2) Calculate $\iint r^3 \, dr \, d\theta$ over the area included b/w the circles $r = 2\sin\theta$, $r = 4\sin\theta$

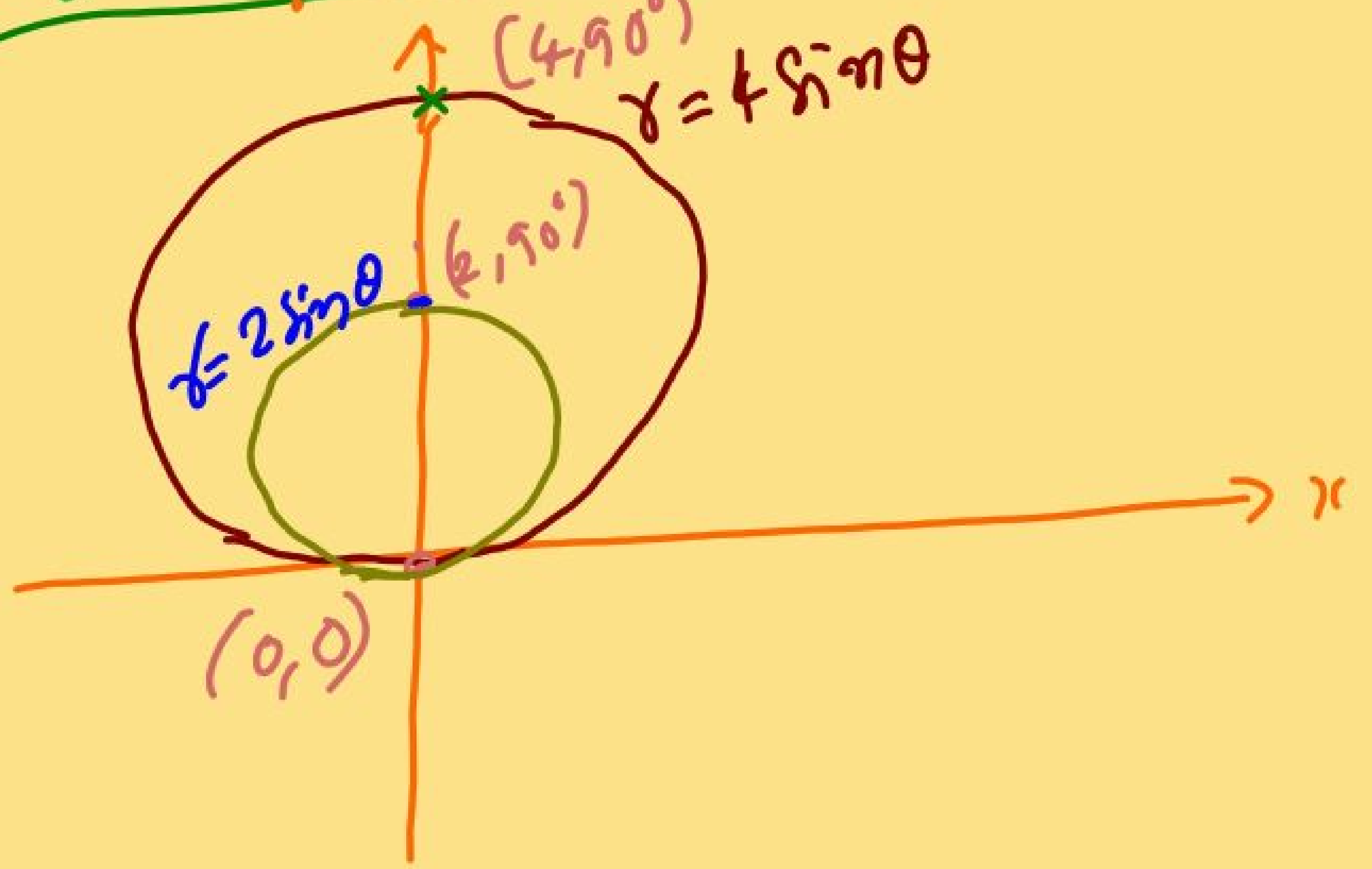
Sol:- Given circles $r = 2\sin\theta$ and $r = 4\sin\theta$

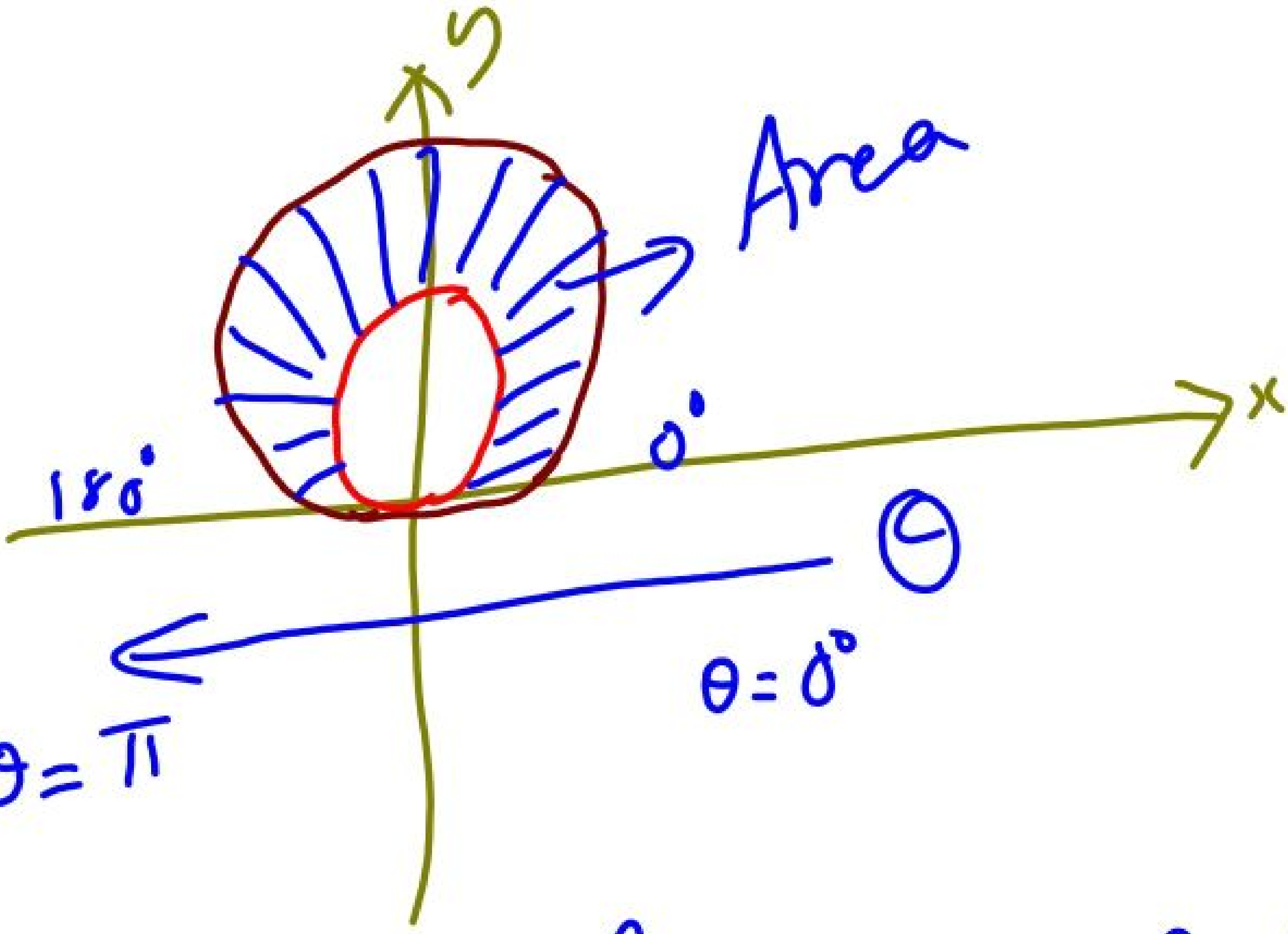
$r = 2\sin\theta$

θ	0	30°	90°
r	0	1	2

$r = 4\sin\theta$

θ	0	30°	90°
r	0	2	4





$\therefore$ r is varies from $2\sin\theta$ to $4\sin\theta$

and θ is varies from 0 to π

$$\iint r^3\,dr\,d\theta = \int_{\theta=0}^{\pi}\int_{r=2\sin\theta}^{4\sin\theta} r^3\,dr\,d\theta$$

