# Final Project Report, Combinatorial Games

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**Abstract**

In this report, we present our understanding of combinatorial games as a part of our course project in CS201A - Discrete Mathematics. The report covers definitions of combinatorial games, game positions, options, trees, impartial games, sums, comparisons, outcome classes and other elementary vocabulary pertaining to the field and a proof of the Fundamental Theorem of Combinatorial Games. We list some basic strategies of winning simple games, as outlined in the referenced book; followed by an analysis of some basic combinatorial games - Nim, Hex, Tic Tac Toe and 2xn Domineering. We mention the origin and rules of each game, followed by a winning strategy or a proof of termination, as required.

## 1. Introduction

"We learnt to play when we were kids, now we play to learn." - A thought that sprang up in our idle minds.

Man is a social and selfish animal, and one of the ways in which his predilection towards company and ego-inflation are manifested is through playing and (most importantly) winning games. Also, games serve as a source of entertainment and a healthy exercise for the mind or the body or both.

'Combinatorial games' is a special subclass of games. All of us have played them in our childhood, and some we continue to play even when we grow old. Thus, the study of these games is appealing, not just from the theoretical point of view, but also because of its practical usefulness.

There are some features that all combinatorial games share. A game is a combinatorial game if the following conditions are satisfied:

- It is a two player game, and the players take turns making their moves.
- The state of the game is known to both the players at all points of the game.
- There is no element of chance involved in the game.
- The game can't end in a draw, and usually the player who makes the final valid move wins the game.

Many popular games such as Nim, Hex, and chess fall into this category, and techniques developed to analyze these games can be used to derive important results about Tic Tac Toe, Dots and Boxes, and various other games as well. Some of the strategies of winning games that we read about, have been described below, followed by an introductory mathematical treatment of games.

## 2. Basic Techniques

Heuristic approach to analysis of games allows giving logical arguments about why a certain move might be better. There are plenty of techniques which work well in many games without much forethought:

### 2.1 Greedy

One of the simplest and most intuitive technique, a player following this strategy always chooses the move that maximizes or minimizes some quantity related to the game position after the move has been made without caring for its long term effects. It is generally not a good strategy except a few simple games like those where winner is decided by a score which accumulates while you play. This is what we unknowingly do while capturing a queen in chess with a pawn, trading a smaller value piece for a larger value one.

### 2.2 Symmetry

Another intuitively obvious strategy. A simple case of this is the Tweedledum-Tweedledee strategy where whenever your opponent does something on one part of the board you mimic this move in another part, giving you a counter move for every move of opponent.

### 2.3 Change The Game

Often a tough-to-crack game can be observed to be inherently similar to some other game, for which a winning strategy can be obtained. This often requires a deep observation and is not as intuitive as the previous strategies.

### 2.4 Parity

A number's parity is whether the number is even or odd. With the normal play convention that the last player with a legal move wins, it is always the objective of the first player to play to ensure that the game lasts an odd number of moves, while the original second player is trying to ensure that it lasts an even number of moves. So, parity is indeed a critical concept. Some games, like 'she loves me she loves me not', rely on an initial parity (i.e. whether the initial number of items is odd or even).

### 2.5 Give Them Enough Rope

When stuck in a losing position, this strategy advises to make the position as complicated as you can with your next move. Confusing the opponent gives you time for analysis. This is not a concrete strategy, but may end up in a position where a strategy mentioned above can be used.

### 2.6 Don't Give Them Any Rope

It involves restricting the other person's moves. If you do not know whether you are losing or winning the game, then a very good strategy is to move so as to restrict the number of options your opponent has and increase the number of your own options. For example, the initial moves in Connect-4 are not very specific towards winning and one can simply keep blocking the person to take the game to a simpler state.

### 2.7 Strategy Stealing

"If an enemy is annoying you by playing well, consider adopting his strategy."

-Chinese proverb

If a person is in a losing position, that implies that the opponent has a winning strategy. If it is possible in the game, the first person can shift the game in his favor by using the strategy of the second person in the remaining game.

## 3. Elementary Theory

Basic theory about combinatorial games. Fundamental theorem of combinatorial games.

