6. FINITE IMPULSE RESPONSE FILTERS

6.1 Introduction

In the previous chapter we studied the design of IIR filter and its properties. In many digital processing applications FIR filters are preferred over their IIR counterparts. The following are the main advantages of the FIR filter over IIR filter.

1. FIR filters are always stable.
2. FIR filters with exactly linear phase can easily be designed.
3. FIR filters can be realized in both recursive and non-recursive structures.
4. FIR filters are free of limit cycle oscillations, when implemented on a finite-word length digital system.
5. Excellent design methods are available for various kinds of FIR filters.

The disadvantages of FIR filter are

1. The implementation of narrow transition band FIR filters are very costly, as it requires considerably more arithmetic operations and hardware components such as multipliers, adders and delay elements.
2. Memory requirement and execution time are very high.

6.2 Linear phase FIR filters

The transfer function of a FIR causal filter is given by

$$H(z) = \sum_{n=0}^{N-1} h(n)\, z^{-n} \qquad \text{... (6.1)}$$

where $h(n)$ is the impulse response of the filter

The Fourier transform of $h(n)$ is

$$H(e^{j\omega}) = \sum_{n=0}^{N-1} h(n)\, e^{-j\omega n} \qquad \text{--- (6.2)}$$

which is periodic in frequency with period 2π

$$H(e^{j\omega}) = \pm \mid H(e^{j\omega}) \mid e^{j\theta(\omega)} \qquad \text{--- (6.3)}$$

where $\mid H(e^{j\omega}) \mid$ is magnitude response and $\theta(\omega)$ is phase response

We define the phase delay and group delay of a filter as

$$\tau_p = \frac{-\theta(\omega)}{\omega} \text{ and } \tau_g = \frac{-d\theta(\omega)}{d\omega} \qquad \text{--- (6.4)}$$

For FIR filters with linear phase we can define

$$\theta(\omega) = -\alpha\omega \quad -\pi \le \omega \le \pi \qquad \text{--- (6.5)}$$

where α is a constant phase delay in samples

Substituting Eq. (6.5) in Eq. (6.4) we have

$\tau_p = \tau_g = \alpha$, which means that α is independent of frequency.

We can write

$$\sum_{n=0}^{N-1} h(n)\, e^{-j\omega n} = \pm \mid H(e^{j\omega}) \mid e^{j\theta(\omega)} \qquad \text{--- (6.6)}$$

which gives us

$$\sum_{n=0}^{N-1} h(n) \cos \omega n = \pm \mid H(e^{j\omega}) \mid \cos \theta(\omega) \qquad \text{--- (6.7)}$$

and

$$-\sum_{n=0}^{n-1} h(n) \sin \omega n = \pm \mid H(e^{j\omega}) \mid \sin \theta(\omega) \qquad \text{--- (6.8)}$$

By taking ratio of Eq. (6.8) to Eq. (6.7) we obtain

$$\frac{\sum_{n=0}^{N-1} h(n) \sin \omega n}{\sum_{n=0}^{N-1} h(n) \cos \omega n} = \frac{\sin \alpha\omega}{\cos \alpha\omega} \qquad \text{... (6.9)}$$

$$\boxed{[\because \theta(\omega) = -\alpha\omega]}$$

After simplifying Eq. (6.9) we have

$$\sum_{n=0}^{N-1} h(n) \sin (\alpha - n)\, \omega = 0 \qquad \text{... (6.10)}$$

The Eq. (6.10) will be zero when

$$h(n) = h(N - 1 - n) \qquad \text{... (6.11)}$$

and

$$\alpha = \frac{N-1}{2} \qquad \text{... (6.12)}$$

Therefore, FIR filters will have constant phase and group delays when the impulse response is symmetrical about $\alpha = \frac{N-1}{2}$

The impulse response satisfying the Eq. (6.11) and Eq. (6.12) for odd and even values of N is shown in Fig. 6.1. When $N = 7$ the centre of symmetry of the sequence occurs at third sample and when $N = 6$, the filter delay is 2½ samples.

If only constant group delay is required, and not the phase delay we may write

$$\theta(\omega) = \beta - \alpha\omega \qquad \text{... (6.13)}$$

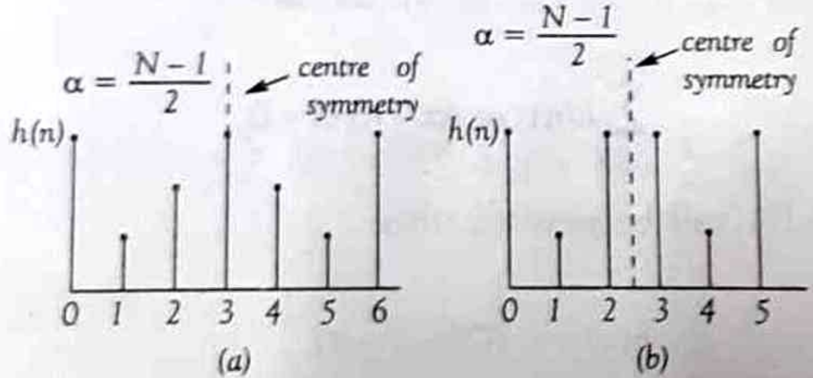

Fig. 6.1 Impulse-response sequences of symmetric sequences for (a) N odd (b) N even

