Problems on vibrating string with non-zero initial velocity:

① A tightly stretched string with fixed end points $x=0$ and $x=l$ is initially at rest in its equilibrium position. If it is set vibrating giving each point a velocity $3x(l-x)$, find the displacement.

Soln:-

The displacement $y(x,t)$ is governed by

$$\frac{\partial^2 y}{\partial t^2} = a^2 \frac{\partial^2 y}{\partial x^2}$$

From the given problem we get the following boundary and initial conditions,

(i) $y(0,t)=0$ for $t \geq 0$

(ii) $y(l,t)=0$ for $t \geq 0$

(iii) $y(x,0)=0$ for $0 \leq x \leq l$

(iv) $\left.\frac{\partial y}{\partial t}\right|_{t=0} = 3x(l-x)$ for $0 \leq x \leq l$.

Now, the suitable solution which satisfies our boundary conditions is given by

$$y(x,t) = (A\cos\lambda x + B\sin\lambda x)(C\cos\lambda at + D\sin\lambda at) \longrightarrow ①$$

Applying boundary condition (i) in eqn, we get

$$y(0,t) = (A(1))(C\cos\lambda at + D\sin\lambda at) = 0.$$

either $A=0$ (or) $C\cos\lambda at + D\sin\lambda at = 0$.

here, $C\cos\lambda at + D\sin\lambda at \neq 0$ ($\because$ it is defined $\forall t$)

$\therefore$, $\boxed{A=0}$

Substitute $A=0$ in eqn ①, we get

$$y(x,t) = B\sin\lambda x\,(C\cos\lambda at + D\sin\lambda at) \longrightarrow ②$$

Applying boundary condition (ii) in eqn ②, we get

$$y(l,t) = B\sin\lambda l\,(C\cos\lambda at + D\sin\lambda at) = 0.$$

here, $C\cos\lambda at + D\sin\lambda at \neq 0$ ($\because$ it is defined $\forall t$)

$\therefore$ either $B=0$ or $\sin\lambda l = 0$.

Suppose, we take $B=0$ and already we have $A=0$

then we get a trivial solution.

$\therefore B \neq 0$.

The only possibility is $\sin\lambda l = 0$.

$\sin\lambda l = \sin n\pi$, where n is any integer.

$$\lambda l = n\pi$$

$$\boxed{\lambda = \frac{n\pi}{l}}$$

Sub. $\lambda = \frac{n\pi}{l}$ in eqn ②, we get

$$y(x,t) = B\sin\frac{n\pi x}{l}\left(C\cos\frac{n\pi at}{l} + D\sin\frac{n\pi at}{l}\right) \longrightarrow ③$$

Applying boundary condition (iii) in eqn ③, we get.

$$y(x,0) = B\sin\frac{n\pi x}{l}\,(C(1) + D(0)) = 0.$$

$$\Rightarrow B\sin\frac{n\pi x}{l}\,C = 0$$

here, $B \neq 0$ ($\because$ If $B=0$, we already explained)

$\sin\frac{n\pi x}{l} \neq 0$ ($\because$ it is defined for all x)

$\therefore \boxed{C=0}$

Substitute $C=0$ in equation ③, we get

$$y(x,t) = B \sin\frac{n\pi x}{l} \, D \sin\frac{n\pi at}{l}.$$

$$= B_n \sin\frac{n\pi x}{l} \sin\frac{n\pi at}{l}$$ where $BC = B_n$, where n is any integer & B_n is any Constant.

