Inverse Z-Transform:

The inverse z-transform of $Z\{x(n)\} = X(z)$ is defined as $Z^{-1}[X(z)] = \{x(n)\}$.

Method: 1 Long Division Method:

Since Z-transform is defined by the series $X(z) = \sum_{n=0}^{\infty} x(n) z^{-n}$, to find the inverse Z-transform $x(n) = Z^{-1}[X(z)]$ of $X(z)$, expand $X(z)$ in the proper power series and collect the coefficient of z^{-n} to get $x(n)$.

Find inverse Z-transform of

(1) $\dfrac{10z}{(z-1)(z-2)}$

(2) $\dfrac{(z+2)z}{z^2+2z+4}$

(3) $\dfrac{2z(z^2-1)}{(z^2+1)^2}$

(4) $\dfrac{z}{z^2+7z+10}$ (Exercise).

1) $$X(z) = \frac{10z}{(z-1)(z-2)} = \frac{10z}{z^2 - 3z + 2} = \frac{10z}{z^2\left[1 - \frac{3z}{z^2} + \frac{2}{z^2}\right]}$$

$$= \frac{10z^{-1}}{1 - 3z^{-1} + 2z^{-2}}$$

$$
\begin{array}{r|l}
 & 10z^{-1} + 30z^{-2} + 70z^{-3} + 150z^{-4} + \cdots \\
\hline
1 - 3z^{-1} + 2z^{-2} & \cancel{10z^{-1}} \\
 & 10z^{-1} - 30z^{-2} + 20z^{-3} \\
 & (-) \quad (+) \quad (-) \\
\hline
 & 30z^{-2} - 20z^{-3} \\
 & 30z^{-2} - 90z^{-3} + 60z^{-4} \\
 & (-) \quad (+) \quad (-) \\
\hline
 & 70z^{-3} - 60z^{-4} \\
 & 70z^{-3} - 210z^{-4} + 140z^{-5} \\
 & (-) \quad (+) \quad (-) \\
\hline
 & 150z^{-4} - 140z^{-5}
\end{array}
$$

we know that, $$X(z) = \sum_{n=0}^{\infty} x(n) z^{-n}$$

$$10z^{-1} + 30z^{-2} + 70z^{-3} + 150z^{-4} + \cdots =$$

$$x(0) + x(1)z^{-1} + x(2)z^{-2} + x(3)z^{-3} + \cdots$$

$\therefore$ $x(0) = 0$

$x(1) = 10$

$x(2) = 30$

$x(3) = 70$

$x(4) = 150$

$\vdots$

In general, $x(n) = 10(2^n - 1)$, $n = 0, 1, 2, \ldots$

(ii) $X(z) = \dfrac{z^2+2z}{z^2+2z+4} = \dfrac{\cancel{z^2}\left(1+\frac{2z}{z^2}\right)}{\cancel{z^2}\left(1+\frac{2z}{z^2}+\frac{4}{z^2}\right)} = \dfrac{1+2z^{-1}}{1+2z^{-1}+4z^{-2}}$

$$
\begin{array}{r|l}
 & 1 - 4z^{-2} + 8z^{-3} - 32z^{-5} + \cdots \\
\hline
1+2z^{-1}+4z^{-2} & 1+2z^{-1} \\
 & 1+2z^{-1}+4z^{-2} \\
 & (-)\quad(-)\quad(-) \\
\hline
 & -4z^{-2} \\
 & -4z^{-2}-8z^{-3}-16z^{-4} \\
 & (+)\quad(+)\quad(+) \\
\hline
 & 8z^{-3}+16z^{-4} \\
 & 8z^{-3}+16z^{-4}+32z^{-5} \\
 & (-)\quad(-)\quad(-) \\
\hline
 & -32z^{-5}
\end{array}
$$

WKT, $X(z) = \sum_{n=0}^{\infty} x(n) z^{-n}$

$1 - 4z^{-2} + 8z^{-3} - 32z^{-5} + \cdots = x(0) + x(1)z^{-1} + x(2)z^{-2} + x(3)z^{-3} + \cdots$

$\therefore$ $x(0)=1$, $x(1)=0$, $x(2)=-4$, $x(3)=8$, $x(4)=0$, $x(5)=-32$, ...

