2 Solution to Question 2

Question 2: To show, $h(x) = \text{sign}\left(\|\Phi(x) - c_-\|^2 - \|\Phi(x) - c_+\|^2\right)$ can be re-written as.

$$\text{sign}\left(\sum_{i=1}^{m} \alpha_i K(x, x_i) + b\right).$$

we know that $\|\Phi(x) - c_-\|^2 = \langle \Phi(x) - c_-, \Phi(x) - c_- \rangle$.

so $h(x) = \text{sign}\left(\langle \Phi(x) - c_-, \Phi(x) - c_- \rangle - \langle \Phi(x) - c_+, \Phi(x) - c_+ \rangle\right).$

Using the property of linearity of inner product according to which.
$\langle x+y, z \rangle = \langle x, z \rangle + \langle y, z \rangle$. and $\langle x, y+z \rangle = \langle x, y \rangle + \langle x, z \rangle$.

$$\begin{aligned} h(x) &= \text{sign}\left(\langle \Phi(x), \Phi(x) - c_- \rangle - \langle c_-, \Phi(x) - c_- \rangle - \left\{\langle \Phi(x), \Phi(x) - c_+ \rangle - \langle c_+, \Phi(x) - c_+ \rangle\right\}\right) \\ &= \text{sign}\left(\langle \Phi(x), \Phi(x) \rangle - \langle \Phi(x), c_- \rangle - \left\{\langle c_-, \Phi(x) \rangle - \langle c_-, c_- \rangle\right\} - \left\{\langle \Phi(x), \Phi(x) \rangle - \langle \Phi(x), c_+ \rangle\right\}\right. \\ &\qquad \left. + \left\{\langle c_+, \Phi(x) \rangle - \langle c_+, c_+ \rangle\right\}\right\}\Big) \\ &= \text{sign}\Big(\cancel{\langle \Phi(x), \Phi(x) \rangle} - \langle \Phi(x), c_- \rangle - \langle c_-, \Phi(x) \rangle + \langle c_-, c_- \rangle - \cancel{\langle \Phi(x), \Phi(x) \rangle} + \langle \Phi(x), c_+ \rangle \\ &\qquad + \langle c_+, \Phi(x) \rangle - \langle c_+, c_+ \rangle \end{aligned}$$

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also $\langle a, b\rangle \equiv \langle b, a\rangle$.

$$\therefore\ h(x) = \mathrm{sign}\left(2\langle \phi(x), c_+\rangle - 2\langle \phi(x), c_-\rangle + \langle c_-, c_-\rangle - \langle c_+, c_+\rangle\right)$$

$$= \mathrm{sign}\left(2\langle \phi(x), c_+ - c_-\rangle + \langle c_-, c_-\rangle - \langle c_+, c_+\rangle\right) \quad - ①$$

$$\cancel{= \mathrm{sign}\left(2\langle \phi(x), c_+ \ c_-\rangle +\right.}$$

c_- and c_+ are vectors. $\therefore$ $\langle c_-, c_-\rangle = \left[\frac{1}{m_-}\sum_{i\in I^-}\phi(x_i)\right]^T\left[\frac{1}{m_-}\sum_{i\in I^-}\phi(x_i)\right]$

$$= \frac{1}{m_-^2}\sum_{i,j\in I^-}\phi(x_i)^T\cdot\phi(x_j) = \frac{1}{m_-^2}\sum_{i,j\in I^-}\phi(x_{I_i})^T\phi(x_{I_j})$$

~~where $x_I = x_i \ \forall i\in I^-$~~

$$= \frac{1}{m_-^2}\sum_{i,j\in I^-}\left\langle \phi(x_{I_i}), \phi(x_{I_j})\right\rangle$$

$$= \frac{1}{m_-^2}\sum_{i,j\in I^-} K(x_i, x_j).$$ using kernel trick.

$- ②$

similarly $\langle c_+, c_+\rangle = \frac{1}{m_+^2}\sum_{i,j\in I^+} K(x_i, x_j). \quad - ③.$

and

$$\langle \phi(x), c_+ - c_-\rangle = \phi(x)^T\left[\frac{1}{m_+}\sum_{i\in I^+}\phi(x_i) - \frac{1}{m_-}\sum_{i\in I^-}\phi(x_i)\right]$$

$$= \frac{1}{m_+}\sum_{i\in I^+}\phi(x)^T\phi(x_i) - \frac{1}{m_-}\sum_{i\in I^-}\phi(x)^T\phi(x_i). \quad - ①$$

$$\cancel{\text{if } y_i = +1} = \sum_{i\in m}\alpha_i\phi(x)^T\phi(x_i).$$

$$\alpha_i = \begin{cases}\frac{1}{m_+} & \text{if } y_i = +1\\ -\frac{1}{m_-} & \text{if } y_i = -1\end{cases}$$

$$\therefore\ h(x) = \mathrm{sign}\left(2\sum_{i=1}^{m}\alpha_i\ \phi(x)^T\phi(x_i) + \underbrace{\frac{1}{m_+^2}\sum_{i,j\in I^+}K(x_i, x_j) + \frac{1}{m_-^2}\sum_{i,j\in I^-}K(x_i, x_j)}_{b}\right)$$

$$h(x) = \mathrm{sign}\left(\cancel{2}\sum_{i=1}^{m}\alpha_i\ \phi(x)^T\phi(x_i) + b\right).$$

taking the 2 out, common doesn't affect the sign.

$$b = \frac{1}{2m_+^2}\sum_{i,j\in I^+}K(x_i, x_j) + \frac{1}{2m_-^2}\sum_{i,j\in I^-}K(x_i, x_j)$$

$$\alpha_i = \begin{cases}\frac{1}{m_+} & \text{if } y_i = +1.\\ \frac{-1}{m_-} & \text{if } y_i = -1.\end{cases}$$