Chapter 6

Multi-rate Digital Signal Processing

6.1 Introduction

There are several use of digital representation of signals for transmission and storage. This has created challenges in the area of digital signal processing. Changing in the sampling rate of audio signal is a common task in the signal processing field. For example, in digital audio, the different sampling rates used are 32 KHz for broadcasting, 44.1 KHz for compact disc and 48 KHz for audio tape. In digital video, the sampling rates for composite video signals are 14.3181818 MHz and 17.734475 MHz for NTSC and PAL respectively. In many practical applications of digital signal processing the problem exists while changing the sampling rate of a signal, either by expanding it or reducing it by some quantity. For example, in telecommunication system that transmit and receive different type of signal (eg., teletype, speech, video etc.), there is a demand to process the different signals at different rates to commensurate with the corresponding bandwidth of the signals. The process of changing a signal from a given rate to a different rate is called sampling rate conversion.

The Discrete Time Systems that process the data at more than one sampling rate is called Multi-rate Systems.

Also Multi rate filters, which are used at the interfaces of continuous and sampled data, results in reducing the cost of the analog signal conditioning components and also improves the signal quality.

6.1.1 Application of Multi-rate Signal Processing

There are many cases where Multi-rate signal processing is used, they are

1. Data acquisition and storage systems.
2. In trans- multiplexers.
3. Speech Processing.
4. Telecommunication
5. Sub-band coding
6. Voice privacy using analog phone lines
7. Analog to Digital (A/D) Converters and
8. Digital to Analog converters (D/A).

6.1.2 Advantages of Multi-Rate Signal Processing

- Computational requirements are less
- Storage space for filter coefficients is less
- Finite arithmetic effects are less
- Filter order required in Multi rate application are low
- Sensitivity to filter coefficient lengths are less

6.2 Basic Operation in Multirate Processing

Multi-Rate DSP Consists of Process

1. **Decimation: (Down Sampling)** It is a process to decrease the sampling rate. (It is used to reduce the number of samples)

2. **Interpolation: (Up Sampling)** It is a process to increase the sampling rate. (It is used to increase the number of samples)

6.2.1 Decimation (Down Sampling)

- Basically low pass filters (LPF) are used for decimation and for interpolation. When decimating, low pass filters are used to decrease the bandwidth of a signal initially to decrease the sampling rate. This is done to reduce aliasing due to the reduction in the sampling rate. When decimating, the bandwidth of a signal is decreased to a suitable value so that minimal aliasing occurs when decreasing the sampling rate
- "Down sampling" is a process of removing some samples. A signal is down sampled only when it is over sampled. (i.e. sampling rate > Nyquist rate).
- Down sampler is a primary sampling rate alteration appliance used to reduce the sampling rate by an integer factor. A down-sampler with a down-sampling factor M, where M is a positive integer, develops an output series $y[n]$ with a sampling rate that is $(\frac{1}{M})^{\text{th}}$ of that of the input sequence $x[n]$. The down sampler is shown in Fig.6.1.
- The sampling rate of a Discrete Time Signal $x(n)$ can be reduced by a factor M, we must keep every M^{th} sample as it is and remove the $(M-1)$ samples in between.
- The output signal $y(n)$ is a down sampled signal of the input signal $x(n)$ and can be represented by $y(n) = x(Mn)$

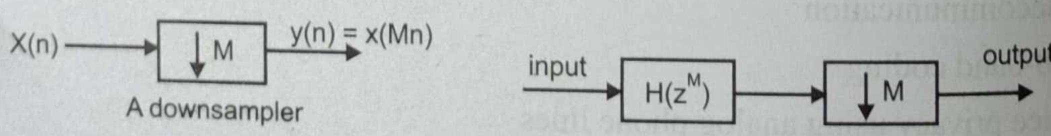


Fig .6.1 Decimation down sampling

The z-transform of output sequence is

$$y(z) = \sum_{-\infty}^{\infty} x(n) z^{-n}$$

$$y(z) = \sum_{-\infty}^{\infty} x(nM) z^{-n}$$

Using Scaling Property of z- transform we can write,

$$y(z) = X'(z^{-1/M})$$

For example, In fig 6.2 the signal $x(n)$ is downsampled by factor 2 to get the signal $y(n) = x(2n)$

$$x(n) = \{1, -2, 3, 7, 9, 4, 5, -6, 8, 2, 7, 1, -4, 0, 9\}$$

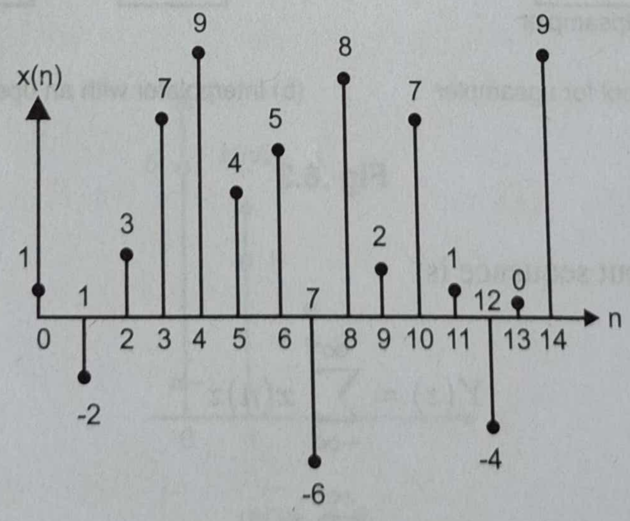
Take $M = 2$ We are keep every M^{th} sample and remove the $(M - 1)$ samples

$$X_d(n) = \{1, 3, 9, 5, 8, 7, -4, 9\}$$

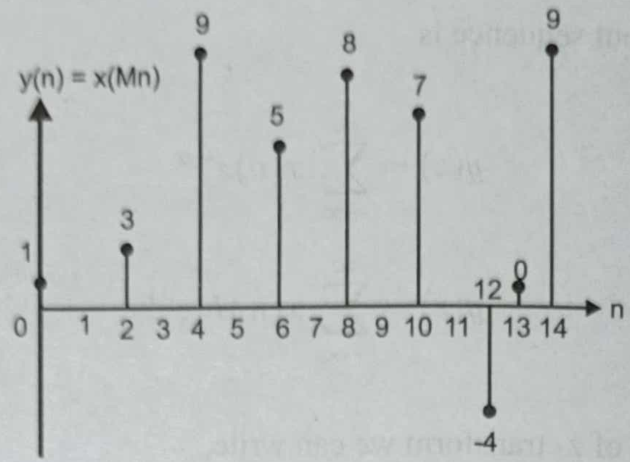
Remove the $(M - 1) = (2 - 1) = 1$ samples

Remove $(M - 1)$ samples

(i.e) 1 sample is remove between two successive samples



(a) Signal x(n)



(b) Decimated o/p signal decimation factor 2

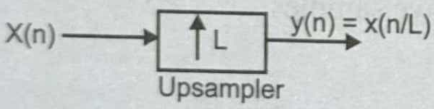
Fig .6.2 Decimation Process with factor 2

6.2.2 Interpolation (Up Sampling)

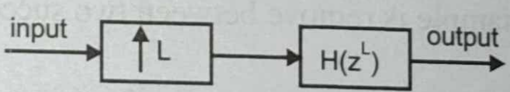
- Interpolation is the exact opposite of decimation. It is an information preserving operation, in that all samples of $x[n]$ are present in the expanded signal $y[n]$."Up sampling" is a process of adding some zeros in between two successive samples.
- The sampling rate of a discrete time signal can be increased by a factor L by placing $(L-1)$ equally spaced zeros between each pairs of samples.

  It is represented by $y(n) = x(n/L)$

The symbol for upsampler is shown in Fig 6.3(a) and Interpolator with an upsampler is shown in Fig 6.3(b).



(a) Symbol for upsampler



(b) Interpolator with an upsampler

Fig .6.3

The z-transform of output sequence is

$$Y(z) = \sum_{-\infty}^{\infty} x(n)z^{-n}$$

$$Y(z) = \sum_{-\infty}^{\infty} x(\frac{n}{L})z^{-n}$$

sing Scaling Property of z-transform we can write,

$$Y(z) = X(z^L)$$

ake $L = 3$, Add $(L-1)$ zeros (i.e) 2 zeros are Add between two successive samples.

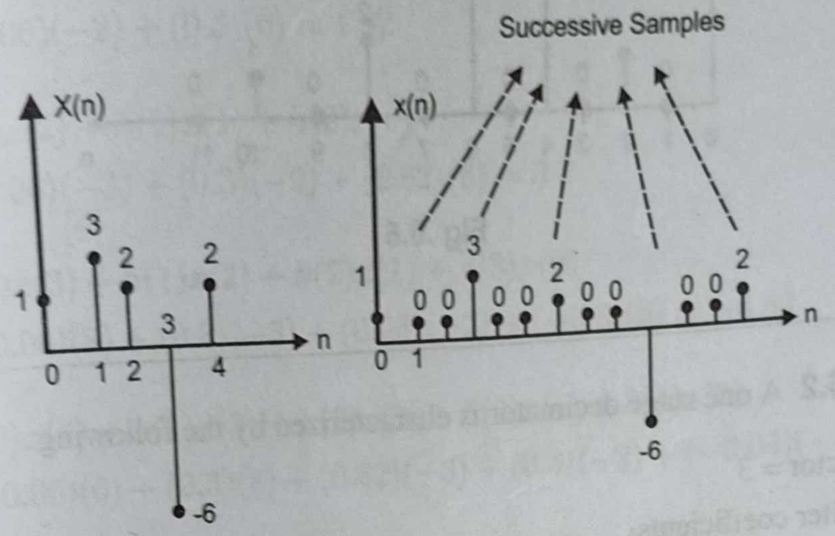


Fig .6.4 Interpolation Process with factor 3

Fig.6.4. shows the signal $x(n)$ and its three fold upsampled signal $y(n)$.

For example, $x(n) = \{1, 3, 2, -6, 2\}$

Take $L = 3$ We are keep every L^{th} sample and Add the (L-1) zeros

$$X_I(n) = \{1, 0, 0, 3, 0, 0, 2, 0, 0, -6, 0, 0, 2\}$$

Add the $(L-1)$ zeros $= (3-1) = 2$ zeros of every L^{th} Sample.

Example 6.1 A signal $x(n) = \{6, 1, 5, 7, 2, 1\}$ Find $(1) x(\frac{n}{2}) (2) x(2n)$.

(2) $x(2n) = \{6, 5, 2\}$

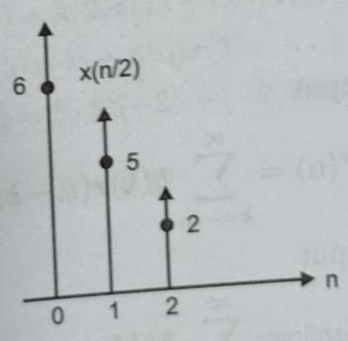


Fig .6.5

(1) $x(n/2) = \{6, 0, 1, 0, 5, 0, 7, 0, 2, 0, 1\}$

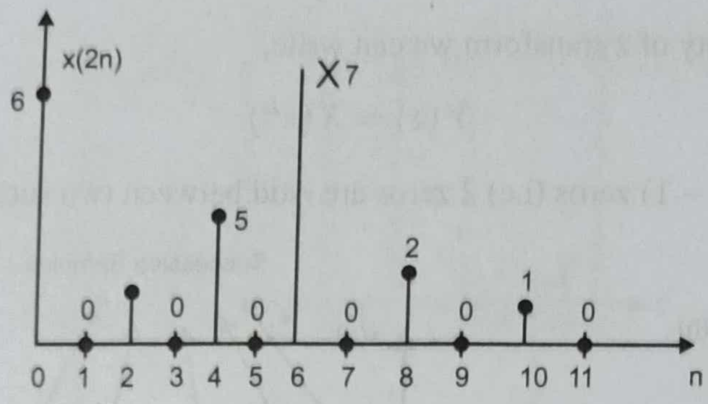


Fig .6.6

■ Example 6.2 A one stage decimator is characterized by the following

Decimation factor = 3

Antialiasing filter coefficients

$$h(0) = -0.06 = h(4),$$
$$h(1) = 0.30 = h(3),$$
$$h(2) = 0.62$$

Given

The data, $x(n)$ with successive values $[6, -2, -3, 8, 6, 6, 4, -2]$ calculation and list the filtered output, $w(n)$ and the output of the decimator $y(m)$.

