coding theory

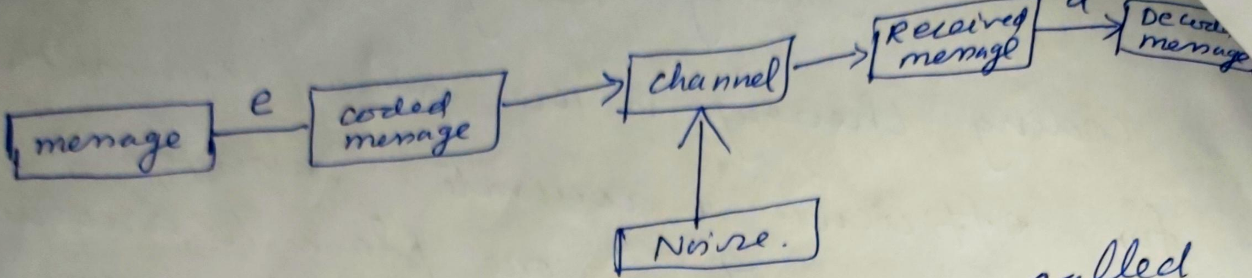
coding theory is the study of methods for efficient & accurate transfer of information from one place to another.

coding theory deals with the problems of detecting & correcting transmission errors caused by noise on the channel.

<u>Channel</u> :- The physical medium through which the information is transmitted is called a channel.

<u>Eg</u> :- Telephone lines & the atmosphere are Examples of channel.

The following diagram provides a rough idea of a general information transmission system



Noise :-

undesirable disturbance called noise, may cause the information received to differ from what was transmitted.

Noise may be caused by

(i) sun spots
(ii) lightning (flash of light electric discharge in the atmosphere.
(iii) folds in a magnetic tape
(iv) meteor showers,
(v) poor typing, poor hearing, poor speech or many other things.

The most important part of the diagram as far as we are concerned is the noise for without it there would be no need for the theory.

De word message

In many cases, the information to be sent is transmitted by a sequence of zeros & ones, we call a 0 or 1 a digit.

Word :- A word is a sequence of digits.

Length of a word :-
is the number of digits in the word.

Eg : 0110101 is a word of length 7.

* A word is transmitted by sending its one after the other across a binary channel.

The term binary refers to the fact that only two digits 0 & 1 are used.

Encoding :-
Encoding is the process of converting data from one form to another.

Decode :- Convert code into original language.

* By using a suitable encoder & decoder it may be possible to detect the error in the messages due to noise in the channel & to correct them.

* The encoder in the binary channel will transform an input message into a binary string consisting of the symbols 0 & 1. Decoding is only the inverse operation of encoding.

Group code :-

A group structure can be given to the set of binary strings of length n.

Let $B = \{0, 1\}$

Then $B^n = \{(x_1, x_2 \cdots x_n) \mid x_i \in B,\ i = 1, 2 \cdots n\}$

B^n is a group under the binary operation of addition modulo 2 denoted by $\oplus$. This group $(B^n, \oplus)$ is called a group code. $\oplus$ is defined by

$(x_1, x_2 \cdots x_n) \oplus (y_1, y_2 \cdots y_n)$

$= ((x_1 +_2 y_1), (x_2 +_2 y_2) \cdots (x_n +_2 y_n))$

under the binary sum

$0 +_2 0 = 0 = 1 +_2 1$; $1 +_2 0 = 1 = 0 +_2 1$

$(B^n, \oplus)$ is an abelian group.

In general, any code which is a group under the operation $\oplus$ is called a group code.

$(0, 0, 0 \cdots 0)$ is the identity elements & the inverse of $(x_1, x_2 \cdots x_n)$ is $(x_1, x_2 \cdots x_n)$ itself.

Hence forth, we write the n-tuple $(x_1, x_2 \cdots x_n)$ as the string $x_1 x_2 \cdots x_n$.

