Practical - I

Limits & Continuity

Q.

1] $\lim_{x \to a} \left[\frac{\sqrt{a+2x} - \sqrt{3x}}{\sqrt{3a+x} - 2\sqrt{x}} \right]$

2] $\lim_{y \to 0} \left[\frac{\sqrt{a+y} - \sqrt{a}}{y\sqrt{a+y}} \right]$

3] $\lim_{x \to \frac{\pi}{6}} \left[\frac{\cos x - \sqrt{3}\sin x}{\pi - 6x} \right]$

4] $\lim_{x \to \infty} \frac{\sqrt{x^2+5} - \sqrt{x^2-3}}{\sqrt{x^2+3} - \sqrt{x^2+1}}$

$\Rightarrow$ 1] $\lim_{x \to a} \left[\frac{\sqrt{a+2x} - \sqrt{3x}}{\sqrt{3a+x} - 2\sqrt{x}} \times \frac{\sqrt{a+2x} + \sqrt{3x}}{\sqrt{a+2x} + \sqrt{3x}} \times \frac{\sqrt{3a+x} + 2\sqrt{x}}{\sqrt{3a+x} + 2\sqrt{x}} \right]$

$\lim_{x \to a} \frac{(a+2x-3x)\ (\sqrt{3a+x} + 2\sqrt{x})}{(3a+x-4x)\ (\sqrt{a+2x} + \sqrt{3x})}$

$\lim_{x \to a} \frac{(a-x)(\sqrt{3a+x} + 2\sqrt{x})}{(3a-3x)(\sqrt{a+2x} + \sqrt{3x})}$

$\frac{1}{3} \lim_{x \to a} \frac{(a-x)\ (\sqrt{3a+x} + 2\sqrt{x})}{(a-x)\ (\sqrt{a+2x} + \sqrt{3x})}$

$\frac{1}{3} \frac{\sqrt{3a+a} + 2\sqrt{a}}{\sqrt{a+2a} + \sqrt{3a}}$

$\frac{1}{3} \times \frac{\sqrt{4a} \cancel{+2\sqrt{a}}}{\sqrt{3a} + \sqrt{3a}}.$

$$\frac{1}{3} \times \frac{2\sqrt{a}+2\sqrt{a}}{\sqrt{3a}+\sqrt{3a}}$$

$$\frac{1}{3} \times \frac{\cancel{4}^{2}\sqrt{\cancel{a}}}{\cancel{2}\sqrt{3\cancel{a}}}$$

$$= \frac{2}{3\sqrt{3}}$$

2] $\lim_{y \to 0} \left[\frac{\sqrt{a+y}-\sqrt{a}}{y\sqrt{a+y}} \times \frac{\sqrt{a+y}+\sqrt{a}}{\sqrt{a+y}+\sqrt{a}} \right)$

$$\lim_{y \to 0} \frac{a+y-a}{y\sqrt{a+y}(\sqrt{a+y}+\sqrt{a})}$$

$$\lim_{y \to 0} \frac{y}{y\sqrt{a+y}(\sqrt{a+y}+\sqrt{a})}$$

$$\frac{1}{\sqrt{a+0}(\sqrt{a+0}+\sqrt{a}}$$

$$\frac{1}{\sqrt{a}(\sqrt{a}+\sqrt{a})} = \frac{1}{2a}$$

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3) $\lim_{x \to \pi/6} \frac{\cos x - \sqrt{3} \sin x}{\pi - 6x}$

By substituting $x - \frac{\pi}{6} = h$

$x = h + \frac{\pi}{6}$

where $h \to 0$.

$\lim_{h \to 0} \frac{\cos\left(h + \frac{\pi}{6}\right) \sqrt{3} \sin\left(h + \frac{\pi}{6}\right)}{\pi - 6\left(h + \frac{\pi}{6}\right)}$

using
$\cos(A+B) = \cos A \cdot \cos B - \sin A \sin B$
$\sin(A+B) = \sin A \cos B + \cos A \sin B.$

$\lim_{h \to 0} \frac{\cosh \cdot \cos\frac{\pi}{6} - \sin h \sin\frac{\pi}{6} - \sqrt{3} \sin h \cos\frac{\pi}{6} + \cos h \sin\frac{\pi}{6}}{\pi - \cancel{6}\left(\frac{6h + \pi}{\cancel{6}}\right)}$

$\cos\frac{\pi}{6} = \cos 30^\circ = \sqrt{3}/2$

$\sin\frac{\pi}{6} = \sin 30^\circ = \frac{1}{2}$

$\lim_{h \to 0} \frac{\cos h \cdot \frac{\sqrt{3}}{2} - \sin h \cdot \frac{1}{2} - \sqrt{3}\left(\sin h \frac{\sqrt{3}}{2} + \cos h \cdot \frac{1}{2}\right)}{\cancel{\pi} - 6h + \cancel{\pi}}$

$$\lim_{h \to 0} \frac{\cos \cancel{\frac{\sqrt{3}}{2}h} - \sin \frac{h}{2} - \sin \frac{3h}{2} - \cancel{\cos \frac{\sqrt{3}h}{2}}}{-6h}$$

$$\lim_{h \to 0} \frac{\cancel{-} \sin \frac{4h}{2}}{\cancel{-} 6h}$$

$$\lim_{h \to 0} \frac{\sin \cancel{4}h}{3 \cancel{12} h}$$

$$\frac{1}{3} \lim_{h \to 0} \frac{\sin h}{h} = \frac{1}{3} \times 1 = \frac{1}{3}$$

4] By rationalizing Numerator & Denominator both

$$\lim_{x \to \infty} \left[\frac{\sqrt{x^2+5} - \sqrt{x^2-3}}{\sqrt{x^2+3} - \sqrt{x^2+1}} \times \frac{\sqrt{x^2+5} + \sqrt{x^2-3}}{\sqrt{x^2+5} + \sqrt{x^2-3}} \times \frac{\sqrt{x^2+3} + \sqrt{x^2+1}}{\sqrt{x^2+3} + \sqrt{x^2+1}} \right]$$

$$\lim_{x \to \infty} \left[\frac{(\cancel{x^2}+5-\cancel{x^2}+3)(\sqrt{x^2+3}+\sqrt{x^2+1})}{(\cancel{x^2}+3-\cancel{x^2}-1)(\sqrt{x^2+5}+\sqrt{x^2-3})} \right]$$

$$\lim_{x \to \infty} \frac{4\cancel{8}(\sqrt{x^2+3}+\sqrt{x^2+1})}{1\cancel{2}(\sqrt{x^2+5}+\sqrt{x^2-3})}$$

$$4 \lim_{x \to \infty} \frac{\sqrt{x^2\left(1+\frac{3}{x^2}\right)} + \sqrt{x^2\left(1+\frac{1}{x^2}\right)}}{\sqrt{x^2\left(1+\frac{5}{x^2}\right)} + \sqrt{x^2\left(1-\frac{3}{x^2}\right)}}$$

After applying limit we get,

$= 4$

5) $f(x) = \dfrac{\sin 2x}{\sqrt{1-\cos 2x}}$, for $0 \le x \le \pi/2$

$= \dfrac{\cos x}{\pi - 2x}$, for $\frac{\pi}{2} < x < \pi$ $\Bigg\}$ at $x = \pi/2$

$f(\pi/2) = \dfrac{\sin 2\left(\frac{\pi}{2}\right)}{\sqrt{1-\cos^2(\pi/2)}}$ $\quad \therefore f(\pi/2) = 0$

f at $x = \pi/2$ define.

$\lim_{x \to \pi/2^+} f(x) = \lim_{x \to \pi/2^+} \dfrac{\cos x}{\pi - 2x}$

By substituting Method

$x - \frac{\pi}{2} = h$

$x = h + \pi/2$

where $h \to 0$

$\lim_{h \to 0} \dfrac{\cos\left(h + \frac{\pi}{2}\right)}{\pi - 2\left(h + \frac{\pi}{2}\right)}$

$\lim_{h \to 0} \dfrac{\cos\left(h + \frac{\pi}{2}\right)}{\pi - 2\left(\frac{2h + \pi}{2}\right)}$

$\lim_{h \to 0} \dfrac{\cos\left(h + \frac{\pi}{2}\right)}{-2h}$ Using $\cos(A+B) = \cos A \cos B - \sin A \cdot \sin B$

$\lim_{h \to 0} \dfrac{\cos h \cdot 0 - \sin h}{-2h}$

$\lim_{h \to 0} \dfrac{\cancel{-}\sin h}{\cancel{-}2h} = \dfrac{1}{2} \lim_{h \to 0} \dfrac{\sin h}{h} = \dfrac{1}{2}$

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iv) $\lim_{x \to \frac{\pi}{2}^-} f(x) = \lim_{x \to \frac{\pi}{2}^-} \frac{\sin 2x}{\sqrt{1-\cos 2x}}$

Using $\sin 2x = 2\sin x.$ $\cos x$.

$\lim_{x \to \frac{\pi}{2}} \frac{2\sin x \cdot \cos x}{\sqrt{2\sin^2 x}}$

$\lim_{x \to \pi/2^-} \frac{2\sin x \cdot \cos x}{\sqrt{2}\sin x}$

$\lim_{x \to \pi/2^-} = \frac{2\cos x}{\sqrt{2}}$

$\frac{2}{\sqrt{2}} \lim_{x \to \pi/2^-} \cos x$

$\therefore$ L.H.L $\neq$ R.H.L.

$\therefore f$ is not continuou at $x = \pi/2$

Q. ii) $f(x) = \frac{x^2-9}{x-3}$ $\quad 0 < x < 3$

$= x+3$ $\quad 3 \le x \le 6$

$= \frac{x^2-9}{x+3}$ $\quad 6 \le x < 9$

at $x = 3$ & $x = 6$

at $x = 2$

i) $f(3) = x^2 - 9 = 0$

f at $x = 3$ define.

ii) $\lim_{x \to 3^+} f(x) = \lim_{x \to 3^+} x+3$

$f(3) = x+3 = 3+3 = 6$

f is define at $x = 3$

$\lim_{x \to 3^+} f(x) = \lim_{x \to 3^+} (x+3) = 6.$

$\lim_{x \to 3^-} f(x) = \lim_{x \to 3^-} \frac{x^2-9}{x-3} = \frac{\cancel{(x-3)}(x+3)}{\cancel{(x-3)}}$

L.H.L = R.H.L

f is continuous at $x=3$

for $x=6$

$f(6) = \frac{x^2-9}{x+3} = \frac{36-9}{6+3} = \frac{27}{9} 3 = 3$

$\lim_{x \to 6^+} \frac{x^2-9}{x+3}$

$\lim_{x \to 6^+} \frac{(x-3)\cancel{(x+3)}}{\cancel{(x+3)}}$

$\lim_{x \to 6^+} (x-3) = 6-3 = 3$

$\lim_{x \to 6^-} x+3 = 3+6 = 9$

$\therefore$ L.H.L $\neq$ R.H.L.

function is not continous.

