Unit-1

Mathematical Logic

Logic: Methods of reasoning.

The aim of logic is to provide rules by which we can determine whether a particular reasoning or argument is valid.

Propositions: A declarative sentence (or assertion) which is true or false, but not both is called a proposition (or statement). Sentences which are exclamatory, interrogative or imperative in nature are not propositions.

Examples:

1. New Delhi is the capital city of India
   ↳ proposition.

2. How beautiful is Rose?
   ↳ Not a proposition

3. $2 + 2 = 6$

↳ proposition

4. Take a cup of coffee.

↳ Not a proposition.

If a proposition is true, we say that the truth value of that proposition is true. It is denoted by T or 1.

If a proposition is false, the truth value is said to be false. It is denoted by F or 0.

Atomic propositions:

Propositions which do not contain any of the logical operators or connectives are called atomic (primary or primitive) propositions.

Compound propositions

Many mathematical statements which can be constructed by combining one or more atomic statements using connectives are called molecular or compound propositions.

Connectives

Definition 1: when p and q are any two propositions, the proposition "p and q" denoted by $p \wedge q$ and called the conjunction of p and q is defined as the compound ~~pos~~ proposition that is true when both p and q are true and is false otherwise.

Truth Table

p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

Definition 2: when p and q are any two propositions, the proposition "p or q" denoted by $p \vee q$ and called the disjunction of p and q is defined as the compound proposition that is false when both p and q are false and is True otherwise.

Truth Table:

p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

Definition 3: Given any proposition p, another proposition formed by writing, "It is not the case that" or "It is false that" before p or by inserting the word "not" suitably in p is called the negation of p and denoted by $\neg p$.

Truth Table

p	$\neg p$
T	F
F	T

Definition 4: If p and q are propositions, the compound proposition "If p, then q", that is denoted by $p \to q$ is called a conditional proposition, which is false when p is true and q is false and true otherwise.

Truth Table

p	q	$p \to q$
T	T	T
T	F	F
F	T	T
F	F	T

Example: Let us consider the statement.

"If I get up at 5 A.M, I will go for a walk".

Definition 5: If p and q are propositions, the compound proposition "p if and only if q", that is denoted by $p \leftrightarrow q$, is called a biconditional proposition, which is true when p and q have the same truth values and is false otherwise.

Truth Table

p	q	$p \leftrightarrow q$
T	T	T
T	F	F
F	T	F
F	F	T

Definition 6: A compound proposition $P = P(p_1, p_2, \dots, p_n)$, where $p_1, p_2, \dots, p_n$ are variables (elemental propositions), is called a tautology, if it is true for every truth assignment for $p_1, p_2, \dots, p_n$.

P is called a contradiction, if it is false for every truth assignment for $p_1, p_2, \ldots, p_n$.

Example: Truth Table

P	$\neg P$	$P \vee \neg P$	$P \wedge \neg P$
T	F.	T	F
F	T	T	F

Definition 7: Two compound propositions $A(p_1, p_2, \ldots, p_n)$ and $B(p_1, p_2, \ldots, p_n)$ are said to be logically equivalent or simply equivalent, if they have identical truth tables.

Example: $\neg(P \vee q) \equiv \neg P \wedge \neg q$.

P	q	$P \vee q$	$\neg(P \vee q)$	$\neg P$	$\neg q$	$\neg P \wedge \neg q$
T	T	T	F	F	F	F
T	F	T	F	F	T	F
F	T	T	F	T	F	F
F	F	F	T	T	T	T

Duality law:

The dual of a compound proposition that contains only the logical operators $\vee, \wedge$ and $\neg$ is the proposition obtained by replacing each $\vee$ by $\wedge$, each $\wedge$ by $\vee$, each T by F and each F by T, where T and F are special variables representing compound propositions that are tautologies and contradictions respectively. The dual of a proposition A is denoted by A^*.

1. Construct a truth table for each of the following compound propositions.

(a) $(p \vee q) \longrightarrow (p \wedge q)$

(b) $(q \rightarrow \neg p) \longleftrightarrow (p \longleftrightarrow q)$.

(c) $(\neg p \longleftrightarrow \neg q) \longleftrightarrow (q \longleftrightarrow r)$.

(a) $(p \vee q) \longrightarrow (p \wedge q)$.

p	q	$p \vee q$	$p \wedge q$	$(p \vee q) \rightarrow (p \wedge q)$
T	T	T	T	T
T	F	T	F	F
F	T	T	F	F
F	F	F	F	T

(b) $(q \rightarrow \neg p) \longleftrightarrow (p \longleftrightarrow q)$

p	q	$\neg p$	$q \rightarrow \neg q$	$p \leftrightarrow q$	$(q \rightarrow \neg p) \leftrightarrow (p \leftrightarrow q)$
T	T	F	F	T	F
T	F	F	T	F	F
F	T	T	T	F	F
F	F	T	T	T	T

(c) $(\neg p \longleftrightarrow \neg q) \longleftrightarrow (q \longleftrightarrow r)$

p	q	r	$\neg p$	$\neg q$	$(\neg p \leftrightarrow \neg q) \equiv a$	$q \leftrightarrow r \equiv b$	$a \leftrightarrow b$.
T	T	T	f	f	T	T	T
T	T	f	f	f	T	f	f
T	f	T	f	T	f	f	T
T	f	f	f	T	f	T	f
f	T	T	T	f	f	T	f
f	T	f	T	f	f	f	T
f	f	T	T	T	T	f	f
f	f	f	T	T	T	T	T

2. Determine which of the following compound propositions are tautologies and which of them are contradictions, using truth tables:

(a) $(\neg q \wedge (p \rightarrow q)) \rightarrow \neg p$

(b) $\neg(q \rightarrow r) \wedge r \wedge (p \rightarrow q)$

(a) $(\neg q \wedge (p \rightarrow q)) \rightarrow \neg p$.

p	q	$\neg p$	$\neg q$	$(p \rightarrow q)$	a $\neg p \wedge (p \rightarrow q)$	$a \rightarrow \neg p$
T	T	f	f	T	f	T
T	f	f	T	f	f	T
f	T	T	f	T	f	T
f	f	T	T	T	T	T

(b) $\neg(q \rightarrow r) \wedge r \wedge (p \rightarrow q)$

p	q	r	$p \rightarrow q$	$q \rightarrow r$	a $\neg(q \rightarrow r)$	b $a \wedge r$	$b \wedge (p \rightarrow q)$
T	T	T	T	T	f	f	f
T	T	f	T	f	T	f	f
T	f	T	f	T	f	f	f
T	f	f	f	T	f	f	f
f	T	T	T	T	f	f	f
f	T	f	T	f	T	f	f
f	f	T	T	T	f	f	f
f	f	f	T	T	f	f	f

Algebra of propositions.

Laws of Algebra of propositions.

Name of the law	Primal form	Dual form
1. Idempotent law	$P \vee P \equiv P$	$P \wedge P \equiv P$
2. Identity law	$P \vee F \equiv P$	$P \wedge T \equiv P$
3. Dominant law	$P \vee T \equiv T$	$P \wedge F \equiv F$
4 Complement law	$P \vee \neg P \equiv T$	$P \wedge \neg P \equiv F$
5. Commutative law	$P \vee q \equiv q \vee P$	$P \wedge q \equiv q \wedge P$
6. Associative law	$(P \vee q) \vee r \equiv P \vee (q \vee r)$	$(P \wedge q) \wedge r \equiv P \wedge (q \wedge r)$
7. Distributive law	$P \vee (q \wedge r) \equiv (P \vee q) \wedge (P \vee r)$	$P \wedge (q \vee r) \equiv (P \wedge q) \vee (P \wedge r)$
8. Absorption law	$P \vee (P \wedge q) \equiv P$	$P \wedge (P \vee q) \equiv P$.
9. Demorgan's law	$\neg(P \vee q) \equiv \neg P \wedge \neg q$	$\neg(P \wedge q) \equiv \neg P \vee \neg q$.

Note:

1. $p \rightarrow q \equiv \neg p \vee q$

2. $p \leftrightarrow q \equiv (p \rightarrow q) \wedge (q \rightarrow p)$.

1.

p	q	$p \rightarrow q$	$\neg p$	$\neg p \vee q$
T	T	T	f	T
T	f	f	f	f
f	T	T	T	T
f	f	T	T	T

2.

p	q	$p \rightarrow q$	$q \rightarrow p$	$(p \rightarrow q) \wedge (q \rightarrow p)$	$p \leftrightarrow q$
T	T	T	T	T	T
T	f	f	T	f	f
f	T	T	f	f	f
f	f	T	T	T	T

1. Without using truth tables, prove the following

(a) $(\neg P \vee q) \wedge (P \wedge (P \wedge q)) \equiv P \wedge q$

(b) $P \rightarrow (q \rightarrow P) \equiv \neg P \rightarrow (P \rightarrow q)$

(a) $(\neg P \vee q) \wedge (P \wedge (P \wedge q))$

$\equiv (\neg P \vee q) \wedge (P \wedge P) \wedge q$ [Associative law]

$\equiv (\neg P \vee q) \wedge (P \wedge q)$ [Idempotent law]

$\equiv (P \wedge q) \wedge (\neg P \vee q)$ [commutative law]

$\equiv ((P \wedge q) \wedge \neg P) \vee ((P \wedge q) \wedge q)$ [Distributive law]

$\equiv (\neg P \wedge (P \wedge q)) \vee ((P \wedge q) \wedge q)$ [Commutative law]

$\equiv ((\neg P \wedge P) \wedge q) \vee (P \wedge (q \wedge q))$ [Associative law]

$\equiv (F \vee q) \vee (P \wedge q)$ [complement and idempotent law]

$\equiv F \vee (P \wedge q)$ [Dominant law]

$\equiv P \wedge q$ [Dominant law]

(b) $p \to (q \to p) \equiv \neg p \vee (q \to p)$

$\equiv \neg p \vee (\neg q \vee p)$

$\equiv \neg p \vee (p \vee \neg q)$ [Commutative and Associative]

$\equiv \neg p \vee T$ [Complement law]

$\equiv T.$ [Dominant law]

$\neg p \to (p \to q) \equiv p \vee (p \to q)$

$\equiv p \vee (\neg p \vee q)$ [$p \to q \equiv \neg p \vee q$]

$\equiv (p \vee \neg p) \vee q$ [Associative law]

$\equiv T \vee q$ [Complement law]

$\equiv T$ [Dominant law].

which implies $p \to (q \to p) \equiv \neg p \to (p \to q)$

2. Prove the following equivalence by proving the equivalence of the dual.

$$\neg((\neg p \wedge q) \vee (\neg p \wedge \neg q)) \vee (p \wedge q) \equiv p$$

The dual of the given equivalence is

$$\neg((\neg p \vee q) \wedge (\neg p \vee \neg q)) \wedge (p \vee q) \equiv p.$$

Let us now prove the dual equivalence.

$\neg((\neg p \vee q) \wedge (\neg p \vee \neg q)) \wedge (p \vee q)$

$\equiv \neg(\neg p \vee (q \wedge \neg q)) \wedge (p \vee q)$ [Distributive law]

$\equiv \neg(\neg p \vee F) \wedge (p \vee q)$ [complement law]

$\equiv \neg(\neg p) \wedge (p \vee q)$ [Identity law]

$\equiv p \wedge (p \vee q)$

$\equiv p$ [Absorption law]

Tautological Implication :

A compound proposition $A(p_1, p_2, \dots, p_n)$ is said to tautologically imply or simply imply the compound proposition $B(p_1, p_2, \dots, p_n)$ if B is true whenever A is true or equivalently if and only if $A \rightarrow B$ is a tautology. This is denoted by $A \Rightarrow B$ read as "A implies B".

Example: $p \Rightarrow p \vee q$.

Truth table

p	q	$p \vee q$	$p \to (p \vee q)$
T	T	T	T
T	f	T	T
f	T	T	T
f	f	f	T

1. Prove the following implication without using truth table

$$\underbrace{((p \vee \neg p) \to q) \to ((p \vee \neg p) \to r)}_{A} \Rightarrow \underbrace{q \to r}_{B}$$

$A \Rightarrow B$; $A \to B$ is a tautology

$$[((p \vee \neg p) \to q) \to ((p \vee \neg p) \to r)] \to (q \to r)$$

$$\equiv [(T \to q) \to (T \to r)] \to (q \to r)$$

$$\equiv [(\neg T \vee q) \to (\neg T \vee r)] \to (q \to r)$$

$$\equiv [(F \vee q) \to (F \vee r)] \to (q \to r)$$

$$\equiv (q \to r) \to (q \to r)$$

$$\equiv (\neg q \vee r) \to (\neg q \vee r)$$

$$\equiv \neg(\neg q \vee r) \vee (\neg q \vee r) \equiv T.$$

1. Prove the following equivalences.

(a) $\neg(p \leftrightarrow q) \equiv (p \wedge \neg q) \vee (\neg p \wedge q)$

(b) $(p \wedge q) \vee (p \wedge \neg q) \equiv p$.

2. Write down the dual of the following statements

(a) $\neg p \rightarrow (p \rightarrow q)$.

[Hint : $\neg p \rightarrow (\neg p \vee q) \equiv \neg\neg p \vee (\neg p \vee q)$
$\equiv p \vee (\neg p \vee q)$]

(b) $(p \wedge q) \rightarrow (p \rightarrow q)$.

3. Construct the truth table for each of the following compound propositions :

(a) $(p \leftrightarrow q) \vee (\neg q \leftrightarrow r)$

(b) $((p \rightarrow q) \wedge (q \rightarrow r)) \rightarrow (p \rightarrow r)$

4. Without using truth tables, prove the following equivalences.

(a) $(p \vee q) \wedge (\neg p \wedge (\neg p \wedge q)) \equiv (\neg p \wedge q)$

(b) $(p \vee q) \rightarrow r \equiv (p \rightarrow r) \wedge (q \rightarrow r)$

5. Write down the dual of the following equivalence and prove the dual without using truth table.

$$(p \to r) \wedge (q \to r) \equiv (p \vee q) \to r.$$

6. Prove the following implication without using truth table.

$$(p \vee q) \wedge (p \to r) \wedge (q \to r) \Rightarrow r$$

7. Determine which of the following statements are tautologies or contradictions:

(a) $(p \wedge q) \wedge \neg(p \vee q)$

(b) $((p \to (q \to r)) \to ((p \to q) \to (p \to r)).$

Theory of Inference

Inference theory is concerned with the inferring of a conclusion from certain hypotheses or basic assumptions called premises by apply certain principles of reasoning called rules of inference. When a conclusion derived from a set of premises by using rules of inference, the process of such derivation is called a formal proof. The rules of inference are only means used to draw a conclusion from a set of premises in a finite sequence of steps, called argument. In this section we deal with the rules of inference by which conclusions are derived from premises. Any conclusion which is arrived at by following these rules is called a valid conclusion and the argument is called a valid argument.

Rules of inference:

The following are the two basic rules of inference called rules P and T.

Rule P: A premise may be introduced at any step in the derivation.

Rule T: A formula S may be introduced in the derivation, if S is tautologically implied by one or more preceding formulas in the derivation.

Rules of Inference.

Name of the Rule	Rule in Tautological form.
1. Simplification	$(P \wedge q) \rightarrow P$ $[P \wedge q \Rightarrow P]$ $(P \wedge q) \rightarrow q$
2. Addition	$P \rightarrow (P \vee q)$ $q \rightarrow (P \vee q)$
3. Conjunction.	$((P) \wedge (q)) \rightarrow (P \wedge q)$
4. Modus ponens	$[P \wedge (P \rightarrow q)] \rightarrow q$

5 Modus tollens $[\neg q \wedge (p \to q)] \to \neg p$

6. Hypothetical Syllogism $[(p \to q) \wedge (q \to r)] \to (p \to r)$

7. Disjunctive Syllogism $[(p \vee q) \wedge \neg p] \to q$

8. Resolution $[(p \vee q) \wedge (\neg p \vee r)] \to (q \vee r)$

9. Dilemma $[(p \vee q) \wedge (p \to r) \wedge (q \to r)] \to r$

Rule CP or Rule of conditional proof

If a formula s can be derived from another formula r and a set of premises, then the statement $(r \to s)$ can be derived from the set of premises alone.

[If the conclusion is of the form $r \to s$, we will take r as an additional premise and derive s using the given premises and r]

Inconsistent premises:

A set of premises $H_1, H_2, \ldots, H_n$ is said to be inconsistent, if their conjunction implies a contradiction.

Indirect Method of proof:

In order to show that a conclusion C follows from the premises $H_1, H_2, \ldots, H_n$ by this method, we assume that C is false and include $\neg C$ as an additional premise. If the new set of premises is inconsistent leading to a contradiction, then the assumption that $\neg C$ is true does not hold good. Hence C is true whenever $H_1 \wedge H_2 \wedge \ldots \wedge H_n$ is true. Thus C follows from $H_1, H_2, \ldots, H_n$.

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1. Direct Method
2. Indirect Method
3. Conditional proof
4. Inconsistent premises.