Definition :-

Hamming Weight :-

Let $x \in B^n$ $(x_1, x_2 \cdots x_n)$

The numbers of 1's in x is called the weight of x denoted $|x|$

Ex :- Let $x = 101011$

$|x| = 4$.

Hamming distance :-

Let $x = x_1 x_2 \cdots x_n$ &

$y = y_1 y_2 \cdots y_n$ be binary

Strings. The Hamming distance between x & y denoted $H(x, y)$ is defined by $H(x, y) = |x \oplus y|$

Equivalently $H(x, y)$ is the number of positions for which $x_i \neq y_i$

Ex :- Let $x = 1011$ & $y = 0101$

$x \oplus y = 1110$ $\quad |x \oplus y| = 3$

Number of positions for which $x_i \neq y_i$ $= 3$.

Definition:-

An (m, n) encoding function $e : B^m \longrightarrow B^n$ (where $n > m$) is one-one function. If $b \in B^m$ then $e(b)$ is called the code word represented by b.

$$\begin{array}{ccc} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{array}$$

m - tuple of message

n - tuple of codeword.

If $m = 3 \Rightarrow |B^3| = 2^3 = 8$ unique element

(8 possible 3 bits) representation of 3 tuple which is generated by 0, 1

(set of messages or cardinality of messages.

$(000, 100, 010, 001, 110, 011, 101, 111)$.

$n = 4$. set of codeword.

$B^4 = \{$ collection of codeword which is generated by all 4 tuples $\}$.

parity check code :-

A function $e : B^m \longrightarrow B^{m+1}$ defined by $e(b_1 b_2 \cdots b_m) = b_1 b_2 \cdots b_{m+1}$

where $b_{m+1} = \begin{cases} 0 & |b| \text{ is even} \\ 1 & |b| \text{ is odd.} \end{cases}$

e is called the parity $(m, m+1)$ check code.

Note :- $|e(b)|$ is always even.

Ex :- The code word representing 101 under $(3, 4)$ parity check is given by.

$$e(101) = 1010$$

If e is an $(m, 3m)$ encoding function defined by.

$$e(b_1 b_2 \cdots b_m) = b_1 b_2 \cdots b_m b_1 b_2 \cdots b_m b_1 b_2 \cdots b_m$$

ie the message b is repeated three times.

Eg :- $e : B^3 \to B^4$

s.t $e(b_1 b_2 b_3) = b_1 b_2 b_3 b_4$.

where $b_4 = \begin{cases} 0 & \text{if } |b| \text{ even} \\ 1 & \text{if } |b| \text{ is odd.} \end{cases}$

$\therefore$ $e(001) = 0011$

$e(011) = 0110$.

Result :-

① An (m,n) encoding function $e: B^m \to B^n$ can detect k or fewer errors iff its minimum Hamming distance is atleast $k+1$ (i.e. $\geq k+1$)

② An (m,n) encoding function $e; B^m \to B^n$ can correct k or fewer errors iff its minimum Hamming distance is atleat $2k+1$ i.e. $\geq 2k+1$

Ex: Given that $E: B^2 \to B^6$ defined by

$e(0,0) = 000000$
$e(0,1) = 010101$
$e(10) = 101010$
$e(11) = 111111$

Find the number of error which can be detected & corrected.

Soln. $d(x_1, x_2) = 3$
$d(x_1, x_3) = 3$
$d(x_1, x_4) = 6$
$d(x_2, x_3) = 6$
$d(x_2, x_4) = 3$
$d(x_3, x_4) = 3$.

minimum H. distance $= 3$

For error detecting case.

$$3 \geq k+1$$

$$k+1 \leq 3$$

$$k \leq 2$$

We can detect 2 or fewer error.

For error correcting case

$$3 \geq 2k+1$$

$$2k \leq 3-1$$

$$2k \leq 2$$

$$k \leq 1$$

We can correct 1 Error only.

