spin current.}}$

Note: (in the prev. lecture, we defined $M^\mu{}_{\nu\lambda} = x_{[\lambda}\Theta^\mu{}_{\nu]} + \Sigma^\mu{}_{\nu\lambda}$, so beware)

Now $\partial_\mu M^\mu_{\ \lambda\nu} = \partial_\mu x_{[\lambda} \Theta^\mu_{\ \nu]} + x_{[\lambda} \cancel{\partial_\mu \Theta^\mu_{\ \nu]}}^{\,0} + \partial_\mu \Sigma^\mu_{\ \lambda\nu}$

(since stress energy tensor is a conserved current)

$$= \Theta_{[\lambda\nu]} + \partial_\mu \Sigma^\mu_{\ \lambda\nu} = 0$$

$$\Rightarrow \Theta_{[\lambda\nu]} = \Theta_{\lambda\nu} - \Theta_{\nu\lambda} = -\partial_\mu \Sigma^\mu_{\ \lambda\nu} \neq 0$$

$\therefore$ the presence of the spin current makes $\Theta_{\lambda\nu}$ non-symmetric.

Another definition for a <u>matter stress tensor</u> coupled to gravity which fixes the symmetric problem -

$$T^{\mu\nu} = \frac{2}{\sqrt{-\det g}} \frac{\delta S}{\delta g_{\mu\nu}} \qquad \text{where } S = \int d^4\underline{x}\, \sqrt{-\det g}\, \mathcal{L}$$

action of the theory coupled to gravity.

The Lagrangian density is generally of the form -

$$\mathcal{L} = \frac{1}{2} g_{\mu\nu} \partial^\mu \phi \partial^\nu \phi - V(\phi)$$

Now, $\underset{\substack{\downarrow \\ \text{by varying } g_{\mu\nu}}}{\delta S} = \int d^4\underline{x}\ \delta\sqrt{-\det g}\, \mathcal{L} + \sqrt{-\det g}\, \delta\mathcal{L}$

Since we are only varying $g_{\mu\nu}$, $\delta\mathcal{L} = \frac{1}{2} \delta g_{\mu\nu} \partial^\mu\phi \partial^\nu\phi$

$$\delta\sqrt{-\det g} = \frac{1}{2\sqrt{-\det g}} (-\delta \det g)$$

and $\delta \det g = \det g \cdot \mathrm{tr}(g^{-1} \cdot \delta g) = \det g \cdot (\underbrace{g^{\nu\mu}}_{= g^{\mu\nu}} \delta g_{\mu\nu})$

$$\Rightarrow \delta S = \int d^4y \left(\frac{-\det g \cdot g^{\mu\nu}}{2\sqrt{-\det g}} \delta g_{\mu\nu}(y) \right) \mathcal{L} + \sqrt{-\det g} \left(\frac{1}{2} \partial^\mu\phi \partial^\nu\phi \right) \delta g_{\mu\nu}(y)$$

$$\Rightarrow \frac{\delta S[\phi]}{\delta g_{\mu\nu}(\underline{x})} = \int d^4y \, \frac{\sqrt{-\det g}}{2} \left(g^{\lambda\sigma} \mathcal{L} + \partial^\lambda \phi \, \partial^\sigma \phi \right) \underbrace{\frac{\delta g^{\lambda\sigma}(\underline{y})}{\delta g^{\mu\nu}(\underline{x})}}_{\delta^4(\underline{x}-\underline{y}) \, \delta^\mu_\lambda \, \delta^\nu_\sigma}$$

$$= \frac{\sqrt{-\det g}}{2} \left(\partial^\mu \phi \partial^\nu \phi + g^{\mu\nu} \mathcal{L} \right)$$

$$\Rightarrow \boxed{T^{\mu\nu} = \partial^\mu \phi \partial^\nu \phi - g^{\mu\nu} \mathcal{L}} \qquad \text{(similar to } \Theta^{\mu\nu}\text{)}$$

$\therefore$ by defn, $\underline{T^{\nu\mu} = T^{\mu\nu}}$ since $g^{\mu\nu}$ is symmetric.

But for non-gravity coupled theories, $\Theta^{\mu\nu} \neq \Theta^{\nu\mu}$ when we have spin.

However, if $\partial_\mu j^\mu = 0$ by Noether's Thm, then $j^{\mu\prime} \equiv j^\mu + Y^\mu$ is also a suitable conserved current provided –

- $\partial_\mu Y^\mu = 0$
- $Q' = \int d^3\vec{x} \, j'^0 \stackrel{!}{=} Q \Rightarrow Q_Y = \int d^3\vec{x} \, Y^0 = 0$

This is possible if Y^0 = divergence of something.

So, if we write $Y^\mu = \partial_\nu X^{\nu\mu}$ s.t. $X^{\nu\mu} = -X^{\mu\nu}$

$$\Rightarrow \partial_\mu Y^\mu = \underbrace{\partial_\mu \partial_\nu}_{S.} \underbrace{X^{\nu\mu}}_{A\text{-}S.} \equiv 0$$

This also $\Rightarrow$ $Y^0 = \partial_0 \underset{\substack{\parallel \\ -X_{00} = 0}}{X^{00}} + \partial_i X^{i0} = \partial_i X^{i0}$

$$\therefore Q_Y = \int_{S_\infty} d^3\vec{x} \; \partial_i X^{i0} = 0$$

This is known as the $\underline{\text{Belinfante procedure}}$ to produce symmetric stress tensors.

Therefore, to do this for $\Theta^{\mu\nu}$,

$$B^{\mu\nu} = \partial_\lambda X^{\lambda,\mu\nu}$$

$$X^{\lambda,\mu\nu} = \frac{1}{2}\left(\Sigma^{\mu,\nu\lambda} + \Sigma^{\nu,\mu\lambda} - \Sigma^{\lambda,\nu\mu}\right)$$

$$X^{\mu,\lambda\nu} = \frac{1}{2}\left(\Sigma^{\lambda,\nu\mu} + \Sigma^{\nu,\lambda\mu} - \Sigma^{\mu,\nu\lambda}\right)$$

$$= \frac{1}{2}\left(\Sigma^{\lambda,\nu\mu} - \Sigma^{\nu,\mu\lambda} - \Sigma^{\mu,\nu\lambda}\right) = -X^{\lambda,\mu\nu}$$

Since $X^{\mu,\lambda\nu} = -X^{\lambda,\mu\nu}$ (antisymmetric in $\mu \leftrightarrow \lambda$)

$$\Rightarrow \partial_\mu B^{\mu\nu} = \underbrace{\partial_\mu \partial_\lambda}_{\text{S. in } (\mu\leftrightarrow\lambda)} \underbrace{X^{\lambda,\mu\nu}}_{\text{A.S. in } (\mu\leftrightarrow\lambda)} \equiv 0.$$

Also, $\int d^3\vec{x}\, B^{0\nu} = \frac{1}{2}\int d^3\vec{x}\left(\partial_i X^{i,0\nu} + \underbrace{\partial_0 X^{0,0\nu}}_{=0}\right)$

$$= \frac{1}{2}\int_{S_\infty} d^3\vec{x}\, \partial_i X^{i,0\nu} \equiv 0$$

$$\Rightarrow \quad \underline{\partial_\mu B^{\mu\nu} = 0} \quad \text{AND} \quad \underline{Q = \int_{S_\infty} d^3\vec{x}\, B^{0\nu} = 0}$$

$\therefore T^{\mu\nu} = \left(\Theta^{\mu\nu} + \alpha B^{\mu\nu}\right)$ will be a conserved current!

Now $T^{[\mu\nu]} = \Theta^{[\mu\nu]} + \alpha B^{[\mu\nu]}$

Now $B^{[\mu\nu]} = B^{\mu\nu} - B^{\nu\mu} = \frac{1}{2}\partial_\lambda\left(\cancel{\Sigma^{\mu,\nu\lambda}} + \cancel{\Sigma^{\nu,\mu\lambda}} - \Sigma^{\lambda,\nu\mu} - \cancel{\Sigma^{\nu,\mu\lambda}} - \cancel{\Sigma^{\mu,\nu\lambda}} + \Sigma^{\lambda,\mu\nu}\right)$

$$= -\partial_\lambda \Sigma^{\lambda,\nu\mu}$$

and $-\partial_\lambda \Sigma^{\lambda\nu\mu} = \Theta^{[\nu\mu]}$

$$\Rightarrow T^{[\mu\nu]} = \Theta^{[\mu\nu]} + \alpha\, \Theta^{[\nu\mu]} = (1-\alpha)\,\Theta^{[\mu,\nu]} \overset{!}{=} 0$$

$$\Rightarrow \underline{\underline{\alpha = 1}}$$

Hence,

$$\boxed{T^{\mu\nu} = \Theta^{\mu\nu} + B^{\mu\nu}}$$ is a symmetric conserved current,

also known as Belinfante-Rosenfeld stress tensor.

where $B^{\mu\nu} = \frac{1}{2}\partial_\lambda\left(\Sigma^{\mu,\nu\lambda} + \Sigma^{\nu,\mu\lambda} - \Sigma^{\lambda,\nu\mu}\right)$.

SCALE TRANSFORMATIONS.

Let's take the standard scalar field action

$$S = \int d^4\underline{x}\,\left(\partial_\mu\phi\,\partial^\mu\phi - (m^2\phi^2 + g\phi^3 + \lambda\phi^4 + \dots)\right]$$

mass dim's. $S \to 0$, $d^4x \to -4$, $\partial_\mu, \phi, \partial^\mu, \phi \to 1, 1, 1, 1$, $m^2, \phi^2 \to 2, 2$, $g, \phi^3 \to 1, 3$, $\lambda, \phi^4 \to 0, 4$

Suppose we scale

$$x \longrightarrow x' = \lambda x \quad , \quad \phi \longrightarrow \phi'(x') = \frac{1}{\lambda}\phi(x) \quad , \quad \partial_\mu \to \partial'_\mu = \frac{1}{\lambda}\partial_\mu$$

but for $\int \cancel{\lambda^4}\, d^4\underline{x}\left(\frac{1}{\cancel{\lambda^4}}(\partial\phi)^2\right)$, $\int \cancel{\lambda^4}\, d\underline{x}^4\left(\lambda\cdot\frac{1}{\cancel{\lambda^4}}\phi\right)$, the scale cancels out

and hence they are scale invariant.

However. $\int d^4\underline{x}\;\lambda^4\left[\left(g\cdot\frac{1}{\lambda^3}\phi^3\right) + \left(m^2\cdot\frac{1}{\lambda^2}\phi^2\right)\right]$ is not scale invariant!

Similarly $\lambda^{(n)}\dfrac{\phi^n}{n!}$ for $n > 4$ also won't be scale invariant

$\therefore$ If we take $\mathcal{L} = \partial_\mu\phi\,\partial^\mu\phi - \lambda\phi^4$, then

$$S = \int d^4\underline{x}\;\cancel{\lambda^4}\left(\frac{1}{\cancel{\lambda^4}}(\partial\phi)^2 - \frac{1}{\cancel{\lambda^4}}\cdot(\lambda\phi^4)\right) = \int d^4\underline{x}\,\left(\partial_\mu\phi\,\partial^\mu\phi - \lambda\phi^4\right)$$

is scale-invariant!

In general, for d-dim^n's, if we scale -

$$x \to x' = \lambda x \ , \quad \phi \to \phi'(x') = \lambda^{-\omega_d} \phi(x)$$

Here ω_D is the canonical dim^n of a Bose field s.t.

$$A = \int d^d\underline{x} \left(\frac{1}{2} (\partial\phi)^2 - V(\phi) \right)$$

$$[\phi] = +\omega_d = +\left(\frac{d-2}{2}\right)$$

For this transformation, $\delta x^\mu = (\delta\lambda)\, x^\mu$

and $\delta\phi = -\delta x^\mu \partial_\mu \phi - \omega_d \, \delta\lambda \, \phi$ because $\lambda^{-\omega_d} = (1+\delta\lambda)^{-\omega_d}$
$= 1 - \omega_d \, \delta\lambda$

Also, for a scale invariant Lagrangian density,

$$d^d\underline{x} \to d^d\underline{x}' = \lambda^d \, d^d\underline{x} \Rightarrow \quad \mathcal{L} \longrightarrow \mathcal{L}' = \lambda^{-d} \mathcal{L} = (1+\delta\lambda)^{-d} \mathcal{L}$$

$$\mathcal{L}'_{(x')} = \mathcal{L}_{(x)} - d\, \delta\lambda \mathcal{L}$$

So, $\mathcal{L}'(x') = \mathcal{L}'(x) + \delta x^\mu \partial_\mu \mathcal{L} = \mathcal{L}(x) - d(\delta\lambda)\mathcal{L}$.

$\Rightarrow \delta\mathcal{L} = -\delta x^\mu \partial_\mu \mathcal{L} - d(\delta\lambda)\mathcal{L}$. and $\delta x^\mu = \delta\lambda \, x^\mu$

So, $\delta\mathcal{L} = -\delta\lambda \left(x^\mu \partial_\mu \mathcal{L} + d\mathcal{L} \right)$

Equating the above to $\delta\mathcal{L}$ eq^n. i.e. $\delta\mathcal{L} = \left(\partial_\phi \mathcal{L} - \partial_\mu \partial_{\partial_\mu\phi} \mathcal{L} \right)^{\to 0} + \partial_\mu \left(\partial_{\partial_\mu\phi} \mathcal{L} \, \delta\phi \right)$

$$\Rightarrow \partial_\mu \left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)} \delta\phi \right) = -\delta\lambda \left(\partial_\mu \mathcal{L} + d\mathcal{L} \right)$$

Notice that $\partial_\mu (x^\mu \mathcal{L}) = \underbrace{(\partial_\mu x^\mu)}_{= d} \mathcal{L} + x^\mu \partial_\mu \mathcal{L} = d\mathcal{L} + x^\mu \partial_\mu \mathcal{L}$

$$\therefore \partial_\mu \left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)} \cdot \left(+\cancel{\delta\lambda} x^\nu \partial_\nu \phi + \omega_d \cancel{\delta\lambda} \phi \right) = \partial_\mu \left(\cancel{-\delta\lambda}\, x^\mu \mathcal{L} \right) \right.$$

Re-arranging the terms

$$\partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \cdot (x^\nu \partial_\nu \phi + \omega_d \phi) - x^\nu \delta^\mu_\nu \mathcal{L} \right) = 0$$

$$\Rightarrow \partial_\mu \Big(\underbrace{x^\nu \Theta^\mu{}_\nu + \pi^\mu \omega_d \phi}_{D^\mu \,\longrightarrow\, \text{scaling current!}} \Big) = 0$$

$\boxed{D^\mu = x^\nu \Theta^\mu{}_\nu + \pi^\mu \omega_d \phi}$ (scaling current) is a conserved current!