$$= \int_{\theta=0}^{\pi} \left(\frac{r^4}{4}\right)_{2\sin\theta}^{4\sin\theta} d\theta$$

$$= \int_{\theta=0}^{2\left(\frac{\pi}{2}\right)} \left[\frac{4^4 \sin^4\theta}{4} - \frac{2^4 \sin^4\theta}{4}\right] d\theta$$

$$= \left(\frac{4^4}{4} - \frac{2^4}{4}\right) 2 \int_{\theta=0}^{\frac{\pi}{2}} \sin^4\theta \, d\theta$$

$$= (64 - 4)\, 2 \left(\frac{3}{4} \times \cancel{\frac{2}{3}} \times \frac{1}{2} \cdot \frac{\pi}{2}\right)$$

$$\boxed{\int_0^{\frac{\pi}{2}} \sin^n\theta \, d\theta = \frac{n-1}{n} \times \cancel{\frac{n-2}{n-1}} \cdots\cdots \cancel{\frac{2}{3}} \times \frac{1}{2} \cdot \frac{\pi}{2}.}$$

n even

$$= \cancel{60}^{\cancel{30}\,15} \times \cancel{2} \times \frac{3}{\cancel{4}_2} \times \frac{1}{\cancel{2}} \cdot \frac{\pi}{\cancel{2}_1}$$

$$= 22.5\,\pi$$

— o —

$$\int_0^{\frac{\pi}{2}} \sin^6 x = \frac{5}{6} \times \frac{3}{4} \times \frac{1}{2} \cdot \frac{\pi}{2}$$

$$\int_0^{\frac{\pi}{2}} \sin^8 x = \frac{7}{8} \times \frac{5}{6} \times \frac{3}{4} \times \frac{1}{2} \quad \frac{\pi}{2}$$

$$\int_0^{2a} f(x)\, dx = 2 \int_0^{a} f(x)\, dx$$

$$\boxed{f(2a+0-x) = f(x)}$$

$$\int_0^{\pi} \sin^4 x \, dx = \int_0^{\pi} \sin^4(\pi+0-x)\, dx$$

$$= \int_0^{\pi} \sin^4(\pi-x)\, dx$$

$$= \int_0^{2\left(\frac{\pi}{2}\right)} \sin^4 x \, dx$$

$$= 2 \int_0^{\frac{\pi}{2}} \sin^4 x \, dx$$

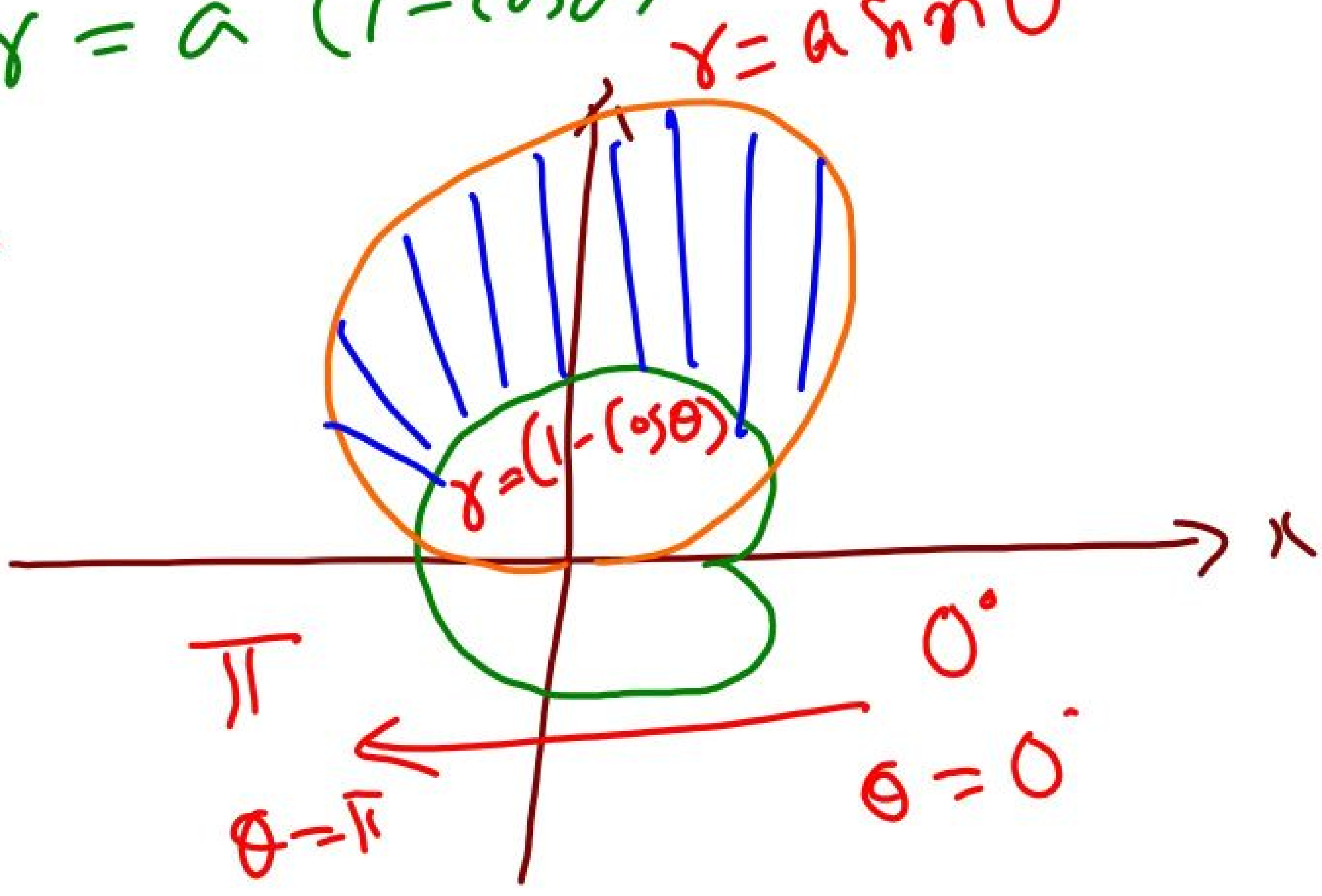
3) Evaluate $\iint r^3 \, dr \, d\theta$

over the area bounded b/w the circles $r = 2\cos\theta$ and $r = 4\cos\theta$

H.W

4) find the area lying inside b/w circle $r = a\sin\theta$ and outside the cardioid $r = a(1-\cos\theta)$

