18MAB201T - Transforms and Boundary Value Problems

Assignment - I

Kunal Keshan
RA2011004010051
ECE - A

1. (i) Form the PDE by eliminating f from $xyz = f(x^2+y^2+z^2)$

Soln

Let $xyz = f(x^2+y^2+z^2)$ — ①

diff ① w.r.t x partially.

$\frac{\partial ①}{\partial x}$: $y\frac{dz}{dx} = f'(x^2+y^2+z^2)\cdot(2x+2zp)$

(or) $yp = f'(x^2+y^2+z^2)(2x+2zp)$

$\frac{yp}{(2x+2zp)} = f'(x^2+y^2+z^2)$ — ②

partially diff ① w.r.t y,

$\frac{\partial ①}{\partial y}$ = $xq = f'(x^2+y^2+z^2)\cdot(2y+2zq)$

(or) $\frac{xq}{(2y+2zq)} = f'(x^2+y^2+z^2)$ — ③

Comparing ② and ③, we get.

$\frac{yp}{(2x+2zp)} = \frac{xq}{(2y+2zq)}$ (or) $\boxed{yp(y+zq) = xq(2x+2zp)}$ — ④

Eqn ④ is the required PDE.

(ii) Form the PDE by eliminating ϕ from $\phi(x^2+y^2+z^2, lx+my+nz) = 0$.

Soln.

Let $\phi(x^2+y^2+z^2, lx+my+nz) = 0$ — ①

It is of the form $f(a, b) = 0$.

Let $a = f(b)$, with ①

$x^2+y^2+z^2 = \phi(lx+my+nz)$ — ②

partially diff ② w.r.t x.

$$\frac{\partial ②}{\partial x} = 2x + 2zp = \phi'(lx+my+nz)(l+np)$$

(or)

$$\frac{2x+2zp}{(l+np)} = \phi'(lx+my+nz) \quad -③$$

partially diff ② w.r.t y.

$$\frac{\partial ②}{\partial y} = 2y+2zq = \phi'(lx+my+nz)(m+nq)$$

(or)

$$\frac{2y+2zq}{(m+nq)} = \phi'(lx+my+nz) \quad -④$$

Comparing ③ and ④, we get.

$$\boxed{(x+zp)(m+nq) = (y+zq)(l+np). \quad - ⑤}$$

Eqs ⑤ is the required PDE.

2. (i) Solve $z = px + qy + \sqrt{p^2+q^2+1}$

Soln: Let $z = px + qy + \sqrt{p^2+q^2+1}$ — ①

It is of the form,

~~Pp + Qq = R~~ $z = px + qy + f(p,q)$.

~~where~~

Let $z = ax + by + \sqrt{a^2+b^2+1}$ — ②, be the complete solution,

where $p = a$ and $q = b$ in ①.

Finding Singular Solution,

$$\frac{\partial z}{\partial a} = 0 = x + \frac{1}{2\sqrt{a^2+b^2+1}} \cdot 2a \quad -③ \quad ; \quad \frac{\partial z}{\partial b} = 0 = y + \frac{1}{2\sqrt{a^2+b^2+1}} \cdot 2b \quad -④.$$

From 3 and 4, we get,

$$x = -\frac{a}{\sqrt{a^2+b^2+1}} \quad \text{and} \quad y = \frac{-b}{\sqrt{a^2+b^2+1}}$$

Subtract and square x and y,

$$1 - x^2 - y^2 = 1 - \frac{a^2}{a^2+b^2+1} - \frac{b^2}{a^2+b^2+1}$$

$$1 - x^2 + y^2 = \frac{1}{a^2+b^2+1} \quad - (5) \quad \text{or} \quad a^2+b^2+1 = \frac{1}{1-x^2-y^2}$$

Substitute (5) in x and y,

we get, $a = \frac{-x}{\sqrt{1-x^2+y^2}}$ and $b = \frac{-y}{\sqrt{1-x^2+y^2}}$

Substitute a and b in eqs (2)

$$z = \frac{-x^2}{\sqrt{1-x^2+y^2}} - \frac{y^2}{\sqrt{1-x^2+y^2}} + \frac{1}{\sqrt{1-x^2+y^2}} \quad - (4)$$

Eqs is the Singular Solution.

