IIR Filter Design by Impulse Invariant Transformation method

The objective of this method is to develop an IIR filter transfer function whose impulse response is the sampled version of the impulse response of the analog filter.

- this tech. is to preserve the freq. response characteristics of the analog filter
- It can be stated that the digital filter with freq. response will possess the freq. response charact. of the corresponding analog filter if the sampling time period T is selected sufficiently small (or sampling freq. f_s should be high) to minimize (or avoid) the effects of aliasing.

Let $h_a(t) \rightarrow$ Impulse response of analog filter

$h(n) \rightarrow$ Impulse response of digital filter

$T \rightarrow$ Sampling period

In this tech, the desired impulse respon of the digital filter is obtained by uniformly sampling the impulse response of the equivalent analog filter. ie

$$h(n) = h_a(t)\Big|_{t=nT} = h_a(nT) \quad \text{— ①}$$

- the laplace t/f of the analog impulse response $h_a(t)$ gives the transfer fn. of analog filters

$$L[h_a(t)] = H_a(s)$$

- the t/f tech can be understood by considering a simple distinct pole case for the anal sys fn.

$$H_a(s) = \sum_{i=1}^{M} \frac{A_i}{s - P_i} \quad \text{— ②}$$

- the impulse response of the sys. specified by eqn ② can be obtained by taking the inverse laplace t/f & it will be of the form

$$h_a(t) = \sum_{i=1}^{M} A_i e^{P_i t} u_a(t) \quad \text{— 3}$$

$u_a(t) \rightarrow$ unit step fn. in continuous time.

The impulse response $h(n)$ of the eqnt. digital filter is obtained by uniformly sampling $h_a(t)$.

$$h(n) = h_a(nT) = \sum_{i=1}^{M} A_i e^{P_i nT} u_a(nT) \quad \text{--- (4)}$$

The sys. response of the digital system of eqn (4) can be obtained by taking the z-t/f.

$$H(z) = \sum_{n=0}^{\infty} h(n) \cdot z^{-n}$$

$$H(z) = \sum_{n=0}^{\infty} \left[\sum_{i=1}^{M} A_i e^{P_i nT} u_a(nT) \right] z^{-n} \quad \text{--- (5)}$$

Interchanging the order of summation,

$$H(z) = \sum_{i=1}^{M} \left[\sum_{n=0}^{\infty} A_i e^{P_i nT} u_a(nT) \right] z^{-n}$$

$$H(z) = \sum_{i=1}^{M} \frac{A_i}{1 - e^{P_i T} z^{-1}} \quad \text{--- (6)}$$

comparing (2) & (6), the mapping formula for the impulse invariant t/f is given by

$$\frac{1}{S - P_i} \longrightarrow \frac{1}{1 - e^{P_i T} z^{-1}} \quad \text{--- (7)}$$

eqn (7) shows that the analog pole at $\boxed{s = P_i}$ mapped into a digital pole at $z = e^{P_i T}$.

∴ the analog poles & the digital poles are related by the relation

$$z = e^{sT} \quad \text{— (8)}$$

The general characteristic of the mapping $z = e^{sT}$ can be obtained by substituting $s = \sigma + j\Omega$ and expressing the complex variable z in the polar form as $z = re^{j\omega}$.

- with these substitutions, eqn. (8) becomes

$$re^{j\omega} = e^{\sigma T} \cdot e^{j\Omega T}$$

$$\begin{array}{l} |z| \\ \angle z \end{array} \boxed{\begin{array}{c} r = e^{\sigma T} \\ \hline \omega = \Omega T \end{array}} \quad \text{— (9)}$$

$\omega = \Omega T$

$$\boxed{\Omega = \frac{\omega}{T}}$$

→ used to compute the digital freq. for a gn. analog freq. & vice

consequently, $\sigma < 0 \Rightarrow 0 < r < 1$

$\sigma > 0 \Rightarrow r > 1$.

$\sigma = 0 \Rightarrow r = 1$.

LH of S-plane is mapped inside the unitcircle in Z-plane

RH of S-plane is mapped outside the unitcircle in Z-plane.

– the $j\Omega$ axis is mapped onto the unitcircle in Z-plane

→ mapping $\omega = \Omega T$ implies that the interval $-\pi/T \leq \Omega \leq \pi/T$ maps into the corresponding values of $-\pi \leq \omega \leq \pi$.

