e).

$$K_1(x_i, x_j) = \phi_1(x_i)^T \phi_1(x_j) = \sum_{l=1}^{d} \phi_{1l}(x_i)\cdot\phi_{1l}(x_j).$$

$$K_2(x_i, x_j) = \phi_2(x_i)^T \phi_2(x_j) = \sum_{m=1}^{d} \phi_{2m}(x_i)\cdot\phi_{2m}(x_j).$$

so $$K(x_i, x_j) = K_1(x_i, x_j)\cdot K_2(x_i, x_j) = \left[\sum_{l=1}^{d} \phi_{1l}(x_i)\cdot\phi_{1l}(x_j)\right]\left[\sum_{m=1}^{d} \phi_{2m}(x_i)\cdot\phi_{2m}(x_j)\right]$$

Expanding out a few terms.

$$= \left[\phi_{11}(x_i)\cdot\phi_{11}(x_j) + \phi_{12}(x_i)\cdot\phi_{12}(x_j) + \dots + \phi_{1d}(x_i)\cdot\phi_{1d}(x_j)\right]$$ has d terms

$*$

$$\left[\phi_{21}(x_i)\cdot\phi_{21}(x_j) + \phi_{22}(x_i)\cdot\phi_{22}(x_j) + \dots + \phi_{2d}(x_i)\cdot\phi_{2d}(x_j)\right]$$ has d terms.

= when we find the product of the above two terms it is going to be a term with d×d. terms.

d terms

$$= \Big[\phi_{11}(x_i)\phi_{11}(x_j)\phi_{21}(x_i)\phi_{21}(x_j) + \phi_{11}(x_i)\phi_{11}(x_j)\phi_{22}(x_i)\phi_{22}(x_j) + \dots + \phi_{11}(x_i)\phi_{11}(x_j)\phi_{2d}(x_i)\phi_{2d}(x_j)$$

$$+ \phi_{12}(x_i)\phi_{12}(x_j)\phi_{21}(x_i)\phi_{21}(x_j) + \dots$$

$$\vdots$$

$$+ \phi_{1d}(x_i)\phi_{1d}(x_j)\cdot\phi_{21}(x_i)\phi_{21}(x_j) + \dots + \phi_{1d}(x_i)\phi_{1d}(x_j)\phi_{2d}(x_i)\phi_{2d}(x_j)\Big]$$

Individual terms among these

~~These d×d terms can be~~ ~~written~~ re-written as.

$$\Big[\left(\overline{\phi_{11}(x_i)\phi_{21}(x_i)}\right)\left(\overline{\phi_{11}(x_j)\phi_{21}(x_j)}\right) + \left(\overline{\phi_{11}(x_i)\phi_{22}(x_i)}\right)\left(\overline{\phi_{11}(x_j)\phi_{22}(x_j)}\right) + \dots + \left(\overline{\phi_{11}(x_i)\phi_{2d}(x_i)}\right)\left(\overline{\phi_{11}(x_j)\phi_{2d}(x_j)}\right)$$

$$+\left(\overline{\phi_{12}(x_i)\phi_{21}(x_i)}\right)\left(\overline{\phi_{12}(x_j)\phi_{21}(x_j)}\right) + \left(\overline{\phi_{12}(x_i)\phi_{22}(x_i)}\right)\cdot\left(\overline{\phi_{12}(x_j)\cdot\phi_{22}(x_j)}\right) + \dots + \left(\overline{\phi_{12}(x_i)\phi_{2d}(x_i)}\right)\left(\overline{\phi_{12}(x_j)\phi_{2d}(x_j)}\right)$$

$$+ \dots$$

$$\vdots$$

$$+\left(\overline{\phi_{1d}(x_i)\phi_{21}(x_i)}\right)\left(\overline{\phi_{1d}(x_j)\phi_{21}(x_j)}\right) + \left(\overline{\phi_{1d}(x_i)\cdot\phi_{22}(x_i)}\right)\left(\overline{\phi_{1d}(x_j)\cdot\phi_{22}(x_j)}\right) + \dots + \left(\overline{\phi_{1d}(x_i)\phi_{2d}(x_i)}\right)\left(\overline{\phi_{1d}(x_j)\phi_{2d}(x_j)}\right)\Big]$$

This sum can be re-written as a dot-product of a vector $\phi(x_i)$ with itself. i.e. $\phi(x_i)^T\phi$

~~such that~~ where $\phi(x_j)$ is defined as follows.

$$\Big[\phi_{11}(x)\cdot\phi_{21}(x), \quad \phi_{11}(x)\phi_{22}(x), \quad \dots, \quad \phi_{11}(x)\phi_{2d}(x),$$

$$\phi_{12}(x)\phi_{21}(x), \quad \phi_{12}(x)\cdot\phi_{22}(x), \quad \dots, \quad \phi_{12}(x)\cdot\phi_{2d}(x),$$

$$\vdots$$

$$\phi_{1d}(x)\cdot\phi_{21}(x), \quad \dots\dots, \quad \dots, \quad \phi_{1d}(x)\cdot\phi_{2d}(x).\Big]^T$$

This is exactly what $\phi(x)$. would be, if it were defined as $\phi_1(x) \otimes \phi_2(x)$. Hence if $K(x_i, x_j) = K_1(x_i, x_j) \times K_2(x_i, x_j)$. then $K(x_i, x_j)$ can be written as $\phi(x_i)^T \phi(x_j)$. where $\phi(x) = \phi_1(x) \otimes \phi_2(x)$.