unit 2

Frequency Realization of Digital filters

Realization of Digital filters

For Recursive realization the current output $y(n)$ is a function of passed outputs, passed and present outputs. This corresponds to Infinite Impulse Response (IIR) filter

For Non-Recursive realization current output $y(n)$ is a function of passed and present inputs. This corresponds to Finite Impulse Response (FIR) filter

IIR filter can be realized using direct form I realization, direct form II realization, Cascade form realization and Parallel form realization

Direct form I realization

General Derivation

Let us consider LTI (Linear Time Invariant) recursive system described by the following different equations

$$y(n) = -\sum_{k=1}^{N} a_k\, y(n-k) + \sum_{k=0}^{M} b_k\, x(n-k)$$

$$= -(a_1 y(n-1) + a_2 y(n-2) \ldots a_N y(n-N) + b_0 x(n) + b_1 x(n-1) \ldots \ldots b_M x(n-M))$$

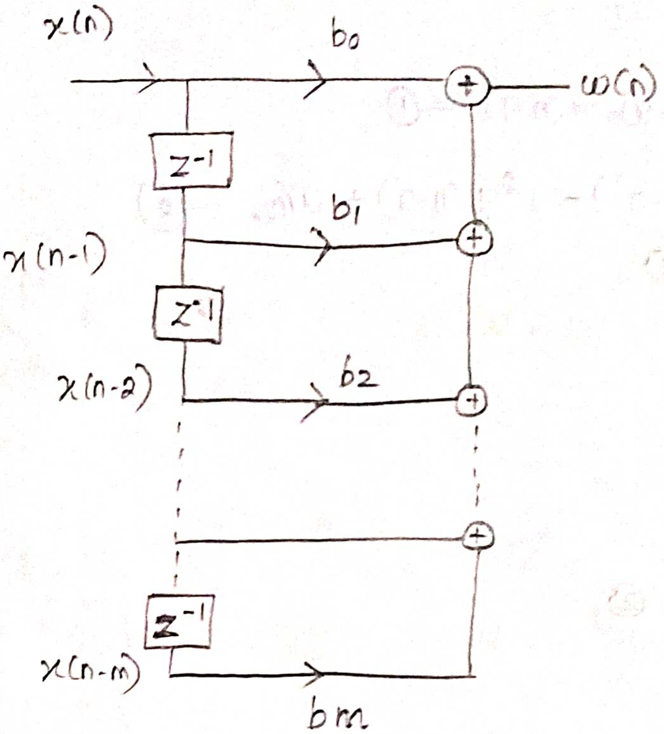
$$= -a_1 y(n-1) - a_2 y(n-2) \ldots \ldots - a_N y(n-N) + b_0 x(n) + b_1 x(n-1) \ldots b_M x(n-M))$$

Let

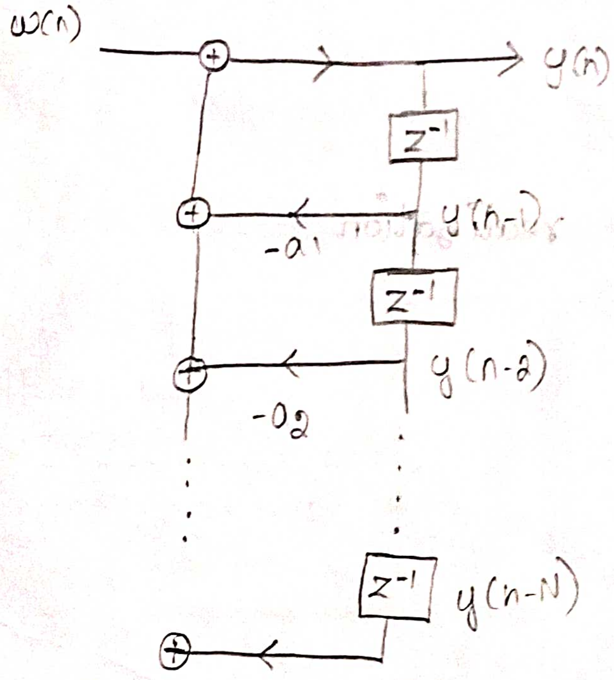
$$w_m = b_0 x(n) + b_1 x(n-1) \ldots b_m x(n-m) \quad -(1)$$

$$y(n) = -a_1 y(n-1) - a_2 y(n-2) \cdots\cdots - a_N y(n-N) + w_m \rightarrow ②$$

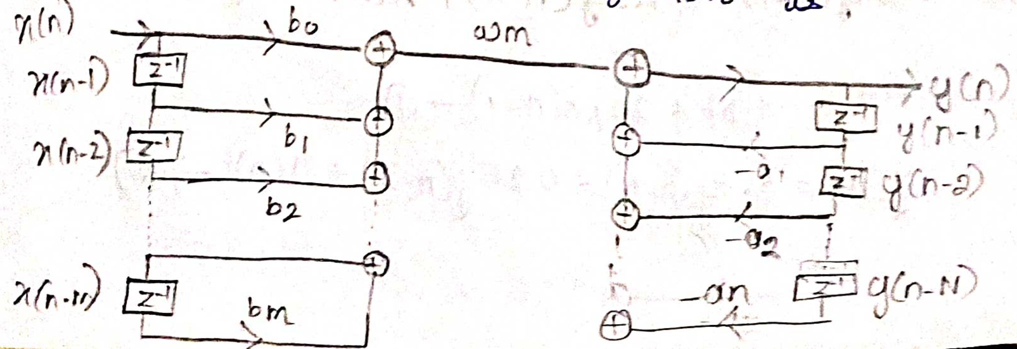
Realization of Eqn ①



Realization of Eqn ②



Final direct form I realization is



Direct form II realization

(1)

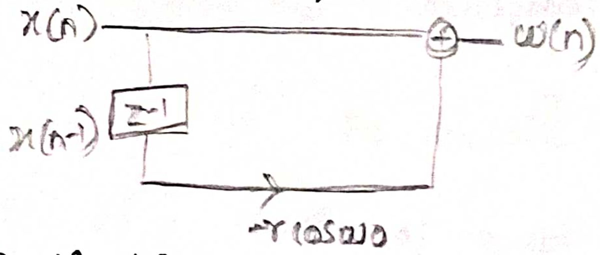
$$y(n) = 2r\cos\omega_0\, y(n-1) - r^2 y(n-2) + x(n) - r\cos\omega_0\, x(n-1)$$

using Direct form I

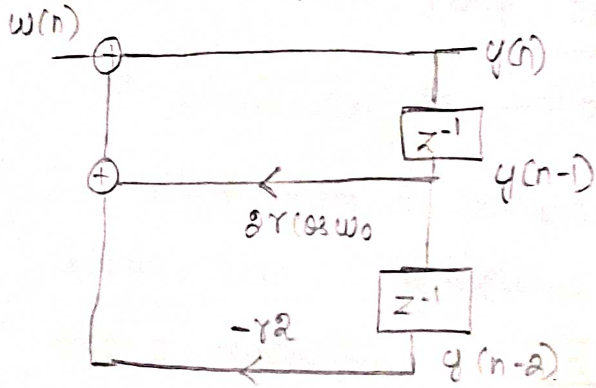
$$\omega n = x(n) - r\cos\omega_0\, x(n-1) \quad —(1)$$

$$y(n) = 2r\cos\omega_0\, y(n-1) - r^2 y(n-2) + \omega(n) \quad —(2)$$

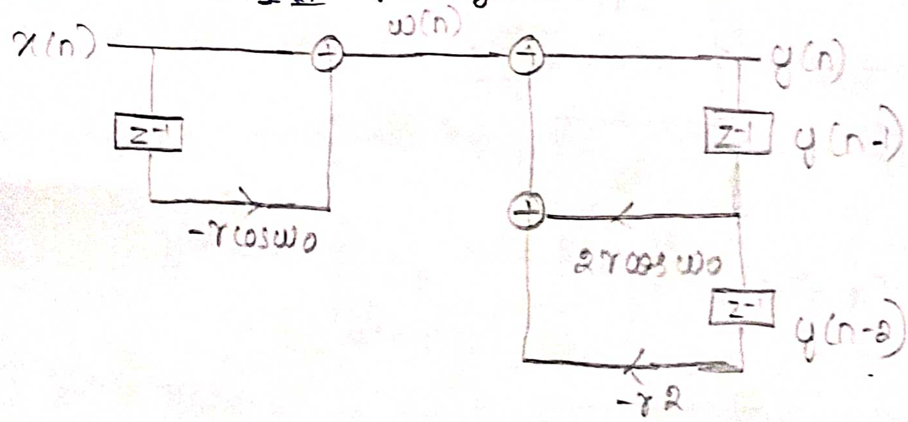
Realization of Eqn (1)



Realization of Eqn (2)



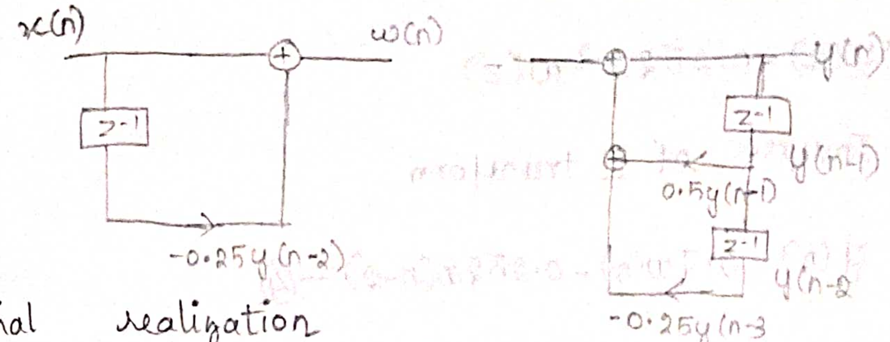
Final Direct form II realization



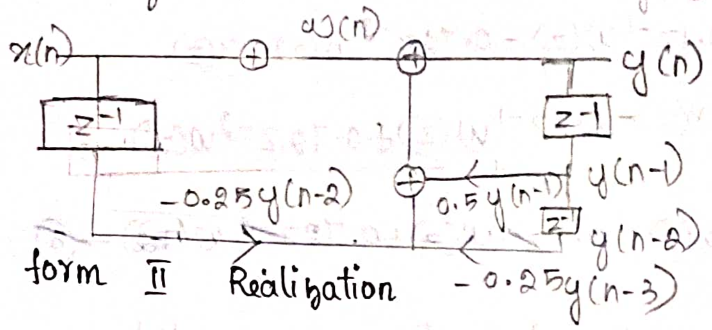
(2) $y(n) = 0.5y(n-1) - 0.25y(n-2) + x(n) + 0.4x(n-1)$

$$\omega n = x(n) + 0.4x(n-1) — (1)$$

$$y(n) = 0.5y(n-1) - 0.25y(n-2) + \omega(n) \rightarrow (2)$$



Final realization



Direct form II Realization

Determine direct form II Realization for the system

$$y(n) = -0.1y(n-1) + 0.72y(n-2) + 0.7x(n) - 0.252x(n-2) \quad \text{—}(1)$$

$$x(n) \overset{Z}{\longleftrightarrow} X(z)$$

$$x(n-k) \overset{Z}{\longleftrightarrow} z^{-k}X(z)$$

Z transform (1)

$$Y(z) = -0.1\cancel{y}\, z^{-1}Y(z) + 0.72z^{-2}Y(z) + 0.7X(z) - 0.252z^{-2}X(z)$$

$$Y(z) + 0.1z^{-1}Y(z) - 0.72z^{-2}Y(z) = 0.7X(z) - 0.252z^{-2}X(z)$$

$$Y(z)\left[1 + 0.1z^{-1} - 0.72z^{-2}\right] = X(z)\left[0.7 - 0.252z^{-2}\right]$$

$$\frac{Y(z)}{X(z)} = \frac{0.7 - 0.252z^{-2}}{1 + 0.1z^{-1} - 0.72z^{-2}}$$

$$\frac{Y(z)}{X(z)} \times \frac{W(z)}{W(z)} = \frac{Y(z)}{W(z)} \times \frac{W(z)}{W(z)} \times \frac{W(z)}{X(z)} = \frac{0.7 - 0.252z^{-2}}{1 + 0.1z^{-1} - 0.72z^{-2}}$$