Now we have

$$H(e^{j\omega}) = \pm \mid H(e^{j\omega}) \mid e^{j(\beta - \alpha\omega)} \quad \text{... (6.14)}$$

The Eq. (6.14) can be expressed as

$$\sum_{n=0}^{N-1} h(n)\, e^{-j\omega n} = \pm \mid H(e^{j\omega}) \mid e^{j(\beta - \alpha\omega)} \quad \text{... (6.15)}$$

which gives us

$$\sum_{n=0}^{N-1} h(n) \cos \omega n = \pm \mid H(e^{j\omega}) \mid \cos (\beta - \alpha\omega) \quad \text{... (6.16)}$$

and

$$-\sum_{n=0}^{N-1} h(n) \sin \omega n = \pm \mid H(e^{j\omega}) \mid \sin (\beta - \alpha\omega) \quad \text{... (6.17)}$$

By taking ratio of Eq. (6.17) to Eq. (6.16) we get

$$\frac{-\sum_{n=0}^{N-1} h(n) \sin \omega n}{\sum_{n=0}^{N-1} h(n) \cos \omega n} = \frac{\sin (\beta - \alpha\omega)}{\cos (\beta - \alpha\omega)}$$

from which we obtain

$$\sum_{n=0}^{N-1} h(n) \sin[\beta - (\alpha - n)\, \omega] = 0 \quad \text{... (6.18)}$$

If $\beta = \pi/2$ the Eq. (6.18) can be written as

$$\sum_{n=0}^{N-1} h(n) \cos (\alpha - n)\, \omega = 0 \quad \text{... (6.19)}$$

The Eq. (6.19) will be satisfied when

$$h(n) = -\, h(N - 1 - n) \quad \text{... (6.20)}$$

$$\alpha = \frac{N-1}{2} \quad \text{... (6.21)}$$

Therefore, FIR filters have constant group delay τ_g, and not constant phase delay when the impulse response is antisymmetrical about $\alpha = \frac{N-1}{2}$.

The impulse response satisfying Eq. (6.20) and Eq. (6.21) is shown in Fig. 6.2. When $N = 7$ the centre of antisymmetry occurs at third sample, and when $N = 6$ the centre of antisymmetry occurs at $2\frac{1}{2}$ sample.

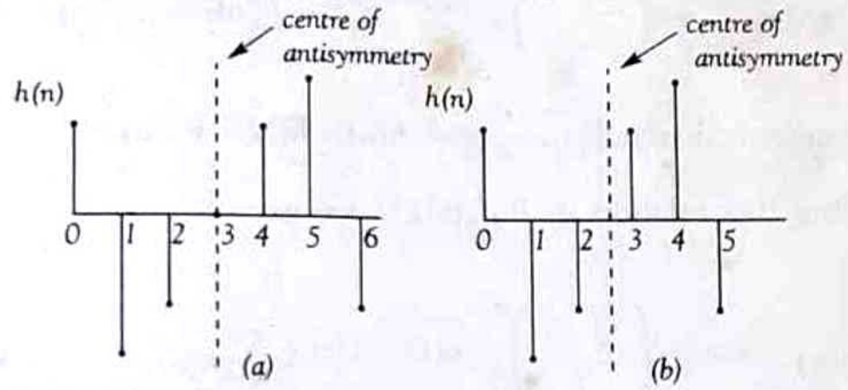

***Fig.6.2** Impulse response sequences satisfying Eq. (6.20) and Eq. (6.21) for (a) $N = 7$ and (b) $N = 6$*

From Fig. 6.2 (a) we find that $h\left(\frac{N-1}{2}\right) = 0$ for antisymmetric odd sequence.

6.3 Frequency response of linear phase FIR filters

Case I: symmetrical impulse response, N odd

The frequency response of impulse response shown in Fig. 6.1a can be written as

$$H(e^{j\omega}) = \sum_{n=0}^{6} h(n)\, e^{-j\omega n}$$

This can be split like $H(e^{j\omega}) = \sum_{n=0}^{2} h(n)\, e^{-j\omega n} + h(3)\, e^{-j3\omega} + \sum_{n=4}^{6} h(n)\, e^{-j\omega n}$

In general, for N samples

$$H(e^{j\omega}) = \sum_{n=0}^{\frac{N-3}{2}} h(n)\, e^{-j\omega n} + h\left(\frac{N-1}{2}\right) e^{-j\omega (N-1)/2} + \sum_{n=\frac{N+1}{2}}^{N-1} h(n)\, e^{-j\omega n} \qquad \text{... (6.21)}$$

Let $n = N - 1 - m$, we have

$$H(e^{j\omega}) = \sum_{n=0}^{\frac{N-3}{2}} h(n)\, e^{-j\omega n} + h\left(\frac{N-1}{2}\right) e^{-j\omega (N-1)/2} + \sum_{m=0}^{\frac{N-3}{2}} h(N-1-m)\, e^{-j\omega(N-1-m)} \quad \cdots (6.22)$$

$$= \sum_{n=0}^{\frac{N-3}{2}} h(n)\, e^{-j\omega n} + h\left(\frac{N-1}{2}\right) e^{-j\omega (N-1)/2} + \sum_{n=0}^{\frac{N-3}{2}} h(N-1-n)\, e^{-j\omega(N-1-n)} \quad \cdots (6.23)$$

