On Some Conditions on Generic Jacobi Theta Function and Associated Contour Integral

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Abstract

This study is an attempt to generalize and evaluate one of the Jacobi theta functions that plays a major role in the derivation of the Riemann Zeta function. The theta function as used in the Riemann Zeta functional equation uses one of the four *theta functions* which were studied by Jacobi. Out of these the third theta function used in conjunction with the Fourier transformation and more specifically the Poisson summation formula leads us to the function equation of the Zeta function. However as seen in the relevant literature [6], a special restriction imposed on this theta function somewhat hides another factor involved in theta function. We here attempt to use the general third - theta function expression and try to see what impact it might have on the further derivation. We also evaluate the integral in the zeta function expression along a simple contour thereafter.

1. Introduction

Theta functions can be used to evaluate numerical results in Elliptic function, which are some doubly periodic functions. The following preliminary derivation that forms the basis of further discussion of our interest [1]

A general form of such a theta function can be written as

$$\Theta(z,q) = \sum_{n=-\infty}^{\infty}(-1)^n q^{n^2} e^{2niz} \qquad \text{---------------------------------- (1)}$$

Here $n > 0$ and $q = e^{\pi i\tau}$ where $i = \sqrt{-1}$, $\mathrm{Im}(\tau) > 0$ where *Im* denotes the imaginary part of the complex number τ

The assumption stated above clearly indicates that | q | <1

If *A* be any positive constant such that | z | ≤ A we have

$$|\ q^{n^2} e^{\pm 2niz}\ | \le |\, q\,|^{n^2}\ e^{2nA} \qquad \text{----------------------------------(2)}$$

We observe that the d'Alembert's ratio for the series $\sum_{n=-\infty}^{\infty} |\, q\,|^{n^2}\ e^{2nA}$ is

$$|\, q\,|^{n^2-(n-1)^2}\ e^{2nA} = |\, q\,|^{2n+1}\ e^{2nA} \qquad \text{---------------------------------- (3)}$$

Since $|q|<1$, d'Alembert's ratio tends to 0 and n tends to ∞.

Thus (2) implies that our original series is uniformly convergent.

---------------------------------- (4)

We note that (1) =>

$$\Theta(z,q) = 1 + 2\sum_{n=1}^{\infty}(-1)^n q^{n^2} \cos 2nz \qquad \text{------------------------------- (5)}$$

We will not go much deeper into other aspects of the Θ function as we are more interested in a particular form of (5) when z is replaced by $z+\frac{\pi}{2}$ i.e. $\Theta(z+\frac{\pi}{2},q)$ which is also denoted by $\Theta_3(z,q)$.

Thus (5) =>

$$\begin{aligned}\Theta_3(z,q) = \Theta(z+\frac{\pi}{2},q) &= 1 + 2\sum_{n=1}^{\infty}(-1)^n q^{n^2} \cos 2n(z+\frac{\pi}{2}) \\ &= 1 + 2\sum_{n=1}^{\infty}(-1)^n q^{n^2} \cos(2nz + n\pi) \\ &= 1 + 2\sum_{n=1}^{\infty} q^{n^2} \cos 2nz\end{aligned}$$

Thus we have

$$\Theta_3(z,q) = \sum_{n=-\infty}^{\infty} q^{n^2} e^{2niz} \qquad \text{----------------------------------- (6)}$$

Our further discussion will be centered on the Θ_3 function and a subsequent functional equation Riemann used for a special case when $z=0$

Then $(6) \Rightarrow \Theta_3(0,q) = \sum_{n=-\infty}^{\infty} q^{n^2}$ -------------------------------- (7)

This condition essentially removed the cosine factor from the terms of the series.

We will attempt to study the impact of the Θ_3 function where $z=0$ and will try to analyze if there is any impact in the functional equation.

2. Some preliminary results

We now discuss some results that would be fruitful to mention beforehand, as these would be used later in our derivations [3].

Result 2.1:

$$\lim_{x\to\infty}\int_0^x e^{-t^2}dt = \frac{1}{2}\sqrt{\pi}\ ,$$

using

$$i.\ \ g'(x) + f'(x) = 0$$

$$ii.\ g(x) + f(x) = \frac{\pi}{4}$$

Proof: Consider the Riemann Integrals

$$f(x) = \left(\int_0^x e^{-t^2}dt\right)^2$$

$$g(x) = \int_0^1 \frac{e^{-x^2(t^2+1)}}{t^2+1}dt$$

Let us denote

$$\alpha(x) = \int_0^x e^{-t^2}dt$$

Let $u = \frac{t}{x} \Rightarrow dt = xdu$

$$\alpha(x) = \int_0^x e^{-t^2}dt = \int_0^1 xe^{-(ux)^2}du = \int_0^1 xe^{-(tx)^2}dt \qquad \text{---------------------------- (a1)}$$

Moreover

$$\alpha(x) = \int_0^x e^{-t^2}dt = \int_0^x e^{-x^2}dx \Rightarrow \alpha'(x) = e^{-x^2} \qquad \text{---------------------------- (a2)}$$

$(a1),(a2) \Rightarrow$

$$f'(x) = 2\alpha'(x)\alpha(x) = 2\int_0^1 xe^{-(t^2+1)x^2}dt$$

---------------------------- (a3)

Now differentiating $g(x)$ under the integration sign gives

$$g'(x) = -2\int_0^1 xe^{-(t^2+1)x^2}\,dt \qquad \text{---------------------------- (a4)}$$

$$\therefore (a3), a(4) \Rightarrow f'(x) + g'(x) = 0$$

Integrating both sides of the above equation gives

$$f(x) + g(x) = C,$$ where C is the constant of integration

$$x = 0 \Rightarrow f(x) = 0 \ \ and$$

$$g(x) = \int_0^1 \frac{1}{t^2+1}\,dt = \left[\tan^{-1} t\right]_0^1 = \frac{\pi}{4}$$

$$\therefore \ f(x) + g(x) = \frac{\pi}{4} \qquad \text{---------------------------- (a5)}$$

Now we observe that g(x) being continuous on [0,1]

$$\because g(x)\, is\, continuous\, on [0,1]$$

$$\lim_{x\to\infty} g(x) = \int_0^1 \lim_{x\to\infty}\left[\frac{e^{-(t^2+1)x^2}}{t^2+1}\,dt\right] = 0$$

$$\therefore x \to \infty, (a5) \Rightarrow \lim_{x\to\infty} f(x) = \frac{\pi}{4}$$

$$\boxed{\lim_{x\to\infty}\int_0^x e^{-t^2}\,dt = \frac{\sqrt{\pi}}{2}}$$

Result 2.2:

$$If \ F(y) = \int_0^\infty e^{-x^2}\cos 2xy dx, y \in R\, then \ F'(y) + 2yF(y) = 0\, and\ deduce\, that$$

$$F(y) = \frac{\sqrt{\pi}e^{-y^2}}{2}$$

Proof: Differentiating $F(y)$ under integration sign being valid as the function is continuous and partial derivatives exist we have

$$F'(y) = -2\int_0^\infty xe^{-x^2}\sin 2xy dx$$

Integrating by parts

$$F'(y) = -\left\{ \left[e^{-x^2} \sin 2xy \right]_0^\infty + 2y \int_0^\infty e^{-x^2} \cos 2xy dx \right\}$$

$$\Rightarrow F'(y) = -(0 + 2yF(y))$$

$$\Rightarrow F'(y) + 2yF(y) = 0$$

Integrating both sides of the above equation

$$\int \frac{dF(y)}{F(y)} + 2\int y dy = 0$$

$$\Rightarrow \ln F(y) + y^2 = C$$

(*C is the constant of integration*) ------------------------------------ (b1)

$$\therefore \; y = 0 \Rightarrow F(y) = \int_0^\infty e^{-x^2} dx = \frac{\sqrt{\pi}}{2}$$ (Using Result 2.1)

Thus $(b1) \Rightarrow C = \ln \frac{\sqrt{\pi}}{2}$

Substituting the value of C back we have

$$\boxed{\int_0^\infty e^{-x^2} \cos 2xy dx = \frac{\sqrt{\pi} e^{-y^2}}{2}}$$

3. Poisson Summation Formula

The Poisson Summation formula forms an essential part of the study of Fourier series and integrals [2].

3.1 Theorem: Let f be a nonnegative function such that the integral $\int_{-\infty}^{\infty} f(x) dx$ exists as an improper Riemann integral. Assume also that f increases on $(-\infty, 0]$ and decreases on $[0, +\infty)$. Then we have

$$\sum_{m=-\infty}^{\infty} \frac{f(m+) + f(m-)}{2} = \sum_{n=-\infty}^{\infty} \int_{-\infty}^{+\infty} f(t) e^{-2\pi \text{int}} dt$$

When f is continuous at m the above formula becomes

$$\boxed{\sum_{m=-\infty}^{+\infty} f(m) = \sum_{n=-\infty}^{+\infty} \int_{-\infty}^{+\infty} f(t) e^{-2\pi \text{int}} dt}$$

4. Functional Equation of the Jacobi Theta Function

The functional equation of the Jacobi Theta function with respect to $\Theta_3(0,q)$ uses the Poisson Summation formula.

