Calculus 3 Formulas

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Partial Derivatives

Domain and range of a multivariable function

A function f of two variables is a rule that assigns to each ordered pair of real numbers (x, y) in a set D a unique real number denoted by $f(x, y)$.

Domain of f: The set D

Range of f: The set of values that f takes on

Level curves of a multivariable function

The level curves of a function f of two variables are the curves with equations $f(x, y) = k$, where k is a constant (in the range of f).

Precise definition of the limit of a multivariable function

Let f be a function of two variables whose domain D includes points arbitrarily close to (a, b). Then the limit of $f(x, y)$ as (x, y) approaches (a, b) is L,

$$\lim_{(x,y)\to(a,b)} f(x, y) = L$$

if for all $\epsilon > 0$ there is a corresponding $\delta > 0$ such that if $(x, y) \in D$ and $0 < \sqrt{(x-a)^2 + (y-b)^2} < \delta$ then

$\left| f(x, y) - L \right| < \epsilon$

Existence of the limit of a multivariable function

If

$f(x, y) \to L_1$ as $(x, y) \to (a, b)$ along a path C_1 and

$f(x, y) \to L_2$ as $(x, y) \to (a, b)$ along a path C_2

where $L_1 \neq L_2$, then $\lim_{(x,y)\to(a,b)} f(x, y)$ does not exist.

Continuity of a multivariable function

A function f of two variables is continuous at (a, b) if

$$\lim_{(x,y)\to(a,b)} f(x, y) = f(a, b)$$

In other words, f is continuous at (a, b) if it's limit at (a, b) is equal to the actual value of the function at (a, b) We say f is continuous on D if f is continuous on every point (a, b) in D.

Definition of the derivative of a multivariable function

Partial derivative with respect to x $\quad f_x(a,b) = \lim_{h \to 0} \frac{f(a+h,b) - f(a,b)}{h}$

Partial derivative with respect to y $\quad f_y(a,b) = \lim_{h \to 0} \frac{f(a,b+h) - f(a,b)}{h}$

Notation of partial derivatives

If $z = f(x,y)$, then we can write the

Partial derivative with respect to x as

$$f_x(x,y) = f_x = \frac{\partial f}{\partial x} = \frac{\partial}{\partial x} f(x,y) = \frac{\partial z}{\partial x} = f_1 = D_1 f = D_x f$$

Partial derivative with respect to y as

$$f_y(x,y) = f_y = \frac{\partial f}{\partial y} = \frac{\partial}{\partial y} f(x,y) = \frac{\partial z}{\partial y} = f_2 = D_2 f = D_y f$$

Rule for finding partial derivatives of $z = f(x,y)$

To find f_x, treat y as a constant and differentiate $f(x,y)$ with respect to x.

To find f_y, treat x as a constant and differentiate $f(x,y)$ with respect to y.

Clairaut's theorem for the mixed second-order partial derivative

Suppose f is defined on a disk D that contains the point (a,b). If the functions f_{xy} and f_{yx} (the mixed second-order partial derivatives) are both continuous on D, then

$$f_{xy}(a,b)=f_{yx}(a,b)$$

Equation of the tangent plane

Suppose f has continuous partial derivatives. An equation of the tangent plane to the surface $z=f(x,y)$ at the point $P(x_0,y_0,z_0)$ is

$$z-z_0=f_x(x_0,y_0)(x-x_0)+f_y(x_0,y_0)(y-y_0)$$

Differentiability of a multivariable function

If $z=f(x,y)$, then f is differentiable at (a,b) if Δz can be expressed as

$$\Delta z=f_x(a,b)\Delta x+f_y(a,b)\Delta y+\epsilon_1\Delta x+\epsilon_2\Delta y$$

where ϵ_1 and $\epsilon_2\to 0$ as $(\Delta x,\Delta y)\to(0,0)$.

or

If the partial derivatives f_x and f_y exist near (a,b) and are continuous at (a,b), then f is differentiable at (a,b).

Total differential

$$dz = f_x(x,y)dx + f_y(x,y)dy = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy$$

Chain rule for multivariable functions

Case 1 - $f[g(t),h(t)]$

Suppose that $z = f(x,y)$ is a differentiable function of x and y, where $x = g(t)$ and $y = h(t)$ are both differentiable functions of t. Then z is a differentiable function of t and

$$\frac{dz}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} \qquad \frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}$$

Case 2 - $f[g(s,t),h(s,t)]$

Suppose that $z = f(x,y)$ is a differentiable function of x and y, where $x = g(s,t)$ and $y = h(s,t)$ are both differentiable functions of s and t. Then

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s} \qquad \frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t}$$

General version

Suppose that u is a differentiable function of n variables x_1, x_2, ... , x_n and each x_i is a differentiable function of m variables t_1, t_2, ... , t_m. Then u is a function of t_1, t_2, ... , t_m and

$$\frac{\partial u}{\partial t_i} = \frac{\partial u}{\partial x_1}\frac{\partial x_1}{\partial t_i} + \frac{\partial u}{\partial x_2}\frac{\partial x_2}{\partial t_i} + \ldots + \frac{\partial u}{\partial x_n}\frac{\partial x_n}{\partial t_i}$$

for each $i = 1, 2, \ \ldots, m$.

Implicit differentiation of a multivariable function

$$\frac{dx}{dy} = -\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial y}} = -\frac{F_x}{F_y}$$

Partial derivatives for implicit differentiation

$$\frac{\partial z}{\partial x} = -\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial z}}$$

$$\frac{\partial z}{\partial y} = -\frac{\frac{\partial F}{\partial y}}{\frac{\partial F}{\partial z}}$$

Directional derivative of a function in two variables

The directional derivative of f at (x_0, y_0) in the direction of a unit vector $\mathbf{u} = \langle a, b \rangle$ is

$$D_{\mathbf{u}}f(x_0, y_0) = \lim_{h \to 0} \frac{f(x_0 + ha, y_0 + hb) - f(x_0, y_0)}{h}$$

if this limit exists.

or

If f is a differentiable function of x and y, then f has a directional derivative in the direction of any unit vector $\mathbf{u} = \langle a, b \rangle$ and

$$D_{\mathbf{u}}f(x, y) = f_x(x, y)a + f_y(x, y)b$$

or

If f is a differentiable function of x and y, then f has a directional derivative in the direction of any unit vector $\mathbf{u} = \langle a, b \rangle$ and

$$D_{\mathbf{u}}f(x, y) = \nabla f(x, y) \cdot \mathbf{u}$$

where $\nabla f(x, y)$ is the gradient of the function and $\mathbf{u}$ is the unit vector.

