## 3.6 FOURIER SERIES METHOD OF FIR FILTER DESIGN

The frequency response of a digital filter is periodic, with period equal to the sampling frequency. From fourier series analysis, we know that any periodic function can be expressed as a linear combination of complex exponentials. Therefore, the desired frequency response of an FIR digital filter can be represented by the fourier series as shown in equation (3.102).

$$H_d(\omega)\Big|_{\omega=\omega T} = H_d(\omega T) = \sum_{n=-\infty}^{+\infty} h_d(n)\, e^{-j\omega nT} \quad .....(3.102)$$

where the fourier coefficients $h_d(n)$ are the desired impulse response sequence of the filter. The samples of $h_d(n)$ can be determined using equation (3.103)

$$h_d(n) = \frac{1}{\omega_s} \int_{\frac{-\omega_s}{2}}^{\frac{\omega_s}{2}} H_d(\omega T)\, e^{j\omega nT}\, d\omega \quad .....(3.103)$$

where, $F_s$ = Sampling frequency in Hz

$\omega_s$ = $2\pi F_s$ = Sampling frequency in rad/sec.

$T$ = $1/F_s$ = Sampling period in sec.

The impulse response obtained from equation (3.103) is an infinite duration sequence. For FIR filters we truncate this infinite impulse response to a finite duration sequence of length N, where N is odd.

$$\therefore h(n) = h_d(n)\ ;\ \text{for } n = -\left(\tfrac{N-1}{2}\right) \text{ to } +\left(\tfrac{N-1}{2}\right) \quad .....(3.104)$$

On taking $\mathcal{Z}$-transform of equation (3.104) we get

$$H(z) = \sum_{n=-(N-1)/2}^{(N-1)/2} h(n)\, z^{-n} \quad .....(3.105)$$

The transfer function of equation (3.105) represents noncausal filter (due to the presence of positive powers of z). Hence the transfer function of equation (3.105) is multiplied by $z^{-(N-1)/2}$

$$\therefore H(z) = z^{-(N-1)/2} \sum_{n=-(N-1)/2}^{(N-1)/2} h(n)\, z^{-n} \quad .....(3.106)$$

$$H(z) = z^{-(N-1)/2}\left[\sum_{n=-(N-1)/2}^{-1} h(n)\, z^{-n} + h(0) + \sum_{n=1}^{(N-1)/2} h(n)\, z^{-n}\right]$$

$$= z^{-(N-1)/2}\left[\sum_{n=1}^{(N-1)/2} h(-n)\, z^{n} + h(0) + \sum_{n=1}^{(N-1)/2} h(n)\, z^{-n}\right]$$

Since $h(n) = h(-n)$ we express H(z) as,

$$H(z) = z^{-(N-1)/2}\left[h(0) + \sum_{n=1}^{(N-1)/2} h(n)\,(z^{n} + z^{-n})\right] \quad .....(3.107)$$

Hence we see that causality is brought about by multiplying the transfer function by the delay factor $\alpha = (N-1)/2$. This modification does not affect the amplitude response of the filter, however the abrupt truncation of the fourier series results in oscillations in the passband and stopband. These oscillations are due to the slow convergence of the fourier series, particularly near the points of discontinuity. This effect is known as **Gibbs Phenomenon.** The undesirable oscillations can be reduced by multiplying the desired impulse response coefficients by an appropriate window function.

The specifications of lowpass, highpass, bandpass and bandstop filters and their desired impulse response for FIR filter design by fourier series method are listed in table 3.2.

## EXAMPLE 3.1

Design a FIR lowpass filter with cutoff frequency 1 kHz and sampling rate of 4 kHz with 11 samples using fourier series method.

### *Solution*

Given that, $f_c = 1$ kHz, $\quad \therefore \omega_c = 2\pi f_c = 2\pi \times 1 \times 10^3 = 6283$ rad/sec

$f_s = 4$ kHz, $\quad \therefore \omega_s = 2\pi f_s = 2\pi \times 4 \times 10^3 = 25132$ rad/sec

Also, $T = 1/f_s = 1/(4\times 10^3) = 0.25 \times 10^{-3}$ sec $= 0.25$ msec

The given filter can be expressed by the following specifications.

$$H_d(\omega T) = \begin{cases} 1\ ; & \text{for } -\omega_c \le \omega \le \omega_c \\ 0\ ; & \text{for } -\dfrac{\omega_s}{2} \le \omega < -\omega_c \\ 0\ ; & \text{for } \omega_c < \omega \le \dfrac{\omega_s}{2} \end{cases}$$

The desired impulse response of the filter is

$$h_d(n) = \frac{1}{\omega_s}\int_{-\omega_c}^{\omega_c} e^{j\omega nT}\, d\omega = \frac{1}{\omega_s}\left[\frac{e^{j\omega nT}}{jnT}\right]_{-\omega_c}^{\omega_c} = \frac{1}{\omega_s}\left[\frac{e^{j\omega_c nT}}{jnT} - \frac{e^{-j\omega_c nT}}{jnT}\right]$$

$$= \frac{2}{\omega_s nT}\left[\frac{e^{j\omega_c nT} - e^{-j\omega_c nT}}{2j}\right] = \frac{2}{\omega_s nT}\sin \omega_c nT$$

$$\boxed{\text{L' Hospital rule} \quad \underset{\theta\to 0}{\text{Lt}} \frac{\sin A\theta}{\theta} = A}$$

When $n \neq 0$; $h_d(n) = \dfrac{2}{\omega_s nT}\sin \omega_c nT = \dfrac{2}{2\pi f_s nT}\sin \omega_c nT = \dfrac{2}{\frac{2\pi}{T} nT}\sin \omega_c nT = \dfrac{\sin \omega_c nT}{\pi n}$

When $n = 0$, the factor $\dfrac{\sin \omega_c nT}{\pi n}$ becomes 0/0, which is indeterminate. Hence using L' hospital rule,

When $n = 0$, $h_d(n) = h_d(0) = \underset{n\to 0}{\text{Lt}} \dfrac{2}{\omega_s nT}\sin \omega_c nT = \dfrac{2}{\omega_s} \underset{n\to 0}{\text{Lt}} \dfrac{\sin \omega_c nT}{nT} = \dfrac{2}{\omega_s}\omega_c = \dfrac{2\omega_c}{\omega_s}$

The impulse response of FIR filter is obtained by truncating $h_d(n)$ to 11 samples.

$$\therefore h(n) = h_d(n) = \frac{\sin \omega_c nT}{n\pi} \quad ; \text{ for } n = 1 \text{ to } (N-1)$$

$$= \frac{2\omega_c}{\omega_s} \quad ; \text{ for } n = 0$$

Here N = 11, $\therefore (N-1) = 10$

When n = 0 ; $h(n) = h(0) = \frac{2\,\omega_c}{\omega_s} = \frac{2 \times 6283}{25132} = \frac{1}{2} = 0.5$

When n = 1 ; $h(n) = h(1) = \frac{\sin \omega_c T}{\pi} = \frac{\sin(6283 \times 0.25 \times 10^{-3})}{\pi} = 0.3183$

When n = 2 ; $h(n) = h(2) = \frac{\sin 2\omega_c T}{2\pi} = \frac{\sin(2 \times 6283 \times 0.25 \times 10^{-3})}{2\pi} = 1.474 \times 10^{-5} \approx 0$

When n = 3 ; $h(n) = h(3) = \frac{\sin 3\omega_c T}{3\pi} = \frac{\sin(3 \times 6283 \times 0.25 \times 10^{-3})}{3\pi} = -0.1061$

When n = 4 ; $h(n) = h(4) = \frac{\sin 4\omega_c T}{4\pi} = \frac{\sin(4 \times 6283 \times 0.25 \times 10^{-3})}{4\pi} = -1.474 \times 10^{-5} \approx 0$

When n = 5 ; $h(n) = h(5) = \frac{\sin 5\omega_c T}{5\pi} = \frac{\sin(5 \times 6283 \times 0.25 \times 10^{-3})}{5\pi} = 0.0637$

When n = 6 ; $h(n) = h(6) = \frac{\sin 6\omega_c T}{6\pi} = \frac{\sin(6 \times 6283 \times 0.25 \times 10^{-3})}{6\pi} = 1.474 \times 10^{-5} \approx 0$

When n = 7 ; $h(n) = h(7) = \frac{\sin 7\omega_c T}{7\pi} = \frac{\sin(7 \times 6283 \times 0.25 \times 10^{-3})}{7\pi} = -0.0455$

When n = 8 ; $h(n) = h(8) = \frac{\sin 8\omega_c T}{8\pi} = \frac{\sin(8 \times 6283 \times 0.25 \times 10^{-3})}{8\pi} = -1.474 \times 10^{-5} \approx 0$

When n = 9 ; $h(n) = h(9) = \frac{\sin 9\omega_c T}{9\pi} = \frac{\sin(9 \times 6283 \times 0.25 \times 10^{-3})}{9\pi} = 0.0354$

When n = 10 ; $h(n) = h(10) = \frac{\sin 10\omega_c T}{10\pi} = \frac{\sin(10 \times 6283 \times 0.25 \times 10^{-3})}{10\pi} = 1.474 \times 10^{-5} \approx 0$

*(Note : Calculate $\sin\theta$ by keeping the calculator in radians mode)*

The transfer function of the digital filter is given by

$$H(z) = z^{-\frac{N-1}{2}} \left[ h(0) + \sum_{n=1}^{\frac{N-1}{2}} h(n)\,(z^n + z^{-n}) \right]$$

Here, N = 11, $\therefore \frac{N-1}{2} = \frac{11-1}{2} = 5$

$$\therefore H(z) = z^{-5}\left[h(0) + \sum_{n=1}^{5} h(n)\,(z^{n} + z^{-n})\right] = h(0)\,z^{-5} + \sum_{n=1}^{5} h(n)\,(z^{-5}z^{n} + z^{-5}z^{-n})$$

$$= h(0)\,z^{-5} + \sum_{n=1}^{5} h(n)\,z^{n-5} + \sum_{n=1}^{5} h(n)\,z^{-(n+5)}$$

$$= h(0)\,z^{-5} + h(1)\,z^{-4} + h(2)\,z^{-3} + h(3)\,z^{-2} + h(4)\,z^{-1} + h(5)\,z^{0}$$
$$+ h(1)\,z^{-6} + h(2)\,z^{-7} + h(3)\,z^{-8} + h(4)\,z^{-9} + h(5)\,z^{-10}$$

$$= h(5) + h(4)\,z^{-1} + h(3)\,z^{-2} + h(2)\,z^{-3} + h(1)\,z^{-4} + h(0)\,z^{-5}$$
$$+ h(1)\,z^{-6} + h(2)\,z^{-7} + h(3)\,z^{-8} + h(4)\,z^{-9} + h(5)\,z^{-10}$$

$$= 0.0637 + 0 - 0.1061\,z^{-2} + 0 + 0.3183\,z^{-4} + 0.5\,z^{-5} + 0.3183\,z^{-6}$$
$$+ 0 - 0.1061\,z^{-8} + 0 + 0.0637\,z^{-10}$$

$$= 0.0637 - 0.1061\,z^{-2} + 0.3183\,z^{-4} + 0.5\,z^{-5} + 0.3183\,z^{-6} - 0.1061\,z^{-8} + 0.0637\,z^{-10}$$