To obtain w(n) we multiply and divide by W(z)

y(n) $$\frac{Y(z)}{W(z)} = 0.7 - 0.252z^{-2} \quad \text{—}(2)$$

w(n) $$\frac{W(z)}{X(z)} = \frac{1}{1 + 0.1z^{-1} - 0.72z^{-2}} \quad \text{—}(3)$$

$$y(z) = 0.7\,w(z) - 0.252\, z^{-2}\, w(z)$$

Inverse of z transform

$$y(n) = 0.7\,w(n) - 0.252\,w(n-2) \quad \text{--- (4)}$$

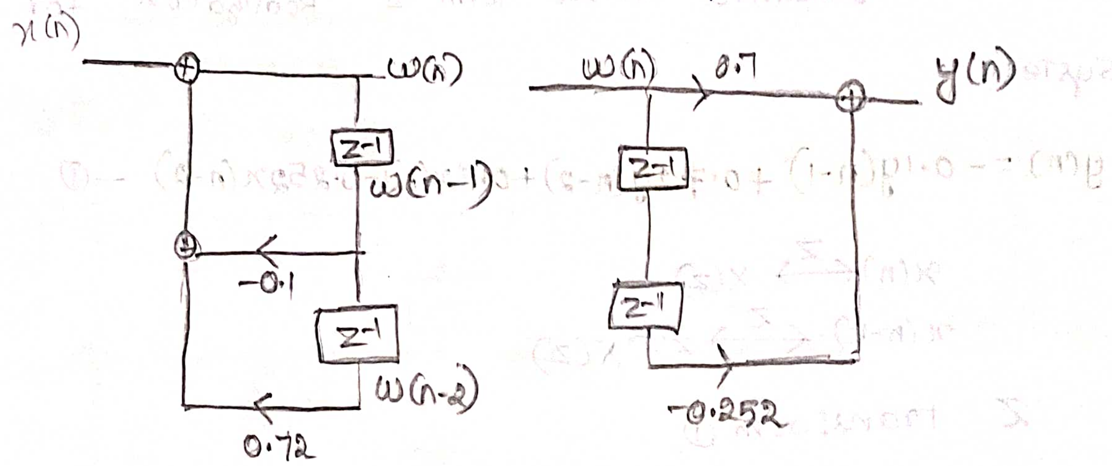
$$w(z) + 0.1 z^{-1} w(z) - 0.72 z^{-2} w(z) = X(z)$$

$$w(z) = X(z) - 0.1 z^{-1} w(z) + 0.72 z^{-2} w(z)$$

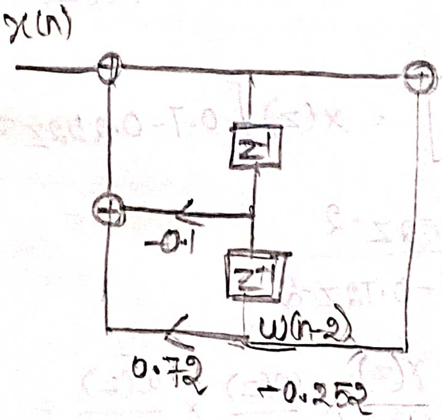
$$z^{-1}$$

$$w(n) = x(n) - 0.1\,\cancel{z^{-1}}\,w(\overset{n-1}{\cancel{z}}) + 0.72\,\cancel{z^{-2}}\,w(n-2) \quad \text{--- (5)}$$

Direct form II realization is represented as



Since the input to the delay element in both the structures are same we can combine the delay elements



Cascade form Realization

Realize the system under cascade form

Realize the system with difference eqn given by

$$y(n) = 3/4\, y(n-1) - 1/8\, y(n-2) + x(n) + 1/3\,(n-1)$$

Solution:

given

$$y(n) = 3/4\, y(n-1) - \frac{1}{8} y(n-2) + x(n) + 1/3\,(n-1)$$

$$y(z) = \frac{3}{4} z^{-1} y(z) - \frac{1}{8} z^{-2} y(z) + x(z) + \frac{1}{3} z^{-1} x(z)$$

$$y(z) - \frac{3}{4} z^{-1} y(z) + \frac{1}{8} z^{-2} y(z) = x(z) + \frac{1}{3} z^{-1} x(z)$$

$$y(z)\left[1 - \frac{3}{4} z^{-1} + \frac{1}{8} z^{-2}\right] = x(z)\left[1 + \frac{1}{3} z^{-1}\right]$$

$$\frac{y(z)}{x(z)} = \frac{1 + \frac{1}{3} z^{-1}}{1 - \frac{3}{4} z^{-1} + 1/8\, z^{-2}}$$

$$1 - \frac{3}{4} z^{-1} + \frac{1}{8} z^{-2}.$$

$$1 - \frac{1}{4} z^{-1} - \frac{1}{2} z^{-1} + \frac{1}{8} z^{-2}$$

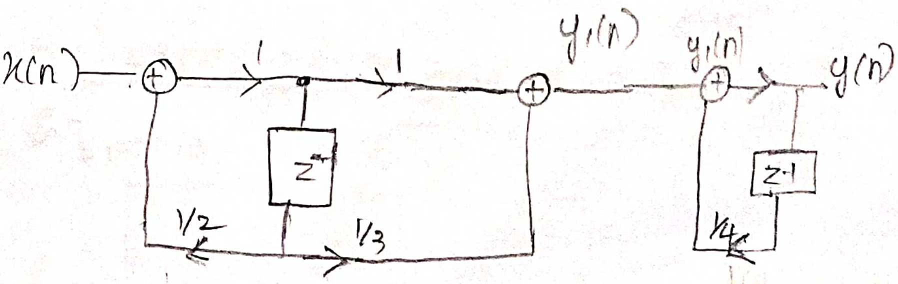
$$= 1\left(1 - \frac{1}{4} z^{-1}\right) - \frac{1}{2} z^{-1}\left(1 - \frac{1}{4} z^{-1}\right)$$

$$= \left(1 - \frac{1}{2} z^{-1}\right)\left(1 - \frac{1}{4} z^{-1}\right)$$

$$\frac{y(z)}{x(z)} = \frac{1 + 1/3\, z^{-1}}{1 - \frac{3}{4} z^{-1} + 1/8\, z^{-2}}$$

$$= \frac{1 + 1/3\, z^{-1}}{\cancel{(1 + 1/2} \left(1 - \frac{1}{2} z^{-1}\right)\left(1 - \frac{1}{4} z^{-1}\right)}$$

$$= \frac{1 + \frac{1}{3} z^{-1}}{\left(1 - \frac{1}{2} z^{-1}\right)} \times \frac{1}{\left(1 - \frac{1}{4} z^{-1}\right)}$$



Parallel realization

$$y(n) = -0.1y(n-1)+0.72y(n-2)+0.7x(n)-0.252x(n-2)$$

Solution

given

$$y(n) = -0.1y(n-1)+0.72y(n-2)+0.7x(n)-0.252x(n-2)$$

$$Y(z) = -0.1z^{-1}y(z)+0.72yz^{-2}y(z)+0.7x(z)-0.252z^{-2}x(z)$$

$$y(z)+0.1z^{-1}y(z)-0.72yz^{-2}y(z) = 0.7x(z)-0.252z^{-2}x(z)$$

$$y(z)(1+0.1z^{-1}-0.72yz^{-2}) = x(z)[0.7-0.252z^{-2}]$$

$$\frac{Y(z)}{X(z)} = \frac{0.7-0.0252z^{-2}}{1+0.1z^{-1}-0.72yz^{-2}}$$

example

$$\frac{7}{3} \quad 3\overline{)7}\ 2,\ \frac{6}{1} \rightarrow 2+\frac{1}{3}$$

Any fraction can be written as $Q+\frac{R}{d}$

$$\begin{array}{r|l} & 0.35 \\ -0.72yz^{-2}+0.1z^{-1}+1 & -0.0252z^{-2}+0.7 \\ & \cancel{-0.0252z^{-2}} \overset{(-)}{+0.035z^{-1}} \overset{(-)}{+0.35} \\ \hline & -0.035z^{-1}+0.35 \end{array}$$

$0.72 \times x = 0.0252$

$x = 0.35$

$$0.35 + \frac{(-0.035z^{-1}+0.35)}{-0.72yz^{-2}+0.1z^{-1}+1}$$

$$\frac{Y(z)}{X(z)} = 0.35+\frac{0.35-0.035z^{-1}}{1+0.1z^{-1}-0.72yz^{-2}} \quad \text{—(1)}$$

$$1+0.1z^{-1}-0.72z^{-2}$$

$$1+\frac{1}{10}z^{-1}-\frac{72}{100}z^{-2}$$

$$\frac{72}{100}=-\frac{8}{10},\frac{9}{10}$$

$$=1\left(1-\frac{8}{10}z^{-1}\right)+\frac{9}{10}z^{-1}\left(1-\frac{8}{10}z^{-1}\right)$$

$$=\left(1+\frac{9}{10}z^{-1}\right)\left(1-\frac{8}{10}z^{-1}\right)$$

$$\frac{0.35-0.035z^{-1}}{(1+0.9z^{-1})(1-0.8z^{-1})}$$

$$=\frac{A}{(1+0.9z^{-1})}+\frac{B}{(1-0.8z^{-1})}$$

$$\frac{0.35-0.035z^{-1}}{(1+0.9z^{-1})(1-0.8z^{-1})}=\frac{A(1-0.8z^{-1})+B(1+0.9z^{-1})}{(1+0.9z^{-1})(1-0.8z^{-1})}$$

$$0.35-0.035z^{-1}=A(1-0.8z^{-1})+B(1+0.9z^{-1})$$

equating constants

$$0.35=A+B \rightarrow (2)$$

equating coefficients of z^{-1}

$$-0.035=-0.8A+0.9B \rightarrow (3)$$

$$(2) \Rightarrow B=0.35-A \rightarrow (4)$$

Subs (4) in (3) and find A,

$$-0.035=-0.8A+0.9(0.35-A)$$

$$-0.035=-0.8A+0.315-0.9A$$

$$=-1.7A+0.315$$

$$0.35=-1.7A \Rightarrow A=-\frac{0.35}{1.7}$$

$$\boxed{A=-0.205}$$

Discrete Fourier Transform (DFT and IDFT)

Inverse Discrete Fourier Transform

DFT is given by

$$X(K) = \sum_{n=0}^{N-1} x(n)\, e^{-\frac{j2\pi nk}{N}} \; ; \; K=0,1,\ldots,N-1$$