The most general solution is

$$y(x,t) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{l} \sin\frac{n\pi at}{l} \longrightarrow ④$$

Before applying b.c (iv), differentiate ④ partially w.r to 't', we get

$$\frac{\partial y}{\partial t}(x,t) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{l} \cdot \cos\frac{n\pi at}{l} \cdot \frac{n\pi a}{l}.$$

Now, applying b.c (iii), we get

$$\left.\frac{\partial y}{\partial t}\right|_{t=0} = \sum_{n=1}^{\infty} B_n \cdot \sin\frac{n\pi x}{l} \cdot (1) \, \frac{n\pi a}{l} = 3x(l-x) \longrightarrow ⑤$$

To find B_n, expand $3x(l-x)$ in a half-range Fourier sine series in the interval $(0,l)$.

$$3x(l-x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l} \text{ where} \longrightarrow ⑥$$

$$b_n = \frac{2}{l}\int_0^l f(x) \sin\frac{n\pi x}{l} \, dx$$

From ⑤ & ⑥, we get

$$B_n \cdot \frac{n\pi a}{l} = b_n.$$

$\therefore$

$$\therefore \quad B_n \cdot \frac{n\pi a}{l} = \frac{2}{l}\int_0^l f(x) \sin\frac{n\pi x}{l}\,dx$$

$$= \frac{2}{l}\int_0^l 3x(l-x) \sin\frac{n\pi x}{l}\,dx$$

$$= \frac{6}{l}\int_0^l (lx-x^2) \sin\frac{n\pi x}{l}\,dx$$

$$= \frac{6}{l}\left[(lx-x^2)\left(\frac{-\cos\frac{n\pi x}{l}}{\frac{n\pi}{l}}\right) - (l-2x)\left(\frac{-\sin\frac{n\pi x}{l}}{\left(\frac{n\pi}{l}\right)^2}\right) + (-2)\left(\frac{\cos\frac{n\pi x}{l}}{\left(\frac{n\pi}{l}\right)^3}\right)\right]_0^l$$

$$= \frac{6}{l}\left[(-2)\frac{l^3}{n^3\pi^3}(-1)^n - \left((-2)\left(\frac{l^3}{n^3\pi^3}\right)(1)\right)\right]$$

$$\Rightarrow \frac{6}{l} \times \frac{-2l^3}{n^3\pi^3}\left((-1)^n - 1\right) = \frac{-12l^3}{l n^3\pi^3}\left((-1)^n - 1\right)$$

$$\Rightarrow \frac{12l^2}{n^3\pi^3}\left(1-(-1)^n\right).$$

$$B_n = \frac{l}{n\pi a}\cdot\frac{12l^2}{n^3\pi^3}\left(1-(-1)^n\right).$$

$$B_n = \frac{12l^3}{n^4\pi^4 a}\left(1-(-1)^n\right)$$

Substitute the value of B_n in eqn (4), we get

$$y(x,t) = \sum_{n=1}^{\infty} \frac{12l^3}{an^4\pi^4}\left(1-(-1)^n\right) \sin\frac{n\pi x}{l} \sin\frac{n\pi at}{l}.$$

2) If a string of length l is initially at rest in equilibrium position and each point of it is given the velocity $\left.\frac{\partial y}{\partial t}\right|_{t=0} = v_0 \sin^3\frac{\pi x}{l}$, $0<x<l$, determine the transverse displacement $y(x,t)$.

Sol:- The displacement $y(x,t)$ is governed by

$$\frac{\partial^2 y}{\partial t^2} = a^2\frac{\partial^2 y}{\partial x^2}$$

From the given problem, we get the following boundary and initial conditions,

(i) $y(0,t)=0$ for $t \geq 0$

(ii) $y(l,t)=0$ for $t \geq 0$

(iii) $y(x,0)=0$ for $0<x<l$

(iv) $\left.\frac{\partial y}{\partial t}\right|_{t=0} = v_0 \sin^3\frac{\pi x}{l}$, $0<x<l$.