Solution:

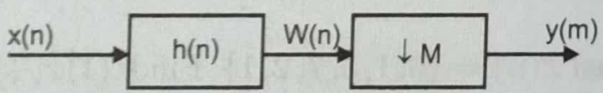


Fig .6.7 Decimation process for the factor M = 3

Filtered output

$$W(n) = \sum_{k=-\infty}^{\infty} h(k)x(n-k)$$

Decimator output

$$y(m) = \sum_{n=-\infty}^{\infty} h(Mm-n)x(n)$$

$$M = 3 \quad x(n) = [6, -2, -3, 8, 6, 4, -2]$$

Filtered output

$$W(0) = h(0)x(0) = (-0.06)(6) = -0.36$$

$$\begin{aligned} W(1) &= h(0)x(1) + h(1)x(0) \\ &= (-0.06)(-2) + (0.3)(6) = 1.92 \end{aligned}$$

$$\begin{aligned} W(2) &= h(0)x(2) + h(1)x(1) + h(2)x(0) \\ &= (-0.06)(-3) + (0.3)(-2) + (0.62)(6) = 3.3 \end{aligned}$$

$$\begin{aligned} W(3) &= h(0)x(3) + h(1)x(2) + h(2)x(1) + h(3)x(0) \\ &= (-0.06)(8) + (0.3)(-3) + (0.06)(-2) + (-0.3)(6) = -0.82 \end{aligned}$$

$$\begin{aligned} W(4) &= h(0)x(4) + h(1)x(3) + h(2)x(2) + h(3)x(1) + h(4)x(0) \\ &= (-0.06)(6) + (0.3)(8) + (0.62)(-3) + (0.3)(-2) + (-0.06)(-6) = -0.78 \end{aligned}$$

$$\begin{aligned} W(5) &= h(0)x(5) + h(1)x(4) + h(2)x(3) + h(3)x(2) + h(4)x(1) \\ &= (-0.06) + (0.3)(4) + (0.62)(8) + (0.3)(-3) + (-0.06)(-2) = 5.74 \end{aligned}$$

$$\begin{aligned} W(6) &= h(0)x(6) + h(1)x(5) + h(2)x(4) + h(3)x(3) + h(4)x(2) \\ &= (-0.06)(-2) + (0.3)(4) + (0.62)(6) + (0.3)(8) + (-0.06)(-3) = 7.62 \end{aligned}$$

$$\begin{aligned} W(7) &= h(0)x(7) + h(1)x(6) + h(2)x(5) + h(3)x(4) + h(4)x(3) + h(5)x(2) \\ &= (0.3)(-2) + (0.62)(4) + (0.3)(6) + (-0.06)(8) = 3.2 \end{aligned}$$

$$\begin{aligned} W(8) &= h(0)x(8) + h(1)x(7) + h(2)x(6) + h(3)x(5) + h(4)x(4) + \\ &\qquad h(5)x(3) + h(6)x(2) + h(7)x(1) + h(8)x(0) \\ &= (-0.62)(-2) + (0.3)(4) + (-0.06)(6) = -0.4 \end{aligned}$$

$$\begin{aligned} W(9) &= [h(0)x(9) + h(1)x(8) + h(2)x(7) + h(3)x(6) + \\ &\qquad h(4)x(5) + h(5)x(4) + h(6)x(3) \ldots\ldots] \\ &= h(3)x(6) + h(4)x(5) = (0.3)(-2) + (-0.06)(4) = -0.84 \end{aligned}$$

$$W(10) = h(4)x(6) = (-0.06)\ (-2) = 0.12$$

Decimator output

$$y(m) = \sum_{n=-\infty}^{\infty} h(Mm - n)x(n) \quad M = 3,\ x(n) = [6, -2, -3, 8, 6, 4, -2]$$

$$y = 0 = h(0)x(0) = -0.36$$

$$y(1) = h(3)x(0) + h(2)x(1) + h(1)x(2) + h(0)x(3)$$
$$= (0.3)(6) + (0.62)(-2) + (0.3)(-3) + (-0.06)(8) = -0.82$$

$$y(2) = h(6)x(0) + h(5)x(1) + h(4)x(2) + h(3)x(3)$$
$$+ h(2)x(4) + h(1)x(5) + h(0)x(6)$$
$$= (-0.06(-3) + (0.3)(8) + (0.62)(6) + (0.3)(4) + (0.06)(-2) = 7.62$$

$$y(3) = h(9)x(0) + h(8)x(1) + h(7)x(2) + h(6)x(3) + h(5)x(4) + h(4)x(5)$$
$$+ h(3)x(6) + [h(2)x(7) + h(1)x(8) + h(0)x(9)]$$
$$= (-0.06)(4) + (0.3)(-2) = -0.84$$

Filtered output,

$$W(n) = [-0.36, 1.92, 3.3., -0.82, -0.78, 5.74, 7.62, 3.2, -0.4, -0.84, 0.12]$$

Decimator output $y(n) = [-0.36, -0.82, 7.62, -0.84,]$

Alternate Method

$$W(n) = x(n) * h(n)$$

Taking z transforms

$$z(W(n)) = z\{x(n) * h(n)\} = z\{x(n)\}$$
$$x(n) = \{6, -2, -3, 8, 6, 4, -2\}$$

z-transform,

$$X(z) = 6 - 2z^{-1} - 3z^{-2} + 8z^{-3} + 6z^{-4} + 4z^{-5} - 2z^{-6}$$
$$h(n) = \{-0.06, 0.3, 0.62, 0.3, -0.06\}$$

z-transform,

$$H(z) = -0.06 + 0.3z^{-1} + 0.62z^{-2} + 0.3z^{-3} - 0.06z^{-4}$$
$$W(z) = X(z) * H(z)$$

Taking inverse z-transform

$$W(n) = \{-0.36, 1.92, 3.3, -0.82, -0.78, 5.74, 7.62, 3.2, -0.4, -0.84, 0.12\}$$

Decimating by factor 3 $y(m) = \{-0.36, -0.82, 7.62, -0.84\}$

6.3 Frequency Transforms of Decimated Sequences

The analysis of decimation is better understood by assessing the frequency spectrum using the Fourier transform.

The implications of aliasing caused by decimation are very similar to those in the case of sampling a continuous-time signal. In general, if the Fourier transform of a signal, $X(\theta)$, occupies the entire bandwidth from $[-\pi, \pi]$, then the Fourier transform of the decimated signal, $X(\downarrow M)(\theta)$, will be aliased. This is due to the superposition of the M shifted and frequency-scaled transforms.

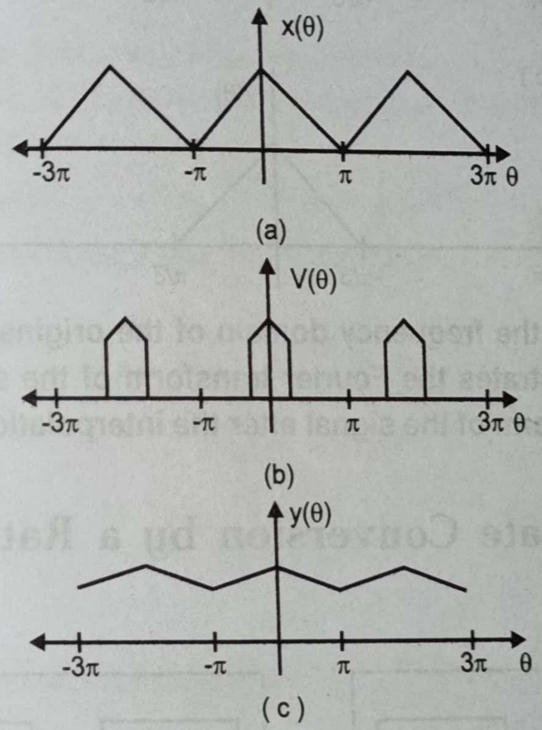


Fig .6.8 Aliasing caused by decimation; (a) Fourier Transform of the original signal; (b) After decimation Filtering; (c) Fourier transform of the decimated signal

Figure 6.8 (a) shows the Fourier transform of the original signal. Fig 6.8 (b) shows the signal after low pass filtering. Figure 6.8(c) shows the expanded spectrum after decimation.

6.4 Frequency Transforms of Expanded Sequences

The effect of expansion on a signal in the frequency domain is illustrated in Figure6.9. below. Fig.6.9(a) shows the Fourier transform of the original signal; Fig.6.9 (b) illustrates the Fourier transform of the signal with zeros added $\omega(\theta)$; and Fig.6.9(c) shows the Fourier transform of the signal after the interpolation filter. It is clearly visible that the shape of the Fourier transform is compressed by a factor L in the frequency axis and is also repeated L times in the range of $[-\pi, \pi]$.

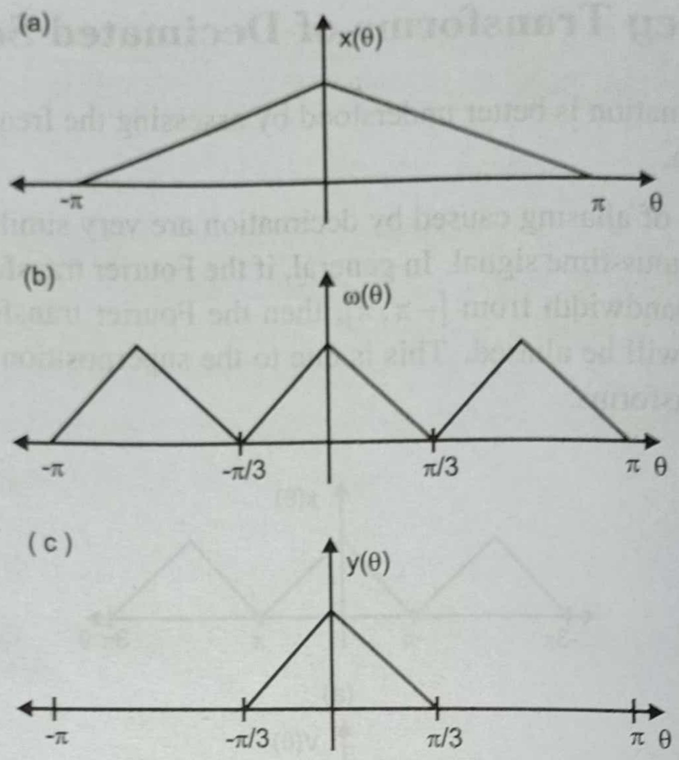


Fig .6.9 Expansion in the frequency domain of the original signal (a) and the expanded signal (b) Illustrates the Fourier transform of the signal with zeros added $\omega(\theta)$ (c) Fourier transform of the signal after the interpolation filter

6.5 Sampling Rate Conversion by a Rational Factor I/D (or) L/M

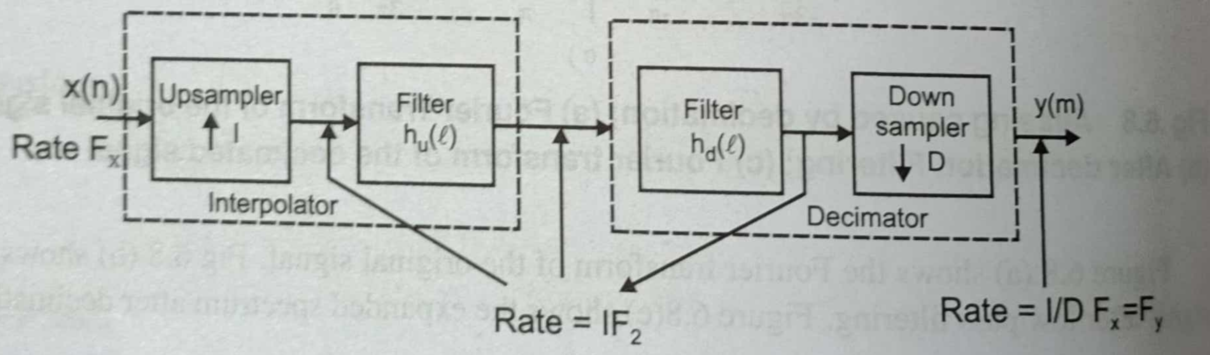


Fig .6.10

Basically, we can achieve this sampling rate conversion by first performing interpolation by the factor I (or) (L) and then decimating the output of the interpolator by the factor D (or) (M). In other words, a sampling rate conversion by the rational factor I/D is accomplished by cascading an interpolator with a decimator, as illustrated in Fig.6.10.

Method for sampling rate conversion by a factor I/D and Cascading of sample rate converters.

The interpolation by a factor I is obtained first to increase the sampling rate to IF_x.

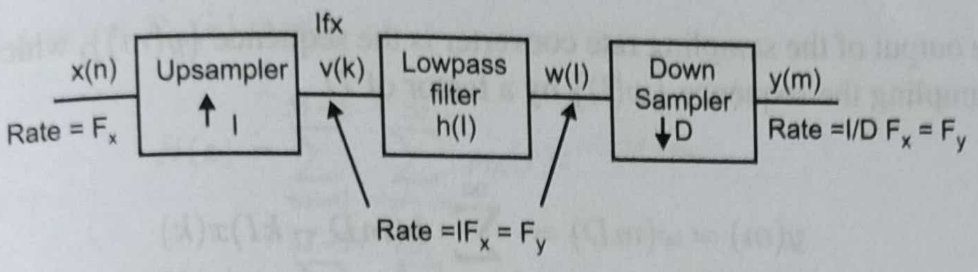


Fig .6.11 Sampling rate conversion by a factor I/D

The output of interpolator is then decimated by a factor D, so that the final output rate is $F_y = I/DF_x$.

In figure 6.10. the two filters with impulse response $[h_u(l)$ and $(h_d(l))]$ are operated at the same rate, namely IF_x, and hence can be combined into a single low pass filter with impulse response $h(l)$ as illustrated in Fig.6.11.

The overall cutoff frequency will be minimum of the two cut-off frequencies.

The frequency response of anti-imaging filter is given as

$$\mathrm{H_u}(\omega) = \begin{cases} c & -\dfrac{\pi}{\mathrm{I}} \le \omega \le \dfrac{\pi}{I} \\ 0 & \text{otherwise} \end{cases}$$

$$= \begin{cases} I & -\dfrac{\pi}{\mathrm{I}} \le \omega \le \dfrac{\pi}{I} \\ 0 & \text{otherwise} \end{cases}$$

c = 1 (Scaling factor)

The frequency response of anti-aliasing filter is given as

$$\mathrm{H_d}(\omega) = \begin{cases} I & -\dfrac{\pi}{\mathrm{D}} \le \omega \le \dfrac{\pi}{D} \\ 0 & \text{otherwise} \end{cases}$$

The overall cascading effect of low pass filter will have a cut off frequency which is minimum of $\frac{\pi}{I}$ and $\frac{\pi}{D}$. Hence we can write the frequency response of combined filter as,

$$H(\omega) = \begin{cases} I & |\omega| \le \min\left(\dfrac{\pi}{D'}\dfrac{\pi}{I}\right) \\ 0 & \text{otherwise} \end{cases}$$

Derivation For Output y(m)

In the time domain, the output of the up sampler is the sequence and the output of the linear time-invariant filter is

$$w(\ell) = \sum_{k=-\infty}^{\infty} h(\ell - k)v(k) = \sum_{k=-\infty}^{\infty} h(\ell - kI)x(k)$$

Finally, the output of the sampling rate converter is the sequence $\{y(m)\}$, which is obtained by down sampling the sequence $\{w(l)\}$ by a factor of D.

Thus

$$y(m) = \omega(mD) = \sum_{k=-\infty}^{\infty} h(mD - kI)x(k)$$

This is the equation for output sequence.

6.6 Polyphase Structure of Decimator

The transfer function $H(z)$ of the polyphase FIR filter is decomposed into M branches given by

$$H(z) = \sum_{m=0}^{M-1} z^{-m} p_m(z^M)$$

Where

$$p_m(z) = \sum_{m=0}^{\frac{N+1}{M}} h(Mn + m)z^{-n}$$

The z-transform of an infinite sequence is given by

$$H(z) = \sum_{n=-\infty}^{\infty} h(n)z^{-n}$$

In this case $H(z)$ an be decomposed into $M-$branches as

$$H(z) = \sum_{m=0}^{M-1} z^{-m} p_m(z^M)$$

$$x_m(r) = x(rM - m)$$

$$x_m(-r) = x(-rM - m)$$

$$x_m(n - r) = x(n - (rM + m))$$

Where

$$p_m(z) = \sum_{r=-\infty}^{\infty} h(rM + m)z^{-r}$$

$$H(z) = \sum_{m=0}^{M-1} \sum_{r=-\infty}^{\infty} z^{-m}(rM + m)z^{-rM}$$

$$H(z) = \sum_{m=0}^{M-1} \sum_{r=-\infty}^{\infty} h(rM + m)z^{-(rM+m)}$$

Let $h(Mn+m) = P_m(r)$ then

$$H(z) = \sum_{m=0}^{M-1} \sum_{r=-\infty}^{\infty} p_m(r) z^{-(rM+m)}$$

$$Y(z) = \sum_{M=0}^{M-1} \sum_{r=-\infty}^{\infty} p_m(r) X(z) z^{-(rM+m)}$$

$$y(n) = \sum_{m=0}^{M-1} \sum_{r=-\infty}^{\infty} p_m(r) x[n-(rM+m)]$$

Let $x_m(r) = x(rM-m)$

$$y(n) = \sum_{m=0}^{M-1} \sum_{r=-\infty}^{\infty} p_m(r) x_m(n-r)$$

$$y(n) = \sum_{m=0}^{M-1} p_m(n) * x_m(n)$$

$$y(n) = \sum_{m=0}^{M-1} y_m(n)$$

Where $\quad y_m(n) = p_m(n) * x_m(n)$

The operation $p_m(n)^* x_m(n)$ is known as polyphase convolution, and the overall process is polyphase filtering $x_m(n)$ is obtained first delaying $x(n)$ by m units then down sampling by a factor M. Next $y_m(n)$ can be obtained by convolving $x_m(n)$ with $p_m(n)$. The structure of a polyphase decimator with 3 branches and a sampling rate reduction by a factor 3 is shown in Fig. 6.12.

For a general case of M branches and a sampling rate reduction by a factor M, the structure of polyphase decimator is shown in Fig.6.13.

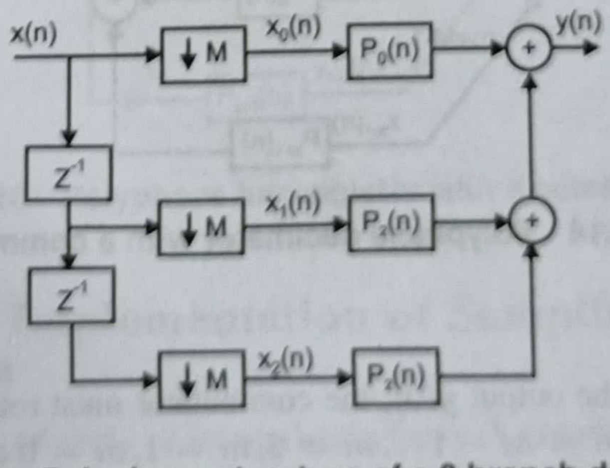


Fig .6.12 Polyphase structure of a 3 branch decimator

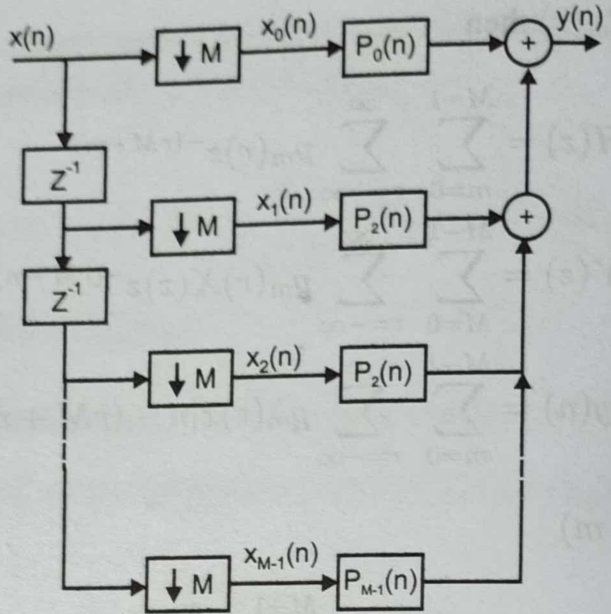


Fig .6.13 Polyphase structure of a M branch decimator

The splitting of $x(n)$ into the low rate sub sequence $x_0(n), x_1(n) \ldots\ldots x_{M-1}(n)$ is often represented by a commutator. In the configuration shown in Fig.6.13. The input values $x(n)$ enter the delay chain at high rate. Then the M down sampler sends the group of M input values of M filters at time $n = mM$.

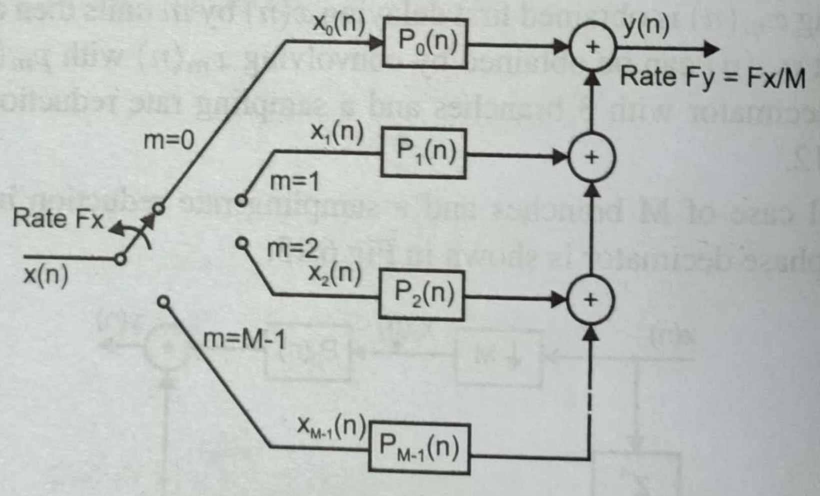


Fig .6.14 Polyphase decimator with a commutator

In Fig.6.14 to produce the output $y(0)$, the commutator must rotate in counter-clockwise direction starting from $m = M-1 \ldots m = 2, m = 1, m = 0$ and give the input values $x(-M+1) \ldots x(-2), x(-1), x(0)$ to the filters $P_{M-1}(n) \ldots P_2(n), P_1(n), P_0(n)$.

6.7 Polyphase Structure of Interpolator

By transposing the decimator structure shown in Fig 6.14, we can obtain the polyphase structure for interpolator, which consists of a set of L sub filters connected in parallel as shown in Fig 6.15. Here the polyphase components of impulse response are given by

$$P_m(n) = h(nL + m) \quad m = 0, 1, 2, \ldots L - 1$$

Where $h(n)$ is the impulse response of anti-imaging filter. The output of L sub filters can be represented as

$$y_m(n) = x(n)p_m(n) \qquad m = 0, 1, 2, \ldots L - 1$$

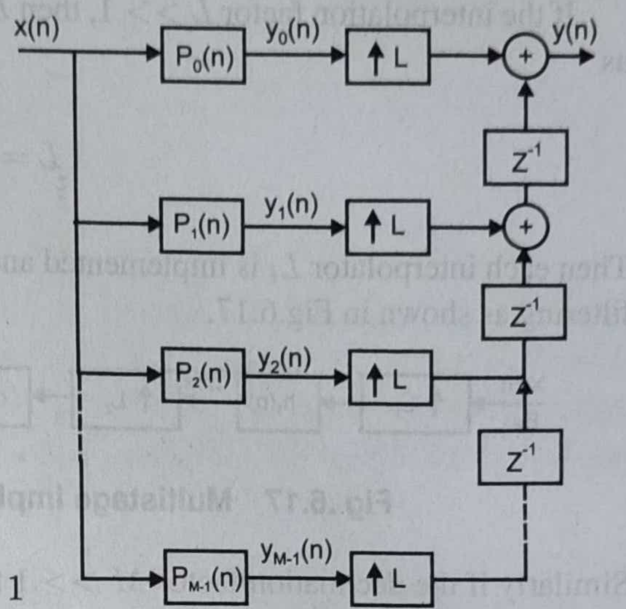


Fig .6.15 Polyphase structure of a M branch interpolator

By up sampling with a factor L and adding a delay z^{-m} the polyphase components are produced from $y_m(n)$. These polyphase components are all added together to produce the output signal $y(n)$.

The output $y(n)$ also can be obtained by combining the signals $x_m(n)$ using a commutator as shown below in Fig.6.16.

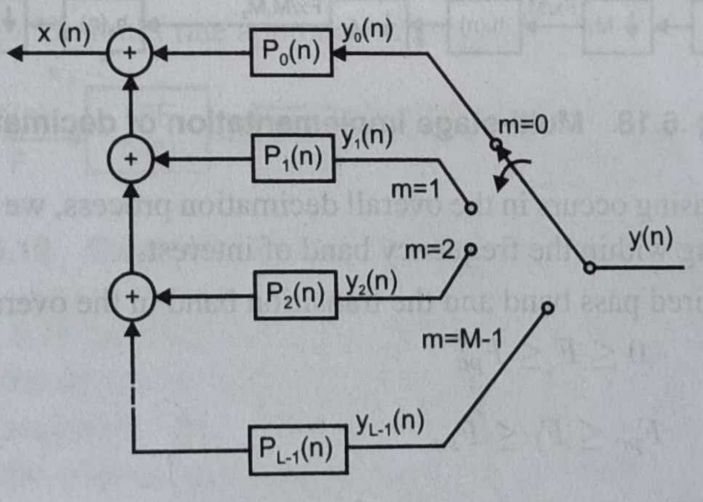


Fig .6.16 Polyphase interpolator with a commutator

6.8 Multistage Implementation of Sampling Rate Conversion

If the decimation factor M and / or interpolation factor L are much larger than unity, the implementation of sampling rate conversion in a single stage is computationally inefficient.

Therefore for performing sampling rate conversion for either $M >> 1$ and / or $L >> 1$ the multistage implementation is preferred.

If the interpolation factor $L >> 1$, then L is expressed into a product of positive integers as

$$L = \prod_{i=1}^{N} L_i$$

Then each interpolator L_i is implemented and cascaded to get N stages of interpolation and filtering as shown in Fig.6.17.

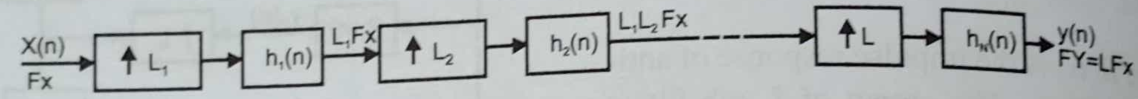


Fig .6.17 Multistage implementation of interpolation

Similarly if the decimation factor $M >> 1$ then M is expressed into a product of positive integers as

$$M = \prod_{i=1}^{N} M_i$$

Each decimator M_i is implemented and cascaded to get N stages of filtering and decimators as shown in Fig.6.18.

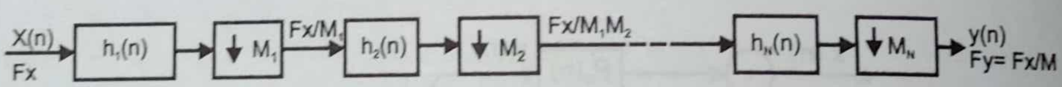


Fig .6.18 Multi-stage implementation of decimation

To ensure that no aliasing occurs in the overall decimation process, we can design each filter stage to avoid aliasing within the frequency band of interest.

Let us define the desired pass band and the transition band in the overall decimator as

Pass band: $0 \leq F \leq F_{pc}$

Transition band: $F_{pc} \leq F_1 \leq F_{sc}$

Where $F_{sc} \leq \frac{F_x}{2D}$. Then, aliasing in the band $0 \leq F \leq F_{sc}$, is avoided by selecting the frequency bands of each filter stage as follows:

Pass band: $0 \leq F \leq F_{pc}$

Transition band: $F_{pc} \leq F_1 \leq F_t - F_{sc}$

Stop band: $F_1 - F_{sc} \leq F \leq \frac{F_1 - 1}{2}$

For example, in the first filter stage we have $F_l = \dfrac{F_x}{D_1}$, and the filter is designed to have the following frequency bands:

Pass band: $0 \le F \le F_{pc}$

Transition band: $F_{pc} \le F \le F_t - F_{sc}$

Stop band: $F_1 - F_{sc} \le F \le \dfrac{F_0}{2}$

After decimation by D_l, there is aliasing from the signal components that fall in the filter transition band. But the aliasing occurs at frequencies above F_{sc}. Thus there is no aliasing in the frequency band $0 \le F \le F_{sc}$. By designing the filters in the subsequent stages to satisfy the specifications given in the above equation. we ensure that no aliasing occurs in the primary frequency band $0 \le F \le F_{sc}$.

6.9 Applications of Multi Rate Signal Processing

There are numerous practical applications of Multi rate signal processing. In this section we describe a few of these applications.

.9.1 Implementation of Narrow Band LPF

A narrow band LPF is characterized by a narrow pass band and a narrow transition band. It requires very large number of coefficients. Due to high value of N it is susceptible to finite word length effects. In addition the number of computations and memory locations required are very high. So multi rate approach of designing LPF overcomes this problem.

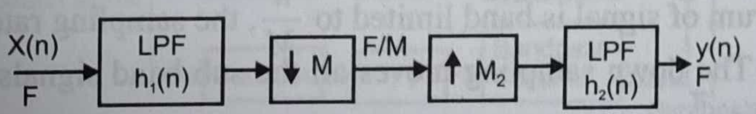


Fig .6.19 Cascaded stage of interpolator and decimator

In the above Fig 6.19 the interpolator and decimator are in cascaded stage. The filters $h_1(n)$ and $h_2(n)$ in the decimator and interpolator are low pass filters. The sampling frequency of the input sequence is first reduced by a factor M and then low pass filtering is performed. Finally the original sampling frequency of the filtered data is obtained using interpolator.

To meet the desired specifications of a narrow band LPF, the filters $h_1(n)$ and $h_2(n)$ are identical, with pass band ripple $\dfrac{\delta p}{2}$ and stop band ripple δ_s.

6.9.2 Digital filter banks

A digital filter bank is a set of band pass filters. The digital filter banks can be classified into two types. They are,

- Analysis filter bank
- Synthesis filter bank

Analysis Filter Bank

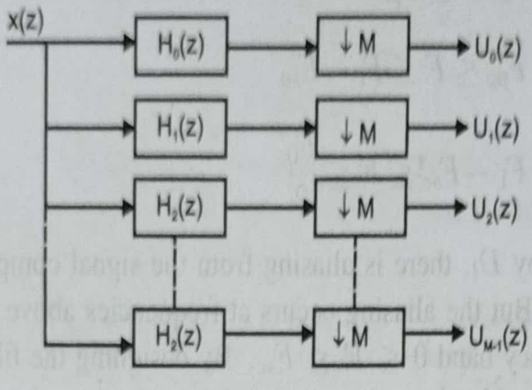


Fig .6.20 Analysis filter bank

- It consists of M sub-filters. The individual sub-filter $H_k(z)$ is known as analysis bank. An analysis filter bank is shown in Fig. 6.20.
- All the sub-filters are equally spaced in frequency and each have the same band width.
- The spectrum of the input signal $X(e^{j\omega})$ lies in the range $0 \leq \omega \leq \pi$.
- The filter bank splits the signal into number of sub bands each having a band width of $\frac{\pi}{M}$.
- The filter $H_o(z)$ is low pass, $H_1(z)$ to $H_{M-2}(z)$ are band pass and $H_{M-1}(z)$ is high pass.
- As the spectrum of signal is band limited to $\frac{\pi}{M}$, the sampling rate can be reduced by a factor M. The down sampling moves all the sub band signals into the base band range $0 \leq \omega \leq \frac{\pi}{2}$.

Synthesis Filter Bank

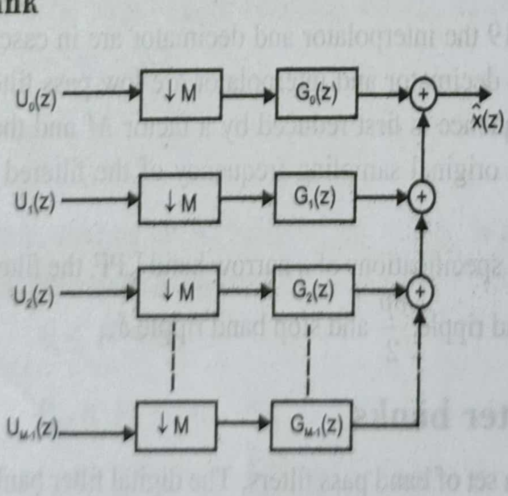


Fig .6.21 Synthesis Filter Bank

The M channel synthesis filter bank is dual of M channel analysis filter bank as shown in Fig.6.21. In this case $U_m(z)$ is fed to an up sampler. The up sampling process produces the signal $U_m(z^M)$.

These signals are applied to filters $G_m(z)$ and finally added to get the output signal $\hat{X}(z)$. The filters $G_0(z)$ to $G_{M-1}(z)$ have the same characteristics as the analysis filters $H_0(z)$ to $H_{M-1}(z)$.

6.9.3 Sub Band Coding of Speech Signals

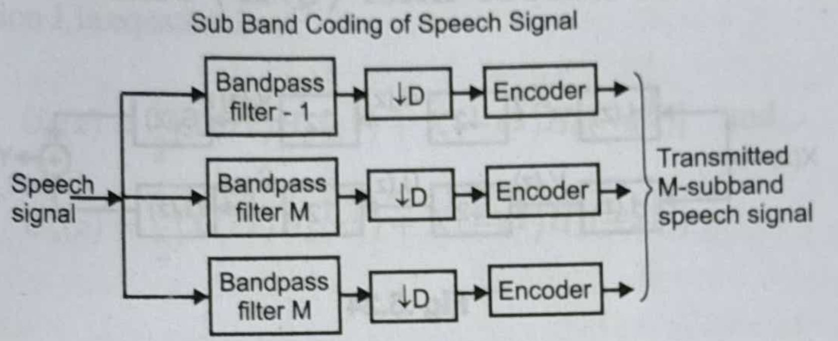


Fig .6.22 Sub band coding of speech signals

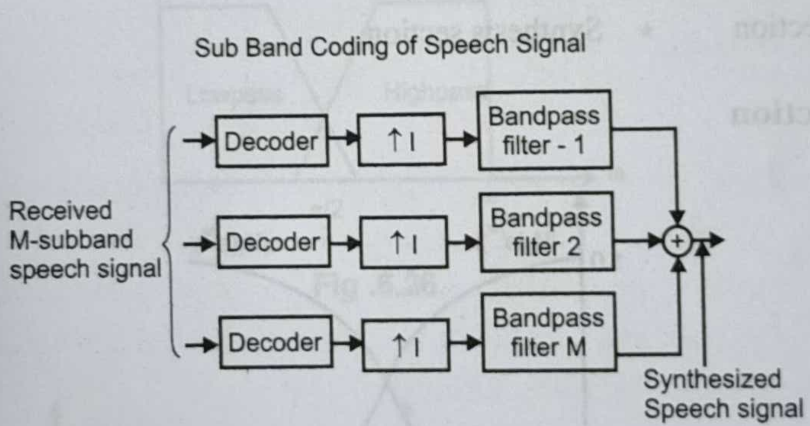


Fig .6.23 Sub band coding of speech signal

In sub band coding of speech signals, the speech signal is divided into sub bands, decimated, encoded and transmitted to the receiver system.

On the receiver side the sub band signals are decoded, interpolated and synthesized into the original speech signal.

In the transmission side,the input signal is split into M numbers of non-overlapping frequency bands using an analysis filter bank consisting of $M-$numbers of band pass filters.

The output of each band pass filter is decimated by a factor of D. The output of decimators are encoded and transmitted as shown in Fig.6.22.

On the reception side, the received sub band signals are decoded and then interpolated to recover the missing samples.

The output of interpolators are applied to a synthesis filter bank consisting of M–numbers of bandpass filters to recover the original signal as shown in Fig.6.23.

In sub band coding process,the following components are used,

- Filter Bank
- Coder
- Multiplexer
- Demultiplexer
- Decoder

6.10 Quadrature mirror filter (QMF) bank

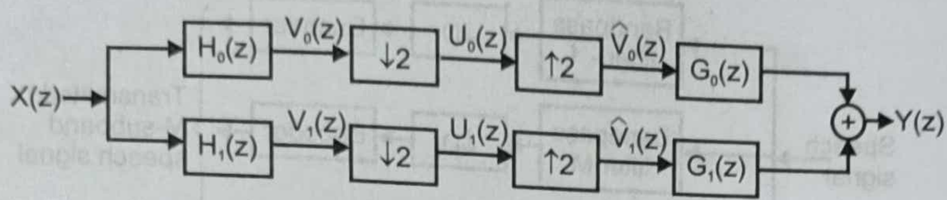


Fig .6.24

It is a two-channel sub band coding filter bank with complementary frequency responses as shown in Fig.6.24. It consists of two sections.

⋆ Analysis section ⋆ Synthesis section

Analysis section

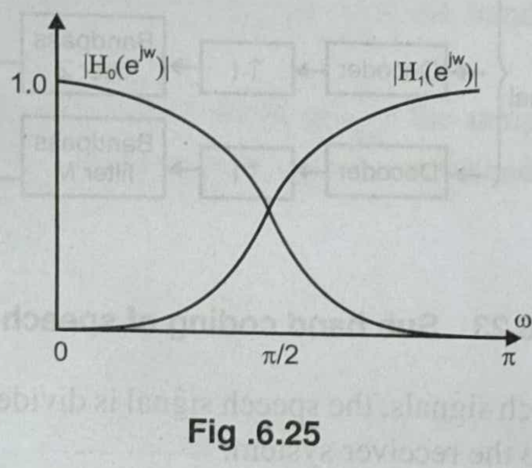


Fig .6.25

- The analysis section is a two channel analysis filter bank as shown in Fig.6.25.
- The signal $x(n)$ is fed to a LPF $H_0(z)$ and a HPF $H_1(z)$ simultaneously. Hence the input signal $x(n)$ is decomposed into high frequency component and low frequency component
- Since the normalized frequency range is $\omega = 0$ and $\omega = \pi$, the out off frequency of HPF and LPF are chosen as $\frac{\pi}{2}$.

The output of low pass and high pass filters are

$$V_0(z) = X(z)H_0(z) \quad \text{and} \quad V_1(z) = X(z)H_1(z) \tag{1}$$

Down sampling with $M = 2$, yields the sub band signals

$$U_0(z) = \frac{1}{2}[V_0(z^{\frac{1}{2}}) + V_1(-z^{\frac{1}{2}})] \quad \text{and} \quad U_1(z) = \frac{1}{2}[V_1(z^{\frac{1}{2}}) + V_1(-z^{\frac{1}{2}})] \tag{2}$$

Substitute equation 1 in equation 2

$$U_0(z) = \frac{1}{2}[X(z^{\frac{1}{2}})H_0(z^{\frac{1}{2}}) + X(-z^{\frac{1}{2}})H_0(-z^{\frac{1}{2}})] \quad \text{and}$$

$$U_1(z) = \frac{1}{2}[X(z^{\frac{1}{2}})H_1(z^{\frac{1}{2}}) + X(-z^{\frac{1}{2}})H_1(-z^{\frac{1}{2}})]$$

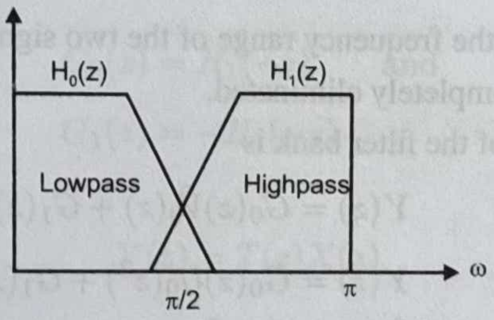


Fig .6.26

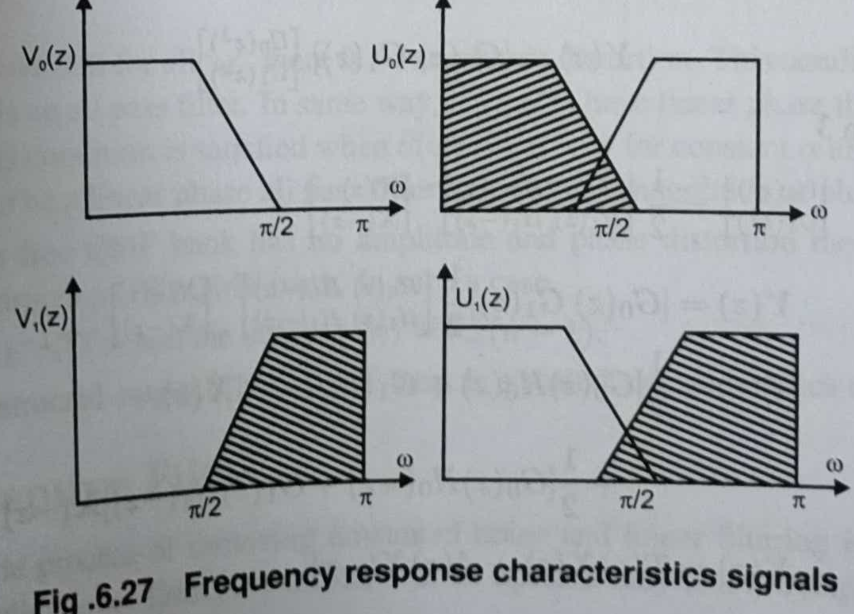


Fig .6.27 Frequency response characteristics signals

In matrix form

$$\begin{bmatrix} U_0(z) \\ U_1(z) \end{bmatrix} = \frac{1}{2}\begin{bmatrix} H_0\left(z^{\frac{1}{2}}\right) & H_0\left(-z^{\frac{1}{2}}\right) \\ H_1\left(z^{\frac{1}{2}}\right) & H_1\left(-z^{\frac{1}{2}}\right) \end{bmatrix}\begin{bmatrix} X\left(z^{\frac{1}{2}}\right) \\ X\left(-z^{\frac{1}{2}}\right) \end{bmatrix} \tag{6.3}$$

Fig.6.26 and Fig.6.27 shows the frequency response characteristic of the signal. $x(n)$ is a white noise input signal. The frequency spectra of $V_0(z)$ have two components one is the original spectrum that depends on $X(z^{1/2})$ lies in the baseband and the other is periodic repetition that is function of $x(z^{1/2})$. The high pass signal $U_1(z)$ drops into the baseband $0 \leq \omega \leq \pi$ and is reversed in frequency. Since the filtered signals are not properly band limited to π, alias signals appear in baseband.

Synthesis section

The signals $U_0(z)$ and $U_1(z)$ are fed to the synthesis filter bank. Here the signals $U_0(z)$ and $U_1(z)$ are up sampled and then passed through two filters $G_0(z)$ and $G_1(z)$ respectively. The filter $G_0(z)$ is a low pass filter and eliminates the image spectrum of $U_0(z)$ in the range $\frac{\pi}{2} < \omega < \pi$. Meanwhile the high pass filter $G_1(z)$ eliminates most of the image spectra in the range $0 \leq \omega \leq \frac{\pi}{2}$. As the frequency range of the two signals $U_0(z)$ and $U_1(z)$ overlap, the image spectra is not completely eliminated.

The reconstructed output of the filter bank is

$$Y(z) = G_0(z)\hat{V}_0(z) + G_1(z)\hat{V}_1(z)$$

$$Y(z) = G_0(z)U_0(z^2) + G_1(z)U_1(z^2) \tag{6.4}$$

Where $\hat{V}_0(z) = U_1(z^2)$

$$\hat{V}_1(z) = U_1(z^2)$$

Equation 4 can be written in matrix form as

$$Y(z) = [G_0(z)\ G_1(z)]\begin{bmatrix} U_0(z^2) \\ U_1(z^2) \end{bmatrix}$$

From equation 3

$$\begin{bmatrix} U_0(z^2) \\ U_1(z^2) \end{bmatrix} = \frac{1}{2}\begin{bmatrix} H_0(z) & H_0(-z) \\ H_1(z) & H_1(-z) \end{bmatrix}\begin{bmatrix} X(z) \\ X(-z) \end{bmatrix}$$

$$Y(z) = [G_0(z)\ G_1(z)]\frac{1}{2}\begin{bmatrix} H_0(z) & H_0(-z) \\ H_1(z) & H_1(-z) \end{bmatrix}\begin{bmatrix} X(z) \\ X(-z) \end{bmatrix}$$

$$Y(z) = \frac{1}{2}[G_0(z)H_0(z) + G_1(z)H_1(z)]X(z)$$

$$+ \frac{1}{2}[G_0(z)H_0(-z) + G_1(z)H_1(-z)]X(-z)$$

$$Y(z) = T(z)X(z) + A(z)X(-z) \tag{6.5}$$

Where

$$T(z) = \frac{1}{2}[G_0(z)H_0(z) + G_1(z)H_1(z)]$$

$$A(z) = \frac{1}{2}[G_0(z)H_0(-z) + G_1(z)H_1(-z)]$$

The function $T(z)$ describes the transfer function of the filter and is called distortion transfer function. The function $A(z)$ is due to aliasing components.

Alias free filter bank

To obtain an alias free filter bank, we can choose the synthesis filter such that $A(z) = 0$.

$$\text{i.e.}\quad A(z) = \frac{1}{2}[G_0(z)H_0(-z) + G_1(z)H_1(-z)] = 0$$

$$G_0(z)H_0(-z) + G_1(z)H_1(-z) = 0$$

A simple sufficient condition for alias cancellation is

$$G_0(z) = H_1(-z) \quad \text{and}$$

$$G_1(z) = -H_0(-z)$$

Then equation 5 becomes

$$Y(z) = T(z)X(z)$$

Substituting $z = e^{j\omega}$ yields

$$Y(e^{j\omega}) = T(e^{j\omega})\, X(e^{j\omega})$$

$$= |T(e^{j\omega})|e^{j\theta(\omega)}\, X(e^{j\omega})$$

If $|T(e^{j\omega})|$ is constant for all 'ω', there is no amplitude distortion. This condition is satisfied when $T(e^{j\omega})$ is an all pass filter. In same way, if $T(e^{j\omega})$ have linear phase there is no phase distortion. This condition is satisfied when $\theta(\omega) = \alpha\omega + \beta$ for constant α and β. Therefore $T(e^{j\omega})$ need to be a linear phase all pass filter to avoid any magnitude or phase distortion.

If an alias free QMF bank has no amplitude and phase distortion then it is called a perfect reconstruction (PR) QMF bank. In such a case.

$Y(z) = kz^{-1}X(z)$ and the output $y(n) = kx(n-1)$.

i.e., the reconstructed output of a PRQMF bank is a scaled, delayed replica of the output.

6.11 Adaptive Filter

Filtering is the process of removing unwanted noise and linear filtering is required in a variety of applications. Generally a filter will be optimal only if it is designed with some knowledge about the input data. If this information is not known, then adaptive filters are used. The adjustable parameters in filter are assigned with values based on the estimated statistical nature of the signals. So, adaptive filters are adaptable to the changing environment.

"A filter with adjustable coefficients is called adaptive filter." It is shown in Fig.6.28.

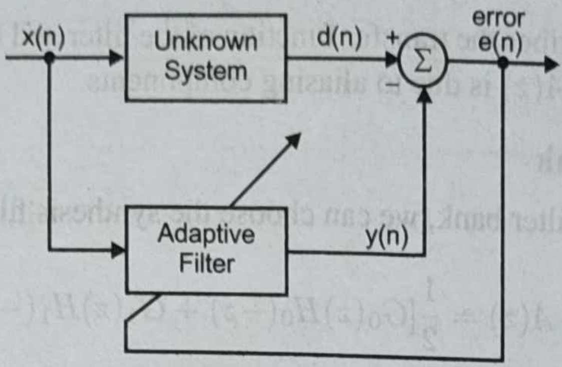


Fig .6.28 Adaptive Filter

Where

$x(n)$ is the signal,

$d(n)$ is the response signal of the unknown system.

$y(n)$ is the output signal from the adaptive filter.

$e(n)$ is the error signal. i.e $e(n) = d(n) - y(n)$.

During each iteration, the adaptive filter adjusts the filter coefficients to minimize $e(n)$.

Applications of Adaptive Filter

Adaptive filters are having the following application areas,

- Adaptive Noise Cancellation
- Frequency Tracking
- Adaptive Channel Equalization
- Adaptive Echo Cancellation
- Adaptive Line Enhancing

6.12 Examples of Adaptive Filtering

The FIR filter in direct form with adjustable coefficients $h(0), h(1) \ldots h(M-1)$ is shown in Fig. 6.29.

In adaptive filtering, the adjustable filter parameters are to be optimised. The criteria arrived at for optimization should consider the filter performance and realisability.

The following examples use the direct form FIR filter structure.

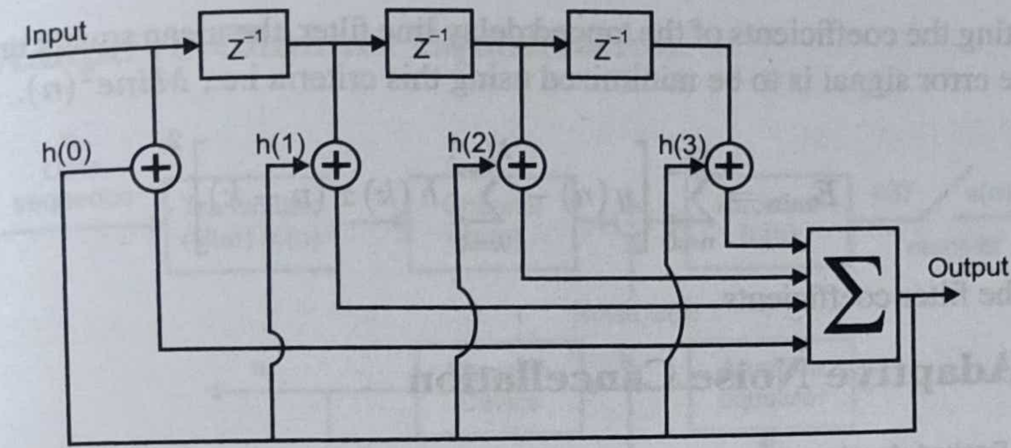


Fig .6.29 Direct - Form adaptive FIR filter

6.12.1 System Modelling

Consider an unknown system which has to be identified. The model of the system is shown below in Fig.6.30.

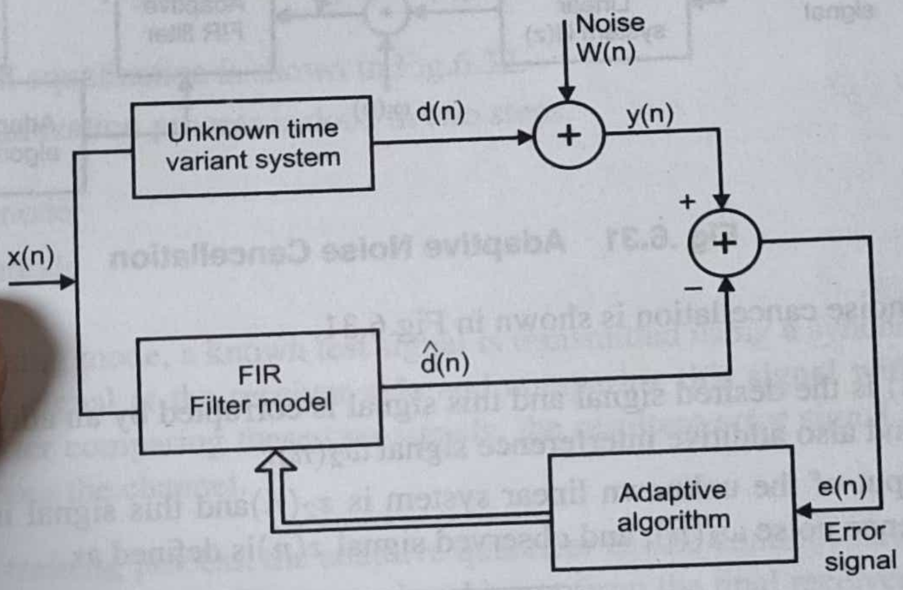


Fig .6.30 System Identification

The requirement is to develop an FIR filter i.e tapped delay line filter with adjustable coefficients. Let the unknown system and the FIR filter be excited by the input signal $x(n)$, the output from the dynamic system be $y(n)$ and $\hat{y}(n)$ be the output of the FIR filter.

$$\hat{y}(n) = \sum_{k=0}^{M-1} h(k)\, x(n-k)$$

The error signal is given by

$$e(n) = y(n) - \hat{y}(n)$$

For calculating the coefficients of the tapped delay line filter, the mean square error criterion is used. The error signal is to be minimized using this criteria i.e., $Mine^2(n)$.

$$E_{\min} = \sum_{n=0}^{\infty} \left[y(n) - \sum_{k=0}^{M-1} h(k)\, x(n-k) \right]^2$$

and select the filter coefficients.

6.12.2 Adaptive Noise Cancellation

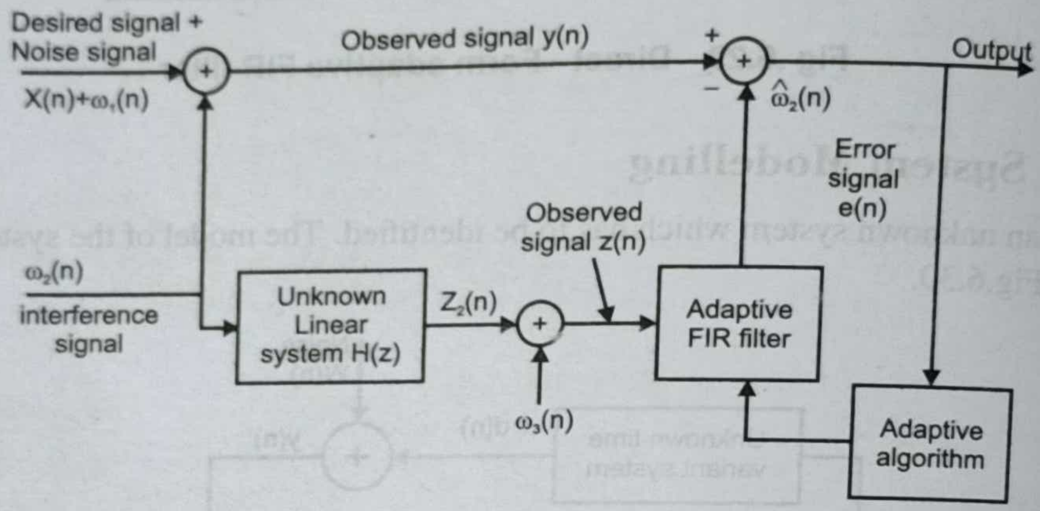


Fig .6.31 Adaptive Noise Cancellation

Adaptive noise cancellation is shown in Fig.6.31.

- Let $x(n)$ is the desired signal and this signal is corrupted by an additive noise signal $\omega_1(n)$ and also additive interference signal $\omega_2(n)$.
- The output of the unknown linear system is $z_2(n)$and this signal is added with the interference noise $\omega_3(n)$, and observed signal $z(n)$is defined as

$$\boxed{z(n) = z_2(n) + \omega_3(n)}$$

- The estimate of the interference signal $\omega_2(n)$ is obtained from an FIR filter. The estimate of the desired signal $x(n)$ is the error signal given by

$$e(n) = y(n) - \hat{\omega_2}(n)$$

$$= y(n) - \sum_{k=0}^{M-1} h(k) z(n-k)$$

Where $\hat{\omega_2}(n)$is the estimate of $\omega_2(n)$.

- The error signal is used for adjusting the FIR filter coefficients.

6.12.3 Adaptive Channel Equalization

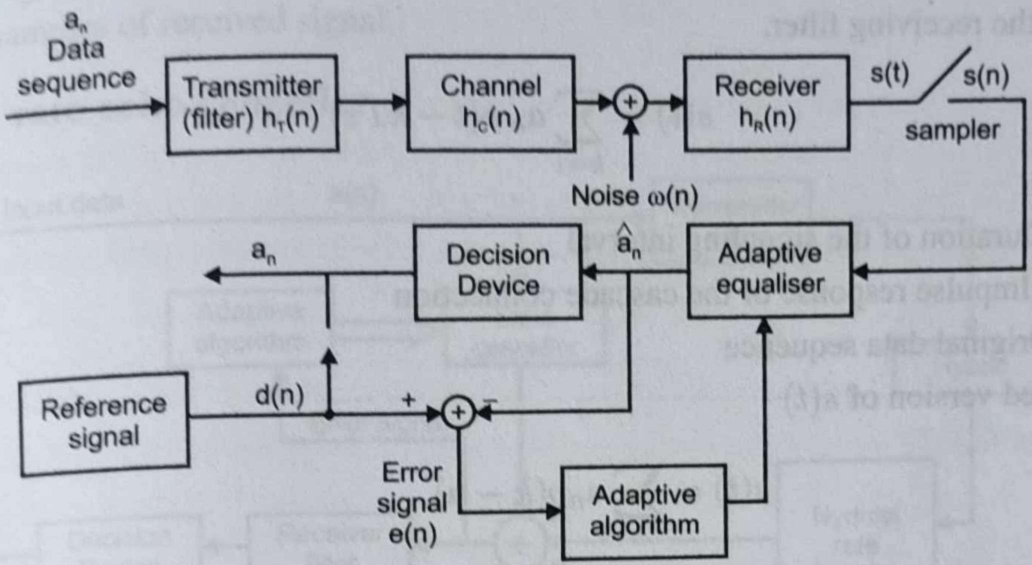


Fig .6.32 Adaptive Channel Equalization

Adaptive channel equalization is shown in Fig.6.32.

The adaptive equalization process is done in two steps.

1. Training mode
2. Tracking mode

- In the training mode, a known test signal is transmitted using a synchronized version of the test signal at the receiver side and comparing this signal with the receiving signal. After comparing these two signals, the resultant error signal gives the information about the channel.
- After the training process, the adaptive quantizer can be continuously adjusted in the decision directed mode. The error signal is got from the final receiver estimate.
- The output of the adaptive equalizer is send to the decision device receiver to get the estimate.
- The error signal estimate is used to adjust the coefficients of the adaptive equalizer.
- The bandwidth of the channel has to be used efficiently. So the basic requirement is to design a reliable system with data to be transmitted at a higher rate.
- There are two factors affect the data in the channel such as

  (i) Inter-symbol interference

  (ii) Thermal noise

Adaptive equalizer is mainly used for compensating the channel distortion so that the detected signal will be reliable.

Output of the receiving filter,

$$s(t) = \sum_{k=1} a_k\, p(t - kTs)$$

Ts – Duration of the signaling interval
$p(t)$ – Impulse response of the cascade connection
a_k – Original data sequence
Sampled version of $s(t)$

$$s(t) = \sum_{n} a_n p(k - n)$$

$$s(t) = a_k p(0) + \sum_{\substack{n \\ n \neq k}} a_n\, p(k - n)$$

6.12.4 Adaptive Echo Cancellation

In any kind of signal transmission, we will get noise, it may be in the form of echo also. The echo signal will reduce the performance of channel and increase the bandwidth, so the echo signal has to be reduced.

The echo signal can be reduced by using echo suppressor/echo canceller. Echo canceller is having different kinds of configurations.

1. Symbol rate echo canceller
2. Nyquist rate echo canceller

Symbol Rate Echo Canceller

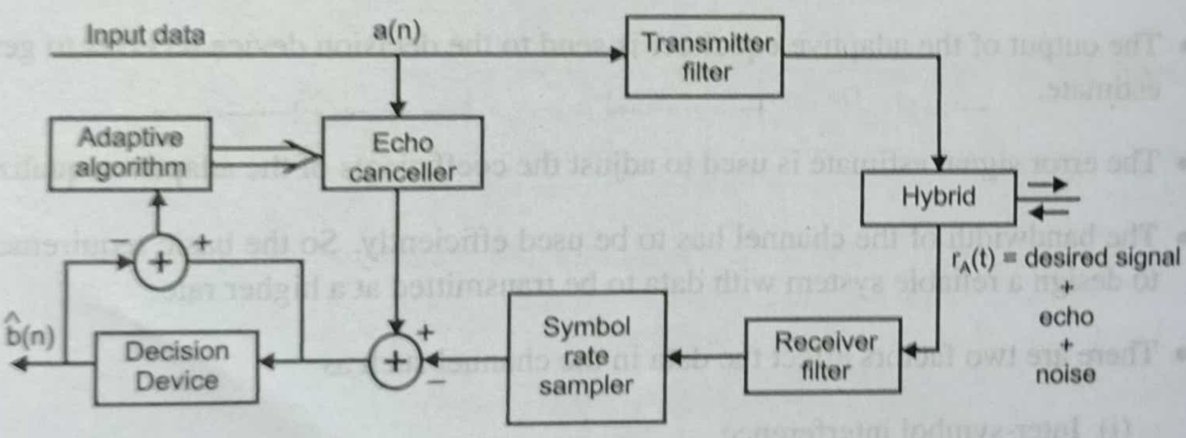


Fig .6.33 Symbol rate echo canceller

symbol rate echo canceller is shown in Fig.6.33.

In the symbol rate echo canceller symbol rate is considered. It will subtract the noise om the samples of received signal.

yquist rate echo canceller

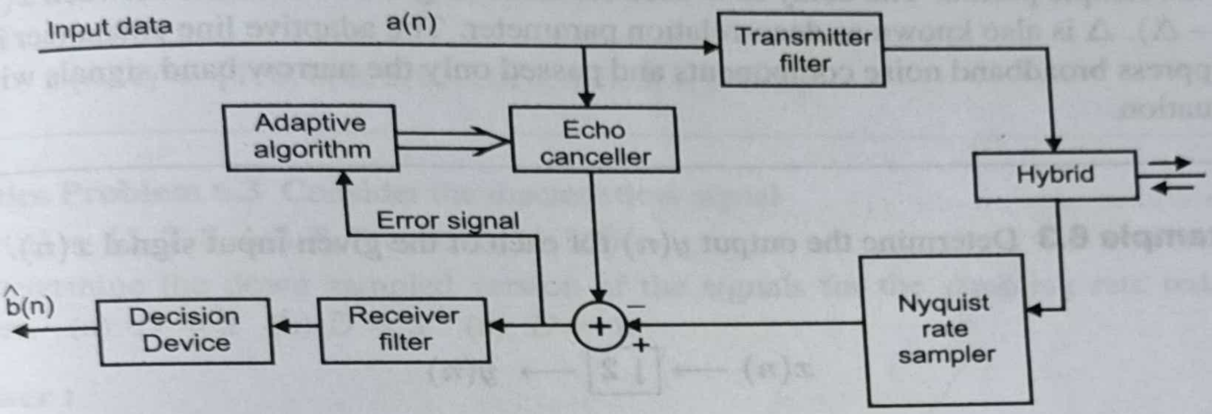


Fig .6.34 Nyquist rate echo canceller

Nyquist rate echo canceller is also having the same diagram of symbol rate echo canceller as shown in Fig.6.34. Instead of symbol rate sampler circuit, we can use the Nyquist rate sampler circuit. The difference between the two is in the Nyquist rate echo cancellers, the echo canceller works at the Nyquist rate, but in the symbol rate echo canceller symbol rate is considered.

6.12.5 Adaptive line enhancer

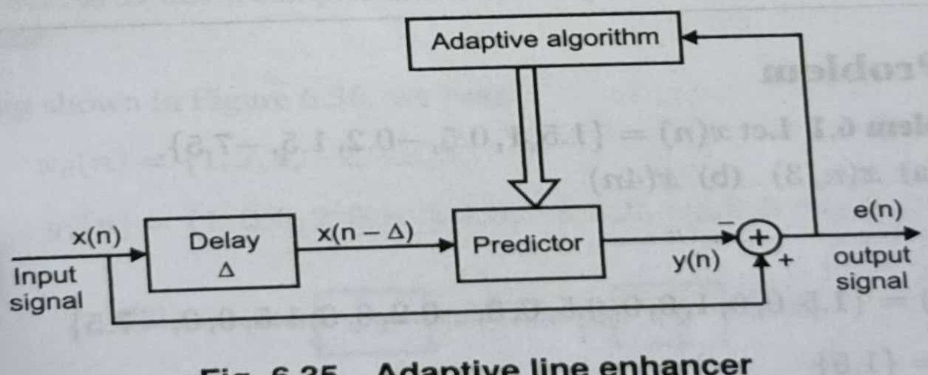


Fig .6.35 Adaptive line enhancer

An adaptive line enhancer is shown in Fig.6.35.

Let the input signal be $x(n)$.

The signal is passed through a delay element with a delay Δ,and its output is $x(n-\Delta)$.

This is passed through a predictor.

The predictor output $y(n)$ is subtracted from the input signal resulting in the error signal

$$e(n) = x(n) - y(n)$$

The error signal controls the predictor coefficients. The delay Δ is equal to or greater than the sample period. The delay Δ is used for removing the correlation between $x(n)$ and $x(n-\Delta)$. Δ is also known as decorrelation parameter. The adaptive line enchancer is used to suppress broadband noise components and passed only the narrow band signals with less attenuation.

Example 6.3 Determine the output $y(n)$ for each of the given input signal $x(n)$.

$$x(n) \longrightarrow \boxed{\downarrow 2} \longrightarrow y(n)$$

(i) $x(n) = \delta(n-4)$

Answer : $y(n) = x(2n) = \delta(2n-4)$

(ii) $x(n) = exp(j0.2n\Pi)u(n)$

Answer :

$$y(n) = x(2n)$$
$$= e^{j0.2\times 2n\pi}u(2n) = e^{j0.4n\pi} \quad \text{at} \quad n = 0, 2, 4, 6,$$

Practice Problem

Practice Problem 6.1 Let $x(n) = \{1.5, 1, 0.5, -0.2, 1.5, -7.5\}$.

Compute (a) $x(n/3)$ (b) $x(4n)$

Answer :

(a) $x(n/3) = \{1.5, 0, 0, 1, 0, 0, \underset{\uparrow}{0.5}, 0, 0, -0.2, 0, 0, 1.5, 0, 0, -7.5\}$

(b) $x(4n) = \{\underset{\uparrow}{1.5}\}$

Practice Problem 6.2 Consider the discrete time signal $x(n) = \{1, 2, 3, 4\}$

Determine the upsampled version of the signals for the sampling rate multiplication factor, (a) $I = 2$ (b) $I = 3$ (c) $I = 4$.

Answer :

(a) $I = 2$

$$x(n/2) = \{1, 0, 2, 0, 3, 0, 4, 0\}$$

(b) $I = 3$

$$x(n/3) = \{1, 0, 0, 2, 0, 0, 3, 0, 0, 4, 0, 0\}$$

(c) $I = 4$

$$x(n/4) = \{1, 0, 0, 0, 2, 0, 0, 0, 3, 0, 0, 0, 4, 0, 0, 0\}$$

Practice Problem 6.3 Consider the discrete time signal

$$x(n) = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}$$

Determine the down sampled version of the signals for the sampling rate reduction factors. (a) $D = 2$ (b) $D = 3$ (c) $D = 4$.

Answer :

(a) $D = 2$

$$x(2n) = \{1, 3, 5, 7, 9, 11\}$$

(b) $D = 3$

$$x(3n) = \{1, 4, 7, 10\}$$

(c) $D = 4$

$$x(4n) = \{1, 5, 9\}$$

---

mple 6.4 Considering an example $x(n) = \{1, 3, 2, 5, 4, -1, -2, 6, -3, 7, 8, 9, \cdots\}$ w that a cascade of D down sampler and I up sampler is interchangeable only when D and I are co-prime.

For the cascading shown in Figure 6.36. we have

$$x_d(n) = \{1, 2, 4, -2, -3, 8, \ldots\}$$

$$y_1(n) = \{1, 0, 0, 2, 0, 0, 4, 0, 0, -2, 0, 0, -3, 0, 0, 8 \ldots\}$$

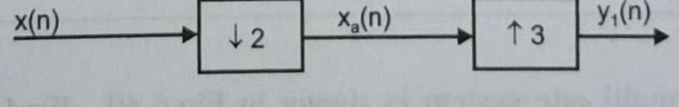


Fig .6.36 Cascading of $D = 2$ and $I = 3$

Interchanging the cascading as shown in Figure.6.37. we have

$$x_u(n) = \{1, 0, 0, 3, 0, 0, 2, 0, 0, 5, 0, 0, 4, 0, 0, -1, 0, 0, -2, 0, 0, 6, 0, 0, -3, 0, 0, 7, 0, 0, 8 \ldots\}$$

$$y_2(n) = \{1, 0, 0, 2, 0, 0, 4, 0, 0, -2, 0, 0, -3, 0, 0, 8 \ldots\}$$

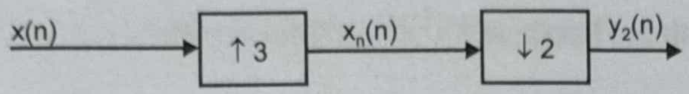


Fig .6.37 Cascading of $I = 3$ and $D = 2$

Now $y_1(n) = y_2(n)$. This shows that the cascade of an I up sampler and a D down sampler are interchangeable when I and D are co-prime.

Let $D = 2$ and $I = 4$. Here D and I are not co-prime.

For the cascading shown in Figure.6.38 we have

$$x_d(n) = \{1, 2, 4, -2, -3, 8, \ldots\}$$
$$y_3(n) = \{1, 0, 0, 0, 2, 0, 0, 0, 4, 0, 0, 0, -2, 0, 0, 0, -3, 0, 0, 0, -8 \ldots\}$$

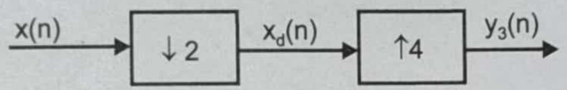


Fig .6.38 Cascading of $D = 2$ and $\ell = 4$

Interchanging the cascading as shown in Figure.6.39. we have

$$x_u(n) = \{1, 0, 0, 0, 3, 0, 0, 0, 2, 0, 0, 0, 5, 0, 0, 0, 4, 0, 0, 0 - 1, \ldots\}$$
$$y_4(n) = \{1, 0, 3, 0, 2, 0, 5, 0, 4, 0, -1 \ldots\}$$

Now, $y_3(n) \neq y_4(n)$.

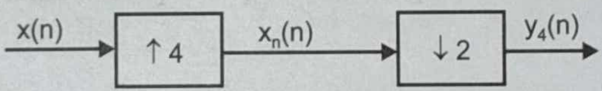


Fig .6.39 Cascading of $I = 3$ and $D = 2$

This shows that the cascading of up sampler and down sampler is not interchangeable when D and I are not co-prime. i.e., when D and I have a common factor.

■ Example 6.5 A multi rate system is shown in Fig.6.40. Find the relation between $x(n)$ and $y(n)$.

In fig. after down sampling $x(n)$, we get

$$V(n) = x(2n)$$

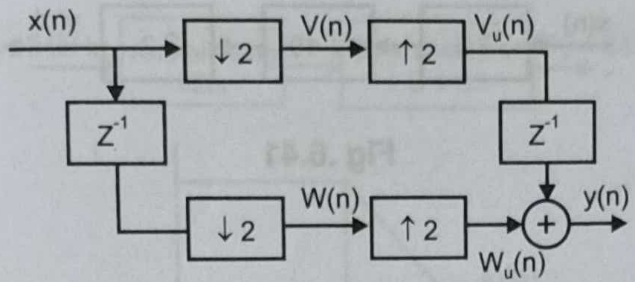


Fig .6.40

If use up sample $V(n)$ by 2, we get

$$V_u(n) = v\left(\frac{n}{2}\right) = \begin{cases} x(n), & \text{for } n=0,\pm 2,\pm 4 \ldots\ldots \\ 0 & \text{otherwise} \end{cases}$$

If we delay $x(n)$ and down sample, we get

$$V(n) = \{x(0), x(2), x(4), x(6) \ldots\ldots\}$$

$$V_u(n) = \{x(0), 0, x(2), 0, x(4), 0, x(6) \ldots\ldots\}$$

$$V_u(n-1) = \{0, x(0), 0, x(2), 0, x(4), 0, x(6) \ldots\ldots\} \quad (1)$$

$$W(n) = \{x(-1), x(1), x(3), x(5), x(7) \ldots\ldots\}$$

$$W_u(n) = \{x(-1), 0, x(1), 0, x(3), 0, x(5), 0 \ldots\ldots\} \quad (2)$$

If we up sample $w(n)$, we get

$$W(n) = x(2n-1)$$

$$W_u(n) = W\left(\frac{n}{2}\right) = \begin{cases} x(n-1), & \text{for } n=0,\pm 2,\pm 4 \ldots\ldots \\ 0 & \text{otherwise} \end{cases}$$

$$y(n) = V_u(n-1) + W_u(n)$$

$$V_u(n) = \{x(0), 0, x(2), 0, x(3) \ldots\}$$

$$V_u(n-1) = \{0, x(0), 0, x(2), 0, x(3), \ldots\}$$

$$W_u(n) = \{x(-1), 0, x(1), 0, x(3), 0 \ldots\}$$

$$W_u(n) + V_u(n-1) = \{x(-1), x(0), x(1), x(2) \ldots\}$$

$$y(n) = x(n-1)$$

Result:

$$(1) + (2) = \{x(-1), x(0), x(1), x(2), x(3) \ldots\ldots\}$$

■ **Example 6.6** Develop expression for the output $y(n)$ as a function of the input $x(n)$ for the multi rate structure of Figure.6.41.

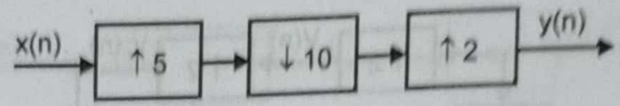


Fig .6.41

Solution

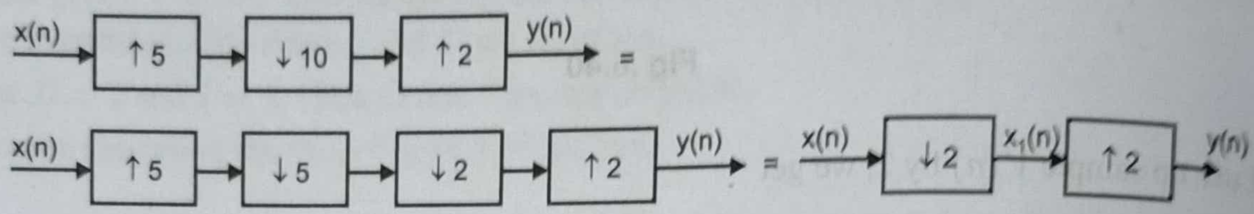


Fig .6.42

Hence $x_1[n] = x[2n]$ and $y[n] = \begin{cases} x_1[n/2], & \text{for } n = 2r \\ 0, & \text{otherwise} \end{cases} = \begin{cases} x[n] & \text{for } n = 2r \\ 0, & \text{otherwise} \end{cases}$

Therefore, $$y[n] = \begin{cases} x_1[n], & \text{for } n = 2r \\ 0, & \text{otherwise} \end{cases}$$

Practice Problem 6.4 For the multi-rate system shown in Fig.6.43 develop an expression for the output $y(n)$ as a function of the input $x(n)$.

$$x(n) \longrightarrow \boxed{\uparrow 4} \longrightarrow \boxed{\downarrow 12} \longrightarrow \boxed{\uparrow 3} \longrightarrow y(n)$$

Fig.6.43

Answer :

$$y(n) = \begin{cases} x(n) & \text{for } n = 3k \\ 0 & \text{otherwise} \end{cases}$$

Practice Problem 6.5 A signal $x(n)$, at a sampling frequency of 2.048 KHz is to be decimated by a factor of 32 to yield a signal at sampling frequency of 64 Hz. The signal band of interest extends from 0 to 30 Hz. The anti- aliasing filter should satisfy the following specifications.

Passband deviation 0.01 dB

Stopband deviation 80dB

Pass band 0 - 30 Hz

Stop band 32 - 64Hz

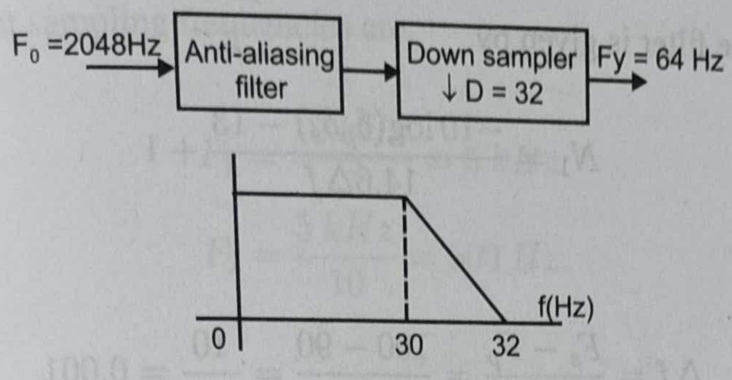


Fig .6.44

The signal component in the range from 30 to 32 Hz should be protected from aliasing. Design a suitable one - stage decimator.

Answer:

$$\Delta\ f = 0.0009765$$
$$N = 3979$$

Example 6.7 Design one-stage and two stage interpolators to meet the following specifications $I = 20$

(a) Pass band : $0 \leq F \leq 90$

(b) Transition band: $90 \leq F \leq 100$

(c) Input sampling rate : $10,000Hz$

(d) Ripple : $\delta_p = 10^{-2}, \delta_s = 10^{-3}$.

Solution:

Interpolation factor $I = 20$
Pass band frequency $F_p = 90Hz.$
Stop band frequency $F_s = 100Hz.$
Input sampling rate $F_{\text{in}} = 10kHz.$
Pass band ripple $\delta_p = 10^{-3}$
Stop band ripple $\delta_s = 10^{-3}$

Single stage interpolator

To design an interpolator, initially design a decimator with a same specifications and then transpose the design to get on interpolator structure.

$$\therefore\ \ D = 20$$

Then, the order of the filter is given by

$$N_1 = \frac{-10\log(\delta_p\delta_s) - 13}{14.6\Delta f} + 1$$

Where

$$\Delta f = \frac{F_s - F_p}{F_{\text{in}}} = \frac{100 - 90}{10k} = \frac{10}{10k} = 0.001$$

$$N_1 = \frac{-10\log(10^{-2} \times 10^{-3}) - 13}{(14.6)(0.001)} + 1$$

$$= \frac{50 - 13}{0.0146} + 1 = 2535.25$$

Rounding the order to the next higher integer value, we get $N_1 = 2536$.

The output sampling freq is given by

$$F_0 = \frac{F_{\text{in}}}{D} = \frac{10k}{20} = 500Hz.$$

Then, the single stage implementation of decimation by a factor. $D = 20$ is shown in Fig.6.45.

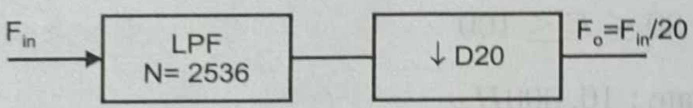


Fig .6.45 Single stage decimator

The transpose Figure 6.45 gives the single stage interpolator ekt of $I = 20$ as shown in Fig.6.46.

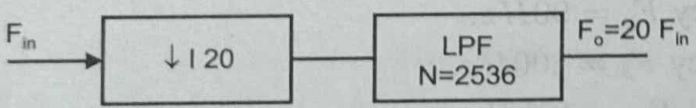


Fig .6.46 Single stage interpolator

Two stage interpolator

Consider two decimation factors as

$$D_1 = 2, \quad D_2 = 10$$

The respective output sampling frequencies are,

$$F_1 = \frac{10\,kHz}{2} = 5\,kHz$$
$$F_2 = \frac{5\,kHz}{10} = 500\,Hz.$$

Fist stage

$$\text{Pass band} : 0 \le F \le 90Hz$$
$$\text{Transition band} : 90Hz \le F \le F_1 - F_{sc}$$
$$: 90Hz \le F \le 4.91kHz$$

$$\Delta f_1 = \frac{4.91kHz - 90}{10kHz} = 0.482$$
$$\delta'_p = \frac{\delta_p}{2} = \frac{10^{-2}}{2} = 0.005$$

Then, the order of the filter is given by

$$N_1 = \frac{-10\,\log(\delta'_p \delta_s) - 13}{14.6 \times 0.482} + 1$$
$$= \frac{40.01}{14.6 \times 0.482} + 1$$
$$= 6.685 \simeq 7$$

Second stage:

$$\text{Pass band} : 0 \le F \le 90Hz$$

$$\text{Transition band} : 90Hz \le F \le 400Hz.$$

$$\Delta f_2 = \frac{400 - 90}{10k} = 0.031$$

$$\delta'_s = \frac{\delta_s}{2} = \frac{10^{-3}}{2}$$
$$= 0.5 \times 10^{-3}$$

Then the order of the filter is

$$N_2 = \frac{-10\ \log(\delta_p \delta_s') - 13}{14.6\Delta f_2} + 1$$

$$= \frac{-10 \log(10^{-2} \times 0.5 \times 10^{-3}) - 13}{14.6 \times 0.031} + 1$$

$$= \frac{40.01}{14.6 \times 0.031} + 1$$

$$= 89.4 = 90$$

The second stage implementation of decimator by factors $D_1 = 2, \quad D_2 = 10$ is shown in Fig.6.47.

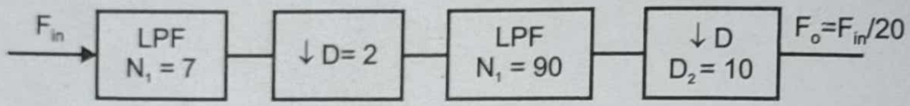


Fig .6.47 Two stage decimator

The transpose of Fig.6.47 gives two stage interpolator with $I_1 = 10$ and $I_2 = 2$ as shown in Fig.6.48.

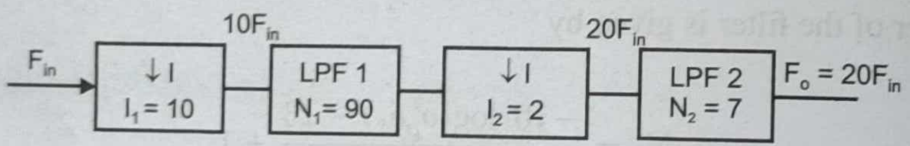


Fig .6.48 Two stage Interpolator

■ **Example 6.8** Implement a two-stage decimator for the sampling rate of the input signal equal to $20KHz$, decimator factor $M = 100$, Pass band equal to 0 to $40Hz$, transition band equal to 40 to $50Hz$, pass band ripple equal to 0.01 and stop band ripple equal to 0.002.

Answer : N = 4656, Δ & 1 = 0.0155. $N_1 = 164$, $N_2 = 845$

■ **Example 6.9** Obtain the polyphase decomposition of the IIR system with transfer function.

$$H(z) = \frac{1 - 4z^{-1}}{1 + 5z^{-1}}$$

The polyphase decomposition of a system is given by

$$H(z) = X_0(z^2) + z^{-1} X_1(z^2)$$

Where $X_0(z^2)$ and $X_1(z^2)$ are the polyphase components.

$$H(z) = \frac{1-4z^{-1}}{1+5z^{-1}}$$

Taking conjugate

$$H(z) = \frac{1-4z^{-1}}{1+5z^{-1}} \times \frac{1-5z^{-1}}{1-5z^{-1}}$$

$$= \frac{1-9z^{-1}+20z^{-2}}{1-25z^{-2}}$$

$$= \frac{1+20z^{-2}}{1-25z^{-2}} + z^{-1}\frac{-9}{1-25z^{-2}}$$

Hence the polyphase components are

$$X_0(z^2) = \frac{1+20z^{-2}}{1-25z^{-2}} \text{ and}$$

$$X_1(z^2) = \frac{-9}{1-25z^{-2}}$$

Two marks

Questions and Answers

1. **What are single-rate systems?**

   The systems that use single sampling rate from A/D converter to D/A converter are known as single rate systems.

2. **What are multi-rate systems?**

   The discrete-time systems that process data at more than one sampling rate are known as multi-rate systems.

3. **What is meant by multi-rate signal processing?** **(AU,May/June 2014)**

   The theory of processing signals at different sampling rates is called multi-rate signal processing. The process of converting a signal from a given rate to a different rate is called sampling rate conversion. Systems that employ multiple sampling rates in the processing of digital signals are called multi rate digital signal processing.

4. **What are the advantages of multi-rate signal processing?**

   - Computational requirements are less
   - Storage space for filter coefficients is less

- Finite arithmetic effects are less
- Filter order required in multi rate applications is low
- Sensitivity to filter coefficient lengths is low

5. **Define down sampling**

Down sampling a sequence x(n) by a factor M is the process of picking every M^{th} sample and discarding the rest.

(Or)

The process of reducing the sampling rate by an integer factor (D) is called decimation of the sampling rate. It is also called down sampling by factor(D).Decimator consists of decimation filter to band limit the signal and down sampler to decrease the sampling rate by an integer factor (D).

6. **What is meant by up sampling?**

Up sampling a sequence $x(n)$ by factor L is the process of inserting $L-1$ zeros between two consecutive samples.

(Or)

Increasing sampling rate of a signal by an integer factor I is known as Interpolation or up-sampling. An increase in the sampling rate by an integer factor I may be done by interpolating $(I-1)$ new samples between successive values of the signals.

7. **Define the basic operations in multi-rate signal processing.**

(AU,April/May 2015)

The basic operations in multi-rate signal processing are

(1) Decimation

(2) Interpolation

Decimation is a process of reducing the sampling rate by a factor D, i.e., down-sampling. Interpolation is a process of increasing the sampling rate by a factor I, i.e., up-sampling.

8. **Give some applications of multi rate signal processing.**

(AU,Nov/Dec 2014) (AU,Nov/Dec 2013) (AU,May/June 2013)

(1) Design of phase shifters

(2) Interfacing of digital systems with different sampling rates

(3) Implementation of narrow band LPF & implementation of Digital Filter Bank

(4) Sub band coding of speech signals & Quadrature mirror filter

(5) Trans multiplexers & Over sampling of A/D and D/A conversion

9. **Define Decimator and Interpolator.** **(AU,Nov/Dec 2014)**

Decimation of the sampling rate is the process of reducing the sampling rate by an integer factor D. It is also called down sampling by factor D.

Interpolation is the process of replacing the zero valued samples inserted by up sampler with approximated values using some type of filtering process.i.e interpolation is the complete process of up sampling and filtering to remove image spectra.

10. **If** $x(n) = \{1, -1, 3, 4, 0, 2, 5, 1, 6, 9, \ldots\ldots\}$ **Find.** $y(n) = x(2n), y(n) = x(3n)$?

$y(n) = x(2n) = \{1, 3, 0, 5, 6 \ldots\}$

$y(n) = x(3n) = \{1, 4, 5, 9 \ldots\}$

11. **If** $x(n) = \{1, 2, 3, 7, 4, -1, 5, \ldots\ldots\}$ **Find** $y(n) = x\left(\frac{n}{2}\right), y(n) = x\left(\frac{n}{3}\right)$?

$y(n) = x\left(\frac{n}{2}\right) = \{1, 0, 2, 0, 3, 0, 7, 0, 4, 0, -1, 0, 5, 0, \ldots\}$

$y(n) = x\left(\frac{n}{3}\right) = \{1, 0, 0, 2, 0, 0, 3, 0, 0, 7, 0, 0, 4, 0, 0, -1, 0, 0, 5, \ldots\}$

12. **What is a decimator ? If the input to the decimator is** $x(n) = 1, 2, -1, 4, 0, 5, 3, 2,$ **What is the output?** **(AU, Nov/Dec 2012)**

Decimation is a process of reducing the sampling rate by a factor D, i.e., down-sampling.

$x(n) = 1, 2, -1, 4, 0, 5, 3, 2$

$D = 2$

Output $y(n) = 1, -1, 0, 3$

13. **Write the need for decimation.** **(AU,May/June 2014)**

Decimation is used for reducing the sampling rate in a multi-rate digital signal processing system.

14. **Give the steps in multistage sampling rate converter design.** **(AU,Nov/Dec 2013)**

To perform sampling rate conversion for either $M >> 1$ or $I >> 1$, then only, we go for multistage implementation.

(i) If interpolation factor $I >> 1$, then express I as $I = \Sigma_{i=1}^{N} Ii$

(ii) Then each interpolator Ii is implemented and cascaded to get N stage of interpolation and filtering.

(iii) Similarly, if decimation factor $M >> 1$,then express M as $I = \Sigma_{i=1}^{N} Mi$

(iv) Then each decimator Mi is implemented and cascaded to get N stage of decimation and filtering.

15. **What is an anti-imaging filter?** **(AU,May/June 2013)**

The image signal is due to the aliasing effect. In case of decimation by M, there will be $M-1$ additional images of the input spectrum. Thus, the input spectrum $X(\omega)$ is band limited to the low pass frequency response. An anti-aliasing filter eliminates the spectrum of $X(\omega)$ in the range $(\pi/D \leq \omega \leq \pi)$.

The anti-aliasing filter is LPF whose frequency response$H_{LPF(\omega)}$ is given by

$$\mathrm{H_{LPF}}(\omega) = \begin{cases} 1, |\omega| \leqslant \frac{\pi}{\mathrm{M}} \\ = 0, \text{otherwise} \end{cases} \qquad D \rightarrow \text{Decimator}$$

16. **What is echo cancellation?** **(AU,April/May 2011)**

A process which removes unwanted echoes from the signal on a telephone line. Echoes are usually caused by impedance mismatches along an analogue line.

17. **What is sub-band coding?** **(AU,Nov/Dec 2010)**

The speech signal is applied to an analysis filter bank consisting of a set of Q band pass filters. This digital filtration divides the speech signal into a non overlapping frequency bands. These filter banks are contiguous in frequency. Hence, by additive recombination of the set of sub band signals, one can approximately generate the original speech signal.

18. **If the spectrum of a sequence $x(n) is X(exp(j\omega))$, then what is the spectrum of a signal down sampled by factor 2**

$$Y(exp(j\omega)) = 0.5\left[X\left(exp\left(\frac{j\omega}{2}\right)\right) + X\left(exp\left(j\left(\frac{\omega}{2-\pi}\right)\right)\right)\right]$$

19. **What is the need for anti aliasing filter prior to down sampling?**

The spectra obtained after down sampling a signal by a factor M is the sum of all the uniformly shifted and stretched version of original spectrum scaled by a factor $\frac{1}{M}$, then down sampling will cause aliasing. In order to avoid aliasing the signal $x(n)$ is to be band limited to plus or minus $\frac{\pi}{M}$. This can be done by filtering the signal $x(n)$ with a low pass filter with a cutoff frequency of $\frac{\pi}{M}$. This filter is known as anti aliasing filter.

20. **What is meant by adaptive filter?**

A filter with adjustable coefficients is called adaptive filter. The adjustable parameters in filter are assigned with values based on the estimated statistical nature of the signals.

21. **What are the applications of adaptive filter?** **(AU,Nov/Dec 2017)**

- Adaptive noise cancelling
- Line enhancing
- Frequency tracking
- Channel equalizations

22. **What is meant by echo cancellation?**

Echo suppression and echo cancellation are methods in telephony to improve voice quality by preventing echo from being created or removing it after it is already present.

23. **What are the different configurations of echo canceller?**

* Symbol rate echo canceller * Nyquist rate echo canceller

24. **What is the need for adaptive equalisation in a digital communication system?**

The need of adaptive equaliser or equalisation is for compensating the channel distortion so that the detected signal will be reliable. The adaptive equalisation process is done in two steps.

* Training mode * Tracking mode

25. **Define sampling rate conversion.**

Sample-rate conversion is the process of changing the sampling rate of a discrete signal to obtain a new discrete representation of the underlying continuous signal.

26. **What are the factors that influence selection of DSPs?**

(a) Architectural features

(b) Execution speed

(c) Type of arithmetic

(d) Word length

27. **How can a sampling rate conversion by a factor of I/D achieved?**

A sampling rate conversion by a factor I/D can be achieved by first performing interpolation by a factor I and then performing decimation by a factor D.

28. **What do you mean by aliasing?**

The overlapping of the spectra at the output of the down sampler due to lack of band limiting of the signal fed to the down sampler is called aliasing.

29. Define local loop.

The local communication link between the subscriber and the local central telephone office is a two wire line, called the local loop

30. Define hybrid.

At the central office, the subscriber two wire lines is connected to the main four wire telephone channels that interconnect different central offices, called trunk lines, by a device called a hybrid.

31. What is meant by echo suppressor?

To mitigate the echoes in voice transmission, the telephone companies employ a device an echo suppressor.

32. What is meant by adaptive noise cancelling?

In echo cancellation, the suppression of narrow band interference in a wideband signal and the ALE are related to another form of adaptive filtering called Adaptive noise cancelling.

33. What is anti-aliasing filter?

The LPF used at the input of decimator is called anti-aliasing filter. It is used to limit the bandwidth of an input signal to $\frac{\pi}{D}$ in order to prevent the aliasing of output spectrum of decimator for decimation by D.

34. What is an anti-imaging filter?

The LPF used at the output of an interpolator is called anti-imaging filter. It is used to eliminate the multiple images in the output spectrum of the interpolator.

35. What are the various coding techniques for images?

Waveform coding, Transform coding, Frequency band coding and parametric methods.

36. What is meant by image compression?

Image compression is the process of reducing the redundancy present in the image. The number of bits required to store the image information are reduced.

37. What are the various image compression techniques for images?

Transform coding, Predictive signal coding and sub-band coding

16 Marks Questions

1. Explain in detail the two basic operations in multi-rate signal processing. **(April/May 2015) (16)**

2. A signal $x(n)$ is given $x(n) = \{0, 1, 2, 3, 4, 5, 6, 0, 1, 2, 3, .\}$(1)Obtain the decimated signal with a factor of 2.(2) Obtain the interpolated signal with a factor of 2. **(May/June 2013)(8)**

3. Explain the sampling rate conversion by a rational factor and derive input output relation in both frequency and time domain. **(May/June 2013) (16)**

4. State the applications of Multi-rate signal processing. **(6)**

5. Explain the multistage implementation of sampling rate conversion. **(Nov/Dec 2013) (6)**

6. Explain the polyphase structure for decimator and interpolator. **(16)**

7. Explain the efficient transversal structure for decimator and interpolator. **(16)**

8. Discuss the procedure to implement digital filter bank using multirate signal processing. **(16)**

9. Explain the application of sampling rate conversion in sub band coding. **(April/May 2015)(8)**

10. What is the need for adaptive equalization in a digital communication system? Explain. **(8)**

11. With a neat block diagram explains about adaptive equalization. **(8)**

12. Explain about echo cancellation. **(16)**

13. With a neat block diagram explains about Adaptive noise cancelling. **(8)**

14. Explain about LPC of speech signals. **(8)**

15. Explain the following:
    (1) Symbol rate echo canceller (2) Nyquist rate echo canceller. **(16)**

16. Explain any four applications of Adaptive filters. **(16)**