Recall that from our earlier discussion the theta function can be expressed as follows

$$\Theta_3(0,q) = \sum_{n=-\infty}^{\infty} q^{n^2} \qquad \text{(using (7))}$$

Since $q = e^{\pi i \tau}$ with the necessary condition that *Im*(τ) > 0 let us express $\tau = ix$ where x >0

Then the above expression takes the form

$$\Theta(x) = \Theta_3(0,x) = \sum_{n=-\infty}^{\infty} e^{-\pi x n^2}$$

The functional equation of the theta function states that

$$\Theta(x) = \frac{1}{\sqrt{x}}\Theta(\frac{1}{x})$$

We present here the proof of the functional equation so that it becomes easy to comprehend the derivation for the generalized Θ_3 function afterwards where z≠ 0.

Proof:

Let us consider the function $f(x) = e^{-\alpha x^2}$ for fixed α > 0.

This function satisfies the pre conditions required for the Poisson Summation formula, which

implies $$\sum_{n=-\infty}^{\infty} e^{-\alpha n^2} = \sum_{n=-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-\alpha t^2} e^{-2i\pi n t} dt$$

---------------------------------- (c1)

Note that the left side of (c1) is $\Theta(\frac{\alpha}{\pi})$

$$\int_{-\infty}^{\infty} e^{-\alpha t^2} e^{-2i\pi n t} dt = 2\int_{0}^{\infty} e^{-\alpha t^2} \cos 2\pi n t dt \qquad \text{---------------------------------- (c2)}$$

Substitute $x = t\sqrt{\alpha} \Rightarrow dx = \sqrt{\alpha}dt$

$$\therefore (c2) \Rightarrow \int_{-\infty}^{\infty} e^{-\alpha t^2} e^{-2i\pi n t} dt = \frac{2}{\sqrt{\alpha}}\int_{0}^{\infty} e^{-x^2} \cos\frac{2\pi n x}{\sqrt{\alpha}} dx = \frac{2}{\sqrt{\alpha}} F(\frac{\pi n}{\sqrt{\alpha}}) \qquad \text{------------------------------- (c3)}$$

where $F(y) = \int_{0}^{\alpha} e^{-x^2} \cos 2xy dx$

However, from result 2.2, $F(y) = \dfrac{\sqrt{\pi}e^{-y^2}}{2}$

$$\therefore (c3) \Rightarrow \int_{-\infty}^{\infty} e^{-\alpha t^2} e^{-2i\pi n t} dt = \left(\frac{2}{\sqrt{\alpha}}\right)\left(\frac{\sqrt{\pi}}{2}\right) e^{-\frac{\pi^2 n^2}{\alpha}}$$

$$\Rightarrow \Theta\left(\frac{\alpha}{\pi}\right) = \sqrt{\frac{\pi}{\alpha}} e^{-\frac{\pi^2 n^2}{\alpha}}$$

Substituting back $\alpha = \pi x$, we have

$$\boxed{\Theta(x) = \left(\frac{1}{\sqrt{x}}\right)\Theta\left(\frac{1}{x}\right)}$$

5. Main Result

By now we have covered all the aspects required to generalize the above functional equation.

We now try to deduce a similar kind of result for $\Theta_3(z,q)$ where z≠ 0

Recall expression (6) in the beginning

$$\Theta_3(z,q) = \sum_{n=-\infty}^{\infty} q^{n^2} e^{2niz}$$

Proceeding as per the derivation for functional equation above let us define a function f as

$$f(z,x) = \Theta_3(z,x) = \sum_{n=-\infty}^{\infty} e^{-\pi x n^2} (\cos 2nz + i \sin 2nz)$$

-------------------------------- (8)

Let us consider the kth term where of this series viz.

$$e^{-\pi x k^2} (\cos 2kz + i \sin 2kz) = e^{-\pi x k^2} \cos 2kz + i e^{-\pi x k^2} \sin 2kz = u(x,z) + iv(x,z),$$
$$where \quad u(x,z) = e^{-\pi x k^2} \cos 2kz \ and \ v(x,z) = e^{-\pi x k^2} \sin 2kz$$

Now for $f'(z,x)$ to exist at any point the *Cauchy-Riemann (CR) equations* must be satisfied so that we have $u_z = v_x$ *and* $u_x = -v_z$ where the subscripts denote the variable with respect to which the function is partially differentiated.

We now rearrange the variables in such a manner that the CR equations hold good.

$$f(x,z) = e^{-\pi z k^2} (\cos \pi x k^2 + i \sin \pi x k^2),$$
$$where \ \ u(x,z) = e^{-\pi z k^2} \cos \pi x k^2 \ \ and \ \ v(x,z) = e^{-\pi z k^2} \sin \pi x k^2$$

$$\left[\begin{array}{l} Here\ \ u_x = -\pi k^2 e^{-\pi z k^2} \sin \pi x k^2,\ \ v_z = -\pi k^2 e^{-\pi z k^2} \sin \pi x k^2 \\ u_z = -\pi k^2 e^{-\pi z k^2} \cos \pi x k^2,\ \ v_x = \pi k^2 e^{-\pi z k^2} \cos \pi x k^2,\ which\, satisfies\, the\, CR\ Equations \end{array}\right]$$

So modifying (8) accordingly we have,

$$\Theta_3(x,z) = \sum_{n=-\infty}^{\infty} e^{-\pi z k^2} (\cos \pi x k^2 + i \sin \pi x k^2)$$

-------------------------------- (9)

Let us allow x = y so that (6) =>

$$\Theta_3(x,x) = \sum_{k=-\infty}^{\infty} e^{-\pi x k^2} (\cos \pi x k^2 + i \sin \pi x k^2) = \sum_{k=-\infty}^{\infty} e^{-\pi x k^2} e^{i \pi x k^2}$$

It will be worthwhile to mention that by (4) the above series is uniformly convergent which is a prerequisite for applying the Poisson summation formula. (*The fact that f decreases on [0, ∞) is used to establish this condition*)

For fixed $\alpha > 0$, let us define $f(x)$ as follows

$$f(x) = e^{-\alpha x^2} e^{i \alpha x^2}$$

On applying the Poisson Summation formula we have

$$\sum_{m=-\infty}^{\infty} e^{-\alpha m^2} e^{i \alpha m^2} = \sum_{n=-\alpha}^{\infty} \int_{-\infty}^{\infty} \left(e^{-\alpha t^2} e^{i \alpha t^2}\right) e^{-2 i \pi n t} dt$$

------------------------------------ (10)

$Now \;\; I =$

$$\int_{-\infty}^{\infty} e^{-\alpha t^2} e^{i\alpha t^2} e^{-2i\pi n t} dt = \int_{-\infty}^{\infty} e^{-\alpha t^2}\left(\cos\alpha t^2 + i\sin\alpha t^2\right)\left(\cos 2\pi n t + i\sin 2\pi n t\right)dt$$

$$= \int_{-\infty}^{0} e^{-\alpha t^2}\left(\cos\alpha t^2 + i\sin\alpha t^2\right)\left(\cos 2\pi n t + i\sin 2\pi n t\right)dt +$$

$$\int_{0}^{\infty} e^{-\alpha t^2}\left(\cos\alpha t^2 + i\sin\alpha t^2\right)\left(\cos 2\pi n t + i\sin 2\pi n t\right)dt$$

$$= I_1 + I_2,$$

$$where \quad I_1 = \int_{-\infty}^{0} e^{-\alpha t^2}\left(\cos\alpha t^2 + i\sin\alpha t^2\right)\left(\cos 2\pi n t + i\sin 2\pi n t\right)dt \qquad and$$

$$I_2 = \int_{0}^{\infty} e^{-\alpha t^2}\left(\cos\alpha t^2 + i\sin\alpha t^2\right)\left(\cos 2\pi n t + i\sin 2\pi n t\right)dt$$

-------------------------- (11)

$Substituting \quad t = -u \Rightarrow dt = -du, \; we\, have$

$$I_1 = \int_{0}^{\infty} e^{-\alpha u^2}\left(\cos\alpha u^2 + i\sin\alpha u^2\right)\left(\cos 2\pi n u - i\sin 2\pi n u\right)du$$

$$= \int_{0}^{\infty} e^{-\alpha t^2}\left(\cos\alpha t^2 + i\sin\alpha t^2\right)\left(\cos 2\pi n t - i\sin 2\pi n t\right)dt$$

$\therefore (11) \Rightarrow$

$$I = 2\int_{0}^{\infty} e^{-\alpha t^2}\left(\cos\alpha t^2 + i\sin\alpha t^2\right)\cos 2\pi n t dt$$

-------------------------- (12)

Now using (10) and (12) and comparing the real imaginary parts

$$\sum_{m=-\infty}^{\infty} e^{-\alpha m^2} \;\cos\alpha m^2 = 2\int_{0}^{\infty} e^{-\alpha t^2} \;\cos\alpha t^2 \;\cos 2\pi n t dt$$

-------------------------- (13)

Let $t\sqrt{\alpha} = v \Rightarrow \alpha t^2, dv = dt\sqrt{\alpha}$, so that the integral in (13) becomes

$$\int_0^\infty e^{-\alpha t^2} \cos \alpha t^2 \cos 2\pi ntdt = \frac{1}{\sqrt{\alpha}} \int_0^\infty e^{-v^2} \cos v^2 \cos \frac{2\pi nv}{\sqrt{\alpha}} dv$$

$$= \frac{1}{\sqrt{\alpha}} F\left(\frac{\pi n}{\sqrt{\alpha}}\right), \quad where\ F(y) = \int_0^\infty e^{-x^2} \cos x^2 \cos 2xydx$$