Directional derivative of a function in three variables

The directional derivative of f at (x_0, y_0, z_0) in the direction of a unit vector $\mathbf{u} = \langle a, b, c \rangle$ is

$$D_{\mathbf{u}}f(x_0, y_0, z_0) = \lim_{h \to 0} \frac{f(x_0 + ha, y_0 + hb, z_0 + hc) - f(x_0, y_0, z_0)}{h}$$

if this limit exists.

or

If f is a differentiable function of x, y and z, then f has a directional derivative in the direction of any unit vector $\mathbf{u} = \langle a, b, c \rangle$ and

$$D_{\mathbf{u}} f(x, y, z) = f_x(x, y, z)a + f_y(x, y, z)b + f_z(x, y, z)c$$

or

If f is a differentiable function of x, y and z, then f has a directional derivative in the direction of any unit vector $\mathbf{u} = \langle a, b, c \rangle$ and

$$D_{\mathbf{u}} f(x, y, z) = \nabla f(x, y, z) \cdot \mathbf{u}$$

where $\nabla f(x, y, z)$ is the gradient of the function and $\mathbf{u}$ is the unit vector.

Gradient of a multivariable function

If f is a function of two variables x and y, then the gradient of f is the vector function ∇f defined by

$$\nabla f(x, y) = \langle f_x(x, y), f_y(x, y) \rangle = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j}$$

Gradient vector of a multivariable function

$$\nabla f = \langle f_x, f_y, f_z \rangle = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} + \frac{\partial f}{\partial z}\mathbf{k}$$

Maximizing the directional derivative

Suppose f is a differentiable function of two or three variables. The maximum value of the directional derivative $D_{\mathbf{u}}f(\mathbf{x})$ is $|\nabla f(\mathbf{x})|$ and it occurs when $\mathbf{u}$ has the same direction as the gradient vector $\nabla f(\mathbf{x})$.

Tangent plane to the level surface

$$F_x(x_0, y_o, z_0)(x - x_0) + F_y(x_0, y_o, z_0)(y - y_0) + F_z(x_0, y_o, z_0)(z - z_0) = 0$$

Local and global extrema of a multivariable function

For a function of two variables x and y,

If $f(x, y) \le f(a, b)$ when (x, y) is near (a, b), then f has a local maximum at (a, b) and

$f(a, b)$ is a local maximum value, unless

the inequality is true for all points (x, y) in the domain of f, in which case f has an absolute maximum at (a, b).

If $f(x, y) \ge f(a, b)$ when (x, y) is near (a, b), then f has a local minimum at (a, b) and

$f(a, b)$ is a local minimum value, unless the inequality is true for all points (x, y) in the domain of f, in which case f has an absolute minimum at (a, b).

Second derivatives test

Suppose the second partial derivatives of f are continuous on a disk with center (a, b) and suppose that $f_x(a, b) = 0$ and $f_y(a, b) = 0$ [that is, (a, b) is a critical point of f]. Let

$$D = D(a, b) = f_{xx}(a, b)f_{yy}(a, b) - [f_{xy}(a, b)]^2$$

If $D > 0$ and $f_{xx}(a, b) > 0$, then $f(a, b)$ is a local minimum

If $D > 0$ and $f_{xx}(a, b) < 0$, then $f(a, b)$ is a local maximum

If $D < 0$ and $f(a, b)$ is not a local maximum or minimum

$((a, b), f(a, b))$ is called a saddle point

If $D = 0$, the test is inconclusive

It can't be used to characterize the critical point $((a, b), f(a, b))$

Extreme value theorem for multivariable functions

If f is continuous on a close, bounded set D in $\mathbb{R}^2$, then f attains an absolute maximum value $f(x_1, y_1)$ and an absolute minimum value $f(x_2, y_2)$ at some points (x_1, y_1) and (x_2, y_2) in D.

Steps to identify global extrema

To find the absolute maximum and minimum values of a continuous function f on a closed, bounded set D:

Find the values of f at the critical points of f in D.

Find the extreme values of f on the boundary of D.

The largest of the values is the absolute maximum value; the smallest of these values is the absolute minimum value.

Method of Lagrange multipliers

To find the maximum and minimum values of $f(x, y, z)$ subject to the constraint $g(x, y, z) = k$ [assuming that these extreme values exist and $\nabla g \neq 0$ on the surface $g(x, y, z) = k$]:

Find all values of x, y, z and λ such that

$$\nabla f(x, y, z) = \lambda \nabla g(x, y, z) \text{ and}$$

$$g(x, y, z) = k$$

Evaluate f at all points (x, y, z) that result from the step above. The largest of these values is the maximum value of f; the smallest is the minimum value of f.

Multiple Integrals

Double integrals to find volume over the rectangular region

If $f(x,y) \geq 0$, then the volume V above the rectangle R and below the surface $z = f(x,y)$,

using **any point** in the sub-rectangle is

$$V = \iint_R f(x,y)dA = \lim_{m,n \to \infty} \sum_{i=1}^{m} \sum_{j=1}^{n} f(x_{ij}^*, y_{ij}^*)\Delta A$$

using the **upper right-hand corner** of the sub-rectangle is

$$V = \iint_R f(x,y)dA = \lim_{m,n \to \infty} \sum_{i=1}^{m} \sum_{j=1}^{n} f(x_i, y_i)\Delta A$$

using the **midpoint** of the sub-rectangle is

$$V = \iint_R f(x,y)dA = \lim_{m,n \to \infty} \sum_{i=1}^{m} \sum_{j=1}^{n} f(\bar{x}_i, \bar{y}_i)\Delta A$$

Fubini's theorem for double integrals

If f is continuous over the rectangle $R = \{(x,y) \mid a \leq x \leq b,\ c \leq y \leq d\}$, then

$$\iint_R f(x,y)dA = \int_a^b \int_c^d f(x,y)\ dy\ dx = \int_c^d \int_a^b f(x,y)\ dx\ dy$$

Double integrals when $f(x, y) = g(x)h(y)$

When $f(x, y)$ sits above the rectangle defined by $R = [a, b] \times [c, d]$ and when $f(x, y)$ can be factored as the product of a function of x only and a function of y only, $f(x, y) = g(x)h(y)$, then the double integral of f can be written as

$$\iint_R f(x, y)dA = \iint_R g(x)h(y)dA = \int_a^b g(x)\ dx \int_c^d h(y)\ dy$$

Double integrals to find volume over the general region

The volume above the region D and below the surface $z = f(x, y)$ is

$$V = \iint_D f(x, y)dA = \iint_R F(x, y)dA$$

where F is given by

$$F(x, y) = \begin{cases} f(x, y) & \text{if } (x, y) \text{ is in } D \\ 0 & \text{if } (x, y) \text{ is in } R \text{ but not in } D \end{cases}$$

and R is a rectangular region around D that encloses D.