Sol:-



$$\int_{\theta}\int_{r} dr\,d\theta = \int_{\theta=0}^{\pi}\int_{r=a(1-\cos\theta)}^{r=a\sin\theta} 1\, dr\,d\theta$$

$$= \int_{\theta=0}^{\pi} \left(r\right)_{a(1-\cos\theta)}^{a\sin\theta} d\theta$$

$$= \int_{\theta=0}^{\pi} \left(a\sin\theta - a(1-\cos\theta)\right) d\theta$$

$$= a\left(-\cos\theta\right)_0^{\pi} - a\left(\theta - \sin\theta\right)_0^{\pi}$$

$$= a\left(-\cos\pi - (-\cos 0)\right)$$
$$- a\left(\pi - \sin\pi - 0 + \sin 0\right)$$

$$= a\left(-(-1) + 1\right) - a\left(\pi - 0\right)$$

$$= 2a - a\pi$$
$$= a(2-\pi)$$

Triple integrals

a function $f(x, y, z)$
in 3-D finite region V,

$$\iiint_V f(x, y, z)\, dV$$ is triple integral

$$\int_{x_1}^{x_2} \int_{y_1}^{y_2} \int_{z_1}^{z_2} f(x, y, z)\, dx\, dy\, dz$$

1) Evaluate $$\int_{-1}^{1} \int_{0}^{z} \int_{x-z}^{x+z} (x+y+z)\, dx\, dy\, dz$$

Sol:-

$$\int_{z=-1}^{1}\int_{x=0}^{z}\int_{y=x-z}^{x+z}(x+y+z)\,dx\,dy\,dz$$

$$\int_{z=-1}^{1}\int_{x=0}^{z}\left\{\int_{y=x-z}^{x+z}\left(y+(x+z)\right)dy\right\}dx\,dz$$

$$\int_{z=-1}^{1}\int_{x=0}^{z}\left[\frac{y^2}{2}+(x+z)\,y\right]_{x-z}^{x+z}dx\,dz$$

$$\int_{z=-1}^{1}\int_{x=0}^{z}\left\{\frac{(x+z)^2}{2}+(x+z)(x+z)-\frac{(x-z)^2}{2}-(x+z)(x-z)\right\}dx\,dz$$

$$\int_{z=-1}^{1}\int_{x=0}^{z}\left\{\frac{1}{2}\left((x+z)^2-(x-z)^2\right)+(x+z)^2\right.$$

$$-\left(x^2 - z^2\right)\Big\}\,dx\,dz$$

$$\boxed{\because (a+b)^2 - (a-b)^2 = 4ab}$$

$$\int_{z=-1}^{1}\int_{x=0}^{z}\left(\frac{1}{\cancel{2}}\left(\cancel{4}^{2}xz\right) + \cancel{x^2} + z^2 + 2xz - \cancel{x^2} + z^2\right)dx\,dz$$

$$\int_{z=-1}^{1}\int_{x=0}^{z}\left(4xz + 2z^2\right)dx\,dz$$

$$\int_{z=-1}^{1}\left(4z\left(\frac{x^2}{2}\right) + 2z^2(x)\right)_0^z dz$$

$$\int_{z=-1}^{1}\left(4z\,\frac{z^2}{2} + 2z^2(z) - 0\right)dz$$

$$\int_{z=-1}^{1}\left(2z^3 + 2z^3\right)dz$$

$$\int_{z=-1}^{1} (4z^3)\, dz$$

$$4\left(\frac{z^4}{4}\right)_{-1}^{1}$$

$$4\left(\frac{1^4}{4} - \frac{(-1)^4}{4}\right)$$

$$4\left(\frac{1}{4} - \frac{1}{4}\right)$$

$$4(0)$$

$$0$$

2) $\int_0^a \int_0^b \int_0^c (x^2+y^2+z^2)\,dx\,dy\,dz$

Sol:- $\int_0^a \int_0^b \left((x^2+y^2)(z) + \frac{z^3}{3}\right)_{z=0}^{c} dx\,dy$

$$\int_0^a \int_0^b \left(\left(x^2+y^2\right) c + \frac{c^3}{3} - 0 \right) dx\,dy$$

$$= \int_0^a \int_0^b \left\{ \left[c x^2 + \frac{c^3}{3} \right] + c y^2 \right\} dy\,dx$$

$$= \int_0^a \left[\left(c x^2 + \frac{c^3}{3} \right) (y) + c \frac{y^3}{3} \right]_{y=0}^{b} dx$$

$$= \int_0^a \left(\left(c x^2 + \frac{c^3}{3} \right) b + c \frac{b^3}{3} - 0 \right) dx$$

$$= \int_0^a \left\{ c b x^2 + \left[\frac{b c^3}{3} + \frac{c b^3}{3} \right] \right\} dx$$

$$= \left[c b \left(\frac{x^3}{3} \right) + \left(\frac{b c^3}{3} + \frac{c b^3}{3} \right) (x) \right]_{x=0}^{a}$$

$$= cb\,\frac{a^3}{3} + \left(\frac{bc^3}{3} + \frac{cb^3}{3}\right)a - 0$$

$$= \frac{a^3bc}{3} + \frac{abc^3}{3} + \frac{ab^3c}{3}$$

$$= \frac{abc}{3}\left(a^2 + b^2 + c^2\right)$$

3) $$\int_0^1 \int_{y^2}^1 \int_0^{1-x} x \; dz\,dx\,dy$$

Sol:- $$\int_{y=0}^1 \int_{x=y^2}^1 \int_{z=0}^{1-x} x \; dz\,dy\,dz$$

$$\int_{y=0}^1 \int_{x=y^2}^1 x \left\{\int_{z=0}^{1-x} 1 \; dz\right\} dx\,dy$$

$$\int_{y=0}^1 \int_{x=y^2}^1 x \; (z)_0^{1-x} \; dx\,dy$$

$$\int_{y=0}^{1} \int_{x=y^2}^{1} x\,(1-x-0)\,dx\,dy$$

$$\int_{y=0}^{1} \left\{ \int_{x=y^2}^{1} (x-x^2)\,dx \right\} dy$$

$$\int_{y=0}^{1} \left[\frac{x^2}{2} - \frac{x^3}{3} \right]_{x=y^2}^{1} dy$$

$$\int_{y=0}^{1} \left[\frac{1^2}{2} - \frac{1^3}{3} - \frac{(y^2)^2}{2} + \frac{(y^2)^3}{3} \right] dy$$

$$\int_{y=0}^{1} \left(\frac{3-2}{6} - \frac{y^4}{2} + \frac{y^6}{3} \right) dy$$

$$\left[\frac{1}{6}\,y - \frac{1}{2}\,\frac{y^5}{5} + \frac{1}{3}\,\frac{y^7}{7} \right]_0^1$$

$$\frac{1}{6}(1) - \frac{1}{10}(1)^5 + \frac{1}{21}(1)^7 - 0$$

$$\frac{1}{6} - \frac{1}{10} + \frac{1}{21}$$

$$\begin{array}{r|l} 3 & 6, 10, 21 \\ \hline 2 & 2, 10, 7 \\ \hline & 1, 5, 7 \end{array}$$

$$\frac{35 - 21 + 10}{210}$$

$$\frac{24}{210}$$

$$\frac{4}{35}$$

<u>HW</u>

4) $\int_{-c}^{c} \int_{-b}^{b} \int_{-c}^{c} (x^2 + y^2 + z^2)\, dx\, dy\, dz$

5) $\int_{0}^{a} \int_{0}^{x} \int_{0}^{x+y} e^{x+y+z}\, dz\, dy\, dx$

5) Sol: Given

$$\int_{x=0}^{a}\int_{y=0}^{x}\int_{z=0}^{x+y} e^{x}\cdot e^{y}\cdot e^{z}\, dz\, dy\, dx$$