This means $\partial_\mu (\pi^\mu \omega_d \phi) = - \partial_\mu (x^\nu \Theta^\mu{}_\nu) = - \Theta^\mu{}_\mu$

In cases where $\Theta^\mu{}_\nu$ is traceless, classical scale invariance $\pi^\mu \omega_d \phi$ is itself conserved.

Stress energy tensor for $\mathcal{L}_{\text{Dirac}}$.

$\mathcal{L} = \bar{\psi}(i\gamma^\mu \partial_\mu - m)\psi$

Under $U(1)$ transf., $\psi \to e^{i\theta}\psi$, $\bar{\psi} \to e^{-i\theta}\bar{\psi} \Rightarrow j^\mu = \bar{\psi}\gamma^\mu \psi$ is the conserved qty.

What about the stress energy tensor?

$$\Theta^\mu{}_\nu = \pi^\mu \partial_\nu \psi - \delta^\mu_\nu \mathcal{L}$$

$$\pi^\mu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \psi)} = i\bar{\psi}\gamma^\mu \implies \pi^0 = i\bar{\psi}\gamma^0 = i\psi^\dagger \gamma^0 \gamma^0 = i\psi^\dagger$$

Equal time Poisson Brackets. (Review)

$$\{O(\vec{x},t), O'(\vec{y},t)\} = \int d^3\vec{z} \left(\frac{\delta O(\vec{x},t)}{\delta \phi(\vec{z},t)} \frac{\delta O'(\vec{y},t)}{\delta \pi(\vec{z},t)} - (O \leftrightarrow O') \right)$$

$$\Rightarrow \{\phi(\vec{x},t), \pi(\vec{y},t)\} = \int d^3\vec{z} \left(\underbrace{\frac{\delta \phi(\vec{x},t)}{\delta \phi(\vec{z},t)}}_{\delta^{(3)}(\vec{x}-\vec{z})} \underbrace{\frac{\delta \pi(\vec{y},t)}{\delta \pi(\vec{z},t)}}_{\delta^{(3)}(\vec{y}-\vec{z})} - 0 \right) = \delta^{(3)}(\vec{x}-\vec{y})$$

Implement the action of charges on fields as follows-

$$\delta\phi \equiv \{Q(t), \phi(\vec{x},t)\}$$

So, if $Q = \alpha \int \pi(\vec{y},t)\, \hat{O}\, \phi(\vec{y},t)\, d^3\vec{y}$

first quantized operator

then $\{Q, \phi(\vec{x},t)\} = \alpha \int \{\pi, \phi\}\, \hat{O}\, \phi(\vec{y},t)\, d^3\vec{y}$

$$= -\alpha \int d^3\vec{y}\; \delta^{(3)}(\vec{x}-\vec{y})\, \hat{O}\, \phi(\vec{y},t)$$

$$= -\alpha\, \hat{O}\, \phi(\vec{y},t)$$

Going from classical $\longrightarrow$ quantum, $\{Q, \phi(\vec{x},t)\} \longrightarrow \dfrac{[\hat{Q}, \hat{\phi}(\vec{x},t)]}{i}$

Therefore, we expect $[\hat{Q}, \hat{\phi}(\vec{x},t)] = -i\alpha\, \hat{O}\, \hat{\phi}$

Now $\hat{O}_H(t) = U^\dagger O_s U$. If $U = e^{i\theta Q}$

$$\Rightarrow \hat{O}_H(t) = e^{i\theta Q}\, \hat{O}\, e^{-i\theta Q} = \hat{O} + i\theta[Q, \hat{O}] + O(\theta^2)$$

$$\Rightarrow \delta\hat{O} = i[Q, \hat{O}] = \alpha\, \hat{O}\, \phi$$

Lecture-30.
(11-11-21)

FIELD QUANTIZATION.

Real scalar field $\phi(\underline{x})$

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\,\partial^\mu\phi - \underset{-\frac{m^2}{2}\phi^2}{V(\phi)} \longrightarrow S = \int d^4\underline{x}\ \mathcal{L}(\phi(\underline{x}), \partial_\mu\phi(\underline{x}))$$

$\Rightarrow$ Eq^{n}s of motion - $\partial_\mu \dfrac{\partial\mathcal{L}}{\partial\partial_\mu\phi} = \dfrac{\partial\mathcal{L}}{\partial\phi}$

and in this case, it gives us $\quad \partial_\mu\partial^\mu\phi + m^2\phi = -\dfrac{\partial V}{\partial\phi}$

For a free field, $V=0$

$\Rightarrow (\Box + m^2)\,\phi(\vec{x},t) = 0 \quad$ and $\quad \pi^\mu = \dfrac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)} = \partial^\mu\phi$

$$\mathcal{H} = \pi\dot{\phi} - \mathcal{L} = \pi\cdot\pi - \left(\frac{\pi^2}{2} - \frac{(\vec{\nabla}\phi)^2}{2} - \frac{m^2\phi^2}{2}\right)$$

$$\Rightarrow \mathcal{H} = \frac{1}{2}(\pi^2 + \vec{\nabla}\phi^2) + \frac{1}{2}m\phi^2$$

Everything upto now was classical.

To quantize the system, we impose equal time commutation relations.

We know $\{\phi(\vec{x},t), \pi(\vec{y},t)\} = \delta^{(3)}(\vec{x}-\vec{y})$

$\{\phi(\vec{x},t), \phi(\vec{y},t)\} = 0 = \{\pi(\vec{x},t), \pi(\vec{y},t)\}$

For quantization, $\phi \longrightarrow \hat{\phi}$, $\pi \longrightarrow \hat{\pi}$, $\{\ \} \longrightarrow \dfrac{[\ ]}{i}$

$$\left.\begin{aligned} \Rightarrow [\hat{\phi}(\vec{x},t), \hat{\pi}(\vec{y},t)] &= i\,\delta^{(3)}(\vec{x}-\vec{y}) \\ [\hat{\phi}(\vec{x},t), \hat{\phi}(\vec{y},t)] = 0 &= [\hat{\pi}(\vec{x},t), \hat{\pi}(\vec{y},t)] \end{aligned}\right\}$$ CANONICAL EQUAL TIME COMMUTATORS.

Equations of motion for the quantized KG field -

$$(\square + m^2)\,\hat{\phi}(\vec{x},t) = 0$$

$$\hat{\phi}(\vec{x},t) = \int d^3\vec{p}\;\hat{\tilde{\phi}}(\vec{p},t)\,e^{i\vec{p}\cdot\vec{x}} \longrightarrow \text{Fourier transform.}$$

Subbing into KG eqⁿ.

$$(\partial_t^2 - \vec{\nabla}^2 + m^2)\,\hat{\phi}(\vec{x},t) = 0$$

$$\Rightarrow \int d^3\vec{p}\,(\partial_t^2 - \nabla^2 + m^2)\,e^{i\vec{p}\cdot\vec{x}}\,\hat{\tilde{\phi}}(\vec{p},t) = \int d^3\vec{p}\,(\partial_t^2 + \vec{p}^{\,2} + m^2)\,e^{i\vec{p}\cdot\vec{x}}\,\tilde{\phi}(\vec{x},t) \overset{!}{=} 0$$

$$\Rightarrow (\partial_t^2 + E_{\vec{p}}^{\,2})\,\hat{\tilde{\phi}}(\vec{p},t) = 0$$

$$\underline{\hat{\tilde{\phi}}(\vec{p},t) = A_{\vec{p}}^{\pm}\,e^{\pm i E_{\vec{p}} t}}$$

So, $\hat{\phi}(\vec{x},t) = \int d^3\vec{p}\;A_{\vec{p}}^{-}\,e^{-iE_p t + i\vec{p}\cdot\vec{x}} + A_{\vec{p}}^{+}\,e^{iE_p t + i\vec{p}\cdot\vec{x}}$

$$= \int d^3\vec{p}\;A_{\vec{p}}^{-}\,e^{-i\underline{p}\cdot\underline{x}} + \int d^3\vec{p}\;A_{\vec{p}}^{+}\,e^{iE_p t}\,e^{i\vec{p}\cdot\vec{x}} \quad (\text{do } \vec{p} \longrightarrow -\vec{p})$$

under $\vec{p} \to -\vec{p}$, $d^3\vec{p} \longrightarrow -d^3\vec{p}$ but $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \longrightarrow \int_{\infty}^{-\infty}\int_{\infty}^{-\infty}\int_{\infty}^{-\infty} = -\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}$

$$\Rightarrow \int d^3\vec{p} \longrightarrow \int d^3\vec{p} \quad \text{under } \vec{p} \longrightarrow -\vec{p}$$

So, $\hat{\phi}(\vec{x},t) = \int d^3\vec{p}\;A_{\vec{p}}^{-}\,e^{-i\underline{p}\cdot\underline{x}} + A_{-\vec{p}}^{+}\,e^{+i\underline{p}\cdot\underline{x}}$

Since we are quantizing a real field, $\hat{\phi}(\underline{x}) \overset{!}{=} \phi^{\dagger}(\underline{x})$

So, $\hat{\phi}^{\dagger}(\vec{x},t) = \int d^3\vec{p}\;A_{\vec{p}}^{-\dagger}\,e^{+i\underline{p}\cdot\underline{x}} + A_{-\vec{p}}^{+\dagger}\,e^{-i\underline{p}\cdot\underline{x}}$

$$\Rightarrow A_{-\vec{p}}^{(+)\dagger} = A_{\vec{p}}^{(-)} \quad \text{and} \quad A_{\vec{p}}^{(-)\dagger} = A_{-\vec{p}}^{(+)}$$

If we drop the minus sign,

$$\underline{A_{-\vec{p}}^{(+)} = A_{\vec{p}}^{\dagger}}$$

So, we can write the quantized field as-

$$\boxed{\hat{\phi}(\vec{x},t) = \int d^3\vec{p}\ \left(A_{\vec{p}}\, e^{-ip\cdot\underline{x}} + A^{\dagger}_{\vec{p}}\, e^{+ip\cdot\underline{x}}\right)}$$

π (field mom^n) here can be found as $\pi = \dot{\phi}$

$$\Rightarrow \pi(\vec{x},t) = \int d^3\vec{p}\ (iE_{\vec{p}})\left(-A_{\vec{p}}\, e^{-ip\cdot\underline{x}} + A^{\dagger}_{\vec{p}}\, e^{+ip\cdot\underline{x}}\right)$$

Let us now find what happens if we impose canonical commutation relations -

$$[\hat{\phi}(\vec{x},t), \hat{\phi}(\vec{y},t)] = 0$$

$$= \int d^3\vec{p} \int d^3\vec{q}\ [A_{\vec{p}}, A_{\vec{q}}]\, e^{-i(p_0+q_0)t}\, e^{i(\vec{p}\cdot\vec{x}+\vec{q}\cdot\vec{y})} + [A^{\dagger}_{\vec{p}}, A^{\dagger}_{\vec{q}}]\, e^{i(p_0+q_0)t}\, e^{-i(\vec{p}\cdot\vec{x}+\vec{q}\cdot\vec{y})}$$
$$+ [A_{\vec{p}}, A^{\dagger}_{\vec{q}}]\, e^{-i(p_0-q_0)t}\, e^{i(\vec{p}\cdot\vec{x}-\vec{q}\cdot\vec{y})} + [A^{\dagger}_{\vec{p}}, A_{\vec{q}}]\, e^{i(p_0-q_0)t}\, e^{-i(\vec{p}\cdot\vec{x}-\vec{q}\cdot\vec{y})}$$

$$\stackrel{!}{=} 0$$

Similarly

$$[\hat{\phi}(\vec{x},t), \hat{\pi}(\vec{y},t)] = i\,\delta^{(3)}(\vec{x}-\vec{y})$$

$$= iE_{\vec{q}} \int d^3\vec{p} \int d^3\vec{q}\ [-A_{\vec{p}}, A_{\vec{q}}]\, e^{-i(p_0+q_0)t}\, e^{i(\vec{p}\cdot\vec{x}+\vec{q}\cdot\vec{y})} + [A^{\dagger}_{\vec{p}}, A^{\dagger}_{\vec{q}}]\, e^{i(p_0+q_0)t}\, e^{-i(\vec{p}\cdot\vec{x}+\vec{q}\cdot\vec{y})}$$
$$+ [A_{\vec{p}}, A^{\dagger}_{\vec{q}}]\, e^{-i(p_0-q_0)t}\, e^{i(\vec{p}\cdot\vec{x}-\vec{q}\cdot\vec{y})} - [\vec{A}^{\dagger}_{p}, \vec{A}_{q}]\, e^{i(p_0-q_0)t}\, e^{-i(\vec{p}\cdot\vec{x}-\vec{q}\cdot\vec{y})}$$

$$\stackrel{!}{=} i\,\delta^{(3)}(\vec{x}-\vec{y})$$

The thing which solves the above commutation rel^n eq^{n}s is -

$$\boxed{[A_{\vec{p}}, A_{\vec{q}}] = 0 = [A^{\dagger}_{\vec{p}}, A^{\dagger}_{\vec{q}}]} \quad \text{and} \quad \boxed{[A_{\vec{p}}, A^{\dagger}_{\vec{q}}] = \alpha_{\vec{p}}\, \delta^{(3)}(\vec{p}-\vec{q})}$$

Let's check if this gives the right result for $[\hat{\phi}(\vec{x},t), \hat{\pi}(\vec{y},t)]$

$$[\hat{\phi}(\vec{x},t), \hat{\pi}(\vec{y},t)] = \int d^3\vec{p}\int d^3\vec{q}\ (+iE_q)\cdot$$

$$\Big\{ [A_{\vec{p}}, A_{\vec{q}}^\dagger]\, e^{-i(p_0-q_0)t}\, e^{i(\vec{p}\cdot\vec{x}-\vec{q}\cdot\vec{y})} - [A_{\vec{p}}^\dagger, A_{\vec{q}}]\, e^{i(p_0-q_0)t}\, e^{-i(\vec{p}\cdot\vec{x}-\vec{q}\cdot\vec{y})} + \_\_\ \cancelto{0}{[A_{\vec{p}}, A_{\vec{q}}]} + \_\_\ \cancelto{0}{[A_{\vec{p}}^\dagger, A_{\vec{q}}^\dagger]} \Big\}$$

$$= \int d^3\vec{p}\int d^3\vec{q}\ iE_{\vec{q}}\Big(e^{-i(p_0-q_0)t} e^{i(\vec{p}\cdot\vec{x}-\vec{q}\cdot\vec{y})}\, \delta^{(3)}(\vec{p}-\vec{q})\,\alpha_{\vec{p}} - e^{i(p_0-q_0)t} e^{-i(\vec{p}\cdot\vec{x}-\vec{q}\cdot\vec{y})} \cdot (-\alpha_{\vec{p}}\delta^{(3)}(\vec{p}-\vec{q}))$$

$\vec{p}=\vec{q}$
$\Rightarrow p_0 = \sqrt{\vec{p}^2+m^2} = \sqrt{\vec{q}^2+m^2} = q_0$

$$= i\int d^3\vec{p}\ E_{\vec{p}}\left(e^{-i\vec{p}\cdot(\vec{x}-\vec{y})} + e^{+i\vec{p}\cdot(\vec{x}-\vec{y})}\right)\alpha_{\vec{p}}$$

$\vec{p}\to -\vec{p}$

$$= i\int d^3\vec{p}\ E_{\vec{p}}\left(2\alpha_{\vec{p}}\, e^{+i\vec{p}\cdot(\vec{x}-\vec{y})}\right)$$

Now $i\int d^3\vec{p}\ E_{\vec{p}}\left(2\alpha_{\vec{p}}\, e^{i\vec{p}\cdot(\vec{x}-\vec{y})}\right) \overset{!}{=} i\,\delta^{(3)}(\vec{x}-\vec{y}) \overset{!}{=} i\int \frac{d^3\vec{p}}{(2\pi)^3}\, e^{i\vec{p}\cdot(\vec{x}-\vec{y})}$

$\Rightarrow$ if $2E_{\vec{p}}\alpha_{\vec{p}} = \frac{1}{(2\pi)^3}$ or if $\alpha_{\vec{p}} = \frac{1}{2E_{\vec{p}}(2\pi)^3}$, then it works.