Error correction using matrices :-

When $m, n \in Z^+$ & $n > m$ the encoding function $e : B^m \longrightarrow B^n$ where $B = (0,1)$ is given by a $m \times n$ matrix G over B. This matrix G is called the generator matrix for the code & is of the form

$$G = \left[I_m \mid A \right] \text{ ie } \left[I_{m\times m} \mid A_{m \times n-m} \right]_{m \times n}$$

where I_m is the $m \times m$ unit matrix.
& A is an $m \times (n-m)$ matrix to be chosen suitably.

Relation between message, codeword & G

$$\underset{1\times n}{C} = \underset{1\times m}{m} \underset{m\times n}{G}$$ [dimension of codeword $= 1 \times n$.

* with the help of G. Matrix, we can generate all the codewords.

Parity check matrix :-

with the help of this we can check whether the code word are correctly transmitted or not.

$$H = \left[A^T_{(n-m)\times m} \;\middle|\; I_{n-m \times n-m} \right]_{n-m \times n}$$

Note :- The codeword c is valid or we can say correctly transmitted
if $CH^T = 0$ or $HC^T = 0$.
if $HC^T \neq 0$. code are not transmitted correctly.

Problems :-

Given that

$$G = \begin{pmatrix} 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 & 1 \end{pmatrix}$$. Find all

the code words.

(ii) Minimum Hamming Weight & Minimum Hamming distances.

(iii) How many errors we can detect & correct.

(iv) Parity check matrix.

(v) check the following codes are correctly transmitted or not.

a = 01000   b = 11010.

soln.

$$G = \begin{bmatrix} 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 & 1 \end{bmatrix}_{2\times 5}$$

$m=2$
$n=5$.

$$G = \left[I_{m\times m} \mid A_{m\times n-m} \right]_{m\times n}$$

$$= \left[I_{2\times 2} \mid A_{2\times 3} \right]_{2\times 5}.$$

(i) $G = \left[\begin{array}{cc|ccc} 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 & 1 \end{array}\right]$.

$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}$

if $m=2$, $n=5$, then the coding function is defined by

$e : B^2 \longrightarrow B^5$.

$|B^2| = 2^2 = 4$ [all possible 4 message of 2 tuple which is generated by 0 & 1]

$= (00, 10, 01, 11)$

We know that $c = mG$.

$c_1 = \underset{1\times 2}{\overrightarrow{(0\ 0)}} \underset{2\times 5}{\begin{pmatrix} 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 & 1 \end{pmatrix}}$

$= (0.1+0.0 \quad 0.0+0.1 \quad 0.1+0.1 \quad 0.1+0.0 \quad 0.1+0.1)$

$= (0+0 \quad 0+0 \quad 0+0 \quad 0+0 \quad 0+0)$ w.r.t addition modulo 2

$= (0 \ 0 \ 0 \ 0 \ 0)$

$c_2 = (1\ 0) \begin{pmatrix} 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 & 1 \end{pmatrix}$

$= (1+0 \quad 0+0 \quad 1+0 \quad 1+0 \quad 1+0) = (1\ 0\ 1\ 1\ 1)$

$$x_3 = (0\ 1)\,G = (0\ 1)\begin{pmatrix} 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 & 1 \end{pmatrix}$$

$$= (0\ 1\ 1\ 0\ 1)$$

$$x_4 = (1\ 1)\,G = (1\ 1\ 0\ 1\ 0).$$

Minimum humming distance between the codewords

$d(x_1, x_2) = 4$

$d(x_1, x_3) = 3$

$d(x_1, x_4) = 3$

$d(x_2, x_3) = 3$

$d(x_2, x_4) = 3$

$d(x_3, x_4) = 4$

minimum Humming distance $= 3$.

For error detecting case

$3 \geq k+1$

$k \leq 2$

$\therefore$ we can detect 2 or fewer error here.

III^{ly} For error correcting case

$3 \geq 2k+1$

$2k+1 \leq 3$

$2k \leq 2$

$k \leq 1$

we can correct 1 error here.