In Eqs (2), let $b = \phi(a)$.

$$z = ax + \phi(a)y + \sqrt{a^2+\phi^2(a)+1} \quad - (5)$$

partially diff (5), w.r.t a.

$$0 = x + \phi'(a)y + \frac{1}{2\sqrt{a^2+\phi^2(a)+1}}(2a + 2\phi(a)\phi'(a)) \quad - (6)$$

Using (5) and (6), a is eliminated and the General Solution is found.

(ii) Solve $z^2(p^2+q^2+1)=1$ — (1)

Soln

Let $z = ax+by+c$ be the complete soln.

$p=a$ and $q=b$, sub in (1).

$z^2(a^2+b^2+1)=1$ — (2)

$a^2+b^2+1 = \frac{1}{z^2}$

$b^2 = \frac{1}{z^2} - (a^2+1)$ (or)

Simplify with (2)

$z^2 = \frac{1}{(a^2+b^2+1)}$ — (2)

diffe w.r.t a,

$\frac{\partial (2)}{\partial a} = 0 = \frac{-1}{(a^2+b^2+1)} 2a$

diffe w.r.t b,

$\frac{\partial (2)}{\partial b} = 0 = \frac{-1}{(a^2+b^2+1)} 2b$

(ii) Solve $z^2(p^2+q^2+1)=1$

Soln.

Let $z^2(p^2+q^2+1)=1$ — (1), Form, $F(z,p,q)=0$

Assume $z = f(x+ay)$ and $u = x+ay$.

then $p = \frac{dz}{du}$ and $q = a\frac{dz}{du}$

Sub in (1)

$$z^2\left[\left(\frac{dz}{du}\right)^2 + a^2\left(\frac{dz}{du}\right)^2 + 1\right] = 1.$$

$$z^2\left[(1+a^2)\left(\frac{dz}{du}\right)^2 + 1\right] = 1$$

$$(1+a^2)\left(\frac{dz}{du}\right)^2 = \frac{1}{z^2} - 1 \quad ; \quad \left(\frac{dz}{du}\right)^2 = \frac{1-z^2}{z^2(1+a^2)}$$

(or) $$\frac{dz}{du} = \frac{\sqrt{1-z^2}}{z\sqrt{1+a^2}}$$

$$\frac{z}{\sqrt{1-z^2}}\,dz = \frac{1}{\sqrt{1+a^2}}\,du$$

Integrating,

Let $1-z^2 = t$

$\frac{dt}{dz} = 0-2z$

$\frac{-dt}{2z} = dz$

$$-\frac{1}{2}\int \frac{1}{\sqrt{t}}\,dt = \int \frac{1}{\sqrt{1+a^2}}\,du$$

$$-\frac{1}{2}\,\frac{\sqrt{t}}{1/2} = \frac{1}{\sqrt{1+a^2}}\,u + b.$$

$$-\sqrt{1-z^2} = \frac{1}{\sqrt{1+a^2}}(x+ay) + b.$$

$$-\sqrt{1-z^2} = \frac{x+ay}{\sqrt{1+a^2}} + b \quad \text{—} (2)$$

is the Complete Solution

Singular soln.

In (2), partially diff w.r.t b,

$0 = 1$, which is not possible.

Therefore no Singular soln.

General soln.

