19/01/2021
19 January 2021 09:06 AM

Unit – V (Sequences & Series)

(I) Sequence of Real no's.

(II) Series of Real numbers
- series of the Real numbers.
- series of alternating +ve and −ve numbers.

Sequence of Real numbers:

Let $\{a_n\}_{n=1}^{\infty}$ be a sequence of Real numbers.

$\Rightarrow a_1, a_2, a_3, a_4 \ldots a_n, \ldots$

Rough.
A.P
$a, a+d, a+2d, \ldots$
nth term will be $a+(n-1)d$

(Eg): $a_n = \frac{1}{n}$.

Here $a_1 = \frac{1}{1} = 1$
$a_2 = \frac{1}{2} = \frac{1}{2}$
$a_3 = \frac{1}{3}$ . . . $\{\frac{1}{n}\}_{n=1}^{\infty}$.

(Eg). $a_n = k$

$a_1 = k,\ a_2 = k,\ a_3 = k \ldots$
constant sequence. $\{k\}_{n=1}^{\infty}$.

Def: Limit of sequence $\{a_n\}_{n=1}^{\infty}$ (or)

$\{a_n\}_{n=1}^{\infty}$ Converges to l.

$-5, -4, -3, -2, -1, \ldots$
$a_1 = -5$
$a_2 = -4$
$\vdots$
$a_5 = -1$

Let $\{a_n\}_{n=1}^{\infty}$ be a sequence of Real numbers. and if given $\epsilon > 0$ (small, but nearer to 0). there exist ... such that nearer to 0). There exists

$N > 0$, (integer) such that

✓ $|a_n - l| < \epsilon \quad \forall\, n \geq N$

(OR)

✓ $\lim_{n \to \infty} a_n = l$

$\epsilon \to 0$
$a_n = l$

(Eg) (i) $a_n = \left\{\frac{1}{n}\right\}_{n=1}^{\infty}$

$a_1 = 1, \quad a_2 = \frac{1}{2}, \quad a_3 = \frac{1}{3} \ldots$

$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1}{n} \to 0$

$\therefore$ limit of $\left\{\frac{1}{n}\right\}_{n=1}^{\infty}$ is 0

$\{1, 0.5, 0.33, 0.25, 0.2, 0.166, \ldots \ldots 0\}$

(ii) $a_n = \{K\}_{n=1}^{\infty}$

$\lim_{n\to\infty} a_n = \lim_{n\to\infty} K = K =$ finite.

limit of $\{K\}_{n=1}^{\infty}$ is K.

Def:

(i) Bounded Below:

Let $\{a_n\}_{n=1}^{\infty}$ be a sequence of Real nos.

If we say $\{a_n\}_{n=1}^{\infty}$ is said to have bounded below, then $a_n \geq m$, m – real number

(eg): $\left\{\frac{1}{n}\right\}_{n=1}^{\infty}$ have the bounded below as 0. $\forall\, n$

(ii) Bounded above: ... real no/.

(2) Bounded above :

$a_n \le m, \quad m$ — real no/.

$\forall n$.

(eg): $\left\{\frac{1}{n}\right\}_{n=1}^{\infty}$, have bounded above as $\underline{\underline{1}}$.

Def: Bounded sequence $\{a_n\}_{n=1}^{\infty}$.

if $\{a_n\}_{n=1}^{\infty}$ have both bounded above and bounded below

ie, $m_1 \le a_n \le m_2, \quad \forall n.$

$m_1, m_2 \rightarrow$ real nos/.

(eg): (1) $\left\{\frac{1}{n}\right\}_{n=1}^{\infty}$ is bounded, since

Bdd below $= 0$
Bdd above $= \underline{1}$

$0 \le \frac{1}{n} \le 1$

(2) $\{n^3\}_{n=1}^{\infty} \Rightarrow 1^3, 2^3, 3^3, \ldots$

This sequence is not bounded, since only bounded below $= 1$

Def: Divergent sequence:

(1) $\{a_n\}_{n=1}^{\infty}$ diverges to $+\infty$ if given positive real no/. M, then there exist positive integer N such that $a_n > M$ for all $n \ge N$.

(OR)

$$\lim_{n\to\infty} a_n = +\infty$$

(ii) $\{a_n\}_{n=1}^{\infty}$ diverges to $-\infty$.

$$\lim_{n\to\infty} a_n = -\infty.$$

(Eg): (i) $\{a_n\}_{n=1}^{\infty} = \{n^2\}_{n=1}^{\infty}$ is dgs to ∞.

(ii) $\{a_n\}_{n=1}^{\infty} = \{(-n)\}_{n=1}^{\infty}$ dgs to $-\infty$

Def:

Monotonically increasing sequence:

A sequence $\{a_n\}_{n=1}^{\infty}$ is said to be monotonically increasing if

$a_n \le a_{n+1}$ for all n.

ie, $a_1 \le a_2 \le a_3 \le a_4 \le \dots$

Monotonically decreasing sequence:

$a_n \ge a_{n+1}$ for all n.

ie, $a_1 \ge a_2 \ge a_3 \ge \dots$

(Eg): $\{\frac{1}{n}\}_{n=1}^{\infty}$ is monotonically decreasing

$\{n^2\}_{n=1}^{\infty}$ is monotonically increasing.

Results:

(i) A monotonically increasing sequence with the ... is a convergent

(i) A monotonically increasing sequence with the bounded above will be a convergent sequence.

$1, 2, 3, 4 \ldots$ ∄

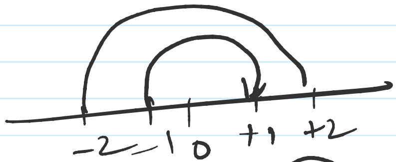
(2) A monotonically decreasing sequence will be convergent with bounded below sequence.

(i) $\frac{1}{1}, \frac{1}{2}, \frac{1}{3} \ldots\ldots$ ‖

(ii) $10, 9, 8, 7, 6, \ldots 0, -1, -2, \ldots$ ‖

Def: Oscillatory sequence:

(eg): $-1, +1, -2, +2, -3, +3, \ldots\ldots$



i.e, does not converge (or) does not diverge to $+\infty$ and $-\infty$.

(eg) $a_n = \{(-1)^n\}_{n=1}^{\infty}$.

$-1, +1, -1, +1 \ldots\ldots$

$(-1)^n$ is oscillating sequence.

Results:

(i) If $|x| < 1$, then $\lim_{n\to\infty} x^n = 0$

(ii) If $x > 1$, then $\lim_{n\to\infty} x^n = \infty$.

(iii) If $\lim_{n\to\infty} a_n = a$, $\lim_{n\to\infty} b_n = b$, then

(eg) $\lim_{n\to\infty} \left(\frac{1}{2}\right)^n = \frac{1}{\infty} = 0$

(iii) If $\lim_{n\to\infty} a_n = \ldots$, $\ldots$, | (eg) $\lim_{n\to\infty} \left(\frac{1}{2}\right)^n$ $= \lim_{n\to\infty} \frac{1}{2^n} = \frac{1}{\infty} = 0$

then

$$\{a_n \pm b_n\}_{n=1}^{\infty} \to a \pm b$$

$$\{a_n b_n\}_{n=1}^{\infty} \to ab$$

$$\left\{\frac{a_n}{b_n}\right\}_{n=1}^{\infty} \to \frac{a}{b}, \text{ provided } b \neq 0.$$

Problems:

1. Find the limit of $\left\{\frac{n+1}{2n+7}\right\}_{n=1}^{\infty}$ (or)

Check the $\left\{\frac{n+1}{2n+7}\right\}_{n=1}^{\infty}$ is converges/ diverges.

Sol:

$$a_n = \frac{n+1}{2n+7}$$

$$\lim_{n\to\infty} \frac{n+1}{2n+7} = \lim_{n\to\infty} \frac{\cancel{n}(1+\frac{1}{n})}{\cancel{n}(2+7/n)}$$

$$= \lim_{n\to\infty} \frac{1+1/n}{2+7/n} = \frac{1}{2} = \text{finite}.$$

Here, converges to the limit $\frac{1}{2}$.

2. $\{a_n\}_{n=1}^{\infty} = \{x^{1/n}\}$ where $x > 0$.

$$\lim_{n\to\infty} x^{1/n} = x^0 = 1.$$ | $\lim_{n\to\infty} 2^{1/n} = 2^0 = 1$.

Converges to 1.

Cauchy's General Principle of Convergence

Let $\{a_n\}_{n=1}^{\infty}$ be a sequence of Real numbers, ... a positive

Let $\{a_n\}_{n=1}^{\infty}$ be a sequence of Real ...
Given $\epsilon > 0$ (small), there exist a positive integer N, such that

$$|a_n - a_m| < \epsilon \quad \forall\, n, m \geq N \;\&\; n > m.$$

Rough.
$|a_n - l| < \epsilon$
$\forall\, n \geq N$

Then, $\{a_n\}_{n=1}^{\infty}$ be a convergent sequence.

Infinite Series

If $\{a_n\}_{n=1}^{\infty}$ be sequence of Real nos/:

Then $\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + a_4 + \cdots$

Sequence
$a_1, a_2, a_3, a_4 \ldots$

Def: $\sum_{n=1}^{\infty} a_n \to$ converges.

Let $\{a_n\}_{n=1}^{\infty}$ be a sequence

$$S_1 = a_1$$
$$S_2 = a_1 + a_2$$
$$S_3 = a_1 + a_2 + a_3$$
$$\vdots$$
$$S_n = a_1 + a_2 + a_3 + \cdots + a_n$$
$$\vdots$$

Now, (i) $\sum_{n=1}^{\infty} a_n$ converges only if $\{S_n\}_{n=1}^{\infty}$ converges.

$\Downarrow$

$\lim_{n \to \infty} S_n \to$ finite

(ii) $\sum_{n=1}^{\infty} a_n$ diverges if $\{S_n\}_{n=1}^{\infty}$ diverges

(iii) $\sum_{n=1}^{\infty} a_n$ oscillates if $\{S_n\}_{n=1}^{\infty}$ oscillates.

Eg: $\sum_{n=1}^{\infty} a_n = 1+2+3+4+\cdots$

$S_1 = 1$,

$S_2 = 1+2 = 3$

$S_3 = 1+2+3 = 6$

$S_4 = 1+2+3+4 = 10$

$\vdots$

$S_n = 1+2+3+4+\cdots+n$

$\lim_{n\to\infty} S_n = \lim_{n\to\infty} (1+2+3+\cdots+n)$

$= \infty$. diverges.

$\sum_{n=1}^{\infty} a_n = 1+2+3+4+\cdots = $ diverges.

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Series with positive terms.

$\sum_{n=1}^{\infty} a_n$, Here $a_n > 0$.

Theorem: If $\sum_{n=1}^{\infty} a_n$ cgs, then $\lim_{n\to\infty} a_n = 0$

But the converse is not true.

converse: If $\lim_{n\to\infty} a_n = 0 \not\Rightarrow \sum_{n=1}^{\infty} a_n$ cgs.

Eg: $\sum_{n=1}^{\infty} \frac{1}{n}$

Here, $\lim_{n\to\infty} \frac{1}{n} = 0$, but

$\sum_{n=1}^{\infty} \frac{1}{n}$ does not converges.

Recall.

$\sum_{n=1}^{\infty} a_n \to$ cgs/dgs

we will formulate the partial sum of sequence

$S_1 = a_1,\ S_2 = a_1 + a_2 \ldots$

$S_n = a_1 + a_2 + \cdots + a_n$

$\{S_n\}_{n=1}^{\infty}$

(1) $\sum_{n=1}^{\infty} a_n$ cgs if $\{S_n\}_{n=1}^{\infty}$ cgs

(2) $\sum_{n=1}^{\infty} a_n$ dgs if $\{S_n\}$ dgs.

Tests for positive termed series:

(1) Integral Test

(2) Comparision test

(3) Ratio test $\to$ D'Alembert'z ratio Test

(4) Rabbe's test

(5) Root test $\to$ Cauchy's Root Test.

(6) Logarithmic test

Integral test:

if $\sum f(n) = f(1) + f(2) + f(3) + \ldots$, also, $f(n)$ decreases as n increases, then

$\sum f(n)$ cgs if $\int_1^{\infty} f(x)dx$ is finite

$\sum f(n)$ dgs if $\int_1^{\infty} f(x)dx$ is infinite

Problem: 1

By using Integral test, Show that $\sum \frac{1}{n^p}$ cgs if $p > 1$, and dgs if $p \le 1$.

Sol: Let $f(n) = \frac{1}{n^p}$, $f(1) = \frac{1}{1^p}$, $f(2) = \frac{1}{2^p}$

$f(n)$ decreases as n increases.

case (i) : $p \neq 1$, $f(n) = \frac{1}{n^p}$, $f(x) = \frac{1}{x^p}$

$$\int_1^{\infty} f(x)dx = \int_1^{\infty} \frac{1}{x^p} dx$$

$$= \lim_{N\to\infty} \int_1^{N} \frac{1}{x^p} dx$$

$$= \lim_{N\to\infty} \left[\frac{x^{-p+1}}{-p+1}\right]_1^N$$

$$= \lim_{N\to\infty} \left[\frac{x^{1-p}}{1-p}\right]_1^N = \lim_{N\to\infty}\left[\frac{N^{1-p} - 1}{1-p}\right]$$

$$= 0 - \frac{1}{1-p} = \left\{\frac{1}{p-1} \text{ if } p > 1, \infty \text{ if } p < 1.\right.$$

$\int x^n dx = \frac{x^{n+1}}{n+1}$

$N^{1-p} = \frac{N}{N^p}$

$P \neq 1$, $\int_1^\infty f(x)\,dx = \begin{cases} \frac{1}{p-1} & \text{if } p>1 \Rightarrow \text{cgs} \\ \infty & \text{if } p<1. \Rightarrow \text{dgs.} \end{cases}$

<u>case (ii)</u> : $p=1$.

$$\int_1^\infty \frac{1}{x}\,dx = (\log x)_1^\infty = (\log x)^\infty - (\log x)_1^0 = \infty$$

$$\Rightarrow \sum_{n=1}^\infty \frac{1}{n^p} \text{ dgs if } p=1.$$

<u>Conclusion</u>: $\sum_{n=1}^\infty \frac{1}{n^p}$ $\begin{cases} \text{converges if } p>1 \\ \text{diverges if } p\le 1 \end{cases}$

<u>Results</u>: (i) $\sum_{n=1}^\infty \frac{1}{n}$ diverges., $p=1$

(ii) $\sum_{n=1}^\infty \frac{1}{n^2}$ converges, $p=2$

(iii) $\sum_{n=1}^\infty \frac{1}{\sqrt{n}}$ diverges since $p=\frac{1}{2}$

(iv) $\sum_{n=1}^\infty \frac{1}{n^{p+q-1}}$ $\begin{cases} \text{cgs if } (p+q-1)>1 \rightarrow p+q>2 \\ \text{dgs if } (p+q-1)\le 1 \rightarrow p+q\le 2 \end{cases}$

(II) comparision test

If $\sum_{n=1}^\infty u_n$ and $\sum_{n=1}^\infty v_n$ are two positive termed series and if $\boxed{u_n \le k v_n}$ for all n, then

(i) $\sum_{n=1}^\infty u_n$ will converge if $\sum_{n=1}^\infty v_n$ cgs.

(ii) $\sum_{n=1}^\infty u_n$ will diverge if $\sum_{n=1}^\infty v_n$ dgs.

<u>Other forms of comparision test.</u>

if $\sum_{n=1}^\infty u_n$ and $\sum_{n=1}^\infty v_n$ are two positive termed series, and if $\frac{u_n}{v_n} \rightarrow (\text{finite} \neq 0)$ as $n\to\infty$.

$\sum_{n=1}^\infty u_n$ $\begin{cases} \text{(i) cgs if } \Sigma v_n \text{ cgs.} \\ \text{(ii) dgs if } \Sigma v_n \text{ dgs.} \end{cases}$

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Recall:

Comparision Test

(i) $\sum_{n=1}^{\infty} u_n$, $\sum_{n=1}^{\infty} v_n$ are +ve termed

(ii) $u_n \leq k v_n$ (or) $\lim_{n\to\infty} \frac{u_n}{v_n} \Rightarrow = \text{finite} (\neq 0)$

Then

$\sum u_n$ cgs if $\sum v_n$ cgs

$\sum u_n$ dgs if $\sum v_n$ dgs.

Problems:

Test the convergance of the series

$$\sum_{n=1}^{\infty} \frac{1}{(n+1)(2n+1)}$$

Sol:

Let $u_n = \frac{1}{(n+1)(2n+1)}$

Let $v_n = \frac{\text{the degree of } n \text{ in } Nr}{\text{the degree of } n \text{ in } Dr}$

$$= \frac{n^0}{n^2} = \frac{1}{n^2}$$

$$\lim_{n\to\infty} \frac{u_n}{v_n} = \lim_{n\to\infty} \frac{\frac{1}{(n+1)(2n+1)}}{1/n^2}$$

$$= \lim_{n\to\infty} \frac{\frac{1}{\cancel{n^2}(1+1/n)(2+\frac{1}{n})}}{\cancel{1/n^2}}$$

$$= \frac{1}{(1+0)(2+0)} = \frac{1}{2} \neq 0$$

Rough:

$$\sum \frac{\sqrt{n}}{2n} = \frac{n^{1/2}}{2n} = \frac{1}{2n^{1/2}}$$

$$v_n = \frac{1}{n^{1/2}}$$

By Comparision Test, $\sum_{n=1}^{\infty} \frac{1}{(n+1)(2n+1)}$ converges, since $\sum_{n=1}^{\infty} \frac{1}{n^2}$ cgs

② Test the convergance of the series

(i) $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n^2+1}}$ (ii) $\sum_{n=1}^{\infty} \left(\sqrt{n^2+1} - n\right)$

Sol:

(i) Let $u_n = \frac{1}{\sqrt{n^2+1}}$, and $v_n = \frac{n^0}{n^1} = \frac{1}{n}$

$$\lim_{n\to\infty} \frac{u_n}{v_n} = \lim_{n\to\infty} \frac{\frac{1}{\sqrt{n^2+1}}}{1/n}$$

$$= \lim_{n\to\infty} \frac{\frac{1}{\cancel{n}\sqrt{1+\frac{1}{n^2}}}}{\cancel{1/n}}$$

$$\frac{1}{\infty} = 0$$

$$= \frac{1}{\sqrt{1+0}} = 1 \neq 0$$

since $\sum_{n=1}^{\infty} \frac{1}{n}$ dgs, $\Rightarrow \sum_{n=1}^{\infty} \frac{1}{\sqrt{n^2+1}}$ dgs.

(ii) $\sum_{n=1}^{\infty} \left(\sqrt{(n^2+1)} - n\right) \Rightarrow u_n = \sqrt{n^2+1} - n$

try to make u_n, to be in $\frac{Nr}{Dr}$.

$$u_n = \sqrt{n^2+1} - n = \frac{(\sqrt{n^2+1} - n)(\sqrt{n^2+1} + n)}{(\sqrt{n^2+1} + n)}$$

$$= \frac{(\cancel{n^2}+1) - \cancel{n^2}}{(\sqrt{n^2+1} + n)} = \frac{1}{(\sqrt{n^2+1} + n)}$$

$$u_n = \frac{1}{(\sqrt{n^2+1} + n)}$$

Let $v_n = \frac{1}{n^1} = \frac{1}{n}$

$$\lim_{n\to\infty} \frac{u_n}{v_n} = \lim_{n\to\infty} \frac{\frac{1}{(\sqrt{n^2+1} + n)}}{1/n}$$

$$= \lim_{n\to\infty} \frac{\frac{1}{\cancel{n}(\sqrt{1+1/n^2} + 1)}}{1/\cancel{n}}$$

$$= \frac{1}{(\sqrt{1+0} + 1)} = \frac{1}{2} \neq 0$$

By comparision Test,

since $\sum_{n=1}^{\infty} \frac{1}{n}$ dgs $\Rightarrow \sum_{n=1}^{\infty} (\sqrt{n^2+1} - n)$ dgs.

(3) test the convergance of $\sum_{n=1}^{\infty} \frac{1}{n} \sin\left(\frac{1}{n}\right)$

Sol:

$$u_n = \frac{1}{n}\sin\left(\frac{1}{n}\right) = \frac{1}{n}\left(\left(\frac{1}{n}\right) - \frac{(1/n)^3}{3!} + \cdots\right)$$

$$= \frac{1}{n^2} - \frac{1}{3!} \cdots$$

$$v_n = \frac{1}{n^2}$$

Rough

$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots$

$\sin\left(\frac{1}{n}\right) = \left(\frac{1}{n}\right) - \frac{(1/n)^3}{3!} + \cdots$

$$\lim_{n\to\infty} \frac{u_n}{v_n} = \lim_{n\to\infty} \frac{\left(\frac{1}{n}\right)\sin(1/n)}{1/n^2}$$

$$= \lim_{n\to\infty} \frac{\sin(1/n)}{(1/n)}$$

$\theta = \frac{1}{n}$

$n\to\infty \Rightarrow \theta\to 0$

$$= \lim_{\theta\to 0} \frac{\sin\theta}{\theta}$$

$$= 1 \neq 0$$

By comparision test,

since $\sum_{n=1}^{\infty} \frac{1}{n^2}$ cgs $\Rightarrow \sum_{n=1}^{\infty} \frac{1}{n}\sin\left(\frac{1}{n}\right)$ cgs.

(ii) $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} \tan\left(\frac{1}{n}\right)$

sol:
Let $u_n = \frac{1}{\sqrt{n}} \tan\left(\frac{1}{n}\right)$ | $\frac{1}{n^{1/2}}\left(\left(\frac{1}{n}\right) - \cdots\right) = \frac{1}{n^{3/2}}$

$v_n = \frac{1}{n^{3/2}}$

$$\lim_{n\to\infty} \frac{u_n}{v_n} = \lim_{n\to\infty} \frac{\frac{1}{n^{1/2}} \tan\left(\frac{1}{n}\right)}{1/n^{3/2}}$$

$$= \lim_{n\to\infty} \frac{\tan(1/n)}{(1/n)^{3/2-1/2}} = \lim_{n\to\infty} \frac{\tan(1/n)}{(1/n)}$$

$$= 1 \neq 0$$

$\lim_{\theta\to 0} \frac{\tan\theta}{\theta} = 1 \neq 0$

By comparision test,

since $\sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$ cgs $\Rightarrow \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} \tan\left(\frac{1}{n}\right)$ cgs.

$\sum \frac{1}{n^p}$ cgs if $p > 1$

problem: Discuss the convergance of $\sum \frac{1}{(n+a)^p (n+b)^q}$

sol:
Let $u_n = \frac{1}{(n+a)^p (n+b)^q}$

$v_n = \frac{1}{n^{(p+q)}}$

$$\lim_{n\to\infty} \frac{u_n}{v_n} = \lim_{n\to\infty} \frac{\frac{1}{(n+a)^p (n+b)^q}}{\frac{1}{n^{(p+q)}}}$$

$$= \lim_{n\to\infty} \frac{\frac{1}{\cancel{n^p}\left(1+\frac{a}{n}\right)^p \cancel{n^q}\left(1+\frac{b}{n}\right)^q}}{\frac{1}{\cancel{n^{p+q}}}}$$

$\neq 0$ (finite)

By comparision test,

$$\sum_{n=1}^{\infty} \frac{1}{(n+a)^p (n+b)^q} \rightarrow \begin{cases} \text{cgs if } (p+q) > 1 \\ \text{dgs if } (p+q) \leq 1. \end{cases}$$

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problems:

Recall: Comparision

(i) $u_n \geq 0$.

(ii) $\lim_{n\to\infty} \frac{u_n}{v_n} = \text{finite} \neq 0$

Then

(i) Σu_n cgs if Σv_n cgs

(ii) Σu_n dgs if Σv_n dgs

① Check the Convergance of the series

$$\frac{1}{1.2.3} + \frac{3}{2.3.4} + \frac{5}{3.4.5} + \dots \infty$$

sol: Let $u_n = \frac{(2n-1)}{(n).(n+1)(n+2)}$

$$v_n = \frac{n}{n^3} = \frac{1}{n^2}$$

$$\lim_{n\to\infty} \frac{u_n}{v_n} = \lim_{n\to\infty} \frac{(2n-1)}{n(n+1)(n+2)} \times \frac{1}{1/n^2}$$

$$= \lim_{n\to\infty} \frac{\cancel{n}(2-1/n)}{\frac{n^3(1+1/n)(1+2/n)}{1/n^2}}$$

$$= 2 \neq 0.$$

By comparision test,

since $\sum_{n=1}^{\infty} \frac{1}{n^2}$ cgs, $\Rightarrow \sum_{n=1}^{\infty} u_n$ cgs.

Rough:

1, 3, 5, ...

$t_n = a+(n-1)d$

$= 1+(n-1)2$

$= 2n+1-2$

$= 2n-1$

1, 2, 3, ...

$t_n = a+(n-1)d$

$= 1+(n-1)1$

$= \cancel{1}+n-\cancel{1} = n$

2, 3, 4, ...

$t_n = 2+(n-1)1$

$= 2+n-1 = n+1$

② Test the cgance of $\frac{1.2}{3.4.5} + \frac{2.3}{4.5.6} + \frac{3.4}{5.6.7} + \dots$

sol: $u_n = \frac{(n)(n+1)}{(n+2)(n+3)(n+4)}$ ← n=3

3, 4, 5, ...

$t_n = 3+(n-1)1$

$= 3+n-1$

$= n+2$

$$v_n = \frac{n^2}{n^3} = \frac{1}{n}$$

$$\lim_{n\to\infty} \frac{u_n}{v_n} = \text{finite} \neq 0,$$

Since $\Sigma v_n = \Sigma \frac{1}{n}$ dgs $\Rightarrow \sum_{n=1}^{\infty} u_n$ dgs.

③ $\sum_{n=1}^{\infty} e^{-n^2}$ cgs / dgs.

sol: $\sum_{n=1}^{\infty} \frac{1}{e^{n^2}} \Rightarrow u_n = \frac{1}{e^{n^2}}$

Form 2 of comparision.

(i) $u_n \leq k v_n$ as $n \to \infty$.

$e^{n^2} > n^2$

$$e^{n^2} = 1 + \frac{n^2}{1!} + \frac{(n^2)^2}{2!} + \dots$$

$$\frac{1}{e^{n^2}} < \frac{1}{n^2}$$

$e^{n^2} > n^2$

$\Rightarrow \frac{1}{e^{n^2}} < \frac{1}{n^2}$ as $n \to \infty$,

By comparision test,

$\Sigma \frac{1}{n^2}$ cgs $\Rightarrow \Sigma \frac{1}{e^{n^2}}$ cgs.

D'Alembert's Ratio Test (or) RATIO test.

Rule: If

(i) $u_n \geq 0$,

(ii) $\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = L$

then, the series $\sum_{n=1}^{\infty} u_n$ → cgs if $L<1$, → dgs if $L>1$, → Test fails if $L=1$.

Problems:

① Test the convergance of

$$\sum_{n=1}^{\infty} \frac{n^3+1}{2^n+1}$$

sol:

$$u_n = \frac{n^3+1}{2^n+1}$$

$$u_{n+1} = \frac{(n+1)^3+1}{2^{n+1}+1}$$

$$\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = \lim_{n\to\infty} \frac{(n+1)^3+1}{2^{n+1}+1} \times \frac{2^n+1}{n^3+1}$$

$$= \lim_{n\to\infty} \frac{\cancel{n^3}\left[\left(1+\frac{1}{n}\right)^3+\frac{1}{n^3}\right]}{\cancel{2^n}\left(2+\frac{1}{2^n}\right)} \times \frac{\cancel{2^n}\left(1+\frac{1}{2^n}\right)}{\cancel{n^3}\left(1+\frac{1}{n^3}\right)}$$

$$= \frac{(1+0)}{(2+0)} \times \frac{(1+0)}{(1+0)} = \frac{1}{2} = L$$

Here, $L = \frac{1}{2} < 1$, $\sum_{n=1}^{\infty} u_n$ cgs.

② Test the series of $\frac{x}{1} + \frac{1}{2}\frac{x^2}{3} + \frac{1.3}{2.4}\frac{x^3}{5} + \frac{1.3.5}{2.4.6}\frac{x^4}{7} + \ldots\infty$, $x>0$

sol:

Let $u_n = \frac{1.3.5\ldots\boxed{(2n-3)}}{2.4.6.8\ldots(2n-2)} \cdot \frac{x^n}{(2n-1)}$

$1,3,5,\ldots$
$1+(n-1)2$
$1+2n-2$

$2+(n-1)2$
$2+2n-2$

$$u_n = \frac{1.3.5\ldots(2n-3)}{2.4.6\ldots(2n-2)} \frac{x^n}{(2n-1)}$$

$n=4$

$\frac{1.3.5}{2.4.6} x^4$

$$u_{n+1} = \frac{1.3.5\ldots(2n-3)(2n-1)}{2.4.6\ldots(2n-2)(2n)} \times \frac{x^{n+1}}{(2n+1)}$$

$(n-2)$
$n-1$
$n+1$

$$\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = \lim_{n\to\infty} \frac{[1\cdot3\cdot5\cdots(2n-3)](2n-1)}{[2\cdot4\cdot6\cdots(2n-2)](2n)} \times \frac{x^{n+1}}{(2n+1)} \times \frac{2\cdot4\cdot6\cdots\cancel{(2n-2)}}{1\cdot3\cdot5\cdots\cancel{(2n-3)}} \times \frac{(2n-1)}{\cancel{x^n}}$$

$$= \lim_{n\to\infty} \frac{(2n-1)}{2n} \frac{x}{(2n+1)} \times \frac{(2n-1)}{1}$$

$$= \lim_{n\to\infty} \frac{\cancel{n}(2-1/n)}{\cancel{n}(2)\cancel{n}(2+1/n)} \cancel{n}(2-1/n) \cdot x = \cancel{\frac{4}{4}}x$$

$$\boxed{L = x}$$

By Ratio Test, if $L = x < 1 \Rightarrow \sum_{n=1}^{\infty} u_n$ cgs

$L = x > 1 \Rightarrow \sum_{n=1}^{\infty} u_n$ dgs

$L = x = 1 \Rightarrow$ test fails.

② $\frac{1}{2} + \frac{4}{9}x + \frac{9}{28}x^2 + \cdots \infty$, $x > 0$.

Sol: Let $u_n = \frac{n^2}{n^3+1} x^{n-1}$

$n = 1$

$1 + (n-1)$

$1^2, 2^2, 3^2$

$u_{n+1} = \frac{(n+1)^2}{(n+1)^3+1} x^n$

$$\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = \lim_{n\to\infty} \frac{(n+1)^2}{(n+1)^3+1} x^n \times \frac{n^3+1}{n^2} \times \frac{1}{x^{n-1}}$$

$$= \lim_{n\to\infty} \frac{\cancel{n^2}(1+\frac{1}{n})^2}{\cancel{n^3}\left[(1+\frac{1}{n})^3 + \frac{1}{n^3}\right]} \frac{\cancel{n^3}(1+\frac{1}{n^3})}{\cancel{n^2}} x^{n-n+1}$$

$$L = x^1$$

By Ratio test, $L = x < 1 \Rightarrow \sum u_n$ cgs, if $0 < x < 1$.

$L = x > 1 \Rightarrow \sum u_n$ dgs., if $x > 1$

If $x = 1$, $u_n = \frac{n^2}{n^3+1}$

$v_n = \frac{1}{n}$, $\lim_{n\to\infty} \frac{u_n}{v_n} = \lim_{n\to\infty} \frac{\frac{n^3}{n^3+1}}{1/n} = 1 \neq 0$

By comparision Test, $\sum u_n$ dgs $[x = 1]$

Conclusion: $\sum u_n$ cgs if $0 < x < 1$, dgs if $x \geq 1$

Rabbe's test:

If

(i) $\sum_{n=1}^{\infty} u_n$ be a positive termed series.

(ii) $\lim_{n\to\infty} \left\{ n\left(\frac{u_n}{u_{n+1}} - 1\right) \right\} \to >1$, cgs

$\to <1$, dgs.

Problem:

① $\frac{2}{3.4} + \frac{2.4}{3.5.6} + \frac{2.4.6}{3.5.7.8} + \frac{2.4.6.8}{3.5.7.9.10} + \cdots$

Sol:

$u_n = \frac{2.4.6.8 \cdots (2n)}{3.5.7.9.\cdots (2n+1).(2n+2)}$

$3+(n-1)2$

$3+2n-2$

$(2n+1)$

$u_{n+1} = \frac{2\cdot 4\cdot 6 \cdots (2n)(2n+2)}{3.5.7.9.\cdots (2n+1)(2n+3).(2n+4)}$

$2(n+1) = 2n+2$

$2(n+1)+1$

$= 2n+3$

$\lim_{n\to\infty} \frac{u_{n+1}}{u_n}$

$= \lim_{n\to\infty} \frac{[2.4.6.\cdots \cancel{(2n)}](2n+2)}{[3.5.7.\cdots \cancel{(2n+1)}](2n+3)(2n+4)} \times \frac{[3.5.7.\cdots \cancel{(2n+1)}](2n+2)}{[2.4.6 \cdots \cancel{(2n)}]}$

$= \lim_{n\to\infty} \frac{(2n+2)(2n+2)}{(2n+3)(2n+4)} = \frac{4}{4} = 1$

$\frac{u_{n+1}}{u_n}$

By Ratio test, $L=1 \Rightarrow$ test fails.

$n\left(\frac{u_n}{u_{n+1}} - 1\right) = n\left\{\frac{(2n+3)(2n+4)}{(2n+2)(2n+2)} - 1\right\}$

$= n\left\{\frac{(2n+3)(2n+4) - (2n+2)^2}{(2n+2)^2}\right\}$

$= n\left\{\frac{(\cancel{4n^2}+14n+12) - (\cancel{4n^2}+4+8n)}{(2n+2)^2}\right\}$

Using Rabbe's, >1, cgs

$= n\left\{\frac{6n+8}{(2n+2)^2}\right\}$

$\lim_{n\to\infty} n\left\{\frac{u_n}{u_{n+1}} - 1\right\} = \lim_{n\to\infty} \frac{n(6n+8)}{(2n+2)^2} = \frac{6}{4} = \frac{3}{2} > 1$

28/01/2021 - KCE

Recall:

Ratio Test

$L = \lim_{n\to\infty} \frac{u_{n+1}}{u_n}$

(i) $L<1 \Rightarrow$ cgs
(ii) $L>1 \Rightarrow$ dgs
(iii) $L=1 \Rightarrow$ test fails

Raabe's

$L = \lim_{n\to\infty} n\left\{\frac{u_n}{u_{n+1}} - 1\right\}$

(i) $L<1 \Rightarrow$ dgs
(ii) $L>1 \Rightarrow$ cgs.

Problems.

① Test the convergence of the series

$1 + \frac{a}{b} + \frac{a(a+1)}{b(b+1)} + \frac{a(a+1)(a+2)}{b(b+1)(b+2)} + \dots \infty, \quad a, b > 0.$

Solution:

Let $u_n = \frac{a(a+1)(a+2)\dots(a+n-1)}{b(b+1)\dots(b+n-1)}$ | $a+(n-1)1$, $a+n-1$

$u_{n+1} = \frac{a(a+1)(a+2)\dots(a+n)}{b(b+1)\dots(b+n)}$

Problem can be written as

$1 + \left[\frac{a}{b} + \frac{a(a+1)}{b(b+1)} + \dots\right]$

$\Rightarrow 1 + \underbrace{\sum_{n=1}^{\infty}\left(\frac{a(a+1)\dots(a+n-1)}{b(b+1)\dots(b+n-1)}\right)}$

Rough.

$1 + \frac{x}{1!} + \frac{x^2}{2!} + \dots + \frac{x^{n-1}}{(n-1)!}$

$= 1 + \frac{x}{1} + \frac{x(x+1)}{1.2} + \frac{x(x+1)(x+2)}{1.2.3}$

By Ratio test,

$\frac{u_{n+1}}{u_n} = \frac{[a(a+1)\dots\cancel{(a+n-1)}](a+n)}{[b(b+1)\dots\cancel{(b+n-1)}](b+n)} \times \frac{b(b+1)\dots(b+n-1)}{a(a+1)\dots(a+n-1)}$

$a+(n-1)$
$n \to n+1$
$\Rightarrow a+((n+\cancel{1})-\cancel{1})$

$\frac{u_{n+1}}{u_n} = \frac{(a+n)}{(b+n)}$

$\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = \lim_{n\to\infty} \frac{(a+n)}{(b+n)} = \lim_{n\to\infty} \frac{\cancel{n}\left(\frac{a}{n}+1\right)}{\cancel{n}\left(\frac{b}{n}+1\right)} = 1$

Test fails.

By Raabe's test.

$\lim_{n\to\infty} n\left\{\frac{u_n}{u_{n+1}} - 1\right\} = \lim_{n\to\infty} n\left\{\left(\frac{b+n}{a+n}\right) - 1\right\} =$

$= \lim_{n\to\infty} n\left\{\frac{b+\cancel{n}-a-\cancel{n}}{a+n}\right\}$

$= \lim_{n\to\infty} \frac{n(b-a)}{(a+n)}$

$= \lim_{n\to\infty} \frac{\cancel{n}(b-a)}{\cancel{n}\left(\frac{a}{n}+1\right)} = (b-a)$

Case: (i) $(b-a)>1$, cgs

Case (ii) $(b-a)<1$, dgs.

Case (iii) $(b-a)=1$, $u_n = \frac{a[(a+1)\dots\cancel{(a+n-1)}]}{[(a+1)(a+2)(a+3)\dots](a+n)}$

$\Downarrow$

$b = (a+1)$

$u_n = \frac{a}{a+n}, \; v_n = \frac{1}{n}$

If $(b-a)=1$, $\sum u_n$ dgs.

② Test the convergance of $\sum_{n=1}^{\infty} \frac{1}{n!}$

Sol: $u_n = \frac{1}{n!}$, $u_{n+1} = \frac{1}{(n+1)!}$

$(n+1)! = (n+1)\underbrace{n(n-1)\cdots}_{n!}$

$$\frac{u_{n+1}}{u_n} = \frac{1}{(n+1)!} \times n! = \frac{\cancel{n!}}{(n+1)\cancel{n!}} = \frac{1}{n+1}$$

$$\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = \lim_{n\to\infty} \frac{1}{n(1+1/n)} = 0 < 1$$

∴ The series will converge by Ratio test.

② $\sum_{n=1}^{\infty} \frac{3^n n!}{n^n}$

$$u_n = \frac{3^n n!}{n^n}, \quad u_{n+1} = \frac{3^{n+1}(n+1)!}{(n+1)^{(n+1)}}$$

$$\frac{u_{n+1}}{u_n} = \frac{3^{n+1}(n+1)\cancel{n!}}{(n+1)^{n+1}} \times \frac{n^n}{3^n \cancel{n!}} = \frac{3(n+1)n^n}{(n+1)^{n+1}}$$

$$= \frac{3 n^n \cancel{(n+1)}}{(n+1)^n \cdot \cancel{(n+1)}} = 3\left(\frac{1}{\left(\frac{n+1}{n}\right)^n}\right)$$

$$\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = 3 \lim_{n\to\infty} \underbrace{\frac{1}{(1+1/n)^n}} = 3/e > 1$$

Note: (i) $\lim_{n\to\infty} (1+1/n)^n = e$.

By Ratio test, $\sum u_n$ dgs.

③ Discuss the convergance of the series whose nth term is

(i) $\frac{1}{1+nx^n}$ (ii) $\frac{x^n}{1+x^n}$ (iii) $\frac{x^n}{1+n^2x^{2n}}$, $(x>0)$

Sol: (i) $u_n = \frac{1}{1+nx^n}$, $u_{n+1} = \frac{1}{1+(n+1)x^{n+1}}$

$$\frac{u_{n+1}}{u_n} = \frac{1}{1+(n+1)x^{n+1}} \times \left(\frac{1+nx^n}{1}\right) =$$

$$= \frac{1+nx^n}{1+(n+1)x^{n+1}} = \frac{nx^n\left(\frac{1}{nx^n}+1\right)}{1+(n+1)x^n \cdot x}$$

$$\frac{(n+1)\cancel{x^n}\cdot x}{n\cancel{x^n}} \to \left(\frac{n+1}{n}\right)x \to \left(1+1/n\right)x$$

$$= \frac{\cancel{nx^n}\left(\frac{1}{nx^n}+1\right)}{\cancel{nx^n}\left(\frac{1}{nx^n}+\left(1+\frac{1}{n}\right)x\right)}$$

$$\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = \left(\frac{0+1}{0+(1)x}\right) = \frac{1}{x}$$

$$\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = \frac{1}{x} = L$$

Ratio

$L<1$, cgs

$L>1$, dgs

$L=1$, fails.

(i) If $0<x<1 \Rightarrow$ dgs.

(ii) If $x>1 \Rightarrow$ converges.

(iii) If $x=1 \Rightarrow$ fails.

$$u_n = \frac{1}{1+nx^n} \Rightarrow u_n = \frac{1}{1+n} \rightarrow \text{dgs}.$$

29/01/2021 - ECE.

Root Test (or) Cauchy's Root Test.

If

(i) $u_n \geq 0$

(ii) $L = \lim_{n\to\infty} u_n^{1/n}$

Then (i) $L<1 \Rightarrow$ cgs

(ii) $L>1 \Rightarrow$ dgs.

Recall

Comparison. $u_n \geq 0$.

$u_n \leq k v_n$,

(or)

$\lim_{n\to\infty} \frac{u_n}{v_n} =$ finite $\neq 0$

(1) $\sum u_n$ cgs if $\sum v_n$ cgs

(2) $\sum u_n$ dgs if $\sum v_n$ dgs

Ratio

$L = \lim_{n\to\infty} \frac{u_{n+1}}{u_n}$

(1) $L<1 \Rightarrow$ cgs

(2) $L>1 \Rightarrow$ dgs

Rabbe's

$L = \lim_{n\to\infty} n\left(\frac{u_n}{u_{n+1}} - 1\right)$

(1) $L>1 \Rightarrow$ cgs

(2) $L<1 \Rightarrow$ dgs.

Problems:

1. Test the convergence of $\sum_{n=1}^{\infty} \frac{1}{(\log n)^n}$.

Sol:

Let $u_n = \frac{1}{(\log n)^n}$

$$(u_n)^{1/n} = \left[\frac{1}{(\log n)^n}\right]^{1/n} = \frac{1}{(\log n)}$$

$$L = \lim_{n\to\infty} u_n^{1/n} = \lim_{n\to\infty} \frac{1}{\log n} = \frac{1}{\infty} = 0 < 1$$

By Root Test, $L<1 \Rightarrow \sum_{n=1}^{\infty} u_n$ converges.

2. $\sum_{n=1}^{\infty} \frac{1}{(1+1/n)^{n^2}}$

Sol: $u_n = \frac{1}{(1+1/n)^{n^2}}$

$$(u_n)^{1/n} = \left[\frac{1}{(1+1/n)^{n^2}}\right]^{1/n} = \left[\frac{1}{(1+1/n)^n}\right]$$

$$L = \lim_{n\to\infty} u_n^{1/n} = \lim_{n\to\infty} \frac{1}{(1+1/n)^n} = \frac{1}{e} < 1$$

WKT. $\lim_{n\to\infty} (1+1/n)^n = e$

By Root test, $L<1$, Converges.

3. $\sum_{n=1}^{\infty} \frac{x^n}{(n+1)^n}$

Let $u_n = \frac{x^n}{(n+1)^n}$, $u_n^{1/n} = \frac{x}{(n+1)}$

$$L = \lim_{n\to\infty} u_n^{1/n} = \lim_{n\to\infty} \frac{1}{(n+1)} x = \lim_{n\to\infty} \frac{1}{n(1+1/n)} x$$

$\to 0$ for all values x.

$L = 0 < 1$, By Root Test, $\sum u_n \to$ cgs.

④ $\sum_{n=1}^{\infty} \frac{n^3}{3^n}$

$u_n = \frac{n^3}{3^n}$, $u_n^{1/n} = \left(\frac{n^3}{3^n}\right)^{1/n} = \frac{(n^{1/n})^3}{3}$

$n^{3/n} = (n^{1/n})^3$

$(\ )^0 = 1$

$L = \lim_{n\to\infty} u_n^{1/n} = \lim_{n\to\infty} \frac{1}{3}(n^{1/n}) = \frac{1}{3} < 1$

By Root Test, $L<1$, $\Rightarrow$ cgs.

⑤ $\sum_{n=1}^{\infty} \frac{n!}{n^n}$

Sol:

$u_n = \frac{n!}{n^n}$

$(u_n)^{1/n} = \frac{(1.2.3.4\cdots n)^{1/n}}{(n^n)^{1/n}} = \frac{(1.2.3\ldots n)^{1/n}}{n}$

$L = \lim_{n\to\infty} u_n^{1/n}$

$= \lim_{n\to\infty} \frac{(1.2.3\ldots n)^{1/n}}{n} = \frac{(\ )^0}{\infty}$

$= \frac{1}{\infty} = 0$

<1

$L<1 \Rightarrow$ Root Test $\rightarrow$ cgs

Rough

$u_n = \frac{n!}{n^n}$

$u_{n+1} = \frac{(n+1)!}{(n+1)^{n+1}}$

$L = \lim_{n\to\infty} \frac{u_{n+1}}{u_n}$

$= \lim_{n\to\infty} \frac{\cancel{(n+1)}\,\cancel{n!}}{\cancel{(n+1)}(n+1)^n} \times \frac{n^n}{\cancel{n!}}$

$= \lim_{n\to\infty} \frac{n^n}{(n+1)^n}$

$= \lim_{n\to\infty} \frac{\cancel{n^n}}{\cancel{n^n}(1+1/n)^n}$

$= \frac{1}{e} < 1$

$\Rightarrow$ Ratio Test cgs.

Alternating series:

A series in which the terms are alternatively positive and negative is called alternating series.

$u_1 - u_2 + u_3 - u_4 + u_5 - \cdots = \sum_{n=1}^{\infty} (-1)^{n-1} u_n$

+ve, $\sum_{n=1}^{\infty} u_n$, $u_n \geq 0$

Absolute series:

In the alternating series, $\sum_{n=1}^{\infty} |(-1)^{n-1} u_n|$

$\Rightarrow$ the series will contain all the terms are +ve.

Absolute Convergance of the alternating series:

(i) $\sum_{n=1}^{\infty} u_n$ convergent and $\sum_{n=1}^{\infty} |u_n|$ convergent.

Then, we can say the given alternating series is absolutely converges.

Conditionally converges:

$\sum_{n=1}^{\infty} u_n$ converge, whereas, $\sum_{n=1}^{\infty} |u_n|$ diverges.

Then, the alternating series conditionally converges.

Note:

(i) An absolutely convergent series always convergent

i.e, $\sum_{n=1}^{\infty} |u_n|$ cgs $\Rightarrow \sum_{n=1}^{\infty} u_n$ cgs

Converse is not true.

03/02/2021 – ECE.

part - A.
MCQ.
part - B.
(B)
(i)
(ii)
(iii)
(iv).
Ans
(ii)
justification

A. 30×1=30 part - C
B. 15×2=30
C: 5×3=15

75 marks

Recall

Alternating Series

$$a_1 - a_2 + a_3 - a_4 + \cdots = \sum_{n=1}^{\infty} \underbrace{(-1)^{n-1} a_n}_{u_n}$$

Absolutely cgsance:

$\sum_{n=1}^{\infty} |u_n|$ cgs and $\sum_{n=1}^{\infty} u_n$ cgs $\quad\Big\|\quad |(-1)^{n-1}| = 1$

Then, we can say $\sum_{n=1}^{\infty} u_n$ absolutely cgs.

Conditional convergance.

$\sum_{n=1}^{\infty} u_n$ cgs but $\sum_{n=1}^{\infty} |u_n|$ dgs.

Then $\sum_{n=1}^{\infty} u_n$ conditionally cgs.

Note:

(i) If $\sum_{n=1}^{\infty} u_n$ abs cgs $\Rightarrow \sum_{n=1}^{\infty} u_n$ also cgs.

(ii) But reverse is not true.

i.e., $\sum_{n=1}^{\infty} u_n$ cgs. $\not\Rightarrow \sum_{n=1}^{\infty} |u_n|$ cgs.

Example: 1 $\quad \frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \cdots$ is absolutely cgs or not.

Let $\frac{1}{1^2} - \frac{1}{2^2} + \cdots = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{n^2} = \sum_{n=1}^{\infty} u_n.$

Then $\sum_{n=1}^{\infty} |u_n| = \sum_{n=1}^{\infty} \left|(-1)^{n-1} \frac{1}{n^2}\right| = \sum_{n=1}^{\infty} \frac{1}{n^2}$

$\longrightarrow$ cgs, since $\sum \frac{1}{n^p}$ cgs if $p>1$

By the property,

Hence $\sum_{n=1}^{\infty} |u_n|$ cgs $\Rightarrow \sum_{n=1}^{\infty} u_n$ cgs.

Therefore, $\sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{n^2}$ cgs absolutely

Example: 2 $\quad 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{n}$

check whether absolutely cgs/not. $\quad = \sum_{n=1}^{\infty} u_n$

$$\sum_{n=1}^{\infty} |u_n| = \sum_{n=1}^{\infty} \left|(-1)^{n-1} \frac{1}{n}\right| = \sum_{n=1}^{\infty} \frac{1}{n} \text{ dgs}$$

$\sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{n}$ is not absolutely cgs $\quad\Big\|\quad$ Since $\sum_{n=1}^{\infty} \frac{1}{n^p}$ dgs if $p \le 1$.

Problem:

prove that the exponential series $1+\frac{x}{1!}+\frac{x^2}{2!}+\frac{x^3}{3!}+\dots$ is absolutely convergent and hence convergent for all values of x.

Sol: $1+\frac{x}{1!}+\frac{x^2}{2!}+\dots = \sum_{n=1}^{\infty}\frac{x^{n-1}}{(n-1)!} = \sum_{n=1}^{\infty} u_n.$

Let $u_n = \frac{x^{n-1}}{(n-1)!}$

To check: $\sum_{n=1}^{\infty} |u_n|$ cgs or not.

$$L = \lim_{n\to\infty}\frac{|u_{n+1}|}{|u_n|} = \lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right|$$

Rough: $\sum_{n=1}^{\infty} u_n$, $L = \lim_{n\to\infty}\frac{u_{n+1}}{u_n}$

$L<1 \Rightarrow$ cgs

$L>1 \Rightarrow$ dgs

$$u_n = \frac{x^{n-1}}{(n-1)!}, \quad u_{n+1} = \frac{x^n}{n!}$$

$$\frac{u_{n+1}}{u_n} = \frac{x^{\cancel{n}}}{n!}\times\frac{(n-1)!}{x^{\cancel{n-1}}}$$

$$= \frac{x}{n}$$

$n! = n(n-1)!$

$$L = \lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty}\left|\frac{x}{n}\right|$$

$= 0$ for all value of x.

Here $L = 0 < 1, \Rightarrow \sum_{n=1}^{\infty}|u_n|$ cgs.

Hence By the property. $\sum_{n=1}^{\infty}|u_n|$ cgs absolutely $\Rightarrow \sum_{n=1}^{\infty} u_n$ cgs.

Leibnitz test (For checking the convergance of alternating series).

The alternating series $\sum_{n=1}^{\infty}(-1)^{n-1}u_n$ converges if

(i) $\{u_n\}_{n=1}^{\infty}$ is a monotonically decreasing

$u_1 > u_2 > u_3 > u_4 \dots$

$1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\dots = \sum_{n=1}^{\infty}(-1)^{n-1}\left(\frac{1}{n}\right)$, $u_n = \frac{1}{n}$

otherwords, Each term is numerically less than the preceding term.

(ii) $\lim_{n\to\infty} u_n = 0$

Then, $u_1 - u_2 + u_3 - u_4 + \dots$ cgs by the Leibnitz test.

Note: If $\lim_{n\to\infty} u_n \neq 0$, $u_1 - u_2 + u_3 \dots$ Oscillates (or) not Convergent.

problems:

① Discuss the convergance of $\frac{1}{1\cdot 2} - \frac{1}{3\cdot 4} + \frac{1}{5\cdot 6} \cdots \infty$.

Sol: $\frac{1}{1\cdot 2} - \frac{1}{3\cdot 4} + \frac{1}{5\cdot 6} \cdots = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{(2n-1)(2n)}$

$$= \sum_{n=1}^{\infty} (-1)^{n-1} u_n.$$

$1, 3, 5 \cdots$

$1+(n-1)2$

$1+2n-2$

$2n-1$

$u_n = \frac{1}{(2n-1)(2n)}$, $u_{n+1} = \frac{1}{(2(n+1)-1)(2(n+1))}$

$= \frac{1}{(2n+2-1)(2n+2)}$

$= \frac{1}{(2n+1)(2n+2)}$

$u_n \geq u_{n+1}$

$u_n - u_{n+1} \geq 0$

Here,

$(2n+1)(2n+2) > (2n-1)(2n)$

$\Rightarrow \frac{1}{(2n+1)(2n+2)} \leq \frac{1}{(2n-1)(2n)} \Rightarrow u_{n+1} \leq u_n$

Hence $\{u_n\}_{n=1}^{\infty}$ monotonically decreasing

(ii) $\lim_{n\to\infty} u_n = \lim_{n\to\infty} \frac{1}{(2n-1)(2n)} = 0$

By Leibnitz Test, $\sum_{n=1}^{\infty} (-1)^{n-1} u_n$ cgs.

Discuss the convergance of

(i) $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$

(ii) $1 - \frac{\sqrt{2}}{2} + \frac{\sqrt{3}}{3} - \frac{\sqrt{4}}{4} + \cdots$

(iii) $\sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{n^p}$, $p > 0$

Sol:

(i) $1 - \frac{1}{2} + \frac{1}{3} \cdots = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{n}$

Let $u_n = \frac{1}{n}$, $u_{n+1} = \frac{1}{n+1}$

$n+1 > n \Rightarrow \frac{1}{n+1} < \frac{1}{n} \Rightarrow u_{n+1} < u_n$.

$\{u_n\}_{n=1}^{\infty}$ is monotonically decreasing.

$\lim_{n\to\infty} u_n = \lim_{n\to\infty} \frac{1}{n} = 0$

By Leibnitz test, $\sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{n}$ cgs.

Note: $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$ cgs but

$1 + \frac{1}{2} + \frac{1}{3} + \cdots = \sum_{n=1}^{\infty} |u_n|$ dgs

$\therefore \sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{n}$ conditionally cgs.

(ii) $1-\frac{\sqrt{2}}{2}+\frac{\sqrt{3}}{3}-\cdots = \sum_{n=1}^{\infty}(-1)^{n-1}\cdot\frac{\sqrt{n}}{n}$

Let $u_n = \frac{\sqrt{n}}{n} = \frac{1}{\sqrt{n}}$, $u_{n+1} = \frac{1}{\sqrt{n+1}}$

$$\sqrt{n+1} > \sqrt{n} \Rightarrow \frac{1}{\sqrt{n+1}} < \frac{1}{\sqrt{n}}$$

$$\Rightarrow u_{n+1} < u_n$$

$\therefore \{u_n\}_{n=1}^{\infty}$ is monotonically decreasing.

$$\lim_{n\to\infty} u_n = \lim_{n\to\infty}\frac{1}{\sqrt{n}} = \lim_{n\to\infty}\frac{1}{n^{1/2}} = 0.$$

By Leibnitz test, $\sum_{n=1}^{\infty}(-1)^{n-1}\frac{\sqrt{n}}{n}$ cgs.

(iii) $\sum_{n=1}^{\infty}(-1)^{n-1}\frac{1}{n^p}$, $p>0$

Let $u_n = \frac{1}{n^p}$, $p>0$

$u_{n+1} = \frac{1}{(n+1)^p}$, $p>0$.

Rough

$u_{n+1} < u_n$

$\frac{u_{n+1}}{u_n} < 1$

$$\frac{u_{n+1}}{u_n} = \frac{1}{(n+1)^p}\cdot\frac{n^p}{1}$$

$$= \frac{\cancel{n^p}}{\cancel{n^p}\left(1+\frac{1}{n}\right)^p} = \frac{1}{\left(1+\frac{1}{n}\right)^p}$$

$$\frac{u_{n+1}}{u_n} = \frac{1}{\left(1+\frac{1}{n}\right)^p} < 1 \qquad p>0.$$

$\therefore \{u_n\}_{n=1}^{\infty}$ is monotonically decreasing

$$\lim_{n\to\infty} u_n = \lim_{n\to\infty}\frac{1}{n^p} = 0$$

By Leibnitz test,
$\sum_{n=1}^{\infty}(-1)^{n-1}\frac{1}{n^p}$ cgs, $p>0$.

Problem: Discuss the convergance of the series
$\frac{x}{1+x}-\frac{x^2}{1+x^2}+\frac{x^3}{1+x^3}-\cdots\infty$ if $0<x<1$.

Sol:
$$\sum_{n=1}^{\infty}(-1)^{n-1}\frac{x^n}{1+x^n} = \sum_{n=1}^{\infty}(-1)^{n-1}u_n$$

$u_n = \frac{x^n}{1+x^n}$, $u_{n+1} = \frac{x^{n+1}}{1+x^{n+1}}$

$$u_n - u_{n+1} = \frac{x^n}{1+x^n} - \frac{x^{n+1}}{1+x^{n+1}}$$

$$= \frac{(1+x^{n+1})x^n - x^{n+1}(1+x^n)}{(1+x^n)(1+x^{n+1})}$$

$$= \frac{x^n + \cancel{x^n x^{n+1}} - x^{n+1} - \cancel{x^{n+1} x^n}}{(1+x^n)(1+x^{n+1})}$$

$$= \frac{x^n - x^{n+1}}{(1+x^n)(1+x^{n+1})} = \frac{x^n(1-x)}{(1+x^{n+1})(1+x^n)} > 0.$$

$u_n - u_{n+1} > 0$

$\Rightarrow u_n > u_{n+1}$

monotonically decreasing.

$0 < x < 1$

$$\frac{(1/2)^n (1-1/2)}{(1+1/2^{n+1})(1+1/2)}$$

$$\lim_{n\to\infty} u_n = \lim_{n\to\infty} \frac{x^n}{1+x^n} = 0$$

By Leibnitz Test,

$$\sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{1+x^n} \text{ cgs.}$$

$0 < x < 1$

$\lim_{n\to\infty} x^n =$

$x = 1/2$ $\quad \frac{1}{2^n} = 0.$

problem

$\sum_{n=1}^{\infty} (-1)^n (1 + 1/n)$ converges or not?

$u_n = 1 + \frac{1}{n}$, $\quad u_{n+1} = 1 + \frac{1}{n+1}$

Clearly, $\{u_n\}$ is monotonically decreasing.

$\frac{1}{n} > \frac{1}{n+1}$

$1 + \frac{1}{n} > 1 + \frac{1}{n+1}$

$u_n > u_{n+1}$

$$\lim_{n\to\infty} u_n = \lim_{n\to\infty} \left(1 + \frac{1}{n}\right)$$

$$= 1 \neq 0.$$

$\sum_{n=1}^{\infty} (-1)^n u_n$ does not converge, i.e. oscillates.

$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!}$

$\sum_{n=1}^{\infty} (-1)^n \sin\left(\frac{1}{n}\right)$

Let $u_n = \sin\left(\frac{1}{n}\right) = \left(\frac{1}{n}\right) - \cdots$

$u_{n+1} = \sin\left(\frac{1}{n+1}\right) = \left(\frac{1}{n+1}\right) - \cdots$

$\frac{1}{n} > \frac{1}{n+1} \Rightarrow u_n > u_{n+1}$ for all n.

monotonically decreases.

$$\lim_{n\to\infty} \sin\left(\frac{1}{n}\right) = \lim_{n\to\infty} \left(\frac{1}{n} - \frac{(1/n)^3}{3!} + \cdots\right) = 0$$

Converges.

04/02/2021 ECE

Recall: Leibnitz Test.
(alternating series)
$\sum (-1)^{n-1} u_n$
(i) $u_1 > u_2 > u_3 > \cdots$
monotonically decrease.
i.e, $u_n > u_{n+1}$
$u_n - u_{n+1} > 0.$
(ii) $\lim_{n\to\infty} u_n = 0$
$\sum_{n=1}^{\infty} (-1)^{n-1} u_n$ cgt
otherwise,
$\lim_{n\to\infty} u_n \neq 0 \Rightarrow$ oscillate

① Test the convergance of the series.
$1 - 2x + 3x^2 - 4x^3 + \cdots \infty \quad (x<1)$

sol: $1 - 2x + 3x^2 - \cdots = \sum_{n=1}^{\infty} (-1)^{n-1} \cdot nx^{n-1}$

Let $u_n = nx^{n-1}$, $\underline{x<1}$

(i) $1 \geq 2x > 3x^2 > 4x^3$

(ii) $\lim_{n\to\infty} u_n = \lim_{n\to\infty} nx^{n-1}$

$= \lim_{n\to\infty} \frac{n \cdot x^n}{x}$

$= \lim_{n\to\infty} \frac{n}{x^{-n+1}}$

$= \lim_{n\to\infty} \frac{n}{x^{-n}} \cdot \frac{1}{x}$

$\to 0.$

$\lim_{n\to\infty} \frac{1}{x^{-n}}$
$= \lim_{n\to\infty} x^n \to 0$
$\underline{(0<x<1)}$

By Leibnitz, $\sum_{n=1}^{\infty} (-1)^{n-1} nx^{n-1}$ cgs.

② Test the convergance of $\sum_{n=1}^{\infty} \frac{\cos n\pi}{n^2+1}$

sol: $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2+1}$

$\cos n\pi = (-1)^n$
$\cos 0 = 1$
$\cos \pi = -1$
$\cos 2\pi = 1$

Let $u_n = \frac{1}{n^2+1}$, $u_{n+1} = \frac{1}{(n+1)^2+1}$

$(n^2+1) < (n+1)^2+1$

$\Rightarrow \frac{1}{n^2+1} > \frac{1}{(n+1)^2+1} \Rightarrow u_n > u_{n+1}$ for all $\underline{n}$

monotonically decreasing

(ii) $\lim_{n\to\infty} u_n = \lim_{n\to\infty} \frac{1}{n^2+1} = 0$

By Leibnitz Test, convergent.

Examining the absolute / Conditional Convergance of the alternating series.

① Test the absolute convergance of $1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$

sol. $\sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^{2n-2}}{(2n-2)!}$

$x^{2n-2} = x^0 = \frac{1}{0!}$
$0, 2, 4, 6 \ldots$
$a + (n-1)d$
$0 + (n-1)2$
$2n-2.$

Let $u_n = \frac{x^{2n-2}}{(2n-2)!}$

$\sum_{n=1}^{\infty} \left| (-1)^{n-1} \frac{x^{2n-2}}{(2n-2)!} \right|$ cgs or not → to check the absolutely cgsance.

Ratio Test: (become like the termed series)

$|u_n| = \left| \frac{x^{2n-2}}{(2n-2)!} \right|$, $|u_{n+1}| = \left| \frac{x^{2n}}{(2n)!} \right|$

$$\frac{|u_{n+1}|}{|u_n|} \Rightarrow \left| \frac{u_{n+1}}{u_n} \right| = \left| \frac{x^{2n}}{(2n)!} \times \frac{(2n-2)!}{x^{2n-2}} \right|$$

$$= \left| \frac{x^2}{(2n)(2n-1)\cancel{(2n-2)!}} \times \cancel{(2n-2)!} \right|$$

$$= \left| \frac{x^2}{(2n)(2n-1)} \right|$$

$$\lim_{n\to\infty} \left| \frac{u_{n+1}}{u_n} \right| = \lim_{n\to\infty} \frac{|x^2|}{(2n)(2n-1)}$$

(i) $x \neq 0 \Rightarrow \lim_{n\to\infty} \left| \frac{u_{n+1}}{u_n} \right| = 0 < 1$, Ratio Test converges.

(ii) $x = 0 \Rightarrow \lim_{n\to\infty} \left| \frac{u_{n+1}}{u_n} \right| = 0$, converges.

∴ $\sum_{n=1}^{\infty} \left| (-1)^{n-1} \frac{x^{2n-2}}{(2n-2)!} \right|$ cgs ⇒ absolutely converge.

By the Result, $\sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^{2n-2}}{(2n-2)!}$ also cgs. for all $\underline{\underline{x}}$.

② Test the absolute convergance of $\sum_{n=0}^{\infty} (-1)^n (n+1) x^n$, $x < \frac{1}{2}$

Sol: let $u_n = (n+1)x^n$, $u_{n+1} = (n+2)x^{n+1}$

$\underline{\underline{n \to n+1}}$

To check: $\sum_{n=0}^{\infty} |(-1)^n (n+1) x^n|$ cgs/not.

$$\left| \frac{u_{n+1}}{u_n} \right| = \left| \frac{(n+2)x^{\cancel{n+1}}}{(n+1)\cancel{x^n}} \right| = \left| \frac{\cancel{n}(1+2/n)x}{\cancel{n}(1+1/n)} \right|$$

$$\lim_{n\to\infty} \left| \frac{u_{n+1}}{u_n} \right| = \lim_{n\to\infty} \left| \frac{1+2/n}{1+1/n} \right| \cdot |x|$$

$$= 1 \cdot |x| < \frac{1}{2} < 1.$$

By Ratio Test, $\sum_{n=0}^{\infty} |u_n|$ cgs. ⇒ $\sum_{n=0}^{\infty} u_n$ cgs $\underline{\underline{\text{absolutes}}}$

$\sum_{n=1}^{\infty} u_n$ cgs absolutely ⇒ $\sum_{n=0}^{\infty} (-1)^n (n+1) x^n$, cgs

③ Test the absolute / conditional convergence of

$$\frac{1}{\sqrt[5]{2}} - \frac{1}{\sqrt[5]{3}} + \frac{1}{\sqrt[5]{4}} - \frac{1}{\sqrt[5]{5}} + \cdots + (-1)^n \frac{1}{\sqrt[5]{n}} - \cdots$$

sol: Let $|u_n| = \left|\frac{(-1)^n}{\sqrt[5]{n}}\right| = \frac{1}{\sqrt[5]{n}}$

Rough: $\sum \frac{1}{n^p}$ cgs if $p > 1$, dgs if $p \le 1$.

To check: $\sum |u_n|$ cgs / not.

$\sum_{n=1}^{\infty} \frac{1}{n^{1/5}}$, Here $p = \frac{1}{5} < 1$

$\therefore$ By the result, $\sum_{n=1}^{\infty} \frac{1}{\sqrt[5]{n}}$ dgs.

i.e., $\sum_{n=1}^{\infty} |u_n|$ does not converge.

To check: $\sum_{n=1}^{\infty} (-1)^n \frac{1}{\sqrt[5]{n}}$ cgs / oscillates by Leibnitz test.

(i) $\sqrt[5]{2} < \sqrt[5]{3} < \sqrt[5]{4} < \cdots$

$\Rightarrow \frac{1}{\sqrt[5]{2}} > \frac{1}{\sqrt[5]{3}} > \frac{1}{\sqrt[5]{4}} > \cdots$

monotonically decreases

(ii) $\lim_{n \to \infty} u_n = \lim_{n \to \infty} \frac{1}{(n)^{1/5}} = 0$

By Leibnitz test, $\sum_{n=1}^{\infty} (-1)^n \frac{1}{\sqrt[5]{n}}$ cgs.

Conclusion: $\sum_{n=1}^{\infty} (-1)^n \frac{1}{\sqrt[5]{n}}$ cgs but $\sum_{n=1}^{\infty} \left|(-1)^n \frac{1}{\sqrt[5]{n}}\right|$ dgs.

Hence, the given series conditionally converges.

05/02/2021 - ECE

Recall: Positive termed series

(1) Integral
(2) Comparison
(3) Ratio
(4) Raabe's
(5) Root
(6) Logarithmic

Logarithmic test

Let $\sum_{n=1}^{\infty} u_n$ be the positive termed series, then

$$L = \lim_{n\to\infty} n \log \frac{u_n}{u_{n+1}}$$

(i) If $L>1$, cgs,
(ii) If $L<1$, dgs.

Problem:
Test the convergence of $1+\frac{2x}{2!}+\frac{3^2x^2}{3!}+\frac{4^3x^3}{4!}+\cdots$

Sol: Here, $u_n = \frac{n^{n-1}x^{n-1}}{n!}$

$x^0, x^1, x^2, \ldots$
$1, 2, 3, \ldots$
$0+(n-1)1$
$(n-1)$
$1+(n-1)1$
$\cancel{1}+n-\cancel{1}=n$

$$u_{n+1} = \frac{(n+1)^n x^n}{(n+1)!}$$

$$\frac{u_{n+1}}{u_n} = \frac{(n+1)^n \cancel{x^n}}{(n+1)\cancel{!}} \times \frac{\cancel{n!}}{n^{n-1}\cancel{x^{n-1}}}$$

$$= \frac{(n+1)^n}{n^{n-1}} \frac{1}{(n+1)}(x)$$

$$= \frac{(n+1)^{n-1}}{n^{n-1}} x$$

$$\frac{u_{n+1}}{u_n} = \left(1+\frac{1}{n}\right)^{n-1} x$$

$\frac{x^n}{x^{n-1}} = x^{n-n+1} = x$

$\frac{n!}{(n+1)!} = \frac{\cancel{n!}}{(n+1)\cancel{n!}}$

$\boxed{\frac{(n+1)^n}{n^{n-1}}}$

$$\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = \lim_{n\to\infty} \left(1+\frac{1}{n}\right)^{n-1} x$$

$$\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = e x$$

$= \frac{(n+1)^n}{n^n} \cdot \frac{1}{(n)^{-1}}$

$\boxed{\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e}$

Illy $\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^{n-1} = e$

$\lim_{n\to\infty}\left(1+\frac{k}{n}\right)^n = e^k$.

Case (i) If $ex>1 \Rightarrow x>1/e$
$\sum_{n=1}^{\infty} u_n$ dgs by Ratio Test.

Case (ii) If $ex<1 \Rightarrow x<1/e$
$\sum_{n=1}^{\infty} u_n$ cgs by Ratio test.

Case (iii) If $ex=1 \Rightarrow x=1/e$

$\lim\limits_{n\to\infty} \frac{u_{n+1}}{u_n} = ex = 1 \Rightarrow$ Ratio Test fails, since $\underline{L = 1}$.

$$\frac{u_n}{u_{n+1}} = \frac{1}{\left(1+\frac{1}{n}\right)^{n-1} x} = \frac{\left(1+\frac{1}{n}\right)}{\left(1+\frac{1}{n}\right)^{n}} \cdot \frac{1}{x}$$

(Here $x = \frac{1}{e}$).

In Case (iii), we assumed $x = \frac{1}{e}$

$$\frac{u_n}{u_{n+1}} = \frac{\left(1+\frac{1}{n}\right) e}{\left(1+\frac{1}{n}\right)^{n}}.$$

Trying to apply Logarithmic test,

$$n \log \frac{u_n}{u_{n+1}} = n\left[\log\left(\frac{\left(1+\frac{1}{n}\right) e}{\left(1+\frac{1}{n}\right)^{n}}\right)\right]$$

$$= n\left\{\log\left(e\left(1+\frac{1}{n}\right)\right) - \log\left(1+\frac{1}{n}\right)^{n}\right\}$$

$$= n\left\{\log e + \log\left(1+\frac{1}{n}\right) - n\log\left(1+\frac{1}{n}\right)\right\}$$

$$= n\left\{\log e + \log\left(1+\frac{1}{n}\right)(1-n)\right\}$$

$x = \frac{1}{n}$.

$$= n\left\{1 + (1-n)\left\{\frac{1}{n} - \frac{1}{2n^2} + \frac{1}{3n^3} \cdots\right\}\right\}$$

$x - \frac{x^2}{2} + \frac{x^3}{3}$

$$= n\left\{\cancel{1} + \left(\frac{1}{n} - \frac{1}{2n^2} + \cdots\right) + \left(\cancel{-1} + \frac{1}{2n} - \frac{1}{3n^2} + \cdots\right)\right\}$$

$\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots$

$$= n\left\{\left(\frac{1}{n} + \frac{1}{2n}\right) + \left(-\frac{1}{2n^2} - \frac{1}{3n^2}\right) + (\quad) + \right\}.$$

$-\frac{1}{2} - \frac{1}{3}$

$-\frac{5}{6}$

$$= \left\{\left(1 + \frac{1}{2}\right) + \left(-\frac{1}{2n} - \frac{1}{3n}\right) + \cdots\right\}$$

$$\lim_{n\to\infty} n \log\left(\frac{u_n}{u_{n+1}}\right) = \lim_{n\to\infty}\left[\frac{3}{2} + \left(-\frac{5}{6n}\right) + \left(\frac{\quad}{n}\right) \cdots\right]$$

$$= \frac{3}{2} + 0 + 0 + \cdots + 0 = \frac{3}{2} > 1$$

By Logarithmic Test, $\lim\limits_{n\to\infty} n \log\left(\frac{u_n}{u_{n+1}}\right) = L > 1$, Cgs. if $\underline{x = \frac{1}{e}}$.

Conclusion:

If $x \leq \frac{1}{e}$, given series converges.

If $x > \frac{1}{e}$, given series diverges.

Problem: Discuss the convergence of $\sum_{n=1}^{\infty} \frac{1}{n^p}$.

Sol: Let $u_n = \frac{1}{n^p}$, $u_{n+1} = \frac{1}{(n+1)^p}$

$$\frac{u_n}{u_{n+1}} = \frac{(n+1)^p}{n^p} = \frac{\cancel{n^p}\left(1+\frac{1}{n}\right)^p}{\cancel{n^p}}$$

$$\frac{u_n}{u_{n+1}} = \left(1+\frac{1}{n}\right)^p.$$

$$\log\left(\frac{u_n}{u_{n+1}}\right) = \log\left(1+\frac{1}{n}\right)^p = p\log\left(1+\frac{1}{n}\right)$$

$$\log\left(\frac{u_n}{u_{n+1}}\right) = p\left\{\left(\frac{1}{n}\right) - \frac{\left(\frac{1}{n}\right)^2}{2} + \frac{\left(\frac{1}{n}\right)^3}{3} - \dots\right\}$$

$$= p\left\{\frac{1}{n} - \frac{1}{2n^2} + \frac{1}{3n^3} - \dots\right\}$$

case (i): If $p > 1$,

$$L = \lim_{n\to\infty} n\log\left(\frac{u_n}{u_{n+1}}\right)$$

$$= \lim_{n\to\infty} n\left[p\left(\frac{1}{n} - \frac{1}{2n^2} + \frac{1}{3n^3} - \dots\right)\right]$$

$$= p \cdot \lim_{n\to\infty}\left(1 - \frac{1}{2n} + \frac{1}{3n^2} - \dots\right)$$

$$= p \cdot (1) = p > 1.$$

By Logarithmic test, $L > 1$, $\sum_{n=1}^{\infty} \frac{1}{n^p}$, $p > 1$, cgs.

case (ii) $p < 1$.

$$L = \lim_{n\to\infty} n\log\left(\frac{u_n}{u_{n+1}}\right) = p < 1.$$

$$\Rightarrow \sum_{n=1}^{\infty} \frac{1}{n^p} \text{ dgs if } p < 1.$$

case (iii) $p = 1$, the test fails. go for comparison / Integral tests. proved dgs. by the tests.

conclusion: $\sum_{n=1}^{\infty} \frac{1}{n^p}$ $\rightarrow$ cgs if $p > 1$

$\rightarrow$ dgs if $p \leq 1$.