So,
$$\boxed{\begin{aligned} [A_{\vec{p}}, A_{\vec{q}}^\dagger] &= \frac{1}{2E_{\vec{p}}(2\pi)^3}\,\delta^{(3)}(\vec{p}-\vec{q}) \\ [A_{\vec{p}}, A_{\vec{q}}] &= 0 = [A_{\vec{p}}^\dagger, A_{\vec{q}}^\dagger] \end{aligned}}$$

Our scalar field $\phi(\underline{x})$ is a Lorentz scalar. So, we want a Lorentz scalar fourier expansion.

$$\phi(\underline{x}) = \int d^4\underline{p}\ A'(\underline{p})\, e^{-i\underline{p}\cdot\underline{x}}\, \delta(\underline{p}^2-m^2)$$

$\longrightarrow$ Lorentz dot products $\Rightarrow$ invariant.

$$d^4\underline{p}' = \underset{\overset{\parallel}{+1}}{\det(\Lambda)}\, d^4\underline{p}$$

$\therefore$ Manifestly Lorentz invariant.

Hence $A'(p')$ must also be Lorentz invariant!

Here $\delta(p^2-m^2) = \dfrac{\delta(p_0-E_p)+\delta(p_0+E_p)}{2}$

where $\delta(f(x_i)) = \sum_i \dfrac{\delta(x-x_i)}{|f'(x_i)|}$ where $f(x_i)=0$ ← Property of δ-f^{n}s.

$$\text{So, } \phi(\underline{x}) = \int d^3\vec{p}\; A'(\vec{p},E_p)\, e^{-i\underline{p}\cdot\underline{x}}\Big|_{p_0=E_p} + A'(\vec{p},-E_p)\, e^{-i\underline{p}\cdot x}\Big|_{p_0=-E_p}$$

$$= \int d^3\vec{p}\; \left(A'_{\vec{p}}\, e^{-i\underline{p}\cdot\underline{x}} + A'^{\dagger}_{\vec{p}}\, e^{+i\underline{p}\cdot\underline{x}}\right)\Big|_{p_0=E_p} \qquad (\phi^\dagger=\phi)$$

This is another way to obtain quantized fourier transform of ϕ.

<u>Lorentz invariant measure in 3dimensions.</u>

Claim: $\dfrac{d^3\vec{p}}{E_{\vec{p}}}$ is Lorentz invariant.

Under a boost -

$$p_{\parallel}' = \gamma(p_{\parallel} - \beta E_p) \qquad \text{where} \quad p_{\parallel} \equiv \vec{p}\cdot\hat{\beta}$$
$$E_p' = \gamma(E_p - \beta p_{\parallel}) \qquad\qquad \vec{p}_{\perp} \equiv \vec{p} - p_{\parallel}\hat{\beta}$$

and $\vec{p}_{\perp}' = \vec{p}_{\perp}$

$$\text{Therefore, } \frac{d^3\vec{p}'}{E_{\vec{p}'}} = \frac{d^2\vec{p}_{\perp}'\, dp_{\parallel}'}{E_{\vec{p}'}} \overset{\text{inv.}}{=} \frac{d^2 p_{\perp}\, dp_{\parallel}'}{E_{\vec{p}}'}$$

$E_p = \sqrt{p_{\parallel}^2 + p_{\perp}^2 + m^2}$

Now $dp_{\parallel}' = \gamma(dp_{\parallel} - \beta\, dE_p)$ but $dE_p = \dfrac{1}{2E_p}\Big(2p_{\parallel}dp_{\parallel} + 2p_{\perp}\underbrace{dp_{\perp}}_{\overset{\parallel}{0}\ (\text{why?})}\Big)$

$$\Rightarrow dp_{\parallel}' = \gamma\left(dp_{\parallel} - \frac{\beta}{E_p}p_{\parallel}dp_{\parallel}\right) = \gamma\,\frac{dp_{\parallel}}{E_p}(E_p - \beta p_{\parallel})$$

So, $\frac{d^3\vec{p}'}{E_{p'}} = \frac{d^2\vec{p}_\perp\, dp_{\parallel}'}{E_{p'}} = d^2\vec{p}_\perp \frac{\gamma\, dp_{\parallel} \cancel{(E_p - \beta p_{\parallel})}}{E_p} \cdot \frac{1}{\cancel{\gamma (E_p - \beta p_{\parallel})}}$

$= \frac{d^3\vec{p}}{E_p}$. Nice!

Lecture-31
(12-11-21)

Quantization of K G field (contd.)

We wrote the field expansion in fourier modes as -

$$\hat{\phi}(\vec{x},t) = \int d^3\vec{p}\,\left(\hat{A}_{\vec{p}}\, e^{-ip\cdot x} + \hat{A}_{\vec{p}}^{\dagger}\, e^{+ip\cdot x}\right)$$

$$[\hat{A}_{\vec{p}}, \hat{A}_{\vec{q}}^{\dagger}] = \frac{1}{2E_{\vec{p}}(2\pi)^3}\,\delta^{(3)}(\vec{p}-\vec{q}) \quad , \quad [\hat{A}_{\vec{p}}, \hat{A}_{\vec{q}}] = 0$$

If we define $A_{\vec{p}} \equiv \frac{a_{\vec{p}}}{(2\pi)^3\sqrt{2E_{\vec{p}}}}$ $\Rightarrow$ $[\hat{a}_{\vec{p}}, \hat{a}_{\vec{q}}^{\dagger}] = (2\pi)^3\,\delta^3(\vec{p}-\vec{q})$

$$\Rightarrow \boxed{\hat{\phi}(\vec{x},t) = \int \frac{d^3\vec{p}}{(2\pi)^3}\cdot\frac{1}{\sqrt{2E_p}}\left(\hat{a}_{\vec{p}}\, \underbrace{e^{-ip\cdot x}}_{e^{-iE_pt}\cdots} + \hat{a}_{\vec{p}}^{\dagger}\, \underbrace{e^{+ip\cdot x}}_{e^{-i(-E_pt)}\cdots}\right)}$$

$= \hat{\phi}_+(x) + \hat{\phi}_-(x)$ (positive and negative freq. fields)

SHO oscillator in creation/destruction operators -

$[\hat{a}, \hat{a}^{\dagger}] = \mathbb{1}$ Hilbert space spanned by: $\left\{ \frac{(\hat{a}^{\dagger})^n}{\sqrt{n!}}|0\rangle \right\}_{n=0,1,2,3\cdots}$

$\hat{a}|0\rangle \equiv 0$ Hamiltonian

$$H = \hbar\omega\left(\hat{a}^{\dagger}\hat{a} + \frac{1}{2}\right) = \hbar\omega\left(\hat{N} + \frac{1}{2}\right)$$

Particles		Fields.
$\hat{a}$	$\longrightarrow$	$\{a_{\vec{p}}\}$ $\vec{p} \in \mathbb{R}^3$
$[\hat{a}_i, \hat{a}_j^\dagger] = \delta_{ij}$	$\longrightarrow$	$[a_{\vec{p}}, a_{\vec{q}}^\dagger] = (2\pi)^3 \delta^3(\vec{p}-\vec{q})$

i turns into a continuous index $\vec{p}$.

Fock space: Hilbert space for the excitations of the free field, has a particle basis consisting of -

		# of particles
Vacuum: $\|0\rangle$. Defined by $a_{\vec{p}}\|0\rangle \equiv 0$. $\forall \vec{p} \in \mathbb{R}^3$		0
1-particle states:	$N_{\vec{p}}\, a_{\vec{p}}^\dagger \|0\rangle \equiv \|\vec{p}\rangle$ (one particle state with mom^n $\vec{p}$)	1
2-particle states:	$N\, a_{\vec{p}_1}^\dagger a_{\vec{p}_2}^\dagger \|0\rangle \equiv \|\vec{p}_1, \vec{p}_2\rangle$	2
⋮		
N-particle states:	$\prod_i \frac{(a_{\vec{p}_i}^\dagger)^{n_i}}{\sqrt{n_i!}} \|0\rangle = \|n_1(\vec{p}_1), n_2(\vec{p}_2)\ldots\rangle$	N

$$N = \sum_i n_i$$

$\uparrow$ occupation no. basis.

For 1-particle states, we choose the normalization of states as follows -

If $\underline{|\vec{p}\rangle \equiv \sqrt{2E_p}\, a_{\vec{p}}^\dagger |0\rangle}$

$$\Rightarrow \langle \vec{q} | \vec{p} \rangle = \sqrt{2E_{\vec{p}} 2E_{\vec{q}}}\ \langle 0 | a_{\vec{q}}\, a_{\vec{p}}^\dagger |0\rangle = \langle 0 | \underbrace{[a_{\vec{q}}, a_{\vec{p}}^\dagger]}_{(2\pi)^3 \delta^3(\vec{p}-\vec{q})} |0\rangle \cdot \sqrt{2E_{\vec{p}} 2E_{\vec{q}}}$$

$$\langle \vec{q} | \vec{p} \rangle = 2E_{\vec{p}} (2\pi)^3 \delta^3(\vec{p}-\vec{q})$$

$\hookrightarrow$ Relativistically invariant normalization.

This is because, as we know, the delta f^n integration is just 1 -

$$\int d^3\vec{p}\ \delta^3(\vec{p}-\vec{q}) = 1 \xrightarrow[\text{by } E_{\vec{p}}]{\text{multiply, divide}} \int \left(\frac{d^3\vec{p}}{E_{\vec{p}}}\right)\cdot\left(E_p\, \delta^3(\vec{p}-\vec{q})\right) = 1$$

we showed in last lecture that this is a Lorentz invariant measure. $\Rightarrow$ This term is also Lorentz invariant.

So, the normalization $\underline{|\vec{p}\rangle = \sqrt{2E_p}\ \hat{a}^\dagger_{\vec{p}}}|0\rangle$ is a particularly nice one.

Implementation of Lorentz transformations on Fock space.

QM: All symmetry operators are implemented as unitary operators on $\mathcal{H}$.

We want a vacuum that is relativistically invariant.-

$$U(\Lambda)|0\rangle \equiv |0\rangle$$

However, for a 1-particle state, the Lorentz transf. on 1 particle state -

$$U(\Lambda)|\vec{p}\rangle \overset{!}{=} |\Lambda\vec{p}\rangle$$

$$\Rightarrow \sqrt{2E_{\vec{p}}}\ U(\Lambda)\, a^\dagger_{\vec{p}}|0\rangle \overset{!}{=} \sqrt{2E_{\Lambda\vec{p}}}\ \hat{a}^\dagger_{\Lambda\vec{p}}|0\rangle$$

$$\Rightarrow \sqrt{2E_{\vec{p}}}\ U(\Lambda)\, a^\dagger_{\vec{p}}\, U^\dagger(\Lambda)\underbrace{U(\Lambda)|0\rangle}_{\equiv|0\rangle} \overset{!}{=} \sqrt{2E_{\Lambda\vec{p}}}\ \hat{a}^\dagger_{\Lambda\vec{p}}|0\rangle$$

$$\therefore\ U(\Lambda)\, a^\dagger_{\vec{p}}\, U^\dagger(\Lambda) = \sqrt{\frac{E_{\Lambda\vec{p}}}{E_{\vec{p}}}}\ \hat{a}^\dagger_{\Lambda\vec{p}}$$

Completeness relation in each sector.

$$\mathbb{1}_{\text{1-particle}} = \int \frac{d^3\vec{p}}{(2\pi)^3}\, a_{\vec{p}}^{\dagger}|0\rangle\langle 0| a_{\vec{p}} = \int \frac{d^3\vec{p}}{(2\pi)^3} \cdot \frac{1}{2E_p} |\vec{p}\rangle\langle\vec{p}|$$

↑

identity operator
in 1-particle subspace.

We can check this as follows -

$$\underline{\mathbb{1}.\, a_{\vec{q}}^{\dagger}|0\rangle} = \int \frac{d^3\vec{p}}{(2\pi)^3}\, a_{\vec{p}}^{\dagger}|0\rangle\,\langle 0|a_{\vec{p}}\, a_{\vec{q}}^{\dagger}|0\rangle$$

$\langle 0|a_{\vec{p}}\, a_{\vec{q}}^{\dagger}|0\rangle = \langle 0|[a_{\vec{p}}, a_{\vec{q}}^{\dagger}]|0\rangle$
because $a_{\vec{q}}|0\rangle = 0$

$$= \int \frac{d^3\vec{p}}{\cancel{(2\pi)^3}}\, a_{\vec{p}}^{\dagger}|0\rangle\langle 0|\underbrace{[a_{\vec{p}}, a_{\vec{q}}^{\dagger}]}_{=\cancel{(2\pi)^3}\,\delta^3(\vec{p}-\vec{q})}|0\rangle$$

$$= \underline{\underline{a_{\vec{q}}^{\dagger}|0\rangle}} \;\checkmark$$

Let's now try to find out the quantized momentum operator $\hat{\vec{p}}$ of the quantized KG field.