Put $b = \phi(a)$ in (2)

$$-\sqrt{1-z^2} = \frac{x+ay}{\sqrt{1+a^2}} + \phi(a). \quad \text{—} (3)$$

P.D (3) w.r.t a.

$$0 = \frac{(\sqrt{1+a^2})(a) - (x+ay)\left(\frac{1}{2\sqrt{1+a^2}}\cdot 2a\right)}{1+a^2} + \phi'(a).$$

$$0 = \frac{a\sqrt{1+a^2} - \frac{(x+ay)(a)}{\sqrt{1+a^2}}}{(1+a^2)} + \phi'(a)$$

$$0 = \frac{a(1+a^2) - (x+ay)a}{(1+a^2)^{3/2}} + \phi'(a) - (4)$$

Using (3) and (4) General Solution can be found.

3. (i) Solve $x(y-z)p + y(z-x)q = z(x-y)$

Soln. Lagrange Eqs:

$Pp + Qq = R$ where $P = x(y-z) = xy - xz$

$Q = y(z-x) = yz - yx$

$R = z(x-y) = zx - zy$

Let $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$ and Compare to obtain arbitrary Constant.

$$\frac{dx}{xy-xz} = \frac{dy}{yz-yx} = \frac{dz}{zx-zy}$$

(1.1) (1.2) (1.3)

Method of ~~Grouping~~ Multipliers,

$$\frac{dx + dy + dz}{xy - xz + yz - yx + zx - zy} = 0,$$

$\therefore dx + dy + dz = 0$ — (2)

Integrating (2) we get,

$x + y + z = c_1$

Let another multiplier be, $\frac{1}{x}, \frac{1}{y}, \frac{1}{z}$, respectively

$$\frac{\frac{dx}{x} + \frac{dy}{y} + \frac{dz}{z}}{y - z + z - x + x - y} = 0 \quad — (3)$$

Integrating (3), we get

$\log x + \log y + \log z = \log c_2$

(or)

$xyz = c_2$

Solution = $\phi(u, v) = 0$

$\boxed{\phi(x+y+z, xyz) = 0}$ is the required Solution.

3. (ii) Solve $(mz-ny)p + (nx-lz)q = ly - mx$ — (1)

Soln. It is in the form, $Pp + Qq = R$.

Let,

$P = mz - ny$
$Q = nx - lz$
$R = ly - mx$

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

(or) $$\frac{dx}{mz-ny} = \frac{dy}{nx-lz} = \frac{dz}{ly-mx} \quad -(2)$$

Using Method of Multipliers, and Rule of Ratio,

I Multiplier = l, m, n

(2) becomes,

$$\frac{l\,dx + m\,dy + n\,dz}{lmz - lny + mnx - mlz + lny - mnx} = 0.$$

$\therefore$

$$l\,dx + m\,dy + n\,dz = 0 \quad -(3)$$

Integrating (3), we get

$$lx + my + nz = C_1 \quad -(4)$$

2nd multiplier is x, y, z.

(2) becomes,

$$\frac{x\,dx + y\,dy + z\,dz}{mxz - nxy + nxy - lyz + lyz - mxz} \text{ becomes } = 0.$$

$\therefore$ $$x\,dx + y\,dy + z\,dz = 0 \quad -(5)$$

Integrating (5), we get,

$$\frac{x^2}{2} + \frac{y^2}{2} + \frac{z^2}{2} = k$$

(or) $$x^2 + y^2 + z^2 = C_2 \quad -(6)$$

(7)

The solution of ① is,

$$\boxed{F(lx+my+nz,\ x^2+y^2+z^2)=0}$$

4. (i) Solve $r - 4s + 4t = e^{2x+y}$ — ①

Soln. $r = \frac{\partial^2 z}{\partial x^2}$, $s = \frac{\partial^2 z}{\partial x \partial y}$, $t = \frac{\partial^2 z}{\partial y^2}$

Rearranging ①.

$$[D^2 - 4DD' + 4D'^2]z = e^{2x+y} \quad -②$$

Solution, $z = C.F + P.I$

To find C.F

Let ②, $\phi(D, D') = 0$ and $D = m$, $D' = 1$

$(m^2 - 4m + 4) = 0$ — ③, solving ③.

the roots of m are, $m = 2, 2$.