$$\int_{x=0}^{a} e^{x}\int_{y=0}^{x} e^{y}\left\{\int_{z=0}^{x+y} e^{z}\, dz\right\} dy\, dx$$

$$\int_{x=0}^{a} e^{x}\int_{y=0}^{x} e^{y}\left[e^{z}\right]_{0}^{x+y} dy\, dx$$

$$\int_{x=0}^{a} e^{x}\int_{y=0}^{x} e^{y}\left(e^{x+y}-e^{0}\right) dy\, dx$$

$$\int_{x=0}^{a} e^{\cancel{x}}\int_{y=0}^{x}\left(e^{x}\cdot e^{2y}-e^{y}\right) dy\, dx$$

$$\int_{x=0}^{a} e^{x}\left[e^{x}\frac{e^{2y}}{2}-e^{y}\right]_{y=0}^{x} dx$$

$$\int_{x=0}^{a} e^{x}\left[\frac{e^{x}e^{2x}}{2}-e^{x}-e^{x}\frac{e^{0}}{2}-e^{0}\right]dx$$

$$\int_{x=0}^{a} e^{x}\left[\frac{e^{3x}}{2}-e^{x}-\frac{e^{x}}{2}-1\right]dx$$

$$\int_{x=0}^{a} e^{x}\left[\frac{e^{3x}}{2}-\frac{3e^{x}}{2}-1\right]dx$$

$$\int_{x=0}^{a}\left(\frac{e^{4x}}{2}-\frac{3e^{2x}}{2}-e^{x}\right)dx$$

$$\left(\frac{1}{2}\frac{e^{4x}}{4}-\frac{3}{2}\frac{e^{2x}}{2}-e^{x}\right)_{0}^{a}$$

$$\frac{1}{8}e^{4a}-\frac{3}{4}e^{2a}-e^{a}-\frac{1}{8}e^{0}+\frac{3}{4}e^{0}$$
$$+e^{0}$$

$$\frac{e^{4a}}{8}-\frac{3}{4}e^{2a}-e^{a}-\frac{1}{8}+\frac{3}{4}+1$$

$$\frac{e^{4a}}{8}-\frac{3}{4}e^{2a}-e^{a}+\frac{13}{8}$$

6) $$\int_1^e\int_1^{\log y}\int_1^{e^x}\log z \, dx\, dy\, dz$$

Sol:- Given

$$\int_{y=1}^{e}\int_{x=1}^{\log y}\int_{z=1}^{e^x}\log z \, dz \quad dx\,dy$$

$$\int_{y=1}^{e}\int_{x=1}^{\log y}\left[\log z\int 1\,dz-\int\left(\frac{1}{z}\int 1\,dz\right)dz\right]_{z=1}^{e^x} dx\,dy$$

$$\int_{y=1}^{e}\int_{x=1}^{\log y}\left(\log z\, z-\int 1\,dz\right)_{z=0}^{e^x} dx\,dy$$

$$\int_{y=1}^{e}\int_{x=1}^{\log y}\left(z\left(\log z-1\right)\right)_{z=0}^{e^x} dx\,dy$$

$$\boxed{\int \log x\,dx = x\left(\log x-1\right)}$$

$$\int_{y=1}^{e}\int_{x=1}^{\log y}\left[e^x\left(\log e^x-1\right)-0\right]dx\,dy$$

$$\int_{y=1}^{e}\int_{x=1}^{\log y} e^x\left(x-1\right)\,dx\,dy$$

$$\left(\because \log e^{x} = \log_e e^{x} = x \log_e e = x(1) = x \right)$$

$$\int_{y=1}^{e} \left[(x-1) \int e^{x} dx - \int \left((1-0) \int e^{x} dx \right) dx \right]_{x=1}^{\log y} dy$$

$$\int_{y=1}^{e} \left((x-1) e^{x} - e^{x} \right)_{x=1}^{\log y} dy$$

$$\boxed{\int x e^{x} dx = (x-1) e^{x}}$$

$$\int_{y=1}^{e} \left((\log y - 1) e^{\log y} - e^{\log y} - (1-1) e^{1} + e^{1} \right) dy$$

$$\int_{y=1}^{e} \left(\left(\log y - 1\right) y - y - 0 + e\right) dy$$

$$\boxed{\therefore\ e^{\log x} = e^{\log_e^x} = x}$$

$$\int_{y=1}^{e} \left(y \log y - 2y + e\right) dy$$

$$\left[\log y \int y\, dy - \int \left(\frac{d}{dy}(\log y) \int y\, dy\right) dy - 2\frac{y^2}{2} + ey\right]_{y=1}^{e}$$

$$\left[\log y \frac{y^2}{2} - \int \frac{1}{y} \frac{y^2}{2}\, dy - y^2 + ey\right]_{1}^{e}$$

$$\left[\frac{y^2}{2}\log y - \frac{1}{2}\frac{y^2}{2} - y^2 + ey\right]_1^e$$

$$\left[\frac{y^2}{2}\log y - \frac{5}{4}y^2 + ey\right]_1^e$$

$$\frac{e^2}{2}\log e - \frac{5}{4}e^2 + e(e) - \frac{1^2}{2}\log 1$$

$$- \frac{5}{4}(1)^2 + e(1)$$

$$\frac{e^2}{2} - \frac{5}{4}e^2 + e^2 - 0 + \frac{5}{4} + e$$

$$e^2\left(\frac{1}{2} - \frac{5}{4} + 1\right) + e + \frac{5}{4}$$

$$\frac{e^2}{4} + e + \frac{5}{4}$$

$$\frac{1}{4}\left(e^2 + 4e + 5\right)$$

$a^x = n$

$\therefore$

$\log_a^n = x$

$e^0 = 1$

$\log_e^1 = 0$

$a^{\text{anything}} \neq 0$

$\log_a 0 \neq$ anything

$\log 0$ not defined

$e^1 = e \Rightarrow \log_e e = 1$

7) $$\int_0^{\frac{\pi}{2}} \int_0^{a\sin\theta} \int_0^{\frac{a^2-r^2}{a}} r \, dz\, dr\, d\theta$$

Sol:- Given

$$\int_{\theta=0}^{\frac{\pi}{2}} \int_{r=0}^{a\sin\theta} \int_{z=0}^{\frac{a^2-r^2}{a}} r \;\; dz\; dr\; d\theta$$

$$\int_{\theta=0}^{\frac{\pi}{2}} \int_{r=0}^{a\sin\theta} r \left\{ \int_{z=0}^{\frac{a^2-r^2}{a}} (1)\; dz \right\} dr\, d\theta$$

$$\int_{\theta=0}^{\frac{\pi}{2}} \int_{r=0}^{a\sin\theta} r \left(z\right)_{0}^{\frac{a^2-r^2}{a}} dr\, d\theta$$

$$\int_{\theta=0}^{\frac{\pi}{2}} \int_{r=0}^{a\sin\theta} r \left(\frac{a^2-r^2}{a} - 0\right) dr\, d\theta$$

$$\int_{\theta=0}^{\frac{\pi}{2}} \left\{ \int_{r=0}^{a\sin\theta} \left(a^2 r - r^3\right) dr \right\} d\theta$$

$$\int_{\theta=0}^{\frac{\pi}{2}} \left(a^2 \frac{r^2}{2} - \frac{r^4}{4}\right)_{0}^{a\sin\theta} d\theta$$

$$\int_{\theta=0}^{\frac{\pi}{2}} \left(a^2 \frac{a^2\sin^2\theta}{2} - \frac{a^4\sin^4\theta}{4} - 0\right) d\theta$$