-------------------------- (14)

Now

$$F(y) = \int_0^\infty e^{-x^2} \cos x^2 \cos 2xydx = \frac{1}{2}\int_0^\infty e^{-x^2} \left[\cos\left(x^2 + 2xy\right) + \cos\left(x^2 = 2xy\right)\right]dx$$

$$\Rightarrow 2F'(y) = -2\int_0^\infty xe^{-x^2} \sin\left(x^2 + 2xy\right)dx + 2\int_0^\infty xe^{-x^2} \sin\left(x^2 - 2xy\right)dx$$

$$\Rightarrow F'(y) = -A_1 + A_2 \qquad , where \quad A_1 = \int_0^\infty xe^{-x^2} \sin\left(x^2 + 2xy\right)dx, \quad A_2 = \int_0^\infty xe^{-x^2} \sin\left(x^2 - 2xy\right)dx$$

-------------------------- (15)

Integrating by parts,

$$A_1 = \int_0^\infty xe^{-x^2} \sin\left(x^2 + 2xy\right)dx, \ = \left[-\frac{1}{2}\sin\left(x^2 + 2xy\right)e^{-x^2}\right]_0^\infty + \frac{1}{2}\int_0^\infty 2(x+y)\cos\left(x^2 + 2xy\right)e^{-x^2} dx$$

$$\Rightarrow A_1 = \int_0^\infty xe^{-x^2} \cos(x^2 + 2xy)dx \quad + \quad y\int_0^\infty e^{-x^2} \cos\left(x^2 + 2xy\right)dx$$

------------------------- (16)

Similarly

$$A_2 = \int_0^\infty xe^{-x^2} \cos(x^2 - 2xy)dx - y\int_0^\infty e^{-x^2} \cos(x^2 - 2xy)dx$$

------------------------- (17)

$(15),(16),(17) \Rightarrow$

$$F'(y) = \int_0^\infty xe^{-x^2}\left[\cos(x^2-2xy) - \cos(x^2+2xy)\right]dx - y\int_0^\infty e^{-x^2}\left[\cos(x^2-2xy)+\cos(x^2+2xy)\right]dx$$

$$= 2\left[\int_0^\infty xe^{-x^2}\sin x^2 \sin 2xy dx - y\int_0^\infty e^{-x^2}\cos x^2 \cos 2xy dx\right]$$

$$\therefore \quad F'(y) = 2\int_0^\infty xe^{-x^2}\sin x^2 \sin 2xy dx \; - 2yF(y)$$

$$\Rightarrow F'(y) + 2yF(y) = 2\int_0^\infty xe^{-x^2}\sin x^2 \sin 2xy dx$$

------------------------ (18)

Let us define

$$f(x) = 2xe^{-x^2}\sin x^2 \sin 2xy = xe^{-x^2}\left[\cos\left(x^2-2xy\right) - \cos\left(x^2+2xy\right)\right]$$

Now,

$$f(x) = 0 \Rightarrow \cos\left(x^2+2xy\right) = \cos\left(x^2-2xy\right) \Rightarrow x^2+2xy = 2k\pi \pm x^2 - 2xy$$

$$\Rightarrow x^2 = k\pi \; (k>0), \quad xy = \frac{k\pi}{2}, \quad k \in Z$$

Thus under the following conditions we have LHS of (18) equal to zero. This condition defines a relation between the variables x *and* y so that the final equation is in the same form as it was before.

$$\boxed{xy = \frac{k\pi}{2}, \qquad k \in Z}$$ ------------------------ (19)

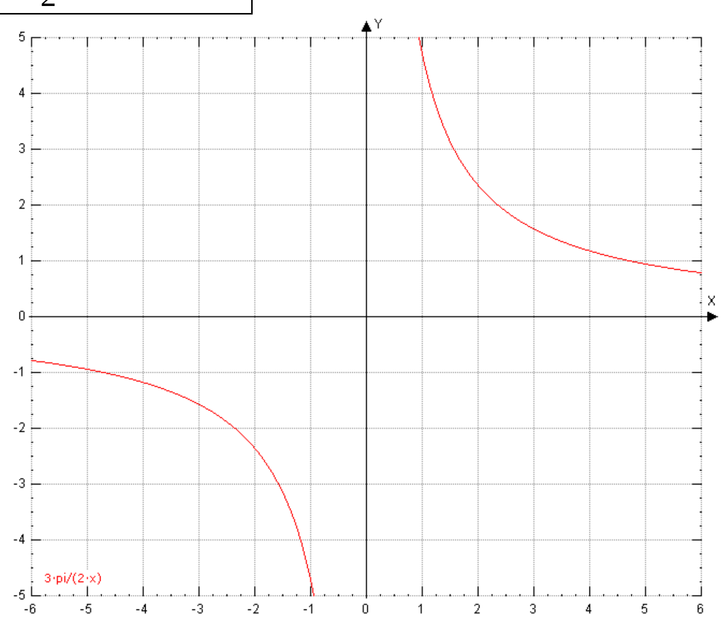


Fig 1. A typical plot of equation (19)

Another interesting fact about the integral can be analyzed as follows.

$\because \qquad -1 \le \sin x^2 \le 1$

$\therefore -2xe^{-x^2} \sin 2xy \le 2xe^{-x^2} \sin x^2 \sin 2xy \le 2xe^{-x^2} \sin 2xy$

$$\Rightarrow -\int_0^\infty 2xe^{-x^2} \sin 2xy \le 2\int_0^\infty xe^{-x^2} \sin x^2 \sin 2xy \le \int_0^\infty 2xe^{-x^2} \sin 2xy \qquad \text{\_\_\_\_\_\_\_\_\_\_\_\_\_\_(20)}$$

$$Now \quad \int_0^\infty 2xe^{-x^2} \sin 2xydx = \left[-e^{-x^2} \sin 2xy\right]_0^\infty + 2y\int_0^\infty e^{-x^2} \cos 2xydx$$

$$= 2y\int_0^\infty e^{-x^2} \cos 2xydx$$

$$= y\sqrt{\pi}e^{-y^2} \,(\,from\;\; 2.2) \qquad \text{\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_(21)}$$

$(18),(20),(21) \Rightarrow$

$$\lim_{y\to\infty} 2\int_0^\infty xe^{-x^2} \sin x^2 \sin 2xy = 0$$

(22) ________________

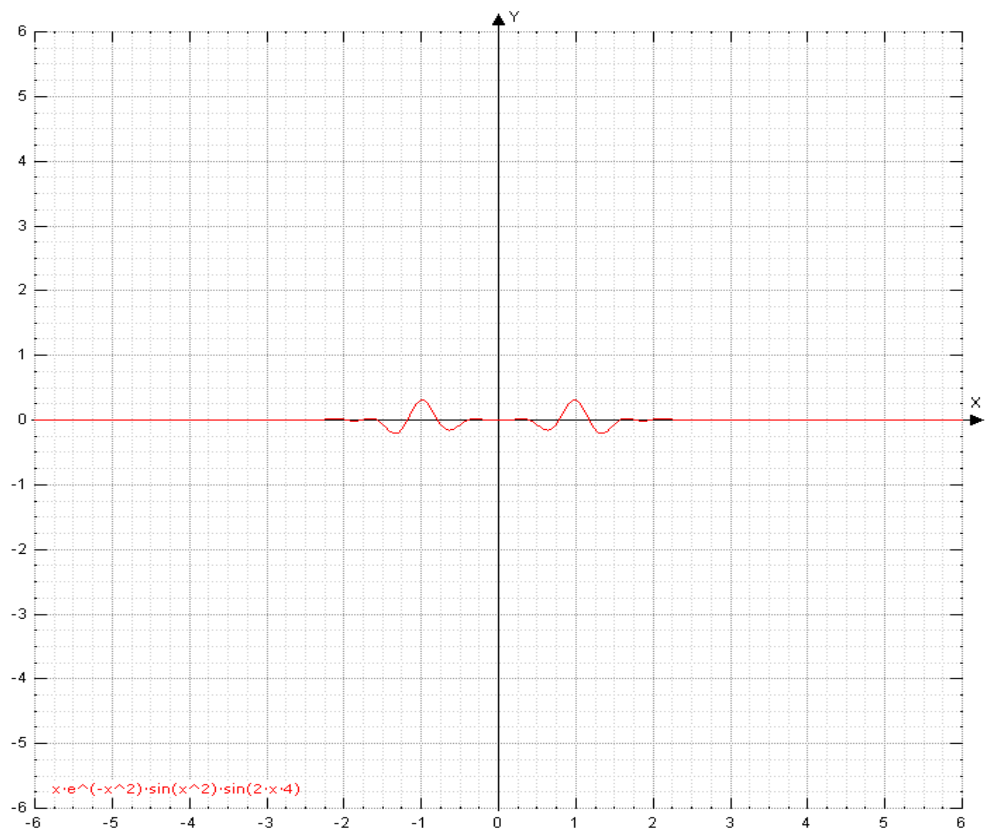


Fig 2. Plot of the integral for small y(=4)

