Time Response of First Order & Second Order Systems:

* Time response analysis is to see the variation of output with respect to time.

* The system stability, accuracy and evaluation is always based on time response analysis.

* In any physical system, output takes some finite time to reach its final value and varies from system to system depending on different factors.

* Time response is obtained in two parts
  1. Transient response
  2. Steady state response.

* Transient response is the response of the system when I/p changes from one state to another. $[C_t(t)]$

* Steady state response is the part of time response which remains after complete transient response $[C_{ss}(t)]$.

* Steady state Error (e_{ss}) is the difference b/w the desired and the actual o/p of the system as $t \to \infty$.

$$e_{ss} = \lim_{t \to \infty} r(t) - c(t) = \lim_{t \to \infty} e(t) = \lim_{s \to 0} s\,E(s)$$

* Response of the system or closed loop TF of system is

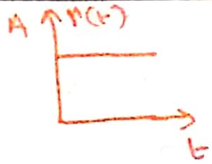
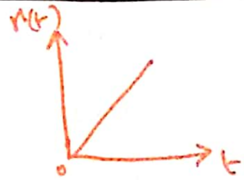
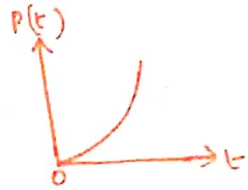
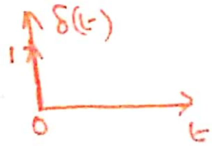
$$M(s) = \frac{C(s)}{R(s)} = \frac{G(s)}{1 \pm G(s)H(s)}$$

$\hookrightarrow$ +ve or -ve feedback.

$$C(s) = M(s)\, R(s).$$

$$c(t) = L^{-1}[C(s)] = L^{-1}[R(s) \cdot M(s)].$$

Test signals:

NAME OF SIGNAL	TIME DOMAIN EQN $r(t)$	L.T of THE SIGNAL $R(s)$	WAVEFORM	LIMIT.
Step / Unit step } $u(t)$	A 1	A/s $1/s$		$t \geq 0$
Ramp / Unit Ramp } $r(t)$	At t	A/s^2 $1/s^2$		$t \geq 0$
Parabolic / Unit Parabolic } $p(t)$	$At^2/2$ $t^2/2$	A/s^3 $1/s^3$		$t \geq 0$
Impulse $\delta(t)$	$\delta(t)$	1		$t = 0$

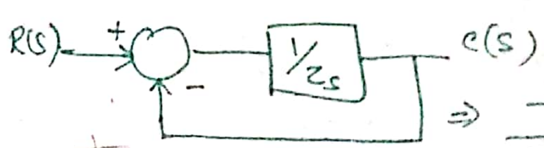
* Order of a system is the maximum power of s in the denominator polynomial of transfer function.

eg: $T.F = \dfrac{s}{2s^2+2s+1}$ ← the order is 2.

* For a first order system the transfer function is given as

$$\boxed{\frac{C(s)}{R(s)} = \frac{1}{1+\tau s}}$$

$\tau = T$



$$\frac{\frac{1}{\tau s}}{1+\frac{1}{\tau s}\times 1}$$

$$\frac{\frac{1}{\tau s}}{\frac{\tau s+1}{\tau s}}$$

$$\Rightarrow \frac{\frac{1}{\tau s}}{1+\frac{1}{\tau s}}$$

TIME RESPONSE OF I ORDER SYSTEM TO UNIT STEP:

* The closed loop transfer function of the 1st order system is

$$\frac{C(s)}{R(s)} = \frac{1}{1+\tau s} \quad \text{and} \quad r(t) = 1 \text{ \& } R(s) = 1/s.$$

* The response in s domain is

$$C(s) = \frac{\frac{1}{s}}{1+\tau s} = \frac{\frac{1}{s}}{\tau(s+1/2)} = \frac{\frac{1}{2}}{s(s+1/2)}.$$

Using partial fraction expansion

$$\frac{\frac{1}{2}}{s(s+1/2)} = \frac{A}{s} + \frac{B}{s+1/2}$$

$$\frac{1}{2} = A(s+1/2) + Bs$$

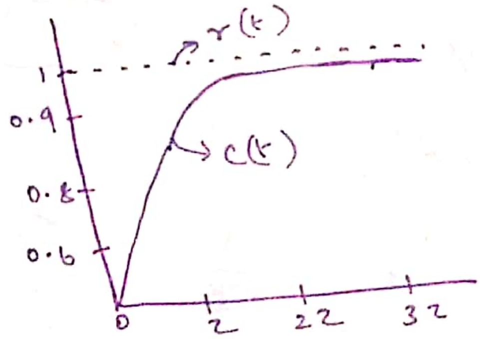
put $s=0 \Rightarrow \frac{1}{2} = \frac{A}{2}.$ $\boxed{A=1}$

put $s=-1/2 \Rightarrow \frac{1}{2} = -\frac{B}{2}.$ $\boxed{B=-1}$

$$C(s) = \frac{1}{s} - \frac{1}{s+1/2}$$

Taking L^{-1} on both sides

$$c(t) = 1 - e^{-t/\tau}$$



when $t=0,\ c(t) = 1-e^{0} = 0$

$t=1\tau,\ c(t) = 1-e^{-1} = 0.632$

$t=2\tau,\ c(t) = 1-e^{-2} = 0.865$

$t=3\tau,\ c(t) = 1-e^{-3} = 0.95$

Steady state error e_{ss} is given by

$$e_{ss} = \lim_{t\to\infty} r(t) - c(t)$$

$$= \lim_{t\to\infty} 1-1+e^{-t/\tau} \Rightarrow \lim_{t\to\infty} e^{-t/\tau} \Rightarrow \boxed{e_{ss}=0}$$

UNIT RAMP:

$$\frac{C(s)}{R(s)} = \frac{1}{1+\tau s}$$

Unit ramp i/p

$r(t) = t \; ; \quad R(s) = \frac{1}{s^2}$

$$C(s) = \frac{\frac{1}{s^2}}{1+2s} = \frac{1}{2s^2(s+1/2)}.$$

Using partial fraction expansion

$$\frac{\frac{1}{2}}{s^2(s+1/2)} = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s+1/2}$$

$$1/2 = As(s+1/2) + B(s+1/2) + Cs^2$$

Put $s=0$

$B/2 = 1/2 \qquad \boxed{B=1}$

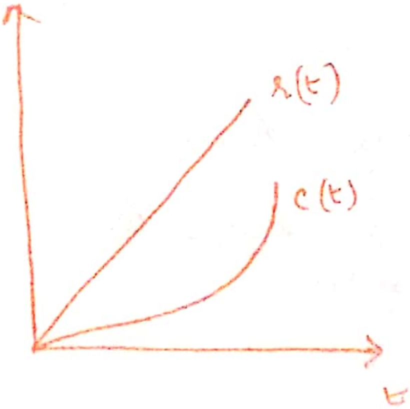
Put $s = -1/2$

$\frac{C}{4} = \frac{1}{2} \qquad \boxed{C = 2}$

Compare coeff of s^2

$A + C = 0$

$\boxed{A = -2}$



$$C(s) = \frac{-2}{s} + \frac{1}{s^2} + \frac{2}{s+1/2}$$

$c(t) = -2 + t + 2e^{-t/2}.$

$\boxed{c(t) = t - 2(1-e^{-t/2})}$

$e_{ss} = \lim_{t\to\infty} r(t) - c(t) = \cancel{t} - \cancel{t} + 2(1-e^{-t/2})$