$\therefore$ The sequence is $1, 0, -4, 8, 0, -32 \ldots$

(iii) $X(z) = \frac{2z(z^2-1)}{(z^2+1)^2} = \frac{2z^3-2z}{z^4+2z^2+1} = \frac{z^3\left[2-\frac{2z}{z^3}\right]}{z^4\left[1+\frac{2z^2}{z^4}+\frac{1}{z^4}\right]}$

$$= \frac{z^{-1}\left[2-2z^{-2}\right]}{1+2z^{-2}+z^{-4}} = \frac{2z^{-1}-2z^{-3}}{1+2z^{-2}+z^{-4}}.$$

$$X(z) = 2z^{-1} - 6z^{-3} + 10z^{-5} - 14z^{-7} + \cdots \quad \text{(Exercise)}.$$

we know that,

$$X(z) = \sum_{n=0}^{\infty} x(n) \cdot z^{-n}$$

$$2z^{-1} - 6z^{-3} + 10z^{-5} - 14z^{-7} + \cdots = x(0) + x(1)z^{-1} + x(2)z^{-2} + x(3)z^{-3} + \cdots$$

$\therefore$ $x(0)=0$, $x(1)=2$, $x(2)=0$, $x(3)=-6$, $x(4)=0$,

$x(5)=10$, $x(6)=0$, $x(7)=-14$, ...

In general, $x(n) = 2n \sin\frac{n\pi}{2}$, $n=0,1,2,\ldots$

Method 2 (Partial Fraction Method)

Find the inverse z-transform of

(1) $\dfrac{z}{z^2+7z+10}$ (2) $\dfrac{z^2+z}{(z-1)(z^2+1)}$ (3) $\dfrac{kTz}{(z-1+kT)(z-1)}$

(4) $\dfrac{z^2}{(z-1/2)(z-1/4)}$ (5) $\dfrac{z}{(z-1)^2(z+1)}$

1. $$X(z) = \frac{z}{z^2+7z+10}.$$

$$\therefore \frac{X(z)}{z} = \frac{1}{z^2+7z+10} = \frac{1}{(z+5)(z+2)}$$

Consider $$\frac{1}{(z+5)(z+2)} = \frac{A}{z+5} + \frac{B}{z+2}$$

$$\frac{1}{(z+5)(z+2)} = \frac{A(z+2)+B(z+5)}{(z+5)(z+2)}$$

$$1 = A(z+2) + B(z+5).$$

Solve, $A = -1/3$; $B = 1/3$.

$$\therefore \frac{X(z)}{z} = \frac{\frac{-1}{3}}{z+5} + \frac{\frac{1}{3}}{z+2}$$

$$X(z) = \frac{-1}{3}\frac{z}{z+5} + \frac{1}{3}\frac{z}{z+2}$$

Taking inverse z- trans. on b.s;

$$Z^{-1}[X(z)] = Z^{-1}\left[\frac{-1}{3}\frac{z}{z+5} + \frac{1}{3}\frac{z}{z+2}\right]$$

$$= \frac{-1}{3} Z^{-1}\left[\frac{z}{z+5}\right] + \frac{1}{3} Z^{-1}\left[\frac{z}{z+2}\right]$$

$$x(n) = \frac{-1}{3}(-5)^n + \frac{1}{3}(-2)^n, \quad n=0,1,2,\ldots$$

WKT,

$$Z[a^n] = \frac{z}{z-a}.$$

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4) Let $X(z) = \dfrac{z^2}{(z-1/2)(z-1/4)}$

$$\frac{X(z)}{z} = \frac{z}{(z-1/2)(z-1/4)}$$

Consider $\dfrac{z}{(z-1/2)(z-1/4)} = \dfrac{A}{z-1/2} + \dfrac{B}{z-1/4}.$

solve, $A=2$, $B=-1$.