$$\frac{a^4}{2}\int_0^{\frac{\pi}{2}} \sin^2\theta \, d\theta - \frac{a^4}{4}\int_0^{\frac{\pi}{2}} \sin^4\theta \, d\theta$$

$$\frac{a^4}{2}\left[\frac{1}{2}\,\frac{\pi}{2}\right] - \frac{a^4}{4}\left[\frac{3}{4}\,\frac{1}{2}\,\frac{\pi}{2}\right]$$

$$\frac{a^4}{4}\,\frac{\pi}{2} - \frac{a^4}{4}\cdot\frac{3}{8}\,\frac{\pi}{2}$$

$$\frac{a^4}{4}\,\frac{\pi}{2}\left(1-\frac{3}{8}\right)$$

$$\frac{a^4}{4}\,\frac{\pi}{2}\left[\frac{5}{8}\right]$$

$$\frac{5a^4\pi}{64}$$

H.W

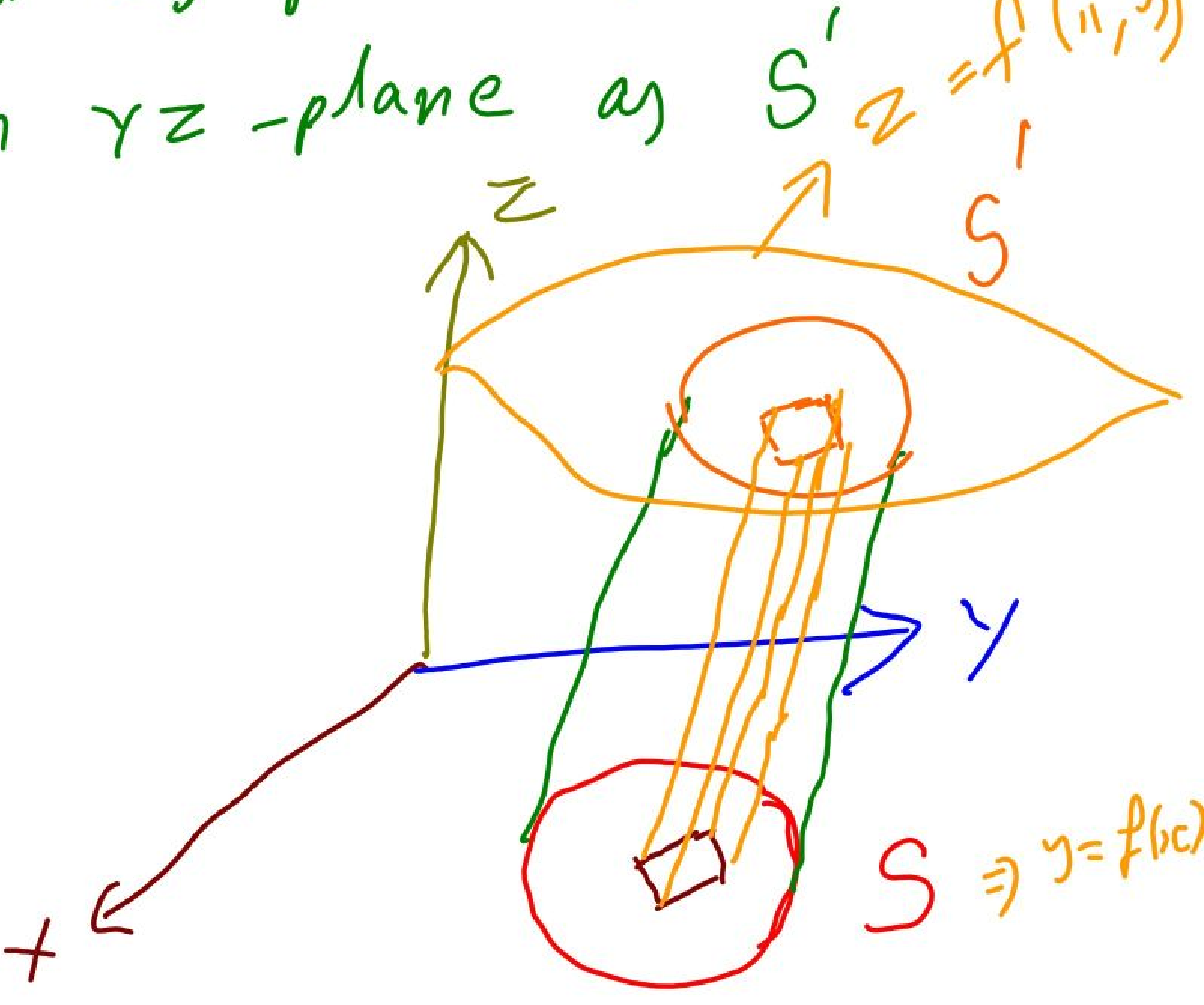
8) $$\int_0^{\log 2}\int_0^{x}\int_0^{x+\log y} e^{x+y+z}\,dx\,dy\,dz$$

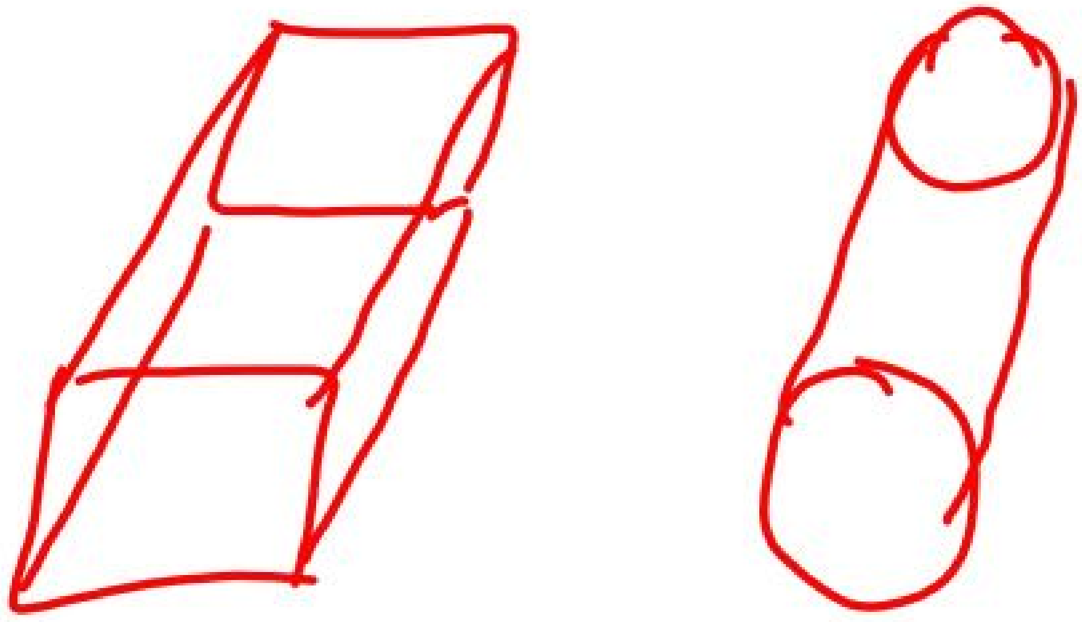
Volumes of solids

By using double integrals

Let a surface is $z = f(x, y)$

Let the image on xz-plane (or) yz-plane, Take the area in xy-plane as S and in yz-plane as S'