For a symmetrical impulse response $h(n) = h(N-1-n)$

Substituting this relation in Eq. (6.23) we get

$$H(e^{j\omega}) = \sum_{n=0}^{\frac{N-3}{2}} h(n)\, e^{-j\omega n} + h\left(\frac{N-1}{2}\right) e^{-j\omega (N-1)/2} + \sum_{n=0}^{\frac{N-3}{2}} h(n)\, e^{-j\omega(N-1-n)}$$

$$= e^{-j\omega (N-1)/2} \left[\sum_{n=0}^{\frac{N-3}{2}} h(n)\, e^{j\omega [(N-1)/2 - n]} + \sum_{n=0}^{\frac{N-3}{2}} h(n)\, e^{-j\omega [(N-1)/2 - n]} + h\left(\frac{N-1}{2}\right) \right]$$

$$= e^{-j\omega (N-1)/2} \left[\sum_{n=0}^{\frac{N-3}{2}} 2h(n) \cos\omega \left(\frac{N-1}{2} - n\right) + h\left(\frac{N-1}{2}\right) \right]$$

Let $\dfrac{N-1}{2} - n = p$, then

$$H(e^{j\omega}) = e^{-j\omega (N-1)/2} \left[\sum_{p=1}^{\frac{N-1}{2}} 2h\left(\frac{N-1}{2} - p\right) \cos\omega p + h\left(\frac{N-1}{2}\right) \right]$$

$$= e^{-j\omega (N-1)/2} \left[\sum_{n=1}^{\frac{N-1}{2}} 2h\left(\frac{N-1}{2} - n\right) \cos\omega n + h\left(\frac{N-1}{2}\right) \right]$$

$$= e^{-j\omega(N-1)/2} \sum_{n=0}^{\frac{N-1}{2}} a(n) \cos \omega n \qquad \text{... (6.24)}$$

$$\text{where } a(0) = h\left(\frac{N-1}{2}\right) \qquad a(n) = 2h\left(\frac{N-1}{2} - n\right) \qquad \text{... (6.25)}$$

$$H(e^{j\omega}) = e^{-j\omega(N-1)/2}\, \overline{H}(e^{j\omega}) = \overline{H}(e^{j\omega})\, e^{j\theta(\omega)}$$

$$\text{where } \overline{H}(e^{j\omega}) = \sum_{n=0}^{\frac{N-1}{2}} a(n) \cos \omega n \qquad \text{... (6.26)}$$

and

$$\theta(\omega) = -\alpha\omega = -\left(\frac{N-1}{2}\right)\omega \qquad \text{... (6.27)}$$

$\overline{H}(e^{j\omega})$ is a real and even function of ω. The magnitude and phase of $H(e^{j\omega})$ are $|H(e^{j\omega})| = |\overline{H}(e^{j\omega})|$ and

$$\angle H(e^{j\omega}) = \theta(\omega) = -\alpha\omega \text{ when } \overline{H}(e^{j\omega}) \geq 0$$

$$= -\alpha\omega + \pi \text{ when } \overline{H}(e^{j\omega}) < 0$$

$\overline{H}(e^{j\omega})$ is called as zero-phase frequency response to distinguish it from the magnitude response. The zero phase-frequency response of the filter may take both positive and negative values, whereas the magnitude response is strictly non negative.

The frequency response of symmetric impulse response for N odd and the relationship between $|H(e^{j\omega})|$ and $\overline{H}(e^{j\omega})$ and between $\theta(\omega)$ and $\angle H(e^{j\omega})$ are shown in Fig.6.3.

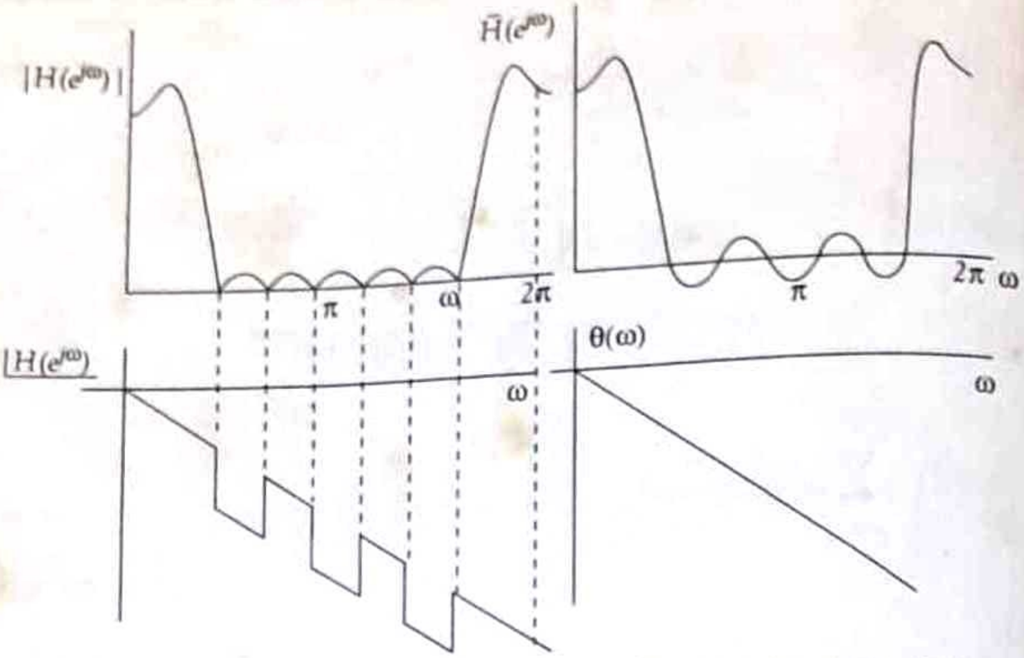

Fig. 6.3 Relation between magnitude response $|H(e^{j\omega})|$ and the zero phase response $\bar{H}(e^{j\omega})$ and between $\underline{|H(e^{j\omega})}$ and $\theta(\omega)$

Case II: Symmetric impulse response for N even

$$H(e^{j\omega}) = \sum_{n=0}^{N-1} h(n)\, e^{-j\omega n}$$

$$= \sum_{n=0}^{\frac{N-2}{2}} h(n)\, e^{-j\omega n} + \sum_{n=\frac{N}{2}}^{N-1} h(n)\, e^{-j\omega n}$$

$$= \sum_{n=0}^{\frac{N-2}{2}} h(n)\, e^{-j\omega n} + \sum_{n=0}^{\frac{N-2}{2}} h(N-1-n)\, e^{-j\omega(N-1-n)}$$

We know $h(n) = h(N-1-n)$

$$H(e^{j\omega}) = \sum_{n=0}^{\frac{N-2}{2}} h(n)\, e^{-j\omega n} + \sum_{n=0}^{\frac{N-2}{2}} h(n)\, e^{-j\omega(N-1-n)}$$

$$= e^{-j\omega(N-1)/2} \left[\sum_{n=0}^{\frac{N-2}{2}} h(n)\, e^{j\omega[(N-1)/2-n]} + \sum_{n=0}^{\frac{N-2}{2}} h(n)\, e^{-j\omega[(N-1)/2-n]} \right]$$

$$= e^{-j\omega(N-1)/2} \left[\sum_{n=0}^{\frac{N-2}{2}} 2h(n) \cos \omega \left(\frac{N-1}{2} - n \right) \right]$$

$$= e^{-j\omega(N-1)/2} \left[\sum_{n=1}^{\frac{N}{2}} 2h\left(\frac{N}{2} - n \right) \cos \left(n - \frac{1}{2} \right) \omega \right]$$

$$= e^{-j\omega(N-1)/2} \sum_{n=1}^{N/2} b(n) \cos \left(n - 1/2 \right) \omega \qquad \text{... (6.28)}$$

where $b(n) = 2h\left(\frac{N}{2} - n \right)$ ... (6.29)

$$H(e^{j\omega}) = e^{-j\omega(N-1)/2}\, \overline{H}(e^{j\omega}) = \overline{H}(e^{j\omega})\, e^{j\theta(\omega)}$$

where $\overline{H}(e^{j\omega}) = \sum_{n=1}^{N/2} b(n) \cos \left(n - \frac{1}{2} \right) \omega$ ... (6.30)

and $\theta(\omega) = -\alpha\omega = -\left(\frac{N-1}{2} \right) \omega$ ... (6.31)

The frequency response of linear phase filter with symmetric impulse response for N even is shown in Fig. 6.4.

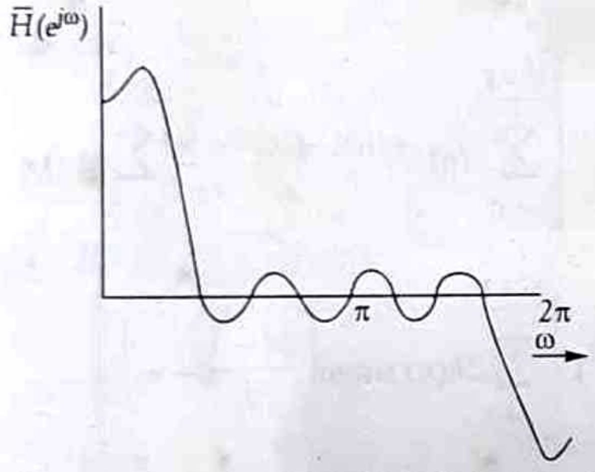

Fig. 6.4 Frequency response of linear phase FIR filter, symmetric impulse response N even

Case III: Antysymmetric N odd

For this type of sequence

$$h\left(\frac{N-1}{2} \right) = 0$$

$$H(e^{j\omega}) = \sum_{n=0}^{N-1} h(n)\, e^{-j\omega n}$$

$$= \sum_{n=0}^{\frac{N-3}{2}} h(n)\, e^{-j\omega n} + h\left(\frac{N-1}{2}\right) e^{-j\omega (N-1)/2} + \sum_{n=\frac{N+1}{2}}^{N-1} h(n)\, e^{-j\omega n}$$

$$= \sum_{n=0}^{\frac{N-3}{2}} h(n)\, e^{-j\omega n} + \sum_{n=\frac{N+1}{2}}^{N-1} h(n)\, e^{-j\omega n}$$

$$= \sum_{n=0}^{\frac{N-3}{2}} h(n)\, e^{-j\omega n} + \sum_{n=0}^{\frac{N-3}{2}} h(N-1-n)\, e^{-j\omega(N-1-n)}$$

we know $h(n) = -h(N-1-n)$, therefore,

$$H(e^{j\omega}) = \sum_{n=0}^{\frac{N-3}{2}} h(n)\, e^{-j\omega n} - \sum_{n=0}^{\frac{N-3}{2}} h(n)\, e^{-j\omega (N-1-n)}$$

$$= e^{-j\omega (N-1)/2} \left[\sum_{n=0}^{\frac{N-3}{2}} h(n)\, e^{j\omega [(N-1)/2 - n]} - \sum_{n=0}^{\frac{N-3}{2}} h(n)\, e^{-j\omega [(N-1)/2 - n]} \right]$$

$$= e^{-j\omega (N-1)/2} j \left[\sum_{n=0}^{\frac{N-3}{2}} 2h(n) \sin \omega\left(\frac{N-1}{2} - n\right) \right]$$

$$= e^{-j\omega (N-1)/2} \; e^{j\pi/2} \sum_{n=1}^{\frac{N-1}{2}} 2h\left(\frac{N-1}{2} - n\right) \sin \omega n$$

$$= e^{-j\omega (N-1)/2} \; e^{j\pi/2} \sum_{n=1}^{\frac{N-1}{2}} c(n) \sin \omega n \qquad \text{... (6.32)}$$

$$\ldots (6.33)$$

where $c(n) = 2h\left(\frac{N-1}{2} - n\right)$

$$H(e^{j\omega}) = \overline{H}(e^{j\omega})\, e^{-j\omega (N-1)/2}\; e^{j\pi/2} = \overline{H}(e^{j\omega})\, e^{j\theta(\omega)}$$

$$\ldots (6.34)$$

$$\overline{H}(e^{j\omega}) = \sum_{n=1}^{\frac{N-1}{2}} c(n) \sin \omega n$$

$$\theta(\omega) = \pi/2 - \alpha\omega = \frac{\pi}{2} - \left(\frac{N-1}{2}\right)\omega \qquad \ldots (6.35)$$

The frequency response of linear phase FIR filter for antisymmetric sequence with N odd is shown in Fig. 6.5.

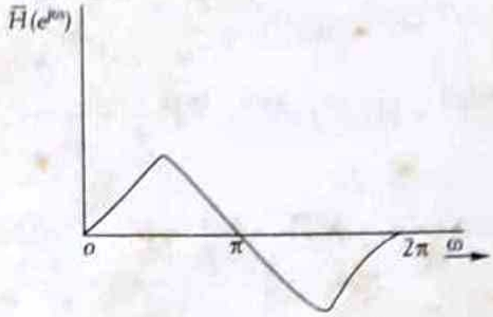

Fig. 6.5 Frequency response of linear phase FIR filter for antisymmetric sequence with N odd

Case IV: N even

$$H(e^{j\omega}) = \sum_{n=0}^{N-1} h(n)\, e^{-j\omega n}$$

$$= \sum_{n=0}^{\frac{N-2}{2}} h(n)\, e^{-j\omega n} + \sum_{n=N/2}^{N-1} h(n)\, e^{-j\omega n}$$

$$= \sum_{n=0}^{\frac{N-2}{2}} h(n)\, e^{-j\omega n} + \sum_{n=0}^{\frac{N-2}{2}} h(N-1-n)\, e^{-j\omega(N-1-n)}$$

We have $h(n) = -h(N-1-n)$, therefore

$$H(e^{j\omega}) = \sum_{n=0}^{\frac{N-2}{2}} h(n)\, e^{-j\omega n} - \sum_{n=0}^{\frac{N-2}{2}} h(n)\, e^{-j\omega(N-1-n)}$$

$$= e^{-j\omega(N-1/2)}\left[\sum_{n=0}^{\frac{N-2}{2}} h(n)\, e^{j\omega[(N-1)/2-n]} - \sum_{n=0}^{\frac{N-2}{2}} h(n)\, e^{-j\omega[(N-1)/2-n]}\right]$$

$$= e^{-j\omega(N-1)/2}\, e^{j\pi/2}\left[\sum_{n=1}^{N/2} 2h\left(\frac{N}{2}-n\right)\sin\omega\left(n-1/2\right)\right]$$

$$= e^{-j\omega(N-1)/2}\, e^{j\pi/2}\left[\sum_{n=1}^{N/2} d(n)\sin\omega\left(n-\frac{1}{2}\right)\right] \quad \text{... (6.36)}$$

$$\text{... (6.37)}$$

where $d(n) = 2h\left(\frac{N}{2}-n\right)$

$$H(e^{j\omega}) = e^{-j\omega(N-1)/2}\, e^{j\pi/2}\, \overline{H}(e^{j\omega}) \quad \text{... (6.38)}$$

$$H(e^{j\omega}) = \overline{H}(e^{j\omega})\, e^{j\theta(\omega)} = \overline{H}(e^{j\omega})\, e^{j(\pi/2-\alpha\omega)} \quad \text{... (6.39)}$$

$$\overline{H}(e^{j\omega}) = \sum_{n=1}^{N/2} d(n)\sin\omega(n-1/2) \quad \text{... (6.40)}$$

$$\theta(\omega) = \pi/2 - \alpha\omega \quad \text{... (6.41)}$$

The frequency response of linear phase filter for antisymmetric sequence with N even is shown in Fig. 6.6.

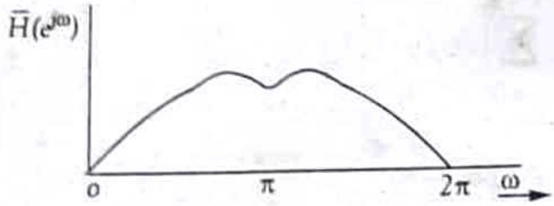

***Fig. 6.6** Frequency response of linear phase filter for antisymmetric impulse response with N even*

The impulse response of symmetric with odd number of samples can be used to design all types of filters.

The symmetric impulse response having even number of samples cannot be used to design highpass filters.

The frequency response of antisymmetric impulse response is imaginary to with in a linear phase factor. This case of filters are most suitable for such filters as Hilbert transformers and differentiators.

Table 6.1 *Summary of characteristics of linear phase FIR filters*

Type	Frequency response $H(e^{j\omega})$	Magnitude response $\lvert \overline{H}(e^{j\omega}) \rvert$	Phase response $\angle H(e^{j\omega})$	Applications
Symmetrical impulse response N odd	$e^{-j\omega (N-1)/2}\left[\sum_{n=0}^{\frac{N-1}{2}} a(n) \cos \omega^{n}\right]$ $a(0) = h\left(\frac{N-1}{2}\right)$ $a(n) = 2h\left[\left(\frac{N-1}{2}\right) - n\right]$	$\left\lvert \sum_{n=0}^{\frac{N-1}{2}} a(n) \cos \omega^{n} \right\rvert$	$-\alpha\omega + \theta$ where $\theta = 0$ for $\overline{H}(e^{j\omega}) > 0$ $\theta = \pi$ for $\overline{H}(e^{j\omega}) < 0$	lowpass, highpass, bandpass, bandstop
Symmetrical impulse response, N even	$e^{-j\omega (N-1)/2}\left[\sum_{n=1}^{\frac{N}{2}} b(n) \cos\left(n - \frac{1}{2}\right)\omega\right]$ $b(n) = 2h\left(\frac{N}{2} - n\right)$	$\left\lvert \sum_{n=1}^{\frac{N}{2}} b(n) \cos\left(n - \frac{1}{2}\right)\omega \right\rvert$	$-\alpha\omega + \theta$ where $\theta = 0$ for $\overline{H}(e^{j\omega}) > 0$ $\theta = \pi$ for $\overline{H}(e^{j\omega}) < 0$	lowpass, bandpass

Type	Frequency response $H(e^{j\omega})$	Magnitude response $\lvert \overline{H}(e^{j\omega}) \rvert$	Phase response $\underline{\lvert H(e^{j\omega})}$	Applications
Antisymmetrical impulse response N odd	$e^{-j\omega(N-1)/2}\, e^{j\pi/2} \sum_{n=1}^{\frac{N-1}{2}} c(n) \sin \omega n$	$\left\lvert \sum_{n=1}^{\frac{N-1}{2}} c(n) \sin \omega n \right\rvert$	$-\alpha\omega + \frac{\pi}{2} + \theta$ where $\theta = 0$ for $\overline{H}(e^{j\omega}) > 0$; $\theta = \pi$ for $\overline{H}(e^{j\omega}) < 0$	differentiator, Hilbert-transformer
Antisymmetrical impulse response N even	$e^{j\pi/2}\, e^{-j\omega\,(N-1)/2} \sum_{n=1}^{\frac{N}{2}} d(n) \sin \omega \left(n - \frac{1}{2}\right)$ $d(n) = \sum_{n=1}^{\frac{N}{2}} 2h\left(\frac{N}{2} - n\right)$	$\left\lvert \sum_{n=1}^{\frac{N}{2}} d(n) \sin \omega \left(n - \frac{1}{2}\right) \right\rvert$	$-\alpha\omega + \frac{\pi}{2} + \theta$ where $\theta = 0$ for $\overline{H}(e^{j\omega}) > 0$; $\theta = \pi$ for $\overline{H}(e^{j\omega}) < 0$	differentiator, Hilbert transformer

6.5 THE FOURIER SERIES METHOD OF DESIGNING FIR FILTERS

The frequency response $H(e^{j\omega})$ of a system is periodic in 2π. From Fourier series analysis we know that any periodic function can be expressed as a linear combination of complex exponentials. Therefore, the desired frequency response of an FIR filter can be represented by the Fourier series

$$H_d(e^{j\omega}) = \sum_{n=-\infty}^{\infty} h_d(n)\, e^{-j\omega n} \quad \dots (6.45)$$

where the Fourier coefficients $h_d(n)$ are the desired impulse response sequence of the filter

$$h_d(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} H_d(e^{j\omega})\, e^{j\omega n}\, d\omega \quad \dots (6.46)$$