The roots are repeating,

$C.F = \phi_1(y+2x) + x\phi_2(y+2x)$ — ④

To find P.I

$$P.I = \frac{e^{2x+y}}{\phi(D,D')} = \frac{e^{2x+y}}{D^2 - 4DD' + 4D'^2} \quad -⑤$$

Let $D = 2$, Coefficient of x
$D' = 1$, Coefficient of y

$$P.I = \frac{e^{2x+y}}{4 - 4(2) + 4} = \frac{e^{2x+y}}{0}.$$

Multiply numerator with x, and P.D Denom w.r.t D. in ⑤

$$P.I = \frac{x(e^{2x+y})}{2D - 4D'} \quad -⑥$$, Let $D = 2$
$D' = 1$

$$P.I = \frac{x(e^{2x+y})}{4-4} = x\frac{e^{2x+y}}{0}.$$

Again multiply x in numerator, and P.D Denominator w.r.t D at (6)

$$P.I = \frac{x^2(e^{2x+y})}{2} \quad -(7)$$

with (4) and (7), the Soltion to the PDE is found.

$Z = C.F + P.I$

$$Z = \phi_1(y+2x) + x\,\phi_2(y+2x) + \frac{x^2(e^{2x+y})}{2}$$

4 (ii) Solve $\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial x\,\partial y} - 6\frac{\partial^2 z}{\partial y^2} = y\cos x$

Soln. Let $(D^2 + DD' - 6D'^2)z = y\cos x \quad -(1)$

Solution, $z = C.F + P.I$

To find C.F,

Let $D = m$ and $D' = 1$ in (1) and $\phi(D, D') = 0$

$(m^2 + m - 6) = 0$; $m = 2, -3$

Roots are Distinct.

$C.F = \phi_1(y+2x) + \phi_2(y-3x) \quad -(2)$

To find P.I,

$$P.I = \frac{y\cos x}{\phi(D, D')} = \frac{y\cos x}{D^2 + DD' - 6D'^2}$$

$D^2 + DD' - 6D'^2 \quad (D-2D')(D+3D')$

$$= \frac{y\cos x}{(D-2D')(D+3D')}$$

$$= \frac{1}{D-2D'}\int(a+3x)\cos x \cdot dx \qquad y = a+3x$$

$$= \frac{1}{D-2D'}\left[(a+3x)\sin x - 3\int \sin x\, dx\right]$$

$$= \frac{1}{D-2D'}\left[(a+3x)\sin x + 3\cos x\right]$$

$$= \frac{1}{D-2D'}\left[y\sin x + 3\cos x\right]$$

$$= \int\left[(a-2x)\sin x + 3\cos x\right]dx \quad \text{where } y = a-2x$$

$$= a-2x(-\cos x) - (-2)(-\sin x) + 3\sin x$$

$$= -y\cos x + \sin x = PI$$

Solution $\cancel{Z} = CF + PI$

$$\boxed{z = \phi_1(y+2x) + \phi_2(y-3x) - y\cos x + \sin x}$$

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5. Solve $(D^2 - 2DD' + D'^2)z = x^2y^2e^{x+y} + 2\cos y - \sin(x-y)$. — ①

Soln

To find CF

in ①, Let $\phi(D, D') = 0$ and $D = m$, $D' = 1$.

we get

$$m^2 - 2m + 1 = 0$$

$$\text{or} \quad (m-1)^2 = 0, \quad m = 1, 1.$$

Roots are repetitive,

$$\therefore CF = \phi_1(y+x) + x\phi_2(y+x).$$

To find P.I:

$$PI = PI_1 + PI_2 + PI_3$$

$$PI_1 = x^2y^2e^{x+y}$$

$$PI_2 = 2\cos y$$

$$PI_3 = -(\sin(x-y))$$

Solving $PI_2 = \frac{2\cos y}{\phi(D,D')} \cdot \frac{2\cos y}{D^2+2DD'+D'^2}$

Let $\cos y = \cos(y+0x)$,

$D^2 = -(0^2)$
$D'^2 = -(1^2)$
$DD' = -1\cdot 0 = 0$.