The volume of solid cylinder having base 'S' and top 'S'' is bounded by the given surface and the path is parallel to the z-axis

$$\lim_{\substack{\delta x \to 0 \\ \delta y \to 0}} \Sigma' \Sigma' z \, \delta x \, \delta y$$

$$\iint z \, dx \, dy$$

(or)

$$\boxed{\iint f(x, y) \, dx \, dy}$$

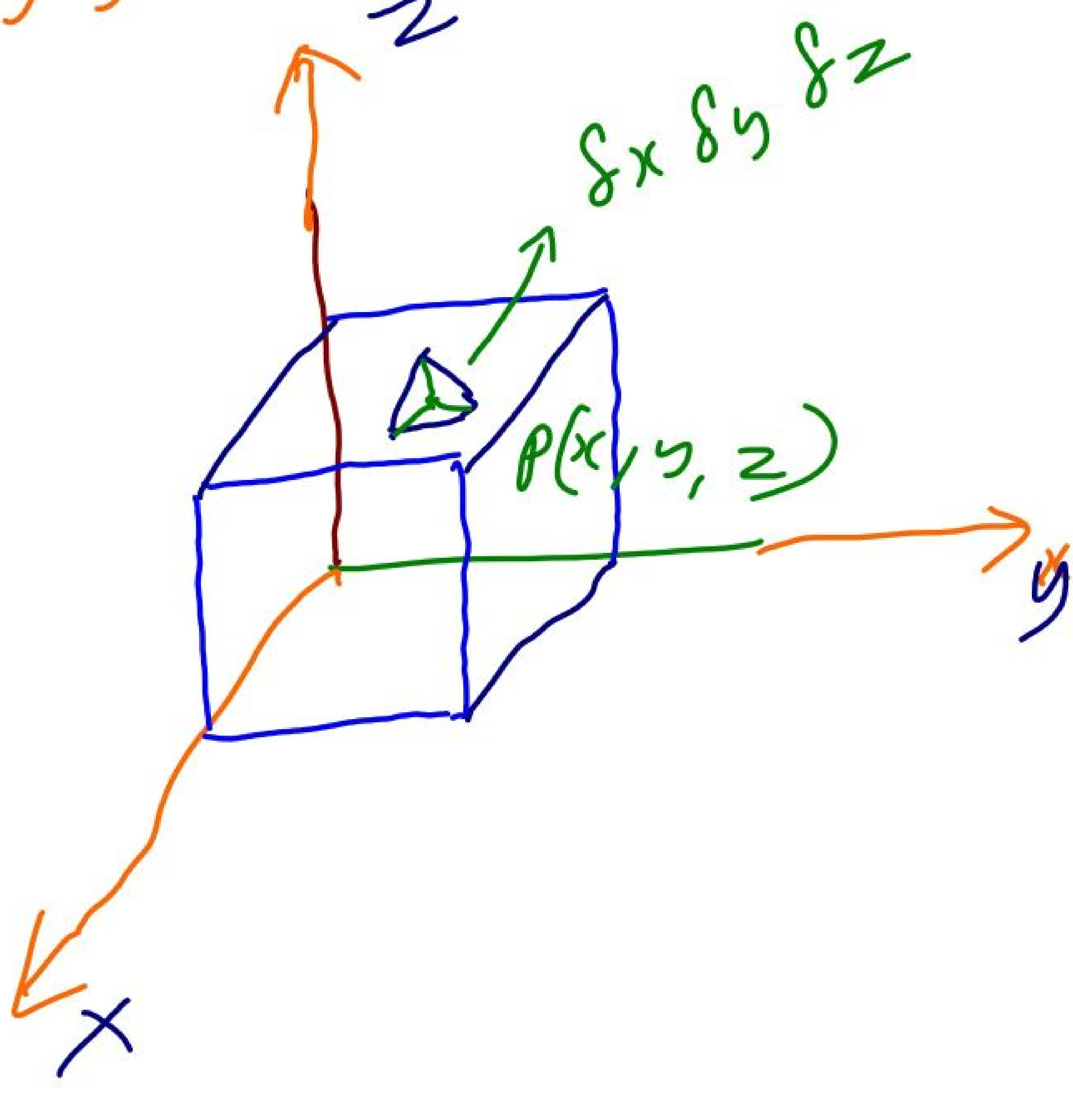
Note:- In polar coordinate system (as $dx\,dy = r\,d\theta\,dr$)

$$\boxed{\iint z \, r \, d\theta \, dr}$$

Volume by using triple Integral

$$\lim_{\substack{\delta x \to 0 \\ \delta y \to 0 \\ \delta z \to 0}} \sum_1 \sum_1 \sum_1 \delta x \, \delta y \, \delta z$$

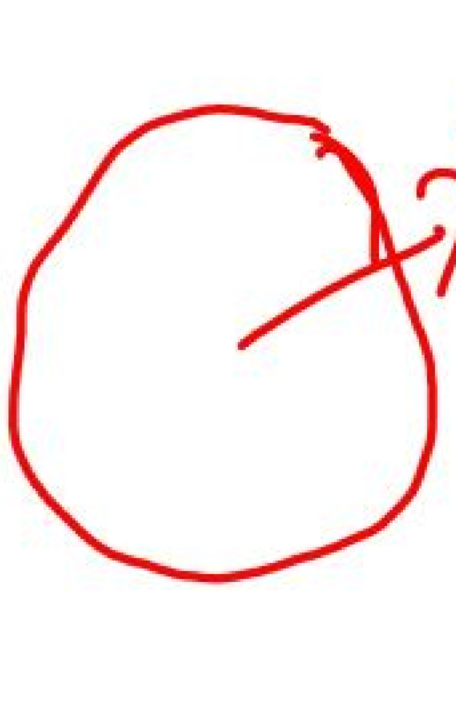
$$= \iiint dx \, dy \, dz$$



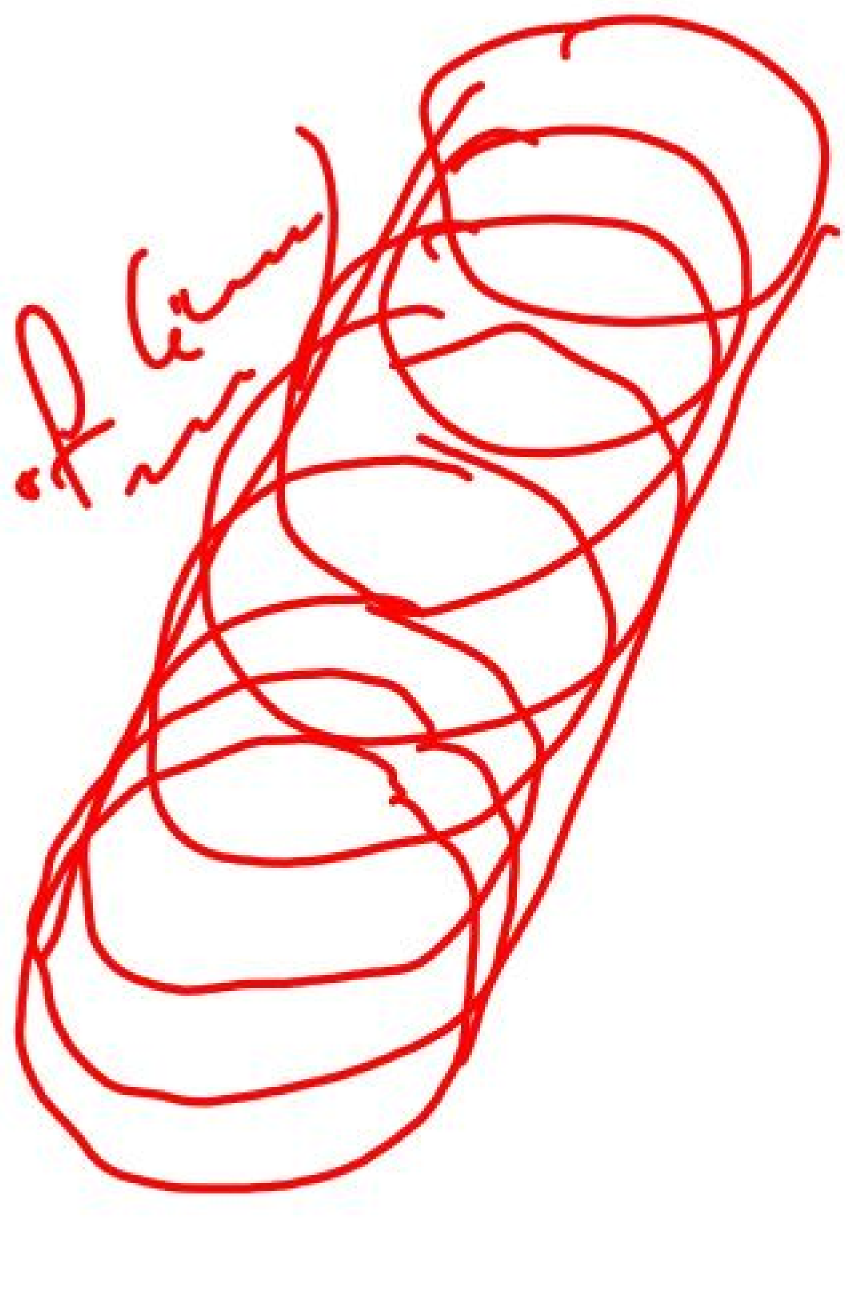
1) Calculate the volume of the solid bounded by the planes $x=0$, $y=0$, $x+y+z=a$ and $z=0$