Parity check matrix :- is used to detection of errors at the receiver side

$$H = \left[A^T_{n-m \times m} \; \middle| \; I_{n-m \times n-m} \right]_{n-m \times n}$$

$$= \left[A^T_{5-2 \times 2} \; \middle| \; I_{5-2 \times 5-2} \right]_{5-2 \times 5}$$

$$= \left[A^T_{3 \times 2} \; \middle| \; I_{3 \times 3} \right]_{3 \times 5}$$

$$= \left[\begin{array}{cc|ccc} 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 1 \end{array} \right]$$

With the help of $Hb^T = 0 \Rightarrow$ code b is correctly transmitted

$Hb^T \neq 0$ code is not valid.

$a = (0\ 1\ 0\ 0\ 0)$

$$H a^T = \begin{pmatrix} 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} 1.0 + 1.1 + 1.0 + 0.0 + 0.0 \\ 1.0 + 0.1 + 0.0 + 1.0 + 0.0 \\ 0 + 1.1 + 0 + 0 + 1.0 \end{pmatrix} = \begin{pmatrix} 0+1+0+0+0 \\ 0+0+0+0+0 \\ 0+1+0+0+0 \end{pmatrix}$$

$$= \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \neq \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

code a is not correctly transmi[tted]

(ii) $b = 11010$

$$Hb^T = \begin{pmatrix} 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} \underbrace{1+1+0+0+0}_{+2} \\ \underbrace{1+0+0+1+0}_{+2} \\ \underbrace{1+1+0+0+0}_{+2} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

code b is correctly transmitted.

(3) Find the code words generated by the encoding functions $e : B^2 \longrightarrow B^5$ with respect to the parity check matrix.

$$H = \begin{pmatrix} 0 & 1 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}_{5 \times 3}.$$

Soln If the encoding function

$e : B^m \longrightarrow B^n$, the generator matrix was assumed as an $m \times n$ matrix

$G = [\, I_m \mid A \,]$ & the parity check matrix was assumed as an $(n-m) \times m$ matrix $H = [\, A^T \mid I_{n-m} \,]$ & as such there was less number of rows & more number of columns in H.

According to this, H is not given in this problem but H^T is given.

Rewriting the given parity check matrix as per our notation.

$$H = \left(\begin{array}{cc|ccc} 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 1 \end{array}\right)_{3 \times 2}$$

Here
$m = 2$
$n = 5$
$n - m = 3$.
$H = 3 \times 2$.

$$= \left(A^T \mid I_{n-m} \right)$$

Generator matrix G is given by

$$G = [\, I_m \mid A \,]$$

$$= \left[\begin{array}{cc|ccc} 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 \end{array}\right]$$

Now $B^2 = (00, 01, 10, 11)$

$2^2 = 4$
4 possible
2 bits
$m \in B^2$.

We know that $C = mG$.
or $e(m) = mG$.

$\therefore \; e(00) = (00)\begin{pmatrix} 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 \end{pmatrix}$

$= (0\,0\,0\,0\,0)$

$e(01) = (0\,1)\begin{pmatrix} 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 \end{pmatrix}$

$= (0\,1\,0\,1\,1)$

$e(10) = (1\,0)\begin{pmatrix} 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 \end{pmatrix}$

$= (1\,0\,0\,1\,1)$

$e(11) = (1\,1)\begin{pmatrix} 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 \end{pmatrix}$

$= (1\,1\,0\,0\,0)$

Hence the code words generated by H are. (00000, 01011, 10011, 11000).

④ Example :-

Show that the (2,5) encoding function from B^2 to B^5 defined by

$e(00) = 00000$; $e(01) = 01110$

$e(10) = 10101$; $e(11) = 11011$

is a group code.

To prove $e(B^2)$ is a subgroup of B^5 $e(B^2) \subseteq B^5$.

We use cayley table.

$\oplus_2$	00000	01110	10101	11011
00000	00000	01110	10101	11011
01110	01110	00000	11011	10101
10101	10101	11011	00000	01110
11011	11011	10101	01110	00000

00000 is the identity & the inverse of any element is itself. Every element in the table belongs to

$e(B^2)$. $\oplus$ is associative

$\therefore e : B^2 \longrightarrow B^m$ is a group code.

(5) Find the code words generated by the parity check matrix, when the encoding function is $e : B^3 \longrightarrow B^6$.

$$H = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Soln

As per our notation, what is given in this problem is not H, but H^T.

$\because$ number of rows is less than the number of column.

$m = 3$

$n = 6$

$I_{n-m} = I_3$.

$$H = \left[A^T \,\middle|\, I_{n-m} \right]$$

$$= \left(\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 1 \end{array} \right)$$

Generator matrix is given by

$$G = [I_m \mid A]$$

$$= [I_3 \mid A]$$

$$= \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 \end{array}\right)$$

Now $B^3 = 2^3 = 8$ possible 3 bits.

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}$$

$\frac{8}{2} = 4$ zeros followed by 4 ones.

$\frac{4}{2} = 2$ zeros followed by 2 ones.

$\frac{2}{2} = 1$ zeros followed by 1 ones.

$\therefore$ we know that $c = mG$.

or $e(m) = mG$.

$$\therefore e(000) = (0\,0\,0) \begin{pmatrix} 1 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 \end{pmatrix}$$

$$= (0\,0\,0\,0\,0\,0)$$

$$e(001) = (001)\begin{pmatrix} 1 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 \end{pmatrix}$$

$$= (001011)$$

$$e(010) = (010)\begin{pmatrix} 1 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 \end{pmatrix}$$

$$= (010101)$$

$$e(100) = (100)\begin{pmatrix} 1 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 \end{pmatrix}$$

$$= (100111)$$

$$e(011) = (011)\begin{pmatrix} 1 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 \end{pmatrix}$$

$$= (011110)$$

$$e(101) = (101)\begin{pmatrix} 1 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 \end{pmatrix}$$

$$= (101100)$$

$$e(110) = (110)\,G = (110010)$$

$$e(111) = (111)\,G = (111001)$$

$\therefore$ The generated code words are

$\{000000, 001011, 010101, 100111, 011110,$
$101100, 110010, 111001\}$.

⑥ **Decoding using generator Matrix :-**

Given the generator matrix

$$G = \begin{pmatrix} 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 & 1 \end{pmatrix}_{3\times 6}$$

m=3
n=

corresponding to the encoding function $e : B^3 \longrightarrow B^6$. Find the corresponding parity check matrix & use it to decode the following received words & hence to find the original message. Are all the words decoded uniquely?

(i) 110101 (ii) 001111 (iii) 110001 (iv) 111111.

Soln

Given $G = [I_m \mid A]_{m\times n}$

$= [I_3 \mid A_{3\times 3}]$

m=3
n=6
n−m=3

$H = [A^T_{3\times 3} \mid I_3]$

$$= \begin{pmatrix} 1 & 0 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{pmatrix}_{3\times 6}.$$

Note :-

With the help of parity check matrix we can detect the error at the receiver side.

After detecting the presence of error in the received vector (r), our aim is to correct it with the help of syndrome decoding. (is used for correcting the error)

The presence of error is detected by

if $H \cdot r^T = 0$ then r (received word) $= C$ (code word)

i.e no error in the received vector

$H \cdot r^T \neq 0$ then $r \neq c$.

We compute the syndrome of each of the received word by using

$$H r^T = \begin{pmatrix} 1 & 0 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \\ 0 \\ 1 \\ 0 \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Since $H r^T = 0$, $\therefore r = c$.

i.e. Received word in this case is the transmitted (encoded) word itself.

Hence the original message is

(1 1 0)

(ii) $r = 0\ 0\ 1\ 1\ \boxed{1}\ 1$

$$H \cdot r^T = \begin{pmatrix} 1 & 0 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \neq \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad \therefore r \neq c.$$

Since, the syndrome $\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ is the same as the fifth column of H, the element in the fifth position of r is changed.

$\therefore$ The decoded word is 001101 & the original message is 001.

(ii) $r = 1\,1\,0\,\boxed{0}\,0\,1$

$$H \cdot r^T = \begin{pmatrix} 1 & 0 & 1 & \boxed{1} & 0 & 0 \\ 1 & 1 & 0 & \boxed{0} & 1 & 0 \\ 0 & 1 & 1 & \boxed{0} & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \neq \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad \therefore r \notin C.$$

Since, the syndrome $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ is the same as the fourth column of H, the fourth component of r is changed to get the decoded word. It is 110101 & the original message is 110.

) $r = 111111$

$$H \cdot r^T = \begin{pmatrix} 1 & 0 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \neq 0 \quad \therefore r \notin C$$

Since, the syndrome $\begin{pmatrix}1\\1\\1\end{pmatrix}$ is not same as any of the column of H, the received word cannot be decoded uniquely.

Since $\begin{pmatrix}1\\1\\1\end{pmatrix} = \begin{pmatrix}1\\1\\0\end{pmatrix} + \begin{pmatrix}0\\0\\1\end{pmatrix}$

$= 1^{st}$ row $+ 5^{th}$ row of H.

Two errors have occurred one in 1^{st} position & one in 5^{th} position.

$\therefore$ It cannot be decided uniquely.

Decoding using Decoding table :-

Construct the decoding table for the group code given by the generator matrix $G = \begin{pmatrix} 1 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 \end{pmatrix}$

Decode the following received wo using decoding table obtaine which of the words could not decoded uniquely ?.

101111, 011010, 101110, 111111.

Soln

Since G is a 3×6 matrix. it corresponds to the encoding function

$e : B^3 \longrightarrow B^6$

Now $B^3 = (000, 001, 010, 100, 011, 101, 110, 111)$

$e(000) = (000)G = (000000)$

$e(001) = (001)G = (001011)$

$e(010) = (010)G = (010101)$

$e(100) = (100)G = (100111)$

$e(011) = (011)G = (011110)$

$e(101) = (101)G = (101100)$

$e(110) = (110)G = (110010)$

$e(111) = (111)G = (111001)$

We form the decoding table by making these encoded words as the elements of the first row & the coset leaders as the elements of the first column.

The coset leaders with only one 1 have been taken in a certain order & then those with two 1's have been taken. The decoding table is given by

codewords →

000000	001011	010101	100111	011110	101100	110010	111001
100000	101011	110101	000111	111110	001100	010010	011001
010000	011011	000101	110111	001110	111100	100010	101001
001000	000011	011101	101111	010110	100100	111010	110001
000100	001111	010001	100011	011010	101000	110110	111101
000010	001001	010111	100101	011100	101110	110000	111011
000001	001010	010100	100110	011111	101101	110011	111000
011000	010011	001101	111111	000110	110100	101010	100001

↑ coset leaders.

(i) Received word 101111

Locate the column in which the received word lies, It is in 4^{th} column. The code word on the top of the column is 100111 ie $e(100) = 100111$

∴ The original message is 100.

(ii) Received word : 011010

It is in 5^{th} column of the table

The corresponding code word on top is

011110.

ie $e(011) = 011110$.

The original message is 011.

(iii) Received word : 101110

It is in 6^{th} column of the table.

The corresponding code word on top is

101100 ie $e(101) = 101100$

$\therefore$ The original message is 101.

(iv) Received word : 111111

It is in 4^{th} column & 8^{th} row,

the coset leader of which contains

two 1's [ie The corresponding coset

leader has weight 2]

$\therefore$ ~~the~~ The received word contains 2

errors. Hence they cannot be

corrected & the code word transmitted

cannot be decoded uniquely.