$$\therefore \frac{X(z)}{z} = \frac{2}{z-1/2} - \frac{1}{z-1/4}.$$

$$X(z) = \frac{2z}{z-1/2} - \frac{z}{z-1/4}$$

Taking inverse - Z trans. on both sides, we get.

$$Z^{-1}[X(Z)] = Z^{-1}[RHS]$$

$$x(n) = 2Z^{-1}\left[\frac{z}{z-1/2}\right] - Z^{-1}\left[\frac{z}{z-1/4}\right]$$

$$x(n) = 2(1/2)^n - (1/4)^n, \quad n=0,1,2,\ldots$$

$$\therefore x(n) = 2(1/2)^n - (1/4)^n, \quad n=0,1,2,\ldots$$

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5) Let $X(z) = \dfrac{z}{(z-1)^2(z+1)}$

$$\frac{X(z)}{z} = \frac{1}{(z-1)^2(z+1)}$$

Consider $\dfrac{1}{(z+1)(z-1)^2} = \dfrac{A}{z+1} + \dfrac{B}{z-1} + \dfrac{C}{(z-1)^2}$

Solve, $A = 1/4$, $B = -1/4$, $C = 1/2$.

$$\therefore \frac{X(z)}{z} = \frac{1/4}{z+1} - \frac{1/4}{z-1} + \frac{1/2}{(z-1)^2}$$

$$X(z) = \frac{1}{4}\frac{z}{z+1} - \frac{1}{4}\frac{z}{z-1} + \frac{1}{2}\frac{z}{(z-1)^2}$$

Taking inverse Z-trans. on b.s, we get

$$Z^{-1}[X(z)] = Z^{-1}(RHS)$$

$$x(n) = \frac{1}{4} Z^{-1}\left(\frac{z}{z+1}\right) - \frac{1}{4} Z^{-1}\left(\frac{z}{z-1}\right) + \frac{1}{2} Z^{-1}\left[\frac{z}{(z-1)^2}\right]$$

$$= \frac{1}{4}(-1)^n - \frac{1}{4} + \frac{1}{2} n, \quad n = 0, 1, 2, \ldots \quad \because Z(n) = \frac{z}{(z-1)^2}$$

Convolution of Sequences:

The Convolution of two sequences $\{x(n)\}$ and $\{y(n)\}$ is defined as

$$w(n) = \sum_{k=-\infty}^{\infty} x(k)\, y(n-k)$$

Note: If it is one sided (right sided) sequence, take $x(k)=0, \; y(k)=0$ for $k<0$.

Convolution Theorem:

If $w(n)$ is the Convolution of two sequences $x(n)$ and $y(n)$, then

$$Z[w(n)] = W(z) = Z[x(n)] \cdot Z[y(n)] = X(z) \cdot Y(z)$$

Problems :-

① Find the Z-transform of the Convolution of $x(n) = u(n)$ & $y(n) = a^n u(n)$.

Solution :-

we know that, $Z[x(n)] = Z(u(n)) = X(z) = \sum_{n=0}^{\infty} u(n) z^{-n}$

$= \sum_{n=0}^{\infty} z^{-n}$

$= \frac{z}{z-1}$ if $|z| > 1$.

$Z[y(n)] = Z[a^n u(n)] = Y(z) = \sum_{n=0}^{\infty} a^n u(n) z^{-n}$

$= \sum_{n=0}^{\infty} a^n z^{-n} = \frac{z}{z-a}$ if $|z| > |a|$.

∴ The Z-transform of the Convolution of $x(n)$ and $y(n)$ is

$W(z) = \frac{z}{z-1} \cdot \frac{z}{z-a} = \frac{z^2}{(z-1)(z-a)}$ if $|z| > \max(|a|, 1)$.

2) Find the Z-transform of the Convolution of $x(n) = a^n u(n)$ and $y(n) = b^n u(n)$.

Solution :-

$Z[x(n)] = Z[a^n u(n)] = X(z) = \frac{z}{z-a}$ if $|z| > |a|$

$Z[y(n)] = Z[b^n u(n)] = Y(z) = \frac{z}{z-b}$ if $|z| > |b|$.