**Table 3.3 : Specification and desired impulse response for FIR filter design by fourier series method**

| Type of filter | Specifications | Impulse response |
|---|---|---|
| Lowpass | $H_d(\omega T)=\begin{cases}1\ ;\ \text{for} & -\omega_c \le \omega \le \omega_c\\ 0\ ;\ \text{for} & -\frac{\omega_s}{2} \le \omega < -\omega_c\\ 0\ ;\ \text{for} & \omega_c < \omega \le \frac{\omega_s}{2}\end{cases}$ | $h_d(n)=\frac{1}{\omega_s}\int_{\frac{-\omega_s}{2}}^{\frac{\omega_s}{2}} H_d(\omega T)\, e^{j\omega nT}\, d\omega = \frac{1}{\omega_s}\int_{-\omega_c}^{\omega_c} e^{j\omega nT}\, d\omega$ <br> $\left(\because H_d(\omega T)=0 \text{ in the range } \frac{-\omega_s}{2} \le \omega < -\omega_c \text{ and } \omega_c < \omega \le \omega_s/2\right)$ |
| Highpass | $H_d(\omega T)=\begin{cases}1\ ;\ \text{for} & -\frac{\omega_s}{2} \le \omega \le -\omega_c\\ 1\ ;\ \text{for} & \omega_c \le \omega \le \frac{\omega_s}{2}\\ 0\ ;\ \text{for} & -\omega_c < \omega < \omega_c\end{cases}$ | $h_d(n)=\frac{1}{\omega_s}\int_{\frac{-\omega_s}{2}}^{\frac{\omega_s}{2}} H_d(\omega T)\, e^{j\omega nT}\, d\omega = \frac{1}{\omega_s}\int_{\frac{-\omega_s}{2}}^{-\omega_c} e^{j\omega nT}\, d\omega + \frac{1}{\omega_s}\int_{\omega_c}^{\frac{\omega_s}{2}} e^{j\omega nT}\, d\omega$ <br> $(\because H_d(\omega T)=0 \text{ in the range } -\omega_c < \omega < \omega_c)$ |
| Bandpass | $H_d(\omega T)=\begin{cases}1\ ;\ \text{for} & -\omega_{c2} \le \omega \le -\omega_{c1}\\ 1\ ;\ \text{for} & \omega_{c1} \le \omega \le \omega_{c2}\\ 0\ ;\ \text{for} & -\frac{\omega_s}{2} \le \omega < -\omega_{c2}\\ 0\ ;\ \text{for} & -\omega_{c1} < \omega < \omega_{c1}\\ 0\ ;\ \text{for} & \omega_{c2} < \omega \le \frac{\omega_s}{2}\end{cases}$ | $h_d(n)=\frac{1}{\omega_s}\int_{\frac{-\omega_s}{2}}^{\frac{\omega_s}{2}} H_d(\omega T)\, e^{j\omega nT}\, d\omega = \frac{1}{\omega_s}\int_{-\omega_{c2}}^{-\omega_{c1}} e^{j\omega nT}\, d\omega + \frac{1}{\omega_s}\int_{\omega_{c1}}^{\omega_{c2}} e^{j\omega nT}\, d\omega$ <br> $\left(\because H_d(\omega T)=0 \text{ in the range } -\frac{\omega_s}{2} \le \omega < -\omega_{c2}\ ;\ -\omega_{c1} < \omega < \omega_{c1} \text{ and } \omega_{c2} < \omega \le \frac{\omega_s}{2}\right)$ |
| Bandstop | $H_d(\omega T)=\begin{cases}1\ ;\ \text{for} & -\frac{\omega_s}{2} \le \omega \le -\omega_{c2}\\ 1\ ;\ \text{for} & -\omega_{c1} \le \omega \le \omega_{c1}\\ 1\ ;\ \text{for} & \omega_{c2} \le \omega \le \frac{\omega_s}{2}\\ 0\ ;\ \text{for} & -\omega_{c2} < \omega < -\omega_{c1}\\ 0\ ;\ \text{for} & \omega_{c1} < \omega < \omega_{c2}\end{cases}$ | $h_d(n)=\frac{1}{\omega_s}\int_{\frac{-\omega_s}{2}}^{\frac{\omega_s}{2}} H_d(\omega T)\, e^{j\omega nT}\, d\omega = \frac{1}{\omega_s}\int_{-\omega_s/2}^{-\omega_{c2}} e^{j\omega nT}\, d\omega + \frac{1}{\omega_s}\int_{-\omega_{c1}}^{\omega_{c1}} e^{j\omega nT}\, d\omega + \frac{1}{\omega_s}\int_{\omega_{c2}}^{\omega_s/2} e^{j\omega nT}\, d\omega$ <br> $(\because H_d(\omega T)=0 \text{ in the range } -\omega_{c2} < \omega < -\omega_{c1} \text{ and } \omega_{c1} < \omega < \omega_{c2})$ |

**Table 3.4 : Window sequences (functions) for FIR filter design**

| Name of window | Window sequence |
|---|---|
| Rectangular | $w_R(n)=\begin{cases}1 & ; -\frac{(N-1)}{2}\le n\le\frac{(N-1)}{2}\\ 0 & ;\text{ otherwise}\end{cases}$ (or) $w_R(n)=\begin{cases}1 & ; 0\le n\le N-1\\ 0 & ;\text{ otherwise}\end{cases}$ |
| Triangular (Barlett) | $w_T(n)=\begin{cases}1-\frac{2\lvert n\rvert}{N-1} & ; -\frac{(N-1)}{2}\le n\le\frac{(N-1)}{2}\\ 0 & ;\text{ otherwise}\end{cases}$ <br> (or) $w_T(n)=\begin{cases}1-\frac{2\lvert n-(N-1)/2\rvert}{N-1} & ; 0\le n\le N-1\\ 0 & ;\text{ otherwise}\end{cases}$ |
| Hanning | $w_C(n)=\begin{cases}0.5+0.5\cos\frac{2n\pi}{N-1} & ; -\frac{(N-1)}{2}\le n\le\frac{(N-1)}{2}\\ 0 & ;\text{ otherwise}\end{cases}$ <br> (or) $w_C(n)=\begin{cases}0.5-0.5\cos\frac{2n\pi}{N-1} & ; 0\le n\le N-1\\ 0 & ;\text{ otherwise}\end{cases}$ |
| Hamming | $w_H(n)=\begin{cases}0.54+0.46\cos\frac{2n\pi}{N-1} & ; -\frac{(N-1)}{2}\le n\le\frac{(N-1)}{2}\\ 0 & ;\text{ otherwise}\end{cases}$ <br> (or) $w_H(n)=\begin{cases}0.54-0.46\cos\frac{2n\pi}{N-1} & ; 0\le n\le N-1\\ 0 & ;\text{ otherwise}\end{cases}$ |
| Blackman | $w_B(n)=\begin{cases}0.42+0.5\cos\frac{2n\pi}{N-1}+0.08\cos\frac{4n\pi}{N-1} & ; -\frac{(N-1)}{2}\le n\le\frac{(N-1)}{2}\\ 0 & ;\text{ otherwise}\end{cases}$ <br> (or) $w_B(n)=\begin{cases}0.42-0.5\cos\frac{2n\pi}{N-1}+0.08\cos\frac{4n\pi}{N-1} & ; 0\le \pi\le N-1\\ 0 & ;\text{ otherwise}\end{cases}$ |
| Kaiser | $w_K(n)=\begin{cases}\dfrac{I_0\left(\alpha\sqrt{1-\left(\frac{2n}{N-1}\right)^2}\right)}{I_0(\alpha)} & ; -\frac{(N-1)}{2}\le n\le\frac{(N-1)}{2}\\ 0 & ;\text{ otherwise}\end{cases}$ <br> (or) $w_K(n)=\begin{cases}\dfrac{I_0\left(\alpha\sqrt{\left(\frac{N-1}{2}\right)^2-\left(n-\frac{N-1}{2}\right)^2}\right)}{I_0\left(\alpha\,\frac{N-1}{2}\right)} & ; 0\le n\le(N-1)\\ 0 & ;\text{ otherwise}\end{cases}$ |

**Table 3.5 : Frequency domain characteristics of some window functions**

| Type of window | Approximate width of mainlobe | Peak sidelobe magnitude (dB) |
|---|---|---|
| Rectangular | $4\pi/N$ | $-13$ |
| Barlett | $8\pi/N$ | $-25$ |
| Hanning | $8\pi/N$ | $-31$ |
| Hamming | $8\pi/N$ | $-41$ |
| Blackman | $12\pi/N$ | $-58$ |

**Note :** *In filter specifications gain & magnitude are same and will be in negative dB. The attenuation is inverse of gain and so it is negative of magnitude or gain in dB. Hence attenuation will be in positive dB.*

## .8 FIR FILTER DESIGN USING WINDOWS

1. Choose the desired frequency response of the filter $H_d(\omega)$.

2. Take inverse fourier transform of $H_d(\omega)$ to obtain the desired impulse response $h_d(n)$. By definition of inverse fourier transform,

$$h_d(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} H_d(\omega)\, e^{j\omega n}\, d\omega$$

**Note :** *Refer table 3.6 for the limits of integration for various type of filter.*

3. Choose a window sequence w(n) (from table 3.4) and determine the product of $h_d(n)$ and $w(n)$. Let this product be $h(n)$

$$\therefore\ h(n) = h_d(n)\, w(n)$$

4. The transfer function H(z) of the filter is obtained by taking $\mathcal{Z}$-transform of h(n). Realize the filter by a suitable structure.

5. Choose a linear phase magnitude function, $|H(\omega)|$ (from table 3.2). Using h(n), obtain an equation for $|H(\omega)|$. Calculate $|H(\omega)|$ for various values of $\omega$ in the range $0 \le \omega \le \pi$ and sketch the graph between $|H(\omega)|$ and $\omega$, which is the frequency response of the filter.

**Table 3.6 : The normalized ideal (desired) frequency response and impulse response for FIR filter design using windows.**

| Type of filter | Ideal (desired) frequency response | Ideal (desired) impulse response |
|---|---|---|
| Lowpass filter | $H_d(\omega) = \begin{cases} e^{-j\omega\alpha} & ; \ -\omega_c \le \omega \le \omega_c \\ 0 & ; \ -\pi \le \omega < -\omega_c \\ 0 & ; \ \omega_c < \omega \le \pi \end{cases}$ | $h_d(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} H_d(\omega)\, e^{j\omega n}\, d\omega = \frac{1}{2\pi} \int_{-\omega_c}^{\omega_c} e^{-j\omega\alpha} e^{j\omega n}\, d\omega$ <br> $(\because H_d(\omega) = 0$ in the range $-\pi \le \omega < -\omega_c$ and $\omega_c < \omega \le \pi)$ |
| Highpass filter | $H_d(\omega) = \begin{cases} e^{-j\omega\alpha} & ; \ -\pi \le \omega \le -\omega_c \\ e^{-j\omega\alpha} & ; \ \omega_c \le \omega \le \pi \\ 0 & ; \ -\omega_c < \omega < \omega_c \end{cases}$ | $h_d(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} H_d(\omega)\, e^{j\omega n}\, d\omega = \frac{1}{2\pi} \int_{-\pi}^{-\omega_c} e^{-j\omega\alpha} e^{j\omega n}\, d\omega + \frac{1}{2\pi} \int_{\omega_c}^{\pi} e^{-j\omega\alpha} e^{j\omega n}\, d\omega$ <br> $(\because H_d(\omega) = 0$ in the range $-\omega_c < \omega < \omega_c)$ |
| Bandpass filter | $H_d(\omega) = \begin{cases} e^{-j\omega\alpha} & ; \ -\omega_{c2} \le \omega \le -\omega_{c1} \\ e^{-j\omega\alpha} & ; \ \omega_{c1} \le \omega \le \omega_{c2} \\ 0 & ; \ -\pi \le \omega < -\omega_{c2} \\ 0 & ; \ -\omega_{c1} < \omega < \omega_{c1} \\ 0 & ; \ \omega_{c2} < \omega \le \pi \end{cases}$ | $h_d(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} H_d(\omega)\, e^{j\omega n}\, d\omega = \frac{1}{2\pi} \int_{-\omega_{c2}}^{-\omega_{c1}} e^{-j\omega\alpha} e^{j\omega n}\, d\omega + \frac{1}{2\pi} \int_{\omega_{c1}}^{\omega_{c2}} e^{-j\omega\alpha} e^{j\omega n}\, d\omega$ <br> $(\because H_d(\omega) = 0$ in the range $-\pi \le \omega < -\omega_{c2}$ ; $-\omega_{c1} < \omega < \omega_{c1}$ and $\omega_{c2} < \omega \le \pi)$ |
| Bandstop filter | $H_d(\omega) = \begin{cases} e^{-j\omega\alpha} & ; \ -\pi \le \omega \le -\omega_{c2} \\ e^{-j\omega\alpha} & ; \ -\omega_{c1} \le \omega \le \omega_{c1} \\ e^{-j\omega\alpha} & ; \ \omega_{c2} \le \omega \le \pi \\ 0 & ; \ -\omega_{c2} < \omega < -\omega_{c1} \\ 0 & ; \ \omega_{c1} < \omega < \omega_{c2} \end{cases}$ | $h_d(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} H_d(\omega)\, e^{j\omega n}\, d\omega = \frac{1}{2\pi} \int_{-\pi}^{-\omega_{c2}} e^{j\omega\alpha} e^{j\omega n}\, d\omega + \frac{1}{2\pi} \int_{-\omega_{c1}}^{\omega_{c1}} e^{-j\omega\alpha} e^{j\omega n}\, d\omega + \frac{1}{2\pi} \int_{\omega_{c2}}^{\pi} e^{-j\omega\alpha} e^{j\omega n}\, d\omega$ <br> $(\because H_d(\omega) = 0$ in the range $-\omega_{c2} < \omega < -\omega_{c1}$ and $\omega_{c1} < \omega < \omega_{c2})$ |

## EXAMPLE 3.2

Design a lowpass filter using rectangular window by taking 9 samples of w(n) and with a cutoff frequency of 1.2 radians/sec.