Sol:-

$\pi r^2 h$

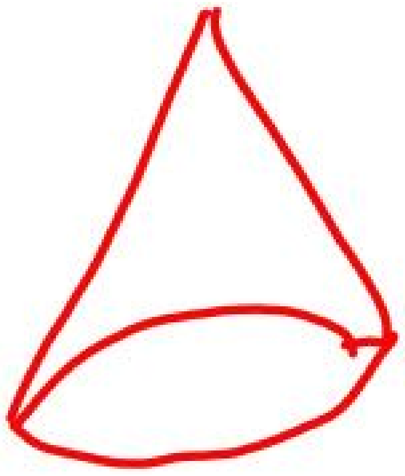


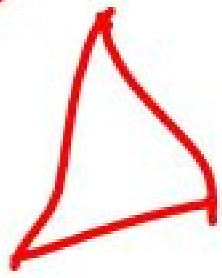
$\rightarrow \pi r^2 \times$ (no. of circles)



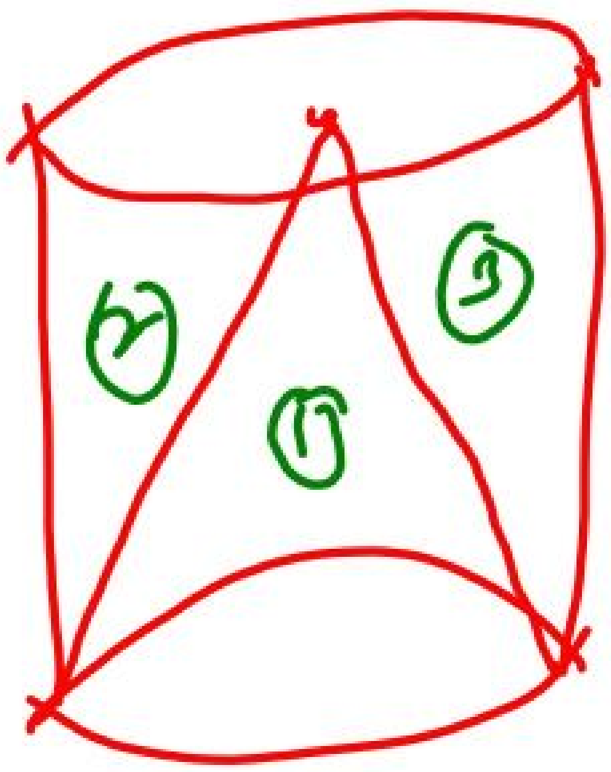
$V = \frac{1}{3}\pi r^2 h$

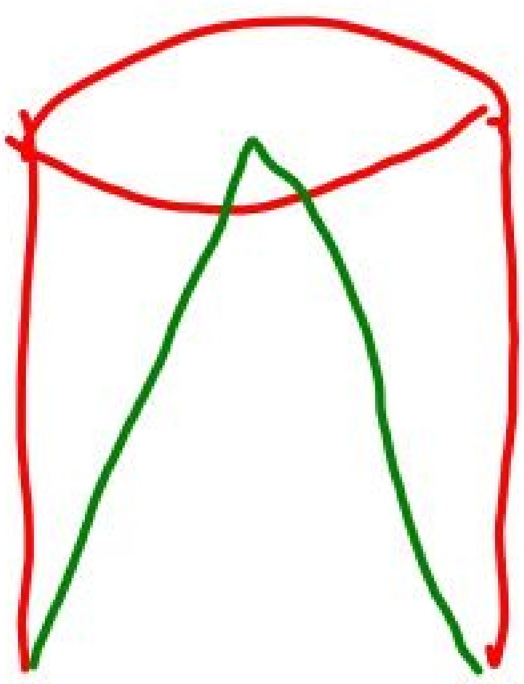
$= \frac{1}{3}$ (cylinder volume)