Now, the suitable solution which satisfies our boundary conditions is given by

$$y(x,t) = (A\cos\lambda x + B\sin\lambda x)(C\cos\lambda at + D\sin\lambda at) \longrightarrow (1)$$

Applying b.c (i) in eqn (1), we get

$$y(x,t) = A(C\cos\lambda at + D\sin\lambda at) = 0$$

either $A=0$ (or) $C\cos\lambda at + D\sin\lambda at = 0$

here, $C\cos\lambda at + D\sin\lambda at \neq 0$ ($\because$ it is defined $\forall t$)

$\therefore \boxed{A=0}$

Sub $A=0$ in eqn ①, we get

$y(x,t) = B\sin\lambda x\,(C\cos\lambda at + D\sin\lambda at) \longrightarrow$ ②

Applying b.c (ii) in eqn ②, we get

$y(l,t) = B\sin\lambda l\,(C\cos\lambda at + D\sin\lambda at) = 0.$

here, $C\cos\lambda at + D\sin\lambda at \neq 0$ ($\because$ it is defined $\forall t$)

$\therefore$ either $B=0$ or $\sin\lambda l = 0$.

Suppose, we take $B=0$ and already we have $A=0$ then we get a trivial solution.

$\therefore B \neq 0$.

The only possibility is $\sin\lambda l = 0$

$\sin\lambda l = \sin n\pi$, where n is any integer.

$\lambda l = n\pi$

$\boxed{\lambda = \frac{n\pi}{l}}$

Sub. $\lambda = \frac{n\pi}{l}$ in eqn ②, we get

$y(x,t) = B\sin\frac{n\pi x}{l}\left(C\cos\frac{n\pi at}{l} + D\sin\frac{n\pi at}{l}\right) \longrightarrow$ ③

Applying b.c (iii) in eqn ③, we get

$y(x,0) = B\sin\frac{n\pi x}{l}\,(C(1) + D(0)) = 0$

$B\sin\frac{n\pi x}{l}\,C = 0.$

here $B \neq 0$ ($\because$ If $B=0$, we already explained)

$\sin\frac{n\pi x}{l} \neq 0$ ($\because$ it is defined for all x)

$\therefore \boxed{C=0}$

Substitute $C=0$ in eqn. (3), we get

$$y(x,t) = B\sin\frac{n\pi x}{l}\, D\sin\frac{n\pi a t}{l}$$

$$= BD\sin\frac{n\pi x}{l}\sin\frac{n\pi a t}{l}$$

$$= B_n\sin\frac{n\pi x}{l}\sin\frac{n\pi a t}{l} \quad \text{where } BD = B_n,$$

n is any integer
B_n is any constant.

$\therefore$ The most general solution is

$$y(x,t) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{l}\sin\frac{n\pi a t}{l} \longrightarrow (4)$$

Before applying b.c (iv) in eqn (4), differenbiate (4) partially w.r to 't', we get.

$$\frac{\partial y}{\partial t}(x,t) = \sum_{n=1}^{\infty} B_n \cdot \sin\frac{n\pi x}{l}\cos\frac{n\pi a t}{l}\cdot\frac{n\pi a}{l}$$

Now, applying bc (iv), we get

$$\left.\frac{\partial y}{\partial t}\right|_{t=0} = \sum_{n=1}^{\infty} B_n\sin\frac{n\pi x}{l}(1)\cdot\frac{n\pi a}{l} = v_0\sin^3\frac{\pi x}{l}.$$

[we know that $\sin 3A = 3\sin A - 4\sin^3 A$

$\Rightarrow \sin^3 A = \frac{3}{4}\sin A - \frac{1}{4}\sin 3A$]

$$\sum_{n=1}^{\infty} B_n\sin\frac{n\pi x}{l}\cdot\frac{n\pi a}{l} = v_0\left[\frac{3}{4}\sin\frac{\pi x}{l} - \frac{1}{4}\sin\frac{3\pi x}{l}\right]$$

$$\Rightarrow B_1\sin\frac{\pi x}{l}\cdot\frac{\pi a}{l} + B_2\sin\frac{2\pi x}{l}\cdot\frac{2\pi a}{l} + B_3\sin\frac{3\pi x}{l}\cdot\frac{3\pi a}{l}$$

$$+ \cdots = \frac{3v_0}{4}\sin\frac{\pi x}{l} - \frac{v_0}{4}\sin\frac{3\pi x}{l}$$

equating the co-efficient on both sides.