### SOLUTION

Given $\omega_c = 1.2$ rad/sec and $N = 9$

For a lowpass filter the desired frequency response is

$$H_d(\omega) = \begin{cases} e^{-j\omega\alpha} & ; \ -\omega_c \le \omega \le \omega_c \\ 0 & ; \ \text{otherwise} \end{cases}$$

The $h_d(n)$ is obtained by taking inverse fourier transform of $H_d(\omega)$

By definition of inverse fourier transform

$e^{a\omega}/a$

$e^{a\omega}$

$$h_d(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} H_d(\omega)\, e^{j\omega n} d\omega = \frac{1}{2\pi}\int_{-\omega c}^{\omega c} e^{-j\omega\alpha}\, e^{j\omega n}\, d\omega$$

$$= \frac{1}{2\pi}\int_{-\omega c}^{\omega c} e^{j\omega(n-\alpha)} d\omega = \frac{1}{2\pi}\left[\frac{e^{j\omega(n-\alpha)}}{j(n-\alpha)}\right]_{-\omega c}^{\omega c}$$

$$= \frac{1}{2\pi}\left[\frac{e^{j(\omega c)(n-\alpha)}}{j(n-\alpha)} - \frac{e^{j(-\omega c)(n-\alpha)}}{j(n-\alpha)}\right] = \frac{1}{\pi(n-\alpha)}\left[\frac{e^{j\omega c(n-\alpha)} - e^{-j\omega c(n-\alpha)}}{2j}\right]$$

$$= \frac{1}{\pi(n-\alpha)} \sin\omega_c(n-\alpha)$$

When $n = \alpha$, the term $\dfrac{\sin\omega_c(n-\alpha)}{(n-\alpha)}$ will become 0 / 0 which is indeterminate.

| L' Hospital rule |
|---|
| $\displaystyle \mathop{Lt}_{\theta\to 0} \frac{\sin A\theta}{\theta} = A$ |

Hence, $h_d(n) = \dfrac{\sin\omega_c(n-\alpha)}{\pi(n-\alpha)}$ ; for $n \ne \alpha$

For $n = \alpha$, $h_d(n)$ can be evaluated using L'Hospital rule.

$$\therefore \text{When } n = \alpha,\ h_d(n) = \mathop{Lt}_{n\to\alpha} \frac{\sin\omega_c(n-\alpha)}{\pi(n-\alpha)} = \frac{1}{\pi}\mathop{Lt}_{n\to\alpha}\frac{\sin\omega_c(n-\alpha)}{(n-\alpha)} = \frac{1}{\pi}\omega_c = \frac{\omega_c}{\pi}$$

The impulse response of FIR filter is obtained by multiplying $h_d(n)$ by window sequence.

$\therefore$ Impulse response, $h(n) = h_d(n)\, w_R(n)$

Rectangular window sequence, $w_R(n) = \begin{cases} 1 & ; \text{ for } n = 0 \text{ to } (N-1) \\ 0 & ; \text{ otherwise} \end{cases}$

$$\therefore h(n) = h_d(n) \ ; \text{ for } n = 0 \text{ to } (N-1)$$

Here, $N = 9$ ; $\omega_c = 1.2$ rad / sec ; $\alpha = \frac{N-1}{2} = \frac{9-1}{2} = 4$

When $n = 0$ ; $h(0) = \frac{\sin(1.2 \times (-4))}{\pi \times (-4)} = -0.0793$

When $n = 1$ ; $h(1) = \frac{\sin(1.2 \times (-3))}{\pi \times (-3)} = -0.0470$

When $n = 2$ ; $h(2) = \frac{\sin(1.2 \times (-2))}{\pi \times (-2)} = 0.1075$

When $n = 3$ ; $h(3) = \frac{\sin(1.2 \times (-1))}{\pi \times (-1)} = 0.2967$

When $n = 4$ ; $h(4) = \frac{1.2}{\pi} = 0.3820$

When $n = 5$ ; $h(5) = \frac{\sin(1.2 \times (1))}{\pi} = 0.2967$

When $n = 6$ ; $h(6) = \frac{\sin(1.2 \times (2))}{\pi \times (2)} = 0.1075$

When $n = 7$ ; $h(7) = \frac{\sin(1.2 \times (3))}{\pi \times (3)} = -0.047$

When $n = 8$ ; $h(8) = \frac{\sin(1.2 \times (4))}{\pi \times (4)} = -0.0793$

The impulse response is satisfying the symmetry condition $h(N-1-n) = h(n)$

Here $N = 9$, $\therefore h(N-1-n) = h(9-1-n) = h(8-n)$

When $n = 5$; $h(5) = h(8-5) \Rightarrow h(5) = h(3)$

When $n = 6$; $h(6) = h(8-6) \Rightarrow h(6) = h(2)$

When $n = 7$; $h(7) = h(8-7) \Rightarrow h(7) = h(1)$

When $n = 8$; $h(8) = h(8-8) \Rightarrow h(8) = h(0)$

The magnitude function of FIR filter when the impulse response is symmetric and N is odd is given below.

$$|H(\omega)| = h\left(\tfrac{N-1}{2}\right) + \sum_{n=1}^{\frac{N-1}{2}} 2h\left(\tfrac{N-1}{2} - n\right) \cos n\omega$$

$$|H(\omega)| = h(4) + 2h(3)\cos\omega + 2h(2)\cos 2\omega + 2h(1)\cos 3\omega + 2h(0)\cos 4\omega$$

$$= 0.382 + 2 \times 0.2967 \times \cos 1\omega + 2 \times 0.1075 \times \cos 2\omega + 2 \times (-0.047) \times \cos 3\omega + 2 \times (-0.0793) \cos 4\omega$$

$$= 0.382 + 0.5934\cos\omega + 0.215\cos 2\omega - 0.094\cos 3\omega - 0.1586\cos 4\omega$$

The transfer function of the filter is given by

$$H(z) = \sum_{n=0}^{N-1} h(n)z^{-n} = \sum_{n=0}^{8} h(n)z^{-n} = \sum_{n=0}^{3} h(n)z^{-n} + h(4)z^{-4} + \sum_{n=5}^{8} h(n)z^{-n}$$

$$= \sum_{n=0}^{3} h(n)z^{-n} + h(4)z^{-4} + \sum_{n=0}^{3} h(8-n)z^{-(8-n)} \qquad \boxed{\because h(N-1-k) = (h(k)}$$

$$= \sum_{n=0}^{3} h(n)z^{-n} + h(4)z^{-4} + \sum_{n=0}^{3} h(n)z^{-(8-n)} = \sum_{n=0}^{3} h(n)\left[z^{-n} + z^{-(8-n)}\right] + h(4)z^{-4} \qquad ..... (3.2.1)$$

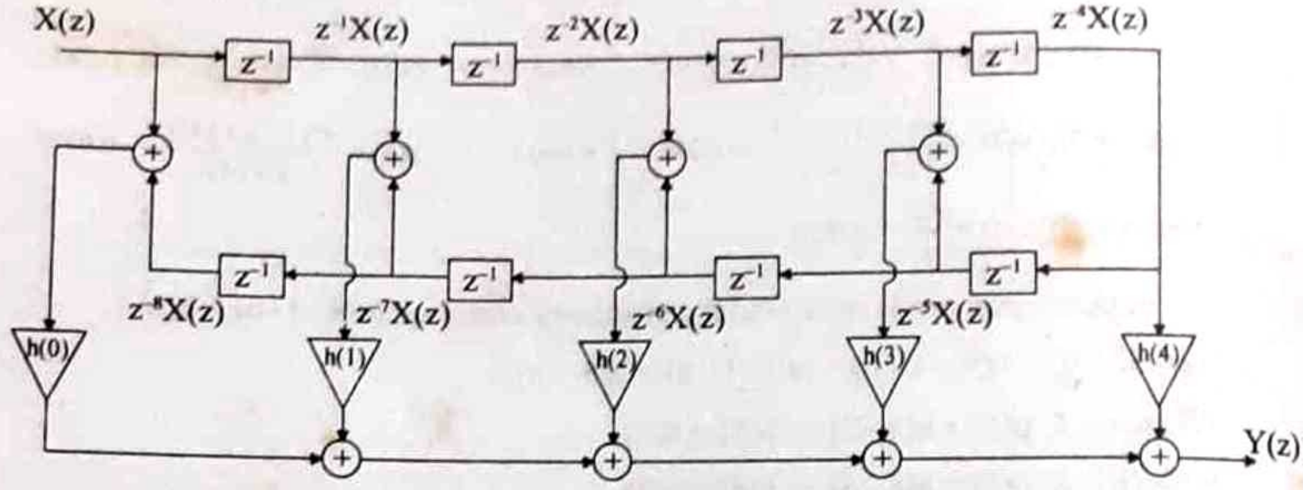
The equation (3.2.1) is the transfer function of FIR lowpass filter.

$$\text{Let } H(z) = \frac{Y(z)}{X(z)} = \sum_{n=0}^{3} h(n)\left[z^{-n} + z^{-(8-n)}\right] + h(4)z^{-4}$$

$$\frac{Y(z)}{X(z)} = h(0)\left[z^{0} + z^{-8}\right] + h(1)\left[z^{-1} + z^{-7}\right] + h(2)\left[z^{-2} + z^{-6}\right] + h(3)\left[z^{-3} + z^{-5}\right] + h(4)z^{-4}$$

$$Y(z) = h(0)[X(z) + z^{-8}X(z)] + h(1)[z^{-1}X(z) + z^{-7}X(z)] + h(2)[z^{-2}X(z) + z^{-6}X(z)]$$
$$+ h(3)[z^{-3}X(z) + z^{-5}X(z)] + h(4)\, z^{-4}X(z) \quad ..... (3.2.2)$$

Using equation (3.2.2) the linear phase realization structure of lowpass FIR filter is obtained as shown below.




## EXAMPLE 3.3

Design a highpass filter using hamming window, with a cut-off frequency of 1.2 radians/sec and N = 9.

### SOLUTION

The desired frequency response for highpass filter is

$$H_d(\omega) = \begin{cases} e^{-j\omega\alpha} & ; \ -\pi \le \omega \le -\omega_c \ \ \& \ \ \omega_c \le \omega \le \pi \\ 0 & ; \ \text{otherwise} \end{cases}$$

The $H_d(n)$ is obtained by inverse fourier transform of $H_d(\omega)$

By definition of inverse fourier transform

$$h_d(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} H_d(\omega)\, e^{j\omega n} d\omega = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-j\omega\alpha}\, e^{j\omega n}\, d\omega = \frac{1}{2\pi}\int_{-\pi}^{-\omega_c} e^{-j\omega\alpha}\, e^{j\omega n}\, d\omega + \frac{1}{2\pi}\int_{\omega_c}^{\pi} e^{-j\omega\alpha}\, e^{j\omega n}\, d\omega$$

$$= \frac{1}{2\pi}\int_{-\pi}^{-\omega_c} e^{j\omega(n-\alpha)} d\omega + \frac{1}{2\pi}\int_{+\omega_c}^{\pi} e^{j\omega(n-\alpha)} d\omega$$

$$= \frac{1}{2\pi}\left[\frac{e^{j\omega(n-\alpha)}}{j(n-\alpha)}\right]_{-\pi}^{-\omega_c} + \frac{1}{2\pi}\left[\frac{e^{j\omega(n-\alpha)}}{j(n-\alpha)}\right]_{\omega_c}^{\pi}$$

$$= \frac{1}{2\pi}\left[\frac{e^{-j\omega_c(n-\alpha)} - e^{-j\pi(n-\alpha)} + e^{j\pi(n-\alpha)} - e^{j\omega_c(n-\alpha)}}{j(n-\alpha)}\right]$$

$$= \frac{1}{\pi(n-\alpha)}\left[\frac{e^{j\pi(n-\alpha)} - e^{-j\pi(n-\alpha)}}{2j} - \left(\frac{e^{j\omega_c(n-\alpha)} - e^{-j\omega_c(n-\alpha)}}{2j}\right)\right]$$

$$= \frac{1}{\pi(n-\alpha)} [\sin(n-\alpha)\pi - \sin \omega_c(n-\alpha)]$$

When $n=\alpha$, the terms $\frac{\sin(n-\alpha)\pi}{(n-\alpha)}$ and $\frac{\sin\omega_c(n-\alpha)}{(n-\alpha)}$ become 0 / 0 which is indeterminate.

Hence, $h_d(n) = \frac{\sin(n-\alpha)\pi - \sin \omega_c(n-\alpha)}{\pi(n-\alpha)}$ ; for $n \neq \alpha$

L' Hospital rule

$$\underset{\theta \to 0}{\text{Lt}} \frac{\sin A\theta}{\theta} = A$$

For $n=\alpha$, $h_d(n)$ can be evaluated using L' Hospital rule.

$$\therefore \text{When } n=\alpha,\ h_d(n) = \underset{n\to\alpha}{\text{Lt}} \frac{\sin(n-\alpha)\pi - \sin\omega_c(n-\alpha)}{\pi(n-\alpha)}$$

$$= \frac{1}{\pi}\left[\underset{n\to\alpha}{\text{Lt}} \frac{\sin\pi(n-\alpha)}{n-\alpha} - \underset{n\to\alpha}{\text{Lt}} \frac{\sin\omega_c(n-\alpha)}{n-\alpha}\right] = \frac{1}{\pi}(\pi-\omega_c) = 1-\frac{\omega_c}{\pi}$$

$$\therefore h_d(n) = 1-\frac{\omega_c}{\pi} \ ; \ \text{for } n=\alpha$$

The window sequence for hamming window is given by

$$w_H(n) = 0.54 - 0.46\cos\left(\frac{2\pi n}{N-1}\right) \ ; \ \text{for } n = 0 \text{ to } (N-1)$$

$$\therefore h(n) = h_d(n)\, w_H(n) = \frac{1}{\pi(n-\alpha)}[\sin\pi(n-\alpha) - \sin(n-\alpha)\omega_c]$$

$$\left[0.54 - 0.46\cos\left(\frac{2\pi n}{N-1}\right)\right] \ ; \ \text{for } n \neq \alpha$$

$$= \left(1-\frac{\omega_c}{\pi}\right)\left(0.54 - 0.46\cos\left(\frac{2\pi n}{N-1}\right)\right) ; \ \text{for } n=\alpha$$

Given that $N = 9$ ; $\omega_c = 1.2$ rad/sec. $\therefore \alpha = \frac{N-1}{2} = \frac{9-1}{2} = 4$

In this example both n and $\alpha$ are integers. Hence $(n-\alpha)$ is also an integer and $(n-\alpha)\pi$ will be an integral multiple of $\pi$.

$\therefore \sin(n-\alpha)\pi = 0$. Also $(N-1) = 8$

$$\therefore h(n) = \frac{-\sin(n-\alpha)\,\omega_c}{\pi(n-\alpha)}\left[0.54 - 0.46\cos\frac{n\pi}{4}\right] ; \ \text{for } n \neq 4$$

$$\text{and } h(n) = \left(1-\frac{\omega_c}{\pi}\right)\left[0.54 - 0.46\cos\frac{n\pi}{4}\right] ; \ \text{for } n = 4$$

$$\text{When } n = 0 \ ; \ h(0) = \frac{-\sin((-4)\times 1.2)}{\pi\times(-4)}[0.54 - 0.46\cos 0] = 0.0063$$

9.

$$\text{When } n = 1 \ ; \ h(1) = \frac{-\sin((-3)\times 1.2)}{\pi\times(-3)}\left[0.54 - 0.46\cos\frac{\pi}{4}\right] = 0.0101$$

$$\text{When } n = 2 \ ; \ h(2) = \frac{-\sin((-2)\times 1.2)}{\pi\times(-2)}\left[0.54 - 0.46\cos\frac{2\pi}{4}\right] = -0.0581$$

$$\text{When } n = 3 \ ; \ h(3) = \frac{-\sin((-1)\times 1.2)}{\pi\times(-1)}\left[0.54 - 0.46\cos\frac{3\pi}{4}\right] = -0.2567$$

$$\text{When } n = 4 \ ; \ h(4) = \left(1-\frac{1.2}{\pi}\right)\left[0.54 - 0.46\cos\frac{4\pi}{4}\right] = 0.6180$$

When n = 5 ; $h(5) = \dfrac{-\sin(1\times 1.2)}{\pi\times 1}\left[0.54 - 0.46\cos\dfrac{5\pi}{4}\right] = -0.2567$

When n = 6 ; $h(6) = \dfrac{-\sin(2\times 1.2)}{\pi\times 2}\left[0.54 - 0.46\cos\dfrac{6\pi}{4}\right] = -0.0581$

When n = 7 ; $h(7) = \dfrac{-\sin(3\times 1.2)}{\pi\times 3}\left[0.54 - 0.46\cos\dfrac{7\pi}{4}\right] = 0.0101$

When n = 8 ; $h(8) = \dfrac{-\sin(4\times 1.2)}{\pi\times 4}\left[0.54 - 0.46\cos\dfrac{8\pi}{4}\right] = 0.0063$

From the above calculations it can be observed that the impulse response is symmetrical with centre of symmetry at n = 4. The magnitude response for linear phase FIR filters, when N is odd and h(n) is symmetrical is given by

$$|H(\omega)| = h\left(\tfrac{N-1}{2}\right) + \sum_{n=1}^{\frac{N-1}{2}} 2h\left(\tfrac{N-1}{2}-n\right)\cos\omega n$$
$$= h(4) + 2h(3)\cos\omega + 2h(2)\cos 2\omega + 2h(1)\cos 3\omega + 2h(0)\cos 4\omega$$
$$= 0.618 + 2\times(-0.2567)\cos\omega + 2\times(-0.0581)\cos 2\omega + 2\times(0.0101)\cos 3\omega$$
$$+ 2\times 0.0063\cos 4\omega$$
$$= 0.618 - 0.5134\cos\omega - 0.1162\cos 2\omega + 0.0202\cos 3\omega + 0.0126\cos 4\omega$$

The transfer function of the filter is given by

$$H(z) = \sum_{n=0}^{N-1} h(n)z^{-n} = \sum_{n=0}^{8} h(n)z^{-n} = \sum_{n=0}^{3} h(n)z^{-n} + h(4)z^{-4} + \sum_{n=5}^{8} h(n)z^{-n}$$
$$= \sum_{n=0}^{3} h(n)z^{-n} + h(4)z^{-4} + \sum_{n=0}^{3} h(8-n)z^{-(8-n)}$$
$$= \sum_{n=0}^{3} h(n)z^{-n} + h(4)z^{-4} + \sum_{n=0}^{3} h(n)z^{-(8-n)}$$
$$= \sum_{n=0}^{3} h(n)\left[z^{-n} + z^{-(8-n)}\right] + h(4)z^{-4} \qquad \text{--- (3.3.1)}$$

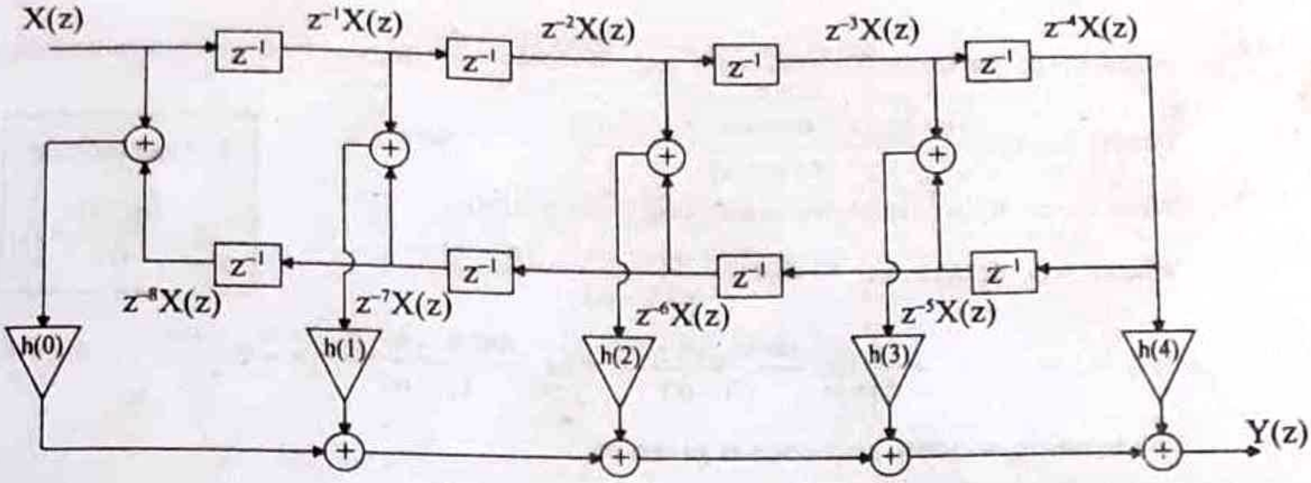
The equation (3.3.1) is the transfer function of FIR highpass filter.

$$\text{Let, } H(z) = \frac{Y(z)}{X(z)} = \sum_{n=0}^{3} h(n)\left[z^{-n} + z^{-(8-n)}\right] + h(4)z^{-4}$$

$$\frac{Y(z)}{X(z)} = h(0)\left[z^{0} + z^{-8}\right] + h(1)\left[z^{-1} + z^{-7}\right] + h(2)\left[z^{-2} + z^{-6}\right] + h(3)\left[z^{-3} + z^{-5}\right] + h(4)z^{-4}$$

$$Y(z) = h(0)\,[X(z) + z^{-8}X(z)] + h(1)\,[z^{-1}X(z) + z^{-7}X(z)] + h(2)\,[z^{-2}X(z) + z^{-6}X(z)]$$
$$+\ h(3)\,[z^{-3}X(z) + z^{-5}X(z)] + h(4)\,z^{-4}X(z) \qquad \text{---- (3.3.2)}$$

Using equation (3.3.2) the linear phase realization of highpass FIR filter is obtained as shown below.




## EXAMPLE 3.4

Design a bandpass filter to pass frequencies in the range 1 to 2 rad/sec using hanning window, with N = 5.