$= 2(1-e^{-t/2})$

$\boxed{e_{ss} = 2}$

Scanned with CamScanner

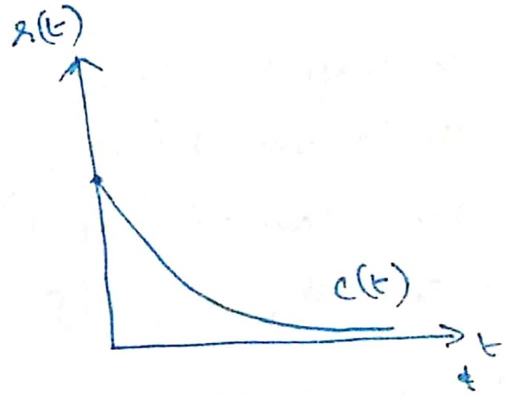
IMPULSE INPUT:

$$\frac{C(s)}{R(s)} = \frac{1}{1+2s}$$

$r(t) = \delta(t) = 1$ for $t=0$. and 0 for $t \neq 0$

$$R(s) = 1.$$

$$C(s) = \frac{1}{1+2s} = \frac{\frac{1}{2}}{(s+\frac{1}{2})}$$



Taking L^{-1} we have

$$c(t) = \frac{1}{2} e^{-t/2}.$$

Steady state error

$$e_{ss} = \lim_{t \to \infty} r(t) - c(t) = \lim_{t \to \infty} \delta(t) - \frac{1}{2} e^{-t/2} = 0 - 0. = 0.$$

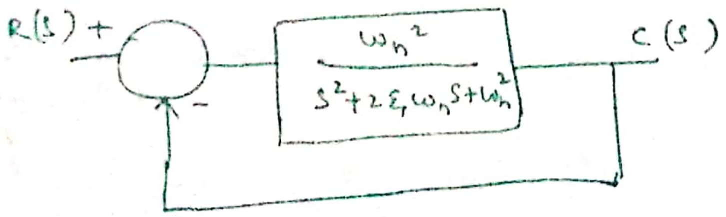
$$\boxed{e_{ss} = 0}$$

SECOND ORDER SYSTEM - TIME RESPONSE:

Transfer function of second order system is

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\xi\omega_n s + \omega_n^2}$$

$\omega_n \rightarrow$ undamped natural frequency (rad/sec).



Damping Ratio (ξ):

It is the ratio of the actual damping to critical damping

$$\xi = \frac{\text{Actual damping}}{\text{Critical damping}}$$

* The response c(t) of 2nd order system depends on the value of damping ratio

* Depending on the value of ξ, the system is classified into 4 cases

(i) Undamped ($\xi = 0$)

(ii) Underdamped ($0 < \xi < 1$)

(iii) overdamped ($\xi > 1$)

(iv) Critically damped ($\xi = 1$)

Characteristic Equation:

The denominator polynomial of closed loop T.F is known as characteristic polynomial and when equated to zero is called the characteristic equation

$$s^2 + 2\xi\omega_n s + \omega_n^2 = 0$$

$$s_1, s_2 = \frac{-2\xi\omega_n \pm \sqrt{4\xi^2\omega_n^2 - 4\omega_n^2}}{2}$$

$$s_1, s_2 = -\xi\omega_n \pm \omega_n\sqrt{\xi^2 - 1}.$$

If $\xi = 0$; $s_1, s_2 = \pm j\omega_n$ → Roots are purely imaginary & S/m is undamped.

If $0<\xi<1$; $s_1, s_2 = -\xi\omega_n \pm j\omega_n\sqrt{1-\xi^2}$ → Roots are complex conjugate & S/m is underdamped

If $\xi = 1$; $s_1, s_2 = -\omega_n$ → Roots are real & Equal & S/m is critically damped

If $\xi \geq 1$, $s_1, s_2 = -\xi\omega_n \pm \omega_n\sqrt{\xi^2-1}$ → Roots are real & unequal & S/m is overdamped

Response of undamped 2nd order S/M for unit step I/p:

The closed loop TF of 2nd order S/M is

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2+2\xi\omega_n s+\omega_n^2}$$

For undamped S/M, $\xi = 0$

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2+\omega_n^2}$$

For unit step $R(s) = 1/s$.

$$C(s) = \frac{\omega_n^2}{s(s^2+\omega_n^2)}$$

Using partial fraction

$$\frac{\omega_n^2}{s(s^2+\omega_n^2)} = \frac{A}{s} + \frac{Bs+C}{s^2+\omega_n^2}$$

$$\omega_n^2 = A(s^2+\omega_n^2) + Bs^2 + Cs \quad \text{—(1)}$$

Put $s=0$

$$\omega_n^2 = A\omega_n^2 \qquad \boxed{A=1}$$

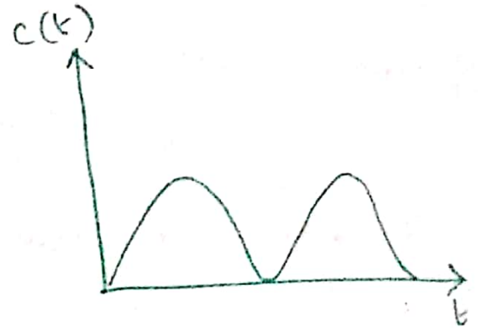
Compare s^2 coefficients on both sides of (1)

$$0 = A+B \Rightarrow 1+B=0 \qquad \boxed{B=-1}$$

Compare s coefficient on both sides of (1)

$$\boxed{C=0}$$

$$C(s) = \frac{1}{s} - \frac{s}{s^2+\omega_n^2} \Rightarrow \boxed{c(t) = 1-\cos\omega_n t}$$



The response is completely oscillatory

Response of underdamped 2^{nd} order S/M for unit step i/p:

For underdamped S/M $0 < \xi < 1$ &

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\xi\omega_n s + \omega_n^2}$$

The roots of characteristic equation are

$$S_1, S_2 = -\xi\omega_n \pm j\omega_n\sqrt{1-\xi^2}$$

$$= -\xi\omega_n \pm j\omega_d$$

For unit step i/p $r(t) = 1$; $R(s) = 1/s$

$$C(s) = \frac{\omega_n^2}{s(s^2 + 2\xi\omega_n s + \omega_n^2)}$$

$$\frac{\omega_n^2}{s(s^2 + 2\xi\omega_n s + \omega_n^2)} = \frac{A}{s} + \frac{Bs + C}{s^2 + 2\xi\omega_n s + \omega_n^2}$$

$$\omega_n^2 = A(s^2 + 2\xi\omega_n s + \omega_n^2) + Bs^2 + Cs \quad \text{— (1)}$$

