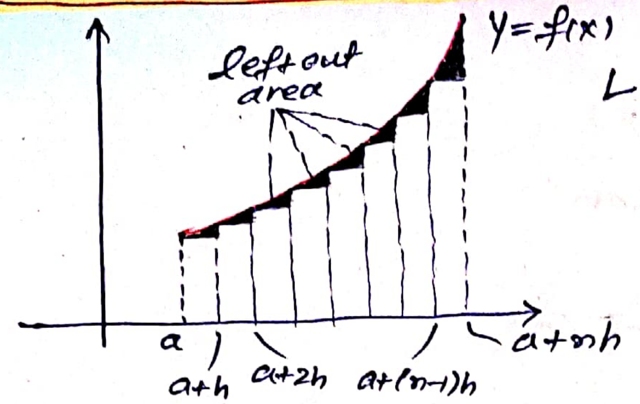
Definite Integration

DEFINITE INTEGRATION AS SUM



$$L = h\left[f(a) + f(a+h) + f(a+2h) - - - - - - + f(a+(n-1)h)\right]$$

$\Downarrow$

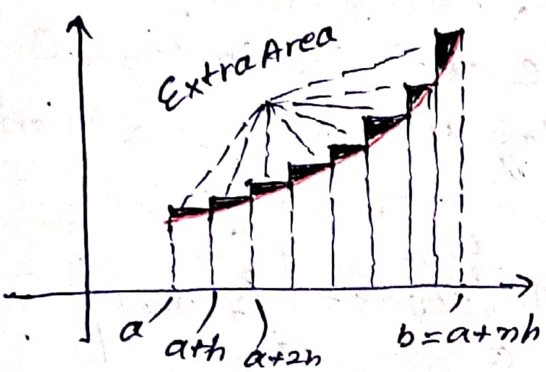
Riemann's lower sum

$\Rightarrow$ No. of strips $= n$

width of each strip $= h$

$nh = b-a$

$h = \dfrac{b-a}{n}$



$$U = h\left(f(a+h) + f(a+2h) + - - - + f(a+nh)\right)$$

$\Downarrow$

Remann's upper sum

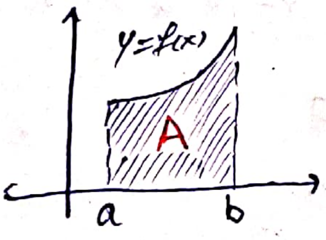
$L < A < U$

as h dec $\Rightarrow$ leftout area decreases

as h dec extra Area dec $\Rightarrow$ upper sum dec.

Now $L < A < U$

By sandwich Thr $\lim\limits_{h\to 0} L = A = \lim\limits_{h\to 0} U$



A = Area b/w $y=f(x)$ from $x=a$ to $x=b$

$$= \lim_{h\to 0} L = \lim_{h\to 0} h\left(f(a) + f(a+h) + - - - + f(a+(n-1)h)\right)$$

$$= \lim_{h\to 0} h \sum_{r=0}^{n-1} f(a+rh)$$

$$= \lim_{h\to 0} U = \lim_{h\to 0} h\left(f(a+h) + f(a+2h) + - - - + f(a+nh)\right)$$

$$= \lim_{h\to 0} h \sum_{r=1}^{n} f(a+2h)$$

$$* \quad \int_a^b f(x)\,dx = A = \lim_{h\to 0} h \sum_{r=0}^{n-1} f(a+rh) = \lim_{h\to 0} h \sum_{r=1}^{n} f(a+rh)$$

$$* \quad \lim_{n\to\infty} \frac{1}{n} \sum_{r_1(n)}^{r_2(n)} f\left(\frac{r}{n}\right) = \int_{\lim\limits_{n\to\infty} \frac{r_1(n)}{n}}^{\lim\limits_{n\to\infty} \frac{r_2(n)}{n}} f(x)\,dx \qquad \left\{\frac{r}{n} \to x\right\} \quad \left\{\frac{1}{n} \to dx\right\}$$

Fundametal Theorem of Calculsus :

* If $f(x)$ is continuous in $[a,b]$ & $F(x)$ be any of it's antiderivative i.e

$$\int f(x)\cdot dx = F(x)$$

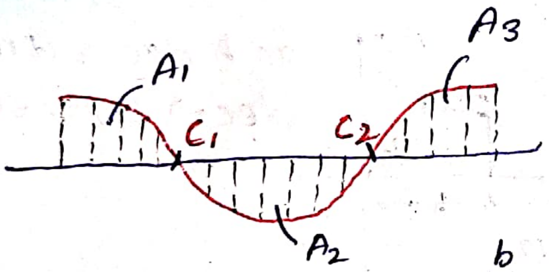
$\Rightarrow F'(x) = f(x) \ \forall \ x \in [a,b]$ then

$$\int_a^b f(x)\cdot dx = F(b) - F(a)$$

$\to$ This theorem is very useful, because it gives us a method of calculating the definite integral more easily, without calculating the limit of a sum

* Geometrical Meaning

$$A = \int_a^{c_1} f(x)\,dx + \left| \int_{c_1}^{c_2} f(x)\cdot dx \right| + \int_{c_2}^{b} f(x)\cdot dx$$



$\Rightarrow \int_a^b f(x)\,dx = A_1 - A_2 + A_3 \Rightarrow \int_a^b f(x)\cdot dx$ gives the net area enclosed by the curve $y = f(x)$, X-Axis & lines $x = a$ & $x = b$

$\to$ if $f(x) > 0 \ \forall \ x \in [a,b]$ then $\int_a^b f(x)\cdot dx > 0$

$\to$ if $f(x) < 0 \ \forall \ x \in [a,b]$ then $\int_a^b f(x)\,dx < 0$

* KAAM KI BAAT :

* The antiderivatives $F(x)$ used in fundamental theorem should be continuous & derivable on $[a,b]$. If a discon. antiderivative is used, it may lead to a wrong result

$$\int_a^b f(x)\cdot dx = F(b) - F(a)$$

where $F(x)$ should be continuous & derivable on $[a,b]$

Change of Variable in Definite integration

$I = \int_a^b f(x) \cdot dx$

$x = g(t)$ → should have a continuous derivative on $[c,d]$ → i.e $g'(t)$ should be continuous on $[c,d]$

$dx = g'(t)dt$

$I = \int_a^d f(g(t)) \cdot g'(t) \cdot dt$

$b, a = g(t)$ ⟶ where $\begin{cases} a = g(c) \\ b = g(d) \end{cases}$

FTOC

$\int_a^b f(x) \cdot dx = F(b) - F(a)$ — F: continuous & derivable on $[a,b]$; $f(x)$: should be continuous on $[a,b]$

EAK KAAM KI BAAT

Something instead of $x = g(t)$ we use the substitution $t = h(x)$ in integrals like

$I = \int_a^b f(h(x)) \cdot h'(x) dx$

here also $h(x)$ should have a continuous derivative on $[a,b]$

$dt = h'(x) \cdot dx \Rightarrow I = \int_{h(a)}^{h(b)} f(t) \cdot dt$

$x^3 + 6x^2 + 11x + 6$
↓
$(x+1)(x+2)(x+3)$

$x^3 - 6x^2 + 11x - 6$
↓
$(x-1)(x-2)(x-3)$

EAK KAAM KI BAAT

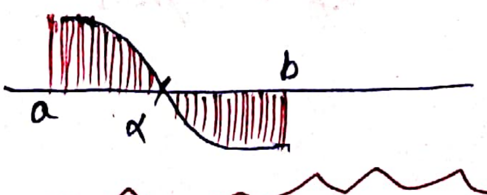
$\int_a^b f(x) \cdot dx = F(b) - F(a)$ & $\int_b^a f(x) dx = F(a) - F(b)$
$= -(F(b) - F(a))$
$= -\int_a^b f(x) dx$

$$\boxed{\int_a^b f(x) dx = -\int_b^a f(x) dx}$$

$$\Rightarrow \boxed{\int_a^b I \cdot II \, dx = \left(I \cdot \int_a^b II \cdot dx\right) - \int_a^b \left(\frac{d(I)}{dx} \cdot \int II \cdot dx\right) \cdot dx}$$

Golden Points to Note:

* If $\int_a^b f(x) \cdot dx = 0$, then the equation $f(x) = 0$ has atleast one root in (a,b) provided f is continuous in $[a,b]$

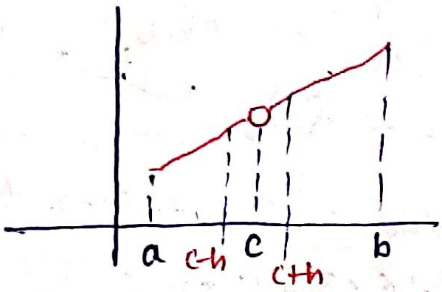


* $$\int_a^b f(x)\cdot d(g(x)) = \int_{g^{-1}(a)}^{g^{-1}(b)} f(x)\cdot g'(x)\cdot dx$$

limit of $g(x)$ — limit of x

$$\int \ln x\,dx = x\ln x - x + C$$

* If f is continuous in $[a,b]$ except at ~~least~~ the point $x=c$ then we have



$$\int_a^b f(x)dx = \int_a^{c^-} f(x)dx + \int_{c^+}^b f(x)\cdot dx$$

or

$$\int_a^b f(x)\,dx = \lim_{h\to 0}\int_a^{c-h} f(x)\,dx + \lim_{h\to 0}\int_{c+h}^b f(x)\,dx$$

A PLATINUM POINT

* If $g(x)$ is the inverse of $f(x)$ and ~~g~~ $f(x)$ has domain $x\in[a,b]$ where $f(a)=c$ and $f(b)=d$ then the value of

$$\int_a^b f(x)\cdot dx + \int_c^d g(y)\cdot dy = (bd-ac)$$

i.e $$\int_a^b f(x)dx + \int_c^d f^{-1}(x)\cdot dx = (bd-ac)$$

Another Golden Point

If the limit of integration are reciprocal of each other try the substitution $x = 1/t$.

NOTE! $\frac{1}{\infty} = 0$

properties of Definite integral :

P(1): Change of variable

$$\int_a^b f(x)\cdot dx = \int_a^b f(t)\cdot dt$$ (change of variable does not change value of integral)

P(2): $\int_a^b f(x)\cdot dx = -\int_b^a f(x)\cdot dx$

P(3): $\int_a^b f(x)\cdot dx = \int_a^c f(x)\cdot dx + \int_c^b f(x)\cdot dx$ (where 'c' may or may not lie in $[a,b]$)

* useful when $f(x)$ is not uniformly defined on $[a,b]$

$\int_0^{\pi/4} \ln(1+\tan x) = \frac{\pi}{8}\ln 2$

YAAD रखें!!

P(4): JACK:

$$\int_{-a}^{a} f(x)\cdot dx = \int_0^a (f(x) + f(-x))\cdot dx$$

$$= \begin{cases} 2\int_0^a f(x)\cdot dx, & f(x) = \text{even fun} \\ 0, & f(x) = \text{odd, func}^n \end{cases}$$

ALL PROPERTIES ARE VERY IMPORTANT

P(5): KING:

$$\int_a^b f(x)\,dx = \int_a^b f(\quad -x)\cdot dx$$

P(6): QUEEN:

$$\int_0^{2a} f(x)\cdot dx = \int_0^a (f(x) + f(2a-x))\cdot dx = \begin{cases} 2\int_0^a f(x)\cdot dx, & f(2a-x) = f(x) \\ 0, & f(2a-x) = -f(x) \end{cases}$$

P(7): ACE:

* $\int_0^{nT} f(x)\,dx = n\int_0^T f(x)\cdot dx$ where $f(T+x) = f(x)$, $n \in I$

* $\int_a^{a+nT} f(x)\cdot dx = n\int_0^T f(x)\cdot dx$, where $n \in I$

Golden Point :

* $\int_0^{\pi/2} \sin x \cdot dx = \int_0^{\pi/2} \cos x \cdot dx = 1$

* $\int_0^{\pi/2} \sin^2 x \cdot dx = \int_0^{\pi/2} \cos^2 x \cdot dx = \pi/4$

* $\int_0^{\pi/2} \sin^3 x \, dx = \int_0^{\pi/2} \cos^3 x \cdot dx = 2/3$

* $\int_0^{\pi/2} \sin^4 x \, dx = \int_0^{\pi/2} \cos^4 x \cdot dx = 3\pi/16$

$$\int_0^{a} \frac{\ln(1+ax)}{1+x^2} = \frac{\tan^{-1}a}{2} \ln(1+a^2)$$
$a \in N$

* $\int_{mT}^{nT} f(x) \cdot dx = (n-m) \int_0^{T} f(x) \cdot dx$, where $n, m \in I$

$$\int_0^{\pi/2} \ln(\sin x) dx = -\frac{\pi}{2} \ln 2$$

* $\int_0^{a+nT} f(x) \cdot dx = \int_0^{nT} f(x) \cdot dx$

* $\int_{a+nT}^{b+nT} f(x) \cdot dx = \int_a^{b} f(x) \, dx$, where $n \in I$

Newton Leibnitz Formula :

→ Differentiation of integral

if $h(x)$ & $g(x)$ are diffʳ functions of x and f is continuous function then

$$\boxed{\frac{d}{dx} \int_{g(x)}^{h(x)} f(t) \cdot dt = f(h(x)) \cdot h'(x) - f(g(x)) \cdot g'(x)}$$

{NOTE : The integrand should not contain a funcⁿ of x }

Walli's Formula :

$$\int_0^{\pi/2} \sin^n x \cdot \cos^m x \cdot dx = \frac{((n-1)(n-3) \cdots\cdots 2 \text{ or } 1)((m-1)(m-3) \cdots 2 \text{ or } 1)}{(m+n)(m+n-2)(m+n-4) \cdots 2 \text{ or } 1} K$$

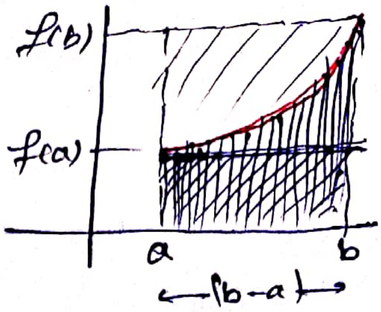
(m, n are non-ve integers)

$$\text{where } K = \begin{cases} \pi/2 , & \text{if } m, n \text{ are both even integers} \\ 1 , & \text{otherwise} \end{cases}$$

ESTIMATION :

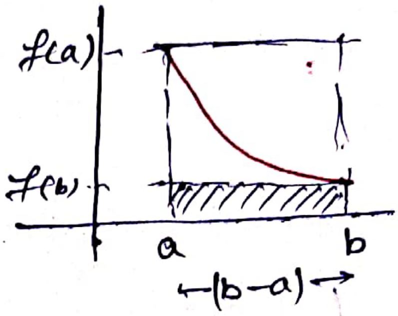
① if f is monotonic inc in $[a,b]$

$$\boxed{(b-a)\cdot f(a) < \int_a^b f(x)\cdot dx < (b-a)\cdot f(b)}$$



② If f is monotonic dec in $[a,b]$

$$\boxed{(b-a) f(b) < \int_a^b f(x)\cdot dx < f(a)\cdot(b-a)}$$

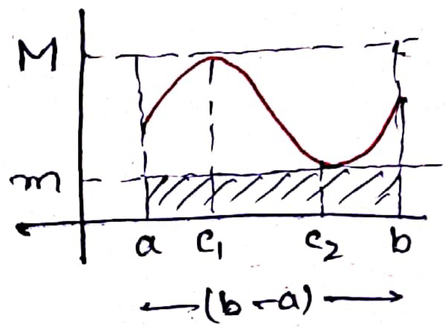


③ If f is non monotonic in $[a,b]$

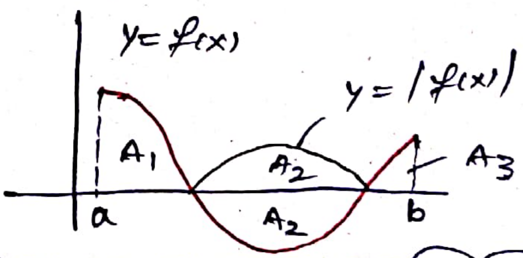
$$\boxed{m(b-a) < \int_a^b f(x) dx < M(b-a)}$$

* m = least value
* M = Greatest value

of a funtion on $[a,b]$



④ $$\boxed{\left|\int_a^b f(x)\cdot dx\right| \leq \int_a^b |f(x)|\cdot dx}$$

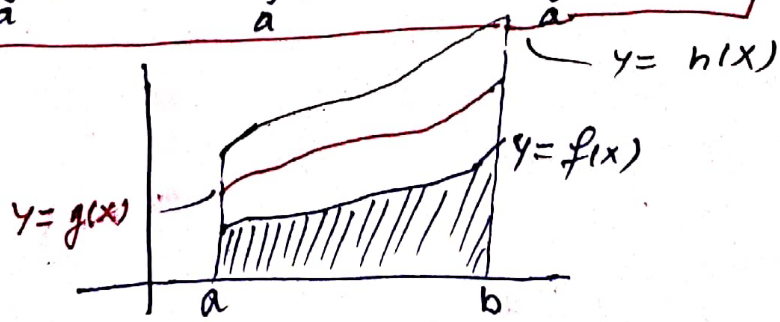


$|A_1 - A_2 + A_3| \leq |A_1| + |A_2| + |A_3|$ (feel)

* Equality hold if $y = f(x)$ lies entirely above/below x-Axis in $[a,b]$

⑤ if $f(x) \leq g(x) \leq h(x)$ in $[a,b]$

$$\Rightarrow \boxed{\int_a^b f(x)\cdot dx \leq \int_a^b g(x)\cdot dx \leq \int_a^b h(x)\cdot dx}$$



Differentiating & Integrating series:

$$(1+x)^n = {}^nC_0 + {}^nC_1 x + {}^nC_2 x^2 + \dots + {}^nC_n x^n, \; n \in N \quad (1)$$

<u>show</u>: $1\cdot{}^nC_1 + 2\cdot{}^nC_2 + \dots + n\cdot{}^nC_n = n\cdot 2^{n-1}$

Diffr (1)

$$n(x+1)^{n-1} = {}^nC_1 + {}^nC_2\cdot 2x + {}^nC_3\cdot 3x^2 + \dots n\cdot{}^nC_n x^{n-1}.$$

<u>put $x=1$</u>

$$n\cdot 2^{n-1} = {}^nC_1 + 2\cdot{}^nC_2 + 3\cdot{}^nC_3 + \dots + n\cdot{}^nC_n$$

Evaluating Integrals Dependent on Parameters:

Differentiate 'I' with respect to the parameter within the sign of integrals taking variable of the integrand as constant. Now evaluate the integral so obtained as usual as a funcn of the parameter and then integrate the result to get 'I'. Constant of Integration to be computed by giving some arbitrary values to the parameter

<u>M(1)</u>

$$I = \int_0^{\pi/2} \sin(x+c)\cdot dx = \left(-\cos(x+c)\right)\Big|_0^{\pi/2}$$

$$= -\left(\cos(\pi/2 + c) - \cos(c)\right)$$

$$= -(-\sin c - \cos c)$$

$$\boxed{I = \sin c + \cos c} \text{ Ans}$$

<u>M(2)</u>

$$\frac{dI}{dc} = \int_0^{\pi/2} \frac{\partial}{\partial c}(\sin(x+c))\cdot dx$$

$$= \int_0^{\pi/2} \cos(x+c)\cdot dx = \sin(x+c)\Big|_0^{\pi/2} = \cos c - \sin c$$

$$\int dI = \int (\cos c - \sin c)\cdot dc$$

$$I = +\sin c + \cos c + K$$

put $c=0$

$$1 = 0 + 1 + K \Rightarrow K = 0$$

$$\boxed{I = \sin c + \cos c} \text{ Ans}$$

$I = \int_0^{\pi/2} \sin(x+c)$

if $c=0$

$I = \int_0^{\pi/2} \sin x \cdot dx = 1$