$W(z) =$ Z - transform of Convolution of $x(n)$ and $y(n)$.

$$= \frac{z^2}{(z-a)(z-b)} \quad \text{if } |z| > \max(|a|, |b|).$$

The Convolution of two Causal sequence $x(n)$ and $y(n)$, we define

$$\{x(n)\} * \{y(n)\} = \sum_{k=0}^{n} x(n-k)\, y(k).$$

Convolution theorem:

If $f(n)$ and $g(n)$ are two Causal sequences,

$$Z\left[\{f(n) * g(n)\}\right] = Z\{f(n)\} \cdot Z\{g(n)\} = F(z) \cdot G(z)$$

Note :-

$$Z^{-1}\left[F(z) \cdot G(z)\right] = f(n) * g(n)$$

$$= \sum_{k=0}^{n} f(n-k)\, g(k)$$

$$= Z^{-1}\left[F(z)\right] * Z^{-1}\left[G(z)\right]$$

Problems :-

1. Find the inverse Z-transform of $\frac{z^2}{(z-a)^2}$ using Convolution theorem.

Solution:-

$$Z^{-1}\left[\frac{z^2}{(z-a)^2}\right] = Z^{-1}\left[\frac{z}{z-a} \cdot \frac{z}{z-a}\right]$$

$$= Z^{-1}\left[\frac{z}{z-a}\right] * Z^{-1}\left[\frac{z}{z-a}\right]$$

$$= a^n * a^n.$$

$$= \sum_{k=0}^{n} a^{n-k} \cdot a^k = \sum_{k=0}^{n} a^n$$

$$= (n+1)\, a^n.$$

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2. Find the inverse Z-transform of $X(z) = \frac{z^2}{(z-1/2)(z-1/4)}$ using Convolution theorem.

Solution:-

$$Z^{-1}\left[X(z)\right] = Z^{-1}\left[\frac{z^2}{(z-1/2)(z-1/4)}\right]$$

$$= Z^{-1}\left[\frac{z}{z-1/2} \cdot \frac{z}{z-1/4}\right]$$

$$= Z^{-1}\left[\frac{z}{z-1/2}\right] * Z^{-1}\left[\frac{z}{z-1/4}\right]$$

$$= (1/2)^n * (1/4)^n$$

$$= \sum_{k=0}^{n} (1/2)^{n-k} \cdot (1/4)^{k}$$

$$= \left(\frac{1}{2}\right)^{n} \sum_{k=0}^{n} (1/2)^{-k} (1/4)^{k}$$

$$= \left(\frac{1}{2}\right)^{n} \sum_{k=0}^{n} (1/2)^{-k} \left((1/2)^{2}\right)^{k}$$

$$= \left(\frac{1}{2}\right)^{n} \sum_{k=0}^{n} (1/2)^{-k} \left(\frac{1}{2}\right)^{2k}$$

$$= \left(\frac{1}{2}\right)^{n} \sum_{k=0}^{n} (1/2)^{k}$$

$$= \left(\frac{1}{2}\right)^{n} \left[1 + (1/2) + (1/2)^{2} + \cdots + \left(\frac{1}{2}\right)^{n}\right]$$

$$= \left(\frac{1}{2}\right)^{n} \left[\frac{1 - (1/2)^{n+1}}{1 - 1/2}\right] \text{ by G.P.}$$

$$\left[a + ar + ar^{2} + \cdots + ar^{n} = \frac{a(1 - r^{n+1})}{1 - r}, \text{ if } r < 1\right]$$

$$= \left(\frac{1}{2}\right)^{n} \left[\frac{1 - (1/2)^{n+1}}{1/2}\right]$$

$$= \left(\frac{1}{2}\right)^{n-1} \left(1 - (1/2)^{n+1}\right)$$

$$= (1/2)^{n-1} - (1/2)^{2n}.$$

3) Find the inverse Z-transform of $\frac{8z^2}{(2z-1)(4z-1)}$ by using Convolution theorem.