6) i) $f(x) = \frac{1-\cos 4x}{x^2}$ $x<0$ $\left.\right\}$ at $x=0$

$= k$ $x=0$

sol f ~~is~~ continous at $x=0$

$\lim_{x \to 0} f(x) = f(0)$

$\lim_{x \to 0} \frac{1-\cos 4x}{x^2} = k$

$$\lim_{x\to 0} \frac{2\sin^2 2x}{x^2} = K$$

$$2\lim_{x\to 0} \frac{\sin^2 2x}{x^2} = K$$

$$2\lim_{x\to 0} \left(\frac{\sin 2x}{x}\right)^2 = K$$

$$2(2)^2 = K$$

$$\therefore K = 8$$

ii) $f(x) = (\sec^2 x)^{\cot^2 x}$ $x \neq 0$
$\quad = K$ $x = 0$ } at $x = 0$

Sol $f(x) = (\sec^2 x)^{\cot^2 x}$

using
$\cancel{\sec^2} \tan^2 x - \sec^2 x = 1$
$\therefore \sec^2 x = 1 + \tan^2 x$
&
$\cot^2 x = \frac{1}{\tan^2 x}$

$$\therefore \lim_{x\to 0} (\sec^2 x)^{\cot^2 x}$$

$$\lim_{x\to 0} (1+\tan^2 x)^{\frac{1}{\tan^2 x}}$$

we know that

$$\lim_{x\to 0} (1+nx)^{\frac{1}{nx}} = e$$

$\therefore e$

$\therefore K = e.$

iii) $f(x) = \dfrac{\sqrt{3} - \tan x}{\pi - 3x}$ $x = \pi/3$, $x = \pi/3$ } at $x = \frac{\pi}{3}$

$= k$

$x - \frac{\pi}{3} = h$

$x = h + \frac{\pi}{3}$

where $h \to 0$

$$f(\pi/3 + h) = \frac{\sqrt{3} - \tan(\pi/3 + h)}{\pi - 3\left(\frac{\pi}{3} + h\right)}$$

$$\lim_{h \to 0} \frac{\sqrt{3} - \tan(\pi/3 + h)}{\pi - 3(\pi/3 + h)}$$

using $\tan(A+B) = \dfrac{\tan A + \tan B}{1 - \tan A \cdot \tan B}$

$$\lim_{h \to 0} \frac{\sqrt{3} - \dfrac{\tan\frac{\pi}{3} + \tan h}{1 - \tan\frac{\pi}{3} \cdot \tan h}}{\cancel{\pi} - \cancel{\pi} - 3h}$$

$$\lim_{h \to 0} \frac{\dfrac{\sqrt{3}\left(1 - \tan\frac{\pi}{3} \cdot \tan h\right) - \left(\tan\frac{\pi}{3} + \tan h\right)}{1 - \tan\frac{\pi}{3} \cdot \tan h}}{-3h}$$

using $\tan\pi/3 = \tan 60 = \sqrt{3}$

$$\lim_{h \to 0} \frac{\dfrac{(\sqrt{3} - \sqrt{3} \times \sqrt{3} \cdot \tan h) - (\sqrt{3} + \tan h)}{1 - \tan\pi/3 \cdot \tan h}}{-3h}.$$

$$\lim_{h\to 0} \frac{\frac{(\sqrt{3}-\sqrt{3}\cdot\tan h)-(\sqrt{3}+\tan h)}{1-\sqrt{3}\cdot\tan h}}{-3h}.$$

$$\lim_{h\to 0} \frac{\frac{(\cancel{\sqrt{3}}-3\tan h-\cancel{\sqrt{3}}-\tan h)}{1-\sqrt{3}\cdot\tan h}}{-3h}$$

$$\lim_{h\to 0} \frac{\cancel{-}4\tan h}{\cancel{-}3h\,(1-\sqrt{3}\tan h)}$$

$$\lim_{h\to 0} \frac{4\tan h}{3h(1-\sqrt{3}\tan h)}$$

$$\frac{4}{3}\lim_{h\to 0}\frac{\tan h}{h}\lim_{h\to 0}\frac{1}{(1-\sqrt{3}\tan h)}$$

$$\frac{\tan h}{h}=1$$

$$=\frac{4}{3}\frac{1}{(1-\sqrt{3}(0))}$$

$$=\frac{4}{3}\left(\frac{1}{1}\right)=\frac{4}{3}$$

7) i) $f(x)=\dfrac{1-\cos 3x}{x\tan x}$ $\quad x\neq 0$

$\quad = 9. \quad x=0$ $\Big\}$ at $x=0$.

$$f(x)=\frac{1-\cos 3x}{x\tan x}$$

$$\lim_{x\to 0}\frac{2\sin^2 3/2\,x}{x\tan x}$$

$$\lim_{x\to 0} = \frac{\dfrac{2\sin^2 \frac{3x}{2x}}{2x} \times x^2}{\dfrac{x^2}{x - \tan x}} \times x^2$$

$$= 2 \lim_{x\to 0} \frac{\left(\frac{3}{2}\right)^2}{1} = \frac{2\times 9}{4^2} = \frac{9}{2}$$

$\lim_{x\to 0} f(x) = \frac{9}{2}$ $\quad g = f(0)$

$\therefore f$ is not continous at $x=0$

Redefine function.

$$f(x) = \begin{cases} \dfrac{1-\cos 3x}{x\tan x} & x \neq 0 \\ \dfrac{9}{2} & x = 0 \end{cases}$$

Now $\lim_{x\to 0} f(x) = f(0)$

f has removable discontinuity at $x=0$

ii) $f(x) = \begin{cases} \dfrac{(e^{3x}-1)\sin x^\circ}{x^2} & x \neq 0 \\ = \pi/6 & x = 0 \end{cases}$ at $x=0$

$$\lim_{x\to 0} \frac{(e^{3x}-1)\sin\left(\frac{\pi x}{180}\right)}{x^2}$$

$$\lim_{x\to 0} \frac{e^{3x}-1}{x} \cdot \lim_{x\to 0} \frac{\sin\left(\frac{\pi x}{180}\right)}{x}$$

$$\lim_{x\to 0} 3 \cdot \frac{e^{3x}-1}{3x} \cdot \lim_{x\to 0} \frac{\sin\left(\frac{\pi x}{180}\right)}{x}$$

$$3 \lim_{x\to 0} \frac{e^{3x}-1}{3x} \cdot \lim_{x\to 0} \frac{\sin\left(\frac{\pi x}{180}\right)}{x}$$

$$\cancel{3} \log e \frac{\pi}{\cancel{180}_{60}} = \frac{\pi}{60} = f(0)$$

f is continuous at $x=0$

8] $f(x) = \frac{e^{x^2} - \cos x}{x^2}$ $x = 0$

is continous at $x=0$.

∴ Given f is continuous at $x=0$

$$\lim_{x\to 0} f(x) = f(0)$$

$$= \lim_{x\to 0} \frac{e^{x^2} - \cos x}{x^2} = f(0)$$

$$\lim_{x\to 0} \frac{e^{x^2} - \cos x - 1 + 1}{x^2}$$

$$\lim_{x\to 0} \frac{(e^{x^2}-1) + (1-\cos x)}{x^2}$$

$$\lim_{x\to 0} \frac{e^{x^2}-1}{x^2} + \lim_{x\to 0} \frac{1-\cos x}{x^2}$$

$$\log e + \lim_{x\to 0} \frac{2\sin^2 x/2}{x^2}$$

$$\log e + 2 \lim_{x\to 0} \left(\frac{\sin x/2}{x}\right)^2$$

Multiply with 2 on Num & Denominator

$= 1 + 2 \times \frac{1}{4 \cdot 2} = \frac{3}{2} = f(0)$ //

No.

9] $f(x) = \frac{\sqrt{2} - \sqrt{1-\sin x}}{\cos^2 x}$ $x \neq \pi/2$

$f(0)$ is continuous at $x = \pi/2$

$\lim_{x \to \pi/2} \frac{\sqrt{2} - \sqrt{1+\sin x}}{\cos^2 x} \times \frac{\sqrt{2} + \sqrt{1+\sin x}}{\sqrt{2} + \sqrt{1+\sin x}}$

$\lim_{x \to \pi/2} \frac{2-1+\sin x}{\cos^2 x(\sqrt{2}+\sqrt{1+\sin x})}$

$= \lim_{x \to \pi/2} \frac{1+\sin x}{1-\sin^2 x(\sqrt{2}+\sqrt{1+\sin x})}$

$= \lim_{x \to \pi/2} \frac{\cancel{1+\sin x}}{(1-\sin x)\cancel{(1+\sin x)}(\sqrt{2}+\sqrt{1+\sin x})}$

$= \lim_{x \to \pi/2} \frac{1}{(1-\sin x)(\sqrt{2}+\sqrt{1+\sin x}}$

$= \frac{1}{2(\sqrt{2}+\sqrt{2})}$

$= \frac{1}{2(2\sqrt{2})} = \frac{1}{4\sqrt{2}}$

$f(\pi/2) = \frac{1}{4\sqrt{2}}$ //

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Practical No: 2

Derivative

Q.1] Show that the following function defined

i) $\cot x$

$f(x) = \cot x$

$Df(a) = \lim_{x \to 0} \frac{f(x) - f(a)}{x - a}$

$= \lim_{x \to 0} \frac{\cot x - \cot a}{x - a}$.

Put $x - a = h$ $\quad x \to a$

$x = h + a$ $\quad h \to 0$

$\lim_{h \to 0} \frac{\cot(x+h) - \cot(a)}{h + a - a}$

$\lim_{h \to 0} \frac{\frac{\cos(a+h)}{\sin(a+h)} - \frac{\cos a}{\sin a}}{h}$.

$\lim_{h \to 0} \frac{\cos(a+h)\sin(a) - \cos a - \sin(a+h)}{\sin(a+h) \cdot \sin a \times h}$

$\lim_{h \to 0} \frac{\sin(a-h-a)}{\sin(a+h) \cdot \sin a \times h}$

$\lim_{h \to 0} \frac{1}{\sin(a+h) \cdot \sin a} - \lim_{h \to 0} \frac{\sin h}{h}$

$= \frac{1}{\sin a \cdot \sin a} \times -1$

$= \frac{-1}{\sin a^2}$

$= -\operatorname{cosec}^2 a$ //

ii) Cosec x

$f(x) = \operatorname{cosec} x$

$$Df(a) = \lim_{x \to 0} \frac{f(x) - f(a)}{x - a}$$

$$\lim_{x \to 0} \frac{\operatorname{cosec} x - \operatorname{cosec}(a)}{x - a}$$

Put $x - a = h$, $h \to 0$

$x \to a$, $x = h + a$

$$\lim_{h \to 0} \frac{\operatorname{cosec}(h+a) - \operatorname{cosec} a}{h + a - a}$$

$$\lim_{h \to 0} \frac{\frac{1}{\sin(h+a)} - \frac{1}{\sin(a)}}{h}$$

$$\lim_{h \to 0} \frac{\sin a - \sin(h+a)}{\sin(h+a) \cdot \sin(a) \times h}$$

$$\lim_{h \to 0} \frac{2 \cos\left(a + \frac{h}{2}\right) \cdot \sin\left(\frac{-h}{2}\right)}{\sin(h+a) \cdot \sin a \times h}$$

$$= 2 \lim_{h \to 0} \frac{\cos\left(a + \frac{h}{2}\right)}{\sin(h+a) \cdot \sin a} \times -\frac{1}{2} \lim_{h \to 0} \frac{\sin\left(-\frac{h}{2}\right)}{-\frac{h}{2}}$$

$$= -\frac{\cos a}{\sin a} \times \sin a$$

$$= -\cot a \cdot \operatorname{cosec} a$$

iii) $\sec x$

$$Df(a) = \lim_{x \to a} \frac{f(x) - f(0)}{x - a}$$

$$\lim_{x \to 0} \frac{\sec x - \sec a}{x - a}$$

Put $x - a = h \quad = a = a + h \quad x \to h$

$h \to 0$

$$\lim_{h \to 0} \frac{\sec(a+h) - \sec a}{h + a - a}$$

$$\lim_{h \to 0} \frac{\frac{1}{\cos(h+a)} - \frac{1}{\cos a}}{h}$$

$$\lim_{h \to 0} \frac{\cos a - \cos(h+a)}{\cos(h+a) \cdot \cos a \times h}$$

$$\lim_{h \to 0} \frac{-2\sin\left(a + \frac{h}{2}\right) \cdot \sin\left(-\frac{h}{2}\right)}{\cos(h+a) \cdot \cos a \times h}$$

$$= \cancel{2} \lim_{h \to 0} \frac{\sin\left(a + \frac{h}{2}\right)}{\cos(h+a) \cdot \cos a} - \frac{1}{\cancel{2}} \lim_{h \to 0} \frac{\sin\left(\frac{-h}{2}\right)}{-\frac{1}{2}}$$

$$= \frac{\sin a}{\cos a} \times \cancel{\cos a}$$

$$= \tan a \cdot \sec a //$$

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Q.2] 38
If $f(x) = 4x+1 \quad x < 2$
$= x^2+5 \quad x > 2$ at $x = 0$ then
find differentiable or not.

$$R.H.D = Df(x^+)$$

$$= \lim_{x \to 2^+} \frac{f(x) - f(a)}{x-a}$$

$$= \lim_{x \to 2^+} \frac{x^2+5-9}{x-2}$$

$$= \lim_{x \to 2^+} \frac{x^2-4}{x-2}$$

$$= \lim_{x \to 2^+} \frac{(x-2)(x+2)}{(x-2)}$$

$$= \lim_{x \to 2^+} x+2$$

$$= 4$$

$$L.H.D = Df(2^-)$$

$$= \lim_{x \to 2^-} \frac{f(x) - f(a)}{x-a}$$

$$= \lim_{x \to 2^-} \frac{4x+1-9}{x-2}$$

$$= \lim_{x \to 2^-} \frac{4x-8}{x-2}$$

$$= \lim_{x \to 2^-} \frac{4\cancel{(x-2)}}{\cancel{(x-2)}}$$

R.H.D = L.H.D.

hence function is differentiable.