### *SOLUTION*

The desired frequency response for bandpass filter is

$$H_d(\omega) = \begin{cases} e^{-j\omega\alpha} & ; \ -\omega_{c2} \le \omega \le -\omega_{c1} \ \& \ \omega_{c1} \le \omega \le \omega_{c2} \\ 0 & ; \ \text{otherwise} \end{cases}$$

The $H_d(n)$ is obtained by inverse fourier transform of $H_d(\omega)$

$$h_d(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} H_d(\omega)\, e^{j\omega n} d\omega$$

$$= \frac{1}{2\pi}\int_{-\omega_{c2}}^{-\omega_{c1}} e^{-j\omega\alpha}\, e^{j\omega n} d\omega + \frac{1}{2\pi}\int_{\omega_{c1}}^{\omega_{c2}} e^{-j\omega\alpha}\, e^{j\omega n} d\omega$$

$$= \frac{1}{2\pi}\int_{-\omega_{c2}}^{-\omega_{c1}} e^{j\omega(n-\alpha)} d\omega + \frac{1}{2\pi}\int_{\omega_{c1}}^{\omega_{c2}} e^{j\omega(n-\alpha)} d\omega$$

$$= \frac{1}{2\pi}\left[\frac{e^{j\omega(n-\alpha)}}{j(n-\alpha)}\right]_{-\omega_{c2}}^{-\omega_{c1}} + \frac{1}{2\pi}\left[\frac{e^{j\omega(n-\alpha)}}{j(n-\alpha)}\right]_{\omega_{c1}}^{\omega_{c2}}$$

$$= \frac{1}{2\pi}\left[\frac{e^{-j\omega_{c1}(n-\alpha)} - e^{-j\omega_{c2}(n-\alpha)}}{j(n-\alpha)}\right] + \frac{1}{2\pi}\left[\frac{e^{j\omega_{c2}(n-\alpha)} - e^{j\omega_{c1}(n-\alpha)}}{j(n-\alpha)}\right]$$

$$= \frac{1}{\pi(n-\alpha)}\left[\frac{e^{j\omega_{c2}(n-\alpha)} - e^{-j\omega_{c2}(n-\alpha)}}{2j} - \frac{e^{j\omega_{c1}(n-\alpha)} - e^{-j\omega_{c1}(n-\alpha)}}{2j}\right]$$

$$= \frac{1}{\pi(n-\alpha)}\left[\sin\omega_{c2}(n-\alpha) - \sin\omega_{c1}(n-\alpha)\right]$$

When $n = \alpha$, the terms $\frac{\sin \omega_{c2}(n-\alpha)}{n-\alpha}$ and $\frac{\sin \omega_{c1}(n-\alpha)}{n-\alpha}$ become 0 / 0 which is indeterminate.

Hence, $h_d(n) = \frac{\sin \omega_{c2}(n-\alpha) - \sin \omega_{c1}(n-\alpha)}{\pi(n-\alpha)}$ ; for $n \neq \alpha$

When $n = \alpha$, $h_d(n)$ can be evaluated using L'Hospital rule.

L' Hospital rule

$$\underset{\theta \to 0}{Lt} \frac{\sin A\theta}{\theta} = A$$

When $n = \alpha$, $h_d(n) = \underset{n \to \alpha}{Lt} \frac{\sin \omega_{c2}(n-\alpha) - \sin \omega_{c1}(n-\alpha)}{\pi(n-\alpha)}$

$$= \frac{1}{\pi}\left[\underset{n \to \alpha}{Lt} \frac{\sin \omega_{c2}(n-\alpha)}{(n-\alpha)} - \underset{n \to \alpha}{Lt} \frac{\sin \omega_{c1}(n-\alpha)}{(n-\alpha)}\right] = \frac{\omega_{c2} - \omega_{c1}}{\pi} \quad ; \text{ for } n = \alpha$$

The hanning window sequence is given by

$$w(n) = 0.5 - 0.5 \cos \frac{2\pi n}{N-1} \quad ; \text{ for } n = 0 \text{ to } (N-1)$$

$$\therefore h(n) = h_d(n)\, w(n) = \left[\frac{\sin \omega_{c2}(n-\alpha) - \sin \omega_{c1}(n-\alpha)}{\pi(n-\alpha)}\right]\left[0.5 - 0.5 \cos \frac{2\pi n}{N-1}\right]; \text{ for } n \neq \alpha$$

$$= \left(\frac{\omega_{c2} - \omega_{c1}}{\pi}\right)\left(0.5 - 0.5 \cos \frac{2\pi n}{N-1}\right); \text{ for } n = \alpha$$

Given that $N = 5$ ; $\omega_{c1} = 1$ rad/sec and $\omega_{c2} = 2$ rad/sec

Here, $\alpha = \frac{N-1}{2} = \frac{5-1}{2} = 2$ and $N - 1 = 5 - 1 = 4$

When $n = 0$ ; $h(0) = \left[\frac{\sin(2 \times (0-2)) - \sin(1 \times (0-2))}{\pi(0-2)}\right][0.5 - 0.5 \cos 0] = 0$

When $n = 1$ ; $h(1) = \left[\frac{\sin(2 \times (1-2)) - \sin(1 \times (1-2))}{\pi(1-2)}\right]\left[0.5 - 0.5 \cos \frac{2\pi}{4}\right] = 0.0108$

When $n = 2$ ; $h(2) = \frac{2-1}{\pi} = 0.3183$

When $n = 3$ ; $h(3) = \left[\frac{\sin(2 \times (3-2)) - \sin(1 \times (3-2))}{\pi(3-2)}\right]\left[0.5 - 0.5 \cos \frac{6\pi}{4}\right] = 0.0108$

When $n = 4$ ; $h(4) = \left[\frac{\sin(2 \times (4-2)) - \sin(1 \times (4-2))}{\pi(4-2)}\right]\left[0.5 - 0.5 \cos \frac{8\pi}{4}\right] = 0$

From the above calculation it can be observed that the impulse response is symmetrical with centre of symmetry at n = 2. The magnitude response for linear phase FIR filters, when N is odd and h(n) is symmetrical is given by

$$|H(\omega)| = h\left(\tfrac{N-1}{2}\right) + \sum_{n=1}^{\frac{N-1}{2}} 2h\left(\tfrac{N-1}{2} - n\right) \cos \omega n$$

$h\left(\frac{N-1}{2}\right) + \sum_{n=1}^{\frac{N-1}{2}} 2h\left(\frac{N-1}{2} - n\right) \cos$

$$= h(2) + 2h(1) \cos \omega + 2h(0) \cos 2\omega = 0.3183 + 2 \times 0.0108 \cos \omega + 0$$

$$= 0.3183 + 0.0216 \cos \omega$$

The transfer function of the filter is given by

$$H(z)=\sum_{n=0}^{N-1}h(n)z^{-n}=\sum_{n=0}^{4}h(n)z^{-n}=h(0)z^{0}+h(1)z^{-1}+h(2)z^{-2}+h(3)z^{-3}+h(4)z^{-4}$$

$$=0.0108z^{-1}+0.3183z^{-2}+0.0108z^{-3}$$

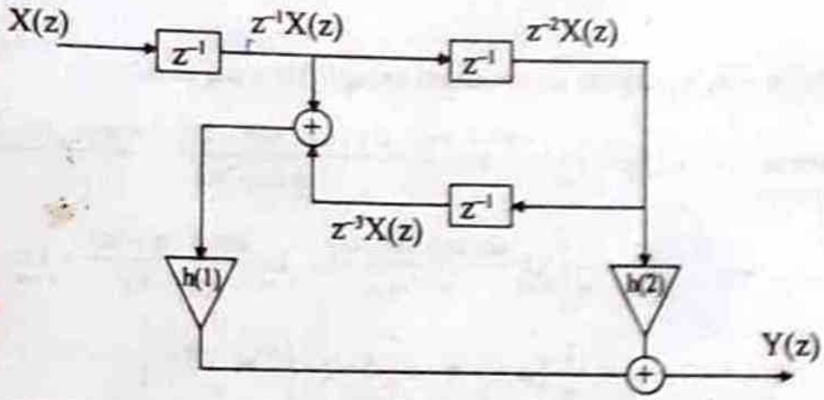
$$=0.0108\left[z^{-1}+z^{-3}\right]+0.3183z^{-2} \qquad \text{..... (3.4.1)}$$

The equation (3.4.1) is the transfer function of FIR filter.

$$\text{Let, } H(z)=\frac{Y(z)}{X(z)}=0.0108\left[z^{-1}+z^{-3}\right]+0.3183z^{-2}$$

$$\therefore Y(z)=\underbrace{0.0108}_{h(1)}\left[z^{-1}X(z)+z^{-3}X(z)\right]+\underbrace{0.3183}_{h(2)}z^{-2}X(z) \qquad \text{..... (3.4.2)}$$

Using equation (3.4.2) the linear phase realization structure of bandpass FIR filter is drawn as shown below.




## EXAMPLE 3.5

Design a bandstop filter to reject frequencies in the range 1 to 2 rad/sec using rectangular window, with N = 7.

### SOLUTION

The desired frequency response for bandstop filter is

$$H_d(\omega)=\begin{cases} e^{-j\omega\alpha} & ;\ -\pi\le\omega\le-\omega_{c2}\ \&\ -\omega_{c1}\le\omega\le\omega_{c1}\ \&\ \omega_{c2}\le\omega\le\pi \\ 0 & ;\ \text{otherwise}\end{cases}$$

The $h_d(n)$ is obtained by inverse fourier transform of $H_d(\omega)$

$$h_d(n)=\frac{1}{2\pi}\int_{-\pi}^{\pi}H_d(\omega)\,e^{j\omega n}d\omega=\frac{1}{2\pi}\int_{-\pi}^{-\omega_{c2}}e^{-j\omega\alpha}\,e^{j\omega n}\,d\omega+\frac{1}{2\pi}\int_{-\omega_{c1}}^{\omega_{c1}}e^{-j\omega\alpha}\,e^{j\omega n}d\omega+\frac{1}{2\pi}\int_{\omega_{c2}}^{\pi}e^{-j\omega\alpha}\,e^{j\omega n}d\omega$$

$$=\frac{1}{2\pi}\int_{-\pi}^{-\omega_{c2}}e^{j\omega(n-\alpha)}\,d\omega+\frac{1}{2\pi}\int_{-\omega_{c1}}^{\omega_{c1}}e^{j\omega(n-\alpha)}\,d\omega+\frac{1}{2\pi}\int_{\omega_{c2}}^{\pi}e^{j\omega(n-\alpha)}\,d\omega$$

$$=\frac{1}{2\pi}\left[\frac{e^{j\omega(n-\alpha)}}{j(n-\alpha)}\right]_{-\pi}^{-\omega_{c2}}+\frac{1}{2\pi}\left[\frac{e^{j\omega(n-\alpha)}}{j(n-\alpha)}\right]_{-\omega_{c1}}^{\omega_{c1}}+\frac{1}{2\pi}\left[\frac{e^{j\omega(n-\alpha)}}{j(n-\alpha)}\right]_{\omega_{c2}}^{\pi}$$

$$=\frac{1}{2\pi}\left[\frac{e^{-j\omega_{c2}(n-\alpha)}-e^{-j\pi(n-\alpha)}}{j(n-\alpha)}+\frac{e^{j\omega_{c1}(n-\alpha)}-e^{-j\omega_{c1}(n-\alpha)}}{j(n-\alpha)}+\frac{e^{j\pi(n-\alpha)}-e^{j\omega_{c2}(n-\alpha)}}{j(n-\alpha)}\right]$$

$$=\frac{1}{\pi(n-\alpha)}\left[\frac{e^{j\omega_{c1}(n-\alpha)}-e^{-j\omega_{c1}(n-\alpha)}}{2j}+\frac{e^{j\pi(n-\alpha)}-e^{-j\pi(n-\alpha)}}{2j}-\frac{e^{j\omega_{c2}(n-\alpha)}-e^{-j\omega_{c2}(n-\alpha)}}{2j}\right]$$

$$=\frac{1}{\pi(n-\alpha)}\left[\sin\omega_{c1}(n-\alpha)+\sin\pi(n-\alpha)-\sin\omega_{c2}(n-\alpha)\right]$$

When $n=\alpha$, the terms $\frac{\sin\omega_{c1}(n-\alpha)}{n-\alpha}$, $\frac{\sin\pi(n-\alpha)}{n-\alpha}$ and $\frac{\sin\omega_{c2}(n-\alpha)}{n-\alpha}$ becomes $0/0$ which is indeterminate.

$$\text{Hence } h_d(n)=\frac{\sin\omega_{c1}(n-\alpha)+\sin\pi(n-\alpha)-\sin\omega_{c2}(n-\alpha)}{\pi(n-\alpha)}\quad;\quad \text{for } n\neq\alpha$$

When $n=\alpha$, $h_d(n)$ can be evaluated using L'Hospital rule.

$$\text{When } n=\alpha,\ h_d(n)=\underset{n\to\alpha}{\mathrm{Lt}}\ \frac{\sin\omega_{c1}(n-\alpha)+\sin\pi(n-\alpha)-\sin\omega_{c2}(n-\alpha)}{\pi(n-\alpha)}$$

$$=\frac{1}{\pi}\left[\underset{n\to\alpha}{\mathrm{Lt}}\frac{\sin\omega_{c1}(n-\alpha)}{(n-\alpha)}+\underset{n\to\alpha}{\mathrm{Lt}}\frac{\sin\pi(n-\alpha)}{(n-\alpha)}-\underset{n\to\alpha}{\mathrm{Lt}}\frac{\sin\omega_{c2}(n-\alpha)}{(n-\alpha)}\right]$$

$$=\frac{1}{\pi}\left[\omega_{c1}+\pi-\omega_{c2}\right]=1-\left(\frac{\omega_{c2}-\omega_{c1}}{\pi}\right)$$

$$\therefore h_d(n)=1-\left(\frac{\omega_{c2}-\omega_{c1}}{\pi}\right)\quad;\quad \text{for } n=\alpha$$

Consider the rectangular window sequence,

$$w_R(n)=\begin{cases}1 & ;\ \text{for } n=0 \text{ to } N-1\\ 0 & ;\ \text{otherwise}\end{cases}$$

$$\therefore h(n)=h_d(n)\,w_R(n)=h_d(n)$$

Given that $N=7$, $\quad\therefore \alpha=\frac{N-1}{2}=\frac{7-1}{2}=3$

In this example both n and $\alpha$ are integers. Hence $(n-\alpha)$ is also an integer and $(n-\alpha)\pi$ will be an integral multiple of $\pi$. $\therefore \sin(n-\alpha)\pi=0$.

$$\text{Hence, } h(n)=\frac{\left[\sin\omega_{c1}(n-\alpha)-\sin\omega_{c2}(n-\alpha)\right]}{\pi(n-\alpha)}\quad;\ \text{for } n\neq\alpha$$

$$=1-\frac{(\omega_{c2}-\omega_{c1})}{\pi}\quad;\quad \text{for } n=\alpha$$

Given that, $\omega_{c1} = 1$ rad / sec and $\omega_{c2} = 2$ rad / sec. Here $\alpha = 3$

When n = 0 ; $h(0) = \left[\dfrac{\sin\ (1 \times (-3)) - \sin\ (2 \times (-3))}{\pi \times (-3)}\right] = 0.0446$

When n = 1 ; $h(1) = \left[\dfrac{\sin\ (1 \times (-2)) - \sin\ (2 \times (-2))}{\pi \times (-2)}\right] = 0.2652$

When n = 2 ; $h(2) = \left[\dfrac{\sin\ (1 \times (-1)) - \sin(2 \times (-1))}{\pi \times (-1)}\right] = -0.0216$

When n = 3 ; $h(3) = 1 - \dfrac{(2-1)}{\pi} = 0.6817$

When n = 4 ; $h(4) = \left[\dfrac{\sin\ (1 \times 1) - \sin\ (2 \times 1)}{\pi \times 1}\right] = -0.0216$

When n = 5 ; $h(5) = \left[\dfrac{\sin\ (1 \times 2) - \sin\ (2 \times 2)}{\pi \times 2}\right] = 0.2652$

When n = 6 ; $h(6) = \left[\dfrac{\sin\ (1 \times 3) - \sin\ (2 \times 3)}{\pi \times 3}\right] = 0.0446$

From the above calculations it can be observed that the impulse response is symmetrical with centre of symmetry at n = 3. The magnitude response for linear phase FIR filters, when N is odd and h(n) is symmetrical is given by

$$|H(\omega)| = h\left(\tfrac{N-1}{2}\right) + \sum_{n=1}^{\frac{N-1}{2}} 2h\left(\tfrac{N-1}{2} - n\right) \cos \omega n$$

$$= h(3) + 2h(2) \cos \omega + 2h(1) \cos 2\omega + 2h(0) \cos 3\omega$$

$$= 0.6817 + 2 \times (-0.0216) \cos \omega + 2 \times 0.2652 \cos 2\omega + 2 \times 0.0446 \cos 3\omega$$

$$= 0.6817 - 0.0432 \cos \omega + 0.5304 \cos 2\omega + 0.0892 \cos 3\omega$$

The transfer function of FIR filter is given by

$$H(z) = \sum_{n=0}^{N-1} h(n) z^{-n} = \sum_{n=0}^{6} h(n) z^{-n}$$

$$= h(0)z^0 + h(1)z^{-1} + h(2)z^{-2} + h(3)z^{-3} + h(4)z^{-4} + h(5)z^{-5} + h(6)z^{-6}$$

$$= h(0) + h(1)z^{-1} + h(2)z^{-2} + h(3)z^{-3} + h(2)z^{-4} + h(1)z^{-5} + h(0)z^{-6}$$

$$\boxed{\text{Here } h(0) = h(6),\ h(1) = h(5),\ h(2) = h(4)}$$

$$= h(0)\,[1 + z^{-6}] + h(1)\,[z^{-1} + z^{-5}] + h(2)\,[z^{-2} + z^{-4}] + h(3)\,z^{-3}$$

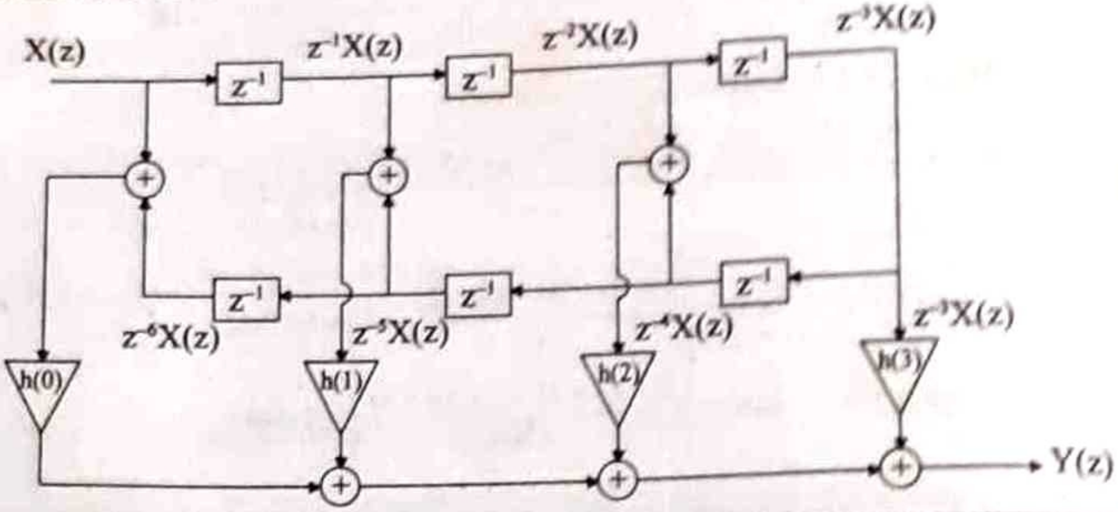
$$= 0.0446\,[1+z^{-6}] + 0.2652\,[z^{-1} + z^{-5}] - 0.0216[z^{-2} + z^{-4}] + 0.6817z^{-2} \quad \text{..... (3.5.1)}$$

The equation (3.5.1) is the transfer function of FIR filter.

$$\text{Let } H(z) = \frac{Y(z)}{X(z)} = h(0)[1 + z^{-6}] + h(1)[z^{-1} + z^{-5}] + h(2)[z^{-2} + z^{-4}] + h(3)z^{-3}$$

$$\therefore Y(z) = h(0)\left[X(z) + z^{-6}X(z)\right] + h(1)\left[z^{-1}X(z) + z^{-5}X(z)\right] + h(2)\left[z^{-2}X(z) + z^{-4}X(z)\right] + h(3)\left[z^{-3}X(z)\right] \quad \text{..... (3.5.2)}$$

Using equation (3.5.2) the linear phase realization structure of bandstop FIR filter is drawn as shown below.




## 3.9 DESIGN OF FIR FILTERS BY FREQUENCY SAMPLING TECHNIQUE

In this method the ideal (desired) frequency response is sampled at sufficient number of points (i.e., N-points). These samples are the DFT coefficients of the impulse response of the filter. Hence the impulse response of the filter is determined by taking inverse DFT.

Let $H_d(\omega)$ = Ideal desired frequency response

$\tilde{H}(k)$ = The DFT sequence obtained by sampling $H_d(\omega)$

$h(n)$ = Impulse response of FIR filter.

The impulse response h(n) is obtained by taking inverse DFT of $\tilde{H}(k)$. For practical realizability the samples of impulse response should be real. This can happen if all the complex terms appear in complex conjugate pairs. This suggest that the terms can be marked by comparing the exponentials. The term $\tilde{H}(k)\ e^{j2\pi nk/N}$ should be matched by the term that has the exponential $e^{-j2\pi nk/N}$ as a factor.

### Procedure for type-1 design

1. Choose the ideal (desired) frequency response $H_d(\omega)$.
2. Sample $H_d(\omega)$ at N-points by taking $\omega = \omega_k = 2\pi k/N$ where k = 0, 1,2 3, ....(N–1), to generate the sequence $\tilde{H}(k)$.

$$\therefore \tilde{H}(k) = H_d(\omega)\Big|_{\omega = \frac{2\pi k}{N}} \quad ; \quad \text{for } k = 0, 1, \ldots\ldots\ (N-1)$$

3. Compute the N samples of h(n) using the following equations.

When N is odd, $h(n) = \frac{1}{N}\left\{\tilde{H}(0) + 2\sum_{k=1}^{\frac{N-1}{2}} \text{Re}\left[\tilde{H}(k)\, e^{j2\pi nk/N}\right]\right\}$

When N is even, $h(n) = \frac{1}{N}\left\{\tilde{H}(0) + 2\sum_{k=1}^{\frac{N}{2}-1} \text{Re}\left[\tilde{H}(k)\, e^{j2\pi nk/N}\right]\right\}$

where "Re" stands for "real part of".

4. Take Z-transform of the impulse response h(n) to get the filter transfer function, H(z)

$$\therefore H(z) = \sum_{n=0}^{N-1} h(n)\, z^{-n}$$

## Procedure for type-2 design

1. Choose the ideal (desired) frequency response $H_d(\omega)$.
2. Sample $H_d(\omega)$ at N-points by taking $\omega = \omega_k = 2\pi(2k+1)/2N$, where k = 0, 1,2 3, ....(N−1), to generate the sequence $\tilde{H}(k)$ .

$$\therefore \tilde{H}(k) = H_d(\omega)\Big|_{\omega=\frac{2\pi(2k+1)}{N}} \;;\; \text{for } k = 0, 1, \ldots\ldots (N-1)$$

3. Compute the N samples of h(n) using the following equations.

When N is odd, $h(n) = \frac{2}{N}\sum_{k=0}^{\frac{N-3}{2}} \text{Re}\left[\tilde{H}(k)\, e^{jn\pi(2k+1)/N}\right]$

When N is even, $h(n) = \frac{2}{N}\, 2\sum_{k=0}^{\frac{N}{2}-1} \text{Re}\left[\tilde{H}(k)\, e^{jn\pi(2k+1)/N}\right]$

where "Re" stands for "real part of".

4. Take Z-transform of the impulse response h(n) to get the filter transfer function, H(z)

$$\therefore H(z) = \sum_{n=0}^{N-1} h(n)\, z^{-n}$$

## Important formulae

The following formulae can be used for calculation of h(n) while designing FIR filter by frequency sampling method.

1. $$\sum_{k=0}^{M-1} \cos k\theta = \frac{\sin\frac{M\theta}{2}\cos\frac{(M-1)}{2}\theta}{\sin\frac{\theta}{2}}$$

2. $$\sum_{k=0}^{M-1} \sin k\theta = \frac{\sin\frac{M\theta}{2}\sin\frac{(M-1)}{2}\theta}{\sin\frac{\theta}{2}}$$

## EXAMPLE 3.6

Determine the coefficients of a linear-phase FIR filter of length N=15 which has a symmetric unit sample response and a frequency response that satisfies the conditions

$$H\left(\frac{2\pi k}{15}\right)=\begin{cases}1 & ; \quad \text{for } k=0,\ 1,\ 2,\ 3\\ 0.4 & ; \quad \text{for } k=4\\ 0 & ; \quad \text{for } k=5,\ 6,\ 7\end{cases}$$

### SOLUTION

For linear phase FIR filter the phase function, $\theta(\omega) = -\alpha\omega$, where $\alpha = (N-1)/2$.

Here $N = 15$, $\therefore \alpha = (15-1)/2 = 7$.

Also, here $\omega = \omega_k = \frac{2\pi k}{N} = \frac{2\pi k}{15}$. Hence we can go for type-1 design.

In this problem the samples of the magnitude response of the ideal (desired) filter are directly given for various values of k.

$$\therefore \tilde{H}(k) = H_d(\omega)\Big|_{\omega=\omega_k} = \begin{cases}1\, e^{-j\alpha\omega_k} & ; \ k=0,\ 1,\ 2,\ 3\\ 0.4\, e^{-j\alpha\omega_k} & ; \ k=4\\ 0 & ; \ k=5,\ 6,\ 7\end{cases}$$

where, $\omega_k = \frac{2\pi k}{15}$

When $k = 0$, $\tilde{H}(0) = e^{-j\alpha\omega_k} = e^{-j7\times\frac{2\pi\times 0}{15}} = 1$

When $k = 1$, $\tilde{H}(1) = e^{-j\alpha\omega_k} = e^{-j7\times\frac{2\pi\times 1}{15}} = e^{-j\frac{14\pi}{15}}$

When $k = 2$, $\tilde{H}(2) = e^{-j\alpha\omega_k} = e^{-j7\times\frac{2\pi\times 2}{15}} = e^{-j\frac{28\pi}{15}}$

When $k = 3$, $\tilde{H}(3) = e^{-j\alpha\omega_k} = e^{-j7\times\frac{2\pi\times 3}{15}} = e^{-j\frac{42\pi}{15}}$

When $k = 4$, $\tilde{H}(4) = 0.4\, e^{-j\alpha\omega_k} = 0.4\, e^{-j7\times\frac{2\pi\times 4}{15}} = 0.4\, e^{-j\frac{56\pi}{15}}$

When $k = 5$, $\tilde{H}(5) = 0$

When k = 6, $\tilde{H}(6) = 0$

When k = 7, $\tilde{H}(7) = 0$

The samples of impulse response are given by

$$h(n) = \frac{1}{N}\left\{\tilde{H}(0) + 2\sum_{k=1}^{\frac{N-1}{2}} \mathrm{Re}\left[\tilde{H}(k)\, e^{j2\pi nk/N}\right]\right\}$$

$$= \frac{1}{15}\left\{\tilde{H}(0) + 2\sum_{k=1}^{7} \mathrm{Re}\left[\tilde{H}(k)\, e^{j2\pi nk/15}\right]\right\}$$

$$= \frac{1}{15}\left\{\tilde{H}(0) + 2\sum_{k=1}^{3} \mathrm{Re}\left[\tilde{H}(k)\, e^{j2\pi nk/15}\right] + 2\ \mathrm{Re}\left[\tilde{H}(4)\, e^{j2\pi n4/15}\right]\right\}$$

$$= \frac{1}{15}\left\{1 + 2\sum_{k=1}^{3} \mathrm{Re}\left[e^{-j7\times\frac{2\pi k}{15}} \times e^{\frac{j2\pi nk}{15}}\right] + 2\ \mathrm{Re}\left[0.4\, e^{-j7\frac{2\pi\times 4}{15}} \times e^{+j\frac{2\pi n4}{15}}\right]\right\}$$

$$= \frac{1}{15}\left\{1 + 2\sum_{k=1}^{3} \mathrm{Re}\left[e^{\frac{j2\pi k}{15}(n-7)}\right] + 2\ \mathrm{Re}\left[0.4\, e^{\frac{j8\pi}{15}(n-7)}\right]\right\}$$

$$= \frac{1}{15}\left\{1 + 2\sum_{k=1}^{3} \cos\frac{2\pi k}{15}(n-7) + 0.8\cos\frac{8\pi}{15}(n-7)\right\}$$

$$= \frac{1}{15} + \frac{2}{15}\cos\frac{2\pi}{15}(n-7) + \frac{2}{15}\cos\frac{4\pi}{15}(n-7) + \frac{2}{15}\cos\frac{6\pi}{15}(n-7) + 0.8\cos\frac{8\pi}{15}(n-7)$$

$$n=0\ ;\ h(0) = \frac{1}{15} + \frac{2}{15}\cos\left(\frac{-14\pi}{15}\right) + \frac{2}{15}\cos\left(\frac{-28\pi}{15}\right) + \frac{2}{15}\cos\left(\frac{-42\pi}{15}\right) + 0.8\cos\left(\frac{-56\pi}{15}\right) = 0.4855$$

$$n=1\ ;\ h(1) = \frac{1}{15} + \frac{2}{15}\cos\left(\frac{-12\pi}{15}\right) + \frac{2}{15}\cos\left(\frac{-24\pi}{15}\right) + \frac{2}{15}\cos\left(\frac{-36\pi}{15}\right) + 0.8\cos\left(\frac{-48\pi}{15}\right) = -0.606$$

$$n=2\ ;\ h(2) = \frac{1}{15} + \frac{2}{15}\cos\left(\frac{-10\pi}{15}\right) + \frac{2}{15}\cos\left(\frac{-20\pi}{15}\right) + \frac{2}{15}\cos\left(\frac{-30\pi}{15}\right) + 0.8\cos\left(\frac{-40\pi}{15}\right) = -0.3333$$

$$n=3\ ;\ h(3) = \frac{1}{15} + \frac{2}{15}\cos\left(\frac{-8\pi}{15}\right) + \frac{2}{15}\cos\left(\frac{-16\pi}{15}\right) + \frac{2}{15}\cos\left(\frac{-24\pi}{15}\right) + 0.8\cos\left(\frac{-32\pi}{15}\right) = 0.6943$$

$$n=4\ ;\ h(4) = \frac{1}{15} + \frac{2}{15}\cos\left(\frac{-6\pi}{15}\right) + \frac{2}{15}\cos\left(\frac{-12\pi}{15}\right) + \frac{2}{15}\cos\left(\frac{-18\pi}{15}\right) + 0.8\cos\left(\frac{-24\pi}{15}\right) = 0.1393$$

$$n=5\ ;\ h(5) = \frac{1}{15} + \frac{2}{15}\cos\left(\frac{-4\pi}{15}\right) + \frac{2}{15}\cos\left(\frac{-8\pi}{15}\right) + \frac{2}{15}\cos\left(\frac{-12\pi}{15}\right) + 0.8\cos\left(\frac{-16\pi}{15}\right) = -0.7484$$

$$n=6\ ;\ h(6) = \frac{1}{15} + \frac{2}{15}\cos\left(\frac{-2\pi}{15}\right) + \frac{2}{15}\cos\left(\frac{-4\pi}{15}\right) + \frac{2}{15}\cos\left(\frac{-6\pi}{15}\right) + 0.8\cos\left(\frac{-8\pi}{15}\right) = 0.2353$$

$$n=7\ ;\ h(7) = \frac{1}{15} + \frac{2}{15}\cos 0 + \frac{2}{15}\cos 0 + \frac{2}{15}\cos 0 + 0.8\cos 0 = \frac{7}{15} + 0.8 = 1.2667$$

For linear phase FIR filters the condition h(N−1−n) = h(n) will be satisfied when $\alpha = N-1/2$.

$\therefore$ When n = 8, $\quad h(15-1-8) = h(6) \Rightarrow h(8) = h(6) = 0.2353$

When n = 9, $\quad h(15-1-9) = h(5) \Rightarrow h(9) = h(5) = -0.7484$

When n = 10, $\quad h(15-1-10) = h(4) \Rightarrow h(10) = h(4) = 0.1393$

When n = 11, $\quad h(15-1-11) = h(3) \Rightarrow h(11) = h(3) = 0.6943$

When n = 12, $\quad h(15-1-12) = h(2) \Rightarrow h(12) = h(2) = -0.3333$

When n = 13, $\quad h(15-1-13) = h(1) \Rightarrow h(13) = h(1) = -0.606$

When n = 14, $\quad h(15-1-14) = h(0) \Rightarrow h(14) = h(0) = 0.4855$

The transfer function of the filter is given by Z-transform of h(n)

$$\left.\begin{array}{l}\therefore \text{The transfer function of} \\ \text{the FIR filter,}\end{array}\right\} H(z) = \sum_{n=0}^{N-1} h(n)\, z^{-n}$$

$$= \sum_{n=0}^{14} h(n)\, z^{-n} = \sum_{n=0}^{6} h(n)\, z^{-n} + h(7)\, z^{-7} + \sum_{n=8}^{14} h(n)\, z^{-n}$$

$$= \sum_{n=0}^{6} h(n)\, z^{-n} + h(7)\, z^{-7} + \sum_{n=0}^{6} h(14-n)\, z^{-(14-n)}$$

$$= \sum_{n=0}^{6} h(n)\left(z^{-n} + z^{-(14-n)}\right) + h(7)\, z^{-7} \quad \boxed{\because h(N-1-k) = h(k)}$$

Using the above transfer function the linear phase realization structure of FIR filter can be obtained.

## EXAMPLE 3.7

Design a linear phase lowpass FIR filter with a cut-off frequency of $\pi/2$ rad/sec. Take N = 17.

### SOLUTION

The frequency response of ideal (desired) lowpass filter with cutoff frequency $\omega_c = \pi/4$ rad/sec can be written as shown below. Also the sketch of ideal frequency response is shown in fig.3.7.1.

$$H_d(\omega) = \begin{cases} e^{-j\alpha\omega} & ; \ 0 \le \omega \le \dfrac{\pi}{2} \\ 0 & ; \ \dfrac{\pi}{2} \le \omega \le 2\pi \end{cases}$$

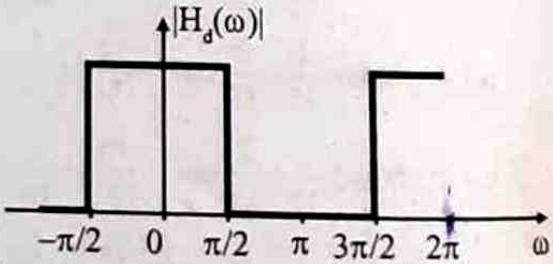



*Fig 3.7.1: Ideal magnitude response of low pass filter.*

where, $\alpha = \dfrac{N-1}{2} = \dfrac{17-1}{2} = 8$ (for linear phase filter)

Let us choose type-1 design

$$\therefore \omega_k = \frac{2\pi k}{N} = \frac{2\pi k}{17}$$

The sequence $\tilde{H}(k)$ is obtained by sampling $H_d(\omega)$ at 17 equidistant points in a period of $2\pi$.

## 3.2 FIR FILTERS (FINITE IMPULSE RESPONSE FILTERS)

The filters designed by using finite number of samples of impulse response are called FIR filters. These finite number of samples are obtained from the infinite duration desired impulse response $h_d(n)$. Here $h_d(n)$ is the inverse fourier transform of $H_d(\omega)$, where $H_d(\omega)$ is the ideal (desired) frequency response. The various methods of designing FIR filters differs only in the method of determining the samples of h(n) from the samples of $h_d(n)$.

**The various steps in designing FIR filter are**

i) Choose an ideal (desired) frequency response, $H_d(\omega)$.

ii) Take inverse fourier transform of $H_d(\omega)$ to get $h_d(n)$ or sample $H_d(\omega)$ at finite number of points (N-point) to get $\tilde{H}(k)$.

iii) If $h_d(n)$ is determined, then convert the infinite duration $h_d(n)$ to a finite duration h(n). (usually h(n) is an N-point sequence) or If $\tilde{H}(k)$ is determined then take N-point inverse DFT to get h(n).

iv) Take Z-transform of h(n) to get H(z), where H(z) is the transfer function of the digital filter.

v) Choose a suitable structure and realize the filter.

vi) Verify the design by simulation.