Double integrals to find volume over type I regions

A plane region D is type I if it lies between the graphs of two continuous functions of x. If the function f is continuous over a type I region D such that

$$D = \{(x, y) \mid a \le x \le b, g_1(x) \le y \le g_2(x)\}$$

$$V = \iint_D f(x, y)dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x, y)\ dy\ dx$$

Double integrals to find volume over type II regions

A plane region D is type II if it lies between the graphs of two continuous functions of y. If the function f is continuous over a type II region D such that

$$D = \{(x, y) \mid c \le y \le d,\ h_1(y) \le x \le h_2(y)\}$$

$$V = \iint_D f(x, y)dA = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x, y)\ dx\ dy$$

Union of regions

If $D = D_1 \cup D_2$ where D_1 and D_2 are non-overlapping regions, then the volume above the region D and below the surface $z = f(x, y)$ is

$$V = \iint_D f(x, y)dA = \iint_{D_1} f(x, y)dA + \iint_{D_2} f(x, y)dA$$

For example, if we need to find the volume over the region D, but part of D is a type I region and the other part is a type II region, preventing us from finding volume of the entire region at once, then we're allowed to find the volume of D_1 separately from D_2, and then add them together to find total volume over D.

Squeeze theorem for double integrals

If $m \leq f(x, y) \leq M$ for all (x, y) in D, then

$$mA(D) \leq \iint_D f(x, y)dA \leq MA(D)$$

Conversion between polar and rectangular coordinates

$$r^2 = x^2 + y^2 \qquad x = r\cos\theta \qquad y = r\sin\theta$$

Double polar integrals to find volume over the rectangular region

If the function f is continuous over a polar region D given by

$0 \leq a \leq r \leq b$ and

$\alpha \leq \theta \leq \beta$, where $0 \leq \beta - \alpha \leq 2\pi$, then

$$V = \iint_R f(x, y)dA = \int_\alpha^\beta \int_a^b f(r\cos\theta, r\sin\theta)r\ dr\ d\theta$$

Double polar integrals to find volume over the general region

If the function f is continuous over a polar region D given by

$$D = \{(r, \theta) \mid \alpha \leq \theta \leq \beta,\ h_1(\theta) \leq r \leq h_2(\theta)\}, \text{ then}$$

$$V = \iint_D f(x, y)dA = \int_{\alpha}^{\beta} \int_{h_1(\theta)}^{h_2(\theta)} f(r\cos\theta, r\sin\theta)r\ dr\ d\theta$$

Properties of the lamina

If a lamina occupies the region D and has density function $\rho(x, y)$, then its

Mass is
$$m = \lim_{k,l \to \infty} \sum_{i=1}^{k} \sum_{j=1}^{l} \rho(x_{ij}^*, y_{ij}^*)\Delta A = \iint_D \rho(x, y)dA$$

Moment about the x-axis is
$$M_x = \lim_{m,n \to \infty} \sum_{i=1}^{m} \sum_{j=1}^{n} y_{ij}^* \rho(x_{ij}^*, y_{ij}^*)\Delta A = \iint_D y\rho(x, y)dA$$

Moment about the y-axis is
$$M_y = \lim_{m,n \to \infty} \sum_{i=1}^{m} \sum_{j=1}^{n} x_{ij}^* \rho(x_{ij}^*, y_{ij}^*)\Delta A = \iint_D x\rho(x, y)dA$$

Center of mass, $(\bar{x}, \bar{y})$ is
$$\bar{x} = \frac{M_y}{m} = \frac{1}{m} \iint_D x\rho(x, y)dA$$

$$\bar{y} = \frac{M_x}{m} = \frac{1}{m} \iint_D y\rho(x, y)dA$$

Moments of inertia

About the x-axis

$$I_x = \lim_{m,n\to\infty} \sum_{i=1}^{m} \sum_{j=1}^{n} (y_{ij}^*)^2 \rho(x_{ij}^*, y_{ij}^*)\Delta A = \iint_D y^2 \rho(x,y)dA$$

About the y-axis

$$I_y = \lim_{m,n\to\infty} \sum_{i=1}^{m} \sum_{j=1}^{n} (x_{ij}^*)^2 \rho(x_{ij}^*, y_{ij}^*)\Delta A = \iint_D x^2 \rho(x,y)dA$$

About the origin

$$I_0 = \lim_{m,n\to\infty} \sum_{i=1}^{m} \sum_{j=1}^{n} \left[(x_{ij}^*)^2 + (y_{ij}^*)^2\right] \rho(x_{ij}^*, y_{ij}^*)\Delta A = \iint_D (x^2 + y^2)\rho(x,y)dA$$

Note: The moment of inertia about the origin is also called the polar moment of inertia, and $I_0 = I_x + I_y$.

Surface area of a multivariable function

To find the area of the surface with equation $z = f(x,y)$ where $(x,y) \in D$ and where f_x and f_y are continuous, use any of these equations:

$$A(S) = \lim_{m,n\to\infty} \sum_{i=1}^{m} \sum_{j=1}^{n} \Delta T_{ij}$$

$$A(S)=\iint_D \sqrt{\left[f_x(x,y)\right]^2+\left[f_y(x,y)\right]^2+1}\,dA$$

$$A(S)=\iint_D \sqrt{1+\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2}\,dA$$

Triple integrals to find volume over the box

If $f(x,y,z)\geq 0$, then the volume V above the box B and below the surface $f(x,y,z)$,

using **any point** in the sub-box is

$$V=\iiint_B f(x,y,z)dV=\lim_{l,m,n\to\infty}\sum_{i=1}^{l}\sum_{j=1}^{m}\sum_{k=1}^{n} f(x_{ijk}^*,y_{ijk}^*,z_{ijk}^*)\Delta V$$

using the **upper right-hand corner** of the sub-box is

$$V=\iiint_B f(x,y,z)dV=\lim_{l,m,n\to\infty}\sum_{i=1}^{l}\sum_{j=1}^{m}\sum_{k=1}^{n} f(x_i,y_j,z_k)\Delta V$$

using the **midpoint** of the sub-box is

$$V=\iiint_B f(x,y,z)dV=\lim_{l,m,n\to\infty}\sum_{i=1}^{l}\sum_{j=1}^{m}\sum_{k=1}^{n} f(\bar{x}_i,\bar{y}_i,\bar{z}_i)\Delta V$$

Fubini's theorem for triple integrals

If f is continuous over the box $B=[a,b]\times[c,d]\times[r,s]$, then

$$\iiint_B f(x,y,z)dV=\int_r^s\int_c^d\int_a^b f(x,y,z)\ dx\ dy\ dz$$

Triple integrals to find volume over type I solid regions

If the function f is a continuous solid region E whose projection onto the xy -plane, D,

is a type I plane region, then

$$E=\{(x,y,z)\ |\ a\le x\le b,\ g_1(x)\le y\le g_2(x),\ u_1(x,y)\le z\le u_2(x,y)\}$$

$$V=\iiint_E f(x,y,z)dV=\int_a^b\int_{g_1(x)}^{g_2(x)}\int_{u_1(x,y)}^{u_2(x,y)} f(x,y,z)\ dx\ dy\ dz$$

is a type II plane region, then

$$E=\{(x,y,z)\ |\ c\le y\le d,\ h_1(y)\le x\le h_2(y),\ u_1(x,y)\le z\le u_2(x,y)\}$$

$$V=\iiint_E f(x,y,z)dV=\int_c^d\int_{h_1(y)}^{h_2(y)}\int_{u_1(x,y)}^{u_2(x,y)} f(x,y,z)\ dz\ dx\ dy$$

Conversion between cylindrical and rectangular coordinates

To convert from cylindrical to rectangular, we use

$$x = r\cos\theta \qquad y = r\sin\theta \qquad z = z$$

To convert from rectangular to cylindrical, we use

$$r^2 = x^2 + y^2 \qquad \tan\theta = \frac{y}{x} \qquad z = z$$

Triple integrals in cylindrical coordinates

$$\iiint_E f(x,y,z)dV = \int_\alpha^\beta \int_{h_1(\theta)}^{h_2(\theta)} \int_{u_1(r\cos\theta, r\sin\theta)}^{u_2(r\cos\theta, r\sin\theta)} f(r\cos\theta, r\sin\theta, z) r \ dz \ dr \ d\theta$$

Conversion between spherical and rectangular coordinates

To convert from spherical to rectangular, we use

$$x = \rho\sin\phi\cos\theta \qquad y = \rho\sin\phi\sin\theta \qquad z = \rho\cos\phi$$

To convert from rectangular to spherical, we use

$$\rho^2 = x^2 + y^2 + z^2$$

Triple integrals in spherical coordinates

If E is a spherical wedge given by

$$E = \{(\rho, \theta, \phi) \mid a \le \rho \le b,\ \alpha \le \theta \le \beta,\ c \le \phi \le d\}$$

$$\iiint_E f(x, y, z)dV = \int_c^d \int_\alpha^\beta \int_a^b f(\rho \sin\phi \cos\theta, \rho \sin\phi \sin\theta, \rho \cos\phi)\rho^2 \sin\phi \ d\rho \ d\theta \ d\phi$$

Jacobian of the transformation in two variables

The Jacobian of the transformation T given by $x = g(u, v)$ and $y = h(u, v)$ is

$$\frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$$

$$= \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u}$$

Jacobian of the transformation in three variables

The Jacobian of the transformation T given by $x = g(u, v)$ and $y = h(u, v)$ is

$$\frac{\partial(x, y, z)}{\partial(u, v, w)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix}$$

$$= \frac{\partial x}{\partial u}\begin{vmatrix} \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix} - \frac{\partial x}{\partial v}\begin{vmatrix} \frac{\partial y}{\partial u} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial w} \end{vmatrix} + \frac{\partial x}{\partial w}\begin{vmatrix} \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} \end{vmatrix}$$

$$= \frac{\partial x}{\partial u}\left(\frac{\partial y}{\partial v}\frac{\partial z}{\partial w} - \frac{\partial z}{\partial v}\frac{\partial y}{\partial w}\right) - \frac{\partial x}{\partial v}\left(\frac{\partial y}{\partial u}\frac{\partial z}{\partial w} - \frac{\partial z}{\partial u}\frac{\partial y}{\partial w}\right) + \frac{\partial x}{\partial w}\left(\frac{\partial y}{\partial u}\frac{\partial z}{\partial v} - \frac{\partial z}{\partial u}\frac{\partial y}{\partial v}\right)$$

Change of variable

Double:

$$\int\int_R f(x,y)dA = \int\int_S f(x(u,v),y(u,v))\left|\frac{\partial(x,y)}{\partial(u,v)}\right| \ du \ dv$$

Triple:

$$\int\int\int_R f(x,y,z)dV = \int\int\int_S f(x(u,v,w),y(u,v,w),z(u,v,w))\left|\frac{\partial(x,y,z)}{\partial(u,v,w)}\right| \ du \ dv \ dw$$

Vectors

Distance formula in three dimensions

The distance $|P_1P_2|$ between the points $P_1(x_1, y_1, z_1)$ and $P_2(x_2, y_2, z_2)$ is

$$|P_1P_2| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

Equation of a sphere

The equation of a sphere

with center (h, k, l) and radius r is $(x - h)^2 + (y - k)^2 + (z - l)^2 = r^2$

with center the origin O and radius r is $x^2 + y^2 + z^2 = r^2$

Definition of vector addition

If $\mathbf{u}$ and $\mathbf{v}$ are vectors positioned so the initial point of $\mathbf{v}$ is at the terminal point of $\mathbf{u}$, then the sum $\mathbf{u} + \mathbf{v}$ is the vector from the initial point of $\mathbf{u}$ to the terminal point of $\mathbf{v}$.

Definition of scalar multiplication

If c is a scalar and $\mathbf{v}$ is a vector, then the **scalar multiple** $c\mathbf{v}$ is the vector whose length is $|c|$ times the length of $\mathbf{v}$ and whose direction is the same as $\mathbf{v}$ if $c > 0$ and is opposite to $\mathbf{v}$ if $c < 0$. If $c = 0$ or $\mathbf{v} = 0$, then $c\mathbf{v} = 0$.

Position vector

Given the points $A(x_1, y_1, z_1)$ and $B(x_2, y_2, z_2)$, the vector a with representation $\overrightarrow{AB}$ is

$$\mathbf{a} = \langle x_2 - x_1, y_2 - y_1, z_2 - z_1 \rangle$$

The magnitude or length of the vector

The length of the two-dimensional vector $\mathbf{a} = \langle a_1, a_2 \rangle$

$$|\mathbf{a}| = \sqrt{a_1^2 + a_2^2}$$

The length of the three-dimensional vector $\mathbf{a} = \langle a_1, a_2, a_3 \rangle$ is

$$|\mathbf{a}| = \sqrt{a_1^2 + a_2^2 + a_3^2}$$

Vector formulas

If $\mathbf{a} = \langle a_1, a_2 \rangle$ and $\mathbf{b} = \langle b_1, b_2 \rangle$, then

$$\mathbf{a} + \mathbf{b} = \langle a_1 + b_1, a_2 + b_2 \rangle$$

$$\mathbf{a} - \mathbf{b} = \langle a_1 - b_1, a_2 - b_2 \rangle$$

$$c\mathbf{a} = \langle ca_1, ca_2 \rangle$$

Similarly, for three-dimensional vectors,

$$\langle a_1, a_2, a_3 \rangle + \langle b_1, b_2, b_3 \rangle = \langle a_1 + b_1, a_2 + b_2, a_3 + b_3 \rangle$$

$$\langle a_1, a_2, a_3 \rangle - \langle b_1, b_2, b_3 \rangle = \langle a_1 - b_1, a_2 - b_2, a_3 - b_3 \rangle$$

$$c\langle a_1, a_2, a_3 \rangle = \langle ca_1, ca_2, ca_3 \rangle$$

Properties of vectors

If $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ are vectors in V_n and c and d are scalar, then

$$\mathbf{a} + \mathbf{b} = \mathbf{b} + \mathbf{a}$$

$$\mathbf{a} + (\mathbf{b} + \mathbf{c}) = (\mathbf{a} + \mathbf{b}) + \mathbf{c}$$

$$\mathbf{a} + 0 = \mathbf{a}$$

$$\mathbf{a} + (-\mathbf{a}) = 0$$

$$c(\mathbf{a} + \mathbf{b}) = c\mathbf{a} + c\mathbf{b}$$

$$(c + d)\mathbf{a} = c\mathbf{a} + d\mathbf{a}$$

$$(cd)\mathbf{a} = c(d\mathbf{a})$$

$$1\mathbf{a} = \mathbf{a}$$

Standard basic vectors

$$\mathbf{i} = \langle 1,0,0 \rangle \qquad \mathbf{j} = \langle 0,1,0 \rangle \qquad \mathbf{k} = \langle 0,0,1 \rangle$$

Dot product

If $\mathbf{a} = \langle a_1, a_2, a_3 \rangle$ and $\mathbf{b} = \langle b_1, b_2, b_3 \rangle$, then the dot product of $\mathbf{a}$ and $\mathbf{b}$ is the number $\mathbf{a} \cdot \mathbf{b}$ given by

$$\mathbf{a} \cdot \mathbf{b} = a_1 b_1 + a_2 b_2 + a_3 b_3$$

Properties of the dot product

If $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ are vectors in V_3 and c is a scalar, then

$$\mathbf{a} \cdot \mathbf{a} = |\mathbf{a}|^2 \qquad (c\mathbf{a}) \cdot \mathbf{b} = c(\mathbf{a} \cdot \mathbf{b}) = \mathbf{a} \cdot (c\mathbf{b})$$

$$\mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{a} \qquad 0 \cdot \mathbf{a} = 0$$

$$\mathbf{a} \cdot (\mathbf{b} + \mathbf{c}) = \mathbf{a} \cdot \mathbf{b} + \mathbf{a} \cdot \mathbf{c}$$

Definition of the dot product

If θ is the angle between the vectors $\mathbf{a}$ and $\mathbf{b}$, then

$$\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}||\mathbf{b}|\cos\theta$$

Corollary

If θ is the angle between the nonzero vectors $\mathbf{a}$ and $\mathbf{b}$, then

$$\cos\theta = \frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}||\mathbf{b}|}$$

Orthogonal

Two vectors $\mathbf{a}$ and $\mathbf{b}$ are orthogonal if and only if $\mathbf{a}\cdot\mathbf{b} = 0$

Projections

The scalar projection of $\mathbf{b}$ onto $\mathbf{a}$ is

$$\text{comp}_{\mathbf{a}}\mathbf{b} = \frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}|}$$

The vector projection of $\mathbf{b}$ onto $\mathbf{a}$ is

$$\text{proj}_{\mathbf{a}}\mathbf{b} = \left(\frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}|}\right)\frac{\mathbf{a}}{|\mathbf{a}|} = \frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}|^2}\mathbf{a}$$

Cross product

If $\mathbf{a} = \langle a_1, a_2, a_3 \rangle$ and $\mathbf{b} = \langle b_1, b_2, b_3 \rangle$, then the cross product of $\mathbf{a}$ and $\mathbf{b}$ is the vector

$$\mathbf{a} \times \mathbf{b} = \langle a_2 b_3 - a_3 b_2, a_3 b_1 - a_1 b_3, a_1 b_2 - a_2 b_1 \rangle$$

The vector $\mathbf{a} \times \mathbf{b}$ is orthogonal to both $\mathbf{a}$ and $\mathbf{b}$.

Angle between two vectors

If θ is the angle between $\mathbf{a}$ and $\mathbf{b}$ (so $0 \leq \theta \leq \pi$), then

$$|\mathbf{a} \times \mathbf{b}| = |\mathbf{a}||\mathbf{b}| \sin \theta$$

Parallel vectors

Two nonzero vectors $\mathbf{a}$ and $\mathbf{b}$ are parallel if and only if

$$\mathbf{a} \times \mathbf{b} = 0$$

Length of the cross product

The length of the cross product $\mathbf{a} \times \mathbf{b}$ is equal to the area of the parallelogram determined by $\mathbf{a}$ and $\mathbf{b}$.

Properties of vector products

$$\mathbf{a} \times \mathbf{b} = -\mathbf{b} \times \mathbf{a}$$

$$(\mathbf{a} + \mathbf{b}) \times \mathbf{c} = \mathbf{a} \times \mathbf{c} + \mathbf{b} \times \mathbf{c}$$

$$(c\mathbf{a}) \times \mathbf{b} = c(\mathbf{a} \times \mathbf{b}) = \mathbf{a} \times (c\mathbf{b})$$

$$\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c}$$

$$\mathbf{a} \times (\mathbf{b} + \mathbf{c}) = \mathbf{a} \times \mathbf{b} + \mathbf{a} \times \mathbf{c}$$

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\mathbf{c}$$

Volume of the parallelepiped

The volume of the parallelepiped determined by the vectors $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ is the magnitude of their scalar triple product,

$$V = |\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})|$$

Vector equation of a line

The vector equation of the line L is

$$\mathbf{r} = \mathbf{r}_0 + t\mathbf{v}$$

where each value of the parameter t gives the position vector $\mathbf{r}$ of a point on L. In other words, as t varies, the line is traced out by the tip of the vector $\mathbf{r}$.

Symmetric equations of the line

$$\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}$$

Parametric equations of the line

The parametric equations of the line L through the point $P_0(x_0, y_0, z_0)$ and parallel to the vector $\mathbf{v} = \langle a, b, c \rangle$ are

$$x = x_0 + at \qquad\qquad y = y_0 + bt \qquad\qquad z = z_0 + ct$$

Each value of the parameter t gives a point (x, y, z) on L.

Vector equation of the line segment

The line segment from $\mathbf{r}_0$ to $\mathbf{r}_1$ is given by the vector equation

$$\mathbf{r}(t) = (1 - t)\mathbf{r}_0 + t\mathbf{r}_1$$ where $0 \le t \le 1$

Vector equation of the plane

$$\mathbf{n} \cdot (\mathbf{r} - \mathbf{r_0}) = 0$$ or $$\mathbf{n} \cdot \mathbf{r} = \mathbf{n} \cdot \mathbf{r_0}$$

Scalar equation of the plane

The scalar equation of the plane through the point $P_0(x_0, y_0, z_0)$ with normal vector $\mathbf{n} = \langle a, b, c \rangle$ is

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$$

Linear equation of the plane

$$ax + by + cz + d = 0$$

Distance from the point to the plane

The distance D from the point P_1 to the plane is equal to the absolute value of the scalar projection of $\mathbf{b}$ onto the normal vector $\mathbf{n} = \langle a, b, c \rangle$. Therefore

$$D = |\text{comp}_{\mathbf{n}}\mathbf{b}| = \frac{|\mathbf{n} \cdot \mathbf{b}|}{|\mathbf{n}|} \qquad \text{or} \qquad D = \frac{|ax_1 + by_1 + cz_1 + d|}{\sqrt{a^2 + b^2 + c^2}}$$

Limit of a vector function

The limit of a vector function $\mathbf{r}$ is defined by taking the limits of its component functions, so if $\mathbf{r}(t) = \langle f(t), g(t), h(t) \rangle$, then

$$\lim_{t \to a} \mathbf{r}(t) = \left\langle \lim_{t \to a} f(t),\ \lim_{t \to a} g(t),\ \lim_{t \to a} h(t) \right\rangle$$

provided the limits of the component function exist.

Definition of the derivative of a vector function

The derivative $\mathbf{r}'$ of a vector function $\mathbf{r}$ is defined as

$$\frac{d\mathbf{r}}{dt} = \mathbf{r}'(t) = \lim_{h\to 0}\frac{\mathbf{r}(t+h)-\mathbf{r}(t)}{h}$$

Derivative of a vector function

If

$$\mathbf{r}(t) = \langle f(t), g(t), h(t)\rangle$$

$$\mathbf{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k}$$

where f, g and h are differentiable functions, then

$$\mathbf{r}'(t) = \langle f'(t), g'(t), h'(t)\rangle$$

$$\mathbf{r}'(t) = f(t)'\mathbf{i} + g'(t)\mathbf{j} + h'(t)\mathbf{k}$$

In other words, to find the derivative of the vector function, just find the derivative of each component separately.

Derivative rules for vector functions

Suppose $\mathbf{u}$ and $\mathbf{v}$ are differentiable vector functions, c is a scalar, and f is a real-valued function. Then

$$\frac{d}{dt}\left[\mathbf{u}(t)+\mathbf{v}(t)\right]=\mathbf{u}'(t)+\mathbf{v}'(t)$$

$$\frac{d}{dt}\left[c\mathbf{u}(t)\right]=c\mathbf{u}'(t)$$

$$\frac{d}{dt}\left[f(t)\mathbf{u}(t)\right]=f'(t)\mathbf{u}(t)+f(t)\mathbf{u}'(t)$$

$$\frac{d}{dt}\left[\mathbf{u}(t)\cdot\mathbf{v}(t)\right]=\mathbf{u}'(t)\cdot\mathbf{v}(t)+\mathbf{u}(t)\cdot\mathbf{v}'(t)$$

$$\frac{d}{dt}\left[\mathbf{u}(t)\times\mathbf{v}(t)\right]=\mathbf{u}'(t)\times\mathbf{v}(t)+\mathbf{u}(t)\times\mathbf{v}'(t)$$

$$\frac{d}{dt}\left[\mathbf{u}(f(t))\right]=f'(t)\mathbf{u}'(f(t))$$ (chain rule)

Definite integral of a vector function

$$\int_a^b \mathbf{r}(t)\ dt=\left(\int_a^b f(t)\ dt\right)\mathbf{i}+\left(\int_a^b g(t)\ dt\right)\mathbf{j}+\left(\int_a^b h(t)\ dt\right)\mathbf{k}$$

Arc length of a vector function

Suppose that the curve has the vector equation $\mathbf{r}(t) = \langle f(t), g(t), h(t) \rangle$ with $a \le t \le b$, or equivalently, the parametric equations $x = f(t)$, $y = g(t)$, and $z = h(t)$ where f', g' and h' are continuous.

If the curve is traversed exactly once as t increases from a to b, then its length is any of the following:

$$L = \int_a^b \sqrt{[f'(t)]^2 + [g'(t)]^2 + [h'(t)]^2}\ dt$$

$$L = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\ dt$$

$$L = \int_a^b |\mathbf{r}'(t)|\ dt$$

Unit vectors

Unit tangent vector $\quad \mathbf{T}(t) = \dfrac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|}$

Unit normal vector $\quad \mathbf{N}(t) = \dfrac{\mathbf{T}'(t)}{|\mathbf{T}'(t)|}$

Binormal vector $\quad \mathbf{B}(t) = \mathbf{T}(t) \times \mathbf{N}(t)$

Curvature

If $\mathbf{T}$ is the unit tangent vector, then curvature is given by any of the following

$$\kappa = \left| \frac{d\mathbf{T}}{ds} \right| \qquad = \qquad \kappa = \frac{\left| \mathbf{T}'(t) \right|}{\left| \mathbf{r}'(t) \right|} \qquad = \qquad \kappa = \frac{\left| \mathbf{r}'(t) \times \mathbf{r}''(t) \right|}{\left| \mathbf{r}'(t) \right|^3}$$

Curvature for a plane equation like $y = f(x)$

For a plane curve $y = f(x)$, we choose x as the parameter and write $\mathbf{r}(x) = x\mathbf{i} + f(x)\mathbf{j}$. Then $\mathbf{r}'(x) = \mathbf{i} + f'(x)\mathbf{j}$ and $\mathbf{r}''(x) = f''(x)\mathbf{j}$. Since $\mathbf{i} \times \mathbf{j} = \mathbf{k}$ and $\mathbf{j} \times \mathbf{j} = 0$, we know that $\mathbf{r}'(x) \times \mathbf{r}''(x) = f''(x)\mathbf{k}$. We also have $|\mathbf{r}'(x)| = \sqrt{1 + \left[f'(x)\right]^2}$ and so

$$\kappa(x) = \frac{\left| f''(x) \right|}{\left[1 + (f'(x))^2\right]^{3/2}}$$

Velocity vector

Suppose a particle moves through space so that its position vector at time t is $\mathbf{r}(t)$. For small values of h, the vector

$$\frac{\mathbf{r}(t+h) - \mathbf{r}(t)}{h}$$

approximates the direction of the particle moving along the curve $\mathbf{r}(t)$. Its magnitude measures the size of the displacement vector per unit time. The formula above gives the average velocity over a time interval of length h and its limit is the velocity vector $\mathbf{v}(t)$ at time t:

$$\mathbf{v}(t) = \lim_{h \to 0} \frac{\mathbf{r}(t+h) - \mathbf{r}(t)}{h} = \mathbf{r}'(t)$$

Therefore, the velocity vector is also the tangent vector and points in the direction of the tangent line.

The speed of the particle at time t is the magnitude of the velocity vector, that is, $|\mathbf{v}(t)|$. This is appropriate because we have

$$|\mathbf{v}(t)| = |\mathbf{r}'(t)| = \frac{ds}{dt}$$ rate of change of distance with respect to time

Parametric equations of the trajectory

$$x = (v_0 \cos \alpha)t$$

$$y = (v_0 \sin \alpha)t - \frac{1}{2}gt^2$$

Tangential and normal components of acceleration

If a_T and a_N are the tangential and normal components of acceleration, then we can write

$$\mathbf{a} = a_T\mathbf{T} + a_N\mathbf{N}$$

where $a_T = v'$ and $a_N = \kappa v^2$

In other words,

$$\mathbf{a} = v'\mathbf{T} + \kappa v^2\mathbf{N}$$

Vector fields in two and three dimensions

Two dimensions

Let D be a set in R^2 (a plane region). A vector field on R^2 is a function $\mathbf{F}$ that assigns to each point (x, y) in D a two-dimensional vector $\mathbf{F}(x, y)$.

Three dimensions

Let E be a subset in R^3. A vector field on R^3 is a function $\mathbf{F}$ that assigns to each point (x, y, z) in E a three-dimensional vector $\mathbf{F}(x, y, z)$.

Definitions for line integrals

Suppose that $\mathbf{F}$ is a continuous vector field in some domain D.

1. $\mathbf{F}$ is a **conservative** vector field if there is a function f such that $\mathbf{F} = \nabla f$. The function f is called a potential function for the vector field. We first saw this definition in the first section of this chapter.

2. $\int_C \mathbf{F} \cdot d\mathbf{r}$ **is independent of path** if $\int_{C_1} \mathbf{F} \cdot d\mathbf{r} = \int_{C_2} \mathbf{F} \cdot d\mathbf{r}$ for any two paths C_1 and C_2 in D with the same initial points and final points.

3. A path C is **closed** if its initial and final points are the same point. As an example, a circle is a closed path.

4. A path C is **simple** if it doesn't cross itself. A circle is a simple curve; a figure 8 is not simple.

5. A region D is **open** if it doesn't contain any of its boundary points.

6. A region D is **connected** if we can connect any two points in the region with a path that lies completely in D.

7. A region D is **simply-connected** if it's connected and contains no holes.

Line integrals

If f is defined on a smooth curve C given by $x = x(t)$, $y = y(t)$, and $a \le t \le b$, then the line integral of f along C can be calculated using either of the following.

$$\int_C f(x, y)\ ds = \lim_{n \to \infty} \sum_{i=1}^{n} f(x_i^*, y_i^*)\Delta s_i$$

$$\int_C f(x, y)\ ds = \int_a^b f\left(x(t), y(t)\right)\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\ dt$$

Line integrals with respect to x and y (with respect to arc length)

The line integral with respect to x:

$$\int_C f(x,y)\ dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*, y_i^*)\Delta x_i$$

$$\int_C f(x,y)\ dx = \int_a^b f\left(x(t), y(t)\right) x'(t)\ dt$$

The line integral with respect to y:

$$\int_C f(x,y)\ dy = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*, y_i^*)\Delta y_i$$

$$\int_C f(x,y)\ dy = \int_a^b f\left(x(t), y(t)\right) y'(t)\ dt$$

Line integral for a continuous vector field

Let $\mathbf{F}$ be a continuous vector field defined on a smooth curve C given by a vector function $\mathbf{r}(t)$, $a \le t \le b$. Then the line integral of F along C is

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_a^b \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)\ dt = \int_C \mathbf{F} \cdot \mathbf{T}\ ds$$

Relationship between line integrals and vector fields

If the vector field $\mathbf{F}$ on R^3 is given in component form by the equation $\mathbf{F} = P\mathbf{i} + Q\mathbf{j} + R\mathbf{k}$, then

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C P\ dx + Q\ dy + R\ dz \qquad \text{where } \mathbf{F} = P\mathbf{i} + Q\mathbf{j} + R\mathbf{k}$$

Fundamental theorem for line integrals

Let C be a smooth curve given by the vector function $\mathbf{r}(t)$, $a \le t \le b$. Let f be a differentiable function of two or three variables whose gradient vector ∇f is continuous on C. Then

$$\int_C \nabla f \cdot d\mathbf{r} = f(\mathbf{r}(b)) - f(\mathbf{r}(a))$$

Independence of path

$\int_C \mathbf{F} \cdot d\mathbf{r}$ is independent of path in D if and only if

$\int_C \mathbf{F} \cdot d\mathbf{r} = 0$ for every closed path C in D.

Conservative vector fields

1. Suppose $\mathbf{F}$ is a vector field that is continuous on an open connected region D. If $\int_C \mathbf{F} \cdot d\mathbf{r}$ is independent of path in D, then $\mathbf{F}$ is a conservative vector field on D; that is, there exists a function f such that $\nabla f = \mathbf{F}$.

2. If $\mathbf{F}(x, y) = P(x, y)\mathbf{i} + Q(x, y)\mathbf{j}$ is a conservative vector field where P and Q have continuous first-order partial derivatives on a domain D, then throughout D we have

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$$

3. Let $\mathbf{F} = P\mathbf{i} + Q\mathbf{j}$ be a vector field on an open simply-connected region D. Suppose that P and Q have continuous first-order derivatives and

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x} \qquad \text{throughout } D$$

Then $\mathbf{F}$ is conservative.

Green's theorem

Let C be a positively oriented, piecewise-smooth, simple closed curve in the plane and let D be the region bounded by C. If P and Q have continuous partial derivatives on an open region that contains D, then

$$\int_C P\ dx + Q\ dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right)\ dA$$

Green’s theorem for the area of the region

The area of the region D is given by any of the following

$$A = \oint_C x\ dy \qquad A = -\oint_C y\ dx \qquad A = \frac{1}{2}\oint_C x\ dy - y\ dx$$

Curl

1. The curl of $\mathbf{F}$ is the vector field on R^3 defined by

$$\text{curl}\ \mathbf{F} = \left(\frac{\partial \mathrm{R}}{\partial \mathrm{y}} - \frac{\partial \mathrm{Q}}{\partial \mathrm{z}}\right)\mathbf{i} + \left(\frac{\partial \mathrm{P}}{\partial \mathrm{z}} - \frac{\partial \mathrm{R}}{\partial \mathrm{x}}\right)\mathbf{j} + \left(\frac{\partial \mathrm{Q}}{\partial \mathrm{x}} - \frac{\partial \mathrm{P}}{\partial \mathrm{y}}\right)\mathbf{k}$$

$$\text{curl}\ \mathbf{F} = \nabla \times \mathbf{F}$$

2. If f is a function of three variables that has continuous second-order partial derivatives, then

$$\text{curl}(\nabla \mathrm{f}) = 0$$

3. If $\mathbf{F}$ is a vector field defined on all of R^3 whose component functions have continuous partial derivatives and $\text{curl}\ \mathbf{F} = 0$, then $\mathbf{F}$ is a conservative vector field.