$PI_2 = \frac{2\cos y}{-1} = -2\cos y$.

$-PI_3 = \frac{\sin(x-y)}{D^2-2DD'+D'^2}$

Let $D^2 = -(1^2) = -1$

$D'^2 = -((-1)^2) = -1$.

$DD' = -1\times(-1) = 1$

$-PI_3 = \frac{\sin(x-y)}{-1-2-1} =$ $PI_3 = \frac{\sin(x-y)}{4}$

$PI_1 = \frac{x^2y^2e^{(x+y)}}{D^2-2DD'+D'^2}$

Taking $e^{(x+y)}$ out of operator, and

let $D = D+1$

$D' = D+1$ in e^{x+y} as coefficients de x and y

$= e^{x+y}\left[\frac{x^2y^2}{(D+1)^2-2(D+1)(D'+1)+(D'+1)^2}\right] = PI_1$

Evaluating the Denominator,

$\Rightarrow D^2+2D+1-2[DD'+D+D'+1]+D'^2+2D'+1$

$\Rightarrow D^2+\cancel{2D}+\cancel{1}-2DD'\cancel{-2D}\cancel{-2D'}\cancel{-2}+D'^2\cancel{+2D'}+\cancel{1}$

$\Rightarrow D^2-2DD'+D'^2$

$\Rightarrow D^2\left[1-\left(\frac{2D'}{D}+\frac{D'^2}{D^2}\right)\right]$, Substituting this in PI_1

$$= e^{x+y}\left[\frac{1}{D^2}\frac{x^2y^2}{\left(1-\frac{2D'}{D}+\frac{D'^2}{D^2}\right)}\right] = e^{x+y}\left[\frac{1}{D^2}\left[1-\frac{2D'}{D}+\frac{D'^2}{D^2}\right]^{-1}x^2y^2\right]$$

$$= \frac{e^{x+y}}{D^2}\left[1+\frac{2D'}{D}+\frac{D'^2}{D^2}+\left(\frac{2D'}{D}+\frac{D'^2}{D^2}\right)^2+\cdots\right]x^2y^2$$

$$= \frac{e^{x+y}}{D^2}\left[x^2y^2+\frac{2\cdot 2y}{D}x^2+\frac{2}{D^2}x^2+\frac{4D'^2}{D^2}x^2y^2+\cdots\right]$$

$$= \frac{e^{x+y}}{D^2}\left[x^2y^2+\frac{4yx^3}{3}+\frac{2\cdot x^4}{3\cdot 4}+\frac{4\cdot 2\, x^4}{3\cdot 4}\right]$$

$$= e^{x+y}\left[\frac{x^4y^2}{3\cdot 4}+\frac{4\cdot y x^5}{3\cdot 4\cdot 5}+\frac{x^6}{6\cdot 5\cdot 6}+\frac{2x^6}{3\cdot 5\cdot 6}\right]$$

$$= e^{x+y}\left[\frac{x^4y^2}{12}+\frac{x^5y}{15}+\frac{x^6}{180}+\frac{x^6}{45}\right]$$

$$= e^{x+y}\left[\frac{x^4y^2}{12}+\frac{x^5y}{15}+\frac{5x^6}{180}\right] = P.I_1$$

Solution,

$$Z = C.F + P.I$$

$$\boxed{Z = \phi_1(y+x) + x\phi_2(y+x) - 2\cos y + \frac{\sin(x-y)}{4} + e^{x+y}\left[\frac{x^4y^2}{12}+\frac{x^5y}{15}+\frac{5x^6}{180}\right]}$$