**Advantages of FIR filter**

1. FIR filters with exactly linear phase can be easily designed.
2. Efficient realizations of FIR filter exist as both recursive and nonrecursive structures.
3. FIR filters realized nonrecursively i.e., by direct convolution are always stable.
4. Roundoff noise, which is inherent in realizations with finite precision arithmetic can easily be made small for nonrecursive realization of FIR filters.

**Disadvantages of FIR filters**

1. The duration of the impulse response should be large (i.e., N should be large) to adequately approximate sharp cutoff filter. Hence a large amount of processing is required to realize such filters when realized via slow convolution.
2. The delay of linear phase FIR filters need not always be an integer number of samples. This non-integral delay can lead to problems in some signal processing applications.

## 3.3 CHARACTERISTICS OF FIR FILTERS WITH LINEAR PHASE

Let h(n) be a causal finite duration sequence defined over the interval $0 \le n \le N-1$ and the samples of h(n) be real.

The Fourier transform of h(n) is

$$H(\omega) = \sum_{n=0}^{N-1} h(n)\, e^{-j\omega n} \qquad .....(3.15)$$

which is periodic in frequency with period $2\pi$.

$$\therefore H(\omega) = H(\omega + 2\pi m)\ ;\ \text{for } m = 0, \pm 1, \pm 2, ......... \qquad .....(3.16)$$

With the restriction that h(n) is real, additional constraints of $H(\omega)$ are obtained as shown below.

Since $H(\omega)$ is complex it can be expressed as magnitude and phase function as shown in equation (3.17).

$$H(\omega) = \pm|H(\omega)|\, e^{j\theta(\omega)} \qquad .....(3.17)$$

The operators ± in the equation (3.17) is necessary since $H(\omega)$ is actually of the form

$$H(\omega) = H_r(\omega)\, e^{j\theta(\omega)} \qquad .....(3.18)$$

where $H_r(\omega)$ is a real function taking on both positive and negative values.

From the property of Fourier transform when h(n) is real we can say that the magnitude function is a symmetric function and the phase function is an antisymmetric function

$$\therefore |H(\omega)| = |H(-\omega)| \qquad .....(3.19)$$

$$|\theta(\omega)| = -|\theta(-\omega)| \qquad .....(3.20)$$

For many practical FIR filter, exact linearity of phase is a desired goal. Let us assume that the phase of $H(\omega)$ is a linear function of $\omega$. Hence $\theta(\omega)$ is directly proportional to $\omega$.

$$\therefore \theta(\omega) \propto \omega \quad \text{or } \theta(\omega) = -\alpha\omega \quad ; \text{ for } \quad -\pi \le \omega \le \pi \qquad .....(3.21)$$

where $\alpha$ is a constant phase delay in samples

From equation (3.15) we get

$$H(\omega) = \sum_{n=0}^{N-1} h(n)\, e^{-j\omega n} \qquad .....(3.22)$$

From equation (3.17) and (3.21) we get

$$H(\omega) = \pm|H(\omega)|\, e^{-j\alpha\omega} \qquad .....(3.23)$$

On equating equations (3.22) and (3.23) we get,

$$\sum_{n=0}^{N-1} h(n)\, e^{-j\omega n} = \pm|H(\omega)|\, e^{-j\alpha\omega}$$

$$\sum_{n=0}^{N-n} h(n)\left[\cos\omega n - j\sin\omega n\right] = \pm|H(\omega)|\left[\cos\alpha\omega - j\sin\alpha\omega\right] \qquad .....(3.24)$$

On equating the real part and imaginary part of equation (3.24) we get

$$\pm|H(\omega)| \cos\alpha\omega = \sum_{n=0}^{N-1} h(n) \cos\omega n \qquad .....(3.25)$$

$$H(\omega) = \pm \left|H(\omega)\right| e^{j(\beta - \alpha\omega)} \quad \text{.....(3.30)}$$

When $H(\omega)$ is expressed in the form of equation (3.30), we can prove that the only possible solution of h(n) exists if,

$$\alpha = \frac{N-1}{2} \;\; ; \quad \beta = \pm\frac{\pi}{2} \;\; \text{and} \;\; h(n) = -h(N-1-n); \text{ for } 0 \le n \le (N-1) \quad \text{.....(3.31)}$$

The filters that satisfy the three conditions of equation (3.31) have a delay of [(N – 1)/2] samples but their impulse responses are antisymmetric around the center of the sequence, as opposed to the true linear phase sequences that are symmetric around the center of the sequence.

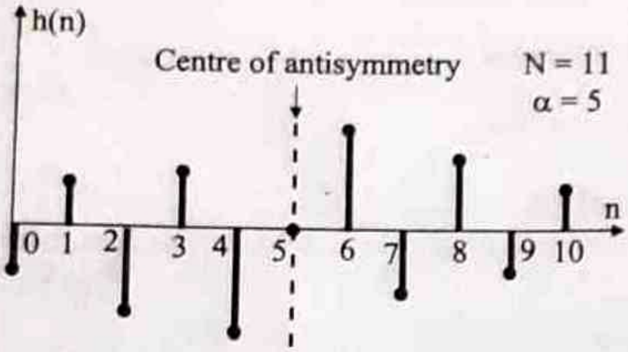



*Fig 3.8 : Example of antisymmetry impulse response for odd N*

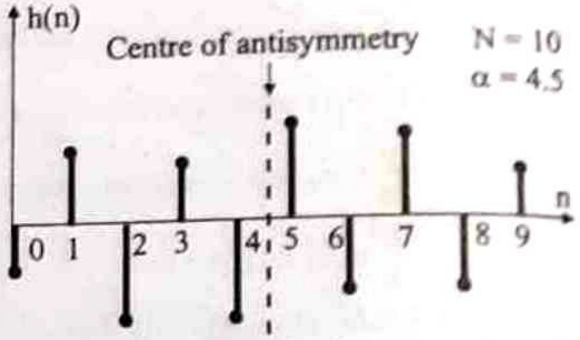



*Fig 3.9 : Example of antisymmetry impulse response for even N*

## 3.4 FREQUENCY RESPONSE OF LINEAR PHASE FIR FILTERS

Depending on the value of N (odd or even) and the type of symmetry of the filter impulse response sequence (symmetric or antisymmetric) there are four possible types of linear phase FIR filters.

The following are the four cases of impulse response for linear phase FIR filters.

***Case I :*** Symmetrical Impulse response when N is odd.

***Case II :*** Symmetrical Impulse response when N is even.

***Case III :*** Antisymmetric Impulse response when N is odd.

***Case IV :*** Antisymmetric Impulse response when N is even.

The frequency response of the filter is the fourier transform of the impulse response. If h(n) is impulse response of FIR filter then fourier transform of h(n) is denoted as $H(\omega)$, which is the frequency response of FIR filter. The $H(\omega)$ is a complex function of $\omega$ and so it can be expressed as, magnitude function $|H(\omega)|$ and phase function $\angle H(\omega)$.

**Table 3.2 : Summary of |H(ω)| for linear phae FIR filters**

| Nature of impulse response | Magnitude function |
|---|---|
| ***Case I :*** Impulse response is symmetric and N is odd | $\lvert H(\omega)\rvert = h\left(\frac{N-1}{2}\right) + \sum_{n=1}^{\frac{N-1}{2}} 2h\left(\frac{N-1}{2}-n\right)\cos\omega n$ |
| ***Case II :*** Impulse response is symmetric and N is even | $\lvert H(\omega)\rvert = \sum_{n=1}^{\frac{N}{2}} 2h\left(\frac{N}{2}-n\right)\cos\left(\omega\left(n-\frac{1}{2}\right)\right)$ |
| ***Case III :*** Impulse response is antisymmetric and N is odd | $\lvert H(\omega)\rvert = \sum_{n=1}^{\frac{N-1}{2}} 2h\left(\frac{N-1}{2}-n\right)\sin\omega n$ |
| ***Case IV :*** Impulse response is antisymmetric and N is even | $\lvert H(\omega)\rvert = \sum_{n=1}^{\frac{N}{2}} 2h\left(\frac{N}{2}-n\right)\sin\left(\omega\left(n-\frac{1}{2}\right)\right)$ |

## 3.5 DESIGN TECHNIQUES FOR LINEAR PHASE FIR FILTERS

There are three well known method of design techniques for linear phase FIR filters

1. Fourier series method and Window method.
2. Frequency sampling method.
3. Optimal filter design methods.

**Design of linear phase FIR filters by fourier series method**

The following two concepts leads to the design of FIR filters by fourier series method.

1. The frequency response of a digital filter is periodic with period equal to sampling frequency.
2. Any periodic function can be expressed as a linear combination of complex exponentials.

In this method the desired frequency response, $H_d(\omega)$ can be converted to a fourier series representation by replacing ω by 2πfT, where T is the sampling time. (1/T = F = Sampling frequency). Then using this expressions the fourier coefficients are evaluated which is the desired impulse response of the filter, $h_d(n)$. On taking Z-transform of $h_d(n)$ we get $H_d(z)$ which is the transfer function of digital filter.

The $H_d(z)$ obtained from $h_d(n)$ will be a transfer function of unrealizable noncausal digital filter of infinite duration. A finite duration impulse response h(n) can be obtained by truncating the infinite duration impulse response $h_d(n)$ to N-samples. Now take Z-transform of h(n) to get H(z) and then multiply H(z) by $z^{-(N-1)/2}$ to get the transfer function of realizable causal digital filter of finite duration.

The abrupt truncation of the fourier series results in oscillations in the passband and stopband. These oscillations are due to slow convergence of the fourier series, particularly near points of discontinuity. This effect is known as the **Gibbs phenomenon.** It can be shown that the undesirable oscillations can be reduced by multiplying the desired impulse response coefficients by an appropriate window function. This leads to the method of FIR filter design using windows.

## Design of linear phase FIR filters using Windows

In this method we begin with the desired frequency response specification $H_d(\omega)$ and determine the corresponding unit sample response $h_d(n)$. The $h_d(n)$ is given by inverse fourier transform of $H_d(\omega)$. The unit sample response $h_d(n)$ will be an infinite sequence and must be truncated at some point say at $n = N-1$ to yield an FIR filter of length N. The truncation is obtained by multiplying $h_d(n)$ by a window sequence $w(n)$. [$w(n)$ is also called window function]. The resultant sequence will be of length N and can be denoted by $h(n)$.

The fourier transform of $h(n)$ is the frequency response of the filter to be implemented in software or in hardware. The frequency response of the filter is denoted by $H(\omega)$. The Z-transform of $h(n)$ will give the filter transfer function $H(z)$. The frequency response of the filter $H(\omega)$ depends on the frequency response of the window function.

The desirable characteristics of the frequency response of window function are the following.

1. The width of the mainlobe should be small and it should contain as much of the total energy as possible.
2. The sidelobes should decrease in energy rapidly as $\omega$ tends to $\pi$.

There have been many windows proposed, that approximates the desired characteristics. In the following sections, the Rectangular window, Hanning window, Hamming window, Blackman window and Kaiser window are discussed.

## Design of linear phase FIR filters by Frequency sampling method

In frequency sampling method of filter design, we begin with the desired frequency response specification $H_d(\omega)$ and it is sampled at N-points to generate a sequence $\tilde{H}(k)$. The N-point inverse DFT of the sequence $\tilde{H}(k)$ gives the impulse response of the filter $h(n)$. The fourier transform of $h(n)$ gives the frequency response, $H(\omega)$ and Z-transform of $h(n)$ gives the transfer function $H(z)$ of the filter.

## Design of optimum equiripple linear-phase FIR filter

The FIR filter design by window and frequency sampling method does not have precise control over the critical frequencies such as $\omega_p$ (passband edge frequency) and $\omega_s$ (stopband edge frequency). This drawback can be overcome by using chebyshev approximation technique. In this method, the weighed approximation error between the desired frequency response and the actual frequency response is spread evenly across the passband and evenly across the stopband of the filter. This results in the reduction of maximum error. The resulting filter have ripples in both the passband and the stopband. This concept of design is called optimum equiripple design criterion.