$$B_1 \cdot \frac{\pi a}{l} = \frac{3v_0}{4} \quad ; \quad B_3 \cdot \frac{3\pi a}{l} = \frac{-v_0}{4},$$

$$B_n = 0 \,, \quad n \neq 1,3.$$

$$B_1 = \frac{3v_0 l}{4\pi a} \quad , \quad B_3 = \frac{-v_0}{4} \times \frac{l}{3\pi a}.$$

$$B_3 = \frac{-v_0 l}{12\pi a}$$

Sub these values in (4), we get

$$y(x,t) = \frac{3v_0 l}{4\pi a} \sin\frac{\pi x}{l} \sin\frac{\pi a t}{l} - \frac{v_0 l}{12\pi a} \sin\frac{3\pi x}{l} \sin\frac{3\pi a t}{l}.$$

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(3) A string is stretched between two fixed points at a distance $2l$ apart and the points of the string are given initial velocities v where $v = \begin{cases} \frac{cx}{l} & \text{in } 0<x<l \\ \frac{c}{l}(2l-x) & \text{in } l<x<2l \end{cases}$

x being the distance from an end point. Find the displacement of the string at any subsequent time.

solution:

The displacement $y(x,t)$ is governed by

$$\frac{\partial^2 y}{\partial t^2} = a^2 \frac{\partial^2 y}{\partial x^2}$$

From the given problem, we get the following boundary and initial conditions, equating the co-efficient on both sides.

$$B_1 \cdot \frac{\pi a}{l} = \frac{3v_0}{4} \quad ; \quad B_3 \cdot \frac{3\pi a}{l} = \frac{-v_0}{4},$$

$$B_n = 0, \quad n \neq 1,3.$$

$$B_1 = \frac{3v_0 l}{4\pi a}, \quad B_3 = \frac{-v_0}{4} \times \frac{l}{3\pi a}$$

$$B_3 = \frac{-v_0 l}{12\pi a}$$

sub these values in (4), we get

$$y(x,t) = \frac{3v_0 l}{4\pi a} \sin\frac{\pi x}{l} \sin\frac{\pi a t}{l} - \frac{v_0 l}{12\pi a} \sin\frac{3\pi x}{l} \sin\frac{3\pi a t}{l}.$$

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(3) A string is stretched between two fixed points at a distance $2l$ apart and the points of the string are given initial velocities v where $v = \begin{cases} \frac{cx}{l} & \text{in } 0<x<l \\ \frac{c}{l}(2l-x) & \text{in } l<x<2l \end{cases}$

x being the distance from an end point. Find the displacement of the string at any subsequent time.

solution:

The displacement $y(x,t)$ is governed by

$$\frac{\partial^2 y}{\partial t^2} = a^2 \frac{\partial^2 y}{\partial x^2}$$

From the given problem, we get the following boundary and initial conditions,

(i) $y(0,t) = 0$ for $t \geq 0$

(ii) $y(2l, t) = 0$ for $t \geq 0$

(iii) $y(x, 0) = 0$ for $0 \leq x \leq 2l$

(iv) $$\left.\frac{\partial y}{\partial t}\right|_{t=0} = \begin{cases} \frac{cx}{l} & \text{in } 0 < x < l \\ \frac{c}{l}(2l - x) & \text{in } l < x < 2l. \end{cases}$$

Now, the suitable solution which satisfies our boundary conditions is given by

$$y(x,t) = (A \cos \lambda x + B \sin \lambda x)(C \cos \lambda at + D \sin \lambda at) \longrightarrow (1)$$

Applying b.c (i) in eqn (1), we get

$$y(0,t) = A(C \cos \lambda at + D \sin \lambda at) = 0$$

either $A = 0$ (or) $C \cos \lambda at + D \sin \lambda at = 0$.

here, $C \cos \lambda at + D \sin \lambda at \neq 0$ ($\because$ it is defined $\forall t$)