IDFT is given by

$$x(n) = \frac{1}{N} \sum_{K=0}^{N-1} X(K)\, e^{\frac{j2\pi nk}{n}} , \; n=0,1,\ldots,N-1$$

N - length of the sequence

1. Find the DFT of the sequence $x(n) = \{\overset{x(0)}{1}, \overset{x(1)}{1}, \overset{x(2)}{0}, \overset{x(3)}{0}\}$

and find IDFT of $y(K) = \{1,0,1,0\}$

Solution

$$X(K) = \sum_{n=0}^{N-1} x(n) e^{-\frac{j2\pi nk}{N}} \; ; \; K=0,1,\ldots N-1$$

$N=4$

$$x(k) = \sum_{n=0}^{3} x(n)\, e^{-\frac{j2\pi nk}{4}} \; ; \; K=0,1,2,3$$

$$x(k) = \sum_{n=0}^{3} x(n)\, e^{-\frac{j\pi nk}{2}} \; ; \; K=0,1,2,3$$

$k=0$

$$x(0) = \sum_{n=0}^{3} x(n) \cdot 1 \Rightarrow x(0) = x(0)+x(1)+x(2)+x(3)$$
$$= 1+1+0+0 = 2$$

$k=1$

$e^{j\theta} = \cos\theta + j\sin\theta$

$e^{-j\theta} = \cos\theta - j\sin\theta$

$$x(1) = \sum_{n=0}^{3} x(n)\, e^{-\frac{j\pi n}{2}} \; ; \; \cancel{K=0,1,2,3}$$

$$= x(0) e^{-j\pi(0)/2} + x(1) e^{-\frac{j\pi}{2}} + 0 + 0$$

$$= 1 + e^{-j\pi/2}$$

$$= 1 + \left[\cos \pi/2 - j\sin \pi/2\right] = \boxed{1-j}$$

iii) $k=2$

$$X(2) = \sum_{n=0}^{3} x(n)\, e^{-j\pi n}$$

$$= \left[x(0)e^{0} + x(1)e^{-j\pi}\right]$$

$$= 1 + [\overset{(-1)}{\cos\pi} - j\overset{(0)}{\sin\pi}]$$

$$= 1 + [-1] = 0$$

$\cos 3\pi/2 = 0$

$\sin 3\pi/2 = -1$

$K=3$

$$X(3) = \sum_{n=0}^{3} x(n)\, e^{-\frac{j\pi 3}{2}} \quad , \cancel{K=0,1,2,3}$$

$$= x(0)e^{0} + x(1)e^{-3/2\, j\pi}$$

$$= 1 + \cos 3\pi/2 - j\sin 3\pi/2$$

$$= 1 + 0 - j(-1)$$

$$\boxed{= 1+j}$$

$$X(K) = \{2, 1-j, 0, 1+j\}$$

IDFT

$$y(n) = 1/N \sum_{K=0}^{N-1} X(K)\, e^{\frac{j2\pi nK}{N}} \quad , n = 0, 1, \ldots, N-1$$

$$Y(K) = \{1, 0, 1, 0\} \qquad N=4$$

$$y(n) = 1/4 \sum_{K=0}^{3} Y(K)\, e^{\frac{j2\pi nK}{2}} \quad , n = 0, 1, 2, 3$$

i) $n=0$

$$y(0) = \frac{1}{4}\sum_{K=0}^{3} y(K)\, e^{0}\ \frac{}{2}, 1$$

$$y(0) = \frac{1}{4}\sum_{K=0}^{3} y(K)$$

~~K=0~~ $K = \cancel{0}1, 0, 1, 0$

$$y(0) = \frac{1}{4}\left[y(0) + y(1) + y(2) + y(3)\right]$$

$$= \frac{1}{4}\left[1+0+1+0\right] = \frac{2}{4} = 0.5$$

ii) $n=1$

$$y(1) = \frac{1}{4}\sum_{K=0}^{3} y(K)\, e^{\frac{j\pi K}{2}}, \cancel{K=0}.$$

$$= \frac{1}{4}\left[y(0)\,e^{0} + y(1)\,e^{j\pi/2} + y(2)\,e^{j\pi}\,4\ y(3)\,e^{j3\pi/2}\right]$$

$$= \frac{1}{4}\left[1+0+e^{j\pi}+0\right]$$

$$= \frac{1}{4}\left[1 + (\cos\pi + j\sin\pi)\right]$$

$$= \frac{1}{4}(1+(-1))$$

$$= 0$$

iii) $n=2$

$$y(2) = \frac{1}{4}\sum_{K=0}^{3} y(K)\quad e^{j\pi K}$$

$$= \frac{1}{4}\left[y(0)\,e^{j\pi\cancel{0}} + y(1)\,e^{j\pi K} + y(2)\,e^{j\pi 2} + y(3)\,e^{j\pi 3}\right]$$

$$= \frac{1}{4}\left[1+0+1\ e^{j\pi} + 0\right]$$

$= \frac{1}{4}(1+e^{j\pi})$

$= \frac{1}{4}\left(1+(\cos\pi + j\sin\pi)\right]$

$= \frac{1}{4}\left(1+(-1+0)\right]$

$= \frac{2}{4}$

$= 0.5$

$n=3$

$y(3) = \frac{1}{4}\sum_{k=0}^{3} y(k)\, e^{j\pi\frac{3k}{2}}$

$y(3) = \frac{1}{4}\left[y(0)e^{0} + y(1)\, e^{j\frac{3}{2}\pi} + y(2)\, e^{j3\pi} + y(3)\, e^{j3\pi} \right]$

$= \frac{1}{4}\left[1+0+e^{j3\pi}+0\right]$

$= \frac{1}{4}\left[1+(\cos 3\pi + j\sin 3\pi)\right]$

$= \frac{1}{4}\left[1+(-1+0)\right]$

$= 0$

$y(n) = \{0.5, 0, 0.5, 0\}$

Find DFT of $x(n) = 1$ for $0 \le n \le 2$

$= 0$ o.w

i) $N=4$ ii) $N=8$

Plot $\underbrace{\text{magnitude of } x(K)}_{|x(K)|}$ and $\angle x(K)$ (Phase response)

Comment on the result

Soln

$$x(n) = \{1, 1, 1, 0\} \quad N=4$$

$$X(K) = \sum_{n=0}^{N-1} x(n)\, e^{\frac{-jn\pi 2k}{N}} \quad , K = 0, 1, \ldots\ldots N-1$$

$N=4$

$$X(K) = \sum_{n=0}^{3} x(n)\, e^{-jn\pi K/2} \quad , K = 0, 1, 2, 3$$

i) $K=0$

$$X(0) = x(0) + x(1) + x(2) + x(3)$$

$$= 1+1+1+0 = 3$$

ii) $K=1$

$$X(1) = \sum_{n=0}^{3} x(n)\, e^{-j\pi K/2}$$

$$= x(0)e^{0} + x(1)e^{-j\pi/2} + x(2)e^{-j\pi} + 0$$

$$= 1 + \overset{\to 0}{\cos \pi/2} - j \overset{\to 1}{\sin \pi/2} + \overset{\to -1}{\cos \pi} - j \overset{\to 0}{\sin \pi}$$

$$= 1 - j - 1 \quad = -j$$

iii) $K=2$

$$X(2) = \sum_{n=0}^{3} x(n)\, e^{-j\pi n}$$

$$= x(0)e^{0} + x(1)e^{-j\pi} + x(2)e^{-2j\pi} + x_3 e^{-j\pi 3}$$

$= x(0) + x(1)e^{-j\pi} + x(2)e^{-j2\pi} + 0$

$= 1 - \cos\pi - j\sin\pi + \cos 2\pi - j\sin 2\pi$

$= 1 - 1 + 1$

$= 1$

iv) $k=3$

$$x(3) = \sum_{n=0}^{3} x(n) e^{-jn\pi\frac{3}{2}} \quad \cancel{k=0,1,2,3}$$

$$x(3) = x(0)e^{0} + x(1)e^{-j3\pi/2} + x(2)e^{-j3\pi} + 0$$

$$= 1 + \cos\frac{3\pi}{2} - j\sin 3\pi/2 + \cos 3\pi - j\sin 3\pi$$

$$= j$$

$x(k) = \{3, -j, 1, j\}$

$\boxed{a+jb}$ $|x(k)| = \sqrt{a^2+b^2}$

$\measuredangle x(k) = \tan^{-1}(b/a)$

θ	$\sin\theta$	$\cos\theta$
0	0	1
$\pi/6$	$1/2$	$\sqrt{3}/2$
$\pi/4$	$1/\sqrt{2}$	$1/\sqrt{2}$
$\pi/3$	$\sqrt{3}/2$	$1/2$
$\pi/2$	1	0

$\tan(-\theta) = -\tan\theta$

$\tan\theta = 0$

$\tan\pi/6 = 1/\sqrt{3}$

$\tan\pi/4 = 1$

$\tan\pi/3 = \sqrt{3}$

$\tan\pi/2 = \infty$

11) $N=8$

$$x(n)=\{1,1,1,0,0,0,0,0\}$$

$$x(k)=\sum_{n=0}^{7} x(n)e^{-\frac{j\pi kn}{4}}, \quad k=0,1,\ldots\ldots 7$$

$k=0$

$$x(0)=\sum_{n=0}^{7} x(n)e^{0}$$

$$= x(0)e^{0}+x(1)e^{0}+x(2)e^{0}+x(3)e^{0}+x(4)e^{0}+x(5)e^{0}+x(6)e^{0}+x(7)e^{0}$$

$$= 1+1+1+0+0+0+0+0 = 3$$

$k=1$

$$x(1)=\sum_{n=0}^{7} x(n)e^{-j\pi n/4}$$

$$= x(0)e^{0}+x(1)e^{-j\pi/4}+x(2)e^{-j\pi/2}+x(3)e^{-j3/4\pi}$$

$$+x(4)\cancel{\,}e^{-j\pi}+x(5)e^{-j5\pi/4}+x(6)e^{-j3/2\pi}$$

$$+x(7)e^{-j\frac{7\pi}{4}}$$

$n=8$ has better resolution

it is computationally complex and costly

$N=4$ has poor resolution it is cost effective and less complex

Fourier Transform pairs

DFT and IDFT are combinedly called as Fourier Transform pairs

i.e

$$x(n) \overset{DFT}{\longleftrightarrow} X(K)$$

Properties of DFT

i) Periodicity

If a signal repeats after n number of samples then it is called periodic signal

consider $x(n)$ to be discrete signal of length N

then i) $x(n+N) = x(n)$

ii) $X(K+N) = X(K)$

Proof

By definition of IDFT

$$x(n) = \frac{1}{N} \sum_{K=0}^{N-1} X(K) e^{\frac{j2\pi nK}{N}}$$

Put $n = n+N$

$$x(n+N) = \frac{1}{N} \sum_{K=0}^{N-1} X(K) e^{\frac{j2\pi(n+N)K}{N}}$$

$$x(n+N) = \frac{1}{N} \sum_{k=0}^{N-1} X(k) e^{\frac{j2\pi(n+N)K}{N}}$$

$$x(n+N) = \frac{1}{N}\sum_{k=0}^{N-1} x(k)\, e^{\frac{j2\pi nk}{k}} \cdot e^{(j2\pi)k}$$