Soln:- Given $\frac{8z^2}{(2z-1)(4z-1)} = \frac{\cancel{8}z^2}{\cancel{2}(z-1/2)\cancel{4}(z-1/4)}$

$$= \frac{z^2}{(z-1/2)(z-1/4)}.$$

$$= \left(\frac{1}{2}\right)^n * \left(\frac{1}{4}\right)^n$$

$$= \left(\frac{1}{2}\right)^{n-1} - \left(\frac{1}{2}\right)^{2n} \quad \left(\text{Refer eg. } 2\right)$$

4) Using Convolution theorem evaluate inverse Z-transform of $\frac{z^2}{(z-1)(z-3)}$

Solution:-

$$Z^{-1}\left[\frac{z^2}{(z-1)(z-3)}\right] = Z^{-1}\left[\frac{z}{z-1} \cdot \frac{z}{z-3}\right]$$

$$= Z^{-1}\left[\frac{z}{z-1}\right] * Z^{-1}\left[\frac{z}{z-3}\right]$$

$$= (1)^n * 3^n$$

$$= \sum_{k=0}^{n} 1^{n-k}\, 3^{k}$$

$$= \sum_{k=0}^{n} 3^{k} = 1+3+3^2+\cdots+3^n$$

$$\left[a+ar+ar^2+\cdots+ar^n = \frac{a(r^{n+1}-1)}{r-1} \text{ if } |r|>1 \right]$$

$$= \frac{(3^{n+1}-1)}{2}$$

Exercise problems :-

① Find $Z^{-1}\left[\dfrac{z^2}{(z-a)(z-b)}\right]$ Ans: $\dfrac{b^{n+1}-a^{n+1}}{b-a}$

② Find $Z^{-1}\left[\dfrac{z^2}{(z-1/2)(z+1/4)}\right]$ Ans: $\dfrac{2}{3}\left(\dfrac{1}{2}\right)^n + \dfrac{1}{3}\left(-1/4\right)^n$

③ Find $Z^{-1}\left[\dfrac{z^2}{(z+a)^2}\right]$ Ans: $(n+1)(-a)^n$

Evaluation of Residues of $f(z)$:-

① Residue of $f(z)$ at its simple pole $z = z_0$ is given by

$$= \lim_{z \to z_0} (z - z_0) f(z).$$

② Residue of $f(z)$ at its pole $z = z_0$ of order n is given by

$$= \lim_{z \to z_0} \left[\frac{1}{(n-1)!} \frac{d^{n-1}}{dz^{n-1}} (z - z_0)^n f(z) \right]$$

Inverse Integral method :

The inverse Z transform of $X(z)$ is given by

$$x(n) = \frac{1}{2\pi i} \oint_C X(z) \cdot z^{n-1} dz$$

$= \Sigma R =$ Sum of the residues of $X(z) z^{n-1}$ at all isolated singularities of $X(z) z^{n-1}$.

C is a circle with centre at origin and Radius R (which is large enough to contain all the singularities)

1. Find the inverse Z-transform of $\frac{z}{(z-1)(z-2)}$.

Solution:

Let $X(z) = \frac{z}{(z-1)(z-2)}$

$$X(z) \cdot z^{n-1} = \frac{z \cdot z^{n-1}}{(z-1)(z-2)} = \frac{z^n}{(z-1)(z-2)}$$

$z=1$ is a simple pole

$z=2$ is also a simple pole.

Residue of $X(z) \cdot z^{n-1}$ at $z=1 = \lim_{z \to 1} \cancel{(z-1)} \cdot \frac{z^n}{\cancel{(z-1)}(z-2)} = -1$

Residue of $X(z) \cdot z^{n-1}$ at $z=2 = \lim_{z \to 2} \cancel{(z-2)} \cdot \frac{z^n}{(z-1)\cancel{(z-2)}} = 2^n$

$\therefore$ $x(n) = \sum R$ where $\sum R$ is the sum of the residues of $X(z) \cdot z^{n-1}$ at the isolated singularities.

ie, $x(n) = 2^n - 1$, $n = 0, 1, 2, \ldots$

(2) Find $Z^{-1}\left[\dfrac{z^2-3z}{(z+2)(z-5)}\right]$

Solution:

Let $X(z) = \dfrac{z^2-3z}{(z+2)(z-5)}$

$$X(z).\,z^{n-1} = \frac{(z^2-3z)\,z^{n-1}}{(z+2)(z-5)}$$

$$= \frac{z^{n+1}-3z^n}{(z+2)(z-5)}$$

$z=-2$ is a simple pole.