Subs $s = 0$

$$\cancel{\omega_n^2} = A\cancel{\omega_n^2} \qquad \boxed{A = 1}$$

Compare s^2 terms coefficients on both sides of (1)

$$0 = A + B \Rightarrow \boxed{B = -1}$$

Compare s terms coefficients on both sides of (1)

$$0 = A\xi\omega_n + C$$

$$\boxed{C = -2\xi\omega_n}$$

Scanned with CamScanner

$$C(s) = \frac{1}{s} + \frac{-s - 2\xi\omega_n}{s^2 + 2\xi\omega_n s + \omega_n^2}$$

$$= \frac{1}{s} - \frac{(s + 2\xi\omega_n)}{s^2 + 2\omega_n\xi s + \xi^2\omega_n^2 - \xi^2\omega_n^2 + \omega_n^2}$$

$$= \frac{1}{s} - \frac{s + 2\xi\omega_n}{(s + \xi\omega_n)^2 + \omega_n^2(1-\xi^2)}$$

$$= \frac{1}{s} - \left[\frac{s + \xi\omega_n + \xi\omega_n}{(s+\xi\omega_n)^2 + \omega_n^2(1-\xi^2)}\right]$$

$$C(s) = \frac{1}{s} - \frac{s + \xi\omega_n}{(s+\xi\omega_n)^2 + \omega_d^2} - \frac{\xi\omega_n \times \omega_d}{\left((s+\xi\omega_n)^2 + \omega_d^2\right)\omega_d} \quad \omega_d = \omega_n\sqrt{1-\xi^2}$$

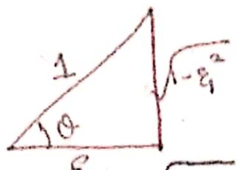
Taking Inverse Laplace on both sides

$$c(t) = 1 - e^{-\xi\omega_n t}\cos\omega_d t - \frac{\xi\omega_n}{\omega_d} e^{-\xi\omega_n t}\sin\omega_d t$$

$$= 1 - e^{-\xi\omega_n t}\left[\cos\omega_d t + \frac{\xi\omega_n}{\omega_d}\sin\omega_d t\right]$$

$$= 1 - e^{-\xi\omega_n t}\left[\cos\omega_d t + \frac{\xi\cancel{\omega_n}}{\cancel{\omega_n}\sqrt{1-\xi^2}}\sin\omega_d t\right]$$

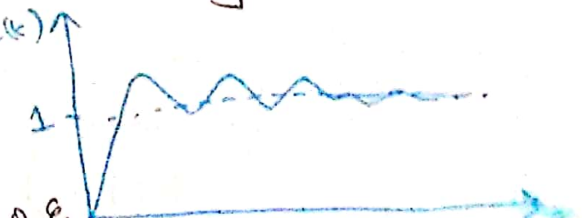
$$c(t) = 1 - \frac{e^{-\xi\omega_n t}}{\sqrt{1-\xi^2}}\left[\sqrt{1-\xi^2}\cos\omega_d t + \xi\sin\omega_d t\right]$$



Take $\sin\theta = \sqrt{1-\xi^2}$; $\cos\theta = \xi$; $\tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{\sqrt{1-\xi^2}}{\xi}$; $\theta = \tan^{-1}\frac{\sqrt{1-\xi^2}}{\xi}$

$$c(t) = 1 - \frac{e^{-\xi\omega_n t}}{\sqrt{1-\xi^2}}\left[\cos\omega_d t \sin\theta + \cos\theta\sin\omega_d t\right]$$

$$\boxed{c(t) = 1 - \frac{e^{-\xi\omega_n t}}{\sqrt{1-\xi^2}}\sin(\omega_d t + \theta)}$$



Oscillations depend of ξ value

Response of overdamped second order system for unit step i/p:

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\xi\omega_n s + \omega_n^2}$$

$$s_1 = \xi\omega_n - \omega_n\sqrt{\xi^2-1}$$
$$s_2 = \xi\omega_n + \omega_n\sqrt{\xi^2-1}$$

For overdamped case $\xi > 1$

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{(s+s_1)(s+s_2)} \qquad R(s) = 1/s.$$

$$C(s) = \frac{1\ \omega_n^2}{s(s+s_1)(s+s_2)}$$

$$\frac{\omega_n^2}{s(s+s_1)(s+s_2)} = \frac{A}{s} + \frac{B}{s+s_1} + \frac{C}{s+s_2}$$

$$\omega_n^2 = A(s+s_1)(s+s_2) + B(s+s_2)s + C(s+s_1)s$$

Put $s = 0$.

$$\omega_n^2 = A s_1 s_2 \qquad \boxed{A = \frac{\omega_n^2}{s_1 s_2}}$$

Put $s = -s_1$

$$\omega_n^2 = -B(s_2 - s_1)s_1$$

$$\boxed{\frac{\omega_n^2}{-s_1(s_2-s_1)} = B}$$

$$A = \frac{\omega_n^2}{(\xi^2\omega_n^2 - \omega_n^2(\sqrt{\xi^2-1})^2)}$$

$$= \frac{\omega_n^2}{\xi^2\omega_n^2 - \omega_n^2(\xi^2-1)}$$

$$A. = \frac{\cancel{\omega_n^2}}{\xi^2\cancel{\omega_n^2} - \cancel{\omega_n^2\xi^2} + \cancel{\omega_n^2}}$$

$$\boxed{A = 1}$$

$$B = \frac{\omega_n^2}{s_1^2 - s_1 s_2} = \frac{-\omega_n^2}{s_1\left(-\xi \cancel{\omega_n} + \omega_n\sqrt{\xi^2-1} + \xi\cancel{\omega_n} + \omega_n\sqrt{\xi^2-1}\right)}$$

$$= \frac{-\omega_n}{2 s_1 \sqrt{\xi^2-1}}$$

$$B = \frac{-\omega_n}{2 s_1 \sqrt{\xi^2-1}}$$

Put $s = -s_2$

$$C(-s_2)(-s_2+s_1) = \omega_n^2$$

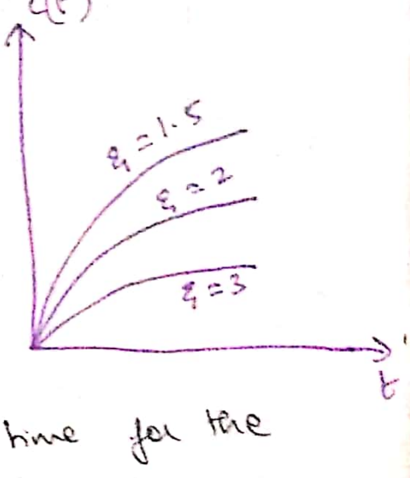
$$C = \frac{-\omega_n^2}{(s_1-s_2)s_2}$$

$$C = \frac{-\omega_n^2}{s_2\left[\xi\cancel{\omega_n} - \omega_n\sqrt{\xi^2-1} - \xi\cancel{\omega_n} - \omega_n\sqrt{\xi^2-1}\right]}$$

$$= \frac{\omega_n}{s_2\left(2\sqrt{\xi^2-1}\right)}$$

$$C(s) = \frac{1}{s} - \frac{\omega_n}{(s+s_1)\, 2 s_1\sqrt{\xi^2-1}} + \frac{\omega_n}{2 s_2\sqrt{\xi^2-1}\,(s+s_2)}$$

$$c(t) = 1 - \frac{\omega_n}{2 s_1\sqrt{\xi^2-1}} e^{-s_1 t} + \frac{\omega_n}{2 s_2\sqrt{\xi^2-1}} e^{-s_2 t}$$



$$c(t) = 1 - \frac{\omega_n}{2\sqrt{\xi^2-1}}\left[\frac{e^{-s_1 t}}{s_1} + \frac{e^{-s_2 t}}{s_2}\right]$$

Response has no oscillations. But takes longer time for the response to reach the final steady state value.