$\therefore \boxed{A = 0}$

Sub $A = 0$ in eqn (1), we get

$$y(x,t) = B \sin \lambda x (C \cos \lambda at + D \sin \lambda at) \longrightarrow (2)$$

Applying b.c (ii) in eqn (2), we get

$$y(2l, t) = B \sin 2l\lambda (C \cos \lambda at + D \sin \lambda at) = 0.$$

$C \cos \lambda at + D \sin \lambda at \neq 0$ ($\because$ it is defined $\forall t$)

$\therefore$ either $B = 0$ (or) $\sin 2l\lambda = 0$.

but $B \neq 0$ (suppose we take $B = 0$ and already we have $A = 0$, then we get a trivial solution)

The only possibility is $\sin 2l\lambda = 0$

$\sin 2\lambda l = \sin n\pi$ where n is any integer

$$2\lambda l = n\pi \Rightarrow \boxed{\lambda = \frac{n\pi}{2l}}$$

Sub $\boxed{\lambda = \frac{n\pi}{2l}}$ in eqn ②, we get

$$y(x,t) = B\sin\frac{n\pi x}{2l}\left(C\cos\frac{n\pi at}{2l} + D\sin\frac{n\pi at}{2l}\right) \longrightarrow ③.$$

Applying b.c (iii) in eqn ③, we get

$$y(x,0) = B\sin\frac{n\pi x}{2l}\left(C(1) + D(0)\right) = 0.$$

$$B\sin\frac{n\pi x}{2l}\, C = 0.$$

here, $B \neq 0$ ($\because$ If $B=0$, we already explained)

$\sin\frac{n\pi x}{2l} \neq 0$ ($\because$ it is defined for all x)

$\therefore \boxed{C=0}$

Sub $C=0$ in eqn ③, we get

$$y(x,t) = B\sin\frac{n\pi x}{2l}\, D\sin\frac{n\pi at}{2l}$$

$$= B_n \sin\frac{n\pi x}{2l}\sin\frac{n\pi at}{2l},$$ where $BD = B_n$, n is any integer, B_n is any constant.

The most general solution is

$$y(x,t) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{2l}\sin\frac{n\pi at}{2l} \longrightarrow ④$$

Before applying b.c (iv), differentiate ④ partially w.r to 't', we get

$$\frac{\partial y}{\partial t}(x,t) = \sum_{n=1}^{\infty} B_n \cdot \sin\frac{n\pi x}{2l} \cos\frac{n\pi at}{2l} \cdot \frac{n\pi a}{2l}$$

Now, applying b.c (iv), we get

$$\left.\frac{\partial y}{\partial t}\right|_{t=0} = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{2l}(1)\frac{n\pi a}{2l} = \begin{cases} \frac{cx}{l} & \text{in } 0<x<l \\ \frac{c}{l}(2l-x) & \text{in } l<x<2l \end{cases} \longrightarrow (5)$$

To find B_n, expand $\begin{cases} \frac{cx}{l} & \text{in } 0<x<l \\ \frac{c}{l}(2l-x) & \text{in } l<x<2l \end{cases}$ in a half-range sine series in the interval $(0,2l)$.

$$\begin{cases} \frac{cx}{l} & \text{in } 0<x<l \\ \frac{c}{l}(2l-x) & \text{in } l<x<2l \end{cases} = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{2l} \text{ where} \longrightarrow (6)$$

$$b_n = \frac{2}{(2l)}\int_0^{2l} f(x)\sin\frac{n\pi x}{2l}\,dx.$$

From (5) & (6), we get

$$\boxed{B_n \cdot \frac{n\pi a}{2l} = b_n}$$

$$\therefore B_n \cdot \frac{n\pi a}{2l} = \frac{2}{2l}\int_0^{2l} f(x)\sin\frac{n\pi x}{2l}\,dx$$