- The stress-energy tensor in classical field theory is given by -

$$\Theta^{\mu\nu} = \partial^{\mu}\phi\,\partial^{\nu}\phi - \eta^{\mu\nu}\mathcal{L}$$

$$\Rightarrow \Theta^{0i} = \partial^0\phi\,\partial^i\phi = \dot{\phi}(-\vec{\nabla}\phi)_i = -\dot{\phi}(\vec{\nabla}\phi)_i = -\pi(\vec{\nabla}\phi)_i \quad \text{for } \mathcal{L}\sim(\partial\phi)^2$$

If we define $\quad P^{\mu} \equiv \int d^3\vec{x}\, \Theta^{0\mu} \quad$ (Lecture 29), then

$$\vec{P} = \int d^3\vec{x}\, \Theta^{0i} = -\int d^3\vec{x}\; \pi\, \vec{\nabla}\phi$$

To find the quantized version of $\vec{P}$, we put in $\hat{\pi}$ and $\hat{\phi}$.

$$\text{So, } \vec{P} = -\int d^3\vec{x} \int \frac{d^3\vec{p}}{(2\pi)^3} \frac{(iE_{\vec{p}})}{\sqrt{2E_p}} \left(-\hat{a}_{\vec{p}}\, e^{-ip\cdot x} + \hat{a}^\dagger_{\vec{p}}\, e^{+ip\cdot x} \right) \cdot \vec{\nabla}\phi$$

$$= -\int d^3\vec{x} \int \frac{d^3\vec{p}\ (iE_{\vec{p}})}{(2\pi)^3\sqrt{2E_p}} \left(-\hat{a}_{\vec{p}}\, e^{-ip\cdot x} + a^\dagger_{\vec{p}}\, e^{+ip\cdot x} \right) \cdot \int \frac{d^3\vec{q}}{(2\pi)^3\sqrt{2E_{\vec{q}}}} \cdot \left[a_{\vec{q}}\, (i\vec{q})\, e^{-iq\cdot x} + a^\dagger_{\vec{q}}\, (-i\vec{q})\, e^{+iq\cdot x} \right]$$

$$= \int d^3\vec{x} \int \frac{d^3\vec{p}\,(-iE_{\vec{p}})}{(2\pi)^3\sqrt{2E_{\vec{p}}}} \left[-\hat{a}_{\vec{p}}\, e^{-ip\cdot x} + \hat{a}^\dagger_{\vec{p}}\, e^{+ip\cdot x} \right] \int \frac{d^3\vec{q}\ (i\vec{q})}{(2\pi)^3\sqrt{2E_{\vec{q}}}} \left[\hat{a}_{\vec{q}}\, e^{-iq\cdot x} - \hat{a}^\dagger_{\vec{q}}\, e^{+iq\cdot x} \right]$$

Let's calculate this term by term-

(a) $\int d^3\vec{x}\ \hat{a}^\dagger_{\vec{p}}\, \hat{a}_{\vec{q}}\, e^{i(p-q)\cdot x}\ \dfrac{(-iE_{\vec{p}}\cdot i\vec{q})}{(2\pi)^6\sqrt{2E_{\vec{p}}2E_{\vec{q}}}}$

$e^{-i(\vec{p}-\vec{q})\cdot\vec{x}}\ e^{i(p_0-q_0)x^0}$

$$= \frac{a^\dagger_{\vec{p}}\, a_{\vec{q}}}{(2\pi)^3} \frac{\cancel{e^{i(p_0-q_0)x^0}}^{\,1}}{\cancel{(2\pi)^3}} \frac{\vec{q}\ E_{\vec{p}}}{\sqrt{2E_{\vec{p}}2E_{\vec{q}}}} \underbrace{\int d^3\vec{x}\ e^{-i(\vec{p}-\vec{q})\cdot\vec{x}}}_{\cancel{(2\pi)^3}\ \delta^3(\vec{p}-\vec{q})}$$

Since $\vec{p}=\vec{q}$

$\Rightarrow p_0 = \sqrt{\vec{p}^2+m^2}$

$= \sqrt{q^2+m^2} = q_0$

$\Rightarrow e^{i(p_0-q_0)x^0} = 1$

$$= \frac{a^\dagger_{\vec{p}}\, a_{\vec{p}}}{(2\pi)^3} \frac{\vec{p}\ \cancel{E_{\vec{p}}}}{2\cancel{E_{\vec{p}}}}\ \delta^3(\vec{p}-\vec{q}) = \underline{\frac{a^\dagger_{\vec{p}}\, a_{\vec{p}}}{(2\pi)^3}\ \delta^3(\vec{p}-\vec{q})\cdot\left(\frac{\vec{p}}{2}\right)}$$

(of course, integrated over $d^3\vec{p}\,d^3\vec{q}$)

(b) $\int d^3\vec{x}\ a_{\vec{p}}\, a^\dagger_{\vec{q}}\ \dfrac{e^{-i(p-q)\cdot x}}{(2\pi)^3}\ \dfrac{(-iE_{\vec{p}}\ i\vec{q})}{\sqrt{2E_{\vec{p}}\ 2E_{\vec{q}}}}$ = same as (a) but with $\underline{a_{\vec{p}}\, a^\dagger_{\vec{p}}}$

(only diff. is $p \leftrightarrow q$ and it doesn't matter)

(c) $\int d^3\vec{q} \int d^3\vec{p} \int d^3\vec{x}\ a_{\vec{p}}\, a_{\vec{q}}\ \dfrac{(-E_{\vec{p}}\ \vec{q})}{\sqrt{2E_{\vec{p}}\ 2E_{\vec{q}}}} \cdot \dfrac{e^{-i(p+q)\cdot x}}{(2\pi)^6} \longrightarrow e^{-i(p_0+q_0)t}\ \underset{\text{integrates to } \delta}{e^{+i(\vec{p}+\vec{q})\cdot\vec{x}}}$

$$= \int d^3\vec{p} \int d^3\vec{q}\ \frac{a_{\vec{p}}\, a_{\vec{q}}}{(2\pi)^3} \frac{(-E_{\vec{p}}\ \vec{q})}{2\sqrt{E_{\vec{p}}\ E_{\vec{q}}}}\ e^{-i(p_0+q_0)t}\ \delta^{(3)}(\vec{p}+\vec{q})$$

$$= \int d^3\vec{p}\ \frac{a_{\vec{p}}\, a_{-\vec{p}}}{(2\pi)^3}\ \frac{\cancel{E_{\vec{p}}}\,\vec{p}}{2\cancel{E_{\vec{p}}}}\ e^{-2iE_{\vec{p}}t}$$

$q_0 = \sqrt{\vec{q}^{\,2}+m^2}$
$= \sqrt{(-\vec{p})^2+m^2}$
$= \sqrt{p^2+m^2} = p_0$
$\Rightarrow p_0 = q_0 = E_{\vec{p}}$

$$= \int d^3\vec{p}\ \frac{\hat{a}_{\vec{p}}\, \hat{a}_{-\vec{p}}}{(2\pi)^3}\ \vec{p}\ e^{-2iE_{\vec{p}}t}$$

$\uparrow$ odd f^n! $\Rightarrow \int d^3\vec{p} \dots = \underline{0}$

<u>ⓓ</u> similar to ⓒ,

$$\int d^3\vec{p} \int d^3\vec{q} \int d^3\vec{x}\ a^{\dagger}_{\vec{p}}\, a^{\dagger}_{\vec{q}}\ \frac{(-E_{\vec{p}}\,\vec{q})}{2\sqrt{E_{\vec{p}}E_{\vec{q}}}}\ \frac{e^{+i(p+q)\cdot x}}{(2\pi)^6}\ \delta^{(3)}(\vec{p}+\vec{q}) = \underline{0}$$

So,

$$\vec{P} = \int d^3\vec{p} \int d^3\vec{q}\ (ⓐ + ⓑ + \cancel{ⓒ}^{\,0} + \cancel{ⓓ}^{\,0}) = \int d^3\vec{p} \int d^3\vec{q}\,(ⓐ + ⓑ)$$

$$= \int d^3\vec{p} \int d^3\vec{q}\ \left[a^{\dagger}_{\vec{p}}\, a_{\vec{p}} + a_{\vec{p}}\, a^{\dagger}_{\vec{p}}\right] \delta^3(\vec{p}-\vec{q}) \cdot \frac{\vec{p}}{2} \cdot \frac{1}{(2\pi)^3}$$

$$= \int \frac{d^3\vec{p}}{(2\pi)^3} \left[a^{\dagger}_{\vec{p}}\, a_{\vec{p}} + a_{\vec{p}}\, a^{\dagger}_{\vec{p}}\right] \cdot \frac{\vec{p}}{2}$$

but $a_{\vec{p}}\, a^{\dagger}_{\vec{p}} = a^{\dagger}_{p}\, a_{\vec{p}} + [a_{\vec{p}}, a^{\dagger}_{\vec{p}}]$, where $[a_{\vec{p}}, a^{\dagger}_{\vec{p}}] = (2\pi)^3\, \delta^3(\vec{p}-\vec{p})$ and $\delta^3(\vec{p}-\vec{p}) = \delta^3(0)$

$$= \int \frac{d^3\vec{p}}{(2\pi)^3} \Big[a^{\dagger}_{\vec{p}}\, a_{\vec{p}} + a^{\dagger}_{\vec{p}}\, a_{\vec{p}} + \underbrace{(2\pi)^3\, \delta^3(\vec{p}-\vec{p})}_{\text{infinite renormalization term}}\Big] \cdot \frac{\vec{p}}{2}$$

$$= \int \frac{d^3\vec{p}}{(2\pi)^3}\ a^{\dagger}_{\vec{p}}\, a_{\vec{p}}\ \frac{\vec{p}}{2} + \text{infinite renormalization term}$$

$$\Rightarrow \boxed{\hat{\vec{P}} = \int \frac{d^3\vec{p}}{(2\pi)^3}\, \vec{p} \cdot \hat{a}^{\dagger}_{\vec{p}}\, \hat{a}_{\vec{p}}}$$

Quantized momentum operator for KG field.

Let's see what we get by operating $\vec{P}$ on a state $|\vec{q}\rangle$

$$\hat{\vec{P}}|\vec{q}\rangle = \int \frac{d^3\vec{p}}{(2\pi)^3}\, \vec{p}\, a^\dagger_{\vec{p}} a_{\vec{p}} |q\rangle = \int \frac{d^3\vec{p}}{(2\pi)^3}\, \vec{p}\, a^\dagger_{\vec{p}} a_{\vec{p}} \sqrt{2E_{\vec{q}}}\, a^\dagger_{\vec{q}} |0\rangle$$

$$= \int d^3\vec{p}\, \frac{\sqrt{2E_{\vec{q}}}}{\cancel{(2\pi)^3}}\, \vec{p}\, a^\dagger_{\vec{p}}\, \underbrace{a_{\vec{p}} a^\dagger_{\vec{q}} |0\rangle}_{\parallel}$$

$$[a_{\vec{p}}, a^\dagger_{\vec{q}}]|0\rangle = \cancel{(2\pi)^3}\, \delta^3(\vec{p}-\vec{q})\, |0\rangle$$

$$= \int d^3\vec{p}\, \sqrt{2E_{\vec{p}}}\, \vec{p}\, a^\dagger_{\vec{p}}\, \delta^3(\vec{p}-\vec{q}) |0\rangle$$

$$= \vec{q}\, \underbrace{\sqrt{2E_{\vec{q}}}\, a^\dagger_{\vec{q}} |0\rangle} = \vec{q}\,|\vec{q}\rangle \qquad \therefore\ \hat{\vec{P}}|\vec{q}\rangle = \vec{q}\,|\vec{q}\rangle$$

So, $|\vec{q}\rangle$ is an eigenstate of the second quantized momentum operator $\hat{\vec{P}}$ with eigenvalue $\vec{q}$.

By extension $\hat{\vec{P}}\,|n_1(\vec{q}_1), n_2(\vec{q}_2)\dots\rangle = \sum_i n_i \vec{q}_i\, |n_1(\vec{q}_1), n_2(\vec{q}_2)\dots\rangle$

Similarly we can find the second quantized Hamiltonian-

$$\hat{H} = \int d^3\vec{x}\, \Theta^{00} = \int d^3\vec{x} \left(\frac{1}{2}\left(\hat{\pi}^2 + (\vec{\nabla}\hat{\phi})^2\right) + \frac{1}{2} m\phi^2\right)$$

Inserting all the expressions for Fourier expansions (do this exercise sometime), -

$$\hat{H} = \frac{1}{2}\int d^3\vec{p}\, \sqrt{\vec{p}^{\,2}+m^2}\, \left(a^\dagger_{\vec{p}} a_{\vec{p}} + a_{\vec{p}} a^\dagger_{\vec{p}}\right) = \frac{1}{2}\int d^3\vec{p}\, E_{\vec{p}} \left(2a^\dagger_{\vec{p}} a_{\vec{p}} + \delta^3(0)\right)$$

zero pt. energy (∞)

$$= \int d^3\vec{p}\, E_{\vec{p}}\, \hat{N}_{\vec{p}} + \text{infinite renormal. term (zero-pt. energy)}$$

$$\Rightarrow \boxed{\hat{P}^\mu = (\hat{H}, \hat{\vec{P}}) = \int d^3\vec{p}\; p^\mu\, \hat{N}_{\vec{p}}}$$

Lecture-32

(15-11-2021)

Scalar field Theory (contd.)

relativistically invariant

1-particle state: $|\vec{p}\rangle = \sqrt{2E_{\vec{p}}}\; a^{\dagger}_{\vec{p}} |0\rangle \Rightarrow \langle \vec{p}|\vec{q}\rangle = 2E_{\vec{p}}\, \delta^3(\vec{p}-\vec{q})\cdot(2\pi)^3$

$$\hat{\phi}(\vec{x},t) = \int \frac{d^3\vec{p}}{(2\pi)^3\sqrt{2E_p}} \left(a_{\vec{p}}\, e^{-i\underline{p}\cdot\underline{x}} + a^{\dagger}_{\vec{p}}\, e^{+i\underline{p}\cdot\underline{x}} \right)\Big|_{p_0=E_p}$$

Sol^{n}s of KG eq^n

$(\Box + m^2)\, e^{\pm i\underline{p}\cdot\underline{x}} = 0$

Let's see what the action of $\hat{\phi}$ is on the vacuum state -

$$\hat{\phi}(\underline{x})|0\rangle = \int \frac{d^3\vec{p}}{(2\pi)^3} \frac{e^{i\underline{p}\cdot\underline{x}}}{\sqrt{2E_{\vec{p}}}}\, a^{\dagger}_{\vec{p}}|0\rangle = \int \frac{d^3\vec{p}}{(2\pi)^3} \frac{e^{i\underline{p}\cdot\underline{x}}}{\sqrt{2E_p}} \cdot \frac{1}{\sqrt{2E_{\vec{p}}}} |\vec{p}\rangle$$

$$= \int \frac{d^3\vec{p}}{2E_p\,(2\pi)^3}\, e^{i(p_0 t - \vec{p}\cdot\vec{x})} |\vec{p}\rangle$$

Then, at time $t=0$

$$\hat{\phi}(\vec{x},0)|0\rangle = \int \frac{d^3\vec{p}}{2E_{\vec{p}}(2\pi)^3}\, e^{-i\vec{p}\cdot\vec{x}} |\vec{p}\rangle$$ $\longrightarrow$ superposition of 1-particle momentum eigenstates.