Scanned with CamScanner

Response of critically damped 2nd order system for unit step i/p:

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\xi\omega_n s + \omega_n^2}$$

critical damping $\xi = 1$ $r(t) = 1$, $R(s) = 1/s$

$$C(s) = \frac{\omega_n^2}{s(s^2 + 2\xi\omega_n s + \omega_n^2)} = \frac{\omega_n^2}{s(s^2 + 2\xi\omega_n + \omega_n^2)} = \frac{\omega_n^2}{s(s + \omega_n)^2}$$

$$\frac{\omega_n^2}{s(s+\omega_n)^2} = \frac{A}{s} + \frac{B}{(s+\omega_n)} + \frac{C}{(s+\omega_n)^2}$$

$$\omega_n^2 = A(s+\omega_n)^2 + B(s+\omega_n)s + Cs$$

Put $s = 0$.

$$\cancel{\omega_n^2} = A\cancel{\omega_n^2} \qquad \boxed{A = 1}$$

Put $s = -\omega_n$

$$\omega_n^2 = -\cancel{\omega_n} C \qquad \boxed{C = -\omega_n}$$

Compare coefficients of s^2

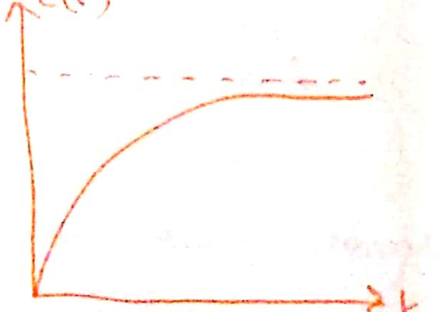
$$A + B = 0 \qquad \boxed{B = -1}$$

$$C(s) = \frac{1}{s} + \frac{-1}{s+\omega_n} + \frac{-\omega_n}{(s+\omega_n)^2} \qquad L^{-1}(t e^{-at}) = \frac{1}{(s+a)^2}$$

Taking L^{-1}

$$c(t) = 1 - e^{-\omega_n t} - \omega_n t e^{-\omega_n t}$$

$$\boxed{c(t) = 1 - e^{-\omega_n t}[1 + \omega_n t]}$$



Response has no oscillations

PERFORMANCE SPECIFICATIONS IN TIME DOMAIN, STEADY STATE ERROR, GENERALIZED ERROR CONSTANTS:

TIME DOMAIN SPECIFICATIONS:

* The performance characteristics of control systems are specified in terms of time domain specifications.
* Systems with energy storage elements cannot respond instantaneously and will exhibit transient responses when subjected to inputs or disturbance.
* The transient response of a practical control system exhibits damped oscillation before reaching steady state.

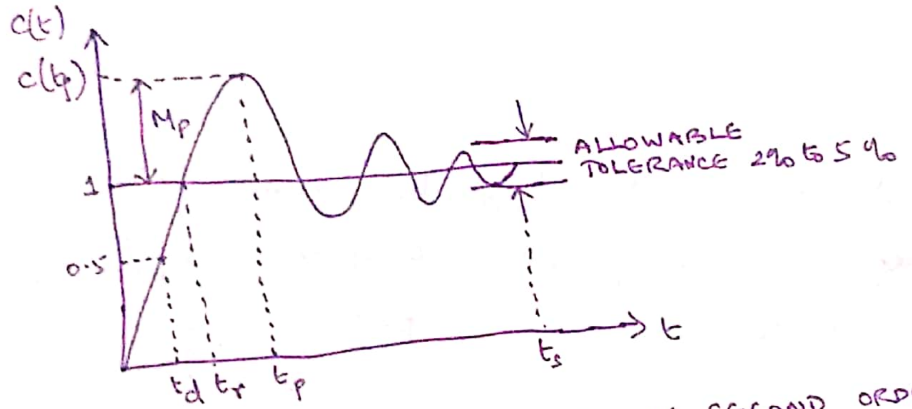


FIG: DAMPED OSCILLATORY RESPONSE OF SECOND ORDER SYSTEM.

to a unit step input

* The transient response characteristics of a control system to a unit step input is specified in terms of foll. time domain specifications

(i) Delay time, t_d (ii) Rise time, t_r (iii) Peak time, t_p
(iv) Maximum overshoot, M_p (v) Settling time, t_s.

(i) Delay Time (t_d):

It is the time taken for response to reach 50% of the final value, for the very first time.

(ii) Rise Time (t_r):

It is the time taken for response to raise from 0 to 100% for very first time. For overdamped it is time taken for response to raise from 10% to 90% & for critically damped, from 5% to 95%.

$$t_r = \frac{\pi - \theta}{\omega_d} \; ; \quad \theta = \tan^{-1}\left(\frac{\sqrt{1-\xi^2}}{\xi}\right) \; ; \quad \omega_d = \omega_n\sqrt{1-\xi^2}$$

(iii) Peak Time (t_p):

It is the time taken for the response to reach peak value or peak overshoot, M_p.

$$\frac{d}{dt} c(t)\Big|_{t=t_p} = 0 \qquad t_p = \frac{\pi}{\omega_n\sqrt{1-\xi^2}} = \frac{\pi}{\omega_d}.$$

(iv) Peak Overshoot (M_p):

It is the ratio of maximum peak value measured from final value to the final value.

$$\% M_p = e^{-\xi\pi/\sqrt{1-\xi^2}} \times 100$$

(v) Settling time (t_s):

* The response of second order system has two components.
  (i) Decaying exponential component : $\exp(-\xi\omega_n t/\sqrt{1-\xi^2})$
  (ii) Sinusoidal Component : $\sin(\omega_d t + \theta)$

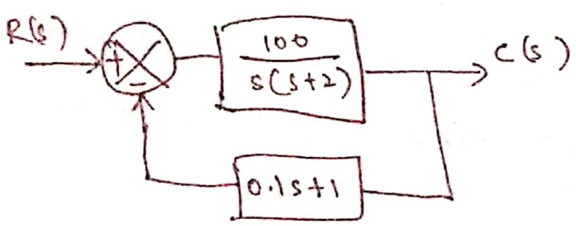
$$t_s = \frac{4}{\xi\omega_n} \quad \text{for } 2\% \text{ error}$$

$$t_s = \frac{3}{\xi\omega_n} \quad \text{for } 5\% \text{ error}.$$

$$t_s = \frac{\ln(\%\,\text{error})}{\xi\omega_n} \quad \text{for generalised error percentage.}$$

EXAMPLE (1):