Divergence

1. If $\mathbf{F} = P\mathbf{i} + Q\mathbf{j} + R\mathbf{k}$ is a vector field on $\mathbb{R}^3$ and $\partial P/\partial x$, $\partial Q/\partial y$, and $\partial R/\partial z$ exist, then the divergence of $\mathbf{F}$ is the function of three variables defined by

$$\text{div } \mathbf{F} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}$$

$$\text{div } \mathbf{F} = \nabla \cdot \mathbf{F}$$

2. If $\mathbf{F}$ is a vector field on $\mathbb{R}^3$, then the curl $\mathbf{F}$ is also a vector field on $\mathbb{R}^3$. Therefore, we can calculate its divergence and say that, if $\mathbf{F} = P\mathbf{i} + Q\mathbf{j} + R\mathbf{k}$ is a vector field on $\mathbb{R}^3$ and P, Q, and R have continuous second-order partial derivatives, then

$$\text{div curl } \mathbf{F} = 0$$

Vector form of green's theorem

$$\oint_C \mathbf{F} \cdot n \ ds = \iint_D \text{div } \mathbf{F}(x, y) \ dA$$

Surface area of the parametric surface

If a smooth parametric surface S is given by the equation

$$\mathbf{r}(u, v) = x(u, v)\mathbf{i} + y(u, v)\mathbf{j} + z(u, v)\mathbf{k} \qquad (u, v) \in D$$

and S is covered just once as (u, v) ranges throughout the parameter domain D then the surface area of S is

$$A(S) = \iint_D |\mathbf{r}_u \times \mathbf{r}_v| \ dA$$

where

$$\mathbf{r}_u = \frac{\partial x}{\partial u}\mathbf{i} + \frac{\partial y}{\partial u}\mathbf{j} + \frac{\partial z}{\partial u}\mathbf{k} \qquad \text{and} \qquad \mathbf{r}_v = \frac{\partial x}{\partial v}\mathbf{i} + \frac{\partial y}{\partial v}\mathbf{j} + \frac{\partial z}{\partial v}\mathbf{k}$$

The surface area formula can also be written as

$$A(S) = \iint_D \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2} \ dA$$

Surface integral

The surface integral of f over the surface S is

$$\iint_S f(x, y, z) \ dS = \lim_{m,n\to\infty} \sum_{i=1}^{m} \sum_{j=1}^{n} f(P_{ij}^*)\Delta S_{ij}$$

$$\iint_S f(x, y, z) \ dS = \iint_D f(\mathbf{r}(u, v))|\mathbf{r}_u \times \mathbf{r}_v| \ dA$$

$$\iint_S f(x, y, z) \ dS = \iint_D f(x, y, g(x, y))\sqrt{\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 + 1} \ dA$$

Surface integral for a continuous vector field

If $\mathbf{F}$ is a continuous vector field defined on an oriented surface S with unit normal vector $\mathbf{n}$, then the surface integral of $\mathbf{F}$ over S is

$$\iint_S \mathbf{F} \cdot \ dS = \iint_S \mathbf{F} \cdot \mathbf{n} \ dS$$

$$\iint_S \mathbf{F} \cdot \ dS = \iint_D \mathbf{F} \cdot (\mathbf{r}_u \times \mathbf{r}_v) \ dA$$

$$\iint_S \mathbf{F} \cdot \ dS = \iint_D \left(-P\frac{\partial g}{\partial x} - Q\frac{\partial g}{\partial y} + R \right) \ dA$$

This integral is also called the flux of $\mathbf{F}$ across S.

Stokes' theorem

Let S be an oriented piecewise-smooth surface that is bounded by a simple, closed, piecewise-smooth boundary curve C with a positive orientation. Let $\mathbf{F}$ be a vector field whose components have continuous partial derivatives on an open region in R^3 that contains S. Then

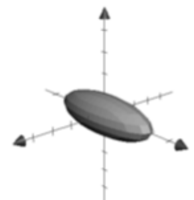
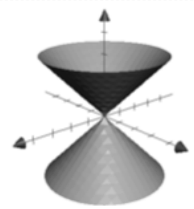
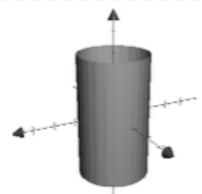
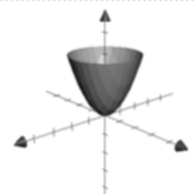
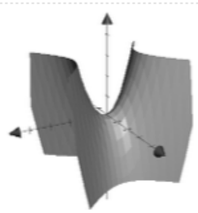
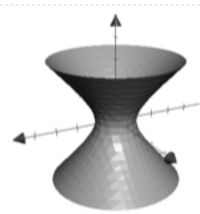
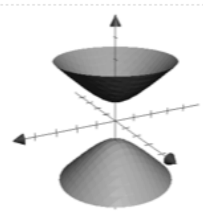
$$\int_C \mathbf{F} \cdot d\mathbf{r} = \iint_S \text{curl } \mathbf{F} \cdot \mathrm{d}\mathbf{S}$$

The divergence theorem

Let E be a simple solid region and let S be the boundary surface of E, given with positive (outward) orientation. Let $\mathbf{F}$ be a vector field whose component functions have continuous partial derivatives on an open region that contains E. Then

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iiint_E \text{div } \mathbf{F} \text{ dV}$$

Quadric surfaces

	surface	standard form	shifted form	traces	axis set by	tips
ellipsoid		$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ center (0,0,0)	$\frac{(x-h)^2}{a^2}+\frac{(y-k)^2}{b^2}+\frac{(z-l)^2}{c^2}=1$ center (h,k,l)	ellipses	major: largest value of a, b and c	a=b=c: the ellipsoid is a sphere
cone		$\frac{x^2}{a^2}+\frac{y^2}{b^2}=\frac{z^2}{c^2}$ center (0,0,0)	$\frac{(x-h)^2}{a^2}+\frac{(y-k)^2}{b^2}=\frac{(z-l)^2}{c^2}$ center (h,k,l)	horizontal: ellipses vertical: hyperbolas x=h or y=k: pairs of lines	the variable by itself on one side of the = sign when all signs are +	
cylinder		$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ center (0,0,0)	$\frac{(x-h)^2}{a^2}+\frac{(y-k)^2}{b^2}=1$ center (h,k)	horizontal: ellipses horizontal, a=b: circles vertical: pairs of lines	the variable not appearing in the equation	a=b: the cylinder is circular
elliptic paraboloid		$\frac{x^2}{a^2}+\frac{y^2}{b^2}=\frac{z}{c}$ center (0,0,0)	$\frac{(x-h)^2}{a^2}+\frac{(y-k)^2}{b^2}=\frac{z-l}{c}$ center (h,k,l)	horizontal: ellipses horizontal, a=b: circles vertical: parabolas	the variable that isn't squared	c: opens up −c: opens down
hyperbolic paraboloid		$\frac{x^2}{a^2}-\frac{y^2}{b^2}=\frac{z}{c}$ center (0,0,0)	$\frac{(x-h)^2}{a^2}-\frac{(y-k)^2}{b^2}=\frac{z-l}{c}$ center (h,k,l)	horizontal: hyperbolas vertical: parabolas	the variable that isn't squared	c: opens up −c: opens down
hyperboloid of one sheet		$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1$ center (0,0,0)	$\frac{(x-h)^2}{a^2}+\frac{(y-k)^2}{b^2}-\frac{(z-l)^2}{c^2}=1$ center (h,k,l)	horizontal: ellipses horizontal, a=b: circles vertical: hyperbolas	the variable with the − sign in front of it when all variables are on the same side	
hyperboloid of two sheets		$-\frac{x^2}{a^2}-\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ center (0,0,0)	$-\frac{(x-h)^2}{a^2}-\frac{(y-k)^2}{b^2}+\frac{(z-l)^2}{c^2}=1$ center (h,k,l)	horizontal: ellipses horizontal, a=b: circles vertical: hyperbolas	the variable with the + sign in front of it when all variables are on the same side	