Unit 2 Realization of Digital filter.

Direct form - I

→ Let us consider an LTI recursive system described by the difference eqn.

$$y(n) = -\sum_{k=1}^{N} a_k y(n-k) + \sum_{k=0}^{M} b_k x(n-k)$$

$$= -a_1 y(n-1) - a_2 y(n-2) \cdots a_{N-1} y(n-N+1)$$

$$- a_N y(n-N) + b_0 x(n) + b_1 x(n-1) \cdots$$

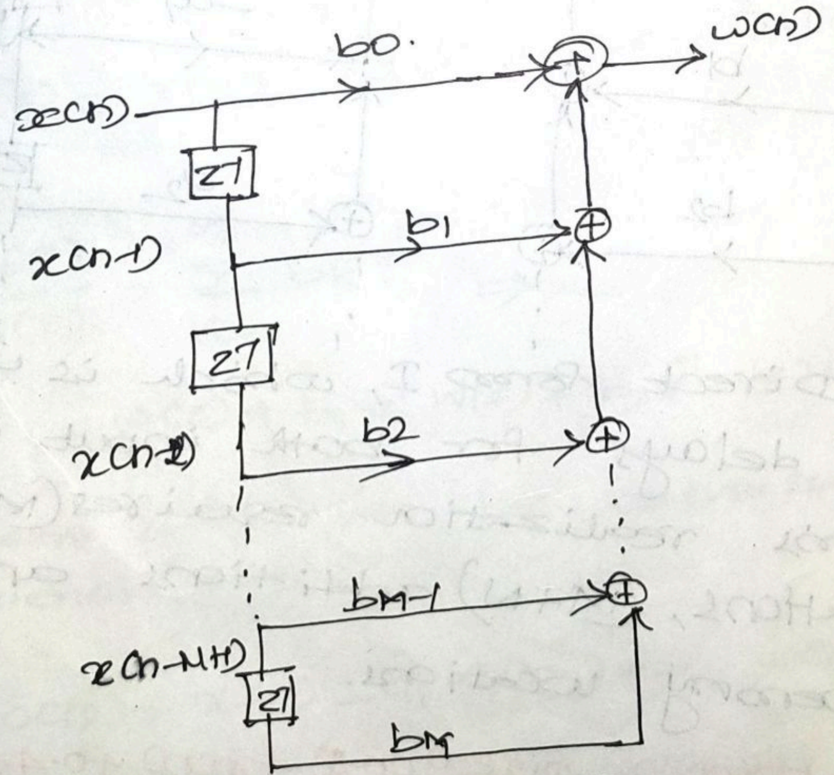
$$+ b_M x(n-M) \longrightarrow (1)$$

Let

$$b_0 x(n) + b_1 x(n-1) + \cdots + b_M x(n-M) = w(n) \rightarrow (2)$$

then $y(n) = -a_1 y(n-1) - a_2 y(n-2) + \cdots - a_N y(n-N) + w(n)$ → (3)

eqn (2) can be realized as.



similarly eqn ③ too. fig 1

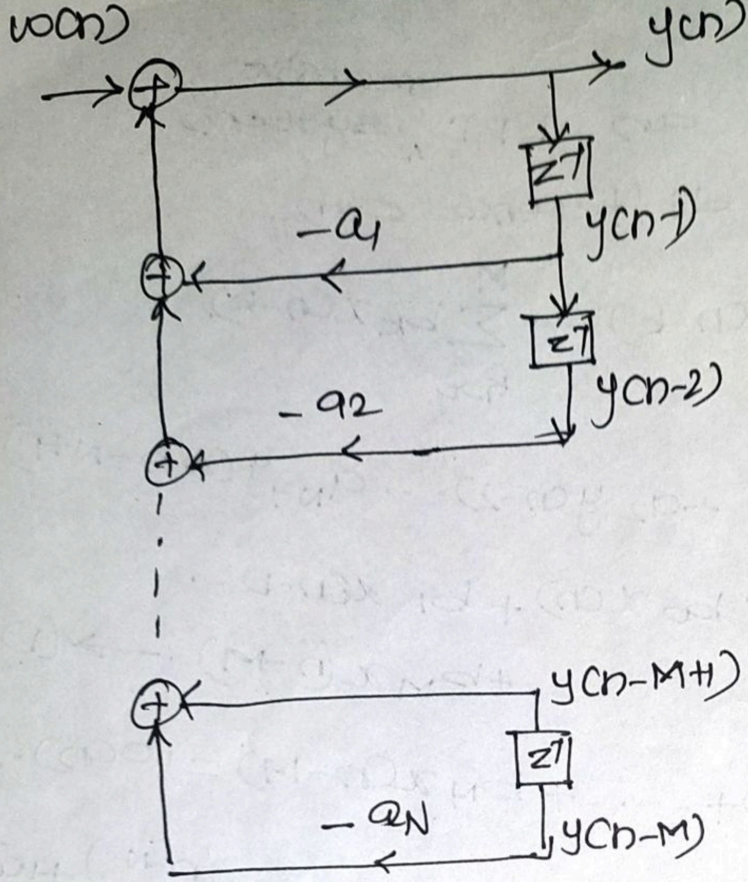


fig (2)

combine fig (1) & (2).

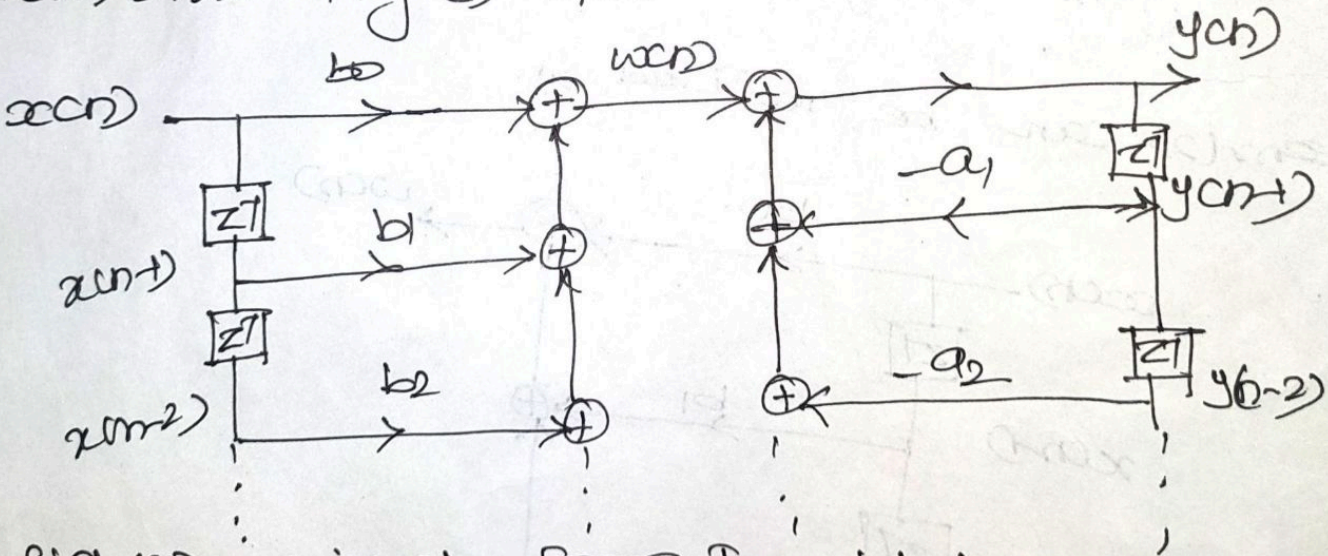


fig (3): Direct form I, which is used separate delays for both input and output. This realization requires (M+N+1) multiplications, (M+N) additions and (M+N+1) memory locations.

Exer: $y(n) = 0.5\,y(n-1) - 0.25\,y(n-2) + x(n) + 0.4\,x(n-1)$

Direct form - II

→ Consider the difference eqn

$$y(n) = -\sum_{k=1}^{N} a_k y(n-k) + \sum_{k=0}^{M} b_k x(n-k) \rightarrow (1).$$

→ The system form of eqn (1) can be expressed as,

$$H(z) = \frac{Y(z)}{X(z)} = \frac{\sum_{k=0}^{M} b_k z^{-k}}{1+\sum_{k=1}^{N} a_k z^{-k}} \rightarrow (2).$$

Let

$$\frac{Y(z)}{X(z)} = \frac{Y(z)}{W(z)} \cdot \frac{W(z)}{X(z)} \quad \text{where}$$

$$\frac{W(z)}{X(z)} = \frac{1}{1+\sum_{k=1}^{N} a_k z^{-k}} \Rightarrow$$

$$W(z) = X(z) - a_1 z^{-1} W(z) - a_2 z^{-2} W(z) \dots - a_N z^{-N} W(z) \rightarrow (3)$$

and

$$\frac{Y(z)}{W(z)} = \sum_{k=0}^{M} b_k z^{-k} \Rightarrow$$

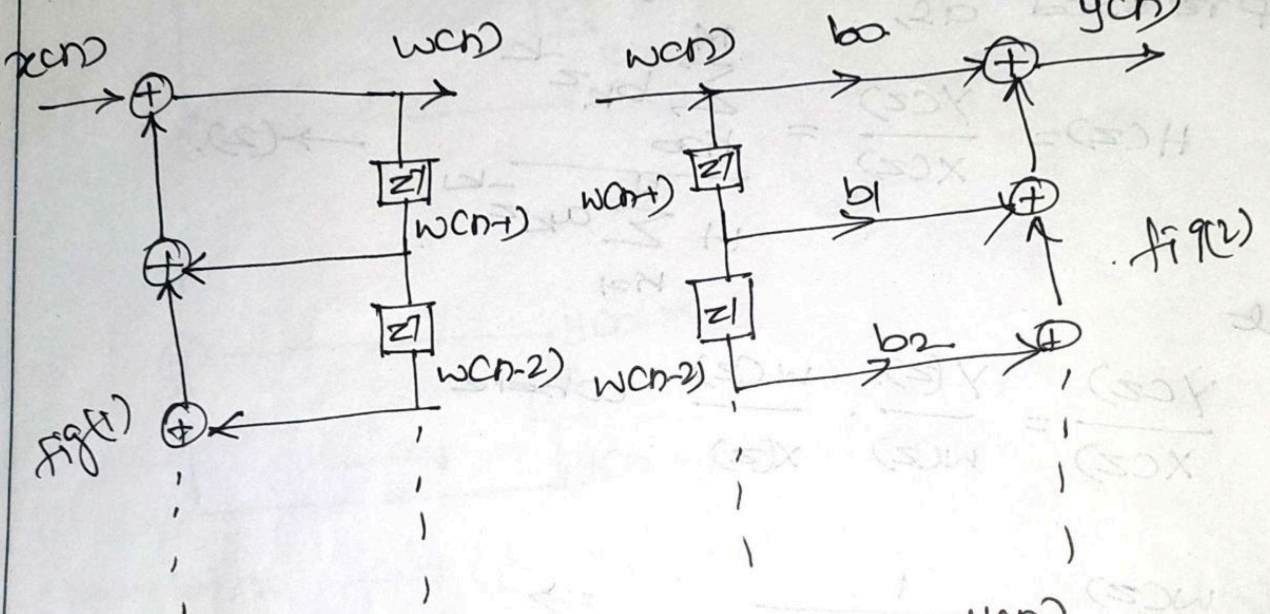
$$Y(z) = b_0 W(z) + b_1 z^{-1} W(z) + b_2 z^{-1} W(z) + \dots + b_M z^{-M} W(z) \rightarrow (4)$$

now eqn (2) & (3) can be expressed in difference eqn

(ii) $w(n) = x(n) - a_1 w(n-1) - a_2 w(n-2) \dots \rightarrow (5)$

$y(n) = b_0 w(n) + b_1 w(n-1) + b_2 w(n-2) \dots \rightarrow (6)$

Eqn (5) & (6) shows that same delay terms used to represent both $w(n)$, $w(n-1)$ ... and $y(n)$. So realization of (5) & (6) $w(n)$ and $y(n)$ are.



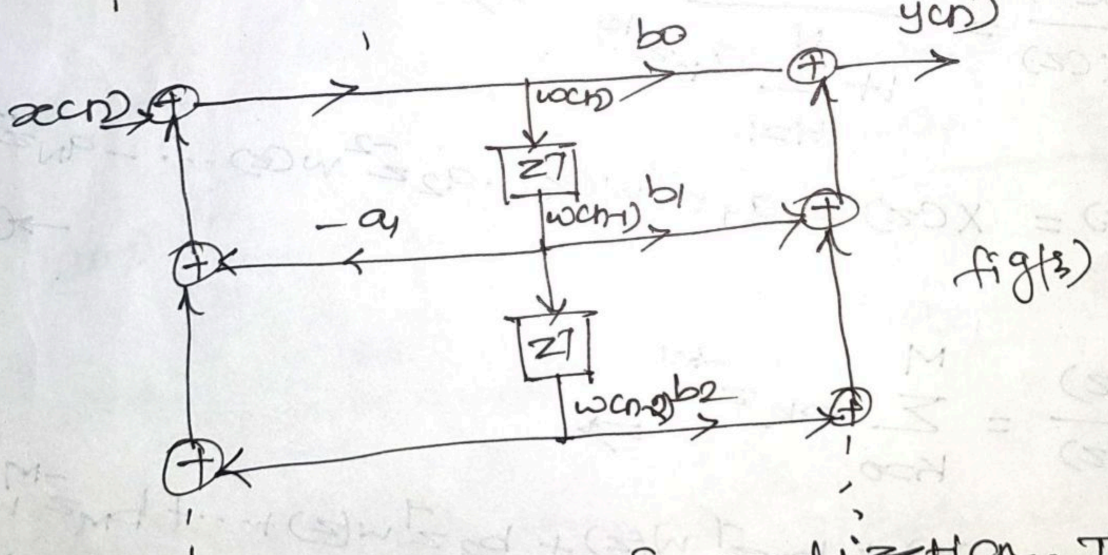


Fig (3) is called DF-II realization. This structure requires $(M+N+1)$ multiplications, $(M+N)$ additions, and the maximum of $\{M,N\}$ memory locations.

∴ the DF-II realization minimizes the number of memory locations, it is said to be canonic.

Determine the DF-II for the following system $y(n) = -0.1\,y(n-1) + 0.72\,y(n-2) + 0.7\,x(n) - 0.252\,x(n-2)$.

The system fun is given by

$$\frac{Y(z)}{X(z)} = \frac{0.7 - 0.252z^{-2}}{1 + 0.1z^{-1} - 0.72z^{-2}} \rightarrow (1)$$

Let

$$\frac{Y(z)}{W(z)} = 0.7 - 0.252z^{-2}$$

$$Y(z) = 0.7\,W(z) - 0.252\,\cancel{z}z^{-2}\,W(z)$$

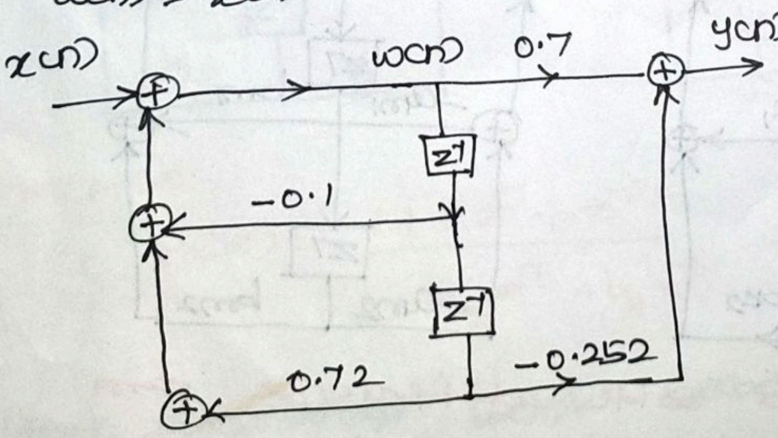
$$y(n) = 0.7\,w(n) - 0.252\,w(n-2). \rightarrow (2).$$

||^{ly}

$$\frac{W(z)}{X(z)} = \frac{1}{1 + 0.1z^{-1} - 0.72z^{-2}}$$

$$W(z) = X(z) - 0.1\,z^{-1}W(z) + 0.72\,z^{-2}\,W(z) \rightarrow (3).$$

$$w(n) = x(n) - 0.1\,w(n-1) + 0.72\,w(n-2) \rightarrow (3).$$



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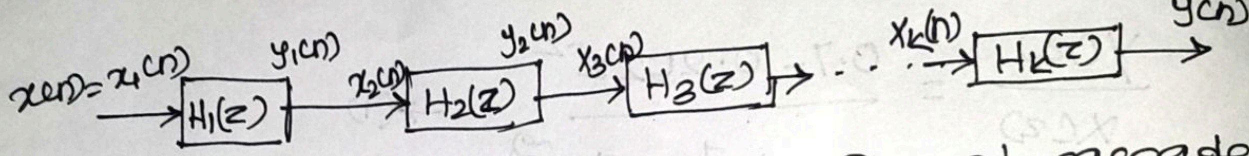
Cascade form:-

→ Let us consider an IIR system with system function

$$H(z) = H_1(z)\, H_2(z) \dots H_K(z) \rightarrow (1)$$ is can

be ~~be~~ represented using block diagrams.



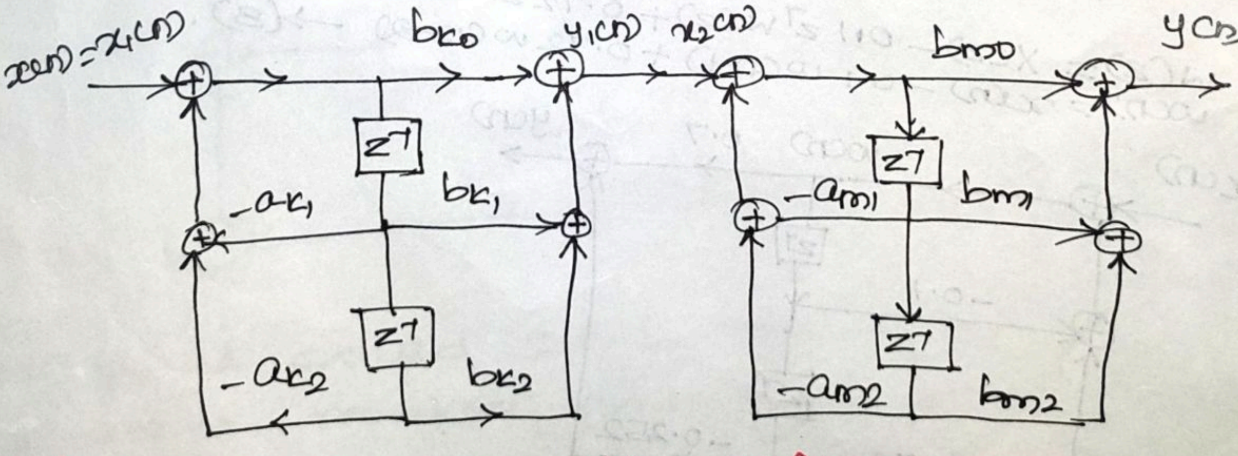
→ Now realize each $H_K(z)$ in DF-II and cascade all structures.

Let us take whole system T.F.

$$H(z) = \frac{(b_{k0} + b_{k1}z^{-1} + b_{k2}z^{-2})(b_{m0} + b_{m1}z^{-1} + b_{m2}z^{-2})}{(1 + a_{k1}z^{-1} + a_{k2}z^{-2})(1 + a_{m1}z^{-1} + a_{m2}z^{-2})} \rightarrow (2)$$

$$H(z) = H_1(z)\, H_2(z).$$

Realize $H_1(z)$ & $H_2(z)$ in DF-2 and cascade to obtain cascade form.



Realize the system with difference eqn

$$y(n) = \frac{3}{4}y(n-1) - \frac{1}{8}y(n-2) + x(n) + \frac{1}{3}x(n-1)$$ in cascade form.

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From the difference eqn

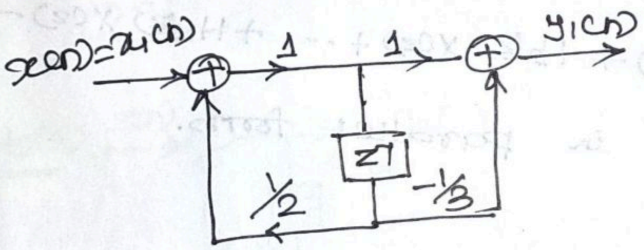
$$H(z) = \frac{Y(z)}{X(z)} = \frac{1+\frac{1}{3}z^{-1}}{1-\frac{3}{4}z^{-1}+\frac{1}{8}z^{-2}}$$

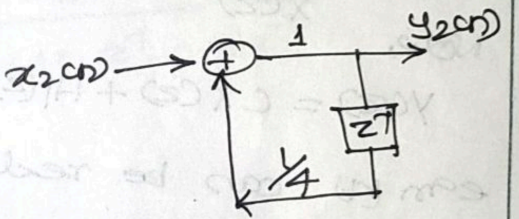
$$= \frac{1+\frac{1}{3}z^{-1}}{(1-\frac{1}{2}z^{-1})(1-\frac{1}{4}z^{-1})} = H_1(z)\,H_2(z).$$

where

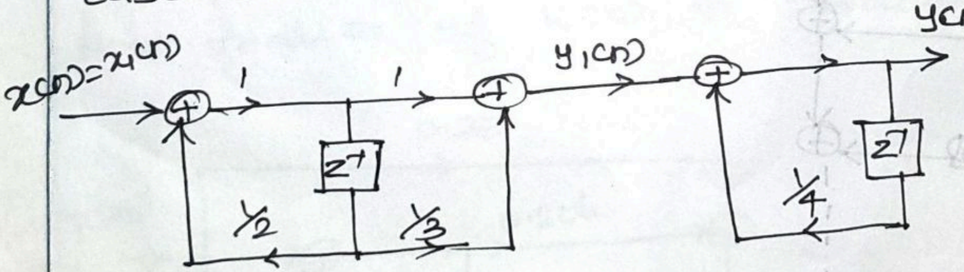
$$H_1(z) = \frac{1+\frac{1}{3}z^{-1}}{1-\frac{1}{2}z^{-1}}, \quad H_2(z) = \frac{1}{1-\frac{1}{4}z^{-1}}$$

$H_1(z) \rightarrow$ DF-II : $H_2(z) \Rightarrow$ DF-II





cascade both



practice:-

$$H(z) = \frac{1+2z^{-1}+z^{-2}}{1-\frac{3}{4}z^{-1}+\frac{1}{8}z^{-2}}$$ obtain cascade form

~~Hint: factorize both numerator & denominator and design $H_1(z)$, $H_2(z)$~~

Parallel form:-

→ A parallel form realization of an IIR system can be obtained by performing a partial expansion of

$$H(z) = c + \sum_{k=1}^{N} \frac{C_k}{1-P_k z^{-1}}, \quad P_k \text{ are poles.} \rightarrow (1)$$

→ eqn (1) can be written as

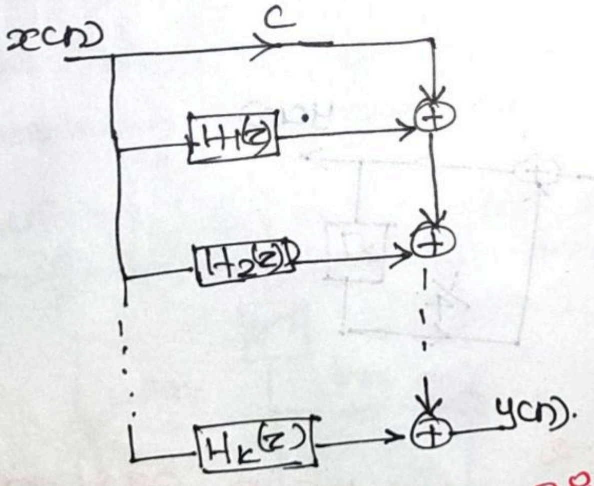
$$H(z) = c + \frac{C_1}{1-P_1 z^{-1}} + \frac{C_2}{1-P_2 z^{-2}} + \cdots + \frac{C_N}{1-P_N z^{-1}} \rightarrow (2)$$

$$H(z) = \frac{Y(z)}{X(z)} = c + H_1(z) + H_2(z) + \cdots + H_N(z) \rightarrow (3)$$

Now,

$$Y(z) = c\,X(z) + H_1(z)\,X(z) + H_2(z)\,X(z) + \cdots + H_N(z)\,X(z) \rightarrow (4)$$

eqn (4) can be realized in parallel form.



Realize the system given by difference eqn

$$y(n) = -0.1\,y(n-1) + 0.72\,y(n-2) + 0.7\,x(n) - 0.252\,x(n-2)$$

in parallel form.

The system fn of the difference eqn is

$$H(z) = \frac{0.7 - 0.252 z^{-2}}{1 + 0.1 z^{-1} - 0.72 z^{-2}}$$

$$= 0.35 + \frac{0.35 - 0.035z^{-1}}{1 + 0.1z^{-1} - 0.72z^{-2}}$$

$$= 0.35 + \frac{0.206}{1 + 0.9z^{-1}} + \frac{0.144}{1 - 0.8z^{-1}}$$

$$\begin{array}{r|l} & -0.35 \;(Q) \\ -0.72z^{-2} + 0.1z^{-1} + 1 \;(\text{Div}) & -0.252z^{-2} + 0.7 \;(\text{Nr}) \\ & 0.252z^{-2} - 0.035z^{-1} - 0.35 \\ \hline & -0.035 + 0.665 \;(R) \end{array}$$

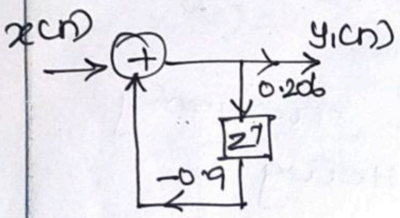
$$= \frac{-0.35(-0.72z^{-2} + 0.1z^{-1} + 1) + (-0.035z^{-1} + 0.665)}{-0.72z^{-2} + 0.1z^{-1} + 1}$$

$$= -0.35 + \frac{-0.035z^{-1} + 0.665}{1 + 0.1z^{-1} - 0.72z^{-2}}$$

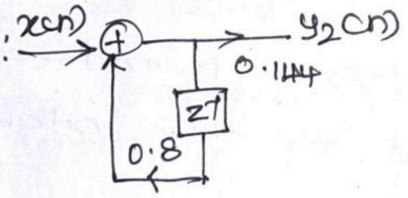
$$= \frac{0.315 - 0.035z^{-1}}{1 + 0.1z^{-1} - 0.72z^{-2}}$$

$H(z) = c + H_1(z) + H_2(z)$.

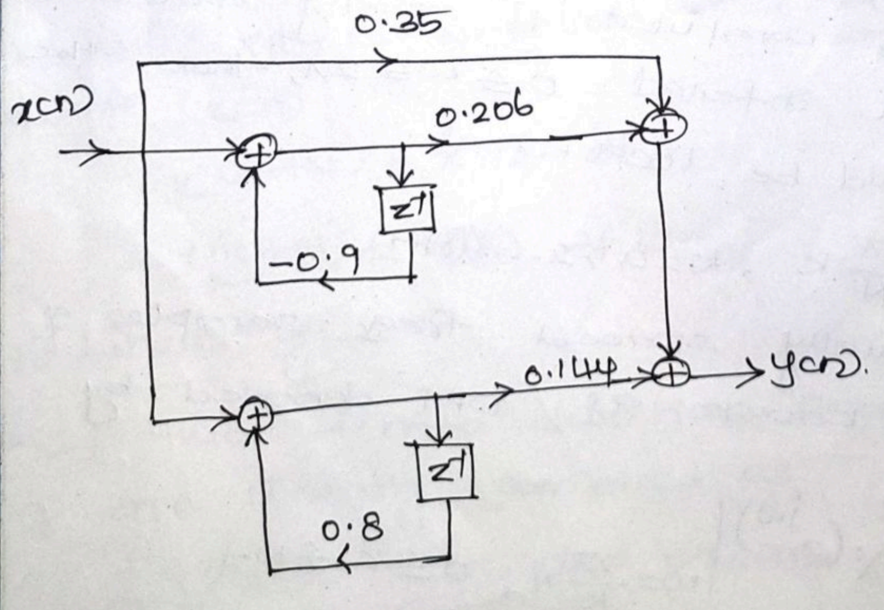
$H_1(z)$ can be realized in DF-II



$H_2(z)$:



Now realize of H(z) in parallel form.



Introduction to DFT:-

→ It is a powerful computational tool to evaluate Fourier transform $X(e^{j\omega})$ on a digital computer or specially designed Hw.

→ It is defined only for sequence of finite length.

∵ $X(e^{j\omega})$ is continuous and periodic, DFT is obtained by sampling one period of the fourier transform at a finite no. of frequency points.

→ It is used in determining the frequency content of a signal, linear filtering in frequency domain.

→ DTFT (Discrete-time Fourier transform) of a sequence is periodic, and we are interested in freq range 0 to 2π.

→ 'ω' may be ∞. and if we use a digital computer to compute 'N' equally spaced points over an interval $0 \le \omega \le 2\pi$, then the 'N' points should be located at.

$$\omega_k = \frac{2\pi}{N} k \quad , k = 0, 1, 2 \ldots (N-1).$$

These 'N' equally spaced freq samples of the DTFT are known as DFT. denoted by $x(k)$,

$$x(k) = X(e^{j\omega}) \Big|_{\omega = \frac{2\pi}{k} N} , \quad 0 \le k \le N-1.$$

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DFT

$$X(k) = \sum_{n=0}^{N-1} x(n)\, e^{-j2\pi kn/N}, \quad 0 \le k \le N-1.$$

IDFT

$$x(n) = \frac{1}{N} \sum_{k=0}^{N-1} X(k)\, e^{j2\pi kn/N}, \quad 0 \le n \le N-1$$

Let $W_N = e^{-j2\pi/N}$. $W_N^k = e^{-\frac{j2\pi k}{N}}$ - twiddle factor.

$$X(k) = \sum_{n=0}^{N-1} x(n)\, W_N^{nk}, \quad 0 \le k \le N-1$$

$$x(n) = \frac{1}{N} \sum_{k=0}^{N-1} X(k)\, W_N^{-nk}, \quad 0 \le n \le N-1.$$

$X(k) = \mathrm{DFT}[x(n)]$, $x(n) = \mathrm{IDFT}[X(k)]$

n - time index, (integer) K = Frequency index. (discrete freq)

properties of DFT:-

(a) periodicity

→ if $X(k)$ is N-point of a finite duration sequence $x(n)$

$$x(n+N) = x(n) \quad \forall n$$

$$X(k+N) = X(k) \quad \forall k.$$

(b) Linearity:-

→ If two finite duration seq $x_1(n)$ and $x_2(n)$ are linearly combined as

$$x_3(n) = a x_1(n) + b x_2(n)$$

then DFT of $x_3(n)$ is

$$X_3(k) = a X_1(k) + b X_2(k).$$

(c) circular shift of a sequence:-

$$x_p(n) = \sum_{l=-\infty}^{\infty} x(n-lN)$$

(d) Time reversal of the sequence.

→ The time reversal of an N-point sequence $x(n)$ is attained by wrapping the seq $x(n)$ around the circle in clockwise direction.

→ It is denoted as $x((-n))_N$ and

$$x((-n))_N = x(N-n), \quad 0 \le n \le N-1$$

If IDFT $[x(n)] = X(k)$ then

$$\text{DFT } x((-n))_N = \text{DFT}[x(N-n)]$$
$$= X((-k))_N = X(N-k).$$

(e) circular frequency shift:-

If DFT $[x(n)] = X(k)$ then

$$\text{DFT}[x(n) e^{j2\pi ln/N}] = X((k-l))_N.$$

(f) Complex conjugate:-

if DFT $[x(n)] = X(k)$ then

$$\text{DFT}[x^*(n)] = X^*(N-k) = X^*((-k))_N.$$

(g) circular convolution:-

Let $x_1(n)$ and $x_2(n)$ are finite duration sequences both of length N with DFTs $X_1(k)$ and $X_2(k)$. Now we find a seq $x_3(n)$ for which the DFT is $X_3(k)$.

$$X_3(k) = X_1(k)\, X_2(k)$$

$$\text{DFT}[x_1(n) \, (N) \, x_2(n)] = X_1(k)\, X_2(k).$$

(h) circular correlation:-

→ For complex-valued seq $x(n)$ and $y(n)$,

if $DFT[x(n)] = X(k)$ and $DFT[y(n)] = Y(k)$.

then

$$DFT[\tilde{r}_{xy}(l)] = DFT\left[\sum_{n=0}^{N-1} x(n)\, y^*((n-l))_N\right]$$

$$= X(k)\, Y^*(k).$$

$\tilde{r}_{xy}(l)$ is the circular crosscorrelation seq.

(i) Multiplication of two sequences:-

if $DFT[x_1(n)] = X_1(k)$ and

$DFT[x_2(n)] = X_2(k)$ then

$$DFT[x_1(n)\, x_2(n)] = \frac{1}{N}[X_1(k) \text{ Ⓝ } X_2(k)]$$

(j) Parseval's theorem.

if $DFT[x(n)] = X(k)$ &

$DFT[y(n)] = Y(k)$

then

$$\sum_{n=0}^{N-1} x(n)\, y^*(n) = \frac{1}{N}\sum_{k=0}^{N-1} X(k)\, Y^*(k).$$

1. Find the DFT of a sequence $x(n) = \{1,1,0,0\}$

soln:-

$L = N = 4.$

$$X(k) = \sum_{n=0}^{N-1} x(n)\, e^{-j2\pi kn/N}, \quad k = 0,1,2\ldots,(N-1)$$

$$X(0) = \sum_{n=0}^{3} x(n) = x(0) + x(1) + x(2) + x(3).$$

$$= 1+1+0+0 = 2$$

$$X(1) = \sum_{n=0}^{3} x(n) e^{-jn\pi/2} = x(0) + x(1) e^{-j\pi/2} + x(2) e^{-j\pi} + x(3) e^{-j3\pi/2}$$

$$= 1 + \cos\pi/2 - j\sin\pi/2$$

$$X(1) = 1 - j$$

$$X(2) = \sum_{n=0}^{3} x(n) e^{-j\pi n} = x(0) + x(1) e^{-j\pi} + x(2) e^{-j2\pi} + x(3) e^{-j3\pi}$$

$$= 1 + \cos\pi - j\sin\pi$$

$$X(2) = 1 - 1 = 0.$$

$$X(3) = \sum_{n=0}^{3} x(n) e^{-j3n\pi/2}$$
$$= x(0) + x(1) e^{-j3\pi/2} + x(2) e^{-j3\pi} + x(3) e^{-j9\pi/2}$$

$$= 1 + \cos 3\pi/2 - j\sin 3\pi/2$$

$$X(3) = 1 + j.$$

$$X(k) = \{2, 1-j, 0, 1+j\}.$$

② Find the DFT of a sequence

$$x(n) = \begin{cases} 1, & \text{for } 0 \le n \le 2 \\ 0, & \text{otherwise.} \end{cases} \quad \text{for } N = 8.$$

<u>soln:-</u> $N = 8$, (add $N-L$ zeros)

$x(0) = x(1) = x(2) = 1$ and.

$x(n) = 0$ for $3 \le n \le 7$.

$$X(k) = \sum_{n=0}^{N-1} x(n)\, e^{-j2\pi kn/N}$$

for $N=8$

$$X(k) = \sum_{n=0}^{7} x(n)\, e^{-j\pi nk/4} \quad , k=0,1,\ldots 7$$

$k=0$

$$X(0) = \sum_{n=0}^{7} x(n)$$

$$= 1+1+1+0+0+0+0+0 = 3.$$

$k=1$

$$X(1) = \sum_{n=0}^{7} x(n)\, e^{-j\pi n/4}$$

$$= x(0) + x(1)e^{-j\pi/4} + x(2)e^{-j\pi/2}$$

$$= 1+0.707 - j0.707 + 0 - j$$

$$X(1) = 1.707 - j1.707$$

$k=2$

$$X(2) = \sum_{n=0}^{7} x(n)\, e^{-j\pi n/2}$$

$$= x(0) + x(1)e^{-j\pi/2} + x(2)e^{-j\pi}$$

$$= 1 + \cos\pi/2 - j\sin\pi/2 + \cos\pi - j\sin\pi$$

$$= 1 - j - 1 = -j.$$

$k=3$

$$X(3) = \sum_{n=0}^{7} x(n)\, e^{-j3\pi n/4}$$

$$= x(0) + x(1)e^{-j3\pi/4} + x(2)e^{-j3\pi/2}$$

$$= 1 + \cos3\pi/4 - j\sin3\pi/4 + \cos3\pi/2 - j\sin3\pi/2$$

$$= 1 - 0.707 - j0.707 + j$$

$$X(3) = 0.293 + j0.293.$$

$k=4$

$$X(4)=\sum_{n=0}^{7} x(n)\, e^{-j\pi n}$$

$$= x(0)+x(1)e^{-j\pi}+x(2)e^{-j2\pi}$$

$$= 1+\cos\pi - j\sin\pi + \cos 2\pi - j\sin 2\pi$$

$$= 1-1+1 = 1$$

$k=5$

$$X(5)=\sum_{n=0}^{7} x(n)\, e^{-j5\pi n/4}$$

$$= x(0)+x(1)e^{-j5\pi/4}+x(2)e^{-j5\pi/2}$$

$$= 0.293 - j0.293$$

$k=6$

$$X(6)=\sum_{n=0}^{7} x(n)\, e^{-j3\pi n/2}$$

$$= x(0)+x(1)e^{-j3\pi/2}+x(2)e^{-j3\pi}$$

$$= 1+\cos 3\pi/2 - j\sin 3\pi/2 + \cos 3\pi - j\sin 3\pi$$

$$= 1+j-1 = j$$

$k=7$

$$X(7)=\sum_{n=0}^{7} x(n)\, e^{-j7\pi n/4}$$

$$= 1+e^{-j7\pi/4}+e^{-j7\pi/2}$$

$$= 1.707 + j1.707$$

$$X(k)=\{3,\ 1.707-j1.707,\ -j,\ 0.293+j0.293,\ 1,\ 0.293-j0.293,\ j,\ 1.707+j1.707\}.$$

Find IDFT of the sequence

$X(k) = \{5, 0, 1-j, 0, 1, 0, 1+j, 0\}$

soln:-

$$x(n) = \frac{1}{N}\sum_{k=0}^{N-1} X(k)\, e^{j2\pi kn/N}, \quad n = 0,1,2\ldots(N-1)$$

For $N=8$

$$x(n) = \frac{1}{8}\sum_{k=0}^{7} X(k)\, e^{j\pi kn/4}, \quad n = 0,1\ldots 7.$$

For $n=0$

$$x(0) = \frac{1}{8}\left[\sum_{k=0}^{7} X(k)\right] = \frac{1}{8}[5+0+1-j+0+1+0+1+j+0]$$

$$= 1$$

$$x(1) = \frac{1}{8}\left[\sum_{k=0}^{7} X(k)\, e^{j\pi k/4}\right]$$

$$= \frac{1}{8}[5+(1-j)j + 1(-1) + (1+j)(-j)]$$

$$= \frac{1}{8}(6) = 0.75$$

$$x(2) = \frac{1}{8}\left[\sum_{k=0}^{7} X(k)\, e^{+j\pi k/2}\right] = \frac{1}{8}[5+(1-j)(-1)+1+(1+j)(-1)]$$

$$= \frac{1}{8}(4) = 0.5$$

$$x(3) = \frac{1}{8}\left[\sum_{k=0}^{7} X(k) e^{j3\pi k/4}\right]$$

$$= \frac{1}{8}[5+(1-j)-j+1(-1)+(1+j)(j)]$$

$$= \frac{1}{8}(2) = 0.25$$

$$x(4) = \frac{1}{8}\left[\sum_{k=0}^{7} X(k)\, e^{j\pi k}\right] = \frac{1}{8}[5+(1-j)1+1+(1+j)]$$

$$= 1.$$

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$$x(5) = \frac{1}{8}\left[\sum_{k=0}^{7} X(k) e^{j5\pi k/4}\right] = \frac{1}{8}\left[5 + (1-j)j - 1 + (1+j)(-j)\right]$$

$$= \frac{1}{8}(6) = 0.75$$

$$x(6) = \frac{1}{8}\left[\sum_{k=0}^{7} X(k) e^{j3\pi k/2}\right] = \frac{1}{8}(4) = 0.5$$

$$x(7) = \frac{1}{8}\left[\sum_{k=0}^{7} X(k) e^{j7\pi k/4}\right]$$

$$= \frac{1}{8}\left[5 + (1-j)(-j) + 1(-1) + (1+j)(j)\right]$$

$$= \frac{1}{8}(2) = 0.25$$

$$x(n) = \{1, 0.75, 0.5, 0.25, 1, 0.75, 0.5, 0.25\}$$

Circular convolution:-

① Concentric circle Method:

→ The circular convolution of two given sequence $x_1(n)$ and $x_2(n)$ can be found by

$x_3(n) = x_1(n) \, (N) \, x_2(N)$.

(i) Graph 'N' samples of $x_1(n)$ as equally spaced points around the outer circle in counter clockwise direction.

(ii) Start at the same point as $x_1(n)$ graph 'N' samples of $x_2(n)$ as equally spaced points around an inner circle in clockwise direction.

(iii) Multiply corresponding samples on the two circles and sum the products to produce output.

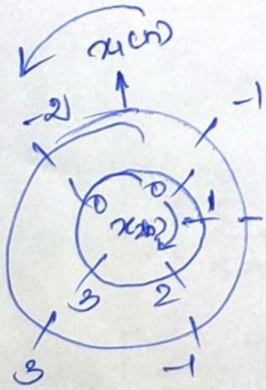
(iv) Rotate the inner circle one sample at a time in counterclockwise direction and go to step 3 to obtain the next value of output.

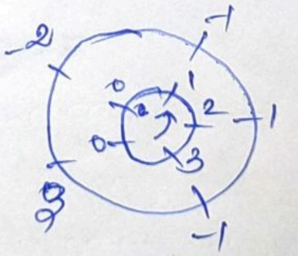
(v) Repeat step 4 until the inner circle 1st sample lines up with the first sample of the exterior circle once again.

Find the circular convolution of two seq $x_1(n) = \{1, -1, -2, 3, -1\}$ and $x_2(n) = \{1, 2, 3\}$.

→ To find the circular convolution, both sequences must be of same length. ∴ we append two zeros to the seq $x_2(n)$ and use concentric circle method.

$x_1(n) = \{1, -1, -2, 3, -1\}$ (indices 0 1 2 3 4) $x_2(n) = \{1, 2, 3, 0, 0\}$ (indices 0 1 2 3 4)

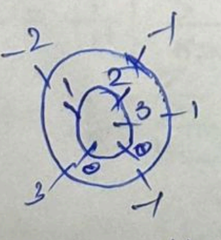




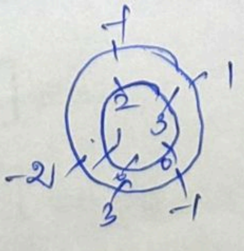
$y(0) = 1 + 0 + 0 + 9 - 2$

$y(0) = 8$

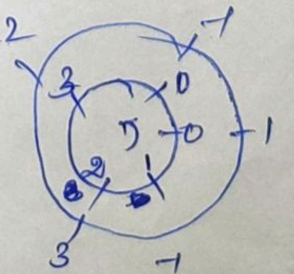
$y(1) = -2$



$y(2) = -1$



$y(3) = -4$



$y(4) = -1$

$y(n) = \{8, -2, -1, -4, -1\}$.

By Matrix Multiplication Method.

$$\begin{bmatrix} x_2(0) & x_2(4) & x_2(3) & x_2(2) & x_2(1) \\ x_2(1) & x_2(0) & x_2(4) & x_2(3) & x_2(2) \\ x_2(2) & x_2(1) & x_2(0) & x_2(4) & x_2(3) \\ x_2(3) & x_2(2) & x_2(1) & x_2(0) & x_2(4) \\ x_2(4) & x_2(3) & x_2(2) & x_2(1) & x_2(0) \end{bmatrix} \begin{bmatrix} x_1(0) \\ x_1(1) \\ x_1(2) \\ x_1(3) \\ x_1(4) \end{bmatrix} = \begin{bmatrix} y(0) \\ y(1) \\ y(2) \\ y(3) \\ y(4) \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 & 3 & 2 \\ 2 & 1 & 0 & 0 & 3 \\ 3 & 2 & 1 & 0 & 0 \\ 0 & 3 & 2 & 1 & 0 \\ 0 & 0 & 3 & 2 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ -2 \\ 3 \\ -1 \end{bmatrix} = \begin{bmatrix} 8 \\ -2 \\ -1 \\ -4 \\ -1 \end{bmatrix}$$

$y(n) = \{8, -2, -1, -4, -1\}$.

Find IDFT of the sequence.

$X(k) = \{5, 0, 1-j, 0, 1, 1+j, 0\}$.

$$x(n) = \frac{1}{N}\sum_{k=0}^{N-1} X(k)\, e^{j2\pi kn/N}, \quad n = 0 \ldots N-1$$

for $N = 8$

$$x(n) = \frac{1}{8}\sum_{k=0}^{N-1} X(k)\, e^{j\pi kn/4}, \quad n = 0 \cdots 7.$$

for $n = 0$

$$x(0) = \frac{1}{8}\left[\sum_{k=0}^{7} X(k)\right] = \frac{1}{8}[5 + 0 + 1 - j + 0 + 1 + 0 + 1 + j + 0] = 1$$

for $n = 1$.

$$x(1) = \frac{1}{8}\left[\sum_{k=0}^{7} X(k)\, e^{j\pi k/4}\right] = \frac{1}{8}[5 + (1-j)\,j + 1(-1) + (1+j)(-j)]$$

$$= 6/8 = 0.75$$

for $n = 2$

$$x(2) = \frac{1}{8}\left[\sum_{k=0}^{7} X(k)\, e^{j\pi k/2}\right] = \frac{1}{8}(4) = 0.5$$

for $n = 3$

$$x(3) = \frac{1}{8}\left[\sum_{k=0}^{7} X(k)\, e^{j3\pi k/4}\right] = 0.25$$

$$x(4) = \frac{1}{8}\left[\sum_{k=0}^{7} X(k)\, e^{j\pi k}\right] = 1$$

$$x(5) = \frac{1}{8}\left[\sum_{k=0}^{7} X(k)\, e^{j5\pi k/4}\right] = 0.75$$

$$x(6) = \frac{1}{8}\left[\sum_{k=0}^{7} X(k)\, e^{j3\pi k/2}\right] = 0.5$$

$$x(7) = \frac{1}{8}\left[\sum_{k=0}^{7} X(k)\, e^{j7\pi k/4}\right] = 0.25$$

$$x(n) = \{1, 0.75, 0.5, 0.25, 1, 0.75, 0.5, 0.25\}$$

Fast Fourier Transform :- (FFT)

→ Proposed by Cooley and Tukey in 1965.

→ Highly efficient procedure for computing the DFT of a finite series and required less number of computations that that of direct evaluation of DFT.

→ DFT coefficient calculation is been carried out "iteratively", bcs of which FFT is used in digital spectral analysis, filter simulation, autocorrelation and pattern recognition.

→ It improves the performance by a factor of 100 or more than DFT.

→ No. of complex multiplication required in

$$DFT = N^2 \qquad FFT = \frac{N}{2}\log_2 N.$$

→ No. of complex addition required in

$$DFT = N(N-1) \qquad FFT = N\log_2 N$$

→ FFT algorithms are based on the fundamental principle of decomposing the computation of DFT of a sequence of length 'N' into successively smaller DFT's.

→ Types. 1. DIT-FFT :- Decimation-in-Time.

2. DIF-FFT :- Decimation-in-frequency.

Decimation-in-time algorithm:-

→ It also called as Radix-2 FFT algorithm.

→ (i) The number of o/p points 'N' can be expressed as a power of 2. $N = 2^M$. (M = integer).

→ Let $x(n)$ is an N-point seq., where N is assumed to be a power of 2.

→ Decimate (or) break this sequence into two sequence of length $N/2$, where one sequence consisting of even-indexed values of $x(n)$ and the other of odd-indexed values of $x(n)$.

(i)

$$\left.\begin{aligned} x_e(n) &= x(2n), \quad n = 0,1 \dots \tfrac{N}{2}-1 \\ x_o(n) &= x(2n+1), \quad n = 0,1,\dots \tfrac{N}{2}-1 \end{aligned}\right\} \rightarrow (1)$$

→ N-point DFT of $x(n)$ can be written as

$$X(k) = \sum_{n=0}^{N-1} x(n) e^{-j2\pi kn/N}, \quad k = 0,1,\dots (N-1). \rightarrow (2)$$

Separating (2) into even & odd indexed values of $x(n)$

$$X(k) = \sum_{\substack{n=0 \\ (\text{Even})}}^{N-1} x(n) W_N^{nk} + \sum_{\substack{n=0 \\ (\text{odd})}}^{N-1} x(n) W_N^{nk}$$

$$= \sum_{n=0}^{\frac{N}{2}-1} x(2n) W_N^{2nk} + \sum_{n=0}^{\frac{N}{2}-1} x(2n+1) W_N^{(2n+1)k}$$

$$= \sum_{n=0}^{\frac{N}{2}-1} x(2n) W_N^{2nk} + W_N^k \sum_{n=0}^{\frac{N}{2}-1} x(2n+1) W_N^{2nk}. \rightarrow (3).$$

$$X(k) = \underbrace{\sum_{n=0}^{\frac{N}{2}-1} x_e(n) W_N^{2nk}}_{N/2\text{-even indexed DFT}} + W_N^k \underbrace{\sum_{n=0}^{\frac{N}{2}-1} x_o(n) W_N^{2nk}}_{N/2\text{ odd indexed DFT}} \rightarrow (4)$$

$W_N^2 = \left(e^{-j2\pi/N}\right)^2 = e^{-j2\pi/(N/2)} = W_{N/2}$ symmetry property twiddle factor.

$W_N^{k+N/2} = -W_N^k$

(ii) $W_N^2 = W_{N/2}$.

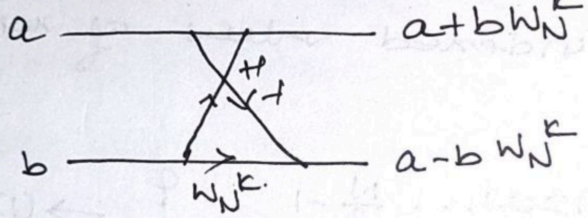
periodicity property

$W_N^{k+N} = W_N^k$

(4) ⇒

$X(k) = \underset{(a)}{X_e(k)} + W_N^k \underset{(b)}{X_O(k)} \longrightarrow (5).$

Flow graph of butterfly diagram.



Bit-reversal:-

i/p sample index	Binary	Bit reversed binary	Bit reversed sample index
0	000	000	0
1	001	100	4
2	010	010	2
3	011	110	6
4	100	001	1
5	101	101	5
6	110	011	3
7	111	111	7

DIT = i/p is bit reversal order

o/p is natural order.

Find the DFT of the sequence $x(n) = \{6, 4\}$ using DIT algorithm.
$\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\;\; 0\;\; 1$

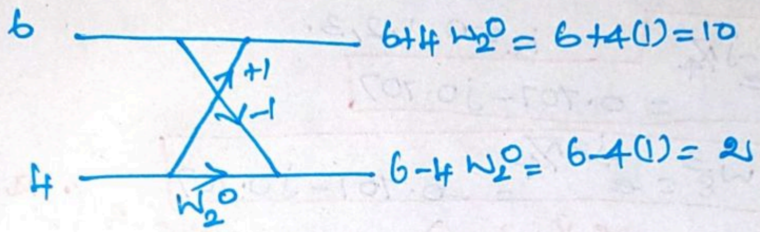
$$W_N^k = e^{-j2\pi k/N}$$

$k = 0 \dots \frac{N}{2} - 1$, $N = 2$.

$W_2^0 = e^0 = 1$

$k = 0, \dots \frac{2}{2} - 1 \; (1-1)$.

$\boxed{k=0}$



$X(k) = \{10, 2\}$.

Compute 4-point DFT of a sequence using DIT algorithm. $x(n) = \{0, 1, 2, 3\}$.

1st stage 2-point DFT $\boxed{W_2^0 = 1}$

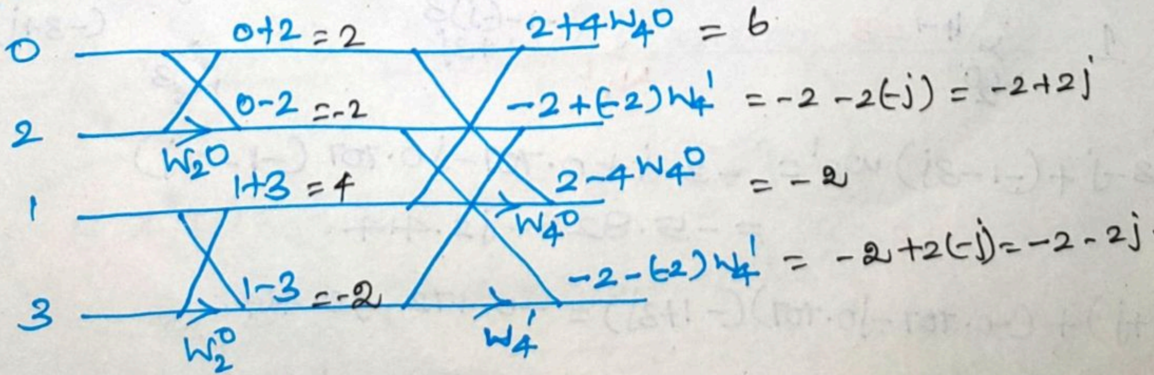
2nd stage 4-point DFT. $W_4^0 = 1$, $W_4^1 = -j$

$W_N^k = e^{-j2\pi k/N}$. $(N=4)$. $k = 0 \dots \frac{N}{2} - 1 = 0 \dots \frac{4}{2} - 1$

$k = 0 \dots 1 \Rightarrow 0, 1$.

$\boxed{W_4^0 = e^0 = 1}$, $W_4^1 = e^{-j2\pi/4} = e^{-j\pi/2} = -j$. $\boxed{W_4^1 = -j}$

0, 2 → Even index 1, 3 – odd index.



$X(k) = \{6, -2+2j, -2, -2-2j\}$

Find DFT of a sequence $x(n) = \{1,2,3,4,4,3,2,1\}$ using DIT algorithm.

0 1 2 3 4 5 6 7

Twiddle factor:

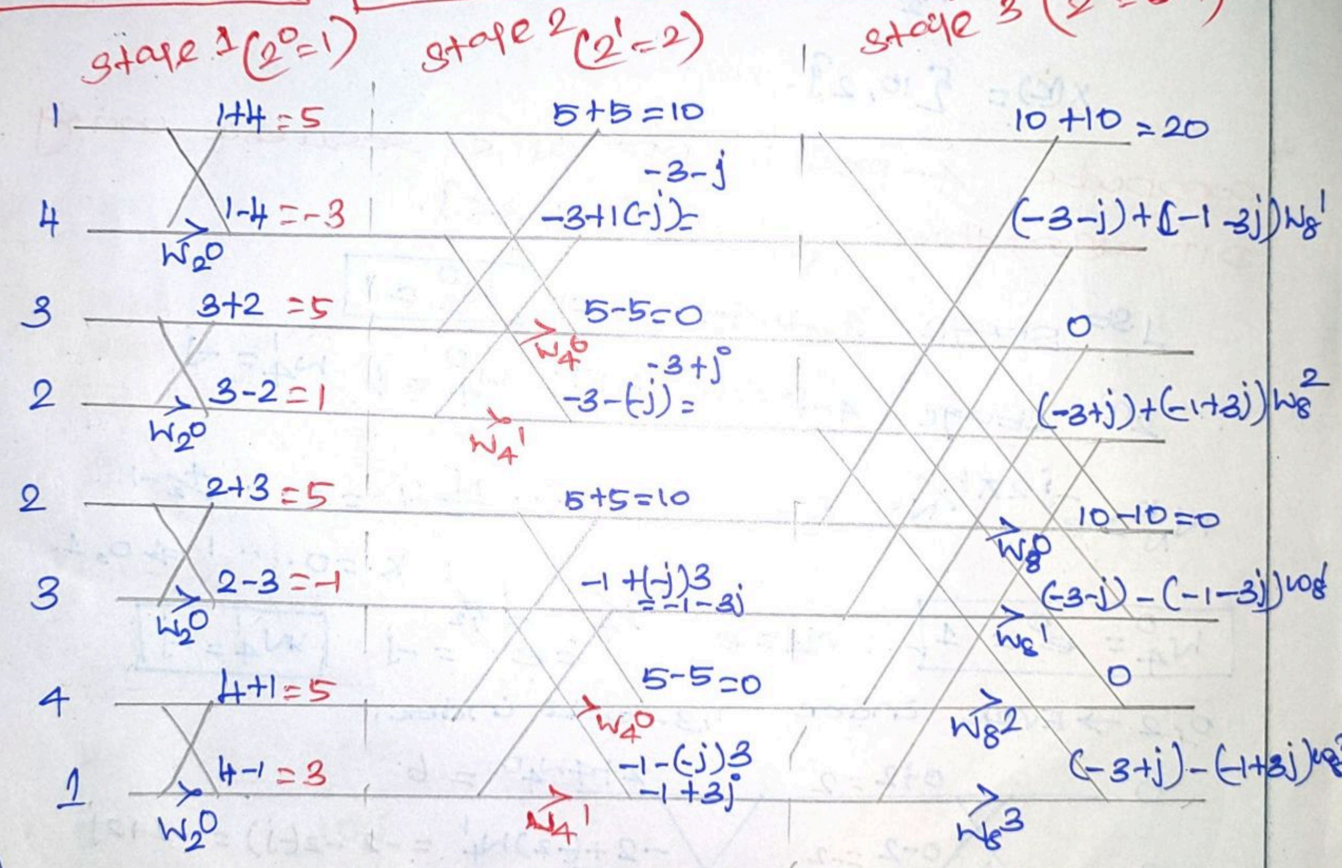
$W_N^k = e^{-j2\pi k/N}$. $\boxed{W_2^0 = 0}$ $\boxed{W_4^0 = 1,\ W_4^1 = -j}$

$W_8^k = e^{-j2\pi k/8} = e^{-j\pi k/4}$. $k = 0 \dots \frac{N}{2} - 1 \quad \left(\frac{8}{2} - 1\right) = (4-1) = 3.$

$k = 0, 1, 2, 3.$

$\boxed{W_8^0 = 1}$ $\boxed{W_8^1 = e^{-j\pi/4} = 0.707 - j0.707}$

$\boxed{W_8^2 = e^{-j\pi/2} = -j}$ $\boxed{W_8^3 = e^{-j3\pi/4} = -0.707 - j0.707}$



① $-3-j+(-1-3j)W_8^1 = -3-j+0.707-j0.707(-1-3j)$
$= -5.828 - j2.414.$

② $(-3+j)+(-0.707-j0.707)(-1+3j) = -0.172 - j0.414.$

③ $-3-j-(0.707-j0.707)(-1-3j) = -0.172+j0.414$

④ $(-3+j)-(-0.707-j0.707)(-1+3j) = -5.828+j2.414.$

$X(k) = \{20, -5.828-j2.414, 0, -0.172-j0.414, 0, -0.172+j0.414, 0, -5.828+j2.414\}.$

SRM

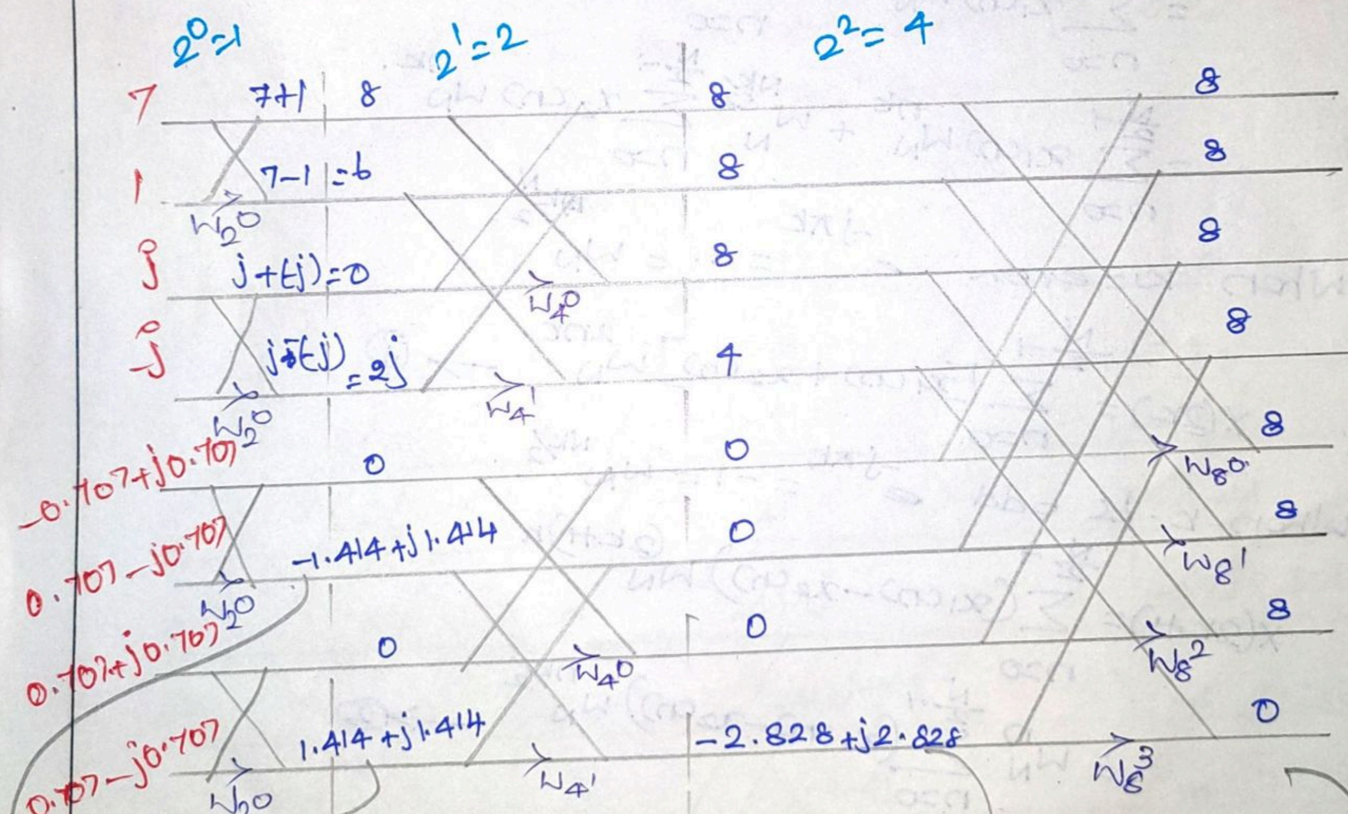
3.2 Find IDFT of the sequence

$X(k) = \{7, -0.707-j0.707, -j, 0.707-j0.707, 1, 0.707+j0.707, j, -0.707+j0.707\}$ using DIT algorithm.

Take complex conjugate of X(k) and apply bit r/v index inputs to 8-point flow graph.

$X^*(k) = \{\underset{0}{7}, \underset{1}{-0.707+j0.707}, \underset{2}{+j}, \underset{3}{0.707+j0.707}, \underset{4}{1}, \underset{5}{0.707-j0.707}, \underset{6}{-j}, \underset{7}{-0.707-j0.707}\}$

E	O
0	1
4	5
2	3
6	7



$N\,x^*(n) = \{8, 8, 8, 8, 8, 8, 8, 0\}$

$x(n) = \frac{1}{N}\{8, 8, 8, 8, 8, 8, 8, 0\} = \{1, 1, 1, 1, 1, 1, 1, 0\}$

$(-0.707+j0.707) - (0.707-j0.707)$ $= -1.414+j1.414$	$(0.707+j0.707) - (-0.707-j0.707)$ $= 1.414+j1.414$	$(-1.414+j1.414) - (-j)(1.414+j1.414)$ $= -2.828+j2.828$	$4 - (-0.707-j0.707)(-2.828+j2.828)$ $= 0$

Decimation-in-Frequency (DIF)

→ The o/p sequence $X(k)$ is divided into smaller and smaller subsequences.

$x_1(n) = x(n), \quad 0 \le n \le \frac{N}{2}-1$ } i/p is divided into

$x_2(n) = x(n+N/2), \quad 0 \le n \le \frac{N}{2}-1$ two '$N/2$' seq

→ The N-point DFT of $x(n)$ can be written as,

$$X(k) = \sum_{n=0}^{\frac{N}{2}-1} x(n) W_N^{nk} + \sum_{n=N/2}^{N-1} x(n) W_N^{nk}$$

$$= \sum_{n=0}^{\frac{N}{2}-1} x_1(n) W_N^{nk} + \sum_{n=0}^{\frac{N}{2}-1} x_2(n) W_N^{(n+N/2)k}$$

$$= \sum_{n=0}^{\frac{N}{2}-1} x_1(n) W_N^{nk} + W_N^{Nk/2} \sum_{n=0}^{\frac{N}{2}-1} x_2(n) W_N^{nk}$$

When k = even $e^{-j\pi k} = 1 = W_N^{kN/2}$.

$$X(2k) = \sum_{n=0}^{\frac{N}{2}-1} [x_1(n) + x_2(n)] W_N^{2nk} \rightarrow ①$$

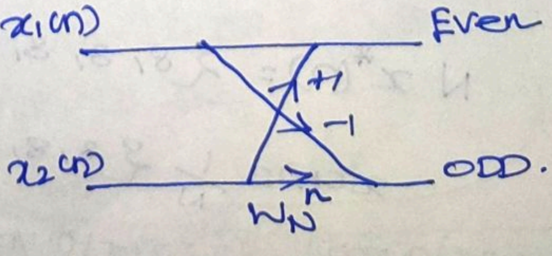
When k is odd $e^{-j\pi k} = -1 = W_N^{Nk/2}$.

$$X(2k+1) = \sum_{n=0}^{\frac{N}{2}-1} (x_1(n) - x_2(n)) W_N^{(2k+1)n}$$

$$= W_N^{n} \sum_{n=0}^{\frac{N}{2}-1} (x_1(n) - x_2(n)) W_N^{nk/2} \rightarrow ②$$

Even $= x_1(n) + x_2(n)$

ODD $= [x_1(n) - x_2(n)] W_N^{n}$

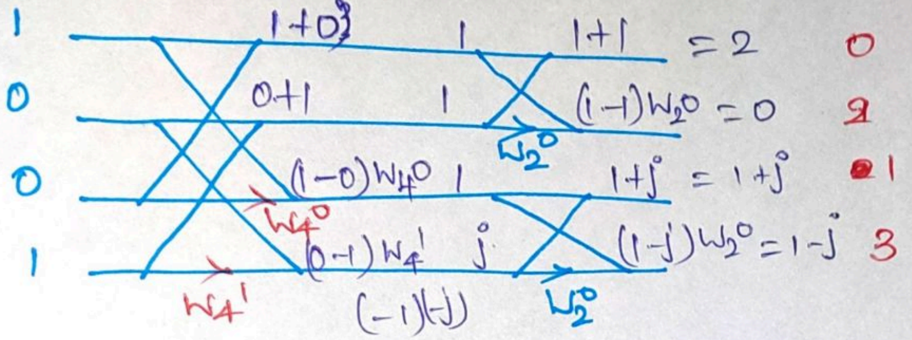


Find the DFT of the seqn $x(n) = \{1, 0, 0, 1\}$ using DIF algorithm.

0 1 2 3

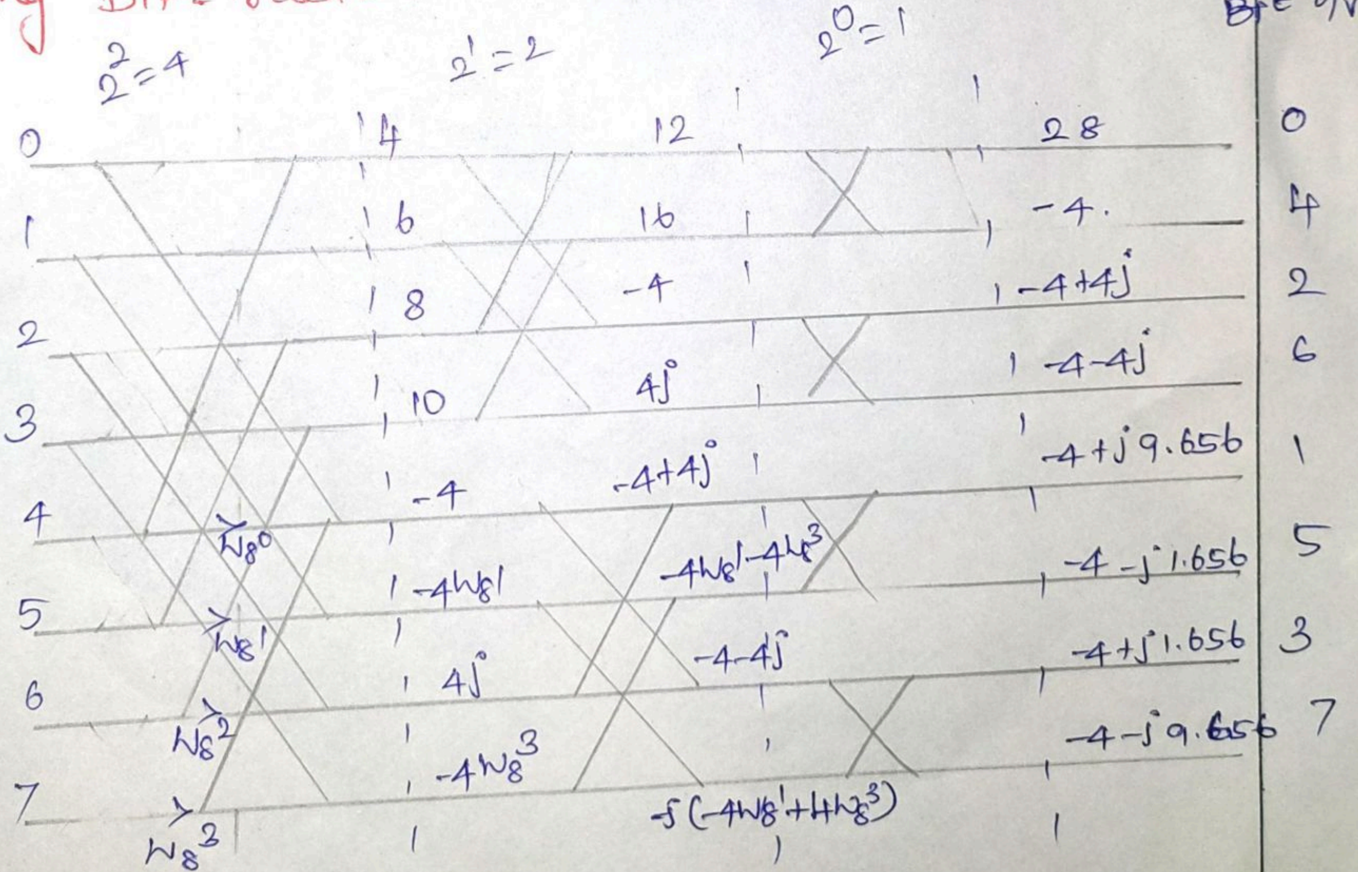
E	O
0	1
4	5
2	3
6	7

$W_2^0 = 1, \quad W_4^0 = 1, \quad W_4^1 = -j.$



$X(k) = \{2, 1+j, 0, 1-j\}.$

Find DFT of the sequence $x(n) = \{0, 1, 2, 3, 4, 5, 6, 7\}$ using DIF-radix-2-FFT algorithm.



$X(k) = \{28, -4+j9.656, -4+4j, -4+j1.656, -4, -4-j1.656, -4-4j, -4-j9.656\}.$