What is the response of the system for unit step input



Solution:

$$\frac{C(s)}{R(s)} = \frac{G(s)}{1+G(s)H(s)}$$

$$\frac{C(s)}{R(s)} = \frac{\frac{100}{s(s+2)}}{1+\frac{100(0.1s+1)}{s(s+2)}}$$

$$\frac{C(s)}{R(s)} = \frac{\frac{100}{\cancel{s(s+2)}}}{\frac{s(s+2)+100(0.1s+1)}{\cancel{s(s+2)}}} = \frac{100}{s(s+2)+10s+100} = \frac{100}{s^2+12s+100}$$

Scanned with CamScanner

$s^2 + 12s + 100$ ← characteristic equation

$$s_1, s_2 = \frac{-12 \pm \sqrt{144 - 4\times1\times100}}{2} = \frac{-12 \pm j16}{2} = -6 \pm j8$$

* The roots are complex conjugate. The system is underdamped

$$C(s) = \left(\frac{100}{s^2+12s+100}\right)\left(\frac{1}{s}\right).$$

$$C(s) = \frac{100}{s(s^2+12s+100)} = \frac{A}{s} + \frac{Bs+C}{s^2+12s+100}$$

$$100 = A(s^2+12s+100) + Bs^2 + Cs$$

Subs $s = 0$.

$100 = A(100)$

$\frac{100}{100} = A$ $\boxed{A=1}$

Compare/equate coefficients of s on both sides

$0 = 12A + C$ $0 = 12 + C$ $\boxed{C = -12}$

Compare/equate coefficients of s^2 on both sides

$0 = A + B$ ⇒ $\boxed{B = -1}$

$$C(s) = \frac{1}{s} + \frac{-s-12}{s^2+12s+100}$$ $2\times6\times s + 6^2.$

$$= \frac{1}{s} - \frac{(s+12)}{s^2+2\times6\times s+36+64}.$$

$$= \frac{1}{s} - \frac{(s+6+6)}{(s+6)^2+8^2}.$$

$$= \frac{1}{s} - \frac{(s+6)}{(s+6)^2+8^2} - \frac{6\times8}{8[(s+6)^2+8^2]}$$

$c(t) = 1 - e^{-6t}\cos 8t - \frac{6}{8}e^{-6t}\sin 8t$ ⇒ $\boxed{c(t) = 1 - e^{-6t}\left[\cos 8t + \frac{6}{8}\sin 8t\right]}$

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\xi\omega_n s + \omega_n^2} = \frac{100}{s^2+12s+100}$$

$$\omega_n^2 = 100 \quad ; \quad \omega_n = 10.$$

$$2\xi\omega_n = 12 \quad ; \quad \xi = \frac{12}{2\omega_n} = \frac{12}{2\times 10} = \frac{12}{20} = \frac{3}{5} = 0.6$$

$$\boxed{\xi = 0.6} \qquad \sqrt{1-\xi^2} = \sqrt{1-0.36} = \sqrt{0.64} = 0.8.$$

$$\cos\theta = 0.6 \qquad \sin\theta = 0.8.$$

$$c(t) = 1 - e^{-6t}\left[\frac{\cos 8t \cdot \sin\theta}{\sin\theta} + \frac{6}{8}\,\frac{\sin 8t \,\cos\theta}{\cos\theta}\right]$$

$$= 1 - e^{-6t}\left[\frac{\cos 8t\,\sin\theta}{0.8} + \frac{6}{8}\,\frac{\sin 8t\,\cos\theta}{0.6}\right].$$

$$= 1 - e^{-6t}\left[\frac{\cos 8t\; 0.8\times 10}{8} + \frac{\cancel{6}\times 10}{8 \times \cancel{6}} \sin 8t\; 0.6\right]$$

$$= 1 - e^{-6t} \times \frac{10}{8}\left[\cos 8t\,\sin\theta + \sin 8t\,\cos\theta\right]$$

$$= 1 - e^{-6t} \times \frac{10}{8}\left[\sin(8t+\theta)\right].$$

$$\theta = \tan^{-1}\left(\frac{\sqrt{1-\xi^2}}{\xi}\right) = \tan^{-1}\left(\frac{0.8}{0.6}\right) = 53^\circ \text{ or } 0.925 \text{ rad}$$

$$\boxed{c(t) = 1 - 1.25 e^{-6t} \sin(8t + 0.925).}$$

EXAMPLE (2):

Measurements conducted on a servomechanism shows the system response to be $c(t) = 1 + 0.2e^{-60t} - 1.2e^{-10t}$ when subjected to unit step input. Obtain an expression for closed loop transfer function. Determine the undamped natural frequency and damping ratio.

$$c(t) = 1 + 0.2e^{-60t} - 1.2e^{-10t}$$

$$C(s) = \frac{600}{s(s+60)(s+10)}$$

$$C(s) = \frac{1}{s}\frac{600}{(s+60)(s+10)} \Rightarrow C(s) = R(s)\frac{600}{(s+60)(s+10)}$$

$$\frac{C(s)}{R(s)} = \frac{600}{s^2 + 10s + 60s + 600} = \frac{600}{s^2 + 70s + 600}$$

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\xi\omega_n s + \omega_n^2}$$

$$\omega_n^2 = 600$$

$$\omega_n = \sqrt{600} = 24.49 \text{ rad/sec}$$

$$2\xi\omega_n = 70.$$

$$2 \times 24.49 \times \xi = 70 \quad ; \quad \xi = 1.43$$

EXAMPLE (3):

The unity feedback system has open loop transfer function $G(s) = \frac{K}{s(s+10)}$. Determine the gain K, so that the system will have a damping ratio of 0.5 for this value of K. Determine ts, Mp, tp for unit step I/p.

Solution:

The T.F for system is $\frac{C(s)}{R(s)} = \frac{G(s)}{1+G(s)H(s)}$ $\quad H(s) = 1$

$$\frac{C(s)}{R(s)} = \frac{\frac{K}{s(s+10)}}{1+\frac{K}{s(s+10)}} = \frac{K}{s(s+10)+K} = \frac{K}{s^2+10s+K}.$$

$R(s) = 1/s.$ So $\quad C(s) = \left(\frac{1}{s}\right)\left(\frac{K}{s^2+10s+K}\right).$

Scanned with CamScanner

For second order system

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2+2\xi\omega_n s+\omega_n^2} = \frac{K}{s^2+10s+K}$$

$\omega_n^2 = K.$ $\quad 2\xi\omega_n = 10$

$\omega_n = \sqrt{K}$; $\quad 2\times 0.5\times\sqrt{K} = 10.$

$\boxed{\omega_n = \sqrt{K}}$ $\quad \sqrt{K} = 10$

$\boxed{K = 100}$

$$\frac{C(s)}{R(s)} = \frac{100}{s^2+10s+100}$$

$\boxed{\omega_n = 10 \text{ rad/sec}}$

$$\% M_p = e^{-\xi\pi/\sqrt{1-\xi^2}} \times 100$$

$\boxed{\% M_p = 16.3\%.}$

$$t_p = \frac{\pi}{\omega_d} = 0.363 \text{ sec.}$$

$\boxed{t_p = 0.363 \text{ sec}}$

EXAMPLE (4) :