$z=5$ is a " ".

Residue of $X(z).\,z^{n-1}$ at $z=-2$ =

$$\lim_{z\to -2} \cancel{(z+2)} \frac{z^{n+1}-3z^n}{\cancel{(z+2)}(z-5)} = \frac{(-2)^{n+1}-3(-2)^n}{(-7)}$$

$$\Rightarrow \frac{(-2)^n[-2-3]}{-7} = \frac{5}{7}(-2)^n$$

Residue of $X(z).\,z^{n-1}$ at $z=5$ is

$$\lim_{z\to 5} \cancel{(z-5)} \frac{z^{n+1}-3z^n}{(z+2)\cancel{(z-5)}} = \frac{5^{n+1}-35^n}{5+2}$$

$$= \frac{5^n[5-3]}{7} = \frac{2}{7}(5)^n$$

$\therefore\ x(n) = \sum R$

$$= 5/7\,(-2)^n + \frac{2}{7}(5)^n, \quad n=0,1,2,\ldots$$

3) Find $Z^{-1}((X(z))$ where $X(z) = \dfrac{4z^2-2z}{z^3-5z^2+8z-4}$.

Soln:

Given $X(z) = \dfrac{4z^2-2z}{z^3-5z^2+8z-4}$

$$X(z).z^{n-1} = \frac{(4z^2-2z)}{z^3-5z^2+8z-4} . z^{n-1}$$

$$= \frac{4z^{n+1} - 2z^n}{z^3-5z^2+8z-4}$$

Poles are given by $z^3-5z^2+8z-4=0$.

$$\begin{array}{r|rrrr} 1 & 1 & -5 & 8 & -4 \\ & 0 & 1 & -4 & 4 \\ \hline & 1 & -4 & 4 & \boxed{0} \end{array}$$

$z=1$ is one of pole.

$z^2 - 4z + 4 = 0$.

$z = 2, 2$.

$\therefore$ $z=1$ is a simple pole

$z=2$ is a pole of order 2.

Residue of $X(z).z^{n-1}$ at $z=1$ $= \lim\limits_{z\to 1} \cancel{(z-1)} \left(\dfrac{4z^{n+1} - 2z^n}{\cancel{(z-1)}(z-2)^2}\right)$

$$= \frac{4(1)^{n+1} - 2(1)^n}{(1-2)^2} = 2$$

$\frac{vu' - uv'}{v^2}$

Residue of $X(z) \cdot z^{n-1}$ at $z=2$ of order 2 =

$$\frac{1}{(2-1)!} \lim_{z \to 2} \frac{d^1}{dz^1} \cancel{(z-2)^2} \frac{4z^{n+1} - 2z^n}{(z-1)\cancel{(z-2)^2}}$$

$$= \frac{1}{1} \lim_{z \to 2} \frac{d}{dz} \left(\frac{4z^{n+1} - 2z^n}{(z-1)} \right) \frac{u}{v}$$

$$= \lim_{z \to 2} \left[\frac{(z-1)\left(4(n+1)z^{n+1-\cancel{1}} - 2nz^{n-1}\right) - (4z^{n+1} - 2z^n)(1)}{(z-1)^2} \right]$$

$$= \left[\frac{(2-1)\left(4(n+1)2^n - 2n\,2^{n-1}\right) - 4\,2^{n+1} + 2\,2^n}{(2-1)^2} \right]$$