$\Rightarrow \langle \vec{q}|\hat{\phi}(\vec{x},0)|0\rangle = e^{-i\vec{q}\cdot\vec{x}}$ $\longrightarrow$ Position space eigenf^n of momentum $\vec{q}$ ($\langle\vec{q}|\vec{x}\rangle$ in NRQM)

Some important properties of $\hat{H}$ and $\hat{\vec{P}}$ (P^μ)

We can prove that - $\phi(\underline{x}) = e^{i\underline{\hat{P}}\cdot\underline{x}}\, \phi(0)\, e^{-i\underline{\hat{P}}\cdot\underline{x}}$

$$\hat{\vec{P}} \equiv \int \frac{d^3\vec{p}}{(2\pi)^3}\, \vec{p}\, \underbrace{a_{\vec{p}}^\dagger a_{\vec{p}}}_{\hat{N}_{\vec{p}}} \qquad \text{and} \qquad \hat{H} \equiv \int \frac{d^3\vec{p}}{(2\pi)^3}\, E_{\vec{p}}\, \underbrace{a_{\vec{p}}^\dagger a_{\vec{p}}}_{\hat{N}_{\vec{p}}}$$

where $E_{\vec{p}} = \sqrt{\vec{p}^{\,2}+m^2}$

$$\Rightarrow \underline{\vec{P}|\vec{q}\rangle} = \int \frac{d^3\vec{p}}{(2\pi)^3}\vec{p}\, a_{\vec{p}}^\dagger a_{\vec{p}} \sqrt{2E_{\vec{q}}}\, a_{\vec{q}}^\dagger|0\rangle = \int \frac{d^3\vec{p}}{(2\pi)^3} a_{\vec{p}}^\dagger \sqrt{2E_{\vec{q}}}\, \vec{p}\, \underbrace{[a_{\vec{p}}, a_{\vec{q}}^\dagger]|0\rangle}_{(2\pi)^3\delta^3(\vec{p}-\vec{q})|0\rangle}$$

$$= \int d^3\vec{p} \sqrt{2E_{\vec{q}}}\, \vec{p}\, a_{\vec{p}}^\dagger|0\rangle\, \delta^3(\vec{p}-\vec{q}) = \vec{q}\, a_{\vec{q}}^\dagger|0\rangle\sqrt{2E_{\vec{q}}}$$

$$= \underline{\vec{q}\,|\vec{q}\rangle}$$

Let's now calculate the following commutator -

$$\underline{[\hat{\vec{P}}, a_{\vec{q}}]} = \int \frac{d^3\vec{p}}{(2\pi)^3}\, \vec{p}\, [a_{\vec{p}}^\dagger a_{\vec{p}}, a_{\vec{q}}] = \int \frac{d^3\vec{p}}{(2\pi)^3}\, \vec{p} \Big(a_{\vec{p}}^\dagger \underbrace{[a_{\vec{p}}, a_{\vec{q}}]}_{0} + \underbrace{[a_{\vec{p}}^\dagger, a_{\vec{q}}]}_{-(2\pi)^3\delta^3(\vec{p}-\vec{q})} a_{\vec{p}} \Big)$$

$$= -\int d^3\vec{p}\cdot\vec{p}\,\delta^3(\vec{p}-\vec{q})\, a_{\vec{p}} = \underline{\underline{-\vec{q}\,\hat{a}_{\vec{q}}}}$$

$$\therefore (\vec{x}\cdot\hat{\vec{P}})a_{\vec{q}} = a_{\vec{q}}\,\vec{x}\cdot\hat{\vec{P}} - \vec{x}\cdot\vec{q}\, a_{\vec{q}}$$

$$= a_{\vec{q}}\,(\hat{\vec{P}}-\vec{q})\cdot\vec{x}$$

$$\Rightarrow (\vec{x}\cdot\hat{\vec{P}})^n a_{\vec{q}} = a_{\vec{q}}\left[(\hat{\vec{P}}-\vec{q})\cdot\vec{x}\right]^n$$

$$\Rightarrow (-i\,\vec{x}\cdot\hat{\vec{P}})^n a_{\vec{q}} = a_{\vec{q}}\left[-i\,(\hat{\vec{P}}-\vec{q})\cdot\vec{x}\right]^n$$

$$\Rightarrow e^{-i\vec{x}\cdot\hat{\vec{P}}} a_{\vec{q}} = a_{\vec{q}}\, e^{-i(\hat{\vec{P}}-\vec{q})\cdot\vec{x}}$$

$$\Rightarrow e^{-i\vec{x}\cdot\hat{\vec{P}}} a_{\vec{q}}\, e^{+i\vec{x}\cdot\hat{\vec{P}}} = a_{\vec{q}}\, e^{-i(\cancel{\hat{\vec{P}}}-\vec{q})\cdot\vec{x}}\, \cancel{e^{+i\vec{x}\cdot\hat{\vec{P}}}}$$

$$\boxed{\begin{aligned} &\Rightarrow e^{-i\vec{x}\cdot\hat{\vec{P}}} a_{\vec{q}}\, e^{+i\vec{x}\cdot\hat{\vec{P}}} = a_{\vec{q}}\, e^{+i\vec{q}\cdot\vec{x}} \\ &\Rightarrow e^{-i\vec{x}\cdot\hat{\vec{P}}} a_{\vec{q}}^\dagger\, e^{+i\vec{x}\cdot\hat{\vec{P}}} = a_{\vec{q}}^\dagger\, e^{-i\vec{q}\cdot\vec{x}} \end{aligned}}$$

Doing this sandwich on $\hat{\phi}(\vec{x}=0,t) = \hat{\phi}(0,t)$

$$\underline{e^{-i\hat{\vec{P}}\cdot\vec{x}}\;\hat{\phi}(0,t)\;e^{+i\hat{\vec{P}}\cdot\vec{x}}} = e^{-i\hat{\vec{P}}\cdot\vec{x}}\int\frac{d^3\vec{p}}{(2\pi)^3\sqrt{2E_{\vec{p}}}}\left(a_{\vec{p}}\,e^{-iE_{\vec{p}}t}+a^{\dagger}_{\vec{p}}\,e^{iE_{\vec{p}}t}\right)\cdot e^{+i\hat{\vec{P}}\cdot\vec{x}}$$

$$= \int\frac{d^3\vec{p}}{(2\pi)^3\sqrt{2E_p}}\left(\underbrace{e^{-i\hat{\vec{P}}\cdot\vec{x}}\,a_{\vec{p}}\,e^{+i\hat{\vec{P}}\cdot\vec{x}}}\,e^{-iE_pt}+\underbrace{e^{-i\hat{\vec{P}}\cdot\vec{x}}\,a^{\dagger}_{\vec{p}}\,e^{+i\hat{\vec{P}}\cdot\vec{x}}}\,e^{iE_pt}\right)$$

$$= \int\frac{d^3\vec{p}}{(2\pi)^3}\left(a_{\vec{p}}\,e^{-iE_pt}\,e^{+i\vec{p}\cdot\vec{x}}+a^{\dagger}_{\vec{p}}\,e^{+iE_pt}\,e^{-i\vec{p}\cdot\vec{x}}\right)$$

using the property on the last page

$$= \int\frac{d^3\vec{p}}{(2\pi)^3}\left(a_{\vec{p}}\,e^{-i\underline{p}\cdot\underline{x}}+a^{\dagger}_{\vec{p}}\,e^{+i\underline{p}\cdot\underline{x}}\right) = \underline{\hat{\phi}(\vec{x},t)}$$

Similarly, we can find $e^{+i\hat{H}t}\,\hat{\phi}(\vec{x},0)\,e^{-i\hat{H}t}$ can be derived by finding the action of $e^{+i\hat{H}t}$ on $a_{\vec{q}}, a^{\dagger}_{\vec{q}}$, i.e.

$$\boxed{\begin{aligned} e^{+i\hat{H}t}\,a_{\vec{q}}\,e^{-i\hat{H}t} &= a_{\vec{q}}\,e^{+iE_{\vec{q}}t} \\ e^{+i\hat{H}t}\,a^{\dagger}_{\vec{q}}\,e^{+i\hat{H}t} &= a^{\dagger}_{\vec{q}}\,e^{-iE_{\vec{q}}t} \end{aligned}}$$

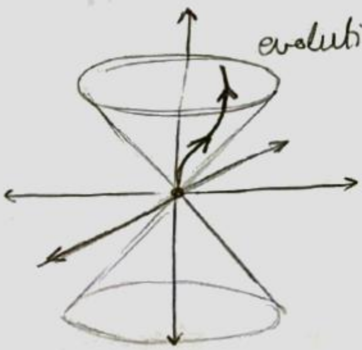
$$\Rightarrow\quad \underline{e^{+i\hat{H}t}\,\hat{\phi}(\vec{x},0)\,e^{-i\hat{H}t}} = \hat{\phi}(\vec{x},t)$$

This implies a very important eq^n for finding the evolution of $\hat{\phi}$

$$\boxed{e^{i\underline{\hat{P}}\cdot\underline{x}}\;\hat{\phi}(\underline{0})\;e^{-i\underline{\hat{P}}\cdot\underline{x}} = \hat{\phi}(\underline{x})}$$

Propogation of fields: the issue of causality.

Consider the quantity



$$\langle 0|\, \phi(\underline{x})\, \phi(\underline{y})\, |0\rangle \equiv D(\underline{x}-\underline{y})$$

This is the amplitude of a particle to propagate from y to $\underline{x}$.

$$D(\underline{x}-\underline{y}) = \int \frac{d^3\vec{p}\; d^3\vec{q}}{(2\pi)^6 \sqrt{2E_{\vec{p}}\, 2E_{\vec{q}}}} \langle 0|\, a_{\vec{p}}\, a^{\dagger}_{\vec{q}}\, |0\rangle\, e^{-i\underline{p}\cdot\underline{x}}\, e^{+i\underline{q}\cdot\underline{y}}$$

rest of the terms $=0$ because $a_{\vec{p}/\vec{q}}\,|0\rangle = 0$

$$= \int \frac{d^3\vec{p}\; d^3\vec{q}}{(2\pi)^6 \sqrt{2E_{\vec{p}}\, 2E_{\vec{q}}}} \langle 0|\, \underbrace{[a_{\vec{p}}, a^{\dagger}_{\vec{q}}]}_{(2\pi)^3\, \delta^3(\vec{p}-\vec{q})}\, |0\rangle\, e^{-i\underline{p}\cdot\underline{x}}\, e^{i\underline{q}\cdot\underline{y}}$$

$$= \int \frac{d^3\vec{p}}{2E_{\vec{p}}\,(2\pi)^3}\, e^{-i\underline{p}\cdot(\underline{x}-\underline{y})} \equiv D(\underline{x}-\underline{y})$$

Let us now evaluate this integral in two different cases –

(a) Timelike separation: $x^0 - y^0 = t$, $\vec{x}-\vec{y} = 0$ (if $\exists$ a frame in which $\underline{x}$ & y are connected by a timelike path, then they are timelike connected $\forall$ frames)

$$D(\underline{x}-\underline{y}) = \frac{1}{8\pi^3}\int_0^{\infty} \frac{dp}{2E_p}\, p^2 \int_{-1}^{+1} \underset{\overset{\parallel}{\xi}}{d(\cos\theta)} \int_0^{2\pi} d\phi\; e^{ip\xi|\vec{x}-\vec{y}|}\, e^{-ip^0(x^0-y^0)}$$

since $|\vec{x}-\vec{y}| = 0$ & $x^0 - y^0 = t$

$$\Rightarrow D(\underline{x}-\underline{y}) = \frac{1}{8\pi^3} \int_0^{\infty} \frac{dp\, p^2}{2E_{\vec{p}}} \int_{-1}^{+1} d\xi \int_0^{2\pi} d\phi\; e^{-iE_p t} = \frac{1}{4\pi^2}\int_0^{\infty} dp\, \frac{p^2 e^{-iE_p t}}{E_{\vec{p}}}$$

⑤

Now $E_{\vec{p}}^2 = \vec{p}^2 + m^2 \Rightarrow E_p dE_p = p\,dp \Rightarrow p^2 dp = \sqrt{E_{\vec{p}}^2 - m^2} \cdot E_p dE_p$

$$D(\underline{x}-\underline{y}) = \frac{1}{4\pi^2}\int_0^\infty dE_{\vec{p}} \frac{\sqrt{E_{\vec{p}}^2 - m^2}\,\cancel{E_p}}{\cancel{E_{\vec{p}}}} e^{-iE_p t} = \frac{1}{4\pi^2}\int_0^\infty dE_{\vec{p}} \sqrt{E_{\vec{p}}^2 - m^2}\, e^{-iE_{\vec{p}} t}$$

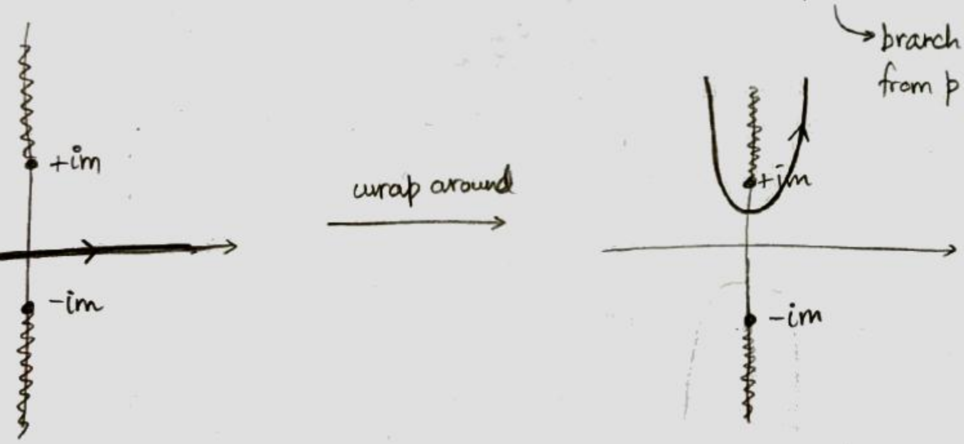
$$D_{\text{timelike}}(\underline{x}-\underline{y}) = \frac{-im}{4\pi^2}\frac{K_1(imt)}{t} \xrightarrow{\lim_{t\to\infty}} \left[\frac{-im}{32\pi^3 t^3}\right]^{1/2} e^{-imt} + \mathcal{O}(t^{-2})$$

(b) <u>Spacelike separation</u>: $x^0 - y^0 = 0,\ \vec{x} - \vec{y} = \vec{r}$

$$D_{\text{spacelike}}(\underline{x}-\underline{y}) = \frac{1}{8\pi^3}\int_0^\infty \frac{p^2 dp}{2E_{\vec{p}}} \int_{-1}^{+1} d\xi \int_0^{2\pi} d\phi\ e^{ip\xi\overset{r}{\overset{\shortparallel}{|\vec{x}-\vec{y}|}}}\ e^{-iE_{\vec{p}}\overset{0}{\overset{\shortparallel}{(x^0-y^0)}}}$$

$$= \frac{1}{8\pi^2}\int_0^\infty dp \frac{p^2}{E_{\vec{p}}} \int_{-1}^{+1} d\xi\ e^{ip\xi r} = \frac{-i}{8\pi^2 r}\int_{-\infty}^{\infty} dp \frac{p\,e^{ipr}}{\sqrt{p^2+m^2}}$$

branch cuts! from $p = \pm im$



$$D_{\text{spacelike}}(\underline{x}-\underline{y}) = \frac{m}{4\pi^2 r} K_1(mr) = D_{\text{timelike}}(\underline{x}-\underline{y})\Big|_{\substack{\vec{x}-\vec{y}=0 \\ x^0-y^0=\boxed{t=-ir}}}$$

$$\xrightarrow[\lim_{r\to\infty}]{\text{in the}} \frac{e^{-mr}}{4\pi^2}\left(\sqrt{\frac{m\pi}{2r^3}} + \mathcal{O}(r^{-2})\right)$$

exponentially damped!