For a unity feedback system with overall transfer function $G(s) = \frac{1}{s(s+1)}$. Find (i) damping factor (ii) undamped natural frequency (iii) Peak time (iv) Maximum Overshoot (v) Unit step response $c(t)$

$$\frac{C(s)}{R(s)} = \frac{G(s)}{1+G(s)H(s)} = \frac{\frac{1}{s(s+1)}}{1+\frac{1}{(s+1)s}\times 1} = \frac{1}{s^2+s+1}$$

$$\frac{\omega_n^2}{s^2+2\xi\omega_n s+\omega_n^2} = \frac{1}{s^2+s+1.}$$

$\omega_n^2 = 1$ $\quad \boxed{\omega_n = 1}$

$2\xi\omega_n = 1.$

$2\xi = 1$

$\boxed{\xi = 1/2 = 0.5}$

Scanned with CamScanner

$$t_p = \frac{\pi}{\omega_d} = \frac{\pi}{\omega_n\sqrt{1-\xi^2}} = \frac{\pi}{\sqrt{1-0.25}} = 3.628 \text{ sec.}$$

$$M_p = e^{-\xi\pi/\sqrt{1-\xi^2}} = e^{-0.5\pi/0.866} = e^{-1.814} = 0.163.$$

$$\boxed{\% M_p = 16.3\ \%}$$

EXAMPLE 5 :

A control system has following T.F $G(s) = \frac{4}{s+2}$; $H(s) = 1/s$

Find damping of system, damping ratio, Time of overshoot, $\% M_p$.

characteristic equation $1 + G(s)\,H(s) = 0.$

$$1 + \frac{4}{(s+2)} \times 1/s = 0.$$

$$1 + \frac{4}{s^2+2s} = 0.$$

$$\frac{s^2+2s+4}{s^2+2s} = 0. \qquad s^2+2s+4 = 0. \quad \text{—— (1)}$$

From standard transfer function

$$s^2 + 2s + 4 = s^2 + 2\xi\omega_n s + \omega_n^2.$$

$$\omega_n^2 = 4. \quad \boxed{\omega_n = 2}$$

$$2\xi\omega_n = 2$$

$$2\xi \times 2 = 2.$$

$$\boxed{\xi = 2/4 = 0.5}$$

$$t_p = \frac{\pi}{\omega_d} = \frac{\pi}{\omega_n\sqrt{1-\xi^2}} = \frac{\pi}{2\sqrt{1-0.25}} = \frac{\pi}{2(0.866)} = 1.814 \text{ sec.}$$

$$\boxed{t_p = 1.814 \text{ sec}}$$

$$M_p = e^{-\xi\pi/\sqrt{1-\xi^2}} = e^{-0.5\pi/0.866} = 0.163$$

$$\boxed{M_p \% = 16.3}$$

EXAMPLE (6):

For a system with $G(s) = \frac{K_1}{s^2}$ & $H_s = 1+K_2 s$. Find K_1 & K_2. so that $M_p = 0.25$ & t_p is 2 sec when step i/p is applied

characteristic equation

$$1+G(s)H(s) = 1+\frac{K_1}{s^2}(1+K_2 s) = 0.$$

$$s^2+2\xi\omega_n s+\omega_n^2 = s^2+K_1+K_2K_1 s = 0$$

$$\omega_n^2 = K_1 \qquad \omega_n = \sqrt{K_1}$$

$$2\xi\omega_n = K_1 K_2$$

$$2\xi\sqrt{K_1} = K_1 K_2$$

$$2\xi = K_2\sqrt{K_1}$$

$$\xi = \frac{\sqrt{K_1}\,K_2}{2} \Rightarrow \xi = \frac{\omega_n K_2}{2}$$

$$\boxed{K_2 = \frac{2\xi}{\omega_n}} \qquad \boxed{K_1 = \omega_n^2}$$

$$M_p = e^{-\xi\pi/\sqrt{1-\xi^2}} = 0.25$$

Taking log on both sides

$$\frac{+\xi\pi}{\sqrt{1-\xi^2}} = \ln(0.25) = +1.386$$

$$\frac{\xi\pi}{\sqrt{1-\xi^2}} = 1.386$$

$$\xi\pi = 1.386\sqrt{1-\xi^2}$$

$$\xi^2\pi^2 = (1.386)^2(1-\xi^2).$$

$$\xi^2\pi^2 = (1.386)^2 - (1.386)^2\xi^2.$$

$$\xi^2\pi^2 + (1.386)^2\xi^2 = (1.386)^2$$

$$\xi^2\pi^2 + 1.9218\,\xi^2 = 1.9218$$

$$(9.8696 + 1.9218)\xi^2 = 1.9218$$

$$\xi^2 = \frac{1.9218}{9.8696 + 1.9218} = 0.1629$$

$$\begin{array}{r} 0.9999 \\ 1.0000 \\ 0.1629 \\ \hline 0.8371 \end{array}$$

$$\boxed{\xi = 0.404}$$

$$t_p = \frac{\pi}{\omega_d} = \frac{\pi}{\omega_n\sqrt{1-\xi^2}} = \frac{\pi}{\omega_n\sqrt{1-(0.404)^2}}$$

$$\frac{\pi}{\omega_n\sqrt{1-0.1629}} = \frac{\pi}{\omega_n\sqrt{0.8371}} = \frac{\pi}{\omega_n\sqrt{0.8371}} = 2.$$

$$2\,\omega_n\sqrt{0.8371} = \pi$$

$$\omega_n = \frac{\pi}{2\sqrt{0.8371}} = 1.717 \text{ rad/sec}$$

$$K_1 = (1.717)^2 = 2.949; \qquad K_2 = \frac{2\times 0.404}{1.717} = 0.471$$

$$\boxed{K_1 = 2.949} \qquad \boxed{K_2 = 0.471}$$

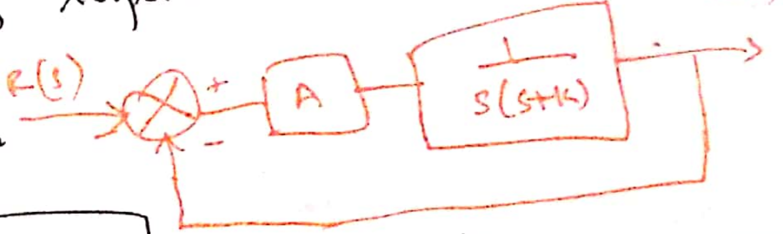
EXAMPLE (4):

A step ~~of~~ 2 is applied to unity feedback system given, find A & K so that $\xi = 0.6$; $\omega_d = 8$ rad/sec; & find the peak value of responce

$\xi = 0.6$; $\omega_d = 8$ rad/sec

$K = 2\xi\omega_n$.