3) If $f(x) = 4x + 7 \quad x < 3$
$= x^2 + 3x + 1 \quad x \geq 3$

Then find f is differentiable or not)

R.H.D $= Df(3^+)$

$$= \lim_{x \to 3^+} \frac{f(x) - f(a)}{x - a}$$

$$= \lim_{x \to 3^+} \frac{x^2 + 3x + 1 - 19}{x - 3}$$

$$= \lim_{x \to 3^+} \frac{x^2 + 3x - 18}{x - 3}$$

$$= \lim_{x \to 3^+} \frac{x^2 - 6x - 3x - 18}{x - 3}$$

$$= \lim_{x \to 3^+} \frac{(x+6)\cancel{(x-3)}}{\cancel{(x-3)}}$$

$$= \lim_{x \to 3^+} x + 6.$$

$= 9.$

L.H.D $= Df(3^-)$

$$\lim_{x \to 3^-} \frac{f(x) - f(a)}{x - a}.$$

$$= \lim_{x \to 3^-} \frac{4x+3-19}{x-3}$$

$$= \lim_{x \to 3^-} \frac{4x-12}{x-3}$$

$$= \lim_{x \to 3} \frac{4\cancel{(x-3)}}{\cancel{(x-3)}}$$

$$= 4.$$

R.H.D $\neq$ L.H.D

Function is not differentiable at $x = 3$

4) If $f(x) = 8x - 5 \quad x < 2$
$3x^2 - 4x + 7 \quad x > 2$ } at $x = 2$

find function is differentiable or not

→ R.H.D = Df(2^+)

$$\lim_{x \to 2^+} \frac{f(x) - f(a)}{x-a}$$

$$\lim_{x \to 2^+} \frac{3x^2-4x+7-11}{x-2}$$

$$\lim_{x \to 2^+} \frac{3x^2-4x-4}{x-2}$$

$$\lim_{x \to 2^+} \frac{3x^2-6x+2x-4}{x-2}$$

$$\lim_{x \to 2^+} \frac{3x(x-2) + 2(x-2)}{x-2}$$

$$\lim_{x \to 2^+} \frac{(x-2)(3x+2)}{(x-2)}$$

$$\lim_{x \to 2^+} 3x+2$$

$$= 3 \times 2 + 2$$

$$= 8$$

L.H.D $Df = (2^-)$

$$= \lim_{x \to 2} \frac{f(x) - f(a)}{x-a}$$

$$= \lim_{x \to 2^-} \frac{8x - 5 - a}{x-2}$$

$$= \lim_{x \to 2^-} \frac{8x - 16}{x-2}$$

$$= \lim_{x \to 2} \frac{8\cancel{(x-2)}}{\cancel{(x-2)}}$$

$$= 8$$

R.H.D = L.H.D

hence is differentiable function.

Practical No. 3

Application of Derivative

Q.1] Find the intervals in which functions is increasing or decreasing.

i) $f(x) = x^3 - 5x - 11$ ii) $f(x) = x^2 - 4x$

iii) $f(x) = 2x^3 + x^2 - 20x + 4$ iv) $f(x) = x^3 - 27x + 5$

v) $f(x) = 69 - 24x - 9x^2 + 2x^3$

Q.2) Find the intervals in which function is concave upwards and concave downwards.

i) $y = 3x^2 - 2x^3$ ii) $y = x^4 - 6x^3 + 12x^2 + 5x + 7$

iii) $y = x^3 - 27x + 5$ iv) $y = 69 - 24x - 9x^2 + 2x^3$

v) $y = 2x^3 + x^2 - 20x + 4$

Solutions

Q.1) i) $\rightarrow$ $f(x) = x^3 - 5x - 11$

$f'(x) = 3x^2 - 5$

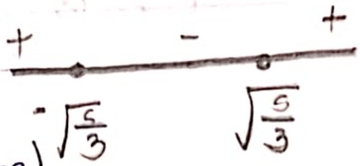
f is increasing iff $f'(x) > 0$

$\therefore 3x^2 - 5 > 0$

$3x^2 > 5$

$x^2 > \frac{5}{3}$

$x > \pm \sqrt{\frac{5}{3}}$



$x \in \left(-\infty, -\sqrt{\frac{5}{3}}\right) \cup \left(\sqrt{\frac{5}{3}}, \infty\right)$

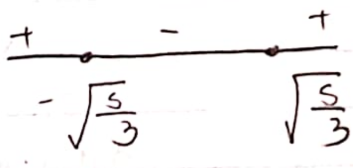
$\therefore$ f is decreasing iff $f'(x) < 0$

$\therefore 3x^2 - 5 < 0$

$3x^2 < 5$

$x^2 < \frac{5}{3}$

$x < \pm \sqrt{\frac{5}{3}}$



$\therefore x \in \left(-\sqrt{\frac{5}{3}}, \sqrt{\frac{5}{3}}\right)$ //

i A

ii) $f(x) = x^2 - 4x$

$f(x) = 2x - 4.$

f is increasing iff $f(x) > 0$

$\therefore 2x - 4 > 0$

$2(x-2) > 0$

$x - 2 > 0$

$\cancel{x > 2}$ $\quad x \in (2, \infty)$

& f is decreasing if $f'(x) < 0$

$\therefore 2x - 4 < 0$

$2(x-2) < 0$

$x - 2 < 0$

$x \in (-\infty, 2)$

iii) $f(x) = 2x^3 + x^2 - 20x + 9$

$f'(x) = 6x^2 + 2x - 20$

$\because f$ is increasing if $f'(x) > 0$

$\therefore 6x^2 + 2x - 20 > 0$

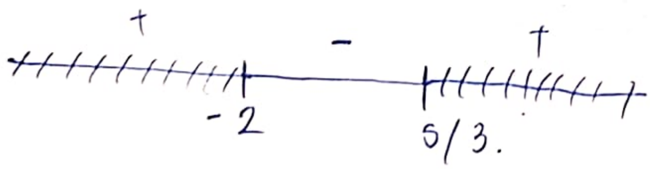
$2(3x^2 + x - 10) > 0$

$3x^2 + x - 10 > 0$

$3x^2 + 6x - 5x - 10 > 0$

$3x(x+2) - 5(x+2) > 0$

$(x+2)(3x-5) > 0.$



$x \in (-\infty, -2) \cup (5/3, \infty)$

& f is decreasing iff $f'(x) < 0$

$\therefore 6x^2 + 2x - 20 < 0$

$2(\cancel{3}x^2 + x - 10) < 0$

$3x^2 + x - 10 < 0$

$3x^2 + 6x - 5x - 10 < 0$

$3x(x+2) - 5(x+2) < 0$

$\therefore (x+2)(3x-5) < 0$

$x \in (-2, 5/3)$

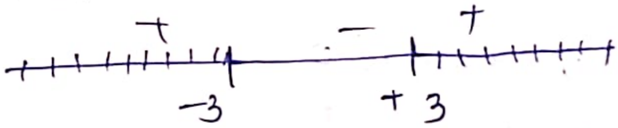
iv) $f(x) = x^3 - 27x + 5$

$f'(x) = 3x^2 - 27$

$\because$ f is increasing if $f'(x) > 0$

$3(x^2 - 9) > 0$

$(x-3)(x+3) > 0$



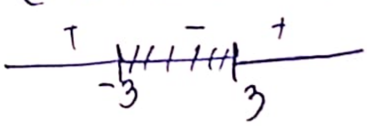
$x \in (-\infty, -3) \cup (3, \infty)$

& f is decreasing if $f'(x) < 0$

$\therefore 3x^2 - 27 < 0$

$3(x^2 - 9) < 0$

$(x+3)(x-3) < 0$



$\therefore x \in (-3, 3)$

v) $f(x) = 2x^3 - 9x^2 - 24x + 69$.

$f'(x) = 6x^2 - 18x - 24$

$\therefore f$ is increasing if $f'(x) > 0$

$\therefore 6x^2 - 18x - 24 > 0$

$6(x^2 - 3x - 4) > 0$

$x^2 - 4x + x - 4 > 0$

$x(x-4) + 1(x-4) > 0$

$(x-4)(x+1) > 0$

$x \in (-\infty, -1) \cup (4, \infty)$

& f is decreasing if $f'(x) < 0$

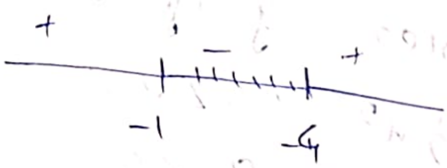
$\therefore 6x^2 - 18x - 24 < 0$

$6(x^2 - 3x - 4) < 0$

$x^2 - 4x + x - 4 < 0$

$x(x-4) + 1(x-4) < 0$

$(x-4)(x+1) < 0$



$x \in (-1, 4)$

2) find the intervals in which function is concave upwards.

i) $y = 3x^2 - 2x^3$

$\therefore f(x) = 3x^2 - 2x^3$

$f'(x) = 6x - 6x^2$

$f''(x) = 6 - 12x$

f is concave upward if $f''(x) > 0$

$(6 - 12x) > 0$

$12(6/12 - x) > 0$

$x - 1/2 > 0$

$x > 1/2$

$\therefore f''(x) > 0$

$x \in (1/2, \infty)$

ii) $y = x^4 - 6x^3 + 12x^2 + 5x + 7$

$f'(x) = 4x^3 - 18x^2 + 24x + 5$

$f''(x) = 12x^2 - 36x + 24$

f is concave upward if $f''(x) > 0$

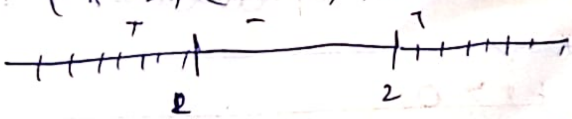
$\therefore 12x^2 - 36x + 24 > 0$

$12(x^2 - 3x + 2) > 0$

$x^2 - 2x - x + 2 > 0$

$x(x-2) + (x-2) > 0$

$(x-2)(x-1) > 0$



$x \in (-\infty, 1) \cup (2, \infty)$

iii) $y = x^3 - 27x + 5$

$f'(x) = 3x^2 - 27$

$f''(x) = 6x$

f is concave upward iff $f''(x) > 0$

$\therefore 6x > 0$

$x > 0$

$x \in (0, \infty)$

iv) $y = 69 - 24x - 9x^2 + 2x^3$

$f(x) = 2x^3 - 9x^2 - 24x + 64$

$f'(x) = 6x^2 - 18x - 24$

$f''(x) = 12x - 18$

f is concave upward iff $f''(x) > 0$

$\therefore 12x - 18 > 0$

$12(x - 18/12) > 0$

$x - 3/2 > 0$

$\therefore x > 3/2$

$\therefore x \in (3/2, \infty)$

v) $y = 2x^3 + x^2 - 20x + 4$

$f(x) = 2x^3 + x^2 - 20x + 4$

$f'(x) = 6x^2 + 12x - 20$

$f''(x) = 12x + 2$

f is concave upward iff $f''(x) > 0$

$f''(x) > 0$

$12x + 2 > 0$

$12(x + 2/12) > 0$

$x + 1/6 > 0$

$x < -1/6$ $\qquad x \in (-1/6, \infty)$

AV
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Practical No. 4

Topic. Application of Derivative & Newton's Method

41

find maxima & minima.

Q.1] find maximum & minimum value of following function.

i) $f(x) = x^2 + \frac{16}{x^2}$ ii) $f(x) = 3 - 5x^3 + 3x^5$.

iii) $f(x) = x^3 - 3x^2 + 1$ is $[-\frac{1}{2}, 4]$

iv) $f(x) = 2x^3 - 3x^2 - 12x + 1$ is $[-2, 3]$

Q.2] find the root of following equation by newton's method (take 4 iteration only) correct upto 4 decimal's.

i) $f(x) = x^3 - 3x^2 - 55x + 9.5$ (take $x_0 = 0$)

ii) $f(x) = x^4 - 4x - 9$ in $[2, 3]$

iii) $f(x) = x^3 - 1.8x^2 - 10x + 17$ in $[1, 2]$

Answers

Q.1]

i] $f(x) = x^2 + \frac{16}{x^2}$.

$f'(x) = 2x - \frac{32}{x^3}$

Now consider,

$f'(x) = 0$

$\therefore 2x - \frac{32}{x^3} = 0$

$2x = \frac{32}{x^3}$

$x^4 = \frac{32}{2}$

$x^4 = 16$

$x = \pm 2$.

$f''(x) = 2 + \frac{96}{x^4}$

$f''(2) = 2 + \frac{96}{2^4}$

$= 2 + \frac{96}{16}$

$= 2 + 6$

$= 8 > 0$

$\therefore$ f has minimum value at $x = 2$

$\therefore f(2) = 2^2 + \frac{16}{2^2}$

$= 4 + \frac{16}{4}$

$= 4 + 4$

$= 8$

$f''(-2) = 2 + \frac{96}{-2^4}$

$= 2 + \frac{96}{16}$

$= 2 + 6$

$= 8 > 0$.

$\therefore$ f has minimum value at $x = 2$.

$\therefore$ function reaches minimum value at $x = 2$ & $x = -2$

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ii) $f(x) = 3 - 5x^3 + 3x^5$

$f'(x) = -15x^2 + 15x^4$

Consider,

$f'(x) = 0$

$-16x^2 + 15x^4 = 0$

$15x^4 = 15x^2$

$x^2 = 1$

$x = \pm 1$

$\therefore f''(x) = -30x + 60x^3$

$f(1) = -30 + 60$

$= 30 > 0$

$\therefore$ f has minium value at $x = 1$

$\therefore f(1) = 3 - 5(1)^3 + 3(1)^5$

$= 6 - 5$

$= 1$

$f''(-1) = -30(-1) + 60(-1)^3$

$= 30 - 60$

$= 30 < 0$

$\therefore$ f has maximum value at $x = -1$

$\therefore f(-1) = 3 - 5(-1)^3 + 3(-1)^5$

$= 3 + 5 \cancel{- 3}$

$= 5$

$\therefore$ f has the maximum value 5 at $x = -1$ & has the minimum value 1 at $x = 1$

No.

iii) $f(x) = x^3 - 3x^2 + 1$

$\therefore f'(x) = 3x^2 - 6x$

Consider,

$f'(x) = 0$

$\therefore 3x^2 - 6x = 0$

$3x(x-2) = 0$

$\therefore 3x = 0$ or $x - 2 = 0$

$\therefore x = 0$ or $x = 2$

$f''(x) = 6x - 6$

$\therefore f''(0) = 6(0) - 6$

$= -6 < 0$

$\therefore f$ has maximum value at $x = 0$

$\therefore f(0) = (0)^3 - 3(0)^2 + 1$

$= 1$

$f''(2) = 6(2) - 6$

$= 12 - 6$

$= 6 > 0$

$\therefore f$ has minimum value at $x = 2$.

$f(2) = (2)^2 - 3(2)^2 + 1$

$= 8 - 3(4) + 1$

$= 9 - 12$

$= -3$

$\therefore f$ has maximum value 1 at $x = 0$ &

f has minimum value -3 at $x = 2$.

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iv) $f(x) = 2x^3 - 3x^2 - 12x + 1$

$f'(x) = 6x^2 - 6x - 12$

Consider,

$f'(x) = 0$

$\therefore 6x^2 - 6x - 12 = 0$

$6(x^2 - x - 2) = 0$

$x^2 - x - 2 = 0$

$x^2 + x - 2x - 2 = 0$

$x(x+1) - 2(x+1) = 0$

$(x-2)(x+1) = 0$

$\therefore x = 2$ or $x = -1$

$\therefore f''(x) = 12x - 6$

$\therefore f''(2) = 12(2) - 6$

$= 24 - 6$

$= 18 > 0$

$\therefore f$ has minimum value at $x = 2$

$\therefore F(2) = 2(2)^3 - 3(2)^2 - 12(2) + 1$

$= 2(8) - 3(4) - 24 + 1$

$= 16 - 12 - 24 + 1$

$= -19$

$f''(-1) = 12(-1) - 6$

$= -12 - 6$

$= -18 < 0$

$\therefore f$ has maximum value at $x = -1$

$\therefore f(-1) = 2(-1)^3 - 3(-1)^2 - 12(-1) + 1$

$= -2 - 3 + 12 + 1$

$= 8.$

$\therefore f$ has maximum value 8 at $x = -1$ &

f has minimum value -19 at $x = 2$.

Q.2] Ans.

i] $f(x) = x^3 - 3x^2 - 55x + 9.5$ $\quad x_0 = 0 \rightarrow$ given

$f'(x) = 3x^2 - 6x - 55$

By Newton's Method,

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$\therefore x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

$$\therefore x_1 = 0 + \frac{9.5}{55}$$

$$\therefore x_1 = 0.1727$$

$$\therefore f(x_1) = (0.1727)^3 - 3(0.1727)^2 - 55(0.1727) + 9.5$$

$$= 0.0051 - 0.0895 - 9.4985 + 9.5$$

$$= -0.0829.$$

$$f'(x_1) = 3(0.1727)^2 - 6(0.1727) - 55$$

$$= 0.0895 - 1.0362 - 55$$

$$= -55.9467$$

$$\therefore x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

$$= 0.1727 - \frac{0.0829}{55.9467}$$

$$= 0.1712.$$

$$f(x_2) = (0.1712)^3 - 3(0.1712)^2 - 55(0.1712) + 9.5.$$

$$= 0.0050 - 0.0879 - 9.416 + 9.5$$

$$= 0.0011$$

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$f'(x_2) = 3(0.1712)^2 - 6(0.1712) \cdot 55$

$= 0.0879 - 1.0272 - 55$

$= -55.9393$

$\therefore x_3 = x_2 - \frac{f(x_2)}{f'(x_2)}$

$= 0.1712 + \frac{0.0011}{55.9393}$

$= 0.1712.$

$\therefore$ The root of the equation is 0.1712.

ii) $f(x) = x^3 - 4x - 9$ $[2, 3]$

$f'(x) = 3x^2 - 4$

$f(2) = 2^3 - 4(2) - 9$

$= 8 - 8 - 9$

$= -9.$

$f(3) = 3^3 - 4(3) - 9$

$= 27 - 12 - 9$

$= 6$

Let $x_0 = 3$ be the initial approximation

$\therefore$ By Newton's method,

$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$

$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$

$= 3 - \frac{6}{23}$

$= 2.7392.$

$$f(x_1) = (2.7392)^3 - 4(2.7392) - 9$$
$$= 20.5528 - 10.9568 - 9$$
$$= 0.596.$$
$$f'(x_1) = 3(2.7392)^2 - 4.$$
$$= 22.5096 - 4$$
$$= 18.5096$$
$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$
$$= 2.7392 - \frac{0.596}{18.5096}.$$
$$= 2.7071$$
$$f(x_2) = (2.7071)^3 - 4(2.7071) - 9$$
$$= 19.8386 - 10.8284 - 9$$
$$= 0.0102.$$
$$f'(x_2) = 3(2.7071)^2 - 4$$
$$= 21.9851 - 4$$
$$= 17.9851.$$
$$\therefore x_3 = x_2 - \frac{f(x_2)}{f'(x_2)}$$
$$= 2.7071 - \frac{0.0102}{17.9851}$$
$$= 2.7071 - 0.0056$$
$$= 2.7015$$
$$f(x_3) = (2.7015)^3 - 4(2.7015) - 9$$
$$= 19.7158 - 10.806 - 9.$$
$$= -0.0901$$
$$f'(x_3) = 3(2.7015)^2 - 4$$
$$= 21.8943 - 4$$
$$= 17.8943.$$

iv) $f(x) = x^3 - 1.8x^2 - 10x + 17$ $[1, 2]$

$f'(x) = 3x^2 - 3.6x - 10.$

$f(1) = (1)^3 - 1.8(1)^2 - 10(1) + 17.$

$= 1 - 1.8 - 10 + 17$

$= 6.2.$

$f(2) = (2)^3 - 1.8(2)^2 - 10(2) + 17$

$= 8 - 7.2 - 20 + 17$

$= -2.2$

let $x_0 = 2$ be initial approximation By Newton's method,

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$x_1 = x_0 - \frac{f(x_2)}{f'(x_2)}$$

$$= 2 - \frac{2.2}{5.2}$$

$= 2 - 0.4230$

$= 1.577.$

$f(x_1) = (1.577)^3 - 1.8(1.577)^2 - 10(1.577) + 17.$

$= 3.9219 - 4.4764 - 15.77 + 17$

$= 0.6755.$

$f'(x_1) = 3(1.577)^2 - 3.6(1.577) - 10.$

$= 7.4608 - 5.6772 - 10$

$= -8.2164.$

$$\therefore x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

$$= 1.577 + \frac{0.6755}{0.2164}$$

$= 1.577 + 0.0822.$

$= 1.6592.$

8.

$f(x_2) = (1.6592)^3 - 1.8\ (1.6592)^2 - 10(1.6592) + 17.$

$= 4.5677 - 4.9553 - 16.592 + 17$

$= 0.0204$

$f'(x_2) = 3(1.6592)^2 - 3.6(1.6592) - 10$

$= 8.2588 - 5.97312 - 10$

$= -7.7143$

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)}$$

$$= 1.6592 + \frac{0.0204}{7.7143}$$

$= 1.6592 + 0.0026$

$= 1.6618$

$x_4\ f(x_3) = (1.6618)^3 - 1.8(1.6618)^2 - 10(1.6618) + 17$

$= 4.5892 - 4.9708 - 16.618 + 17$

$= 0.0004.$

$f'(x_3) = 3(1.6618)^2 - 3.6(1.6618) - 10.$

$= 8.2847 - 5.9824 - 10$

$= -7.6977$

$$x_4 = x_3 - \frac{f(x_3)}{f'(x_3)}$$

$$= 1.8618 + \frac{0.0004}{7.6977}$$

$= 1.6618$

The root of equation is 1.6618

Practical - 5.

i] $\int \frac{dx}{\sqrt{x^2+2x-3}}$

$$I = \int \frac{dx}{\sqrt{x^2+2x-3}}$$

$$= \int \frac{dx}{\sqrt{x^2+2x+1-4}}$$

$$= \int \frac{dx}{\sqrt{(x+1)^2-(2)^2}}.$$

Comparing with $\int \frac{dx}{\sqrt{x^2-a^2}}$; $x^2 = (x+1)^2$, $a^2 = (2)^2$.

$$I = \log |x + \sqrt{x^2-a^2}| + C$$

$$= \log |x+1 + \sqrt{(x+1)^2-(2)^2}| + C.$$

ii) $\int (4e^{3x}+1)\, dx$

$$I = \int (4e^{3x}+1)\, dx.$$

$$= \int 4e^{3x}\, dx + \int 1\, dx.$$

$$= \frac{4e^{3x}}{3} + x + C$$

iii) $\int (2x^2 - 3\sin x + 5\sqrt{x})\, dx$

$$I = \int (2x^2 - 3\sin x + 5\sqrt{x})\, dx$$

$$= 2\int x^2 dx - 3\int \sin x\, dx + 5\int \sqrt{x}\, dx.$$

$$= \frac{2}{3}x^3 + 3 \cos x + 5 \times \frac{2}{3} x^{3/2} + C.$$

$$= \frac{2}{3}x^3 + 3\cos x + \frac{10x^{3/2}}{3} + C.$$

iv) $\int \frac{x^3+3x+4}{\sqrt{x}}\, dx$

$$I = \int \frac{x^3+3x+4}{\sqrt{x}}\, dx$$

$$= \int \left(\frac{x^3}{\sqrt{x}} + \frac{3x}{\sqrt{x}} + \frac{4}{\sqrt{x}}\right) dx$$

$$= \int \left(\frac{x^3}{x^{1/2}} + \frac{3x}{x^{1/2}} + \frac{4}{x^{1/2}}\right) dx$$

$$= \int (x^3 x^{-1/2} + 3x \cdot x^{-1/2} + 4x^{-1/2}) dx.$$

$$= \int x^{5/2} dx + 3 \int x^{1/2} dx + 4 \int x^{-1/2} dx$$

$$= \frac{2}{7} x^{7/2} + 3 \cdot \frac{2}{3} x^{3/2} + 4 x^{1/2} \cdot 2 + C.$$

$$= \frac{2}{7} x^{7/2} + 2x^{3/2} + 8\sqrt{x} + C.$$

v) $\int t^7 \sin(2t^4)\, dt.$

$$I = \int t^7 \sin(2t^4)\, dt.$$

let, $t^4 = x$

$4t^3 dt = dx$

$I = \frac{1}{4}\int 4t^3 \cdot t^4 \sin(2t^4)\,dt$

$= \frac{1}{4}\int x \cdot \sin(2x)\,dx$

$= \frac{1}{4}\left[x \int \sin 2x - \int\left[\int \sin 2x \cdot \frac{d}{dx}(x)\right]\right]$

$= \frac{1}{4}\left[-\frac{x\cos 2x}{2} + \frac{1}{2}\int \cos 2x \cdot 1\right]$

$= \frac{1}{4}\left[-\frac{x\cos 2x}{2} + \frac{1}{4}\sin 2x\right] + C$

$= -\frac{1}{8}x\cos 2x + \frac{1}{16}\sin 2x + C$

Resubstituting $x = t^4$

$\therefore I = \frac{1}{8}t^4\cos(2t^4) + \frac{1}{16}\sin(2t^4) + C.$

vi) $\int \sqrt{x}\,(x^2-1)\,dx$

$I = \int \sqrt{x}\,(x^2-1)\,dx$

$= \int(\sqrt{x}\cdot x^2 - \sqrt{x})\,dx$

$= \int(x^{5/2} - \sqrt{x})\,dx$

$= \int x^{5/2}\,dx - \int \sqrt{x}\,dx.$

$= \frac{2}{7}x^{7/2} - \frac{2}{3}x^{3/2} + C$

vii) $\int \frac{1}{x^3}\sin\left(\frac{1}{x^2}\right)dx$

$I = \int \frac{1}{x^3}\sin\left(\frac{1}{x^2}\right)dx.$

Let

$\frac{1}{x^2} = t.$

$\therefore x^{-2} = t.$

$\therefore \frac{-2}{x^3}dx = dt.$

02

$$I = \frac{1}{-2}\int \frac{-2}{x^3} \sin\left(\frac{1}{x^2}\right) dx$$

$$= -\frac{1}{2}\int \sin t \, dt$$

$$= -\frac{1}{2}\left[-\cos t\right] + C$$

$$= \frac{1}{2}\cos t + C$$

Resubstituting $t = \frac{1}{x^2}$

$$\therefore I = \frac{1}{2}\cos\left(\frac{1}{x^2}\right) + C$$

viii) $\int \frac{\cos x}{\sqrt[3]{\sin^2 x}}$

$$I = \int \frac{\cos x}{\sqrt[3]{\sin^2 x}} dx$$

let $\sin x = t$

$\cos x \, dx = dt$

$$\therefore I = \int \frac{dt}{\sqrt[3]{t^2}}$$

$$\therefore I = \int \frac{dt}{t^{2/3}}$$

$$= \int t^{-2/3} dt$$

$$= 3t^{1/3} + C$$

$$= 3(\sin x)^{1/3} + C$$

$$= 3\sqrt[3]{\sin x} + C$$

ix) $\int e^{\cos^2 x} \sin 2x \, dx.$

$$I = \int e^{\cos^2 x} \sin 2x \, dx$$

let $\cos^2 x = t$

$-2\cos x \sin x \, dx = dt$

$-2\sin 2x \, dx = dt$

$I = -\int -\sin 2x \, e^{\cos^2 x} dx.$

$= \int e^t dt$

$= -e^t + C$

Resubstituting $t = \cos^2 x$

$\therefore I = -e^{\cos^2 x} + C$

x] $\int \left(\frac{x^2 - 2x}{x^3 - 3x^2 + 1} \right) dx$

$I = \int \left(\frac{x^2 - 2x}{x^3 - 3x^2 + 1} \right) dx$

let,

$x^3 - 3x^2 + 1 = t$

$\therefore (3x^2 - 6x) dx = dt$

$3(x^2 - 2x) dx = dt$

$(x^2 - 2x) dx = \frac{dt}{3}$

AV 03/03/2022

$\therefore I = \int \frac{1}{t} \frac{dt}{3}$

$= \frac{1}{3} \int \frac{dt}{t}$

$= \frac{1}{3} \log t + C$

Resubstituting $t = x^3 - 3x^2 + 1$,

$\therefore I = \frac{1}{3} \log(x^3 - 3x^2 + 1) + C$.

Practical - 6

1)

$$L = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt.$$

$$x = t - \sin t \quad \therefore \frac{dx}{dt} = 1 - \cos t$$

$$y = 1 - \cos t \quad \therefore \frac{dy}{dt} = 0 - (-\sin t) = \sin t.$$

$$L = \int_0^{2\pi} \sqrt{(1 - \cos t + \sin t)^2 + (\sin t)^2}\, dt$$

$$= \int_0^{2\pi} \sqrt{1 - 2\cos t + 1}\, dt$$

$$= \int_0^{2\pi} \sqrt{2 - 2\cos t}\, dt = \int_0^{2\pi} \sqrt{2(1 - \cos t)} = \int_0^{2\pi} \sqrt{2 \times 2\sin^2 t/2}$$

$$= \sqrt{4\sin^2 t/2}.$$

$$= \int_0^{2\pi} 2 \cdot \left|\sin \frac{t}{2}\right| dt \quad \therefore \sin^2 \frac{t}{2} = \frac{1 - \cos t}{2}.$$

$$= \int_0^{2\pi} 2 \sin \frac{t}{2}\, dt$$

$$= \left(-4\cos\left(\frac{t}{2}\right)\right)_0^{2\pi} = (-4\cos \pi) - (-4\cos 0)$$

$$= 4 + 4$$

$$= 8 //.$$

ii) $y = \sqrt{4-x^2}$ $x \in [-2,2]$

$$l = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt.$$

$y = \sqrt{4-x^2}$ $\therefore \frac{dy}{dx} = 2\int_0^2 1 + \left(\frac{-x}{\sqrt{4-x^2}}\right)^2 dx.$

$$= 2\int_0^2 \sqrt{\frac{1+x^2}{4-x^2}}\, dx$$

$$= 4\int_0^2 \frac{1}{\sqrt{4-x^2}}\, dx.$$

$$= 4\left(\sin^{-1}(x/2)\right)_0^2$$

$$= 2\pi.$$

iii) $y = x^{3/2}$ in $[0,4]$

$f'(x) = \frac{3}{2}x^{1/2}$

$[f'(x)]^2 = \frac{9}{4}x.$

$$L = \int_0^b \sqrt{1+[f'(x)]^2}\, dx.$$

$$= \int_0^4 \sqrt{1+\frac{9}{4}x}\, dx.$$

$$= \int_0^4 \sqrt{\frac{4+9x}{4}}\, dx.$$

$$= 1/2 \int_0^4 \sqrt{4+9x}\, dx$$

$$= \frac{1}{2}\left[\frac{(4+9x)^{1/2+1}}{1/2+1}\right]_0^4 dx.$$

$$= \frac{1}{27}\left[(4+9x)^{3/2}\right]_0^4.$$

$$= \frac{1}{27}\left[(4+0)^{3/2} - (4+36)^{3/2}\right]$$

$$= \frac{1}{27}\,(4)^{3/2} - (40)^{3/2}]$$

iv) $x = 3\sin t \qquad y = 3\cos t$

$$\frac{dx}{dt} = 3\cos t \qquad \frac{dy}{dt} = -3\sin t$$

$$L = \int_0^{2\pi} \sqrt{(3\cos t)^2 + (-3\sin)^2}\, dt.$$

$$= \int_0^{2\pi} \sqrt{9\sin^2 t + 9\cos^2 t}\, dt$$

$$= \int_0^{2\pi} \sqrt{9(\sin^2 t + \cos^2 t)}\, dt$$

$$= \int_0^{2\pi} \sqrt{9(1)}\, dt$$

$$= \int_0^{2\pi} 3\, dt = 3\int_0^{2\pi} dt = 3[x]_0^{2\pi}$$

$$= 3(2\pi - 0)$$

$$= 6\pi.$$

v) $x = \frac{1}{6}y^3 + \frac{1}{2y} \quad y = (1, 2)$

$$\frac{dx}{dy} = \frac{y^2}{2} - \frac{1}{2y^2} = \frac{y^4 - 1}{2y^2}.$$

$$= \int_1^2 \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy$$

$$= \int_1^2 \sqrt{\frac{(y^4+1)^2}{(2y)^2}}\, dy.$$

$$= \int_1^2 \frac{y^4+1}{2y^2}\, dy.$$

$$= \frac{1}{2}\int_1^2 y^2\, dy + \frac{1}{2}\int_1^2 y^{-2}\, dy$$

$$= \frac{1}{2}\left[\frac{y^3}{3} - \frac{y^{-2}}{1}\right]$$

$$= \frac{1}{2}\left[\frac{8}{3} - \frac{1}{2} - \frac{1}{3} + 1\right]$$

$$= \frac{1}{2}\left[\frac{7}{3} + \frac{1}{2}\right] = \frac{17}{12},$$

Q.2) i) $\int_0^2 e^{x^2}\, dx$ with $n=4$

$a=0$, $b=2$, $n=4$

$h = \frac{2-0}{4} = \frac{1}{2} = 0.5$

x	0	0.5	1	1.5	2
y	1	1.2840	2.7182	9.4877	54.5981
	y_0	y_1	y_2	y_3	y_4

By Simpson's Rule.

$$\int_0^2 e^{x^2}\, dx = \frac{0.5}{3}\left[(1+54.5981)+4(1.2840+9.4877)+2(2.7182+54.5981)\right]$$

$$= \frac{0.5}{3}\left[55.5981+43.0868+114.6326\right]$$

$$= 1.1779.$$

ii) $\int_0^4 x^2\, dx$

$L = \frac{4-0}{4} = 1$

x	0	1	2	3	4
y	0	1	4	9	16

$$\int_0^4 x^2\, dx = \frac{1}{3}\left[16+4(10)+8\right] = 64/3$$

$$\int_0^4 x^2\, dx = 21.533$$

iii) $\int_0^{\pi/3} \sqrt{\sin x}\, dx \quad n=6.$

$$h = \frac{\pi/3 - 0}{6} = \frac{\pi}{18}.$$

x	0	$\pi/18$	$2\pi/18$	$3\pi/18$	$4\pi/18$	$6\pi/18$	$8\pi/18$
y	0	0.4166	0.50	0.70	0.8004	0.8729	0.99(?)

$$\int_0^{\pi/3} \sqrt{\sin x} = \pi/54 \times 12.1163$$

$$= \int_0^{\pi/3} \sqrt{\sin x}\, dx = 0.7049.$$

Practical No. 7

Differential Equation

Q.1] i) $x\frac{dy}{dx} + y = e^x$

$\frac{dy}{dx} + \frac{1}{x}y = \frac{e^x}{x}$

$P(x) = \frac{1}{x}$ $\quad Q(x) = \frac{e^x}{x}$.

$I.F = e^{\int P(x)\,dx}$

$= e^{\int 1/x\,dx}$

$= e^{\ln x}$

$I.F = x$.

$y(I.F) = \int Q(x)(I.F)\,dx + c$

$= \int \frac{e^x}{x} \cdot x \cdot dx + c$

$= \int e^x dx + c$

$xy = e^x + c$

Q.2) $e^x\frac{dy}{dx} + 2e^x y = 1$

$\frac{dy}{dx} + \frac{2\cancel{e^x}}{\cancel{e^x}}y = \frac{1}{e^x}$ $\quad (\div \text{ by } e^x)$

$\frac{dy}{dx} + 2y = \frac{1}{e^x}$

$\frac{dy}{dx} + 2y = e^{-x}$.

5.1

$P(x) = 2$ $\qquad Q(x) = e^{-x}$

$\int P(x)\,dx$

$I.F = e^{\int 2dx}$

$= e^{2x}$

$y \cdot (I.F) = \int Q(x)(I.F)\,dx + c$

$y \cdot e^{2x} \int e^{-x} + 2x\,dx + c$

$= \int e^{x}\,dx + c$

$y \cdot e^{2x} = e^{x} + c.$

iii) $x\frac{dy}{dx} = \frac{\cos x}{x} - 2y.$

$x\frac{dy}{dx} = \frac{\cos x}{x} - 2y$

$\therefore \frac{dy}{dx} + \frac{2y}{x} = \frac{\cos x}{x^2}$

$P(x) = 2(x)$ $\qquad Q(x) = \frac{\cos x}{x^2}$

$I.F = e^{\int P(x)dx}$

$= e^{\int 2/x\,dx}$

$= e^{2\ln x}$

$= \ln x^2$

$I.F = x^2$

$y(I.F) = \int Q(x)(I.F)\,dx + c$

$y = \int \frac{\cos x}{x^2} \cdot x^2\,dx + c$

$= \int \cos x + c$

$x^2 y = \sin x + c.$

iv) $x\frac{dy}{dx} + 3y = \frac{\sin x}{x^2}$.

$\frac{dy}{dx} + \frac{3y}{x} = \frac{\sin x}{x^3}$ ($\div$ by x on both sides)

$P(x) = 3/x$ $Q(x) = \sin x/x^3$

$= e^{\int P(x)dx}$

$= e^{\int 3/x\, dx}$

$= e^{3/x}\, x$

$= e^{\ln x^3}$

$I \cdot F = x^3$.

5) $x = \frac{1}{6}y^3 + \frac{1}{2y}$

$\frac{dx}{dy} = \frac{y^2}{2} = \frac{1}{2y^2}$

$\frac{dx}{dy} = \frac{y^4-1}{2y^2}$

$I = \int_0^2 \sqrt{1+\left(\frac{dx}{dy}\right)^2}\, dy$

$= \int_0^2 \sqrt{1+\frac{(y^4-1)^2}{4y^2}}\, dy$

$= \int_0^2 \sqrt{\frac{(y^4-1)+4\times y^4\times 1}{4y^4}}\, dy$.

$= \cancel{\int_0^2 \sqrt{\frac{(y^4-1)+4\times y^4\times 1}{4y^4}}}\, dy$

$$= \int_0^2 \sqrt{\frac{(y^4+1)^2}{(2y^2)^2}}\, dy$$

$$= \int_0^2 \frac{y^4+1}{2y^2}\, dy.$$

$$= \frac{1}{2}\int_1^2 y^2\, dy + \frac{1}{2}\int_1^2 y^{-2}\, dy$$

$$= \frac{1}{2}\left[\frac{y^3}{3} - \frac{y^{-1}}{1}\right]_1^2$$

$$= \frac{1}{2}\left[\frac{8}{3} - \frac{1}{2} - \frac{1}{3} + 1\right]$$

$$= \frac{1}{2}\left[\frac{17}{16}\right]$$

$$= \frac{17}{12} \text{ unit.}$$

5) $$e^{2x}\frac{dy}{dx} + 2e^{2x}y = 2x$$

$$\frac{dy}{dx} + 2y = \frac{2x}{e^{2x}}.$$

$P(x) = 2 \quad Q(x) = 2e/e^{2x} = 2xe^{-2x}$

$$I.F = e^{\int P(x)dx}$$

$$= e^{\int 2dx}$$

$$= e^{2x}.$$

$$y(I.F) = \int Q(x)(I.F)\, dx + c$$

$$= \int 2xe^{-2x}e^{2x}\, dx + c$$

$$= \int 2x\, dx + c$$

$$ye^{2x} = x^2 + C.$$

6) $\sec^2 x \cdot \tan y \, dx + \sec^2 y \tan x \, dy = 0$

$\sec^2 x \cdot \tan y \, dx = -\sec^2 y \cdot \tan x \, dy.$

$$\frac{\sec^2 x \, dx}{\tan x} = -\frac{\sec^2 y \, dy}{\tan y}.$$

$$\int \frac{\sec^2 x \, dx}{\tan x} = -\int \frac{\sec^2 y \, dy}{\tan y}.$$

$\log|\tan x| = -\log|\tan y| + c.$

$\therefore \log|\tan x| = -\log|\tan y| + c$

$\log|\tan x - \tan y| = c$

$\tan x \cdot \tan y = e^c.$

7) $\frac{dy}{dx} = \sin^2(x - y + 1)$

Put $x - y + 1 = v$

Differentiating on both sides.

$x - y + 1 \cdot v.$

$1 - \frac{dy}{dx} = \frac{dv}{dx}$

$\frac{1 - dv}{dx} = \frac{dy}{dx}$

$\cancel{1 - \frac{dv}{dx}} = \sin^2 v.$

$\frac{dv}{dx} = 1 - \sin^2 v$

$\frac{dv}{dx} = \cos^2 v$

$\frac{dv}{\cos^2 v} = dx.$

$\int \sec^2 v \, dv = \int dx$

$\tan v = x + c$

$\tan(x + y - 1) = x + c$

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8) $\frac{dy}{dx} = \frac{2x+3y-1}{6x+9y+6}$

Put $2x+3y = v$

$2 + 3\frac{dy}{dx} = \frac{dv}{dx}$

$\frac{dy}{dx} = \frac{1}{3}\left(\frac{dv}{dx} - 2\right)$

$\frac{1}{3}\left(\frac{dv}{dx} - 2\right) = \frac{1}{3}\left(\frac{v-1}{v+2}\right)$

$\frac{dv}{dx} = \frac{v-1}{v+2} + 2$

$\frac{dv}{dx} = \frac{v-1+2v+4}{v+2}$

$= \frac{3v+3}{v+2}$

$= \frac{3(v+1)}{v+2}$

$\int\left(\frac{v+2}{v+1}\right) dv = 3\,dx$

$= \int \frac{v+1}{v}\,dx + \int \frac{1}{v+1}\,dv = 3x$

$v + \log|x| = 3x + c$

$2x + 3y + \log|2x+3y+1| = 3x + c$

$3y = x - \log|2x+3y+1| + c_4$.

AK
10/01/2020

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Practical-8

Topic: Euler's Method

1) $\frac{dy}{dx} = y + e^{x} - 2$, $y(0) = 2$, $h = 0.5$, find $y(2)$

2) $\frac{dy}{dx} = 1 + y^2$, $y(0) = 0$, $h = 0.2$. find $y(1)$

3) $\frac{dy}{dx} = \sqrt{\frac{x}{y}}$, $y(0) = 1$, $h = 0.2$ find $y(1)$

4) $\frac{dy}{dx} = 3x^2 + 1$, $y(1) = 2$, find $y(2)$ for $h = 0.5$, $h = 0.25$

5) $\frac{dy}{dx} = \sqrt{xy} + 2$ $y(1) = 1$ find $y(1.2)$ with $h = 0.2$

Answers.

① $\frac{dy}{dx} = y + e^x - 2$

$f(x,y) = y + e^x - 2$, $y_0 = 2$, $x = 0$, $h = 0.5$.

n	x_n	y_n	$f(x_n, y_n)$	y_{n+1}
0	0	2	1	2.5
1	0.5	2.5	2.487	3.57435
2	1	3.5743	4.2925	5.3615.

$y_{n+1} = y_n + h\, f(x_n, y_n)$ //.

n	x_n	y_n	$f(x_n, y_n)$	y_{n+1}
3	1.5	5.3615	7.8431	9.28305
4	2	9.2831		

∴ By Euler's formula,

$y(2) = 9.2831$

② $\frac{dy}{dx} = 1 + y^2$

$f(x,y) = 1 + y^2$, $y_0 = 0$, $x_0 = 0$, $h = 0.2$

using Euler's iteration formula.

$y_{n+1} = y_n + h\ f(x_n, y_n)$

n	x_n	y_n	$f(x_n, y_n)$
0	0	0	1
1	0.2	0.2	1.04
2	0.4	0.408	1.1665
3	0.6	0.6413	1.4113
4	0.8	0.9236	1.8530
5	1	1.2942	

$\therefore$ By Euler's formula,

$y(1) = 1.2942$

3) $\frac{dy}{dx} = \sqrt{\frac{x}{y}}$ $\quad y(0)=1 \quad x_0 = 0, \ h = 0.2$

using Euler's iteration formula,

$y_{n+1} = y_n + h\ f(x_n, y_n)$

n	x_n	y_n	$f(x_n, y_n)$
0	0	1	0
1	0.2	0	
2	0.4		
3	0.6		
4	0.8		
5	1		

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(c) $\frac{dy}{dx} = 3x^2+1$, $y_0 = 2$, $x_0 = 1$, $h = 0$.

for $h = 0.5$

using Euler's iteration formula,

$$y_{n+1} = y_n + h\, f(x_n, y_n)$$

n	x_n	y_n	$f(x_n, y_n)$	y_{n+1}
0	1	2	4	4
1	1.5	4	49	28.5
2	2	28.5		

∴ By Euler's formula,

$y(2) = 28.5$

for $h = 0.25$

n	x_n	y_n	$f(x_n, y_n)$	y_{n+1}
0	1	2	4	3
1	1.25	3	5.6875	4.4219
2	1.5	4.4219	7.75	6.3594
3	1.75	6.3594	10.1815	8.9048
4	2	8.9048		

∴ By Euler's formula

$y(2) = 8.9048$.

(9) $\frac{dy}{dx} = \sqrt{xy} + 2$ $\quad y_0 = 1, x_0 = 1, h = 0.2$

using Euler's iteration formula,

$$y_{n+1} = y_n + h\, f(x_n, y_n)$$

n	x_n	y_n	$f(x_n, y_n)$
0	1	1	3
1	1.2	1.6	

$\therefore$ By Euler's formula,

$$y(1.2) = 1.6$$

12/01/2020

Practical - 9

Limit and Partial order derivative.

Q.1]

i) $\lim_{(x,y)\to(-4,-1)} \dfrac{x^3-3y+y^2-1}{xy+5}$

At $(-4,-1)$ Denominator $\neq 0$

$\therefore$ By applying limit.

$= \dfrac{(-4)^3-3(-1)+(-1)^2-1}{-4(-1)+5}$

$= \dfrac{-64+3+1-1}{4+5}$

$= \dfrac{-61}{9}$

ii) $\lim_{(x,y)\to(2,0)} \dfrac{(y+1)(x^2+y^2-4x)}{x+3y}$

At $(2,0)$, Denominator $\neq 0$

$\therefore$ By applying limit,

$= \dfrac{(0+1)\ ((2)^2+0-4(2))}{2+0}$

$= \dfrac{1\ (4+0-8)}{2}$

$= \dfrac{-4}{2}$

$= -2$

ii) $\lim_{(x,y,z)\to(1,1,1)} \frac{x^2-y^2z^2}{x^3-x^2yz}$

At (1,1,1), Denominator = 0

$\therefore \lim_{(x,y,z)\to(1,1,1)} \frac{x^2-y^2z^2}{x^3-x^2yz}$

$= \lim_{(x,y,z)\to(1,1,1)} \frac{(x+yz)\cancel{(x-yz)}}{x^2\cancel{(x-yz)}}$

$= \lim_{(x,y,z)\to(1,1,1)} \frac{x+yz}{x^2}$

On Applying limit

$= \frac{1+1(1)}{(1)^2}$

$= 2$

Q.2] i) $f(x,y) = xye^{x^2+y^2}$

$\therefore f_x = y(1\cdot e^{x^2+y^2}) + xy(e^{x^2+y^2}\cdot 2x)$

$= ye^{x^2+y^2} + 2x^2y\,e^{x^2+y^2}$

$f_y = x(1\cdot e^{x^2+y^2}) + xy(e^{x^2+y^2}\cdot 2y)$

$= x\cdot e^{x^2+y^2} + 2xy^2 e^{x^2+y^2}$

$\therefore f_x = ye^{x^2+y^2} + 2x^2ye^{x^2+y^2}$

$f_y = xe^{x^2+y^2} + 2xy^2e^{x^2+y^2}$,

ii) $f(x,y) = e^x \cos y$

$f_x = \cos y e^x$

$f_y = e^x - \sin y$

$\therefore f_y = -\sin y e^x$

iii) $f(x,y) = x^3y^2 - 3x^2y + y^3 + 1$

$f_x = y^2 3x^2 - 3y\, 2x + 0 + 0$

$= 3x^2y^2 - 6xy$

$f_y = x^3 2y - 3x^2 + 3y^2$

$= 2x^3y - 3x^2 + 3y^2$

Q.3) Using definition find values of f_x, f_y at $(0,0)$ for $f(x,y) = \dfrac{2x}{1+y^2}$

$$f_x(a,b) = \lim_{h \to 0} \frac{f(a+h,b) - f(a,b)}{h}$$

$$f_y(a,b) = \lim_{h \to 0} \frac{f(a,b+h) - f(a,b)}{h}$$

According to given $(a,b) = (0,0)$.

$$f_x(0,0) = \lim_{h \to 0} \frac{f(h,0) - f(0,0)}{h}$$

$$= \lim_{h \to 0} \frac{2h - 0}{h} = 2$$

$$f_y(0,0) = \lim_{h \to 0} \frac{f(0,h) - f(0,0)}{h}$$

$$= \lim_{h \to 0} \frac{0-0}{h} = 0.$$

$\therefore f_x = 2 \quad , f_y = 0$

Q.4

i) $F(x,y) = \frac{y^2 - xy}{x^2}$

$$F_x = \frac{x^2 \frac{\partial}{\partial x}(y^2 - xy) - (y^2 - xy)\frac{\partial}{\partial x}(x^2)}{(x^2)^2}$$

$$= \frac{x^2(-y) - (y^2 - xy)(2x)}{x^4}$$

$$= \frac{-x^2 y - 2x(y^2 - xy)}{x^4}$$

$$f_y = \frac{2y - x}{x^2}$$

$$F_{xx} = \frac{\partial}{\partial x}\left(\frac{-x^2 y - 2x(y^2 - xy)}{x^4}\right)$$

$$= \frac{x^4\left(\frac{\partial}{\partial x}(-x^2 y - 2xy^2 + 2x^2 y)\right) - (-x^2 y - 2xy + 2x^2 y)\frac{\partial}{\partial x}(x^4)}{(x^4)^2}$$

$$= \frac{x^4(-2xy - \cancel{2y^2 + 4xy}) - 4x^3(-x^2 y - 2xy + 2x^2 y)}{x^6} \quad \text{—(i)}$$

$$f_{yy} = \frac{\partial}{\partial y}\left(\frac{2y - x}{x^2}\right)$$

$$= \frac{2 - 0}{x^2} = \frac{2}{x^2} \quad \text{—(2)}$$

$$f_{xy} = \frac{\partial}{\partial y}\left(\frac{-x^2 y - 2xy^2 + 2x^2 y}{x^4}\right)$$

$$= \frac{-x^2 - 4xy + 2x^2}{x^4} \quad \text{—(3)}$$

$$f_{yx} = \frac{\partial}{\partial x}\left(\frac{2y-x}{x^2}\right)$$

$$= \frac{x^2 \frac{d}{dx}(2y-x) - (2y-x)\frac{d}{dx}(x^2)}{(x^2)^2}$$

$$= \frac{-x^2 - 4xy + 2x^2}{x^4} \quad \text{—(4)}$$

$\therefore$ from ③ & ④

$f_{xy} = f_{yx}$.

ii) $f(x,y) = x^3 + 3x^2y^2 - \log(x^2+1)$

$$f_x = \frac{d}{dx}(x^3 + 3x^2y^2 - \log(x^2+1))$$

$$= 3x^2 + 6xy^2 - \frac{2x}{x^2+1}$$

$$f_y = \frac{\partial}{\partial y}(x^3 + 3x^2y^2 - \log(x^2+1))$$

$$= 0 + 6x^2y - 0$$

$$= 6x^2y$$

$$f_{xx} = 6x + 6y^2 - \left(\frac{x^2+1\frac{d}{dx}(2x) - 2x\frac{d}{dx}(x^2)}{(x^2+1)^2}\right)$$

$$= 6x + 6y^2 - \left(\frac{2(x^2+1) - 4x^2}{(x^2+1)^2}\right) \quad \text{—(1)}$$

$f_{yy} = \frac{d}{dy}(6x^2 y)$

$= 6x^2$ ——— (2)

$f_{xy} = \frac{d}{dy}\left(3x^2 + 6xy^2 - \frac{2x}{x^2+1}\right)$

$= 0 + 12xy - 0$

$= 12xy$ ——— (3)

$f_{yx} = \frac{d}{dx}(6x^2 y)$

$= 12xy$

from (3) & (4)

$\therefore f_{xy} = f_{yx}$ ——— (4)

iii) $f(x,y) = \sin(xy) + e^{x+y}$

$\rightarrow f_x = y\cos(xy) + e^{x+y}(1)$

$= y\cos(xy) + e^{x+y}$

$f_y = x\cos(xy) + e^{x+y}(1)$

$= x\cos(xy) + e^{x+y}$.

$\therefore f_{xx} = \frac{d}{dx}(y\cos(xy) + e^{x+y})$

$= -y\sin(xy)\cdot(y) + e^{x+y}(1)$

$= -y^2\sin(xy) \cancel{+ e^{x+y}}$ ——— (i)

$f_{yy} = \frac{d}{dy}(x\cos(xy) + e^{x+y})$

$= -x\sin(xy)(x) + e^{x+y}(1)$

$= -x^2\sin(xy) + e^{x+y}$ ——— (2)

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$$f_{xy} = \frac{d}{dy}\left(y\cos(xy) + e^{x+y}\right)$$
$$= -y^2\sin(xy) + \cos(xy) + e^{x+y} \quad —(3)$$
$$f_{yx} = \frac{d}{dx}\left(x\cos(xy) + e^{x+y}\right)$$
$$= -x^2\sin(xy) + \cos(xy) + e^{x+y} \quad —(4)$$

$\therefore$ from (3) & (4)

$f_{xy} \neq f_{yx}$.

Q.5)

i) $f(x,y) = \sqrt{x^2+y^2}$ at $(1,1)$

$\rightarrow f(1,1) = \sqrt{(1)^2+(1)^2} = \sqrt{2}$

$$f_x = \frac{1}{2\sqrt{x^2+y^2}}(2x) \qquad f_y = \frac{1}{2\sqrt{x^2+y^2}}(2y)$$
$$= \frac{x}{\sqrt{x^2+y^2}} \qquad = \frac{y}{\sqrt{x^2+y^2}}$$

f_x at $(1,1) = \frac{1}{\sqrt{2}}$ $\qquad$ f_y at $(1,1) = \frac{1}{\sqrt{2}}$

$$\therefore L(x,y) = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b)$$
$$= \sqrt{2} + \frac{1}{\sqrt{2}}(x-1) + \frac{1}{\sqrt{2}}(y-1)$$
$$= \sqrt{2} + \frac{1}{\sqrt{2}}(x-1+y-1)$$
$$= \sqrt{2} + \frac{1}{\sqrt{2}}(x+y-2)$$
$$= \cancel{\sqrt{2}} + \frac{1}{\sqrt{2}}x + \frac{1}{\sqrt{2}}y \cancel{\frac{2}{\sqrt{2}}}$$

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$$= \frac{x+y}{\sqrt{2}}$$

ii) $f(x,y) = 1 - x + y \sin x$ at $\left(\frac{\pi}{2}, 0\right)$

$f\left(\frac{\pi}{2}, 0\right) = 1 - \frac{\pi}{2} + 0 = 1 - \frac{\pi}{2}$

$f_x = 0 - 1 + y\cos x$ $\qquad$ $f_y = 0 - 0 + \sin x$

f_x at $\left(\frac{\pi}{2}, 0\right) = -1 + 0$ $\qquad$ f_y at $\left(\frac{\pi}{2}, 0\right) = \sin\frac{\pi}{2} = 1$

$= -1$

$$L(x,y) = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b)$$

$$= 1 - \frac{\pi}{2} + (-1)\left(x - \frac{\pi}{2}\right) + 1(y - 0)$$

$$= 1 - \cancel{\frac{\pi}{2}} - x + \cancel{\frac{\pi}{2}} + y$$

$$= 1 - x + y.$$

iii) $f(x,y) = \log x + \log y$ at $(1,1)$

$f(1,1) = \log(1) + \log(1) = 0.$

$f_x = \frac{1}{x} + 0$ $\qquad$ $f_y = 0 + \frac{1}{y}$

f_x at $(1,1) = 1$ $\qquad$ f_y at $(1,1) = 1$

$$\therefore L(x,y) = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b)$$

$$= 0 + 1(x-1) + 1(y-1)$$

$$= x - 1 + y - 1$$

$$= x + y - 2$$

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Practical - 10

Q.1]

i) $f(x,y) = x + 2y - 3$ $\qquad a(1,-1)$; $u = 3i - j$

→ Here,

$u = 3i - j$ is not a unit vector

$\bar{u} = 3\bar{i} - \bar{j}$

$|u| = \sqrt{10}$

unit vector along u is $\frac{\bar{u}}{|u|} = \frac{1}{\sqrt{10}}(3\bar{i} - \bar{j})$

$= \frac{1}{\sqrt{10}}(3,-1)$

$= \left(\frac{3}{\sqrt{10}}, \frac{-1}{\sqrt{10}}\right)$

Now,

$f(a+hu) = f\left((1,-1) + h\left(\frac{3}{\sqrt{10}}, \frac{-1}{\sqrt{10}}\right)\right)$

$= f\left(1 + \frac{3h}{\sqrt{10}}, -1 - \frac{h}{\sqrt{10}}\right)$

$= 1 + \frac{3h}{\sqrt{10}} + 2\left(-1 - \frac{h}{\sqrt{10}}\right) - 3$

$= 1 - 2 - 3 + \frac{3h}{\sqrt{10}} - \frac{2h}{\sqrt{10}}$

$= -4 + \frac{h}{\sqrt{10}}$.

$$\therefore D_u f(a) = \lim_{h \to 0} \frac{f(a+hu) - f(a)}{h}$$

$$= \lim_{h \to 0} \frac{-4 + \frac{h}{\sqrt{10}} - (-4)}{h}$$

$$= \lim_{h \to 0} \frac{\cancel{h}}{\cancel{h}\sqrt{10}}$$

$$= \frac{1}{\sqrt{10}}.$$

6.1

ii) $f(x, y) = y^2 - 4x + 1$

$a = (3, 4)$, $u = i + 5j$

→ Here,

$u = i + 5j$ is not a unit vector

$\bar{u} = \bar{i} + 5\bar{j}$

$|\bar{u}| = \sqrt{26}$

$\therefore$ Unit vector along u is $\frac{\bar{u}}{|\bar{u}|} = \frac{1}{\sqrt{26}}(\bar{i} + 5\bar{j})$

$$= \frac{1}{\sqrt{26}}(1, 5)$$

$$= \left(\frac{1}{\sqrt{26}}, \frac{5}{\sqrt{26}}\right)$$

Now,

$$f(a+hu) = f\left((3,4) + h\left(\frac{1}{\sqrt{26}}, \frac{5}{\sqrt{26}}\right)\right)$$

$$= f\left(3 + \frac{h}{\sqrt{26}}, 4 + \frac{5h}{\sqrt{26}}\right)$$

$$= \left(4 + \frac{5h}{\sqrt{26}}\right)^2 - 4\left(3 + \frac{h}{\sqrt{26}}\right) + 1$$

$$= 16 + \frac{25h^2}{26} + \frac{40h}{\sqrt{26}} - 12 - \frac{4h}{\sqrt{26}} + 1$$

$$= \frac{25h^2}{26} + \frac{36h}{\sqrt{26}} + 5$$

$$D_u f(a) = \lim_{h \to 0} \frac{f(a+hu) - f(a)}{h}$$

$$= \lim_{h \to 0} \frac{\frac{25h^2}{26} + \frac{36h}{\sqrt{26}} + 5 - 5}{h}$$

$$= \lim_{h \to 0} \frac{\frac{25h^2}{26} + \frac{36h}{\sqrt{26}}}{h}$$

$$= \lim_{h \to 0} \frac{\cancel{h}\left(\frac{25h}{26} + \frac{36}{\sqrt{26}}\right)}{\cancel{h}}$$

$$= \frac{25(0)}{26} + \frac{36}{\sqrt{26}}$$

$$= \frac{36}{\sqrt{26}}.$$

iii) $f(x,y) = 2x + 3y$ $\quad a = (1,2)$, $u = 3i + 4j$

→ Here,

$u = 3i + 4j$ is not a unit vector

$\bar{u} = 3\bar{i} + 4\bar{j}$

$|\bar{u}| = \sqrt{25} = 5$

∴ Unit vector along $u = \frac{\bar{u}}{|\bar{u}|} = \frac{1}{5}(5\bar{i} + 4\bar{j})$

$= \frac{1}{5}(3,4)$

$= \left(\frac{3}{5}, \frac{4}{5}\right)$

Now,

$f(a+hu) = f\left((1,2) + h\left(\frac{3}{5}, \frac{4}{5}\right)\right)$

$= f\left(1 + \frac{3h}{5}, 2 + \frac{4h}{5}\right)$

$= 2\left(1 + \frac{3h}{5}\right) + 3\left(2 + \frac{4h}{5}\right)$

$= 2 + \frac{6h}{5} + 6 + \frac{12h}{5}$

$= 8 + \frac{18h}{5}$

$\therefore$ Duf $(a) = \lim_{h \to 0} \frac{f(a+hu) - f(a)}{h}$

$= \lim_{h \to 0} \frac{8 + \frac{18h}{5} - 8}{h}$

$= \lim_{h \to 0} \frac{18\cancel{h}}{5\cancel{h}}$

$= \frac{18}{5}.$

Q.2]

i) $f(x,y) = x^y + y^x \qquad a = (1,1)$

$f_x = y(x^{y-1}) + y^x \log y$

$f_y = x(y^{x-1}) + x^y \log x$

$\nabla f(x,y) = (f_x, f_y)$

$= (y x^{y-1} + y^x \log y, x y^{x-1} + x^y \log x)$

$\nabla f(x,y)$ at $(1,1)$

$= (1(1)^0 + 1^1 \log(1), 1(1)^{(1-1)} + 1 \log 1)$

iii) $f(x,y) = (\tan^{-1} x) \cdot y^2 \quad , \quad a = (1,-1)$

$f_x = y^2 \left(\frac{1}{1+x^2}\right) = \frac{y^2}{1-x^2}$

$f_y = 2y \tan^{-1} x$

$\nabla f(x,y) = (f_x, f_y)$

$= \left(\frac{y^2}{1+x^2}, 2y\tan^{-1} x\right)$

$\nabla f(x,y)$ at $(1,-1)$

$$= \left(\frac{(-1)^2}{1+(1)^2} , 2(-1)\tan^{-1}(1) \right)$$

$$= \left(\frac{1}{2} , \frac{-2\pi}{4} \right)$$

$$= \left(\frac{1}{2} , -\frac{\pi}{2} \right)$$

iii) $f(x,y,z) = xyz - e^{x+y+z}$ $\qquad a = (1,-1,0)$

$f_x = yz - e^{x+y+z}$

$f_y = xz - e^{x+y+z}$

$f_z = xy - e^{x+y+z}$

$\nabla f(x,y,z) = (f_x, f_y, f_z)$

$= (yz - e^{x+y+z}, xz - e^{x+y+z}, xy - e^{x+y+z})$

$\nabla f(x,y,z)$ at $(1,-1,0)$

$= (-1(0) - e^{1-1+0}, 1(0) - e^{1-1+0}, 1(-1) - e^{1-1+0})$

$= (0-1, 0-1, -1, -1)$

$= (-1, -1, -2)$

Q.3)

i) $\cancel{x^2} \cos y + e^{xy} = 2$ $\qquad$ at $(1,0)$.

$f(x,y) = x^2\cos y + e^{xy} - 2.$

$f_x = 2x\cos y + y e^{xy}$

$f_y = -x^2 \sin y + x e^{xy}$

$(x_0, y_0) = (1,0)$

f_x at $(1,0) = 2(1)\cos 0 + 0$

$= 2$

f_y at $(1,0) = -(1)^2 \sin(0) + 1(e)^{(0)}$
$= 1$

$f_x(x - x_0) + f_y(y - y_0) = 0$
$2(x-1) + 1(y-0) = 0$
$2x - 2 + y = 0$
$2x + y - 2 = 0$ → Equation of Target
$2x + y - 2 = 0$

now,
for equation of Normal,
$bx + ay + d = 0$
$\therefore x + 2y + d = 0$ At $(1,0)$
$\therefore (1) + 2(0) + d = 0$
$1 + d = 0$
$d = -1$ → Equation of Normal.
$\therefore x + 2y - 1 = 0$

ii) $x^2 + y^2 - 2x + 3y + 2 = 0$ at $(2,-2)$
$f(x,y) = x^2 + y^2 - 2x + 3y + 2$
$f_x = 2x + 0 - 2 + 0 + 0$
$= 2x - 2$
$f_y = 0 + 2y - 0 + 3 + 0$
$= 2y + 3$

$\therefore$ f_x at $(2,-2) = 2(2) - 2$
$= 2$

f_y at $(2,-2) = 2(-2) + 3$
$= -1$

$\therefore$ Equation of target
$f_x(x - x_0) + f_y(y - y_0) = 0$
$2(x-2) + (-1)(y+2) = 0$ → Equation of Target
$2x - 4 - y - 2 = 0$

$bx + ay + d = 0$

$-x + 2y + d = 0$

$-(2) + 2(-2) + d = 0$ at $(2, -2)$

$-2 - 4 + d = 0$

$d = +6$

$\therefore -x + 2y + 6 = 0$ → Equation of Normal

Q.4] i) $x^2 - 2yz + 3y + xz = 7$ at $(2, 1, 0)$

$f(x, y, z) = x^2 - 2yz + 3y + xz - 7$

$f_x = 2x - 0 + 0 + z - 0$

$= 2x + z$

$\therefore f_x$ at $(2, 1, 0) = 2(2) + 0 = 4$

$f_y = -2z + 3 + 0 - 0$

$= -2z + 3$

f_y at $(2, 1, 0) = -2(0) + 3 = 3$

$f_z = 0 - 2y + 0 + x - 0$

$= -2y + x$

f_z at $(2, 1, 0) = -2(1) + 2 = 0$

Equation of tangent;

$f_x(x - x_0) + f_y(y - y_0) + f_z(z - z_0) = 0$

$4(x - 2) + 3(y - 1) + 0(z - 0) = 0$

$4x - 8 + 3y - 3 = 0$

$4x + 3y - 11 = 0$ — Equation of tangent

Equation of normal,

$$\frac{x - x_0}{f_y} = \frac{y - y_0}{f_y} = \frac{z - z_0}{f_z}$$

$$\frac{x - 2}{4} = \frac{y - 1}{3} = \frac{z - 0}{0}$$ — Equation of normal

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ii) $3xyz - x - y + z = -4$

$f(x,y,z) = 3xyz - x - y + z + 4$

$f_x = 3yz - 1 - 0 + 0 + 0$
$= 3yz - 1$

$f_y = 3xz - 0 - 1 + 0 + 0$
$= 3xz - 1$

$f_z = 3xy - 0 - 0 + 1 - 0$
$= 3xy + 1$

at $(1,-1,2)$

f_x at $(1,-1,2) = 3(-1)(2) - 1$
$= -7$

f_y at $(1,-1,2) = 3(1)(2) - 1$
$= 5$

f_z at $(1,-1,2) = 3(1)(-1) + 1$
$= -2$

Equation of tangent

$f_x(x - x_0) + f_y(y - y_0) + f_z(z - z_0) = 0$

$-7(x-1) + 5(y+1) + (-2)(z-2) = 0$

$-7x + 7 + 5y + 5 - 2z + 4 = 0$

$-7x + 5y - 2z + 16 = 0$ — Equation of tangent.

Equation of normal.

$$\frac{x - x_0}{f_x} = \frac{y - y_0}{f_y} = \frac{z - z_0}{f_z}$$

$$\frac{x-1}{-7} = \frac{y+1}{5} = \frac{z-2}{-2}$$ — Equation of normal

Q.5] i) $f(x,y) = 3x^2 + y^2 - 3xy + 6x - 4y$

$\therefore f_x = 6x + 0 - 3y + 6 - 0$
$\therefore 6x - 3y + 6$ —(1)

$f_y = 2y - 3x + 0 - 4$
$= 2y - 3x - 4$ —(2)

$$f_x = 0$$
$$6x - 3y + 6 = 0$$
$$3(2x - y + 2) = 0$$
$$2x - y + 2 = 0$$
$$2x - y = 2 \quad -(3)$$

$$f_y = 0$$
$$2y - 3x - 4 = 0$$
$$2y - 3x = 4 \quad -(4)$$

Multiplying (3) by 2 & Substracting (4) from (3),

$$\therefore 4x - \cancel{2y} = -4$$
$$\underline{\cancel{2y} - 3x = 4}$$
$$7x = 0$$
$$x = 0.$$

Subsituting value of x in (3)

$$\therefore 2(0) - y = -2$$
$$-y = -2$$
$$y = 2$$

$\therefore$ Critical Points are $(0,2)$

Now,

$$r = f_{xx} = 6$$
$$t = f_{yy} = 2$$
$$s = f_{xy} = -3$$
$$rt - s^2 = 12 - 9$$
$$= 3 > 0$$

Here $r > 0$ & $rt - s^2 > 0$

$\therefore f$ has minimum at $(0,2)$

$$\therefore f(0,2) = 3(0)^2 + (2)^2 - 3(0)(2) + 6(0) - 4(2)$$
$$= 0 + 4 - 0 + 0 - 8$$
$$= -4.$$

ii) $f(x,y) = 2x^4 + 3x^2y - y^2$.

$f_x = 8x^3 + 6xy - 0$
$= 8x^3 + 6xy$

$f_y = 0 + 3x^2 - 2y$
$= 3x^2 - 2y$.

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Now,

$f_x = 0$

$8x^3 + 6xy = 0$
$2x(4x^2 + 6xy) = 0$
$4x^2 + 6xy = 0$ —(1)

$f_y = 0$.
$3x^2 - 2y = 0$
$3x^2 - 2y = 0$
$3x^2 - 2y = 0$ —(2)

Multiply eq (1) by 3 & (2) by (4) & subtracting (2) from (1)

$\cancel{12x^2} + 18y = 0$
$-\cancel{12x^2} - 8y = 0$

$24y = 0$
$y = 0$ —(3)

Substituting (3) in (2)

$3x^2 - 2(0) = 0$
$3x^2 = 0$
$x^2 = 0$
$x = 0$ —(4)

Critical points are (0,0)

Now,
$r = f_{xx} = 24x^2 + 6y$
$t = f_{yy} = 6 - 2$
$s = f_{xy} = 6x$.
$rt - s^2 = (24x^2 + 6y)(-2) - (6x)^2$
$= -48x^2 - 12y - 36x^2$
$= -84x^2 - 12y$

At (0,0)
$r = 24(0)^2 + 6(0)$
$= 0$
$s = 6(0) = 0$
$rt - s^2 = -84(0)^2 - 12(0) = 0$
$rt = 0$ & $rt - s^2 = 0$
∴ Nothing can be said

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