The z-transform of the sequence is given by

$$H(z) = \sum_{n=-\infty}^{\infty} h_d(n)\, z^{-n} \quad \dots (6.47)$$

The Eq. (6.47) represents a non-causal digital filter of infinite duration. To get an FIR filter transfer function, the series can be truncated by assigning

$$h(n) = h_d(n) \text{ for } |\, n \,| \le \frac{N-1}{2}$$

$$= 0 \qquad \text{otherwise} \quad \dots (6.48)$$

Then

$$H(z) = \sum_{h=-\left(\frac{N-1}{2}\right)}^{\frac{N-1}{2}} h(n)\, z^{-n} \qquad \text{... (6.49)}$$

$$= h\left(\frac{N-1}{2}\right) z^{-(N-1)/2} + \ldots h(1)\, z^{-1} + h(0) + h(-1)\, z + h(-2)\, z^2 + \ldots$$

$$+ h\left[-\left(\frac{N-1}{2}\right)\right] z^{(N-1)/2}$$

$$= h(0) + \sum_{n=1}^{\frac{N-1}{2}} \left[h(n)\, z^{-n} + h(-n)\, z^{n} \right] \qquad \text{... (6.50)}$$

For a symmetrical impulse response having symmetry at $n = 0$

$$h(-n) = h(n) \qquad \text{... (6.51)}$$

Therefore, the Eq. (6.50) can be written as

$$H(z) = h(0) + \sum_{n=1}^{\frac{N-1}{2}} h(n) \left[z^{n} + z^{-n} \right] \qquad \text{... (6.52)}$$

The above transfer function is not physically realizable. Realizability can be brought by multiplying the Eq. (6.52) by $z^{-(N-1)/2}$ where $\frac{N-1}{2}$ is delay in samples.

$$H'(z) = z^{-(N-1)/2}\, H(z)$$

$$= z^{-(N-1)/2} \left[h(0) + \sum_{n=1}^{\frac{N-1}{2}} h(n) \left(z^{n} + z^{-n} \right) \right] \qquad \text{... (6.53)}$$

Example 6.1: *Design an ideal lowpass filter with a frequency response*

$$H_d(e^{j\omega}) = 1 \quad \text{for} \quad -\frac{\pi}{2} \le \omega \le \frac{\pi}{2}$$

$$= 0 \quad \text{for} \quad \frac{\pi}{2} \le |\omega| \le \pi$$

Find the values of $h(n)$ for $N = 11$. Find $H(z)$. Plot the magnitude response

Solution

The frequency response of lowpass filter with $\omega_c = \frac{\pi}{2}$ is shown in Fig. 6.8.

Given

$$H_d(e^{j\omega}) = 1 \quad \text{for} \; \frac{-\pi}{2} \le \omega \le \frac{\pi}{2}$$

$$= 0 \quad \text{for} \; \frac{\pi}{2} \le |\omega| \le \pi$$

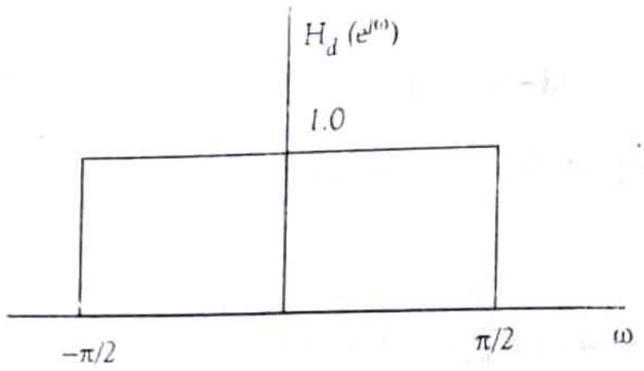

Fig. 6.8 Ideal frequency response of Example 6.1

From the frequency response we can find that $\alpha = 0$. Therefore, we get noncausal filter coefficients symmetrical about $n = 0$, i.e., $h_d(n) = h_d(-n)$. The filter coefficients can be obtained by using the formula given in table 6.2 for zero phase frequency response (or) we can proceed as follows.

$$\text{We know } h_d(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} H_d(e^{j\omega}) \, e^{j\omega n} \, d\omega$$

$$= \frac{1}{2\pi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} e^{j\omega n} \, d\omega \qquad \ldots (6.54)$$

$$= \frac{1}{2\pi j n} e^{j\omega n} \Big|_{-\pi/2}^{\pi/2}$$

$$= \frac{1}{\pi n\,(2j)} \left[e^{j\pi n/2} - e^{-j\pi n/2} \right]$$

$$= \frac{\sin \frac{\pi}{2} n}{\pi n} \quad -\infty \le n \le \infty \qquad \text{... (6.55)}$$

Truncating $h_d(n)$ to 11 samples

we have

$$h(n) = \frac{\sin \frac{\pi}{2} n}{\pi n} \text{ for } |n| \le 5$$

$$= 0 \qquad \text{otherwise} \qquad \text{... (6.56)}$$

For $n = 0$ the Eq. (6.56) becomes indeterminate

So $h(0) = \lim\limits_{n \to 0} \dfrac{\sin \frac{\pi}{2} n}{\pi n} = \dfrac{1}{2} \lim\limits_{n \to 0} \dfrac{\sin \frac{\pi}{2} n}{\frac{\pi n}{2}}$ $\qquad \boxed{\therefore \lim\limits_{\theta \to 0} \frac{\sin \theta}{\theta} = 1}$

$$= \frac{1}{2}$$

(or)