$$(e^{2})^{3} = e^{6}$$

Consider

$$e^{j2\pi k} = (e^{j2\pi})^{k}$$

$$= (\cos 2\pi + j\sin 2\pi)^{k}$$

$$= 1^{k} = 1$$

$$x(n+N) = \frac{1}{N}\sum_{k=0}^{N-1} x(k)\, e^{\frac{j2\pi nk}{N}} \times 1$$

DFT

$$x(k) = \sum_{n=0}^{N-1} x(n)\, e^{\frac{-j2\pi nk}{N}}$$

Put

$$k = k+N$$

$$x(k+N) = \sum_{n=0}^{N-1} x(n)\, e^{\frac{-j2\pi n(k+N)}{N}}$$

$$x(k+N) = \sum_{n=0}^{N-1} x(n)\, e^{\frac{-j2\pi nk}{N}}\; e^{-j2\pi n}$$

$$x(k+N) = \sum_{n=0}^{N-1} x(n)\, e^{\frac{-j2\pi nk}{N}} (e^{-j2\pi})^{n}$$

$$\hookrightarrow \cos 2\pi - j\sin 2\pi$$

$$= (1)^{n} - 1$$

$$= 1$$

$$x(k+N) = \sum_{n=0}^{N-1} x(n)\, e^{\frac{-j2\pi nk}{N}} \cdot 1$$

2) Linearity

$$x(n) \xleftrightarrow{DFT} X_1(K)$$

$$x_2(n) \xleftrightarrow{DFT} X_2(K)$$

Then,

$$a_1 x_1(n) + a_2 x_2(n) \xleftrightarrow{DFT} a_1 X_1(K) + a_2 X_2(K)$$

Proof

$$DFT[x(n)] = X(K) = \sum_{n=0}^{N-1} x(n) e^{\frac{-j2\pi nk}{N}}$$

DFT

$$X(K) = \sum_{n=0}^{N-1} x(n) e^{\frac{-j2\pi nk}{N}}$$

$$DFT[a_1 x_1(n) + a_2 x_2(n)] = \sum_{n=0}^{N-1} (a_1 x_1(n) + a_2 x_2(n)) e^{\frac{-j2\pi nk}{N}}$$

$$DFT[a_1 x_1(n) + a_2 x_2(n)] = \sum_{n=0}^{N-1} a_1 x_1(n) e^{\frac{-j2\pi nk}{N}} + \sum_{n=0}^{N-1} a_2 x_2(n) e^{\frac{-j2\pi nk}{N}}$$

$$= a_1 X_1(K) + a_2 X_2(K)$$

3.) Conjugate symmetry

If DFT $[x(n)] = X(k)$; then,

$X(-k) = X^*(k)$

Proof:

$$DFT\,[x(k)] = \sum_{n=0}^{N-1} x(n)\, e^{-j2\pi nk/N}$$

$$[x(-k)] = \sum_{n=0}^{N-1} x(n)\, e^{-j2\pi n(-k)/N}$$

$$[x(-k)] = \sum_{n=0}^{N-1} x(n)\, e^{j2\pi nk/N} \qquad \left\{ \begin{matrix} a+jb \\ a-jb \end{matrix} \right.$$

$$= X^*(k)$$

Convolution

Linear convolution

Circular convolution

*) Linear convolution | circular convolution

Linear convolution	circular convolution
1) $y(n) = x_1(n) * x_2(n)$	$y(n) = x_1(n) \circledast x_2(n)$
2) Formula: $y(n) = \sum_{n=-\infty}^{\infty} x_1(k)\, x_2(n-k)$	$y(n) = \sum_{k=0}^{N-1} x_1(k)\, x_2((n-k))_N$
Length of $x_1(n) = L$, $x_2(n) = M$, $y(N) = L+M-1$	Length of $y(n) = \max(L, M)$

2 mark

1) What is zeropadding cond its uses

If a sequence $x(n)$ has length L then to find N point DFT (where N is greater than L) we must add N-L zeros to the sequence $x(n)$

uses

1) Better display of frequency spectrum
2) with zero appending DFT can be used in linear filtering

Circular convolution

Procedure

1) Draw two concentric circles
2) Mark all the points of $x_1(N)$ in outer circle & in anticlockwise direction mark all points of $x_2(N)$ starting from same point in the inner circle in clockwise direction multiply the corresponding points and then add
3) Rotate the inner circle in anticlockwise direction

and repeate the some procedure until the inner circle reaches it's original position

Problem

1) Find the circular convolution of $x_1(n)=\{1,-1,-2,3,-1\}$ $x_2(n)=\{1,2,3\}$ ~~and~~

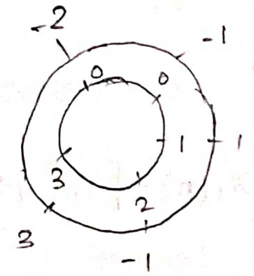
Solution

$x_1(n)=\{1,-1,-2,3,-1\}$

$x_2(n)=\{1,2,3\}$

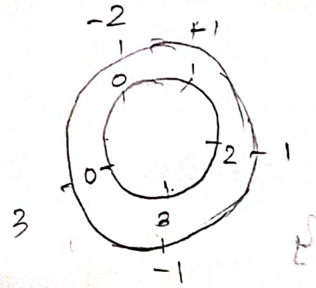
and

$x_2(n)=\{1,2,3,0,0\}$



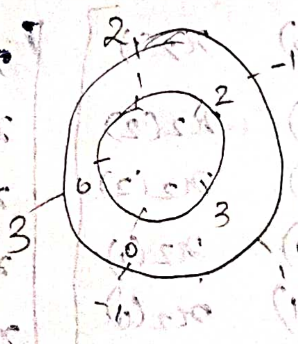
$y(0)=1(1)+0(-1)+0(-2)+3(3)+2(-1)$

$y(0)=8$



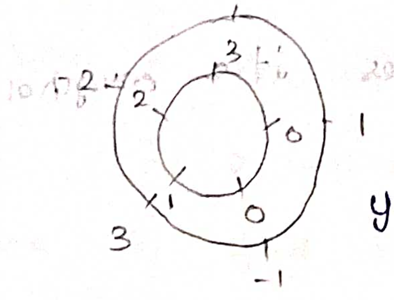
$y(1)=2(1)+1(-1)+0(2)+0(3)+3(-1)$

$y(1)=-2$



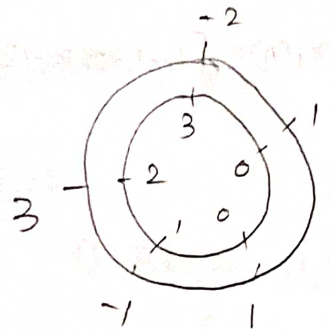
$y(2)=3(1)+2(-1)+1(-2)+0(3)+0(-1)$

$y(2)=-1$



$y(3) = 0(1) + 3(-1) + 2(-2) + 1(3) + 0(-1)$

$=$



$y(4) = 0(1) + 0(1) + 3(-2) + 2(3) + 1(-1)$

$=$

$y(n) = \{8, -2, -1, -4, -1\}$

Matrix convolution

Find the matrix convolution of $x_1(n) = \{1, -1, -2, 3, 4\}$

$x_2(n) = \{1, 2, 3\}$

Solution

$x_1(n) = \{1, -1, -2, 3, 4\}$

$x_2(n) = \{1, 2, 3\}$

and $x_2(n) = \{1, 2, 3, 0, 0\}$

$$\begin{bmatrix} x_2(0) & x_2(4) & x_2(3) & x_2(2) & x_2(1) \\ x_2(1) & x_2(0) & x_2(4) & x_2(3) & x_2(2) \\ x_2(2) & x_2(1) & x_2(0) & x_2(4) & x_2(3) \\ x_2(3) & x_2(2) & x_2(1) & x_2(0) & x_2(4) \\ x_2(4) & x_2(3) & x_2(2) & x_2(1) & x_2(0) \end{bmatrix} \begin{bmatrix} x_1(0) \\ x_1(1) \\ x_1(2) \\ x_1(3) \\ x_1(4) \end{bmatrix} = \begin{bmatrix} y(0) \\ y(1) \\ y(2) \\ y(3) \\ y(4) \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 & 3 & 2 \\ 2 & 1 & 0 & 0 & 3 \\ 3 & 2 & 1 & 0 & 0 \\ 0 & 3 & 2 & 1 & 0 \\ 0 & 0 & 3 & 2 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ -2 \\ 3 \\ -1 \end{bmatrix} = \begin{bmatrix} 8 \\ -2 \\ -1 \\ -4 \\ -1 \end{bmatrix}$$

$y(n) = \{8, -2, -1, -4, -1\}$

Fast Fourier transform

FFT is a method for computing DFT with reduced number of calculations. The computational efficiency is achieved by employing divide and conquer approach

Comparison of no. of. computations (computational efficiency) in Direct DFT and FFT

$\log_2 2^3 = 3$

No of Points N $N = 2^m$	Direct computation: complex addition $N(N-1)$	Direct computation: complex multiplication N^2	RADIX-2 FFT: complex addition $N \log_2 N$	RADIX-2 FFT: complex multiplication $N/2 \log_2 N$
$4 = 2^2$	$4(4-1) = 12$	$4^2 = 16$	$4 \log_2 2^2 = 4(2) = 8$	$\frac{4}{2} \log_2 2^2 = 2(2) = 4$
$8 = 2^3$	$8(8-1) = 56$	$8^2 = 64$	$8 \log_2 2^3 = 8(3) = 24$	$\frac{8}{2} \log_2 2^3 = 12$
$16 = 2^4$	$16(16-1) = 240$	$16^2 = 256$	64	32
$32 = 2^5$	$32(32-1) = 992$	$32^2 = 1024$	160	80
$64 = 2^6$	$64(64-1) = 4032$	$64^2 = 4096$	384	192