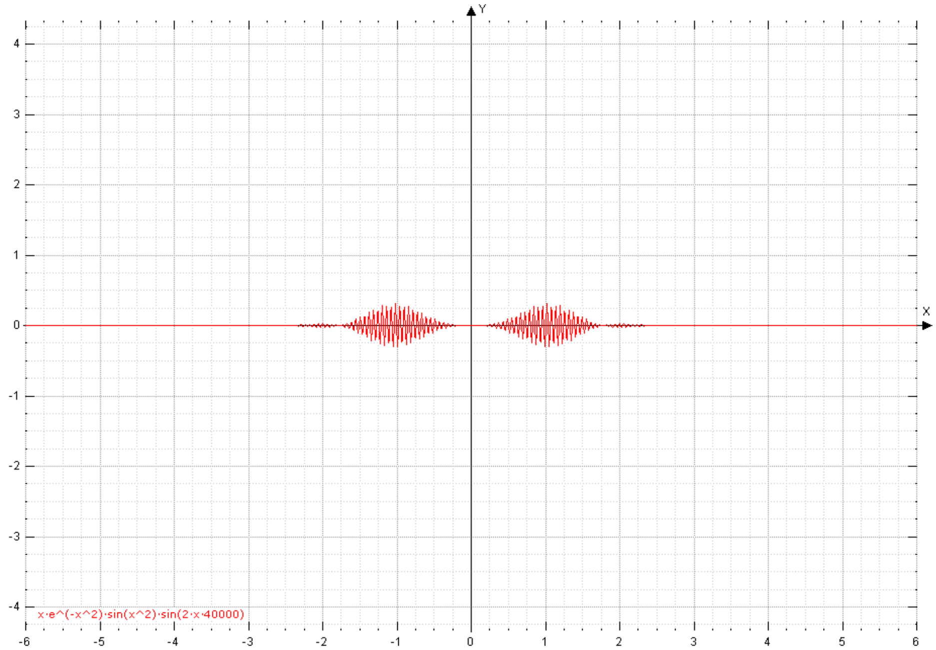


Fig 3. Plot of the integral for higher value of y(=40000)

So that we have using (19),(22)

$$\frac{1}{2}F'(y)+yF(y)=0$$

Using Result 2.2 above we have

$$F(y)=\frac{\sqrt{\pi}e^{-y^2}}{2} \qquad \text{------------------------- (20)}$$

Now, $(13),(14),(20)\Rightarrow$

$$\sum_{m=-\infty}^{\infty}e^{-\alpha m^2}\cos\alpha m^2 = 2\int_0^\infty e^{-\alpha t^2}\cos\alpha t^2\cos 2\pi n t dt$$

$$=\frac{2}{\sqrt{\alpha}}F\left(\frac{\pi n}{\sqrt{\alpha}}\right)$$

$$=\sqrt{\frac{\pi}{\alpha}}e^{-\frac{\pi^2 n^2}{\alpha}}$$

Substituting $\alpha=\pi x$ as earlier we get

$$\sum_{m=-\infty}^{\infty}e^{-\pi x m^2}\cos\pi x m^2 = \frac{1}{\sqrt{x}}e^{-\frac{\pi n^2}{x}} = \sum_{m-\infty}^{\infty}e^{-\pi x m^2}$$

Using the notations we have used and under the conditions specified in (19) we have

$$\boxed{\Theta_3(x,x)=\Theta_3(x,0)=\Theta_3(x)}$$ ------------------------- (21)

Expression (21) can be regarded as the justification for using the theta function without the cosine factors being included in the series expression.

Now that we have a relation defined under a suitable condition as per (21) we will analyze what further impact it might have and together we try to do a further analysis on the expression for zeta function and attempt a contour integration on the expression.

Following is the expression for the zeta function-

$$\zeta(s)=\frac{\pi^{\frac{s}{2}}}{\Gamma\left(\frac{s}{2}\right)}\left\{\frac{1}{s(s-1)}+\int_1^{\infty}\left(x^{\frac{s}{2}-1}+x^{-\frac{s}{2}-\frac{1}{2}}\right)\left(\frac{\Theta(x)-1}{2}\right)dx\right\}$$

------------------------- (22)

But $\Theta(x,x)=1+2\sum_{m=1}^{\infty}e^{-\pi xm^2}\cos\pi xm^2=\Theta(x)$ (using (21))

We assume (21) as valid under the conditions specified and analyze the integrals involved in the zeta function.

Moreover from (4) as discussed above we have the theta function series as uniformly convergent so that (22) may be expressed as

$$\zeta(s)=\frac{\pi^{\frac{s}{2}}}{\Gamma\left(\frac{s}{2}\right)}\left\{\frac{1}{s(s-1)}+\sum_{m=1}^{\infty}\int_1^{\infty}\left(x^{\frac{s}{2}-1}+x^{-\frac{s}{2}-\frac{1}{2}}\right)e^{-\pi xm^2}\cos(\pi xm^2)dx\right\}$$

$$\Rightarrow\zeta(s)=\frac{\pi^{\frac{s}{2}}}{\Gamma\left(\frac{s}{2}\right)}\left\{\frac{1}{s(s-1)}+\sum_{m=1}^{\infty}\left(\int_1^{\infty}x^{\frac{s}{2}-1}e^{-\pi xm^2}\cos(\pi xm^2)dx+\int_1^{\infty}x^{-\frac{s}{2}-\frac{1}{2}}e^{-\pi xm^2}\cos(\pi xm^2)dx\right)\right\}$$

$$\Rightarrow\zeta(s)=\frac{\pi^{\frac{s}{2}}}{\Gamma\left(\frac{s}{2}\right)}\left\{\frac{1}{s(s-1)}+\sum_{m=1}^{\infty}\left(I_{1m}+I_{2m}\right)\right\},$$

$$where\quad I_{1m}=\int_1^{\infty}x^{\frac{s}{2}-1}e^{-\pi xm^2}\cos(\pi xm^2)dx\quad and\quad I_{2m}=\int_1^{\infty}x^{-\frac{s}{2}-\frac{1}{2}}e^{-\pi xm^2}\cos(\pi xm^2)dx,$$

------------------------- (23)

We evaluate the k^{th} terms of the series in (23) for some cases depicted below.

Case 1:

Let $s = a + ib,\ b = 0\ (i.e.\ s \in \Re)$,

$$I_{1k} = \int_1^\infty x^{\frac{s}{2}-1} e^{-\pi x k^2} \cos \pi x k^2 dx = \int_0^\infty x^{\frac{a}{2}-1} e^{-\pi x k^2} \cos \pi x k^2 dx - \int_0^1 x^{\frac{a}{2}-1} e^{-\pi x k^2} \cos \pi x k^2 dx$$

$$\Rightarrow I_{1k} = I_1 - I_2, \quad I_1 = \int_0^\infty x^{\frac{a}{2}-1} e^{-\pi x k^2} \cos \pi x k^2 dx,\ I_2 = \int_0^1 x^{\frac{a}{2}-1} e^{-\pi x k^2} \cos \pi x k^2 dx$$

------------------------- (24)

Recall result 2.3 above

$$\int_0^\infty e^{-Ax} \cos(Bx) x^{m-1} dx = \frac{\Gamma(m)}{r^m} \cos m\theta, \quad where\ r = \sqrt{A^2 + B^2}\ and\ \Theta = \tan^{-1}\left(\frac{B}{A}\right)$$

Comparing the above result with I_1 in (24) we have $A = \pi k^2, B = \pi k^2, m = \dfrac{a}{2}$

$$\therefore \theta = \tan^{-1}\left(\frac{\pi k^2}{\pi k^2}\right) = \frac{\pi}{4},\ r = \sqrt{2\left(\pi k^2\right)^2} = \pi k^2 \sqrt{2}$$

Thus $I_1 = \dfrac{\Gamma\left(\frac{a}{2}\right)}{\left(\pi k^2 \sqrt{2}\right)^{\frac{a}{2}}} \cos\left(\dfrac{\pi a}{8}\right) = \dfrac{\Gamma\left(\frac{a}{2}\right)}{k\left(\pi \sqrt{2}\right)^{\frac{a}{2}}} \cos\left(\dfrac{\pi a}{8}\right)$

------------------------- (25)

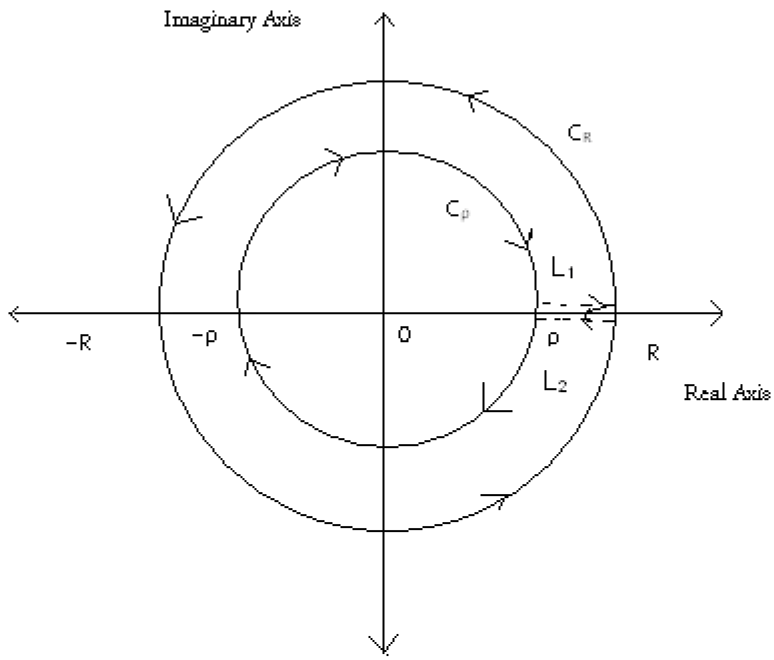
For the second term I_2 in (24), if $\dfrac{a}{2} - 1 > 0$ then on integrating by parts the integral I_2 will generate some integrals with reducing powers of x over a . We are safely ignoring this condition as of now as our area of interest is the critical strip, so, we assume here $\dfrac{a}{2} - 1 < 0, a > 0$

Put $-u = \dfrac{a}{2} - 1,\ u > 0\ and\ A = \pi k^2, B = \pi k^2$

Consider the complex integral $\displaystyle\int_0^1 z^{-u} e^{-Az} e^{iBz} dz,\ where\, z \in C$

The integral above has a simple pole as z = 0.