$(1)^2 = 1$

$$\Rightarrow \quad 4(n+1)2^n - 2n2^{n-1} - 42^{n+1} + 2 \cdot 2^n$$

$$\Rightarrow \quad 4n2^n + 42^n - n2^n - 4(2^n \cdot 2) + 2 \cdot 2^n$$

$$= 4n2^n + 42^n - n2^n - 82^n + 22^n.$$

$$= n3 \cdot 2^n - 2 \cdot 2^n$$

$$= 2^n(3n - 2)$$

$\therefore \; x(n) = \sum R.$

$$x(n) = 2 + 2^n(3n-2), \quad n = 0, 1, 2, \ldots$$

(4) Find the inverse Z-transform of $\frac{z(z+1)}{(z-1)^3}$.

Soln:

Let $X(z) = \frac{z(z+1)}{(z-1)^3}$

$$X(z) \cdot z^{n-1} = \frac{(z^2+z)z^{n-1}}{(z-1)^3}$$

$$X(z) \cdot z^{n-1} = \frac{z^{n+1}+z^n}{(z-1)^3}$$

$z=1$ is a pole of order 3.

Residue of $X(z) \cdot z^{n-1}$ at $z=1$ of order $3 =$

$$\frac{1}{2!} \lim_{z\to 1} \frac{d^{3-1}}{dz^{3-1}} \cancel{(z-1)^3} \frac{z^{n+1}+z^n}{\cancel{(z-1)^3}}$$

$$\Rightarrow \frac{1}{2} \lim_{z\to 1} \frac{d^2}{dz^2} (z^{n+1}+z^n)$$

$$= \frac{1}{2} \lim_{z\to 1} \frac{d}{dz}\left(\frac{d}{dz}(z^{n+1}+z^n)\right)$$

$(n+1)\, z^{n+\cancel{1}-\cancel{1}}$

$$= \frac{1}{2} \lim_{z\to 1} \frac{d}{dz}\left((n+1)z^n + n z^{n-1}\right)$$

$$= \frac{1}{2} \lim_{z\to 1} \left[(n+1)(n) z^{n-1} + n(n-1) z^{n-2}\right]$$

$$= \frac{1}{2}\left[(n+1)n\, 1^{(n-1)} + n(n-1)\, 1^{(n-2)}\right]$$

$$= \frac{1}{2}\left[(n^2+n) + n(n-1)\right] = \frac{1}{2}\left[n^2 + \cancel{n} + n^2 - \cancel{n}\right]$$

$$= n^2$$

(5) Find $Z^{-1}\left[\frac{z}{z^2+1}\right]$ by using Residue method.

$$X(z)\, z^{n-1} = \frac{z^n}{z^2+1}$$

Singular points are given by $z^2+1=0 \Rightarrow z = \pm i$

$+i, -i$ are simple poles.

$$\text{Res}\,(z=i) = \lim_{z\to i} \cancel{(z-i)} \frac{z^n}{(z+i)\cancel{(z-i)}} = \frac{i^n}{2i}$$

$$\text{Res}\,(z=-i) = \lim_{z\to -i} \cancel{(z+i)} \frac{z^n}{\cancel{(z+i)}(z-i)} = \frac{(-i)^n}{(-2i)}$$

$$\therefore Z^{-1}\left[\frac{z}{z^2+1}\right] = \text{Sum of the residues}$$

$$= \frac{i^n}{2i} + \frac{(-i)^n}{-2i}$$

$$i = \cos\frac{\pi}{2} + i\sin\frac{\pi}{2} \qquad -i = \cos\frac{\pi}{2} - i\sin\frac{\pi}{2}$$

$$i^n = \cos\frac{n\pi}{2} + i\sin\frac{n\pi}{2} \qquad (-i)^n = \cos\frac{n\pi}{2} - i\sin\frac{n\pi}{2}$$

$$i^n - (-i)^n = 2i\sin\frac{n\pi}{2}$$

$$Z^{-1}\left[\frac{z}{z^2+1}\right] = \frac{i^n - (-i)^n}{2i} = \frac{\cancel{2i}\sin\frac{n\pi}{2}}{\cancel{2i}} = \sin\frac{n\pi}{2}.$$