FFT

Two types

(DIT) (Desimation in Time) · DIF

I/P bit reversal order · O/P bit reversal order

Normal		Bit reverse
0 - 0 0 0	→	0 0 0 (0)
1 - 0 0 1	→	1 0 0 (4)
2 - 0 1 0	→	0 1 0 (2)
3 - 0 1 1	→	1 1 0 (6)
4 - 1 0 0	→	0 0 1 (1)
5 - 1 0 1	→	1 0 1 (5)
6 - 1 1 0	→	0 1 1 (3)
7 - 1 1 1	→	1 1 1 (7)

bit reversal order

Twiddle factor (or) face factor

$$X(k) = \sum_{n=0}^{N-1} x(n) e^{\frac{-j2\pi nk}{N}}, \quad \text{for } k = 0, 1, \dots N-1$$

whatever in denominator will come over here

$W_N \rightarrow$ Twiddle factor $= e^{\frac{-j2\pi}{N}}$

$$X(k) = \sum_{n=0}^{N-1} x(n) W_N^{nk}, \quad k = 0, 1, \dots N-1$$

Properties of Twiddle factor

i) symmetry $W_N^{K+\frac{N}{2}} = -W_N^K$

ii) Periodicity $W_N^{K+N} = W_N^K$

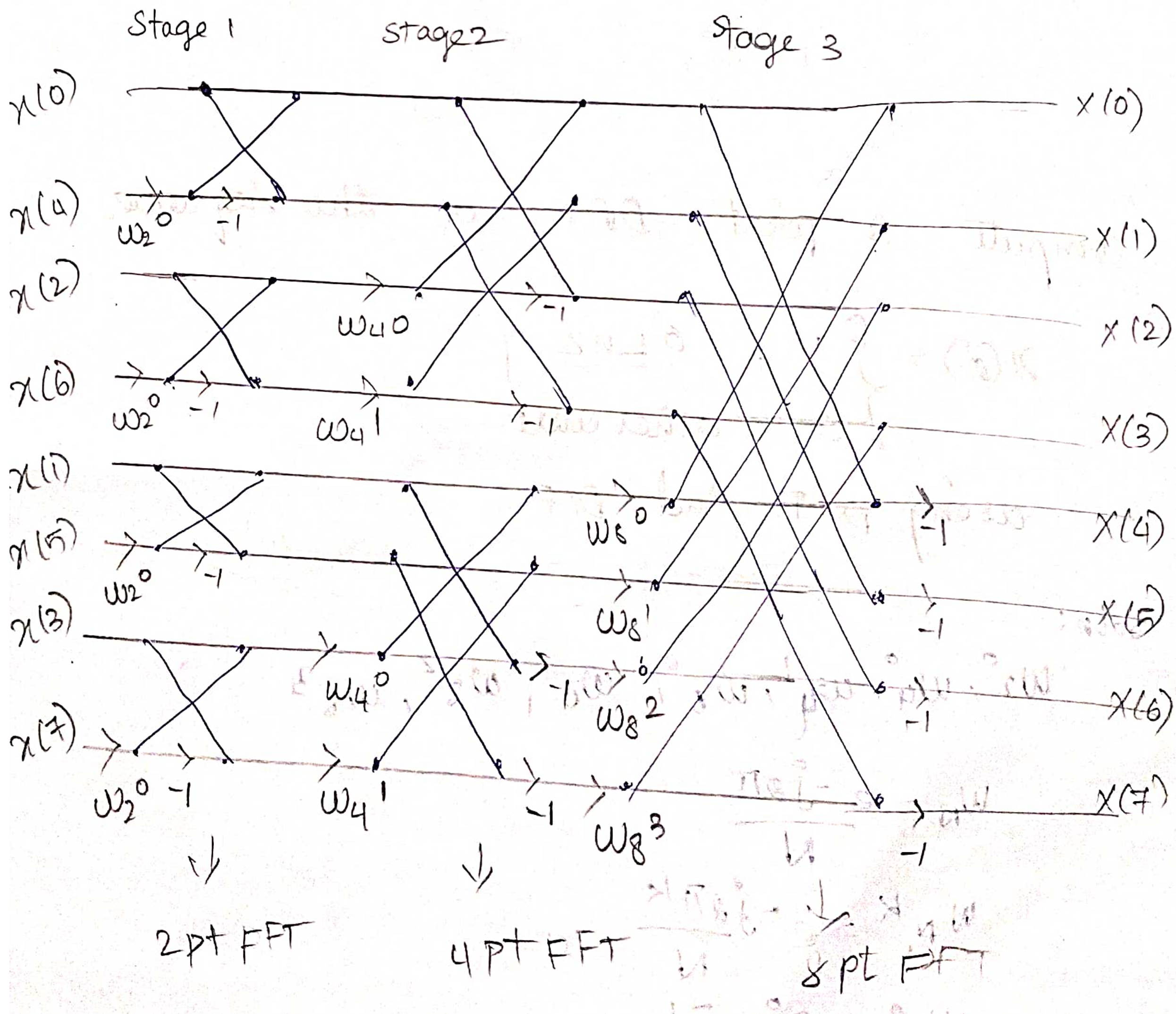
For example

8-point DFT, $(N=8)$

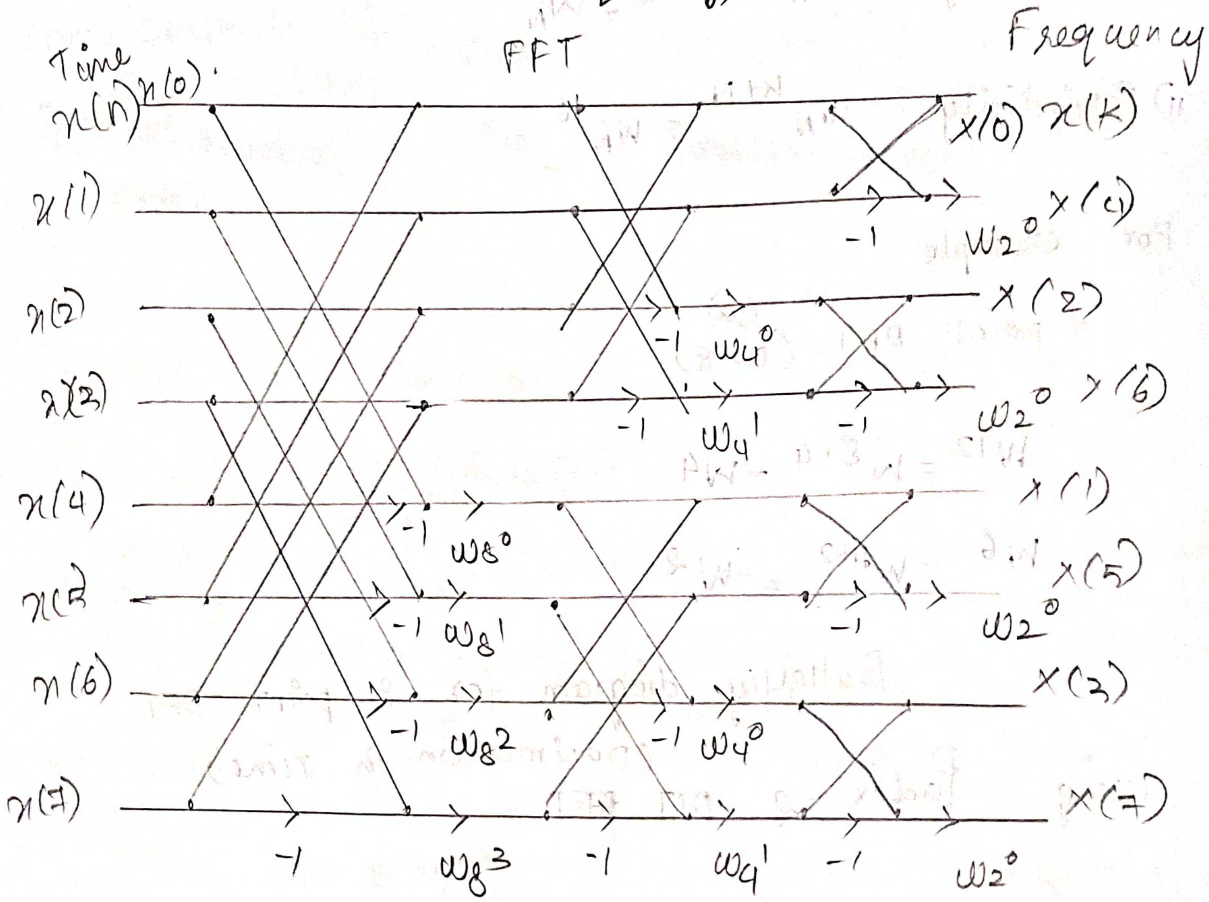
$W^{12} = W^{8+4} = W^4$ (Periodicity)

$W^6 = W^{4+2} = -W^2$ (symmetry

Butterfly diagram for 8 point DFT using Radix-2 DIT ~~DFT~~ (Decimation in Time)



Butterfly diagram of 8 point DFT using radix 2 DIF (Desimation in Frequency)



compute 8 point DFT of the sequence

$$x(n) = \begin{cases} 1, & 0 \le n \le 7 \\ 0, & \text{other wise} \end{cases}$$

using DIT and DIF

soln:

$W_2^0, W_4^0, W_4^1, W_8^0, W_8^1, W_8^2, W_8^3$

$$W_N = e^{\frac{-j2\pi}{N}}$$

$$W_N^K = e^{\frac{-j2\pi K}{N}}$$

$$W_2^0 = e^0 = 1$$

$$W_4^0 = e^0 = 1$$

$$W_4^1 = e^{\frac{-j2\pi}{4}}$$

$$= e^{-j\pi/2}$$

$$= \cos \pi/2 - j \sin \pi/2$$

$$= -j$$

$$W_8^0 = e^0 = 1$$

$$W_8^1 = e^{\frac{-j2\pi}{8}}$$

$$= e^{-j\pi/4}$$

$$= \cos \pi/4 - j \sin \pi/4$$

$$= 0.707 - j0.707$$

$$W_8^2 = e^{\frac{-j2\pi 2}{8}}$$

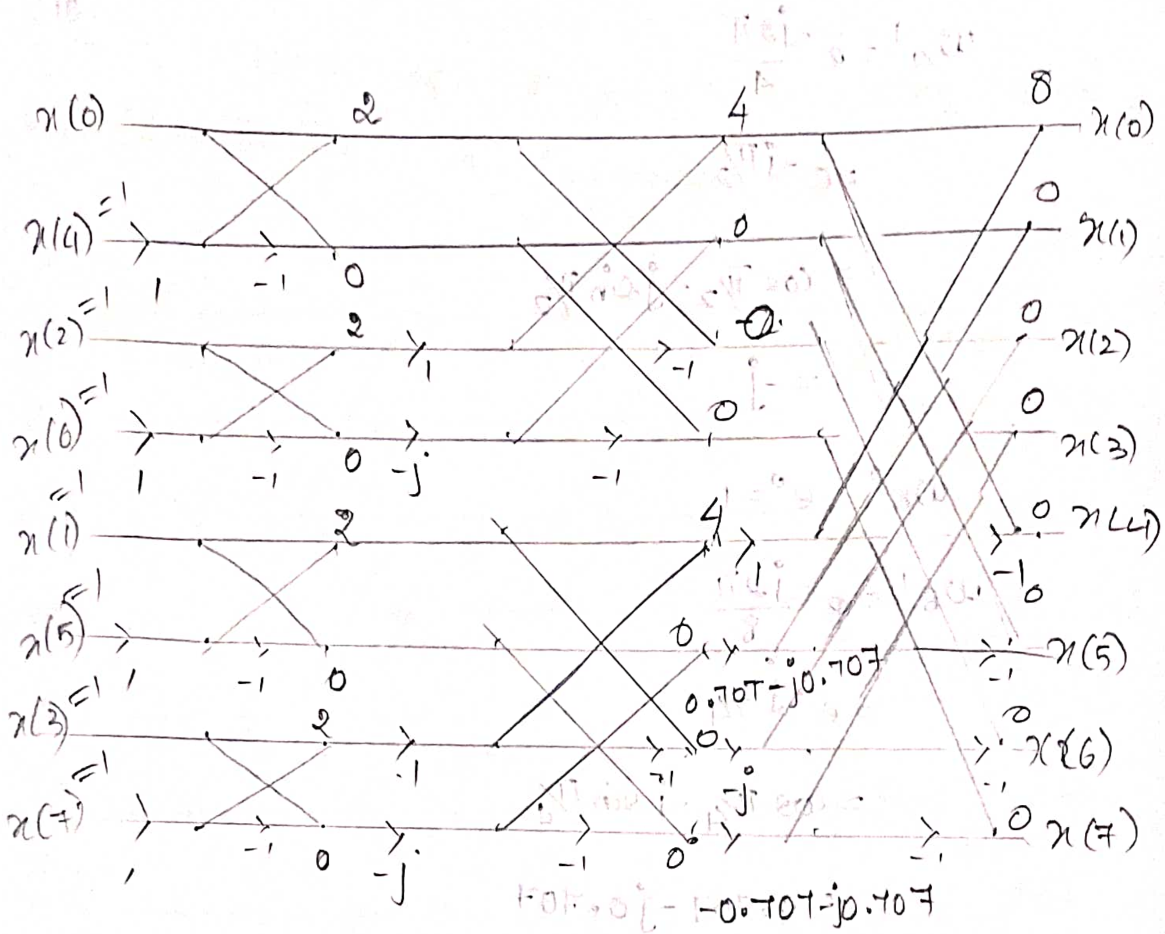
$$= e^{-j\pi/2}$$

$$= \cos \pi/2 - j \sin \pi/2$$

$$= -j$$

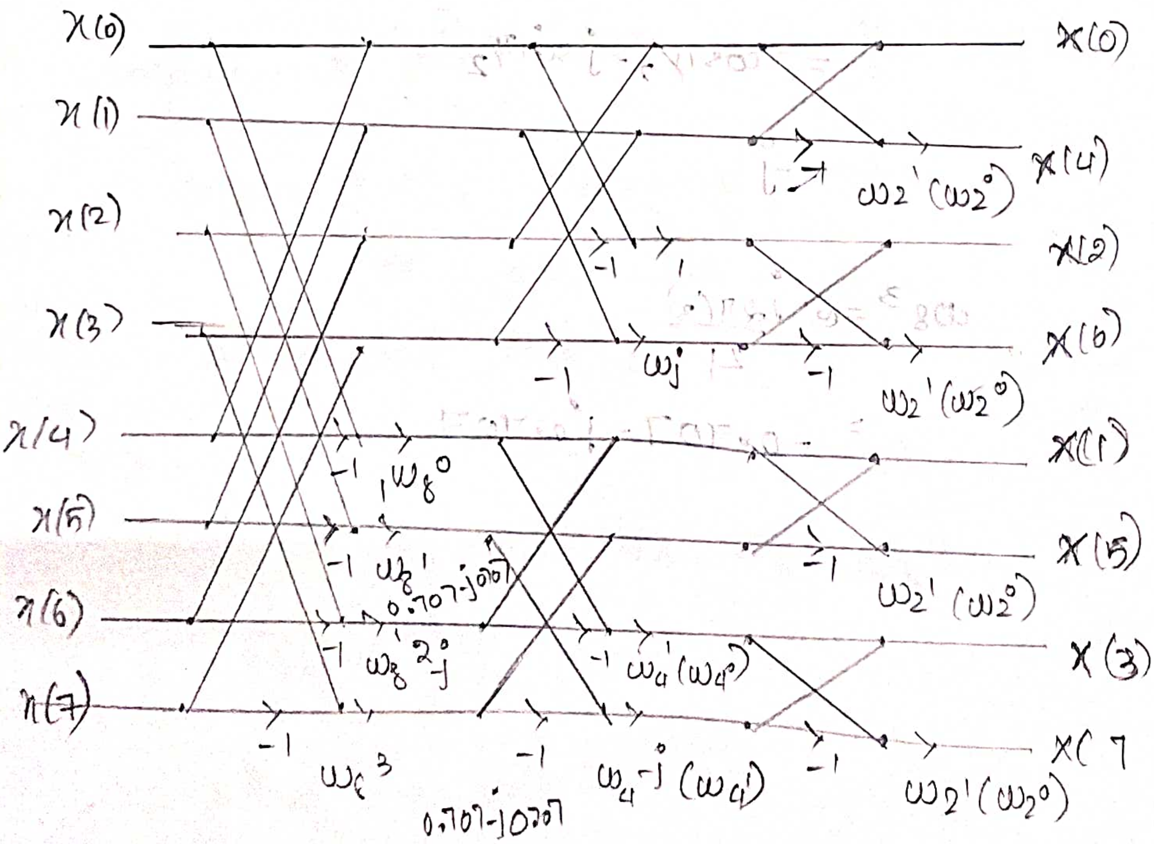
$$W_8^3 = e^{\frac{-j2\pi(3)}{4}}$$

$$= -0.707 - j0.707$$



$X(K) = \{8, 0, 0, 0, 0, 0, 0, 0\}$

$x(n) = \{1, 1, 1, 1, 1, 1, 1, 1\}$



IDFT

DFT $X(K) = \sum_{n=0}^{N-1} x(n) W^{nK}$

IDFT $x(n) = \frac{1}{N} \sum_{n=0}^{N-1} X(K) W^{-nK}$

$$x^*(n) = \frac{1}{N} \sum_{n=0}^{N-1} X^*(K) W^{nK}$$

$$N x^*(n) = \sum_{n=0}^{N-1} X^*(K) W^{nK}$$

Note 1

For IDFT Input is $X^*(K)$ Output is $N x^*(n)$

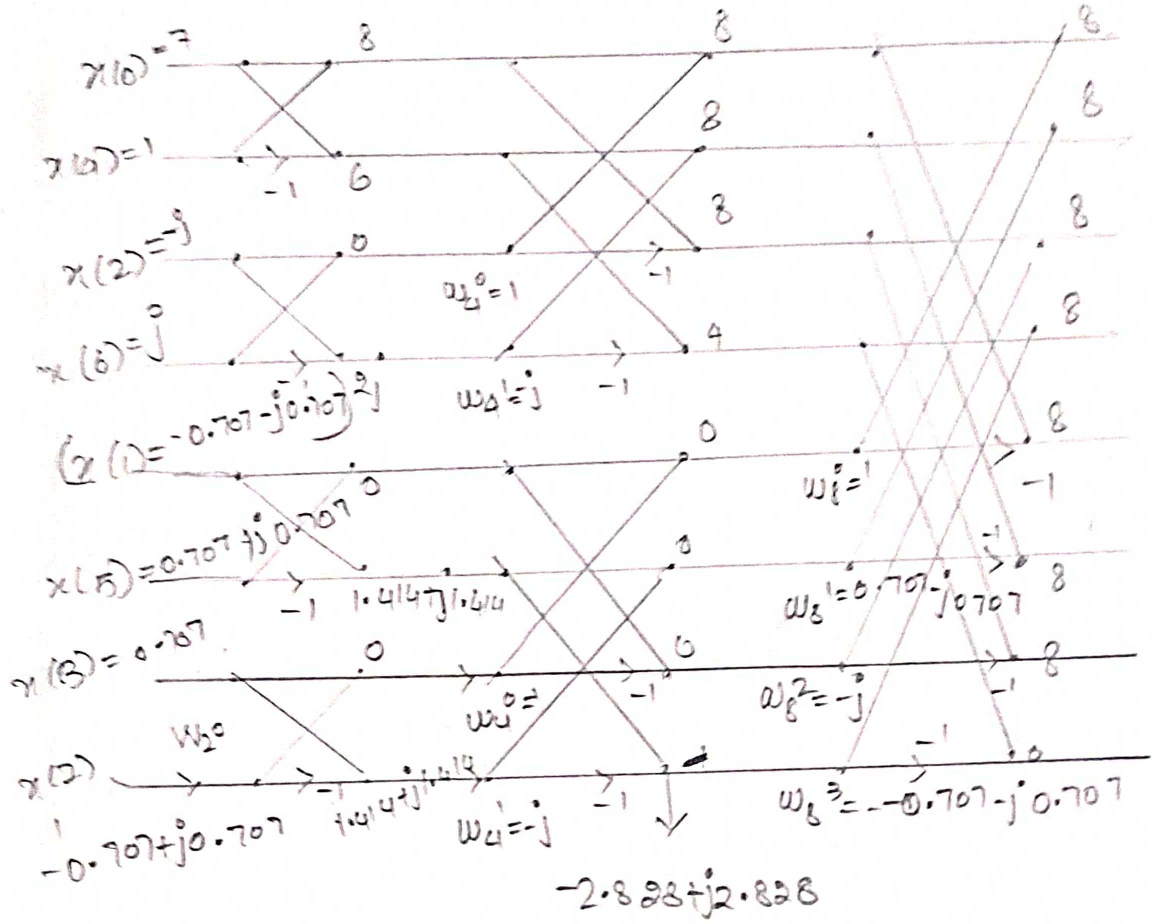
To find $x(n)$ apply conjugate to the o/p of butterfly diagram and divide by N

Note 2

For DIT Input - bit reversal
o/p - Normal order

For DIF Input - Normal order
o/p - bit reversal order

Compute IDFT of the sequence $X(k) = \{7, -0.707 - j0.707, -j, 0.707, 1, 0.707 + j0.707, j, -0.707 + j0.707\}$

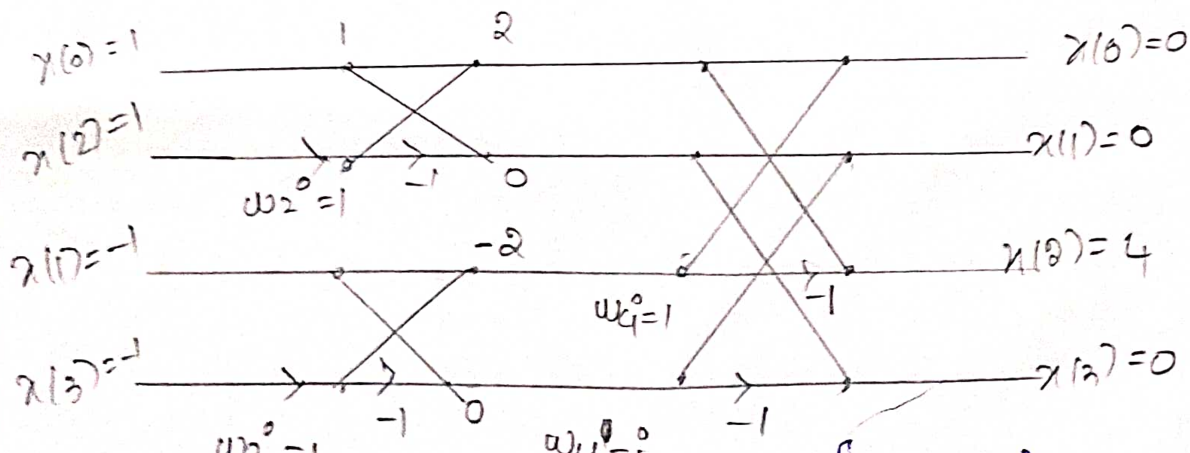


$$N x^*(n) = \{8, 8, 8, 8, 8, 8, 8, 0\}$$

$$x^*(n) = \{1, 1, 1, 1, 1, 1, 1, 0\}$$

$$x^*(n) = \{1, 1, 1, 1, 1, 1, 1, 0\}$$

Compute DFT of a sequence $x(n) = \{1, -1, 1, -1\}$ using DIT algorithm and DIF algorithm (4 point)



$$X(K) = \{0, 0, 4, 0\}$$

Unit 3

FIR Filters

Fourier series method of Designing FIR Filter

Low pass Filter
High pass Filter
Band pass Filter
Band Reject Filter

1. Design an ideal low pass Filter with a frequency response $H_d(e^{j\omega}) = 1$ for $-\pi/2 \le \omega \le \pi/2$

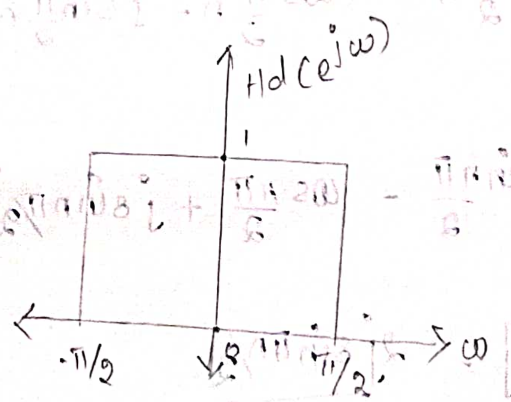
$= 0$ for $\pi/2 \le |\omega| \le \pi$.