### 3.1 Game positions and Options

A game, in its simplest terms, is a list of possible "moves" that two players, called left and right, can make. The game position resulting from any move can be considered to be another game. This idea of viewing games in terms of their possible moves to other games leads to a recursive mathematical definition of games that is standard in combinatorial game theory.

**Definition 3.1.** Game Options: the *options* of a game are the all the positions that arise from the different moves that can be made by the Left or the Right player. These are named *left options* and *right options* respectively. The set of all these options can be obtained by the union of these two sets

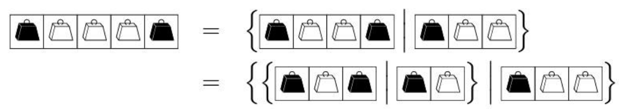


**Figure 1.** Left and Right Options of Clobber, separated by braces.

**Definition 3.2.** Game Position: A *Game Position* **G** is defined by its options, G = $\{G^L|G^R\}$ ,where $G^L$ and $G^R$ are the sets of the left and right options respectively.

### 3.2 Fundamental Theorem of Combinatorial Games

*Fix a game G played between A and B, with A moving first. Either A can force a win moving first, or B can force a win moving second, but not both.*

Every possible move A can make will be one of the following two types. Either his move lands the game to a position from where B can force a win; or it can land to a position from where he can force a win.
If all his moves are of the first type, then he's already lost the game.
In case he has a move of the latter type available, he'll choose that, and then by the definition of the move, he can force a win in his next turn. This is but an inductive definition of a winning move.
Thus either A has a winning move playing first, or B has a winning move playing second.

### 3.3 Outcome classes

The above theorem inspires the idea of games, or more specifically game positions, whose outcome can be forced. We formalize that as the outcome class a game can fall

into.
An outcome class is a class of games with similar predictable outcomes. There are 4 broad categories

| | | |
|---|---|---|
| The next player to play a move wins, irrespective of whether he is $\mathscr{L}$ or $\mathscr{R}$ | Fuzzy | $\mathscr{N}$ |
| The one who played the previous move, or the one who's going to play the next to next move can force a win irrespective of anything else | Zero | $\mathscr{P}$ |
| The Left player can force a win irrespective of what position he plays from | Positive | $\mathscr{L}$ |
| The right player can force a win irrespective of what position he plays from | Negative | $\mathscr{R}$ |

The $\mathscr{N}$ and $\mathscr{P}$ game positions are also called $\mathscr{N}$-positions and $\mathscr{P}$-positions respectively for obvious reasons.

The above definitions help us find some powerful results about combinatorial games. It turns out that classifying the available game options from a position into the various outcome classes can help us determine the outcome class of the game.

$$\begin{array}{lcll} \exists\, G^L \in \mathscr{L} \cup \mathscr{P} & \text{and} & \exists\, G^R \in \mathscr{R} \cup \mathscr{P} & \mathscr{N} \\ \exists\, G^L \in \mathscr{L} \cup \mathscr{P} & \text{and} & \forall\, G^R \in \mathscr{L} \cup \mathscr{N} & \mathscr{L} \\ \forall\, G^L \in \mathscr{R} \cup \mathscr{N} & \text{and} & \exists\, G^R \in \mathscr{R} \cup \mathscr{P} & \mathscr{R} \\ \forall\, G^L \in \mathscr{R} \cup \mathscr{N} & \text{and} & \forall\, G^R \in \mathscr{L} \cup \mathscr{N} & \mathscr{P} \end{array}$$

*Proof of the second entry*
If the game is $\mathscr{L}$ then Left wins if he plays first. Thus, the resulting game $G^L$ will be a win for Left, thus he has a winning move playing second in $G^L$. So left has option in the game which belongs to $\mathscr{L} \cup \mathscr{P}$. Also, the above means that Right cannot force a win. So all his moves would be in $\mathscr{L} \cup \mathscr{N}$. That proves the second entry.
The remaining entries can be proved similarly.

### 3.4 Game Trees

In game theory, a *game tree* is a directed graph whose nodes are positions in a game and whose edges are moves. The complete game tree for a game is the game tree starting at the initial position and containing all possible moves from each position; the complete tree is the same tree as that obtained from the extensive-form game representation.

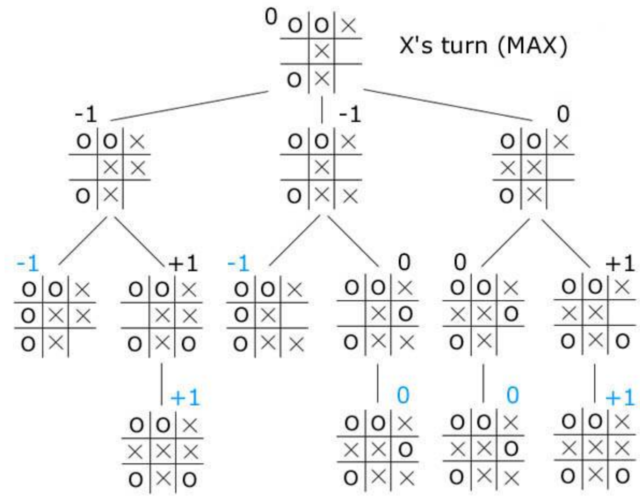



**Figure 2.** An Example of a game tree from a Tic Tac Toe position.

### 3.5 Impartial Games

Impartial Games are defined as the games in which both the players have the same set of options from any position.

Examples of impartial games include Nim and subtraction games. A fundamental theorem characterizing impartial games is:

Every impartial game G is in either N or P.

Proof: Suppose an impartial game G is in L. Then L can win the game by making the next move. But since the set of moves available to R are the same as those available to L, R can also use the same strategy and win by moving first. Thus, there is a contradiction to our assumption that G is in L. Similarly, we can construct an argument to conclude that G cannot be in R. Thus, G can only be in N or in P.

### 3.6 Sum and comparison

#### 3.6.1 Sum of games

*Lessons in play* describes the game sum as:
A sum of two or more game positions is the position obtained by placing the game positions side by side. When it is your move, you can make a single move in a summand of your choice. As usual, the last person to move wins.

John Conway, an English mathematician made some observations that since game sums appear to be central to so many games, following natural notions can be given:

- A move is from a position to a sub-position; the last player to play wins.

- In a sum of games, a player can move on any summand.
- In the negative of a game, the players' roles are reversed.
- Position A is at least as good as B (for Left) if Left is always satisfied when B is replaced by A in any sum.

These are popularly known as Conway's Axioms. The above description of the axioms includes references to addition, negation, and comparison.

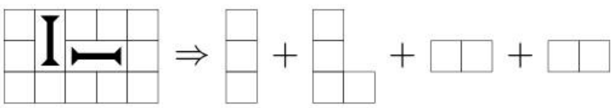


**Figure 3.** A domineering game represented as sum of many smaller games

#### 3.6.2 Comparison of games

Some observations that can be made after having a basic understanding of combinatorial games are as follows:

- Let G be any game and let Z ∈ P be any game that is a second- player win. The outcome classes of G and of G + Z are the same. That is, if Z is any game which has the property that the outcome classes of G and G + Z are always the same, then in particular the outcome classes of the empty game and the sum of the empty game and Z are the same, and so Z must be a second-player win.
- Let G be any game and let X ∈ P ∪ L be any game which Left wins moving second. The outcome class of G + X is at least as favorable for Left as that of G.

That is, if Left wants to win in G + Z he/she need only exploit the fact that he/she wins G and wins moving second on Z.

### 3.7 Equality and Identity

From every standpoint except an aesthetic one, two games are the same and can be considered equal when they have identical game trees.

An utter pragmatist looks at games differently. To him/her, when faced with a game, the only question of interest is: "Who will win?". According to such a person, two games are equal if and only if they have the same outcome class.

The way Combinatorial Game Theory views this is that two games might be considered equal if they can be freely substituted for one another in any context (i.e. when their sums are same) without changing the outcome type. This provides an outline of a strategy for analyzing complex games — decompose the game into sums, and replace complicated summands by simpler ones that they are equal to.

## 4. Nim

Nim is an old, two-player game that originated in China, but its name was coined by Charles L. Bouton, who also solved the game fully in 1901.

Nim consists of a finite number of piles, where every pile has a certain number of sticks. Players take turns making moves. A move consists of removing any number of sticks from a chosen pile, as long as the pile already has that many sticks. The game ends when there are no sticks left to remove. In normal play, the one who plays the last move wins, while in misere play, he loses.

Nim's analysis is slightly involved. So, we first proceeded to study two-pile normal play Nim.

### 4.1 Two Pile Nim

Intuitively, two pile Nim is a second player win if the piles have the same number of sticks: it lies in P. If the number of sticks are unequal, it lies in N. This can be inductively proved.

If the number of sticks in the two piles are unequal, the next player can always remove the number of sticks amounting to the difference between their numbers from the higher pile and make their numbers equal. In the base case, two piles with zero sticks is a win for the previous player.

Further analysis of Nim requires a definition of Nim-sums. A Nim-sum of two numbers is simply application of the XOR operation on their bit representations. For example, $3 \oplus 2 = 1$.

### 4.2 Bouton's Theorem

Bouton's Theorem: A position $(x_1, x_2, \ldots, x_k)$ is a P position if and only if the nim-sum of its components is zero, i.e. $x_1 \oplus x_2 \oplus \ldots \oplus x_k = 0$.

Proof: A terminal position is of the form (0, 0,. . . ,0) which is in P, and its nim-sum is zero. The position just before reaching the terminal position has one of the xi's as non-zero, and the rest as zero. Thus, the nim-sum of the penultimate position is non-zero.

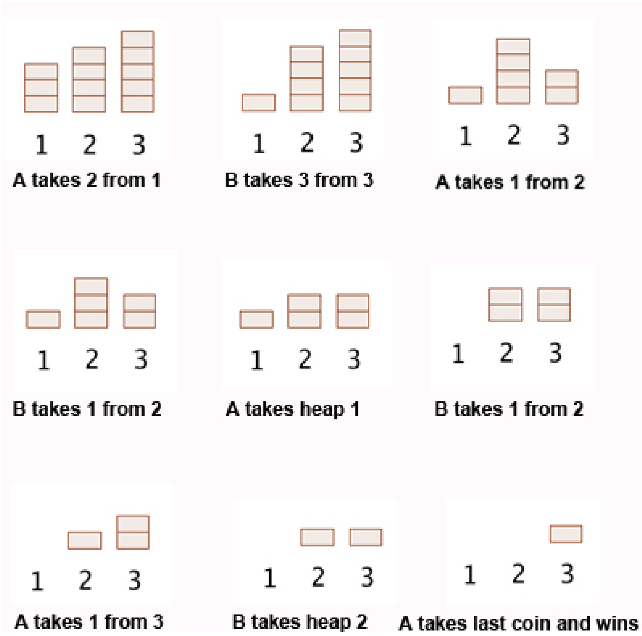



**Figure 4.** A (3, 4, 5) Nim played optimally by Player 1 at each step.

If the position $(x_1, x_2, \ldots, x_k)$ has nim-sum of its components non-zero, then the next player can play a move which takes the nim-sum to zero. This is a winning move, as, now the second player will have to make a move that takes the nim-sum of the components to a non-zero value, which by induction is a first player win.

Why is this valid? Firstly, there always exists a move from a non-zero nim-sum situation to a zero nim-sum situation. To achieve a nim-sum of zero, we need to perform nim-sum on the current nim-sum and the already present components. In effect, to comply to the rules of the game, the component of one of the piles will be transformed after doing its nim-sum with the current sum. Take the left-most non-zero bit in the current nim-sum and choose a number which has a non-zero bit in this position as well. Such a component exists because it must have been the source of the non-zero bit in the nim-sum. When these two numbers are operated upon by nim-sum, we essentially get a number which is smaller than the initial value of the component, as the non-zero bit being talked about above is now a zero, and maybe more bits to the right of it have undergone this transformation. Some bits will be transformed from 0 to 1 by this sum, but since $2^d <$ sum of $2^s$ for all s $<$ d, the resulting component will still be smaller than the one we started with.

Finally, from a position of zero nim-sum, any move will enforce one of the bits in the nim-sum to be changed to 1, as we are altering only one of the components. Thus, a zero position always leads to a non-zero position after a move.

Nim holds a lot of significance in the class of impartial games, mainly due to the Sprague Grundy Theorem, which relates every impartial game to a nim heap of a certain size.

## 5. Hex

The credit for the invention of this popular game goes to Piet Hein, and it has been analyzed and popularized by John Nash and Martin Gardner subsequently.

Hex is a simple two player game played usually on a rhombus which is an nxn grid of hexagons. The goal of a player is to connect a set of two opposite sides of the board by making a path of hexagons colored by his color, before the other player does (who has been allocated the remaining two sides). Players take turns picking hexagons to be colored, generally trying to make their way, or blocking those of their opponents.

There are many variants of the game, the most popular ones being on 11x11, 14x14 and 19x19 boards. A variation of Hex involves playing on an nx(n+1) board, while another variation allows the second player to switch positions with the first after the first player has made his move.

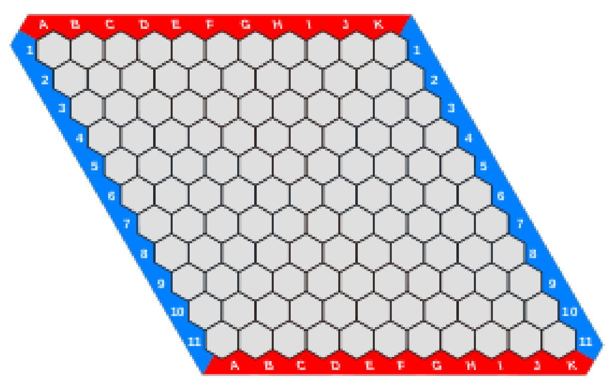


**Figure 5.** An 11X11 Hex Board

### 5.1 Argument for termination

Since there are finite number of hexagons on a board, and each player colors one hexagon on his/her turn, the game will end in a finite number of moves. Also, the game can't end in a draw.

To see this, suppose blue is trying to connect the left side with the right and red is trying to connect the top with the bottom. Assume that the game ended in a draw. As red hasn't won, there does not exist a single path that connects the top and the bottom sides. Thus, blue can always find a path going from left to right which is unobstructed by red hexagons. So, blue has won, which is a contradiction.

### 5.2 nxn Hex lies in N

John Nash gave a non-intuitive, non-constructive proof that the first player can always force a win in nxn Hex, by a technique that is now popularly called, "Strategy Stealing", mentioned earlier in the report. Below, the proof has been put in our own words:

Either the first player has a winning strategy, or the second player has one. If the former is true, then we are done. Assume that the second player has a winning strategy. Player 1 makes his first move in a random hexagon, and then forgets about it. Now he is the second player, and has a winning strategy. He also has an extra hexagon filled up with his color, which can't harm him in any case. If the strategy involves making a move on the hexagon in which he made his first move, he can simply make an arbitrary move anywhere on the board, now counting this move as the random move. So, he is impersonating the second player and playing his winning strategy.

Thus, the first player can always force a win in Hex. In an nx(n+1) Hex, Player 2 can do so.

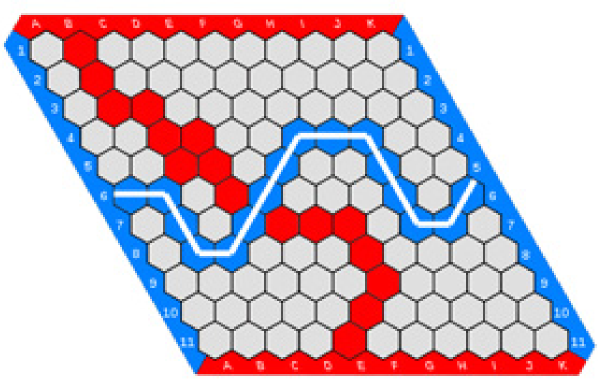


**Figure 6.** A finished game of Hex, won by Blue

## 6. Tic-tac-toe

Tic-tac-toe (also known as Noughts and crosses or Xs and Os), a classical two player game. The objective is to be the first to place 3 of your marks in a horizontal, vertical or diagonal row to win the game.

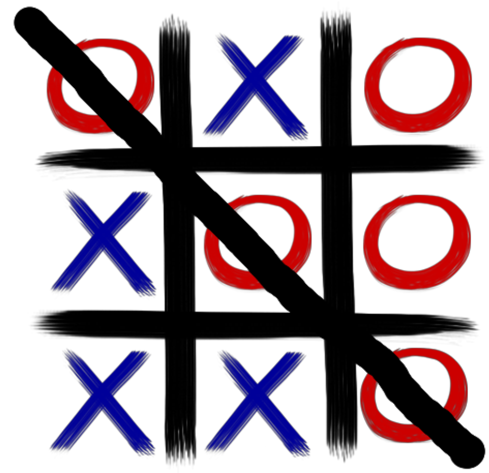


**Figure 7.** Tic Tac Toe

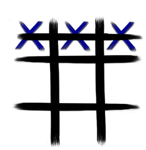
This seemingly simple game has some interesting combinatorial facts worth analyzing. Basic counting tells that there are $3^9$ possible board layouts (3 possibilities for each block - X, O or blank). Detailed analysis through game trees shows that there are only 138 distinct terminal positions after considering board symmetries (i.e. rotations and reflections). If the first player moves a X, then out of those 138 positions:

- 91 positions are won by X.
- 44 positions are won by O.
- 3 positions are drawn.

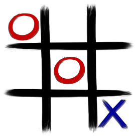
### 6.1 Winning Strategy Ideas

To play a perfect game (to win or draw), there is a recommended sequence of moves to follow based on their availability:

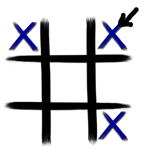
- If you already have 2 in a row, mark the 3rd one to win.



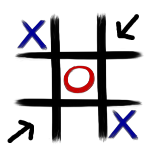
- If the opponent has 2 in a row, block his move by moving your mark on the 3rd position.



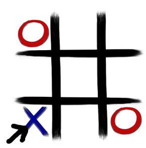
- If the above 2 situations with obvious response are not there, try to create a 'fork' i.e. create 2 threats for your opponent by creating two non-blocked lines of 2 of your mark in just one move.



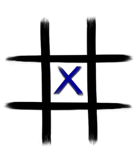
- The next priority should be to block the opponent's work by:
  - Either create two in a row to force the opponent into defending, as long as it doesn't result in him creating a fork. For example, if X has a corner, O has the center, and X has the opposite corner as well, playing an O in the corner crates a fork for X to win.



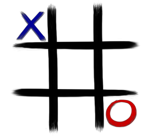
  - Or if there is a configuration where the opponent can fork, block that fork.



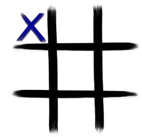
- If center is available, place your mark there (although if it's the first move of the game, placing on corner is the better choice, reason being playing the corner gives the opponent the smallest choice of squares as per game tree which must be played to avoid losing)



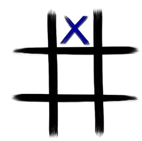
- Move on the corner opposite to opponent's corner (if there is any)



- Otherwise play on any random corner.

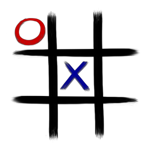


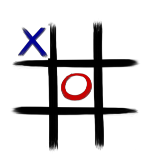
- Last thing would be to move in the middle of any random side.



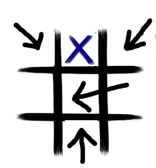
Two tips regarding response to a starting move which we found useful (in these points, assume that your opponent has marked a X in the first move):

- Respond to a corner opening with a center mark, and to a center opening with a corner mark.





- Respond to an edge opening either with a center mark or a corner mark next to the X or an edge mark opposite the X as any other responses will allow X to force the win.



### 6.2 Implementation

A simple computer program can be implemented for playing Tic-tac-toe. The crux of the algorithm lies in the Minimax algorithm.

#### 6.2.1 Algorithm

Every time the computer has to pick a move, he tries to minimize the human's best possible payoff, or in other words tries to maximize his payoff. The above 2 statements are equivalent by the Minimax theorem for zero-sum games, owing to the fact that Tic-tac-toe is a zero sum game. So one player's loss is the opponent's gain by an equal amount.
At every choice given to the computer, it will simulate a random move. For each random move, it will then play the next move from the human's side as if the computer was playing from the human's side. This goes on recursively. Thus, at the end, the computer will have a list of all possible endings for every move he has at his disposal.
It can then easily choose which move to make, according to which move guarantees him the highest payoff. Payoff is higher for the AI if both player proceed to play ideally after that move it has higher chances of winning. In one implementation, if a possible ending is a human-win situation, that ending can have a score of 10; and if it's an AI win, its score can be -10. The score of a draw game is 0. The AI tries to minimize the possible resultant score of the game, while the human tries to maximize it.

#### 6.2.2 Analysis

In effect the computer looks at all possible outcomes and always chooses the best one. So a game between 2 computers running the same algorithm would always end in a draw.
Though this algorithm is basically a brute force algorithm, based on the Minimax theorem; it does not mean that this method can work for other complex games like Connect-4 etc, because of the sheer number of possible moves at every step. In this case we just had $3^9$ moves as explained above.

## 7. Domineering Rectangles

This game comprises of a rectangular grid, where players alternately take turns to place $1 \times 2$ and $2 \times 1$ dominoes on that grid, hoping to be the last person who has a move left. This is not an impartial game; Right has horizontally aligned dominoes ($1 \times 2$) and Left has vertical dominoes ($2 \times 1$). Needless to say, they aren't allowed to rotate their dominoes, that'll just be plain cheating.

First, we notice that in a game of domineering, it'll happen often that the game gets split into different regions. We can treat the remaining game as a sum of multiple games, and try to win each of them individually. The following figure illustrates that.

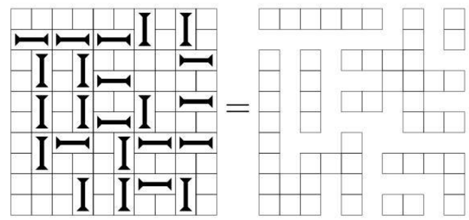


**Figure 8**

We will analyze $2 \times n$ domineering games briefly.

### 7.1 One-Hand-Tied-Principle

This theorem forms the bedrock for the analysis of this game which follows. It states that
*If it is true that Right wins some game G when he promises not to make certain types of moves, then he has a winning strategy in G itself.*
This sounds pretty much intuitive, if I can win with some restrictions, I can win it without restrictions as well. (Having one hand tied up wouldn't exactly be a disadvantage here, but it explains the idea.)

### 7.2 Analysis of $2 \times n$ Domineering

We first start with the statement that a $2 \times 4$ domineering game belongs to $\mathscr{R}$. The following images show winning replies by Right to every possible move by Left. Also,

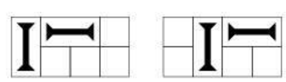


**Figure 9**

right can win by playing first too, as the following image shows: Another simple analysis shows that a $2 \times 3$ game

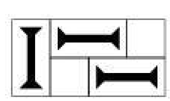


**Figure 10**

is in $\mathscr{N}$. So whoever plays first wins it.

The above observations, clubbed with the One-hand-tied principle yield very powerful deductions.
Firstly, if Right can win a $2 \times 4$ game, he can as well win a $2 \times 8$ game by pretending it is made by 2 $2 \times 4$ games. That is exactly what the sum of games is. He is pretending that the original game is the sum of 2 smaller $2 \times 4$ games, which is infact true for this game. He can then split the game into 2 games, and reply to whatever move Left makes by the reply he'd have given if it was a $2 \times 4$ game. He is effectively restricting himself from choosing some moves which cross the mental boundary between the 2 split games, but he can still win. So he wins in the original game by the One-hand-tied-principle. So he can win all $2 \times 4n$ games by the same strategy.

Another observation is noticed for $2 \times (4n+3)$ games, for all integers n. He can split the game into a game of size $2 \times 3$, and then n games of size $2 \times (4n)$. Now if Right is allowed to play first, he will surely manage to win in the $2 \times 3$ game, and he can as well win in the remaining $2 \times (4n)$ subgames too as shown above.

## 8. Future Plans

Most of us have been reasonably stimulated towards Combinatorial Game Theory while reading the basics of this subject. We have begun our course towards a proper mathematical treatment of the field, and will continue to understand games better, and deal with some really interesting concepts like values of games, their treatment as abelian groups, birthdays of games, canonical forms and more.

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## References

- **Lessons in play:** An introduction to combinatorial game theory by ***M. Albert, R.Nowakowski, D. Wolfe***
- **Wikipedia** articles on ***Combinatorial Game Theory, Nim, Hex*** and ***Tic-Tac-Toe***.
- Google Images