$A = \omega_n^2$



$\boxed{\omega_n = 10 \text{ rad/sec.}}$

$\omega_n\sqrt{1-\xi^2} = 8$ rad/sec

$\boxed{K = 12}$

$$M_p = 2e^{\frac{-\xi\pi}{\sqrt{1-\xi^2}}}$$

$\boxed{M_p = 0.1896}$ $\qquad M_p = c(t_p) - 1 \qquad$ $\boxed{C(t_p) = 2.1896}$ peak response

STEADY STATE ERROR ANALYSIS [e_{ss}]:

* Important measure of accuracy
* It is the error after the transient response has decayed leaving only the continuous response.
* It indicates the error between the actual o/p and desired o/p as $t \to \infty$

$$e_{ss} = \lim_{s \to 0} s\,E(s) = \lim_{s \to 0} s \frac{R(s)}{1+G(s)H(s)}$$

$$\therefore E(s) = \frac{C(s)}{G(s)} = \frac{R(s)}{1+G(s)H(s)}$$

* The final value theorem states,

$$\lim_{t \to \infty} e(t) = \lim_{s \to 0} s\,E(s).$$

* The e_{ss} can be classified into two
  1. Static error constants
  2. Dynamic error constants (or) generalized error constants.

Static Error Constants:

Position error Constant:
It is a measure of e_{ss} b/w the i/p & o/p when the i/p is a unit step function.

$$K_p = \lim_{s \to 0} G(s)H(s) \quad , \quad e_{ss} = \frac{1}{1+K_p}$$

Velocity Error constant:
It is a measure of steady state error b/w the i/p & o/p, when i/p is a ramp function

$$K_v = \lim_{s \to 0} s\,G(s)H(s) \quad ; \quad e_{ss} = \frac{1}{K_v}$$

Accelaration Error Constant:
It is a measure of steady state error b/w the I/p & O/p, when the I/p is a parabolic function.

Scanned with CamScanner

$$K_a = \lim_{s\to 0} s^2 G(s)\,H(s) \quad , \quad e_{ss} = \frac{1}{K_a}$$

Type Number of Control System:

* The number of poles of open loop transfer function lying at origin is the type number of the system.
* If the open loop transfer function is

$$G(s)\,H(s) = K \frac{(s+z_1)(s+z_2)\,\text{------}}{s^N (s+p_1)(s+p_2)\,\text{-----}}$$

$z_1, z_2, \text{----}$ are zeroes

$p_1, p_2 \text{-----}$ are poles

N – no. of poles at origin

K – Constant.

If $N=0$; system is type 0

If $N=1$; system is type 1

If $N=2$; system is type 2 and so on.

Static Position Error Constant (K_p):

Steady state error $\quad e_{ss} = \lim_{s\to 0} s\,E(s)$

$$E(s) = \frac{R(s)}{1+G(s)\,H(s)}$$

for position Error constant, $R(s) = 1/s$

$$\therefore e_{ss} = \lim_{s\to 0} \frac{\cancel{s} \times \frac{1}{\cancel{s}}}{1+G(s)\,H(s)} = \lim_{s\to 0} \frac{1}{1+G(s)\,H(s)}$$

$$\frac{1}{\frac{1+K_p+1}{1+K_p}} = \frac{1+K_p}{2+K_p}$$

$$e_{ss} = \frac{1}{1+\lim_{s\to 0} G(s)\,H(s)} = \frac{1}{1+K_p}.$$

$$\left[\therefore K_p = \lim_{s\to 0} G(s)\,H(s)\right].$$

$$\boxed{e_{ss} = \frac{1}{1+K_p}}$$

For a type 0 system

$$K_p = \underset{s \to 0}{Lt} \frac{K(s+z_1)\ldots}{s^0 .(s+p_1)(s+p_2)\ldots}$$

$\boxed{K_p = K.}$ $\boxed{e_{ss} = \frac{1}{1+K.}}$ ~~e_{ss}~~

For a type 1 system

$$K_p = \underset{s \to 0}{Lt} \frac{K(s+z_1)\ldots}{s(s+P_1)\ldots} = \infty$$

$$K_p = \infty \qquad e_{ss} = \frac{1}{1+\infty} = 0.$$

$\boxed{K_p = \infty}$ $\boxed{e_{ss} = 0}$

Static velocity Error Constant (K_v):

$$e_{ss} = \underset{s \to 0}{Lt}\, s\, E(s)$$

$$E(s) = \frac{R(s)}{1+G(s)\,H(s)}$$

For velocity error constant, $R(s) = \frac{1}{s^2}$

$$e_{ss} = \underset{s \to 0}{Lt}\, \cancel{s} \times \frac{1}{s^{\cancel{2}}(1+G(s)\,H(s))} = \underset{s \to 0}{Lt} \frac{1}{s+s\,G(s)\,H(s)}$$

$$e_{ss} = \frac{1}{0+\underset{s \to 0}{Lt}\, s\,G(s)\,H(s)} = \frac{1}{K_v.} \qquad \left[\because K_v = \underset{s \to 0}{Lt}\, s\,G(s)\,H(s)\right]$$

$\boxed{e_{ss} = \frac{1}{K_v}}$

For type 0 system

$$K_v = \underset{s \to 0}{Lt} \frac{s\,K(s+z_1)}{s^0(s+P_1)} = 0.$$

$\boxed{e_{ss} = \infty}$ $\boxed{K_v = 0}$

For type 1 system,

$$K_v = \lim_{s \to 0} s \frac{K(s+z_1)}{s^1(s+P_1)} = K.$$

$$\boxed{K_v = K} \qquad \boxed{e_{ss} = \frac{1}{K}}$$

Static Accelaration Error Constant (K_a)

$$e_{ss} = \frac{1}{K_a}.$$

For type 2 system

$$K_v = \lim_{s \to 0} s \frac{K(s+z_1)\ldots}{s^2(s+P_1)\ldots}$$

$$\boxed{K_v = \infty} \qquad \boxed{e_{ss} = 0}$$

System Type	Unit Step I/P		Ramp I/P		Parabolic I/P	
	K_p	e_{ss}	K_v	e_{ss}	K_a	e_{ss}
Type 0	K	$\frac{1}{1+K}$	0	∞	0	∞
Type 1	∞	0	K	$\frac{1}{K}$	0	∞
Type 2	∞	0	∞	0	K	$\frac{1}{K}$
Higher Type	∞	0	∞	0	∞	0

Dynamic Error Constants (OR) Generalized Error Constant:

$$e(t) = C_0 r(t) + C_1 r'(t) + \frac{C_2}{2!} r''(t) + \frac{C_3}{3!} r'''(t) + \ldots$$

$$C_0 = \lim_{s \to 0} F(s) \qquad C_2 = \lim_{s \to 0} \frac{d^2}{ds^2} F(s)$$

$$C_1 = \lim_{s \to 0} \frac{d}{ds} F(s) \qquad C_3 = \lim_{s \to 0} \frac{d^3}{ds^3} F(s)$$

C_0, C_1, C_2, C_3 — Dynamic coefficients

$$e_{ss} = \lim_{t \to \infty} e(t)$$

$F(s)$ is the transfer function

$$\frac{C(s)}{R(s)} = \frac{1}{1+G(s)H(s)}$$

EXAMPLE (4):

A unity feedback system has $G(s) = \frac{K}{s^2(s+2)(s+5)}$ find the (i) type of system (ii) error constants (iii) e_{ss} for unit step, ramp & parabolic i/p.