Find the values of $h(n)$ for $N = 11$. Find $H(z)$. Plot the magnitude response.

Soln:

$H_d(e^{j\omega}) = 1$ for $-\pi/2 \le \omega \le \pi/2$

$= 0$ for $\pi/2 \le |\omega| \le \pi$



From the frequency response we infer that the filter is symmetrical about $N = 0$. that is

$Hd(-n) = hd(-n)$

Therefore,

$$hd(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} Hd(e^{j\omega})\, e^{j\omega n}\, d\omega$$

$$\boxed{hd(n) = 1/2\pi \int_{-\pi}^{\pi} Hd(e^{j\omega})\, e^{j\omega n}\, d\omega}$$

$\boxed{Hd(e^{j\omega}) = 1}$; $-\pi/2 \le \omega \le \pi/2$

$$hd(n) = \frac{1}{2\pi}\int_{-\pi/2}^{\pi/2} (1)\, e^{j\omega n}\, d\omega$$

$$= \frac{1}{2\pi}\left[\frac{e^{j\omega n}}{jn}\right]_{-\pi/2}^{\pi/2} = \frac{1}{2\pi}\left[\frac{e^{j\pi n/2}}{jn} - \frac{e^{-j\frac{\pi n}{2}}}{jn}\right]$$

$$= \frac{1}{2\pi}\left[\cos\frac{\pi}{2}n + j\sin\frac{\pi}{2}n - \left(\cos\frac{\pi}{2}n - j\sin\frac{\pi}{2}n\right)\right]$$

$$= \frac{1}{2\pi}\left[\cos\frac{n\pi}{2} + j\sin\frac{n\pi}{2} - \cos\frac{n\pi}{2} + j\sin n\pi/2\right]$$

$$= 1/2\pi\left[\frac{2j\sin n\pi}{2jn}\right] = \frac{2j\sin \pi n/2}{2n\pi}$$

$$= \frac{\sin n\pi/2}{n\pi} \quad -\infty \le n \le \infty$$

Truncating hd(n) to 11 samples ~~are~~ we have

$$h(n) = \begin{cases} \sin \frac{\pi n}{2} & \text{for } |n| \le 5 \\ 0 & \text{for otherwise} \end{cases}$$

$$h(n) = \begin{cases} \frac{\sin \pi n/2}{\pi n} & ; |n| \le 5 \\ 0 & ; \text{otherwise} \end{cases}$$

if $N = \cancel{9} \Rightarrow |n| \le \cancel{4}\ 3$

if $N = 7 = |n| \le 3$

$$hd(n) = \frac{1}{2\pi} \int_{-\pi/2}^{\pi/2} e^{j\omega n}\, d\omega \quad \therefore \boxed{e^{j\omega n} = 1}$$

$$= \frac{1}{2\pi} [1]_{-\pi/2}^{\pi/2}$$

$$= 1/2\pi \left[\pi/2 + \pi/2 \right]$$

$$= \frac{1}{2\pi} \left[2\pi/2 \right] = 1/2$$

$$h(0) = 1/2 = 0.5$$

$$h(1) = \frac{\sin \pi/2}{\pi} = \frac{1}{\pi} = 0.3183 \cdot h(-1)$$

$\sin n\pi = 0$

$$h(2) = \frac{\sin 2\pi/2}{2\pi} = \sin \pi/2\pi = 0 = h(-2)$$

$$h(3) = \frac{\sin 3\pi/2}{3\pi} = -1/3\pi = -0.106 = h(-3)$$

$$h(4) = \frac{\sin 4\pi/2}{4\pi} = \frac{\sin 2\pi}{4\pi} = 0 = h(-4)$$

$$h(5) = \frac{\sin 5\pi/2}{5\pi} = \frac{1}{5\pi} = 0.0636 = h(-5)$$

$\boxed{h(-n) = h(n)} \Rightarrow$ Symmetric

Transfer function of filter is given by

$$H(z) = h(0) + \sum_{n=1}^{N-1/2} \left[h(n) \left(z^n + z^{-n} \right) \right]$$

$$= 0.5 + \sum_{n=1}^{5} \left[h(n) \left(z^n + z^{-n} \right) \right]$$

$$= 0.5 + h(1) \left[z^1 + z^{-1} \right] + h(2) \left[z^2 + z^{-2} \right]$$

$$+ h(3) \left[z^3 + z^{-3} \right] + h(4) \left[z^4 + z^{-4} \right] + h(5) \left[z^5 + z^{-5} \right]$$

$$= 0.5 + 0.3183 \left[z^1 + z^{-1} \right] - 0.106 \left[z^3 + z^{-3} \right] + 0.06366 \left[z^5 + z^{-5} \right]$$

Transfer function of the Realizable filter is

$$H'(z) = z^{-\frac{(N-1)}{2}} H(z)$$

$$= z^{-5} H(z)$$

$$= z^{-5} \left[0.5 + 0.3183 z^1 + 0.3183 z^{-1} - 0.106 z^3 - 0.106 z^{-3} + 0.06366 z^5 + 0.06366 z^{-5} \right]$$

$$= 0.5 z^{-5} + 0.318 z^1 z^{-5} + 0.3183 z^{-1} z^{-5} - 0.106 z^3 z^{-5} - 0.106 z^{-3} z^{-5} + 0.06366 z^5 z^{-5} + 0.06366 z^{-5} z^{-5}$$

$$= 0.5 z^{-5} + 0.318 z^{-4} + 0.3183 z^{-6} - 0.106 z^{-2} - 0.106 z^{-8} + 0.06366 z^0 + 0.06366 z^{-10}$$

$$= 0.5z^{-5} + 0.318z^{-4} + 0.3183z^{-6} - 0.106z^{-2}$$

$$- 0.106z^{-8} + 0.06366 + 0.06366z^{-10}.$$

The filter coefficients $h(n)$ is the coefficient of z^{-n}

$h(0) = 0.06366 = h(10)$ $\quad h(0) = h(N-1) = h(10)$

$h(1) = 0 = h(9)$ $\quad h(1) = h(9)$

$h(2) = h(8) = -0.106$ $\quad h(2) = h(8)$

$h(3) = h(7) = 0$

$h(4) = h(6) = 0.3183$

$h(5) = 0.5$

The frequency response is given by

$$|H(e^{j\omega})| = \sum_{n=0}^{N-1/2} a(n) \cos\omega n$$

where,

$$a(0) = h(N-1/2)$$

$$a(n) = 2h\left(\frac{N-1}{2} - n\right)$$

$N=11$

$a(0) = h(5) = 0.5$

$$a(1) = 2h\left(\frac{11-1}{2} - 1\right)$$

$= 2h(4) = 2 \times 0.3183$

$= 0.6366.$

$a(2) = 2 \times h(5-2)$

$= 0$

$a(3) = 2h(5-3)$

$= 2h(2)$

$= 2 \times -0.106$

$= -0.212$

$a(4) = 2h(5-4)$

$= 2 \times h(1)$

$= 0$

$a(5) = 2h(0)$

$= 2 \times 0.06366$

$= 0.12732$

$$|\bar{H}(e^{j\omega})| = \sum_{n=0}^{5} a(n) \cos \omega n$$

$$|\bar{H}(e^{j\omega})| = a(0)\cos 0 + a(1)\cos\omega + \underbrace{a(2)}_{\to 0}\cos 2\omega + a(3)\cos 3\omega + \underbrace{a(4)}_{\to 0}\cos 4\omega + a(5)\cos 5\omega$$

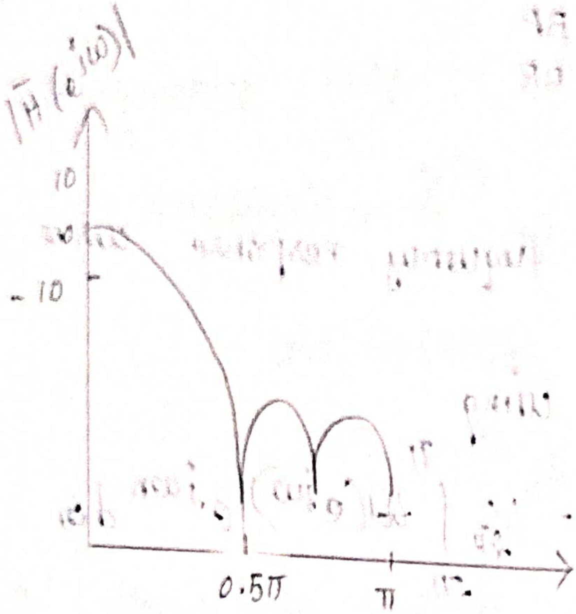
$= 0.5\cos 0 + 0.6366\cos\omega - 0.212\cos 3\omega + 0.12732\cos 5\omega$

$= 0.5 + 0.6366\cos\omega - 0.212\cos 3\omega + 0.12732\cos 5\omega$

Magnitude response in dB

$$|\bar{H}(e^{j\omega})|_{dB} = 20\log|\bar{H}(e^{j\omega})|$$

ω (in degree)	0	10	20		180
$\lvert \bar{H}(e^{j\omega}) \rvert_{dB}$	0.4	0.21			-26



Design of FIR Filters

- Fourier series method (LPF, HPF, BPF, BRF)
- Windowing method (Rectangular, Hamming, Hanning, Blackmann)
- Frequency sampling method → not in syllabus

Procedure for fourier Series

Given

$Hd(e^{j\omega})$ or LP
HP
BP
BR

Step 1.