$$= \frac{2}{2l}\left[\int_0^{l}\frac{cx}{l}\sin\frac{n\pi x}{2l}\,dx + \int_l^{2l}\frac{c}{l}(2l-x)\sin\frac{n\pi x}{2l}\,dx\right]$$

$$= \frac{c}{l^2}\left[x\left(\frac{-\cos\frac{n\pi x}{2l}}{\frac{n\pi}{2l}}\right) - (1)\left(\frac{-\sin\frac{n\pi x}{2l}}{\left(\frac{n\pi}{2l}\right)^2}\right)\right]_0^l +$$

$$(2l-x)\left(\frac{-\cos\frac{n\pi x}{2l}}{\frac{n\pi}{2l}}\right)-(-1)\left(\frac{-\sin\frac{n\pi x}{2l}}{\left(\frac{n\pi}{2l}\right)^2}\right)\Bigg]_{l}^{2l}$$

$$=\frac{c}{l^2}\left[l\left(\frac{2l}{n\pi}\right)\left(-\cos\frac{n\pi(l)}{2l}\right)+\frac{4l^2}{n^2\pi^2}\sin\frac{n\pi(l)}{2l}-0\right]+$$

$$\left[0-\left((l)\left(\frac{2l}{n\pi}\right)\left(-\cos\frac{n\pi(l)}{2l}\right)-\frac{4l^2}{n^2\pi^2}\sin\frac{n\pi(l)}{2l}\right)\right]$$

$$=\frac{c}{l^2}\left[\cancel{-\frac{2l^2}{n\pi}\cos\frac{n\pi}{2}}+\frac{4l^2}{n^2\pi^2}\sin\frac{n\pi}{2}+\cancel{\frac{2l^2}{n\pi}\cos\frac{n\pi}{2}}+\frac{4l^2}{n^2\pi^2}\sin\frac{n\pi}{2}\right]$$

$$=\frac{c}{l^2}\left[\frac{8l^2}{n^2\pi^2}\sin\frac{n\pi}{2}\right]\Rightarrow\frac{8c}{n^2\pi^2}\sin\frac{n\pi}{2}.$$

$$\therefore\ B_n\cdot\frac{n\pi a}{2l}=\frac{8c}{n^2\pi^2}\sin\frac{n\pi}{2}.$$

$$B_n=\frac{16cl}{n^3\pi^3a}\sin\frac{n\pi}{2}.$$

Sub. the value of B_n in eqn ④, we get

$$y(x,t)=\sum_{n=1}^{\infty}\frac{16lc}{n^3\pi^3a}\sin\frac{n\pi}{2}\sin\frac{n\pi x}{2l}\sin\frac{n\pi at}{2l}.$$

Exercise problems

① If a string of length l is initially at rest in equilibrium position and each of its points is given the velocity $\left[\frac{\partial y}{\partial t}\right]_{t=0} = v_0 \sin\frac{\pi x}{l}$, $0<x<l$, determine the displacement $y(x,t)$.

② A tightly stretched string with fixed end points $x=0$ and $x=l$ is initially at rest in equilibrium position. If it is set vibrating giving each point a velocity $\lambda x(l-x)$, show that

$$y(x,t) = \frac{8\lambda l^3}{a\pi^4} \sum_{n=1}^{\infty} \frac{1}{(2n-1)^4} \sin\frac{(2n-1)\pi x}{l} \sin\frac{(2n-1)\pi a t}{l}$$

3) If a string of length l is initially at rest in its equilibrium position and each of its points is given a velocity v such that $v = \begin{cases} cx & \text{for } 0<x\le l/2 \\ c(l-x) & \text{for } l/2<x<l, \end{cases}$

show that the displacement $y(x,t)$ at any time t is given by

$$y(x,t) = \frac{4l^2 c}{\pi^3 a}\left[\sin\frac{\pi x}{l}\sin\frac{\pi a t}{l} - \frac{1}{3^3}\sin\frac{3\pi x}{l}\sin\frac{3\pi a t}{l} + \dots\right]$$