Consider the contour below where $0 < \rho < R$ where we have isolated the origin.



The function would be analytic inside this contour so that we can apply Cauchy-Goursat theorem.

$$\int_{L_1} z^{-u} e^{(-A+iB)z}\,dz + \int_{C_R} z^{-u} e^{(-A+iB)z}\,dz + \int_{L_2} z^{-u} e^{(-A+iB)z}\,dz + \int_{C_\rho} z^{-u} e^{(-A+iB)z}\,dz = 0$$

------------------------- (26)

Substitute $z = re^{i\theta}$ for the first and the third terms in (26)

Now along L_1 in the contour we have $z = re^{i0} = r$

Similarly along L_2 we find

$z = re^{i2\pi} = r(\cos 2\pi + i\sin 2\pi) = r$, $z^{-u} = e^{-u\ln z} = e^{-u[\ln r + i2\pi]} = r^{-u}e^{-2i\pi u}$

Using the above values along L_1 and L_2 we have,

$$\int_{L_1} z^{-u} e^{(-A+iB)z}\,dz = \int_{\rho}^{R} r^{-u} e^{(-A+iB)r}\,dr$$

$$\int_{L_2} z^{-u} e^{(-A+iB)z}\,dz = \int_{R}^{\rho} r^{-u} e^{-2i\pi u} e^{(-A+iB)r}\,dr$$

------------------------ (27)

$(26),(27) \Rightarrow$

$$\int_{\rho}^{R} r^{-u} e^{(-A+iB)r} dr + \int_{C_R} z^{-u} e^{(-A+iB)z} dz + \int_{R}^{\rho} r^{-u} e^{-2i\pi u} e^{(-A+iB)r} dr + \int_{C_\rho} z^{-u} e^{(-A+iB)z} dz = 0$$

$$\Rightarrow (1-e^{-2i\pi u})\int_{\rho}^{R} r^{-u} e^{(-A+iB)r} dr = -\left(\int_{C_R} z^{-u} e^{(-A+iB)z} dz + \int_{C_\rho} z^{-u} e^{(-A+iB)z} dz \right)$$

------------------------ (28)

We now try to evaluate the second term on the RHS of (28).

We have

$$\left| z^{-u} e^{(-A+iB)z} \right| = \left| z^{-u} \right| \left| e^{(-A+iB)z} \right| = \rho^{-u} \left| e^{i\phi} \right| \quad (where\ i\phi = (-A+iB)z\,,\ |z| = \rho)$$

$$\Rightarrow \left| z^{-u} e^{(-A+iB)z} \right| \ = \rho^{-u} \qquad (\because Euler's\ identity\ is\ valid\ for\ complex\ \mathrm{var}\,iables)$$

$$\therefore \left| \int_{C_\rho} z^{-u} e^{(-A+iB)z} dz \right| \le ML,\ where\ M = \rho^{-u}\ and\ L = 2\pi\rho$$

$$\Rightarrow \left| \int_{C_\rho} z^{-u} e^{(-A+iB)z} dz \right| \le 2\pi\rho^{-u+1} = 2\pi\rho^{\frac{a}{2}} \quad (\because -u = \frac{a}{2} - 1)$$

$$\Rightarrow \lim_{\rho\to 0} \left| \int_{C_\rho} z^{-u} e^{(-A+iB)z} dz \right| = 0 \quad (a > 0)$$

------------------------ (29)

Proceeding in a similar manner we get

$$\lim_{R\to 1} \left| \int_{C_R} z^{-u} e^{(-A+iB)z} dz \right| \le 2\pi$$ ------------------------ (30)

Using (28), (29) and (30) and as $\rho \to 0\, and\ R \to 1$ we get

$$(1-e^{-2i\pi u})\int_{0}^{1} r^{-u} e^{(-A+iB)r} dr = -2\pi \quad (\max imum\ possible\ value\ from\,(30))$$

$$\Rightarrow \int_{0}^{1} r^{-u} e^{(-A+iB)r} dr = \frac{-2\pi}{(1-e^{-2i\pi u})} = \frac{-2\pi}{\left[(1-\cos 2\pi u) + i \sin 2\pi u\right]} = \frac{-2\pi\left[(1-\cos 2\pi u) - i\sin 2\pi u\right]}{\left[(1-\cos 2\pi u)^2 + \sin^2 2\pi u\right]}$$

Comparing the real and imaginary parts in the above equation we get

$$\int_{0}^{1} r^{-u} e^{-Ar} \cos dr = \frac{-2\pi(1-\cos 2\pi u)}{\left[(1-\cos 2\pi u)^2 + \sin^2 2\pi u\right]}$$

-------------------------- (31)

A stricter condition imposed on (30) can modify (31)

$$\int_0^1 r^{-u} e^{-Ar} \cos dr = \frac{(\delta - 2\pi)(1 - \cos 2\pi u)}{\left[(1-\cos 2\pi u)^2 + \sin^2 2\pi u\right]}, 0 \leq \delta < 2\pi$$

-------------------------- (32)

$(24),(25),(32) \Rightarrow$

$$I_{1k} = I_1 - I_2 = \frac{\Gamma\left(\frac{a}{2}\right)\cos\left(\frac{\pi a}{8}\right)}{k\left(\pi\sqrt{2}\right)^{\frac{a}{2}}} - \frac{(\delta - 2\pi)(1-\cos 2\pi u)}{\left[(1-\cos 2\pi u)^2 + \sin^2 2\pi u\right]}$$

Substituting $u = 1 - \frac{a}{2}$ we get

$$I_{1k} = I_1 - I_2 = \frac{\Gamma\left(\frac{a}{2}\right)\cos\left(\frac{\pi a}{8}\right)}{k\left(\pi\sqrt{2}\right)^{\frac{a}{2}}} - \frac{(\delta - 2\pi)(1-\cos 2\pi u)}{\left[(1-\cos 2\pi u)^2 + \sin^2 2\pi u\right]}$$

$$= \frac{\Gamma\left(\frac{a}{2}\right)\cos\left(\frac{\pi a}{8}\right)}{k\left(\pi\sqrt{2}\right)^{\frac{a}{2}}} - \frac{(\delta - 2\pi)(1-\cos(2\pi - a\pi))}{\left[(1-\cos(2\pi - a\pi))^2 + \sin^2(2\pi - a\pi)\right]}$$

$$= \frac{\Gamma\left(\frac{a}{2}\right)\cos\left(\frac{\pi a}{8}\right)}{k\left(\pi\sqrt{2}\right)^{\frac{a}{2}}} - \frac{(\delta - 2\pi)(1-\cos(a\pi))}{\left[(1-\cos(a\pi))^2 + \sin^2(a\pi)\right]}$$

$$= \frac{\Gamma\left(\frac{a}{2}\right)\cos\left(\frac{\pi a}{8}\right)}{k\left(\pi\sqrt{2}\right)^{\frac{a}{2}}} - \frac{(\delta - 2\pi)\sin^2\frac{a\pi}{2}}{\left[2\sin^2\frac{a\pi}{2}\left(\sin^2\frac{a\pi}{2} + \cos^2(\frac{a\pi}{2})\right)\right]}$$

$$\therefore \quad I_{1k} = \frac{\Gamma\left(\frac{a}{2}\right)\cos\left(\frac{\pi a}{8}\right)}{k\left(\pi\sqrt{2}\right)^{\frac{a}{2}}} - \left(\frac{\delta}{2} - \pi\right)$$

----

(33)

Our next task is to evaluate I_{2k} in (23) under the same conditions –

$$I_{2k} = \int_1^{\infty} x^{-\frac{s}{2}-\frac{1}{2}} e^{-\pi x k^2} \cos(\pi x k^2) dx = \int_0^{\infty} x^{-\frac{s}{2}-\frac{1}{2}} e^{-\pi x k^2} \cos(\pi x k^2) dx - \int_0^{1} x^{-\frac{s}{2}-\frac{1}{2}} e^{-\pi x k^2} \cos(\pi x k^2) dx$$

$$\Rightarrow I_{2k} = \int_0^{\infty} x^{-\frac{a}{2}-\frac{1}{2}} e^{-\pi x k^2} \cos(\pi x k^2) dx - \int_0^{1} x^{-\frac{a}{2}-\frac{1}{2}} e^{-\pi x k^2} \cos(\pi x k^2) dx$$

$$\Rightarrow I_{2k} = T_1 - T_2, \quad where\ T_1 = \int_0^{\infty} x^{-\frac{a}{2}-\frac{1}{2}} e^{-\pi x k^2} \cos(\pi x k^2) dx\ and\ \ T_2 = \int_0^{1} x^{-\frac{a}{2}-\frac{1}{2}} e^{-\pi x k^2} \cos(\pi x k^2) dx$$

----------------------------- (34)

Using result 2.3 from above again

$$\int_0^{\infty} e^{-Ax} \cos(Bx) x^{m-1} dx = \frac{\Gamma(m)}{r^m} \cos m\theta, \quad where\ r = \sqrt{A^2 + B^2}\ and\ \Theta = \tan^{-1}\left(\frac{B}{A}\right)$$

Comparing the first term in the above expression with the following result again –

$$A = \pi k^2, B = \pi k^2, m = 1 - \frac{a}{2}, \Theta = \tan^{-1} 1 = \frac{\pi}{4}, r = \pi k^2 \sqrt{2}$$

$$\therefore T_1 = \frac{\Gamma\left(1 - \frac{a}{2}\right) \cos\left(\frac{\pi}{4} - \frac{a\pi}{8}\right)}{\left(\pi k^2 \sqrt{2}\right)^{\left(1 - \frac{a}{2}\right)}}$$

----------------------------- (35)

For term T_2 in (34) we evaluate the following complex integral as earlier

$$\int_0^1 z^{-u} e^{-Az} e^{iBz} dz, \ where\ z \in C\ and\ u = \frac{a+1}{2}$$

$$\because u > 0\ ,\ z^{-u}\ \ has\ a\ \ simple\ pole\ at\ z = 0$$

$$\therefore (32) \Rightarrow$$

$$\int_0^1 r^{-u}e^{-Ar}\cos dr = \frac{(\delta'-2\pi)(1-\cos 2\pi u)}{\left[(1-\cos 2\pi u)^2+\sin^2 2\pi u\right]}, 0\le \delta' < 2\pi$$

$$= \frac{(\delta'-2\pi)(1-\cos(\pi+\pi a)}{\left[(1-\cos(\pi+\pi a))^2+\sin^2(\pi+\pi a)\right]} \left(\because u=\frac{a+1}{2}\right)$$

$$= \frac{(\delta'-2\pi)(1+\cos \pi a)}{\left[(1-\cos(\pi a))^2+\sin^2(\pi a)\right]}$$

$$= \frac{(\delta'-2\pi)(2\cos^2\frac{\pi a}{2})}{4\cos^2\frac{\pi a}{2}\left[\cos^2\frac{\pi a}{2}+\sin^2\frac{\pi a}{2}\right]}$$

$$= \frac{\delta'}{2}-\pi$$

$$\therefore T_2 = \frac{\delta'}{2}-\pi$$

----------------------------- (36)

$(34),(35),(36) \Rightarrow$

$$I_{2k} = \frac{\Gamma\left(1-\frac{a}{2}\right)\cos\left(\frac{\pi}{4}-\frac{a\pi}{8}\right)}{\left(\pi k^2\sqrt{2}\right)^{\left(1-\frac{a}{2}\right)}} - \left(\frac{\delta'}{2}-\pi\right)$$

----------------------------- (37)

Using $(33),(37)$ we get

$$I_{1k}+I_{2k} = \frac{\Gamma\left(\frac{a}{2}\right)\cos\left(\frac{\pi a}{8}\right)}{k\left(\pi\sqrt{2}\right)^{\frac{a}{2}}} + \frac{\Gamma\left(1-\frac{a}{2}\right)\cos\left(\frac{\pi}{4}-\frac{a\pi}{8}\right)}{\left(\pi k^2\sqrt{2}\right)^{\left(1-\frac{a}{2}\right)}} - 2\pi+\varepsilon, 0\le\varepsilon<2\pi$$

----

(38)

$$\therefore (23),(38) \Rightarrow \zeta(s) = \frac{\pi^{\frac{s}{2}}}{\Gamma\left(\frac{s}{2}\right)}\left\{\frac{1}{s(s-1)}+\sum_{m=0}^{\infty}(I_{1m}+I_{2m})\right\}$$

$$\Rightarrow \zeta(a) = \frac{\pi^{\frac{a}{2}}}{\Gamma\left(\frac{a}{2}\right)}\left\{\frac{1}{a(a-1)}+\sum_{k=0}^{\infty}\left(\frac{\Gamma\left(\frac{a}{2}\right)\cos\left(\frac{\pi a}{8}\right)}{k\left(\pi\sqrt{2}\right)^{\frac{a}{2}}} + \frac{\Gamma\left(1-\frac{a}{2}\right)\cos\left(\frac{\pi}{4}-\frac{a\pi}{8}\right)}{\left(\pi k^2\sqrt{2}\right)^{\left(1-\frac{a}{2}\right)}} + 2\pi-\varepsilon\right)\right\}$$

----------------------------- (39)

$$\because 1-\frac{a}{2}>0, a>0$$

$$\therefore \lim_{k\to\infty}[(I_{1k}+I_{2k})-(I_{1(k-1)}+I_{2(k-1)})=0$$

So the sequence of the terms in series in (39) is Cauchy.

Our next thought can be to prove the convergence of the above series and to evaluate the value of ε . *(A quick glance of the numerical values by assigning a=1/2 shows that the cosine terms are approximately 0.89-0.9 and Γ(1/4) = 3.625 and Γ(3/4) = 1.22)*

A plot of the zeta function shows that there is no zero at $s = \frac{1}{2} + i0$. In fact it is the series $\sum_{n=0}^{\infty} \frac{1}{\sqrt{n}}$ which is divergent according to the definition of zeta function from which the meromorphic extension of zeta is derived into complex numbers.

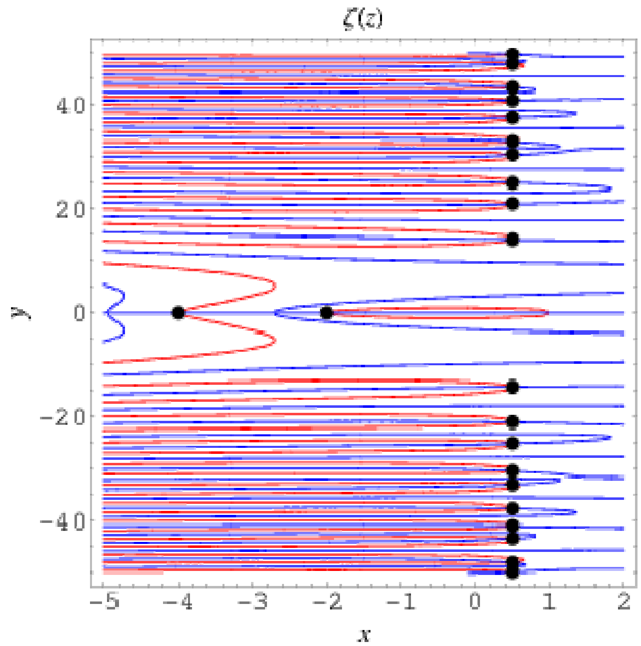


Fig 4. A plot of the zeta function which shows the zeroes as black spots
(Courtesy: http://en.wikipedia.org/wiki/Riemann_zeta_function)

We have done a small evaluation of the integral by assuming that s is real. Now we proceed to the following condition.

<u>Case 2:</u>

$$s = a + ib, a, b > 0, a, b \in \Re, i = \sqrt{-1} \, and \; 0 < a \leq \frac{1}{2}$$

We recall Euler's integral in the complex plane

$$\Gamma(s) = \int_0^{\infty} e^{-x} x^{s-1} dx, \ \mathrm{Re}(s) > 0$$

It will be worthwhile to mention Weirstress approximation formula of Gamma function in this regard

$$\Gamma(s) = \left[s e^{\gamma s} \prod_{r=1}^{\infty} \left(1 + \frac{s}{r}\right) e^{\frac{-s}{r}} \right]^{-1}$$

For $s = -1, -2, \ldots$, the function is not defined and these values contribute to the trivial zeroes of the zeta function. As we have assumed that s, we will not follow the approach of segregating the real and imaginary parts as before. To simplify the integration we will not include the cosine terms in the series as we have done for s when s was real.

We use the following expression using (23) for complex s and exclude the cosine factors from the terms.

$$I_{1k} = \int_1^{\infty} x^{\frac{s}{2}-1} e^{-\pi x k^2} dx = \int_0^{\infty} x^{\frac{s}{2}-1} e^{-\pi x k^2} dx - \int_0^{1} x^{\frac{s}{2}-1} e^{-\pi x k^2} dx$$

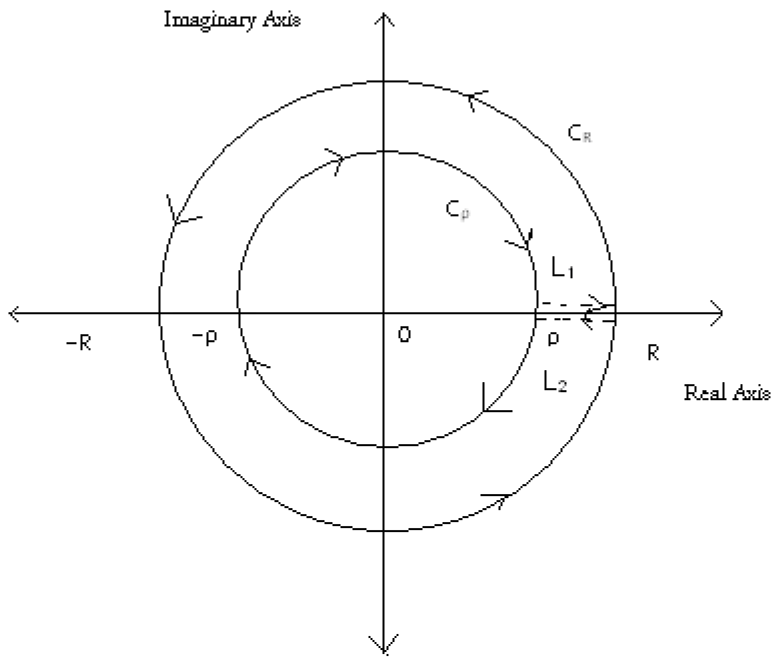
$$\Rightarrow I_{1k} = I_1 - I_2,\ I_1 = \int_0^{\infty} x^{\frac{s}{2}-1} e^{-\pi x k^2} dx\ ,\ I_2 = \int_0^{1} x^{\frac{s}{2}-1} e^{-\pi x k^2} dx \qquad ------------(40)$$

$$Let\ u = \pi x k^2,\ \therefore x = \frac{u}{\pi k^2} \Rightarrow dx = \frac{du}{\pi k^2}$$

$$\therefore I_1 = \int_0^{\infty} x^{\frac{s}{2}-1} e^{-\pi x k^2} dx = \frac{1}{\left(\pi k^2\right)^{\frac{s}{2}}} \int_0^{\infty} u^{\frac{s}{2}-1} e^{-u} du = \frac{\Gamma\left(\frac{s}{2}\right)}{\left(\pi k^2\right)^{\frac{s}{2}}}$$

----------------- (41)

As done earlier let we investigate the complex integral $\int_0^1 z^{\frac{s}{2}-1} e^{-\pi z k^2} dz$ along the same contour by isolating the origin as shown below.



We can apply Cauchy-Goursat theorem as the function is analytic inside the contour.

$$\int_{L_1} z^{\frac{s}{2}-1} e^{-\pi z k^2} dz + \int_{C_R} z^{\frac{s}{2}-1} e^{-\pi z k^2} dz + \int_{L_2} z^{\frac{s}{2}-1} e^{-\pi z k^2} dz + \int_{C_\rho} z^{\frac{s}{2}-1} e^{-\pi z k^2} dz = 0$$

---------------------------- (42)

We substitute $z = re^{i\theta}$ as done previously along L1 and L2 so that we have, along L1

$$z = re^{i0} = r$$

---------------------------- (43)

And along L2

$$z = re^{i2\pi} = r(\cos 2\pi + i \sin 2\pi) = r,$$

$$z^{\frac{s}{2}-1} = e^{\left(\frac{s}{2}-1\right)\ln z} = e^{\left(\frac{s}{2}-1\right)(\ln r + i2\pi)} = r^{\left(\frac{s}{2}-1\right)} e^{\left(\frac{s}{2}-1\right)2i\pi}$$

------------------------------------- (44)

$$\int_{L_1} z^{\frac{s}{2}-1} e^{-\pi z k^2} dz = \int_\rho^R r^{\frac{s}{2}-1} e^{-\pi r k^2} dr \qquad \text{(using (43))}$$

------------------------------------- (45)

$$\int_{L_2} z^{\frac{s}{2}-1} e^{-\pi z k^2} dz = \int_R^\rho r^{\frac{s}{2}-1} e^{\left(\frac{s}{2}-1\right)2i\pi} e^{-\pi r k^2} dr \quad \text{(using (44))}$$

------------------------------------- (46)

For evaluating the integrals along C_R and C_ρ we proceed as follows-

$$\left|z^{\frac{s}{2}-1}e^{-\pi zk^2}\right| = \left|e^{\left(\frac{s}{2}-1\right)\ln z}\right|\left|e^{-\pi k^2 r(\cos\theta + i\sin\theta)}\right| \qquad (where\ z = r(\cos\theta + i\sin\theta), i = \sqrt{-1})$$

$$= \left|e^{\left(\frac{s}{2}-1\right)\ln z}\right|\left|e^{-\pi k^2 r\cos\theta}\right|\left|e^{-\pi k^2 ri\sin\theta}\right| = e^{-\pi k^2 r\cos\theta}\left|e^{\left(\frac{a}{2}-1+\frac{ib}{2}\right)(\ln r + i\theta)}\right|$$

$$= e^{-\pi k^2 r\cos\theta}\left|e^{\left(\left(\frac{a}{2}-1\right)\ln r - \frac{b\theta}{2}\right)}\right|\left|e^{i\left(\frac{b\ln r}{2}+\theta\left(\frac{a}{2}-1\right)\right)}\right| \quad \left(\because \left|e^{-\pi k^2 ri\sin\theta}\right| = 1\right)$$

$$= e^{-\pi k^2 r\cos\theta}e^{\left(\left(\frac{a}{2}-1\right)\ln r - \frac{b\theta}{2}\right)} \qquad \left(\because \left|e^{i\left(\frac{b\ln r}{2}+\theta\left(\frac{a}{2}-1\right)\right)}\right| = 1\right)$$

$$\therefore \left|z^{\frac{s}{2}-1}e^{-\pi zk^2}\right| = e^{-\pi k^2 r\cos\theta}e^{\left(\left(\frac{a}{2}-1\right)\ln r - \frac{b\theta}{2}\right)}$$

-------------------------------------- (47)

$$\therefore \left|\int_{C_\rho} z^{\frac{s}{2}-1}e^{-\pi zk^2}\,dz\right| \le ML,\ where\ M = e^{-\pi k^2\rho\cos\theta}e^{\left(\left(\frac{a}{2}-1\right)\ln\rho - \frac{b\theta}{2}\right)} \ \ (from(45))\ and\ L = 2\pi\rho$$

$$\Rightarrow \left|\int_{C_\rho} z^{\frac{s}{2}-1}e^{-\pi zk^2}\,dz\right| \le e^{-\pi k^2\rho\cos\theta}\rho^{\left(\frac{a}{2}-1\right)}e^{\frac{-b\theta}{2}}2\pi\rho$$

$$\Rightarrow \lim_{\rho\to 0}\left|\int_{C_\rho} z^{\frac{s}{2}-1}e^{-\pi zk^2}\,dz\right| = 0 \qquad (\because a > 0)$$

-------------------------------------- (48)

Similarly

$$\left|\int_{C_R} z^{\frac{s}{2}-1}e^{-\pi zk^2}\,dz\right| \le 2\pi R^{\left(\frac{a}{2}\right)}e^{\frac{-b\theta}{2}}e^{-\pi k^2 R\cos\theta}$$

$$\Rightarrow \lim_{R\to 1}\left|\int_{C_R} z^{\frac{s}{2}-1}e^{-\pi zk^2}\,dz\right| \le 2\pi e^{\frac{-b\theta}{2}}e^{-\pi k^2\cos\theta}$$

$$\Rightarrow \lim_{R\to 1}\left|\int_{C_R} z^{\frac{s}{2}-1}e^{-\pi zk^2}\,dz\right| = 2\pi e^{\frac{-b\theta}{2}}e^{-\pi k^2\cos\theta} - \delta, 0 \le \delta < 2\pi e^{\frac{-b\theta}{2}}e^{-\pi k^2\cos\theta}$$

-------------------------------------- (49)

Using (42),(45),(46),(48),(49) we have

$$\int_0^1 r^{\frac{s}{2}-1} e^{-\pi r k^2} dr + \int_1^0 r^{\frac{s}{2}-1} e^{\left(\frac{s}{2}-1\right)2i\pi} e^{-\pi r k^2} dr + \left(\int_{C_R} z^{\frac{s}{2}-1} e^{-\pi z k^2} dz + \int_{C_\rho} z^{\frac{s}{2}-1} e^{-\pi z k^2} dz \right) = 0$$

$$\Rightarrow \left(1 - e^{\left(\frac{s}{2}-1\right)2i\pi}\right) \int_0^1 r^{\frac{s}{2}-1} e^{-\pi r k^2} dr + \left(0 + 2\pi e^{\frac{-b\theta}{2}} e^{-\pi k^2} - \delta\right) = 0$$

$$\Rightarrow \int_0^1 r^{\frac{s}{2}-1} e^{-\pi r k^2} dr = \frac{\delta - 2\pi e^{\frac{-b\theta}{2}} e^{-\pi k^2 \cos\theta}}{\left(1 - e^{\left(\frac{s}{2}-1\right)2i\pi}\right)}$$

------------------------------------- (50)

Using (40), (41), (50) we have

$$I_{1k} = I_1 - I_2 = \frac{\Gamma\left(\frac{s}{2}\right)}{\left(\pi k^2\right)^{\frac{s}{2}}} - \frac{\delta - 2\pi e^{\frac{-b\theta}{2}} e^{-\pi k^2 \cos\theta}}{\left(1 - e^{\left(\frac{s}{2}-1\right)2i\pi}\right)}$$

------------------------------------- (51)

From (23) we now calculate the following (without the cosine factor as earlier)

$$I_{2k} = \int_1^\infty x^{-\frac{s}{2}-\frac{1}{2}} e^{-\pi x k^2} dx = \int_0^\infty x^{-\frac{s}{2}-\frac{1}{2}} e^{-\pi x k^2} dx - \int_0^1 x^{-\frac{s}{2}-\frac{1}{2}} e^{-\pi x k^2} dx$$

$$\Rightarrow I_{2k} = J_1 - J_2, \ where\ J_1 = \int_0^\infty x^{-\frac{s}{2}-\frac{1}{2}} e^{-\pi x k^2} dx,\ J_2 = \int_0^1 x^{-\frac{s}{2}-\frac{1}{2}} e^{-\pi x k^2} dx$$

----------------------------------- (52)

Using Euler's integral we have

$$J_1 = \int_0^\infty x^{-\frac{s}{2}-\frac{1}{2}} e^{-\pi x k^2} dx = \int_0^\infty x^{\left(\left(\frac{1-s}{2}\right)-1\right)} e^{-\pi x k^2} dx$$

$$\Rightarrow J_1 = \frac{\Gamma\left(\frac{1-s}{2}\right)}{\left(\pi k^2\right)^{\frac{1-s}{2}}} \qquad (\because 0 < \mathrm{Re}(s) < 1, \therefore \mathrm{Re}\left(\frac{1-s}{2}\right) > 0)$$

--------------------------- (53)

As in (45), (46) we use the same contour for integration for J_2.

For the complex integral $\int z^{-\left(\frac{s+1}{2}\right)} e^{-\pi z k^2} dz$ along L_1 and L_2 we have

$$\int_{L_1} z^{-\left(\frac{s+1}{2}\right)} e^{-\pi z k^2} dz = \int_{\rho}^{R} r^{-\left(\frac{s+1}{2}\right)} e^{-\pi r k^2} dr$$

$$\int_{L_2} z^{-\left(\frac{s+1}{2}\right)} e^{-\pi z k^2} dz = \int_{R}^{\rho} r^{-\left(\frac{s+1}{2}\right)} e^{-i\pi(s+1)} e^{-\pi r k^2} dr$$

--------------------------- (54)

For the contours along C_ρ and C_R we have

$$\left| z^{-\left(\frac{s+1}{2}\right)} e^{-\pi z k^2} \right| = \left| e^{-\left(\frac{a+1}{2}+\frac{ib}{2}\right)(\ln r + i\theta)} \right| \left| e^{-\pi k^2 r(\cos\theta + i\sin\theta)} \right|$$

$$\Rightarrow \left| z^{-\left(\frac{s+1}{2}\right)} e^{-\pi z k^2} \right| = e^{-\left(\left(\frac{a+1}{2}\right)\ln r - \frac{b\theta}{2}\right)} e^{-\pi k^2 r\cos\theta}$$

$$\Rightarrow \left| z^{-\left(\frac{s+1}{2}\right)} e^{-\pi z k^2} \right| = r^{-\left(\frac{a+1}{2}\right)} e^{\frac{-b\theta}{2}} e^{-\pi k^2 r\cos\theta}$$

--------------------------- (55)

$$\left| \int_{C_\rho} z^{-\left(\frac{s+1}{2}\right)} e^{-\pi z k^2} dz \right| \le ML \ \ where\ M = \rho^{-\left(\frac{a+1}{2}\right)} e^{\frac{-b\theta}{2}} e^{-\pi k^2 \rho\cos\theta}, L = 2\pi\rho$$

$$\Rightarrow \left| \int_{C_\rho} z^{-\left(\frac{s+1}{2}\right)} e^{-\pi z k^2} dz \right| \le 2\pi\rho^{\left(\frac{1-a}{2}\right)} e^{\frac{-b\theta}{2}} e^{-\pi k^2 \rho\cos\theta}$$

$$\Rightarrow \lim_{\rho\to 0} \left| \int_{C_\rho} z^{-\left(\frac{s+1}{2}\right)} e^{-\pi z k^2} dz \right| = 0 \qquad (a<1)$$

--------------------------- (56)

Similarly,

$$\left| \int_{C_R} z^{-\left(\frac{s+1}{2}\right)} e^{-\pi z k^2} dz \right| \le 2\pi R^{\left(\frac{1-a}{2}\right)} e^{\frac{-b\theta}{2}} e^{-\pi k^2 R\cos\theta}$$

$$\Rightarrow \lim_{R\to 1} \left| \int_{C_R} z^{-\left(\frac{s+1}{2}\right)} e^{-\pi z k^2} dz \right| \le 2\pi e^{\frac{-b\theta}{2}} e^{-\pi k^2 \cos\theta}$$

$$\Rightarrow \lim_{R\to 1} \left| \int_{C_R} z^{-\left(\frac{s+1}{2}\right)} e^{-\pi z k^2} dz \right| = 2\pi e^{\frac{-b\theta}{2}} e^{-\pi k^2 \cos\theta} - \delta' \ \ \left(0 \le \delta' < 2\pi e^{\frac{-b\theta}{2}} e^{-\pi k^2 \cos\theta}\right)$$

--------------------------- (57)

Thus using (54),(56),(57) and applying Cauchy Goursat Theorem we have

$$\int_0^1 r^{-\left(\frac{s+1}{2}\right)} e^{-\pi r k^2} dr + \int_1^0 r^{-\left(\frac{s+1}{2}\right)} e^{-i\pi(s+1)} e^{-\pi r k^2} dr + 0 + 2\pi e^{\frac{-b\theta}{2}} e^{-\pi k^2 \cos\theta} - \delta' = 0$$

$$\Rightarrow \int_0^1 r^{-\left(\frac{s+1}{2}\right)} e^{-\pi r k^2} dr = \frac{\delta' - 2\pi e^{\frac{-b\theta}{2}} e^{-\pi k^2 \cos\theta}}{(1 - e^{-i\pi(s+1)})}$$

$$\Rightarrow J_2 = \frac{\delta' - 2\pi e^{\frac{-b\theta}{2}} e^{-\pi k^2 \cos\theta}}{(1 - e^{-i\pi(s+1)})}$$

$$\therefore I_{2k} = J_1 - J_2$$

$$\Rightarrow \quad I_{2k} = \frac{\Gamma\left(\frac{1-s}{2}\right)}{\left(\pi k^2\right)^{\frac{1-s}{2}}} - \frac{\delta' - 2\pi e^{\frac{-b\theta}{2}} e^{-\pi k^2 \cos\theta}}{(1 - e^{-i\pi(s+1)})}$$

-------------------------------- (58)

From (23), we have

$$\zeta(s) = \frac{\pi^{\frac{s}{2}}}{\Gamma\left(\frac{s}{2}\right)} \left\{ \frac{1}{s(s-1)} + \sum_{k=1}^{\infty} \left(I_{1m} + I_{2m} \right) \right\}$$

$$\Rightarrow \zeta(s) = \frac{\pi^{\frac{s}{2}}}{\Gamma\left(\frac{s}{2}\right)} \left\{ \frac{1}{s(s-1)} + \sum_{k=1}^{\infty} \left(\frac{\Gamma\left(\frac{s}{2}\right)}{\left(\pi k^2\right)^{\frac{s}{2}}} - \frac{\delta - 2\pi e^{\frac{-b\theta}{2}} e^{-\pi k^2 \cos\theta}}{\left(1 - e^{\left(\frac{s}{2}-1\right)2i\pi}\right)} + \frac{\Gamma\left(\frac{1-s}{2}\right)}{\left(\pi k^2\right)^{\frac{1-s}{2}}} - \frac{\delta' - 2\pi e^{\frac{-b\theta}{2}} e^{-\pi k^2 \cos\theta}}{(1 - e^{-i\pi(s+1)})} \right) \right\} \quad (\,from\ (51),(58))$$

$$\Rightarrow \zeta(s) = \frac{\pi^{\frac{s}{2}}}{\Gamma\left(\frac{s}{2}\right)} \left\{ \frac{1}{s(s-1)} + \sum_{k=1}^{\infty} \left(\frac{\Gamma\left(\frac{s}{2}\right)}{\left(\pi k^2\right)^{\frac{s}{2}}} + \frac{\Gamma\left(\frac{1-s}{2}\right)}{\left(\pi k^2\right)^{\frac{1-s}{2}}} + 2\pi e^{\frac{-b\theta}{2}} e^{-\pi k^2 \cos\theta} \left(\frac{1}{(1 - e^{-i\pi(s+1)})} + \frac{1}{\left(1 - e^{\left(\frac{s}{2}-1\right)2i\pi}\right)} \right) - (\delta + \delta') \right) \right\}$$

Conclusion

We have used the contour integration and derived an equivalent expression for the zeta function. This expression can be further studied and the series in this expression can be analyzed. Interestingly we may have a recursive relation of $\zeta(s)$ as the series in the above expression has the term $\frac{1}{k^s}$ which is of basic interest.

We have also studied the general Jacobi theta function and derived some conditions which have to be satisfied for the generic function to be used as done in the zeta function.

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