→ In general, any freq. interval $(2k-1)\pi/T \leq \Omega \leq (2k+1)\frac{\pi}{T}$ where k is an integer, will also map into the interval $-\pi \leq \omega \leq \pi$ in the Z-plane

→ Thus the mapping from the ana. freq. Ω to the digital freq. ω is many to one. → which (dig. domain) reflects the effects of aliasing due to sampling of the impulse response.

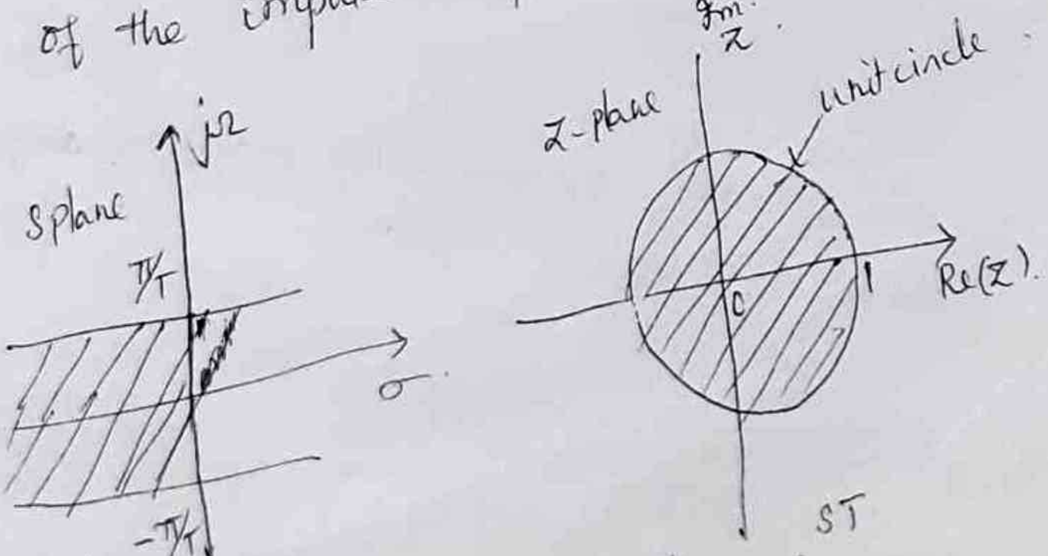

Fig. mapping of $z = e^{sT}$

Some of the properties of the Imp Inv. t/f

$$\frac{1}{(S+S_i)^m} \rightarrow \frac{(-1)^{m-1}}{(m-1)!} \frac{d^{m-1}}{ds^{m-1}} \left[\frac{1}{1-e^{-sT} z^{-1}} \right] ; \quad S \rightarrow$$

$$\frac{S+a}{(S+a)^2+b^2} \rightarrow \frac{1-e^{-aT}(\cos bT) z^{-1}}{1-2e^{-aT}(\cos bT) z^{-1} + e^{-2aT} z^{-2}}$$

$$\frac{b}{(S+a)^2+b^2} \rightarrow \frac{e^{-aT}(\sin bT) z^{-1}}{1-2e^{-aT}(\cos bT) z^{-1} + e^{-2aT} z^{-2}}$$

from eqn ①, In H...

1. If $\sigma < 0$, analog pole $\underline{S}$ lie on $\underline{LH}$ of S-plane. The corresponding digital pole $\underline{z}$ will lie $|z|<1$. hence the corresponding digital pole inside the unit circle in z-plane.

2. If $\sigma > 0$, S lie on the RH of S-plane. $|z|>1$, hence z will lie outside the unit circle in z-plane.

3. If $\sigma = 0$, S lie on Imaginary axis of S-plane. $|z|=1$. hence z will lie on the unit circle in z.

→ above pts. are applicable for mapping any point S-plane to z-plane.

the analog tr fn.

$$H(S) = \frac{1}{(S+1)(S+2)}$$. Determine $H(z)$ using impulse inv tech. Assume $T = 1S$ & $0.1sec$

Soln:

$$H(S) = \frac{1}{(S+1)(S+2)} = \frac{A}{S+1} + \frac{B}{S+2}$$

$$H(S) = \frac{A(S+2) + B(S+1)}{(S+1)(S+2)}$$

$$1 = A(S+2) + B(S+1)$$

letting $S = -2$, $B = -1$

$S = -1$, $A = 1$.

$$\therefore H(S) = \frac{1}{S+1} - \frac{1}{S+2}$$

$$H(z) = \frac{1}{1-e^{-T}z^{-1}} - \frac{1}{1-e^{-2T}z^{-1}}$$

$$H(z) = \frac{z^{-1}\left[e^{-T} - e^{-2T}\right]}{1-\left[e^{-T}+e^{-2T}\right]z^{-1} + e^{-3T}z^{-2}}.$$

Since $T = 1sec$

$$H(z) = \frac{0.2326z^{-1}}{1-0.5032z^{-1}+0.0498z^{-2}}$$

when $T = 0.1sec$

$$H(z) = \frac{0.142z^{-1}}{(1-0.9048z^{-1})(1-0.818z^{-1})}$$

$$\frac{0.086z^{-1}}{1-1.7235z^{-1}+0.7408z^{-2}}$$

2. Convert the analog filter into a digital filter whose sys. fn. is

$$H(S) = \frac{S+0.2}{(S+0.2)^2+9}$$

Use imp. inv tech

Assume $T = 1$ sec

the gn sys. is in the form of

$$H(S) = \frac{S+a}{(S+a)^2+b^2}$$

where $a = 0.2$

$b = 3$.

∴ The sys. response of the digital filter can be obtained using

$$H(z) = \frac{1-e^{-aT}(\cos bT)z^{-1}}{1-2e^{-aT}(\cos bT)z^{-1}+e^{-2aT}z^{-2}}$$

$$= \frac{1-e^{-0.2T}(\cos 3T)z^{-1}}{1-2e^{-0.2T}(\cos 3T)z^{-1}+e^{-0.4T}z^{-2}}$$

$T = 1$ sec

$$H(z) = \frac{1-0.8187(-0.99)z^{-1}}{1-2(0.8187)(-0.99)z^{-1}+0.6703z^{-2}}$$

$$H(z) = \frac{1+0.8105z^{-1}}{1+1.6210z^{-1}+0.6703z^{-2}}$$

Determine $H(z)$ using the impulse inv tech for the analog sys. fn

$$H(S) = \frac{1}{(S+0.5)(S^2+0.5S+2)}$$

Soln:

$H(S)$ can be written as

$$H(S) = \frac{1}{(S+0.5)(S^2+0.5S+2)}$$

$$H(S) = \frac{A}{S+0.5} + \frac{BS+C}{S^2+0.5S+2}.$$

$$A(S^2+0.5S+2) + (BS+C)(S+0.5) = 1$$

Comparing the coeffs. of S^2, S & the constants on either side

$$A+B = 0.$$

$$0.5A + 0.5B + C = 0$$

$$2A + 0.5C = 1.$$

$$A = 0.5, \quad B = -0.5 \quad \& \quad C = 0.$$

$$H(S) = \frac{0.5}{S+0.5} - \frac{0.5S}{S^2+0.5S+2}$$

$$= \frac{0.5}{S+0.5} - 0.5\left[\frac{S}{(S+0.25)^2+(1.39)^2}\right]$$

$$= \frac{0.5}{S+0.5} - 0.5\left[\frac{S+0.25}{(S+0.25)^2+(1.39)^2} - \frac{0.25}{(S+0.25)^2+(\ldots}\right.$$

$$= \frac{0.5}{S+0.5} - 0.5\left[\frac{S+0.25}{(S+0.25)^2+(1.39)^2}\right]$$

$\frac{(0.25}{1.39} \times 1.39)$

$$+ 0.0898\left[\frac{1.39}{(S+0.25)^2+(1.39)^2}\right]$$

$$H(z) = \frac{0.5}{1-e^{-0.5T}z^{-1}} - 0.5\left[\frac{1-e^{-0.25T}(\cos 1.39T)z^{-1}}{1-2e^{-0.25T}(\cos 1.39T)z^{-1} + e^{-0.5T}z^{-2}}\right]$$

$$+ 0.0898\left[\frac{e^{-0.25T}(\sin 1.39T)z^{-1}}{1-2e^{-0.25T}(\cos 1.39T)z^{-1}+e^{-0.5T}z^{-2}}\right]$$

$T = 1\text{sec}$

$$H(z) = \frac{0.5}{1-0.6065z^{-1}} - 0.5\left[\frac{1-\overset{0.77}{0.1385}z^{-1}}{1+\underset{1.54}{0.277}z^{-1}+0.606z^{-2}}\right]$$

$$+ 0.0898\left[\frac{\overset{0.0187}{0.7663}z^{-1}}{1-\underset{1.54}{0.277}z^{-1}+0.606z^{-2}}\right]$$

$$H(z) = \frac{0.3032\overset{z^{-1}}{z^2} + 0.2194\overset{z^{-2}}{z}}{\ldots}$$

IIR Filter design by the Bilinear transformation

The IIR filter design using impulse Inv. method is appropriate for the design of low pass filters & BP filters whose resonant freqs are low

- that method is not suitable for HP or BS filters. this limitation is overcome in the mapping tech. called the Bilinear transformation

→ this transformation is a one - to - one mapping from S- domain to the Z- domain

ie the Bilinear transformation is a conformal mapping that transforms the $j\Omega$ axis into the unit circle in the Z-plane only once, thus avoiding aliasing of freq. components.

$H(s) = \frac{0.5}{} - 0.5$

- also the transformation of a stable analog filter results in a stable digital filter as all the poles in the left half of S-plane are mapped inside the unit circle of the Z-domain.

- the Bilinear transformation is obtained by using the trapezoidal formula for numerical integration.

- Let the sys. fn. of the analog

$$H(s) = \frac{b}{s+a}.$$

- the differential eqn. describing can be obtained from eqn. ①

$$H(s) = \frac{Y(s)}{X(s)} = \frac{b}{s+a}$$

$$sY(s) + aY(s) = bX(s).$$

- taking inv. Laplace t/f

$$\frac{dy(t)}{dt} + a\,y(t) = b \quad \text{—③}$$

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Eqn ③ is integrated b/w the limits $(nT-T)$ and nT.

$$\int_{nT-T}^{nT} \frac{dy(t)}{dt}\,dt + a\int_{nT-T}^{nT} y(t)\,dt = b\int_{nT-T}^{nT} x(t)\,dt \quad \text{—— ④}$$

The trapezoidal rule for numeric integration is gn. by

$$\boxed{\int_{nT-T}^{nT} a(t)\,dt = \frac{T}{2}\left[a(nT) + a(nT-T)\right]}$$

Applying trapezoidal rule in eqn. ④.

$$y(nT) + y(nT-T) + \frac{aT}{2}y(nT) + \frac{aT}{2}y(nT-T) = \frac{bT}{2}x(nT) + \frac{bT}{2}x(nT-T)$$

$$y(n) - y(n-1) + \frac{aT}{2}y(n) + \frac{aT}{2}y(n-1) = \frac{bT}{2}x(n) + \frac{bT}{2}x(n-1)$$

taking Z-transform, the sys fn. of the digital filter is

$$\cancel{H(z)} = \frac{Y(z)}{X(z)}$$

$$Y(z) - z^{-1}Y(z) + \frac{aT}{2}Y(z) + \frac{aT}{2}z^{-1}Y(z) = \frac{bT}{2}X(z) + \frac{bT}{2}z^{-1}X(z)$$

$0.5 \quad 0.5$

$$Y(z) - z^{-1}Y(z) + \frac{aT}{2}Y(z) + \frac{aT}{2}z^{-1}Y(z) = \frac{bT}{2}X(z) + \frac{bT}{2}z^{-1}X(z)$$

$$Y(z)\left[\frac{1-z^{-1}}{1+z^{-1}}\right] + Y(z)\left[\frac{1+z^{-1}}{1+z^{-1}}\right]\frac{aT}{2} = X(z)\cdot\frac{bT}{2}\left[\frac{1+z^{-1}}{1+z^{-1}}\right]$$

$$Y(z)\left[\frac{aT}{2} + \frac{1-z^{-1}}{1+z^{-1}}\right] = \frac{bT}{2}X(z)$$

$$\frac{Y(z)}{X(z)} = \frac{\frac{bT}{2} \times \frac{2}{T}}{\frac{2}{T} \times \frac{aT}{2} + \frac{1-z^{-1}}{1+z^{-1}} \times \frac{2}{T}}$$

$$\frac{Y(z)}{X(z)} = \frac{b}{\frac{2}{T}\left(\frac{1-z^{-1}}{1+z^{-1}}\right) + a}$$

$$S = \frac{2}{T}\left(\frac{1-z^{-1}}{1+z^{-1}}\right) \quad \text{— (6)}$$

omparing eqns. 1 & 6

$$S = \frac{2}{T}\left(\frac{1-z^{-1}}{1+z^{-1}}\right) = \frac{2}{T}\left(\frac{z-1}{z+1}\right)$$

– the general charact. of the mapping $z = e^{ST}$ can be obtained by substituting $S = \sigma + j\Omega$ and expressing the complex variable z in the polar form as

$$z = r \cdot e^{jw}.$$

$$S = \frac{2}{T}\left(\frac{z-1}{z+1}\right) = \frac{2}{T}\left(\frac{r \cdot e^{jw}-1}{r \cdot e^{jw}+1}\right)$$

commerce

$$S = \frac{2}{T}\left(\frac{r^2-1}{1+r^2+2r\cos w} + j\,\frac{2r\sin w}{1+r^2+2r\cos w}\right)$$

$$\sigma = \frac{2}{T}\left(\frac{r^2-1}{1+r^2+2r\cos w}\right) \quad — 7$$

$$\Omega = \frac{2}{T}\left(\frac{2r\sin w}{1+r^2+2r\cos w}\right) \quad — 8$$

from eqn. 7 it can be noted that

if $r<1$, $\sigma<0$.

$r>1$, $\sigma>0$.

— Thus the L.H of S-plane maps inside unit circle in the z-plane and the tran results in a stable digital system. (8)

∴ consider eqn. (8) for unity magnitude

ie ($r=1$), $\sigma=0$ $\quad \Omega = \frac{2}{T}\left(\frac{\sin\omega}{1+\cos\omega}\right)$

$$\Omega = \frac{2}{T}\left(\frac{\sin\omega}{2\cos^2 \omega/2}\right)$$

$$= \frac{2}{T}\left(\frac{2\sin\omega/2\,\cos\omega/2}{2\cos^2\omega/2}\right)$$

$$\Omega = \frac{2}{T}\tan\omega/2 .$$

(or) $$\boxed{\omega = 2\tan^{-1}\frac{\Omega T}{2}} \quad \text{— (9)}$$

This gives the relationship between the frequencies in the 2-domains.

It can be noted that the entire range in Ω is mapped only once into the range $-\pi \le \omega \le \pi$.

...however, as seen in fig. the mapping is nonlinear & the lower freqns in analog domain are expanded in the digital domain, whereas the higher freqns. are compressed.

This is due to the nonlinearity of the arc tangent fn. & called as freq warping.

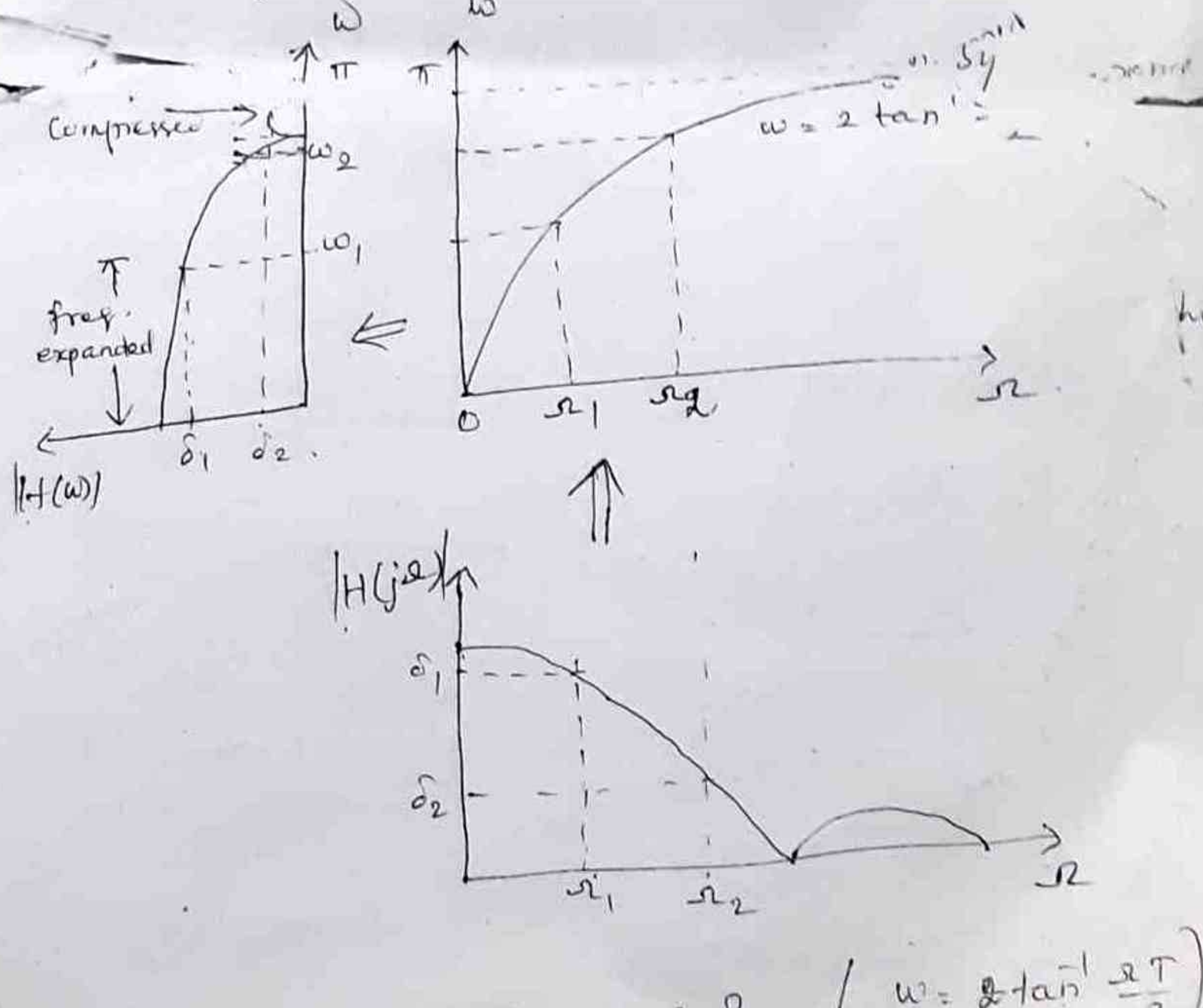
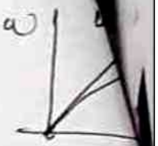


fig: relationship b/w ω & Ω $\left(\omega = 2\tan^{-1}\frac{\Omega T}{2} \right)$

Warping effect

For low Freq. the relation b/w Ω … linear. as a result, the digital filter … same amp. response as the analog filter …

- For HFs, relation b/w ω & Ω becomes non linear. and distortion is introduced in the frq. scale of the digital filter to that of the analog filter.



pre warping:

Warping effect can be eliminated by …warping the analog filter. This can … by ~~finding~~ prewarping analog frequencies using

$$\Omega = \frac{2}{T} \tan \frac{\omega}{2}.$$

$\therefore$ we have

$$\Omega_p = \frac{2}{T} \tan \frac{\omega_p}{2}$$

and

$$\Omega_s = \frac{2}{T} \tan \frac{\omega_s}{2}.$$

digital filter with a 3-dB B.W of 0.25π is to be designed from the analog filter whose sys. response is

$$H(S) = \frac{\Omega_c}{S+\Omega_c}.$$

Using Bilinear t/f. obtain H(Z).

Soln:

$$\Omega_c = \frac{2}{T}\tan\frac{\omega}{2}$$

$$= \frac{2}{T}\tan 0.125\pi = 0.828/T$$

response of the dig. filter is gn. by

$$H(z) = H(s)\Big|_{S=\frac{2}{T}\frac{(z-1)}{(z+1)}}$$

$$= \frac{\Omega_c}{\frac{2}{T}\frac{(z-1)}{z+1}+\Omega_c} = \frac{0.828(z+1)}{2(z-1)+0.828(z+1)}$$

$$= \frac{0.828z+0.828}{2z-2+0.828z+0.828}$$

$$H(z) = \frac{1+z^{-1}}{3.414-1.414z^{-1}}$$

$$= \frac{0.828z+0.828}{2.828z-1.2}$$

$$= \frac{0.828(z+1)}{2.828z-1.2}$$

Warping effect

2) Using Bilinear Hf. obtain H(z)

$H(s) = \frac{1}{(s+1)^2}$. $T = 0.1$ sec.

soln:

$H(z) = H(s) \Big|_{s = \frac{2}{T}\frac{(z-1)}{z+1}}$

$= \frac{1}{\left(\frac{2}{T}\frac{(z-1)}{z+1} + 1\right)^2}$

$T = 0.1$ Sec.

$H(z) = \frac{1}{\left(20\left(\frac{z-1}{z+1}\right) + 1\right)^2} = \frac{(z+1)^2}{(21z-19)^2}$

ur. $H(z) =$

$H(z) = \frac{-}{(1-}$

Bilinear t/f to $H(s) = \frac{2}{(s+1)(s+3)}$, $T=\ldots$

$$H(z) = H(s)\Big|_{s=\frac{2}{T}\left(\frac{z-1}{z+1}\right)}$$

$$= \frac{2}{\left(\frac{2}{T}\frac{(z-1)}{z+1}+1\right)\left[\frac{2}{T}\left(\frac{z-1}{z+1}\right)+3\right]}$$

$$= \frac{2(z+1)^2}{(21z-19)(23z-17)}$$

chara filter

4) Convert the analog filter with sys. fn.

$$H(s) = \frac{s+0.1}{(s+0.1)^2+9}$$ into a digital IIR filter

using Bilinear t/f. The digital filter should have a resonant freq. of $\omega_r = \pi/4$

Soln: from the i/p fn, we note that

$\Omega_c = 3$.

- the sampling period T can be determined

using $\Omega_c = \frac{2}{T} \tan \frac{\omega_r}{2}$.

$$T = \frac{2}{\Omega_c} \tan\frac{\omega}{2} = \frac{2}{3} \tan \pi/8$$

$$= 0.276 \text{ sec}.$$

$$H(z) = H(s)\Big|_{s = \frac{2}{T}\frac{(z-1)}{(z+1)}}$$

$$H(z) = \frac{\frac{2}{T}\frac{(z-1)}{(z+1)} + 0.1}{\left[\frac{2}{T}\left(\frac{z-1}{z+1}\right) + 0.1\right]^2 + 9}$$

$$= \frac{\left(\frac{2}{T}\right)(z-1)(z+1) + 0.1(z+1)^2}{\left[\left(\frac{2}{T}\right)(z-1) + 0.1(z+1)\right]^2 + 9(z+1)^2}$$

$(1-0.904$

$T = 0.276$ sec

$$H(z) = \frac{1 + 0.027z^{-1} - 0.973z^{-2}}{8.572 - 11.84z^{-1} + 8.177z^{-2}}$$

Freq. Trans for Analog filter, Transformation.

Types of trans

LP → LP	$s \to \frac{\Omega_p}{\Omega_p'} s$	$s \to \frac{s}{\Omega_c}$
LP → HP	$s \to \frac{\Omega_p \Omega_p'}{s}$	$s \to \frac{\Omega_c}{s}$
LP → BP	$s \to \Omega_p \frac{s^2 + \Omega_l \Omega_u}{s(\Omega_u - \Omega_l)}$	$s \to \frac{s^2 + \Omega_l \Omega_u}{s(\Omega_u - \Omega_l)}$
LP → BS	$s \to \Omega_p \frac{s(\Omega_u - \Omega_l)}{s^2 + \Omega_u \Omega_l}$	$s \to \frac{s(\Omega_u - \Omega_l)}{s^2 + \Omega_l \Omega_u}$

LP - BP :

$$z^{-1} \rightarrow -\frac{\left[z^{-2} - \frac{2\alpha K}{1+K} z^{-1} + \frac{K-1}{K+1}\right]}{\frac{K-1}{K+1} z^{-2} - \frac{2\alpha K}{K+1} z^{-1} + 1}.$$

$$\alpha = \frac{\cos\left[\frac{(\omega_u + \omega_l)}{2}\right]}{\cos\left[\frac{(\omega_u - \omega_l)}{2}\right]}$$

$$K = \cot\left[\frac{\omega_u - \omega_l}{2}\right] \tan\frac{\omega_p}{2}.$$

$\omega_u \rightarrow$ upper cut off freq.

$\omega_l \rightarrow$ lower cut off freq.

LP - BS

$$z^{-1} \rightarrow \frac{z^{-2} - \frac{2\alpha}{1+K} z^{-1} + \frac{1-K}{1+K}}{\frac{1-K}{1+K} z^{-1} - \frac{2\alpha}{1+K} z^{-1} + 1}$$

$$\alpha = \frac{\cos\left[(\omega_u + \omega_l)/2\right]}{\cos\left[(\omega_u - \omega_l)/2\right]} \quad \Bigg| \quad K = \tan\left[\frac{(\omega_u - \omega_l)}{2}\right] \tan\frac{\omega_p}{2}$$

Freq. transformation in digital domain

A digital LP filter can be converted into a digital HP, BP, & BS or another LP digital filter

LP - LP :

$$z^{-1} \longrightarrow \frac{z^{-1} - \alpha}{1 - \alpha z^{-1}}$$

$\omega_p \rightarrow$ passband freq. of LP

where $\alpha = \dfrac{\sin\left[\dfrac{(\omega_p - \omega_p')}{2}\right]}{\sin\left[\dfrac{(\omega_p + \omega_p')}{2}\right]}$

$\omega_p' \rightarrow$ pass band freq of new filter

LP - HP :

$$z^{-1} = -\left[\frac{z^{-1} + \alpha}{1 + \alpha z^{-1}}\right]$$

$\omega_p \rightarrow$ pass band freq of LP

$$\alpha = -\frac{\cos\left[(\omega_p' + \omega_p)/2\right]}{\cos\left[\dfrac{(\omega_p' - \omega_p)}{2}\right]}$$

$\omega_p' \rightarrow$ pass band freq. of HP filter

Convert the single pole LP filter with system fn.

$$H(z) = \frac{0.5(1+z^{-1})}{1-0.302z^{-1}}$$ into band pass filter

with upper & lower cut off frequ. ω_u & ω_l respectively. The LP filter has 3 dB B.W.

$\omega_p = \frac{\pi}{6}$ & $\omega_u = \frac{3\pi}{4}$. $\omega_l = \pi/4$.

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digital to digital transf. from LP to a BP - filter is

$$z^{-1} \longrightarrow \frac{-\left(z^{-2} - \frac{2\alpha K}{K+1} + \frac{K-1}{K+1}\right)}{\frac{K-1}{K+1} z^{-2} - \frac{2\alpha K}{K+1} z^{-1} + 1}$$

where

$$K = \cot\left[\frac{\omega_u - \omega_l}{2}\right] \tan\frac{\omega_p}{2}$$

$$= \cot\left[\frac{\frac{3\pi}{4} - \frac{\pi}{4}}{2}\right] \tan \pi/12$$

$$= \cot\left(\frac{\pi}{4}\right) \tan\frac{\pi}{12} = 0.268$$

$$\alpha = \frac{\cos \frac{\omega_u + \omega_l}{2}}{\cos \frac{\omega_u - \omega_l}{2}} = \frac{\cos\left[\frac{\frac{3\pi}{4} + \frac{\pi}{4}}{2}\right]}{\cos\left[\frac{\frac{3\pi}{4} - \pi/4}{2}\right]}$$

$$\alpha = \frac{\cos \pi/2}{\cos \pi/4} = 0 .$$

Substituting the values of α and k in the t/f

$$z^{-1} \rightarrow \frac{-\left[z^{-2} + \frac{0.268 - 1}{0.268 + 1}\right]}{\frac{0.268 - 1}{0.268 + 1} z^{-2} + 1}$$

ie
$$z^{-1} \rightarrow \frac{-(z^{-2} - 0.577)}{-0.577 z^{-2} + 1}$$

Now the tr. fn. of BP filter can be obtained by substituting the above t/f. in $H(z)$

$$H(z) = \frac{0.5\left[1 + \frac{-z^{-2} + 0.577}{1 - 0.577z^{-2}}\right]}{1 - 0.302\left[\frac{-z^{-2} + 0.577}{1 - 0.577z^{-2}}\right]}$$

$$= 0.5\left[\frac{1.577(1 - z^{-2})}{0.82575 - 0.275z^{-2}}\right]$$

$$H(z) = \frac{0.955(1 - z^{-2})}{(1 - 0.333z^{-2})}$$