Substitute $n = 0$ in Eq. (6.54) we get

$$h(0) = h_d(0) = \frac{1}{2\pi} \int_{-\pi/2}^{\pi/2} d\omega = \frac{1}{2\pi}\, \omega \Big|_{-\pi/2}^{\pi/2} = \frac{\pi}{2\pi} = \frac{1}{2}$$

For $n = 1$

$$h(1) = h(-1) = \frac{\sin \frac{\pi}{2}}{\pi} = \frac{1}{\pi} = 0.3183$$

Similarly

$$h(2) = h(-2) = \frac{\sin \pi}{2\pi} = 0$$

$$h(3) = h(-3) = \frac{\sin \frac{3\pi}{2}}{3\pi} = -\frac{1}{3\pi} = -0.106$$

$$h(4) = h(-4) = \frac{\sin\frac{4\pi}{2}}{4\pi} = 0$$

$$h(5) = h(-5) = \frac{\sin\frac{5\pi}{2}}{5\pi} = \frac{1}{5\pi} = 0.06366$$

The transfer function of the filter is given by

$$H(z) = h(0) + \sum_{n=1}^{\frac{N-1}{2}} \left[h(n)\left(z^{n} + z^{-n} \right) \right]$$

$$= 0.5 + \sum_{n=1}^{5} h(n)(z^{n} + z^{-n})$$

$$= 0.5 + 0.3183(z^{1} + z^{-1}) - 0.106(z^{3} + z^{-3}) + 0.06366(z^{5} + z^{-5})$$

The transfer function of the realizable filter is

$$H'(z) = z^{-(N-1)/2}\, H(z)$$

$$= z^{-5}\left[0.5 + 0.3183 \;\; (z^{1} + z^{-1}) - 0.106\,(z^{3} + z^{-3}) + 0.06366\,(z^{5} + z^{-5}) \right]$$

$$= 0.06366 - 0.106z^{-2} + 0.3183z^{-4} + 0.5z^{-5} + 0.3183\,z^{-6}$$

$$- 0.106z^{-8} + 0.06366\,z^{-10} \qquad \text{... (6.57)}$$

From the above Eq. (6.57) the filter coefficients of causal filter are given by

$h(0) = h(10) = 0.06366$; $h(1) = h(9) = 0$; $h(2) = h(8) = -0.106$

$h(3) = h(7) = 0 \quad h(4) = h(6) = 0.3183$; $h(5) = 0.5$

The frequency response is given by

$$\overline{H}(e^{j\omega}) = \sum_{n=0}^{5} a(n) \cos \omega n \text{ where}$$

$$a(0) = h\left(\frac{N-1}{2} \right) = h(5) = 0.5$$

$$a(n) = 2h\left(\frac{N-1}{2} - n\right)$$

$$a(1) = 2h(5-1) = 2h(4) = 0.6366$$

$$a(2) = 2h(5-2) = 2h(3) = 0$$

$$a(3) = 2h(5-3) = 2h(2) = -0.212$$

$$a(4) = 2h(5-4) = 2h(1) = 0$$

$$a(5) = 2h(5-5) = 2h(0) = 0.127$$

$$\overline{H}(e^{j\omega}) = 0.5 + 0.6366 \cos\omega - 0.212 \cos 3\omega + 0.127 \cos 5\omega \qquad \text{... (6.58)}$$

The magnitude in dB is calculated by varying ω from 0 *to* π and tabulated below. The magnitude $|H(e^{j\omega})|_{dB} = 20\log|\overline{H}(e^{j\omega})|$

ω (in degrees)	0	10	20	30	40	50	60	70	80
$\left\| H(e^{j\omega}) \right\|_{dB}$	0.4	0.21	−0.26	−0.517	−0.21	0.42	0.77	0.21	−1.79

90	100	110	120	130	140	150	160	170	180
−6	−14.56	−31.89	−20.6	−26	−32	−24.7	−30.55	−32	−26

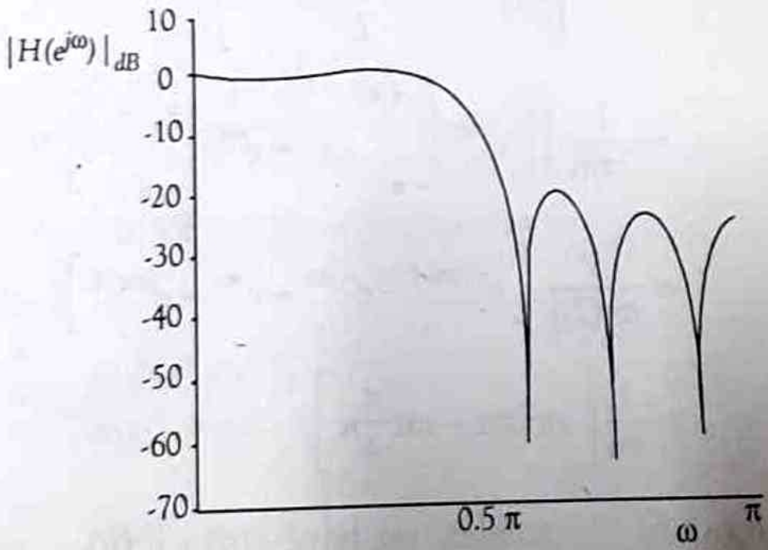

***Fig. 6.9** Frequency response of lowpass filter of example 6.1*