Micro causality

Operators with spacelike separation should commute.

In NRQM, non-commuting operators interfere with each other's measurement.

like- $[\hat{x}, \hat{p}] \neq 0 \Rightarrow \Delta x \Delta p \geq 1/2$

If $[\hat{\phi}(\underline{x}), \hat{\phi}(\underline{y})] = 0$ for $(\underline{x}-\underline{y})^2 < 0$ (spacelike), then measurements of the field at $\underline{x}$ & $\underline{y}$ won't interfere. $\Rightarrow$ Any operators built from ϕ & π will also be non-interfering.

But is this possible?

$$[\phi(\underline{x}), \phi(\underline{y})] = \int \frac{d^3\vec{p}\, d^3\vec{q}}{(2\pi)^6 2\sqrt{E_{\vec{p}} E_{\vec{q}}}} \left(\left[a_{\vec{p}} e^{-i\underline{p}\cdot\underline{x}} + a^\dagger_{\vec{p}} e^{+i\underline{p}\cdot\underline{x}}, a_{\vec{q}} e^{-i\underline{q}\cdot\underline{y}} + a^\dagger_{\vec{q}} e^{+i\underline{q}\cdot\underline{y}} \right] \right)$$

$$= \int \frac{d^3\vec{p}\, d^3\vec{q}}{(2\pi)^6 2\sqrt{E_{\vec{p}} E_{\vec{q}}}} \left(\underbrace{[a_{\vec{p}}, a^\dagger_{\vec{q}}]}_{=(2\pi)^3 \delta^3(\vec{p}-\vec{q})} e^{-i(\underline{p}\cdot\underline{x} - \underline{q}\cdot\underline{y})} + \underbrace{[a^\dagger_{\vec{p}}, a_{\vec{q}}]}_{=-(2\pi)^3 \delta^3(\vec{p}-\vec{q})} e^{+i(\underline{p}\cdot\underline{x} - \underline{q}\cdot\underline{y})} \right)$$

all other $[\,,\,] = 0$.

$$= \int \frac{d^3\vec{p}}{(2\pi)^3 2E_{\vec{p}}} \left(e^{-i\underline{p}\cdot(\underline{x}-\underline{y})} - e^{+i\underline{p}\cdot(\underline{x}-\underline{y})} \right)$$

$$\boxed{[\phi(\underline{x}), \phi(\underline{y})] = \int \frac{d^3\vec{p}}{(2\pi)^3 2E_{\vec{p}}} \left(e^{-i\underline{p}\cdot(\underline{x}-\underline{y})} - e^{-i\underline{p}\cdot(\underline{y}-\underline{x})} \right) = D(\underline{x}-\underline{y}) - D(\underline{y}-\underline{x})}$$

If $(\underline{x}-\underline{y})^2 < 0$, we can go to a frame where $x^0 - y^0 \equiv t = 0$,

$$\Rightarrow [\phi(\underline{x}), \phi(\underline{y})] = \int \frac{d^3\vec{p}}{(2\pi)^3 2E_{\vec{p}}} \left(e^{+i\vec{p}\cdot(\vec{x}-\vec{y})} - e^{-i\vec{p}\cdot(\vec{x}-\vec{y})} \right)$$

$$\Rightarrow \boxed{\begin{array}{c} [\phi(\underline{x}), \phi(\underline{y})] = 0 \\ \text{for } (\underline{x}-\underline{y})^2 < 0 \end{array}}$$

since they cancel out!

Lecture-33

(16-11-2021)

In the last lecture, we found out -

$$[\hat{\phi}(\underline{x}), \hat{\phi}(\underline{y})] = 0 \qquad \text{if } (\underline{x}-\underline{y})^2 < 0 \text{ i.e. } \underline{x} \,\&\, \underline{y} \text{ are connected via spacelike path.}$$

So, we define the following object -

$$D_R(\underline{x}-\underline{y}) = \Theta(x^0-y^0)\,[\hat{\phi}(\underline{x}), \hat{\phi}(\underline{y})] \qquad \begin{array}{l} \neq 0 \text{ if } x^0 \geq y^0 \\ = 0 \text{ if } x^0 < y^0 \end{array}$$

↑ Causal propagator for propagating disturbances forwards in time.

Let us do another computation -

$$(\partial_{\underline{x}}^2+m^2)\,D_R(\underline{x}-\underline{y}) = (\partial_{\underline{x}}^2+m^2)\left(\Theta(x^0-y^0)\,[\phi(\underline{x}), \phi(\underline{y})]\right)$$

$$= \Theta(x^0-y^0)\,\cancelto{0}{\left([\underbrace{(\partial_{\underline{x}}^2+m^2)\hat{\phi}(\underline{x})}_{=0}, \hat{\phi}(\underline{y})]\right)} + 2\,\partial_{\underline{x}}^{\mu}\,\Theta(x^0-y^0)\cdot[\partial_{\underline{x}\mu}\hat{\phi}(\underline{x}), \hat{\phi}(\underline{y})]$$

$$+ \partial_{\underline{x}}^2\,\Theta(x^0-y^0)\cdot[\hat{\phi}(\underline{x}), \hat{\phi}(\underline{y})]$$

$$= 2\,\delta(x^0-y^0)\,[\dot{\hat{\phi}}(\underline{x}), \hat{\phi}(\underline{y})] + \partial_x\delta(x^0-y^0)\cdot[\hat{\phi}(\underline{x}), \hat{\phi}(\underline{y})]$$

Since using delta-f^n only really makes sense under integrals -

$$\int dx^0\,\partial_{x^0}\delta(x^0-y^0)\cdot[\hat{\phi}(\underline{x}), \hat{\phi}(\underline{y})] = \underbrace{\cancelto{0}{\delta(x^0-y^0)\,[\hat{\phi}(\underline{x}), \hat{\phi}(\underline{y})]}\Big|}_{\text{boundary term}} - \int dx^0\,\delta(x^0-y^0) \times [\dot{\hat{\phi}}(\underline{x}), \hat{\phi}(\underline{y})]$$

$$\Rightarrow \partial_x\,\delta(x^0-y^0)\,[\phi(\underline{x}), \phi(\underline{y})] \underset{\substack{\text{upto an}\\ \text{integral}}}{=} -\delta(x^0-y^0)\cdot[\dot{\hat{\phi}}(\underline{x}), \hat{\phi}(\underline{y})] \qquad \dot{\hat{\phi}} = \hat{\pi}$$

$$\Rightarrow (\partial_{\underline{x}}^2+m^2)\,D_R(\underline{x}-\underline{y}) = 2\,\delta(x^0-y^0)\cdot[\hat{\pi}(\underline{x}), \hat{\phi}(\underline{y})] - \delta(x^0-y^0)\,[\hat{\pi}(\underline{x}), \hat{\phi}(\underline{y})]$$

$$\Rightarrow (\partial_{\underline{x}}^2 + m^2) D_R(\underline{x}-\underline{y}) \qquad \delta(x^0-y^0)\,[\hat{\pi}(\underline{x}), \hat{\phi}(\underline{y})]$$

$$[\hat{\pi}(\underline{x}), \hat{\phi}(\underline{y})] = -i\,\delta^3(\vec{x}-\vec{y})$$

$$\Rightarrow \boxed{(\partial_{\underline{x}}^2 + m^2) D_R(\underline{x}-\underline{y}) = -i\,\delta^{(4)}(\underline{x}-\underline{y})}$$

Therefore, we can see that $D_R(\underline{x}-\underline{y})$ is the Green f^n of K.G. eq^n.
Since it vanishes for $x^0 < y^0$, it is the retarted Green's f^n.

$$\underline{\underline{D_R(\underline{x}-\underline{y}) = i\,G_R(\underline{x}-\underline{y})}}$$

If we take the expval. of $[\hat{\phi}(\underline{x}), \hat{\phi}(\underline{y})]$ -

$$\langle 0|\,[\hat{\phi}(\underline{x}), \hat{\phi}(\underline{y})]\,|0\rangle = \int \frac{d^3\vec{p}}{(2\pi)^3}\left(\frac{e^{-i\underline{p}\cdot(\underline{x}-\underline{y})}}{2E_{\vec{p}}} + \frac{e^{+i\underline{p}\cdot(\underline{x}-\underline{y})}}{-2E_{\vec{p}}}\right)$$

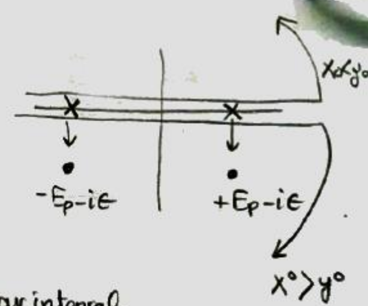
This is exactly what we get on applying retarded B.C.s on KG propagator.
i.e.

$$I = \int \frac{d^3\vec{p}}{(2\pi)^3}\, e^{i\vec{p}\cdot(\vec{x}-\vec{y})} \int \frac{dp_0}{(2\pi i)}\left(\frac{-1}{p_0^2 - E_p^2}\right) e^{-ip_0(x^0-y^0)}$$

$\longrightarrow$ poles at $p_0 = \pm E_p$

Shifting $\pm E_p \longrightarrow \pm E_p - i\epsilon$

$$\Rightarrow \int \frac{dp_0}{(2\pi i)}\left(\frac{-1}{(p_0-(E_p-i\epsilon))(p_0-(-E_p-i\epsilon))}\right) e^{-ip_0(x^0-y^0)}$$



$$= \Theta(x^0-y^0)\cdot\frac{1}{2E_p}\left(e^{-iE_p(x^0-y^0)} - e^{iE_p(x^0-y^0)}\right)$$

after doing the contour integral.

$$\Rightarrow I = \int \frac{d^3\vec{p}}{(2\pi)^3}\cdot\frac{1}{2E_p}\left(e^{-i\underline{p}\cdot(\underline{x}-\underline{y})} - e^{i\underline{p}\cdot(\underline{x}-\underline{y})}\right)\cdot\Theta(x^0-y^0)$$

$\therefore$ Using residue th^m backwards

$$\boxed{\int \frac{d^3\vec{p}}{(2\pi)^3}\left(\frac{e^{-i\underline{p}\cdot(\underline{x}-\underline{y})}}{2E_p} + \frac{e^{i\underline{p}\cdot(\underline{x}-\underline{y})}}{-2E_p}\right) \underset{x^0>y^0}{=} \int \frac{d^3\vec{p}}{(2\pi)^3}\int \frac{dp^0}{(2\pi i)}\left(\frac{-1}{p^2-m^2}\right) e^{-ip_0(x^0-y^0)}}$$

Particle Creation by a Classical Source.

A source for K. G. field, given by $j(\underline{x})$ is turned on for a finite time from $-T$ to T. The initial state is the vacuum $|0\rangle$

$$(\partial^2+m^2)\phi(\underline{x}) = j(\underline{x})$$

$$\partial_\mu\left(\frac{\partial \mathcal{L}_0}{\partial(\partial_\mu\phi)}\right) - \frac{\partial \mathcal{L}_0}{\partial\phi} = +\frac{\partial \mathcal{L}_{int}}{\partial\phi} \qquad \mathcal{L} = \mathcal{L}_0 + j\phi$$

Solving it using Green's f^ns –

$$\phi(\underline{x}) = \underset{\substack{\uparrow \\ \text{free KG} \\ \text{sol}^n}}{\phi_0(\underline{x})} - \int d^4y \; G_R(\underline{x}-\underline{y})\, j(\underline{y})$$

where $(\square_x+m^2)\, G(\underline{x}-\underline{y}) = -\delta^4(\underline{x}-\underline{y})$ $\qquad D_R = iG_R$

$$\Rightarrow \phi(\underline{x}) = \phi_0(\underline{x}) + i\int d^4y \; D_R(\underline{x}-\underline{y})\, j(\underline{y})$$

$$= \phi_0(\underline{x}) + i\int d^4y \int \frac{d^3\vec{p}}{(2\pi)^3}\cdot\frac{1}{2E_{\vec{p}}}\, \Theta(x^0-y^0)\left[e^{-ip\cdot(\underline{x}-\underline{y})} - e^{+ip\cdot(\underline{x}-\underline{y})}\right]\Big|_{p_0=E_p} j(\underline{y})$$

$\phi(\underline{x})$ at $x \gg T$ so field has settled down to free oscillations $(\Theta(x^0-y^0)\approx 1)$?

If we put in F.T. of j, $\tilde{j}(p) = \int d^4y\, e^{ip\cdot y}\, j(y)$ s.t. $p^2=m^2$

$$\Rightarrow \phi(\underline{x}) = \phi_0(\underline{x}) + \int \frac{d^3\vec{p}}{(2\pi)^3}\cdot\frac{1}{\sqrt{2E_{\vec{p}}}}\left(e^{-ip\cdot x}\frac{i\tilde{j}(p)}{\sqrt{2E_{\vec{p}}}} - e^{ip\cdot x}\frac{i\tilde{j}(-p)}{\sqrt{2E_{\vec{p}}}}\right)$$

Since $j(\underline{x}) = j^*(\underline{x}) \Rightarrow \tilde{j}^*(p) = \tilde{j}(-p)$

$$\therefore \phi(\underline{x}) = \int \frac{d^3\vec{p}}{(2\pi)^3 \sqrt{2E_{\vec{p}}}} \left[\underbrace{\left(a_{\vec{p}} + \frac{i\tilde{j}(p)}{\sqrt{2E_p}} \right)}_{\text{new } \tilde{a}_{\vec{p}}} e^{-ip\cdot\underline{x}} + h.c. \right]$$

Now, earlier, our Hamiltonian was of the form

$$\hat{H} = \int \frac{d^3\vec{p}}{(2\pi)^3} E_{\vec{p}}\, a^\dagger_{\vec{p}}\, a_{\vec{p}} \quad + \text{zero energy term.}$$

If we now replace this with our new $\tilde{a}_{\vec{p}}$ and $\tilde{a}^\dagger_{\vec{p}}$, we get -

$$\hat{H} = \int \frac{d^3\vec{p}}{(2\pi)^3} E_{\vec{p}} \left(a^\dagger_{\vec{p}} - \frac{i\tilde{j}(-p)}{\sqrt{2E_{\vec{p}}}} \right)\left(a_{\vec{p}} + \frac{i\tilde{j}(p)}{\sqrt{2E_{\vec{p}}}} \right)$$

At very late times, $x^0 \ggg T$,

$$\langle 0|\hat{H}|0\rangle = \int \frac{d^3\vec{p}}{(2\pi)^3} E_{\vec{p}} \underbrace{\frac{|\tilde{j}(p)|^2}{2E_{\vec{p}}}} = \int \frac{d^3\vec{p}}{(2\pi)^3} \frac{|\tilde{j}(p)|^2}{2}$$

probability for $j(\underline{x})$
to create particles with
$p^2 = m^2$

$\therefore$ # of particles produced in the final state -

$$\int dN = \int \frac{d^3\vec{p}}{(2\pi)^3} \text{Prob.}(\vec{p}) = \int \frac{d^3\vec{p}}{(2\pi)^3} \frac{|\tilde{j}(p)|^2}{2E_{\vec{p}}} = \langle 0|\hat{N}|0\rangle$$

where $\hat{N} = \int \frac{d^3\vec{p}}{(2\pi)^3} a^\dagger_{\vec{p}}\, a_{\vec{p}}$

Feynman Propagator (of the KG field)

$$D_F(\underline{x}-\underline{y}) = i\Delta_F(\underline{x}-\underline{y}) = i\int \frac{d^4 p}{(2\pi)^4} \frac{e^{-ip\cdot(\underline{x}-\underline{y})}}{p^2-m^2+i\epsilon}$$

$$(\Box_x + m^2)\, D_4(\underline{x}-\underline{y}) = -i\,\delta^4(\underline{x}-\underline{y})$$

On calculating the Feynman propagator -

$$D_F(\underline{x}-\underline{y}) = \int \frac{d^3\vec{p}}{(2\pi)^3} \frac{1}{2E_{\vec{p}}} \left(\Theta(x_0-y_0)\, e^{-ip\cdot(\underline{x}-\underline{y})} + \Theta(y_0-x_0)\, e^{+ip\cdot(\underline{x}-\underline{y})} \right)$$

Now, if we recall, the def^n of $D(\underline{x}-\underline{y})$ -

$$D(\underline{x}-\underline{y}) = \langle 0|\,\phi(\underline{x})\,\phi(\underline{y})\,|0\rangle = \int \frac{d^3\vec{p}}{(2\pi)^3} \frac{1}{2E_{\vec{p}}}\, e^{-ip\cdot(\underline{x}-\underline{y})}$$

$$\Rightarrow D(\underline{y}-\underline{x}) = \int \frac{d^3\vec{p}}{(2\pi)^3} \frac{1}{2E_{\vec{p}}}\, e^{+ip(\underline{x}-\underline{y})}$$

$$\therefore D_F(\underline{x}-\underline{y}) = \Theta(x_0-y_0)\, D(\underline{x}-\underline{y}) + \Theta(y_0-x_0)\, D(\underline{y}-\underline{x})$$

$$= \Theta(x_0-y_0)\langle 0|\,\hat{\phi}(\underline{x})\,\hat{\phi}(\underline{y})\,|0\rangle + \Theta(y_0-x_0)\langle 0|\,\hat{\phi}(\underline{y})\,\hat{\phi}(\underline{x})\,|0\rangle$$

We now define the time ordering symbol

$$\mathcal{T}\,\hat{\phi}(\underline{x})\,\phi(\underline{y}) \equiv \Theta(x^0-y^0)\,\hat{\phi}(\underline{x})\,\hat{\phi}(\underline{y}) + \Theta(y^0-x^0)\,\hat{\phi}(\underline{y})\,\hat{\phi}(\underline{x})$$

$$\Rightarrow \boxed{D_F(\underline{x}-\underline{y}) = \langle 0|\,\mathcal{T}\,\hat{\phi}(\underline{x})\,\hat{\phi}(\underline{y})\,|0\rangle} = i\Delta_F(\underline{x}-\underline{y})$$

Complex Klein Gordon Field.

Suppose we have 2 real K G fields with the same mass parameter

$$\mathcal{L} = \frac{1}{2}\left[(\partial\hat{\phi}_1)^2 + (\partial\hat{\phi}_2)^2\right] - \frac{m^2}{2}(\hat{\phi}_1^2 + \hat{\phi}_2^2)$$

Define $\hat{\phi} \equiv \frac{\hat{\phi}_1 + i\hat{\phi}_2}{\sqrt{2}}$, $\hat{\phi}^\dagger = \frac{\hat{\phi}_1 - i\hat{\phi}_2}{\sqrt{2}}$

Then we can define -

$$\mathcal{L} = (\partial\hat{\phi})^\dagger(\partial\hat{\phi}) - m^2\hat{\phi}^\dagger\hat{\phi}$$

Now we'll have

$$\hat{\phi}_1(\underline{x}) = \int \frac{d^3\vec{p}}{(2\pi)^3\sqrt{2E_{\vec{p}}}}\left(\hat{a}_{1\vec{p}}\,e^{-i\underline{p}\cdot\underline{x}} + \hat{a}^\dagger_{1\vec{p}}\,e^{+i\underline{p}\cdot\underline{x}}\right)$$

$$\hat{\phi}_2(\underline{x}) = \int \frac{d^3\vec{p}}{(2\pi)^3\sqrt{2E_{\vec{p}}}}\left(a_{2\vec{p}}\,e^{-i\underline{p}\cdot\underline{x}} + \hat{a}^\dagger_{2\vec{p}}\,e^{+i\underline{p}\cdot\underline{x}}\right)$$

$$\Rightarrow \frac{(\phi_1 + i\phi_2)}{\sqrt{2}}(\underline{x}) = \int \frac{d^3\vec{p}}{(2\pi)^3}\left(\left(\frac{a_1 + ia_2}{\sqrt{2}}\right)_{\vec{p}} e^{-i\underline{p}\cdot\underline{x}} + \left(\frac{a_1^\dagger + ia_2^\dagger}{\sqrt{2}}\right)_{\vec{p}} e^{+i\underline{p}\cdot\underline{x}}\right)$$

no longer the h.c.

Define $$\boxed{\hat{a}_{\vec{p}} \equiv \frac{\hat{a}_1 + i\hat{a}_2}{\sqrt{2}} \quad \text{and} \quad \hat{b}_{\vec{p}} \equiv \frac{\hat{a}_1 - i\hat{a}_2}{\sqrt{2}}}$$

$$\Rightarrow \hat{\phi}(\underline{x}) = \int \underbrace{\frac{d^3\vec{p}}{(2\pi)^3}\frac{1}{\sqrt{2E_{\vec{p}}}}}_{\widetilde{d^3\vec{p}}}\left(\hat{a}_{\vec{p}}\,e^{-i\underline{p}\cdot\underline{x}} + \hat{b}^\dagger_{\vec{p}}\,e^{+i\underline{p}\cdot\underline{x}}\right)$$

$$\hat{\phi}(\underline{x}) = \int \widetilde{d^3\vec{p}}\left(a_{\vec{p}}\,e^{-i\underline{p}\cdot\underline{x}} + b^\dagger_{\vec{p}}\,e^{+i\underline{p}\cdot\underline{x}}\right)$$

$$\Rightarrow \underline{\hat{\phi}^{\dagger}(\underline{x}) = \int \widetilde{d^3\vec{p}} \left(b_{\vec{p}}\, e^{-ip\cdot x} + a^{\dagger}_{\vec{p}}\, e^{+ip\cdot x} \right)}$$

So now since $\mathcal{L} = \dot{\phi}\dot{\phi}^{\dagger} - (\vec{\nabla}\phi^{\dagger})(\vec{\nabla}\phi) - m^2\phi^{\dagger}\phi$

$$\Rightarrow \pi_{\phi} = \frac{\partial \mathcal{L}}{\partial \dot{\phi}} = \underbrace{\dot{\phi}^{\dagger}}_{=\pi} \quad \text{and} \quad \pi_{\phi^{\dagger}} = \frac{\partial \mathcal{L}}{\partial \dot{\phi}^{\dagger}} = \underbrace{\dot{\phi}}_{=\pi^{\dagger}}$$

$$\therefore \mathcal{L} = \pi\pi^{\dagger} - |\vec{\nabla}\phi|^2 - m^2\phi^{\dagger}\phi$$

Similarly, $\hat{P}^{\mu} = \sum_{i=1}^{2} \int \frac{d^3\vec{p}}{(2\pi)^3}\, p^{\mu}\, a^{\dagger}_{\vec{p}i}\, a_{\vec{p}i}$

Since we have

$$a_{\vec{p}} = \frac{a_1 + i a_2}{\sqrt{2}} \qquad b_{\vec{p}} = \frac{a_1 - i a_2}{\sqrt{2}}$$

$$a^{\dagger}_{\vec{p}} = \frac{a_1^{\dagger} - i a_2^{\dagger}}{\sqrt{2}} \qquad b^{\dagger}_{\vec{p}} = \frac{a_1^{\dagger} + i a_2^{\dagger}}{\sqrt{2}}$$

$$\Rightarrow \hat{a}_1 = \frac{\hat{a} + \hat{b}}{\sqrt{2}} \qquad \hat{a}_2 = \frac{\hat{a} - \hat{b}}{\sqrt{2}\, i}$$

$$\Rightarrow \underline{\hat{a}_1^{\dagger}\hat{a}_1 + \hat{a}_2^{\dagger}\hat{a}_2} = \left(\frac{a^{\dagger}_{\vec{p}} + b^{\dagger}_{\vec{p}}}{\sqrt{2}}\right)\left(\frac{\hat{a}_{\vec{p}} + \hat{b}_{\vec{p}}}{\sqrt{2}}\right) + \left(\frac{\hat{a}^{\dagger}_{\vec{p}} - \hat{b}^{\dagger}_{\vec{p}}}{-\sqrt{2}\, i}\right)\left(\frac{a_{\vec{p}} - b_{\vec{p}}}{+\sqrt{2}\, i}\right)$$

$$= \frac{a^{\dagger}_{\vec{p}} a_{\vec{p}} + b^{\dagger}_{\vec{p}} b_{\vec{p}} + \cancel{a^{\dagger}_{\vec{p}} b_{\vec{p}}} + \cancel{b^{\dagger}_{\vec{p}} a_{\vec{p}}} + a^{\dagger}_{\vec{p}} a_{\vec{p}} + b^{\dagger}_{\vec{p}} b_{\vec{p}} - \cancel{b^{\dagger}_{\vec{p}} a_{\vec{p}}} - \cancel{a^{\dagger}_{\vec{p}} b_{\vec{p}}}}{2}$$

$$= \underline{\underline{a^{\dagger}_{\vec{p}} a_{\vec{p}} + b^{\dagger}_{\vec{p}} b_{\vec{p}}}}$$

So, $\boxed{\hat{P}^{\mu} = \int \frac{d^3\vec{p}}{(2\pi)^3}\, p_{\mu} \left(a^{\dagger}_{\vec{p}} a_{\vec{p}} + b^{\dagger}_{\vec{p}} b_{\vec{p}} \right)}$

Also, we know that the creation & destruction operators of the two real fields are independent, so -

$$[a_{\vec{p}_1}, a^\dagger_{\vec{q}_1}] = (2\pi)^3\, \delta^3(\vec{p}-\vec{q}) = [a_{\vec{p}2}, a^\dagger_{\vec{q}2}]$$

$$[a_{\vec{p}_1}, a^\dagger_{\vec{q}_2}] = 0 = [a_{\vec{p}_2}, a^\dagger_{\vec{q}_1}]$$

So, $$\left[\frac{a_{\vec{p}} + b_{\vec{p}}}{\sqrt{2}}, \frac{a^\dagger_{\vec{q}} + b^\dagger_{\vec{q}}}{\sqrt{2}}\right] = \frac{1}{2}\left([a_{\vec{p}}, a^\dagger_{\vec{q}}] + [a_{\vec{p}}, b^\dagger_{\vec{q}}] + [b_{\vec{p}}, a^\dagger_{\vec{q}}] + [b_{\vec{p}}, b^\dagger_{\vec{q}}]\right)$$

$$\overset{!}{=} (2\pi)^3\, \delta^3(\vec{p}-\vec{q})$$

Similarly, $[a_1, a_2^\dagger] = 0$

$$\Rightarrow \left[\frac{a_{\vec{p}} + b_{\vec{p}}}{\sqrt{2}}, \frac{a^\dagger_{\vec{q}} - b^\dagger_{\vec{q}}}{-\sqrt{2}i}\right] = \frac{i}{2}\left([a_{\vec{p}}, a^\dagger_{\vec{q}}] - [b_{\vec{p}}, b^\dagger_{\vec{q}}] + [b_{\vec{p}}, a^\dagger_{\vec{q}}] - [a_{\vec{p}}, b^\dagger_{\vec{q}}]\right)$$

$$\overset{!}{=} 0$$

All of this calculation implies the following relations -

$$\boxed{\begin{gathered} [a_{\vec{p}}, a^\dagger_{\vec{q}}] = [b_{\vec{p}}, b^\dagger_{\vec{q}}] = (2\pi)^3\, \delta^3(\vec{p}-\vec{q}) \\ [a_{\vec{p}}, a_{\vec{q}}] = [a^\dagger_{\vec{p}}, a^\dagger_{\vec{q}}] = [b_{\vec{p}}, b_{\vec{q}}] = [b^\dagger_{\vec{p}}, b^\dagger_{\vec{q}}] = 0 \\ [a_{\vec{p}}, b^\dagger_{\vec{q}}] = [a^\dagger_{\vec{p}}, b_{\vec{q}}] = [a_{\vec{p}}, b_{\vec{q}}] = [a^\dagger_{\vec{p}}, b^\dagger_{\vec{q}}] = 0 \end{gathered}}$$

Conserved current for $\mathbb{C}$- scalar field.

Whenever we have a $\mathcal{L}$ of the form -

$$\mathcal{L} = \partial\phi^\dagger \partial\phi - m^2 \phi^\dagger\phi \Rightarrow \text{it is invariant under} \quad \begin{aligned} \phi' &= e^{i\theta}\phi \\ \phi^{\dagger\prime} &= e^{-i\theta}\phi^\dagger \end{aligned}$$

$\Rightarrow \quad \delta\phi = i\theta\phi \quad$ and $\quad \delta\phi^\dagger = -i\theta\phi$

$\therefore \exists$ a conserved current which can be found as follows -

$$j^\mu(\underline{x}) \sim \frac{\partial \mathcal{L}}{\partial \partial_\mu \phi}\,\delta\phi + \frac{\partial \mathcal{L}}{\partial \partial_\mu \phi^\dagger}\,\delta\phi^\dagger = (\partial^\mu\phi^\dagger)(i\theta\phi) - (\partial^\mu\phi)(i\theta\phi^\dagger)$$

$$\Rightarrow j^\mu(\underline{x}) = i\left(\partial^\mu\phi\cdot\phi^\dagger - \partial^\mu\phi^\dagger\cdot\phi\right).$$

We define the conserved charge as-

$$Q = \int d^3\vec{x}\, j^0 = i\int d^3\vec{x}\left(\phi^\dagger\dot{\phi} - \dot{\phi}^\dagger\phi\right)$$

$$Q = i\int d^3\vec{x}\,\widetilde{d^3\vec{q}}\,\widetilde{d^3\vec{p}}\left[\left(b_{\vec{p}}\,e^{-ip\cdot x} + a^\dagger_{\vec{p}}\,e^{+ip\cdot x}\right)(-iq_0)\left(a_{\vec{q}}\,e^{-iq\cdot x} - b^\dagger_{\vec{q}}\,e^{+iq\cdot x}\right)\right.$$
$$\left. -(-ip_0)\left(b_{\vec{p}}\,e^{-ip\cdot x} - a^\dagger_{\vec{p}}\,e^{ip\cdot x}\right)\left(a_{\vec{q}}\,e^{-iq\cdot x} + b^\dagger_{\vec{q}}\,e^{+iq\cdot x}\right)\right]$$

- If we consider similar combinations like $a^\dagger_{\vec{p}}\,a_{\vec{q}}$, or $b_{\vec{p}}\,b^\dagger_{\vec{q}}$, we'll a coeff. of form-

$$\int d^3\vec{x}\, e^{\pm i(p-q)\cdot x} = \int d^3\vec{x}\, e^{\pm i[(p_0-q_0)x^0 - (\vec{p}-\vec{q})\cdot\vec{x}]}$$
$$= \cancelto{1}{e^{\pm i(p_0-q_0)x^0}}\Big|_{\vec{p}=\vec{q}}(2\pi)^3\,\delta^3(\vec{p}-\vec{q}) = \underline{(2\pi)^3\,\delta^3(\vec{p}-\vec{q})} \longrightarrow \text{time independent}$$

- However, for the cross combinations, like $b_{\vec{p}}\,a_{\vec{q}}$, or $a^\dagger_{\vec{p}}\,b^\dagger_{\vec{q}}$, we'll get a coeff. -

$$\int d^3\vec{x}\, e^{\pm i(\vec{p}+\vec{q})\cdot\vec{x}} = e^{\pm i(p_0+q_0)\cdot x^0}\Big|_{\vec{p}=-\vec{q}}\,(2\pi)^3\,\delta^3(\vec{p}+\vec{q})$$

but since $p_0 = \sqrt{\vec{p}^{\,2}+m^2} = \sqrt{\vec{q}^{\,2}+m^2} = q_0$

$$\Rightarrow \int d^3\vec{x}\, e^{\pm i(p+q)\cdot x} = \underline{e^{\pm 2iE_{\vec{p}}t}(2\pi)^3\,\delta^3(\vec{p}+\vec{q})} \longrightarrow \text{time dependent!}$$

Now $Q = +\int d^3\vec{x}\, \widetilde{d^3\vec{p}}\, \widetilde{d^3\vec{q}} \Big[q_0 \Big(b_{\vec{p}} a_{\vec{q}} e^{-i(p+q)\cdot x} - b_{\vec{p}} b^{\dagger}_{\vec{q}} e^{-i(p-q)\cdot x} + a^{\dagger}_{\vec{p}} a_{\vec{q}} e^{+i(p-q)\cdot x} - a^{\dagger}_{\vec{p}} b^{\dagger}_{\vec{q}} e^{+i(p+q)\cdot x}$

$- p_0 \Big(b_{\vec{p}} a_{\vec{q}} e^{-i(p+q)\cdot x} + b_{\vec{p}} b^{\dagger}_{\vec{q}} e^{-i(p-q)\cdot x} - a^{\dagger}_{\vec{p}} a_{\vec{q}} e^{+i(p-q)\cdot x} - a^{\dagger}_{\vec{p}} b^{\dagger}_{\vec{q}} e^{+i(p+q)\cdot x} \Big) \Big]$

$$= \int \frac{\widetilde{d^3\vec{p}}\, \widetilde{d^3\vec{q}}}{(2\pi)^3} \Big[\delta^3_{(\vec{p}-\vec{q})} \left(a^{\dagger}_{\vec{p}} a_{\vec{q}} - b_{\vec{p}} b^{\dagger}_{\vec{q}} - b_{\vec{p}} b^{\dagger}_{\vec{q}} + a^{\dagger}_{\vec{p}} a_{\vec{q}} \right)\cdot p_0$$

$$\delta^3(\vec{p}+\vec{q}) \left(\cancel{b_{\vec{p}} a_{\vec{q}} e^{-2iE_{\vec{p}}t}} - \cancel{a^{\dagger}_{\vec{p}} b^{\dagger}_{\vec{q}} e^{+2iE_{\vec{p}}t}} - \cancel{b_{\vec{p}} a_{\vec{q}} e^{-2iE_{\vec{p}}t}} - \cancel{a^{\dagger}_{\vec{p}} b^{\dagger}_{\vec{q}} e^{+2iE_{\vec{p}}t}} \right) \Big] \cdot p_0$$

$$= \cancel{(2\pi)^3} \int \frac{d^3\vec{p}}{\cancel{(2\pi)^3}} \frac{d^3\vec{q}}{(2\pi)^3} \frac{E_{\vec{p}}}{2\sqrt{E_{\vec{p}} E_{\vec{q}}}} \left(2a^{\dagger}_{\vec{p}} a_{\vec{q}} - 2 b_{\vec{p}} b^{\dagger}_{\vec{q}} \right) \delta^{(3)}(\vec{p}-\vec{q})$$

$$= \int \frac{d^3\vec{p}}{(2\pi)^3} \left(a^{\dagger}_{\vec{p}} a_{\vec{p}} - b_{\vec{p}} b^{\dagger}_{\vec{p}} \right)$$

We can replace $b_{\vec{p}} b^{\dagger}_{\vec{p}}$ by $b_{\vec{p}} b^{\dagger}_{\vec{p}} = b^{\dagger}_{\vec{p}} b_{\vec{p}} + \underbrace{[b_{\vec{p}}, b^{\dagger}_{\vec{p}}]}_{=(2\pi)^3 \delta^3(0)}$

$$\hat{Q} = \int \frac{d^3\vec{p}}{(2\pi)^3} \Big(a^{\dagger}_{\vec{p}} a_{\vec{p}} - b^{\dagger}_{\vec{p}} b_{\vec{p}} - (2\pi)^3 \delta^3(0) \Big)$$

$\longrightarrow$ renormalizable term (irrelevant)

$$\Rightarrow \boxed{\hat{Q} = \int \frac{d^3\vec{p}}{(2\pi)^3} \left(a^{\dagger}_{\vec{p}} a_{\vec{p}} - b^{\dagger}_{\vec{p}} b_{p} \right)} \longrightarrow \text{conserved charge.}$$

density of particles

density of anti-particles

FOCK SPACE (of $\mathbb{C}$-scalar field)

$a_{\vec{p}} |0\rangle = 0 \qquad b_{\vec{p}} |0\rangle = 0$

$|\vec{p}, Q=+1\rangle = \sqrt{2E_{\vec{p}}}\, a_{\vec{p}} |0\rangle \qquad \hat{Q}(a^{\dagger}_{\vec{p}}|0\rangle) = +1\,(a^{\dagger}_{\vec{p}}|0\rangle)$

$|\vec{p}, Q=-1\rangle = \sqrt{2E_{\vec{p}}}\, b_{\vec{p}} |0\rangle \qquad \hat{Q}(b^{\dagger}_{\vec{p}}|0\rangle) = -1\,(b^{\dagger}_{\vec{p}}|0\rangle)$

QUANTIZATION OF DIRAC FIELD

Bosons : obey Bose-Einstein statistics. (any # of particles can occupy the same state in $\mathcal{H}$)
$(s=0,1,2\ldots)$

Fock space: $\prod_{\vec{p}} \frac{(a^\dagger_{\vec{p}})^{n_{\vec{p}}} (b^\dagger_{\vec{p}})^{\bar{n}_{\vec{p}}}}{\sqrt{n_{\vec{p}}!\,\bar{n}_{\vec{p}}!}} \left(\sqrt{2E_{\vec{p}}}\right)^{n_{\vec{p}}+\bar{n}_{\vec{p}}} |0\rangle$

i.e. $n_{\vec{p}}$ particles & $\bar{n}_{\vec{p}}$ anti-particles with same momentum are allowed.

Fermions: Obey Fermi-Dirac statistics (Pauli exclusion principle must be obeyed)
$(s=\frac{1}{2},\frac{3}{2},\ldots)$ No more than 1 particle can occupy the same quantum state.

Creation operators carry a conserved quantum no. Also, the way we've defined our Hamiltonian, it is positive semidefinite!

Qtys. : $\vec{p}, E_{\vec{p}}, Q\ldots$

Operators : $\hat{\vec{p}}, \hat{H}, \hat{Q} \ldots \leftarrow$ conserved consts. of motion

∴ We don't want a state with same (identical) particles with identical quantum numbers.

If we define the new creation & annihilation operators as $c^\dagger$ and c, then we'd expect

$$c^\dagger|0\rangle \neq 0 \quad , \text{ but } (c^\dagger)^2|0\rangle = 0$$

$$\underbrace{c^\dagger c}_{\text{acts as a no. operator}}|0\rangle = 0 \qquad \text{and} \qquad c^\dagger c\, c^\dagger|0\rangle = c^\dagger|0\rangle$$

These properties are satisfied if -

$$\{\hat{c}, \hat{c}^\dagger\} = \mathbb{1} \quad \text{and} \quad \{\hat{c}, \hat{c}\} = \{\hat{c}^\dagger, \hat{c}^\dagger\} = 0$$

So, with complex Dirac fields to describe charged spin-$\frac{1}{2}$ particles and anti-particles, we must use <u>anti-commutation</u> relations to perform canonical quantization.

A guess:

$$\underbrace{\psi(\underline{x})}_{\begin{pmatrix}\cdot\\\cdot\\\cdot\\\cdot\end{pmatrix}_{\text{4-component object}}} = \sum_{s=1}^{2} \int \frac{d^3\vec{p}}{(2\pi)^3} \cdot \frac{1}{\sqrt{2E_{\vec{p}}}} \left(c_{\vec{p},s}\, u_{\vec{p},s}\, e^{-ip\cdot x} + d_{\vec{p},s}\, v_{\vec{p},s}\, e^{+ip\cdot x} \right)$$

We want $(i\cancel{\partial}-m)\psi = 0$.

But we know that $\longrightarrow$ $\left\{ \begin{array}{l} (\cancel{p}-m)\, u_{\vec{p},s} = 0 \\ -(\cancel{p}+m)\, v_{\vec{p},s} = 0 \end{array} \right\} \Rightarrow (i\cancel{\partial}-m)\psi = 0$ trivially.

<u>Canonical Quantization.</u>

$$S_{\text{Dirac}} = \int d^4\underline{x} \cdot \left(\bar{\psi}\, (i\cancel{\partial}-m)\, \psi \right)$$

$$\underset{\text{upto a B.T.}}{\equiv} \int d^4\underline{x} \left(\bar{\psi} \cdot \frac{i}{2} \overleftrightarrow{\cancel{\partial}} - m \right) \psi$$

$$\delta S = 0 \Rightarrow \int d^4\underline{x}\ \delta\bar{\psi}\, (i\overrightarrow{\cancel{\partial}}-m)\psi + \bar{\psi}\, (i\cancel{\partial}-m)\, \delta\psi = 0$$

$$\Rightarrow \underline{(i\overrightarrow{\cancel{\partial}}-m)\psi(\underline{x}) = 0} \quad \text{and} \quad \underline{\bar{\psi}\, (i\overleftarrow{\cancel{\partial}}+m) = 0}$$

So, $\mathcal{L}_{\text{Dirac}} = \bar{\psi}\, (i\cancel{\partial}-m)\, \psi$ is the correct $\mathcal{L}$.

- $$\underline{\underline{\pi_\alpha}} = \frac{\partial \mathcal{L}}{\partial \dot{\psi}_\alpha} = \frac{\partial}{\partial \dot{\psi}_\alpha} \left(\bar{\psi} \left(i\gamma^0 \partial_0 + i\vec{\gamma}\cdot\vec{\nabla} - m \right) \psi \right)_\alpha = \left(\bar{\psi}\, i\gamma^0 \right)_\alpha$$

$$= \psi^\dagger \gamma^0 i \gamma^0 = \underline{\underline{i\psi^\dagger}}$$

Equal time anti-commutation relations

- $\underline{\{\psi_\alpha(\underline{x}), \psi_\beta(\underline{x}')\}_{t=t'} = 0}$

- $\{\psi_\alpha(\underline{x}), \pi_\beta(\underline{x}')\}_{t=t'} = i\,\delta^3(\vec{x}-\vec{x}')$

$$\Rightarrow \underline{\{\psi_\alpha(\underline{x}), \psi^\dagger_\beta(\underline{x}')\}_{t=t'} = \delta^3(\vec{x}-\vec{x}')}$$

So, now, the fields are given by -

$$\psi(\underline{x}) = \sum_{s=1}^{2} \int \frac{d^3\vec{p}}{(2\pi)^3} \cdot \frac{1}{2E_{\vec{p}}} \left(c_{\vec{p},s}\, u_s(\vec{p})\, e^{-ip\cdot \underline{x}} + d^\dagger_{\vec{p},s}\, v_s(\vec{p})\, e^{+ip\cdot \underline{x}} \right)$$

$$\psi^\dagger(\underline{x}) = \sum_{s=1}^{2} \int \frac{d^3\vec{p}}{(2\pi)^3} \cdot \frac{1}{2E_{\vec{p}}} \left(d_{\vec{p},s}\, v^\dagger_s(\vec{p})\, e^{-ip\cdot \underline{x}} + c^\dagger_{\vec{p},s}\, u^\dagger_s(\vec{p})\, e^{+ip\cdot \underline{x}} \right)$$

Imposing anti-commutator relations -

$$\{\psi_\alpha(\underline{x}), \psi_\beta(\underline{x}')\}_{t=t'} = 0$$

$$\Rightarrow \text{possible if } \{c_{\vec{p},s}, c_{\vec{q},s'}\} = 0 = \{d_{\vec{p},s}, d_{\vec{q},s'}\}$$

$$\{c_{\vec{p},s}, d^\dagger_{\vec{q},s'}\} = 0$$

$$\{\psi_\alpha(\underline{x}), \psi^\dagger_\beta(\underline{x}')\}_{t=t'} = \delta^3(\vec{x}-\vec{x}')$$