$$G(s)H(s) = \frac{K}{s^2(s+2)(s+5)}$$

(i) The T.F has 2 poles at origin so the system is type 2 s/m

(ii) Error constants K_p, K_v, K_a :

$$K_p = \lim_{s\to 0} G(s)\,H(s) = \lim_{s\to 0} \frac{K}{s^2(s+2)(s+5)} = \infty$$

$$K_p = \infty$$

$$K_v = \lim_{s\to 0} sG(s)\,H(s) = \lim_{s\to 0} \frac{\cancel{s}K}{s^{\cancel{2}}(s+2)(s+5)} = \infty$$

$$K_v = \infty$$

$$K_a = \lim_{s\to 0} s^2\,G(s)H(s) = \lim_{s\to 0} \frac{\cancel{s^2}K}{\cancel{s^2}(s+2)(s+5)} = \frac{K}{10}$$

$$K_a = \frac{K}{10}$$

(iii) steady state Error e_{ss} :

e_{ss} for unit step $= \frac{1}{1+K_p} = 0.$ $\boxed{e_{ss}\big|_{step} = 0}$

e_{ss} for unit ramp $= \frac{1}{K_v} = \frac{1}{\infty} = 0$ $\boxed{e_{ss}\big|_{ramp} = 0}$

e_{ss} for Parabolic i/p $= \frac{1}{K_a} = \frac{1}{\frac{K}{10}} = \frac{10}{K}.$ $\boxed{e_{ss}\big|_{parabola} = \frac{10}{K}.}$

EXAMPLE (2):

Find K_p, K_v, K_a for the system whose open loop T.F is $G(s) = \frac{10}{s(s+1)}$ with unity feedback. Find ess for $r(t) = 1+t$

$$G(s)\,H(s) = \frac{10}{s(s+1)}$$

- type 1 system.

Error Constants:

$$K_p = \lim_{s\to 0} G(s)\,H(s) = \lim_{s\to 0} \frac{10}{s(s+1)} = \infty$$

$$K_v = \lim_{s\to 0} s\,G(s)\,H(s) = \lim_{s\to 0} \frac{\cancel{s}\,10}{\cancel{s}(s+1)} = \frac{10}{1} = 10.$$

$$K_a = \lim_{s\to 0} s^2\,G(s)\,H(s) = \lim_{s\to 0} \frac{s^2\,10}{\cancel{s}(s+1)} = 0.$$

ess for $r(t) = 1+t$ $\quad R(s) = \frac{1}{s} + \frac{1}{s^2}$

$$E(s) = \frac{R(s)}{1+G(s)\,H(s)} = \frac{\frac{1}{s}+\frac{1}{s^2}}{1+\frac{10}{s(s+1)}}$$

$$= \frac{\frac{1}{s}+\frac{1}{s^2}}{\frac{s(s+1)+10}{s(s+1)}}$$

$$\frac{\frac{1}{s}+\frac{1}{s^2}\left(\frac{s+1}{s^2}\right)}{\frac{s^2+s+10}{\cancel{s}(s+1)}}$$

$$= \frac{\frac{1}{\cancel{s}}}{\frac{s^2+s+10}{\cancel{s}(s+1)}} + \frac{\frac{1}{\cancel{s^2}}}{\frac{s^2+s+10}{\cancel{s}(s+1)}}$$

$$\frac{s+1}{s} \times \frac{s+1}{s^2+s+10}$$

$$E(s) = \frac{s+1}{s(s^2+s+10)} + \frac{s(s+1)}{s(s^2+s+10)}$$

$$E(s) = \frac{(s+1+s^2+s)}{s(s^2+s+10)} = \frac{(s^2+2s+1)}{s(s^2+s+10)}$$

$$e_{ss} = \lim_{s \to 0} s\,E(s) = \frac{\cancel{s}(s^2+2s+1)}{\cancel{s}(s^2+s+10)} = 0$$

$$= \frac{(s+1)^2}{(s^2+s+10)} = \frac{1}{10} = 0.1$$

$$\boxed{e_{ss} = 0.1}$$

EXAMPLE 3:

The unity feedback system has a forward path T.F $G(s) = \frac{10}{s+1}$. Find e_{ss} and generalized error coefficients for $r(t) = 2t$

$r(t) = 2t$

$r'(t) = 2$

$r''(t) = 0.$

Generalized error series

$$e(t) = C_0\, r(t) + \frac{C_1}{1!} r'(t) + \frac{C_2}{2!} r''(t)$$

$$F(s) = \frac{1}{1+\frac{10}{s+1} \times 1} = \frac{s+1}{s+11} \quad \frac{u}{v} \qquad \frac{d}{dt}\left(\frac{u}{v}\right) = \frac{v\,du - u\,dv}{v^2}.$$

$$C_0 = \lim_{s \to 0} F(s) = \frac{1}{11}.$$

$$C_1 = \lim_{s \to 0} \frac{d}{ds} F(s) = \frac{v\,du - u\,dv}{v^2} \quad \frac{(s+11) - (s+1)}{(s+11)^2} = \frac{10}{11^2} = \frac{10}{121}$$

$C_2 = 0.$

$\therefore e(t) = \frac{1}{11} 2t + \frac{10}{121} \times 2 + 0.$

$$\boxed{e(t) = \frac{2t}{11} + \frac{20}{121}}$$

$$e_{ss} = \lim_{t \to \infty} \frac{2t}{11} + \frac{20}{121} = \infty$$

$$\boxed{e_{ss} = \infty}$$

EXAMPLE 4:

For a unity feedback system the open loop transfer function $G(s) = \frac{10(s+2)}{s^2(s+1)}$. Find the e_{ss} & generalized error constants when the i/p is $R(s) = 3/s - \frac{2}{s^2} + \frac{1}{3s^3}$

$$R(s) = \frac{3}{s} - \frac{2}{s^2} + \frac{1}{3s^3}.$$

Taking L^{-1} $r(t) = 3 - 2t + \frac{1}{3}\frac{t^2}{2} = 3 - 2t + \frac{t^2}{6}.$

$$r'(t) = -2 + \frac{2t}{6} = -2 + \frac{t}{3}.$$

$$r''(t) = 1/3. \qquad r'''(t) = 0.$$

$$F(s) = \frac{1}{1 + \frac{10(s+2)}{s^2(s+1)}} = \frac{s^2(s+1)}{s^3+s^2+10s+20}.$$

Generalized error series

$$e(t) = C_0 r(t) + \frac{C_1}{1!} r'(t) + \frac{C_2}{2!} r''(t)$$

$C_0 = \lim_{s \to 0} F(s) = 0.$; $C_1 = \lim_{s \to 0} \frac{d}{ds} F(s) = \lim_{s \to 0} \frac{d}{ds}\left[\frac{s^2(s+1)}{s^3+s^2+10s+20}\right]$

$C_1 = 0$

$$c_2 = \frac{1}{10}.$$

$$e(t) = 0 \times r(t) + 0 \times r'(t) + \frac{c_2}{2!} r''(t)$$

$$= \frac{1}{10} \times \frac{1}{3} \times \frac{1}{2} = \frac{1}{60}.$$

$$\boxed{e_{ss} = \frac{1}{60}}$$

CONTROLLER: