Problems on Vibrating String with zero initial velocity:

(1) A tightly stretched string with fixed end points $x=0$ and $x=l$ is initially in a position given by $y(x,0) = y_0 \sin^3\left(\frac{\pi x}{l}\right)$. If it is released from rest from this position, find the displacement y at any time and at any distance from the end $x=0$.

Solution:- The displacement y of the particle at a distance x from the end $x=0$ at time t is governed by

$$\frac{\partial^2 y}{\partial t^2} = a^2 \frac{\partial^2 y}{\partial x^2}.$$

From the given problem, we get the following boundary conditions.

(i) $y(0,t) = 0$ for all $t \geq 0$

(ii) $y(l,t) = 0$ for all $t \geq 0$

(iii) $\left.\frac{\partial y}{\partial t}\right|_{t=0} = 0$ for $0 \leq x \leq l$

(iv) $y(x,0) = y_0 \sin^3 \frac{\pi x}{l}$, for $0 \leq x \leq l$.

Now the correct solution which satisfies our boundary conditions is given by

$$y(x,t) = (A\cos\lambda x + B\sin\lambda x)(C\cos\lambda at + D\sin\lambda at) \longrightarrow (1)$$

Applying the boundary condition (i) in (1), we get

$$y(0,t) = A(C\cos\lambda at + D\sin\lambda at) = 0.$$

either $A=0$ or $C\cos\lambda at + D\sin\lambda at = 0$.

$\ldots D\sin\lambda at \neq 0$ ($\because$ it is defined $\forall t$)

$\therefore \boxed{A = 0}$

Sub $A=0$ in eqn ①, we get

$$y(x,t) = B \sin\lambda x \,(C\cos\lambda at + D\sin\lambda at) \longrightarrow ②$$

Applying boundary condition (ii) in eqn ②, we get

$$y(l,t) = B\sin\lambda l\,(C\cos\lambda at + D\sin\lambda at) = 0.$$

here, $C\cos\lambda at + D\sin\lambda at \neq 0$ ($\because$ it is defined for all t)

$\therefore$ either $B = 0$ (or) $\sin\lambda l = 0$.

but, $B \neq 0$ (suppose if we take $B=0$ and already we have $A=0$, then we get a trivial solution)

$\therefore$ we take $\sin\lambda l = 0$

$\sin\lambda l = \sin n\pi$ where n is any integer

$\lambda l = n\pi$

$\therefore \boxed{\lambda = \frac{n\pi}{l}}$

Substitute $\lambda = \frac{n\pi}{l}$ in eqn ②, we get

$$y(x,t) = B\sin\frac{n\pi x}{l}\left(C\cos\frac{n\pi at}{l} + D\sin\frac{n\pi at}{l}\right) \longrightarrow ③.$$

Before applying boundary condition (iii), diff ③ partially w.r.to 't',

$$\frac{\partial y}{\partial t} = B\sin\frac{n\pi x}{l}\left(C\left(-\sin\frac{n\pi at}{l}\cdot\frac{n\pi a}{l}\right) + D\left(\cos\frac{n\pi at}{l}\cdot\frac{n\pi a}{l}\right)\right)$$

Applying boundary condition (iii), we get

$$\left.\frac{\partial y}{\partial t}\right|_{t=0} = B\sin\frac{n\pi x}{l}\left(D\cdot\frac{n\pi a}{l}\right) = 0.$$

here $B \neq 0$ ($\because$ Suppose $B=0$, we have already explained)

$\sin\frac{n\pi x}{l} \neq 0$ ($\because$ it is defined for all x)

~~n~~ $\frac{n\pi a}{l} \neq 0$ ($\because$ all are constants).

$\therefore \boxed{D=0}$.

$\therefore$ Substituting $D=0$ in eqn ③, we get

$$y(x,t) = B \sin\frac{n\pi x}{l} \; C \cos\frac{n\pi at}{l}$$

$$y(x,t) = B_n \sin\frac{n\pi x}{l} \cos\frac{n\pi at}{l} \quad \text{where } BC = B_n, \quad \to (*)$$

n is any integer & B_n is any constant.

Since the partial differential equation (wave equation) is linear any linear combination of solutions (or sum of the solutions) of the form $(*)$ with $n = 1, 2, 3, \ldots$ is also a solution of the equation.

$\therefore$ The most general solution of $(*)$ can be written as

$$y(x,t) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{l} \cos\frac{n\pi at}{l}. \longrightarrow ④$$

Applying the boundary condition (iv) in eqn ④, we get

$$y(x,0) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{l} = y_0 \sin^3\frac{\pi x}{l}.$$

[we know that $\sin 3A = 3\sin A - 4\sin^3 A$.

$\sin 3A - 3\sin A = -4\sin^3 A$.

$-\sin 3A + 3\sin A = 4\sin^3 A$

$\therefore \sin^3 A = \frac{3}{4}\sin A - \frac{1}{4}\sin 3A$]

$$\sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{l} = y_0\left[\frac{3}{4}\sin\frac{\pi x}{l} - \frac{1}{4}\sin\frac{3\pi x}{l}\right]$$

$$\sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{l} = \frac{3y_0}{4}\sin\frac{\pi x}{l} - \frac{y_0}{4}\sin\frac{3\pi x}{l}.$$

ie, $B_1 \sin\frac{\pi x}{l} + B_2 \sin\frac{2\pi x}{l} + B_3 \sin\frac{3\pi x}{l} + \dots = \frac{3y_0}{4}\sin\frac{\pi x}{l} - \frac{y_0}{4}\sin\frac{3\pi x}{l}$

By equating like coefficients on either side, we get

$$B_1 = \frac{3y_0}{4}, \quad B_3 = \frac{-y_0}{4} \quad \text{and} \quad B_n = 0, \text{ for } n \neq 1, 3.$$

Substituting these values of in eqn ④, we get

$$y(x,t) = \frac{3y_0}{4}\sin\frac{\pi x}{l}\cos\frac{\pi at}{l} - \frac{y_0}{4}\sin\frac{3\pi x}{l}\cos\frac{3\pi at}{l}.$$

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2) A string is stretched and fastened to two points l apart. Motion is started by displacing the string into the form $y = k(lx - x^2)$ from which it is released ~~on~~ at time $t=0$. Find the displacement of any point of the string at a distance x from one end at any time t.

Soln:- The displacement $y(x,t)$ is governed by

$$\frac{\partial^2 y}{\partial t^2} = a^2 \frac{\partial^2 y}{\partial x^2}$$

The boundary conditions are

(i) $y(0,t) = 0$ for all $t \geq 0$

(ii) $y(l,t) = 0$ for all $t \geq 0$

(iii) $\left.\frac{\partial y}{\partial t}\right|_{t=0} = 0$ for $0 \leq x \leq l$.

(iv) $y(x,0) = k(lx - x^2)$ for $0 \leq x \leq l$.

The suitable solution which satisfies our boundary conditions is given by

$$y(x,t) = (A\cos\lambda x + B\sin\lambda x)(C\cos\lambda at + D\sin\lambda at) \longrightarrow (1)$$

Applying boundary condition (i) in eqn (1), we get

$$y(0,t) = A(C\cos\lambda at + D\sin\lambda at) = 0.$$

either $A=0$ (or) $C\cos\lambda at + D\sin\lambda at = 0$

here, $C\cos\lambda at + D\sin\lambda at \neq 0$. ($\because$ it is defined $\forall t$)

$$\therefore \boxed{A=0.}$$

Substitute $A=0$ in eqn (1), we get

$$y(x,t) = B\sin\lambda x\,(C\cos\lambda at + D\sin\lambda at) \longrightarrow (2)$$

Applying boundary condition (ii) in eqn (2), we get

$$y(l,t) = B\sin\lambda l\,(C\cos\lambda at + D\sin\lambda at) = 0$$

here, $C\cos\lambda at + D\sin\lambda at \neq 0$ ($\because$ it is defined $\forall t$)

$\therefore$, either $B=0$ or $\sin\lambda l = 0$.

but $B \neq 0$ (suppose, we take $B=0$ and already we have $A=0$, then we get a trivial solution)

$\therefore$ we take $\sin\lambda l = 0$

$\sin\lambda l = \sin n\pi$, where n is any integer

$$\lambda l = n\pi$$

$$\therefore \lambda = \frac{n\pi}{l}$$

Substituting $\lambda = \frac{n\pi}{l}$ in eqn (2), we get

$$y(x,t) = B\sin\frac{n\pi x}{l}\left(C\cos\frac{n\pi at}{l} + D\sin\frac{n\pi at}{l}\right) \longrightarrow (3)$$

Before applying condition (iii), diff (3) partially w.r to 't', we get

$$\frac{\partial y}{\partial t} = B\sin\frac{n\pi x}{l}\left(C\left(-\sin\frac{n\pi at}{l}\cdot\frac{n\pi a}{l}\right) + D\left(\cos\frac{n\pi at}{l}\cdot\frac{n\pi a}{l}\right)\right)$$

Now applying b.c (iii), we get

$$\left.\frac{\partial y}{\partial t}\right|_{t=0} = B \sin\frac{n\pi x}{l}\left(D \cdot \frac{n\pi a}{l}\right) = 0.$$

here, $B \neq 0$ ($\because$ if $B=0$, we already explained)

$\sin\frac{n\pi x}{l} \neq 0$ ($\because$ it is defined for all x)

and $\frac{n\pi a}{l} \neq 0$ ($\because$ all are constants)

$\therefore$ $D=0$

Substituting $D=0$ in eqn ③, we get

$$y(x,t) = B \sin\frac{n\pi x}{l} \cdot C \cos\frac{n\pi at}{l}.$$

$y(x,t) = B_n \sin\frac{n\pi x}{l} \cos\frac{n\pi at}{l}$ where $BC = B_n$, n is any integer & B_n is any constant.

The most general solution is

$$y(x,t) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{l} \cos\frac{n\pi at}{l}. \longrightarrow ④$$

Applying the boundary condition (iv) in eqn ④, we get

$$y(x,0) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{l} = k(lx - x^2) \longrightarrow ⑤$$

To find B_n, expand $k(lx-x^2)$ in a half-range Fourier sine series in the interval $(0,l)$.

$$k(lx-x^2) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l} \quad \text{where}$$

$\longrightarrow ⑥$

$$b_n = \frac{2}{l}\int_0^l f(x) \sin\frac{n\pi x}{l}\, dx$$

From ⑤ & ⑥, we get $\boxed{B_n = b_n}$

$$\therefore B_n = \frac{2}{l}\int_0^l k(lx-x^2)\sin\frac{n\pi x}{l}\,dx$$

$$= \frac{2k}{l}\int_0^l (lx-x^2)\sin\frac{n\pi x}{l}\,dx.$$

$$= \frac{2k}{l}\left[(lx-x^2)\left(\frac{-\cos\frac{n\pi x}{l}}{\frac{n\pi}{l}}\right) - (l-2x)\left(\frac{-\sin\frac{n\pi x}{l}}{\left(\frac{n\pi}{l}\right)^2}\right) + (-2)\left(\frac{\cos\frac{n\pi x}{l}}{\left(\frac{n\pi}{l}\right)^3}\right)\right]_0^l$$

$$= \frac{2k}{l}\left[-0+0-2\left(\frac{l^3}{n^3\pi^3}\right)(-1)^n - \left(-0+0-2\left(\frac{l^3}{n^3\pi^3}\right)(1)\right)\right]$$

$$= \frac{2k}{l}\left[\frac{-2l^3}{n^3\pi^3}(-1)^n + \frac{2l^3}{n^3\pi^3}\right]$$

$$= \frac{2k}{l}\left(\frac{+2l^3}{n^3\pi^3}\right)(-(-1)^n+1)$$

$$= \frac{4kl^2}{n^3\pi^3}(1-(-1)^n).$$

$\therefore$ substitute the value of B_n in eqn ④, we get

$$y(x,t) = \sum_{n=1}^{\infty}\frac{4kl^2}{n^3\pi^3}(1-(-1)^n)\sin\frac{n\pi x}{l}\cos\frac{n\pi at}{l}.$$

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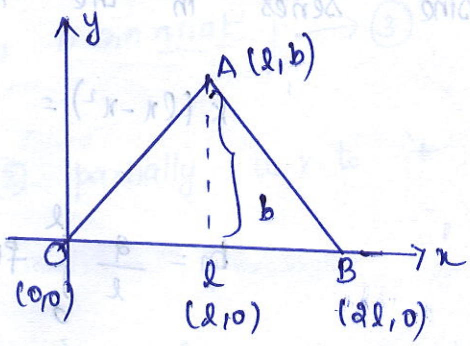
3) A taut string of length $2l$ is fastened at both ends. The mid point of the string is taken to a height b and then released from the rest in that position. Find the displacement of the string.



Soln:-

The boundary conditions are

(i) $y(0,t)=0, \quad t \geq 0$

(ii) $y(2l,t)=0, \quad t \geq 0$

Equation of line along OA, $\underset{x_1\ y_1}{(0,0)}$ $\underset{x_2\ y_2}{(l,b)}$

$$\frac{y-y_1}{y_2-y_1} = \frac{x-x_1}{x_2-x_1} \Rightarrow \frac{y-0}{b-0} = \frac{x-0}{l-0}$$

l

$$\frac{y}{b} = \frac{x}{l}.$$

O

$$\Rightarrow y = \frac{bx}{l}, \quad 0 \le x \le l.$$

Equation of line along AB, $\underset{x_1\ y_1}{(l,b)}$ & $\underset{x_2\ \ y_2}{(2l,0)}$.

$$\frac{y-b}{0-b} = \frac{x-l}{2l-l} \Rightarrow \frac{y-b}{-b} = \frac{x-l}{l}.$$

$$y-b = \frac{-b(x-l)}{l}$$

$$y = b - b\frac{(x-l)}{l} \Rightarrow y = b\left(\frac{2l-x}{l}\right), \quad l \le x \le 2l.$$

$\therefore$ (iv). $y(x,0) = \begin{cases} \dfrac{bx}{l}, & 0 \le x \le l \\ b\left(\dfrac{2l-x}{l}\right), & l \le x \le 2l. \end{cases}$

The suitable solutions which satisfies our boundary conditions is given by

$$y(x,t) = (A\cos\lambda x + B\sin\lambda x)(C\cos\lambda at + D\sin\lambda at) \longrightarrow ①.$$

Applying boundary condition (i) in eqn ①, we get

$$y(x,0) = A(C\cos\lambda at + D\sin\lambda at) = 0.$$

either $A=0$ or $C\cos\lambda at + D\sin\lambda at = 0$.

here, $C\cos\lambda at + D\sin\lambda at \neq 0$ ($\because$ it is defined $\forall t$)

$\therefore$ $A=0$

Substitute $A=0$ in eqn ①, we get

$$y(x,t) = B\sin\lambda x(C\cos\lambda at + D\sin\lambda at) \longrightarrow (2)$$

Applying b.c (ii) in eqn (2), we get

$$y(2l,t) = B\sin\lambda 2l\ (C\cos\lambda at + D\sin\lambda at) = 0.$$

$$y(2l,t) = B\sin 2l\lambda\ (C\cos\lambda at + D\sin\lambda at) = 0.$$

here, $C\cos\lambda at + D\sin\lambda at \neq 0$ ($\because$ it is defined $\forall t$).