Draw ideal frequency response curve

step 2:
find $Hd(n)$ using

$$h_d(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} Hd(e^{j\omega})\, e^{j\omega n}\, d\omega$$

Step 3:

For fourier series method

h(n) is found by Truncating

hd(n) to n samples

Step 4:

Find $h(0), h(1), h(-1), \ldots h(N-1/2), h\left(-\frac{N-1}{2}\right)$

Step 5:

Transfer function of the filter is given by

$$H(z) = h(0) + \sum_{n=1}^{N-1/2}\left[h(n)\left(z^{n}+z^{-n}\right)\right]$$

Step 6: Transfer function of realizable filter is $H'(z) = z^{-(N-1)/2}\, H(z)$

step 7:

From Transfer function find the filter coefficients of causal filter i.e $h(0)$ to $h(N-1)$

Step 8:

Frequency response is given by

$$|\bar{H}(e^{j\omega})| = \sum_{n=0}^{N-1/2} a(n) \cos\omega n, \text{ where}$$

$$a(0) = h(N-1/2)$$

$$a(n) = 2h(N-1/2 - n)$$

Step 9: Magnitude response

$$|\bar{H}(e^{j\omega})|_{dB} = 20 \log |\bar{H}(e^{j\omega})|$$

Step 10:

Draw freq response plot

Design of FIR filter using windowing

1) write $H_d(e^{j\omega})$ for the given filter and draw ideal frequency response curve

2) Find $h_d(n)$ using

$$h_d(n) = \frac{1}{2\pi}\int_{-\infty}^{\infty} H_d(e^{j\omega})\, e^{j\omega n}\, d\omega$$

3) $h(n) = h_d(n) \cdot \omega_{hn}(n)$

4) Find $h(0), h(1), h(2), \ldots h(N-1/2), h(\frac{N-1}{2}-1)$

5) Transfer function of the filter is given by

$$H(z) = h(0) + \sum_{n=1}^{N-1/2} \left[h(n)(z^{n} + z^{-n})\right]$$

6) Transfer function of realizable filter is

$$H(z) = z^{-(N-1)/2}\, H(z)$$ —x— Transfer func

7) From Transfer function find the filter coefficients of Causal filter i.e $h(0)$ to $h(N-1)$

8) Frequency response is given by

$$|\bar{H}(e^{j\omega})| = \sum_{n=0}^{N-1/2} a(n) \cos\omega n$$ → sin (Blackman) ↓ antisymmetric

$$a(0) = h(N-1/2)$$

$$a(n) = 2h(\frac{N-1}{2} - n)$$

9) Magnitude response

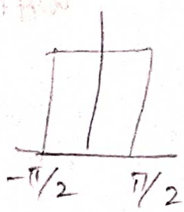
$$|\bar{H}(e^{j\omega})| = 20 \log |\bar{H}(e^{j\omega})|$$

10) Draw frequency response plot

Frequency Response for different Filters

i) LPF

$$H_d(e^{j\omega}) = 1 \quad \text{for} \quad -\pi/2 \le \omega \le \pi/2$$
$$= 0 \quad \text{for} \quad \pi/2 \le |\omega| \le \pi$$

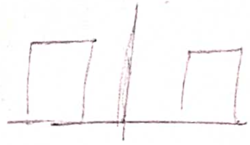


ii) HPF

→ $-\pi/2$ to $-\pi = 1$

$\pi/2$ to $\pi = 1$

$$H_d(e^{j\omega}) = 1 \quad \text{for} \quad \pi/4 \le |\omega| \le \pi$$
$$= 0 \quad \text{for} \quad |\omega| \le \pi/4$$



iii) BPF

$$H_d(e^{j\omega}) = 1 \quad \text{for} \quad \pi/4 \le |\omega| \le 3\pi/4$$
$$= 0 \quad \text{otherwise}$$

iv) BRF

$$H_d(e^{j\omega}) = 1 \quad \text{for} \quad |\omega| \le \pi/3 \quad \text{and} \quad |\omega| \ge 2\pi/3$$
$$= 0 \quad \text{otherwise}$$

Formula for window sequences

1) Rectangular window

$$\omega_R(n) = 1 \; ; \; -(N-1)/2 \le n \le (N-1)/2$$
$$= 0 \quad \text{otherwise}$$

2) Hanning

$$\omega_{Hn}(n) = 0.5 + 0.5 \cos 2\pi n/(N-1) \; ;$$
$$-\frac{(N-1)}{2} \le n \le \frac{(N-1)}{2}$$
$$= 0 \quad \text{otherwise}$$

3) Hamming

$$w_H(n) = 0.54 + 0.46 \cos(2\pi n/_{N-1}) \; ; \; -\frac{(N-1)}{2} \le n \le \frac{(N-1)}{2}$$

$$= 0 \; ; \text{ otherwise}$$

4) Blackmann

$$w_B(n) = 0.42 + 0.5 \cos \frac{2\pi n}{N-1} + 0.08 \cos \frac{4\pi n}{N-1} \; ;$$

$$-\frac{(N-1)}{2} \le n \le \frac{N-1}{2}$$

$$= 0 \; : \text{ otherwise}$$

Examples

Design FIR highpass Filter using hanning window

Find the Transfer function

Soln:

Frequency response for highpass Filter

$$Hd(e^{j\omega}) = 1 \quad \text{for} \quad \pi/_2 < |\omega| < \pi$$

$$= 0 \quad \text{for} \quad |\omega| \le \pi/_4$$

$$hd(n) = \frac{1}{2\pi} \int_{-\infty}^{\infty} Hd(e^{j\omega}) \, e^{j\omega n} \, d\omega$$

$$= 1/_{2\pi} \left[\int_{-\pi/4}^{-\pi/4} 1 \cdot e^{j\omega n} \, d\omega + \int_{\pi/4}^{\pi/4} 1 \cdot e^{j\omega n} \cdot d\omega \right]$$

$$= 1/_{2\pi} \left[\left(\frac{e^{j\omega n}}{jn} \right)_{-\pi}^{-\pi/4} + \left(\frac{e^{j\omega n}}{jn} \right)_{\pi}^{\pi/4} \right]$$

$$= 1/_{2\pi jn} \left[\left(e^{j\omega n} \right)_{\pi}^{-\pi/4} + \left(e^{j\omega n} \right)_{\pi}^{\pi/4} \right]$$

$$= \frac{1}{2\pi jn}\left[e^{-j\pi n/4} - e^{-j\pi n} + e^{j\pi n} - e^{+jn\pi/4}\right]$$

$$= \frac{1}{2\pi jn}\left[e^{j\pi n} - e^{-j\pi n} - \left(e^{j\pi n/4} - e^{-j\pi n/4}\right)\right]$$

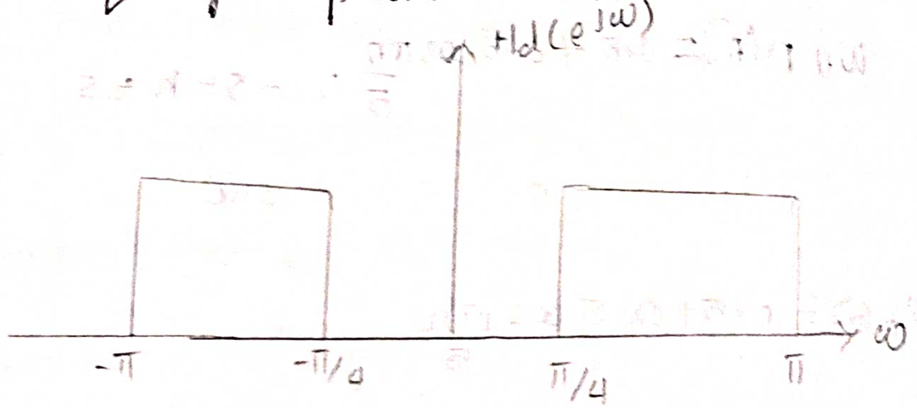
$$\sin\theta = \frac{e^{j\theta} - e^{-j\theta}}{2j}$$

$$\cos\theta = \frac{e^{j\theta} + e^{-j\theta}}{2}$$

$$= \frac{1}{2\pi jn}\left[2j\sin\pi n - 2j\sin\pi n/4\right]$$

$$hd(n) = \frac{1}{\pi n}\left[\sin\pi n - \sin\pi n/4\right], \quad -\infty \le n \le \infty$$

Frequency response



$$h(n) = hd(n) \cdot w_{Hn}(n)$$

$$hd(n) = \frac{1}{\pi n}\left[\sin\pi n - \sin\pi n/4\right]$$

$$hd(0) = \lim_{n\to 0} \frac{1}{2\pi}\left[\int_{-\pi}^{\pi/4} e^{j\omega n}\,d\omega + \int_{\pi/4}^{\pi} e^{j\omega n}\,d\omega\right]$$

$$= \frac{1}{2\pi}\left[\left[-\pi/4 - (-\pi)\right] + (\pi - \pi/4)\right]$$

$$= \frac{1}{2\pi}\left[3\pi/4 + 3\pi/4\right] = \frac{3}{4} = 0.75$$

$$h_d(1) = h_d(-1) = \frac{1}{\pi}\left[\sin\pi(1) - \sin\frac{\pi}{4}(1)\right] = -0.225$$

$$h_d(2) = h_d(-2) = \frac{1}{\pi(2)}\left[\sin\pi(2) - \sin\frac{\pi}{4}(2)\right] = 0.159$$

$$h_d(3) = h_d(-3) = \frac{1}{\pi(3)}\left[\sin\pi(3) - \sin\frac{\pi}{4}(3)\right] = -0.075$$

$$h_d(4) = h_d(-4) = \frac{1}{\pi(4)}\cdot\left[\sin\pi(4) - \sin\frac{\pi}{4}(4)\right] = 0$$

$$h_d(5) = h_d(-5) = \frac{1}{\pi(5)}\left[\sin\pi(5) - \sin\frac{\pi}{4}(5)\right] = 0.045$$

Hanning

$$W_{Hn}(n) = 0.5 + 0.5\cos\frac{\pi n}{5}; \quad -5 \le n \le 5$$

$$0 \quad ; \quad 0.\omega$$

$$\omega_{Hn}(0) = 0.5 + 0.5\cos\frac{\pi n}{5}$$

$$W_{Hn}(1) = W_{Hn}(-1) = 0.5 + 0.5\cos\frac{\pi}{5}(1) = 0.9045$$

$$W_{Hn}(2) = W_{Hn}(-2) = 0.5 + 0.5\cos\frac{\pi}{5}\cdot(2) = 0.655$$

$$W_{Hn}(3) = \omega_{Hn}(-3) = 0.5 + 0.5\cos\frac{\pi}{5}(3) = 0.345$$

$$W_{Hn}(4) = W_{Hn}(-4) = 0.5 + 0.5\cos\frac{\pi}{5}(4) = 0.0954$$

$$W_{Hn}(5) = W_{Hn}(-5) = 0.5 + 0.5\cos\frac{\pi}{5}(5) = 0$$

$h(n) = h_d(n) w_{Hn}(n) \quad -5 \le n \le 5$

$h(0) = h_d(0) \, w_{Hn}(0) = 0.75$

$h(-1) = h(1) = h_d(1) \, w_{Hn}(1)$

$= (-0.225)(0.905) = -0.204$

$h(-2) = h(2) = h_d(2) \, w_{Hn}(2)$

$= (-0.159)(0.655) = 0.104$

$h(-3) = h(3) = h_d(3) \, w_{Hn}(3)$

$= -0.026$

$h(-4) = h(4) = h_d(4) \, w_{Hn}(4) = 0$

$h(-5) = h(5) = h_d(5) \, w_{Hn}(5) = 0$

Transfer func of the filter is

$$H(z) = h(0) + \sum_{n=1}^{5} h(n) \left[z^{-n} + z^{n} \right]$$

$= 0.75 - 0.204 (z + z^{-1}) - 0.104 (z^{2} + z^{-2}) - 0.026 (z^{3} + z^{-3})$

TF of realizable filter

$H(z) = z^{-(N-1)/2} \, H(z)$

$= z^{-5} H(z)$

$= z^{-5} \big[0.75 - 0.204 z - 0.204 z^{-1} - 0.104 z^{2} - 0.104 z^{-2}$

$- 0.026 z^{3} - 0.026 z^{-3} \big]$

$$= 0.75z^{-5} - 0.204z^{-4} - 0.204z^{-6} - 0.104z^{-3} - 0.104z^{-7}$$

$$- 0.026z^{-2} - 0.026z^{-8}$$