Therefore, either $B=0$ or $\sin 2l\lambda = 0$.

Suppose, we take $B=0$ and already we have $A=0$.

then we get a trivial solution.

$\therefore\quad B \neq 0$

The only possibility is $\sin 2l\lambda = 0$.

ie, $\sin 2l\lambda = \sin n\pi$, where n is any integer.

$$2l\lambda = n\pi$$

$$\boxed{\lambda = \frac{n\pi}{2l}}$$

Substitute $\lambda = \frac{n\pi}{2l}$ in eqn (2), we get

$$y(x,t) = B\sin\frac{n\pi x}{2l}\left(C\cos\frac{n\pi at}{2l} + D\sin\frac{n\pi at}{2l}\right) \longrightarrow (3).$$

Before applying boundary condition (iii), diff. eqn (3) partially w.r to t, we get

$$\frac{\partial y}{\partial t} = B\sin\frac{n\pi x}{2l}\left(C\left(-\sin\frac{n\pi at}{2l}\cdot\frac{n\pi a}{2l}\right) + D\left(\cos\frac{n\pi at}{2l}\cdot\frac{n\pi a}{2l}\right.\right.$$

Now, applying boundary condition (iii), we get

$$\left.\frac{\partial y}{\partial t}\right|_{t=0} = B\sin\frac{n\pi x}{2l}\ D\cdot\frac{n\pi a}{2l} = 0.$$

$B \neq 0$ (Since suppose $B=0$, we have already explained)

$\sin \frac{n\pi x}{2l} \neq 0$ ($\because$ it is defined $\forall x$)

$\frac{n\pi x}{2l} \neq 0$ ($\because$ all are constants).

$\therefore \boxed{D=0}$

Sub. $D=0$ in eqn ③, we get.

$$y(x,t) = B \sin \frac{n\pi x}{2l} \, C \cos \frac{n\pi a t}{2l}$$

$$= BC \sin \frac{n\pi x}{2l} \cos \frac{n\pi a t}{2l}$$

$$= B_n \sin \frac{n\pi x}{2l} \cos \frac{n\pi a t}{2l}$$ where $BC = B_n$, n is any integer and B_n is any constant.

The most general solution is

$$y(x,t) = \sum_{n=1}^{\infty} B_n \sin \frac{n\pi x}{2l} \cos \frac{n\pi a t}{2l} \longrightarrow ④$$

Applying b.c (iv) in eqn ④, we get

$$y(x,0) = \sum_{n=1}^{\infty} B_n \sin \frac{n\pi x}{2l} = \begin{cases} \frac{bx}{l}, & 0 \le x \le l \\ b\left(\frac{2l-x}{l}\right), & l \le x \le 2l. \end{cases} \longrightarrow ⑤$$

To find B_n, expand $\begin{cases} \frac{bx}{l}, & 0 \le x \le l \\ b\left(\frac{2l-x}{l}\right), & l \le x \le 2l \end{cases}$ in a half-range Fourier sine series in the interval $(0, 2l)$.

$$\begin{Bmatrix} \frac{bx}{l}, & 0 \le x \le l \\ b\left(\frac{2l-x}{l}\right), & l \le x \le 2l \end{Bmatrix} = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{2l},$$ where

$$b_n = \frac{2}{2l}\int_0^{2l} f(x)\sin\frac{n\pi x}{2l}\,dx$$

From ⑤ & ⑥, we get $\boxed{B_n = b_n}$

$$B_n = \frac{2}{2l}\int_0^{2l} f(x)\sin\frac{n\pi x}{2l}\,dx.$$

$$= \frac{1}{l}\left[\int_0^l \frac{bx}{l}\cdot\sin\frac{n\pi x}{2l}\,dx + \int_l^{2l}\frac{b(2l-x)}{l}\sin\frac{n\pi x}{2l}\,dx\right]$$

$$= \frac{1}{l}\left[\frac{b}{l}\right]\left[\int_0^l x\sin\frac{n\pi x}{2l}\,dx + \int_l^{2l}(2l-x)\sin\frac{n\pi x}{2l}\,dx\right].$$

$$= \frac{b}{l^2}\left[(x)\left(\frac{-\cos\frac{n\pi x}{2l}}{\frac{n\pi}{2l}}\right) - (1)\left(\frac{-\sin\frac{n\pi x}{2l}}{\left(\frac{n\pi}{2l}\right)^2}\right)\right]_0^l +$$

$$\left[(2l-x)\left(\frac{-\cos\frac{n\pi x}{2l}}{\frac{n\pi}{2l}}\right) - (-1)\left(\frac{-\sin\frac{n\pi x}{2l}}{\left(\frac{n\pi}{2l}\right)^2}\right)\right]_l^{2l}$$

$$= \frac{b}{l^2}\left[l\cdot\frac{2l}{n\pi}\left[-\cos\frac{n\pi(l)}{2l}\right] + \frac{4l^2}{n^2\pi^2}\left(\sin\frac{n\pi(l)}{2l}\right) - 0\right] +$$

$$\left[0 - \left((l)\cdot\frac{2l}{n\pi}\left(-\cos\frac{n\pi(l)}{2l}\right) + \left(\frac{4l^2}{n^2\pi^2}\left(-\sin\frac{n\pi(l)}{2l}\right)\right)\right)\right].$$

$$= \frac{b}{l^2}\left[\frac{2l^2}{n\pi}\left(-\cos\frac{n\pi}{2}\right) + \frac{4l^2}{n^2\pi^2}\left(\sin\frac{n\pi}{2}\right)\right.$$

$$\left. - \left[\frac{2l^2}{n\pi}\left(-\cos\frac{n\pi}{2}\right) - \frac{4l^2}{n^2\pi^2}\sin\frac{n\pi}{2}\right]\right]$$

$$= \frac{b}{l^2}\left[\cancel{-\frac{2l^2}{n\pi}\cos\frac{n\pi}{2}} + \frac{4l^2}{n^2\pi^2}\sin\frac{n\pi}{2} + \cancel{\frac{2l^2}{n\pi}\cos\frac{n\pi}{2}}\right.$$

$$\left. + \frac{4l^2}{\ldots}\sin\frac{n\pi}{\ldots}\right]$$

$$= \frac{b}{l^2}\left[\frac{8l^2}{n^2\pi^2}\sin\frac{n\pi}{2}\right] = \frac{8b}{n^2\pi^2}\sin\frac{n\pi}{2}.$$

Sub. value of B_n in eqn ④, we get.

$$y(x,t) = \sum_{n=1}^{\infty}\frac{8b}{n^2\pi^2}\sin\frac{n\pi}{2}\sin\frac{n\pi x}{2l}\cos\frac{n\pi a t}{2l}.$$

Exercise problems.

(1) A tightly stretched string with end points $x=0$ and $x=l$ is initially in a position given by $y(x,0) = y_0 \sin\frac{\pi x}{l}$. If it is released from rest from this position, find the displacement $y(x,t)$ at any point of the string.

(2) A string is stretched and fastened to two points l apart. Motion is started by displacing the string into the form $y = 3(lx - x^2)$ from which it is released at time $t=0$. Find the displacement of any point on the string at a distance of x from one end at any time t.

(3) A tightly stretched string with fixed end points $x=0$ and $x=l$ is initially in a position given by $y(x,0) = k\left(\sin\frac{\pi x}{l} - \sin\frac{2\pi x}{l}\right)$. If it is released from rest from this position, find the displacement y at any distance x from one end at any time t.