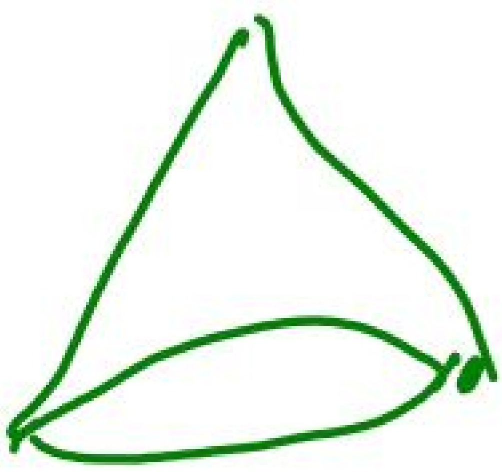


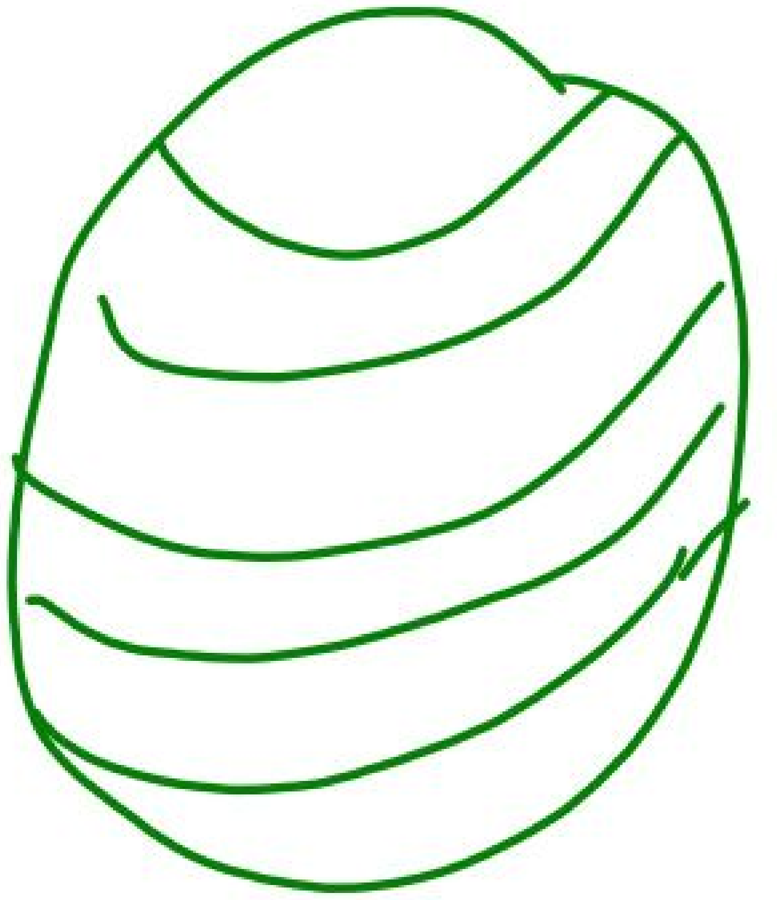
 $\times$

$2\pi r^2 h$

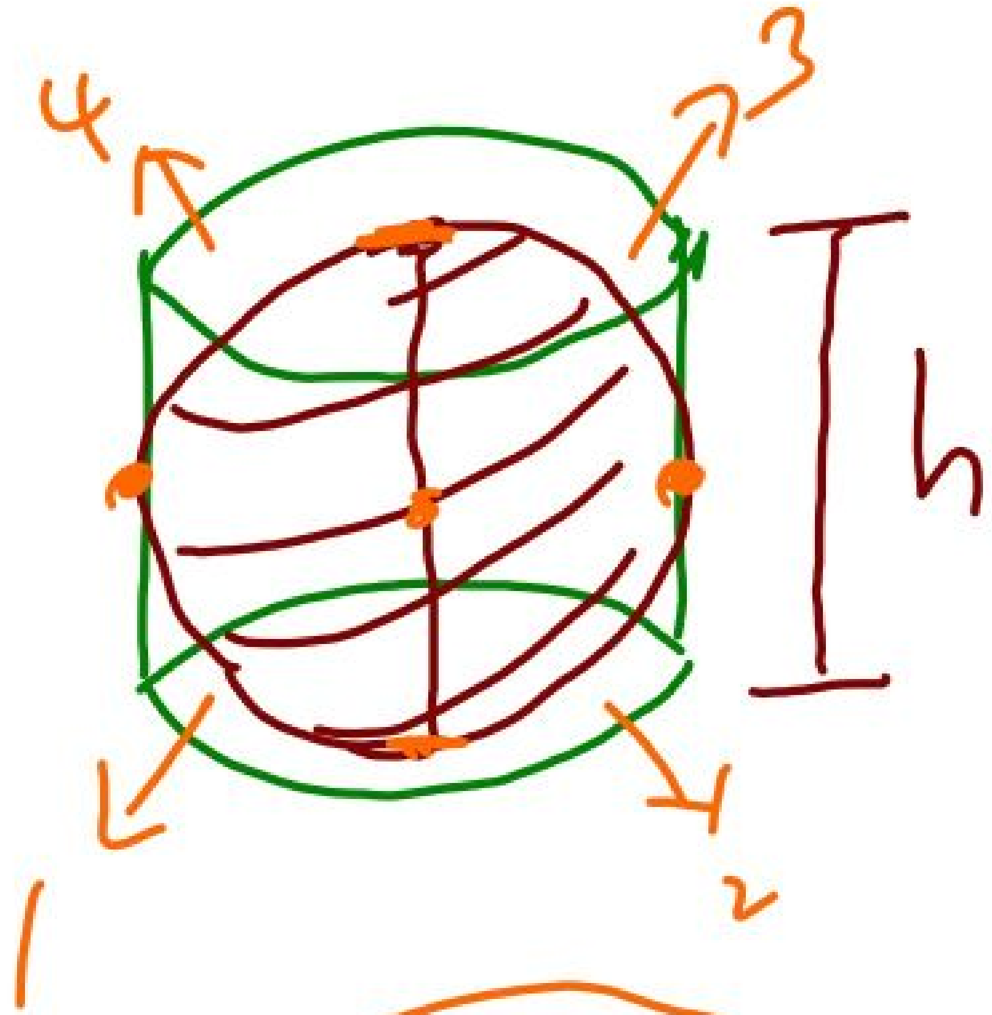


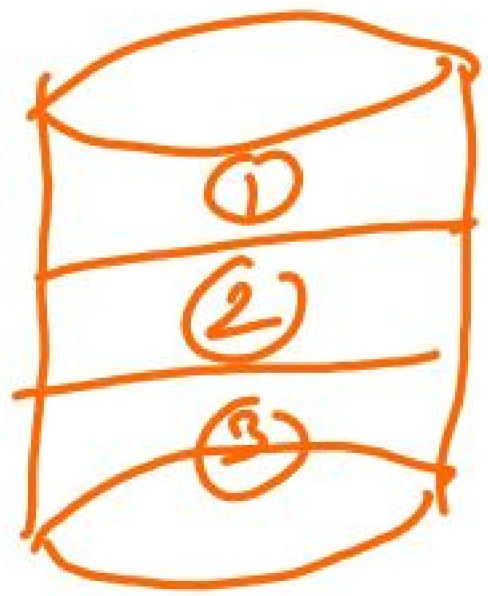






$$V = \frac{4}{3}\pi r^3$$





$\frac{2}{3}$ (cylinder volume)

$$\frac{2}{3}(\pi r^2 h)$$

$$\frac{2}{3}\pi r^2 (r+r)$$

$$\frac{2}{3}\pi r^2 (2r)$$

$$\frac{4}{3}\pi r^3$$

Sol:- Given

Volume of the Solid bound by $x=0$, $y=0$, $x+y+z=a$ and $z=0$

$\because x+y+z=a$

$z = a-x-y$

$\therefore$ z is varies from 0 to $a-x-y$

if $z=0 \Rightarrow 0=a-x-y$

$y=a-x$

$\therefore$ y is varies from 0 to $a-x$

if $y=0 \Rightarrow 0 = a-x$

$x = a$

$\therefore$ x is varies from 0 to a

$x+y+z=a$

Volume of solid $= \iiint dx\,dy\,dz$

$$= \int_{x=0}^{a} \int_{y=0}^{a-x} \int_{z=0}^{a-x-y} dx\,dy\,dz$$

$$= \int_{x=0}^{a} \int_{y=0}^{a-x} (z)_{0}^{a-x-y} \, dy\,dx$$

$$= \int_{x=0}^{a} \int_{y=0}^{a-x} (a-x-y-0) \, dy\,dx$$

$$= \int_{x=0}^{a} \left\{ \int_{y=0}^{a-x} \left((a-x)+y\right) dy \right\} dx$$

$$= \int_{x=0}^{a} \left[(a-x)\,y + \frac{y^2}{2}\right]_0^{a-x} dx$$

$$= \int_{x=0}^{a} \left[(a-x)\,(a-x) + \frac{(a-x)^2}{2} - 0\right] dx$$

$$= \int_{x=0}^{a} \left[(a-x)^2 + \frac{(a-x)^2}{2}\right] dx$$

$$= \left(1+\frac{1}{2}\right) \int_{x=0}^{a} (a-x)^2\, dx$$

$$= \frac{3}{2} \int_{x=0}^{a} (x-a)^2 \, dx$$

$$= \frac{3}{2} \left[\frac{(x-a)^3}{3} \right]_0^a$$

$$= \frac{3}{2} \left[\frac{(a-a)^3}{3} - \frac{(0-a)^3}{3} \right]$$

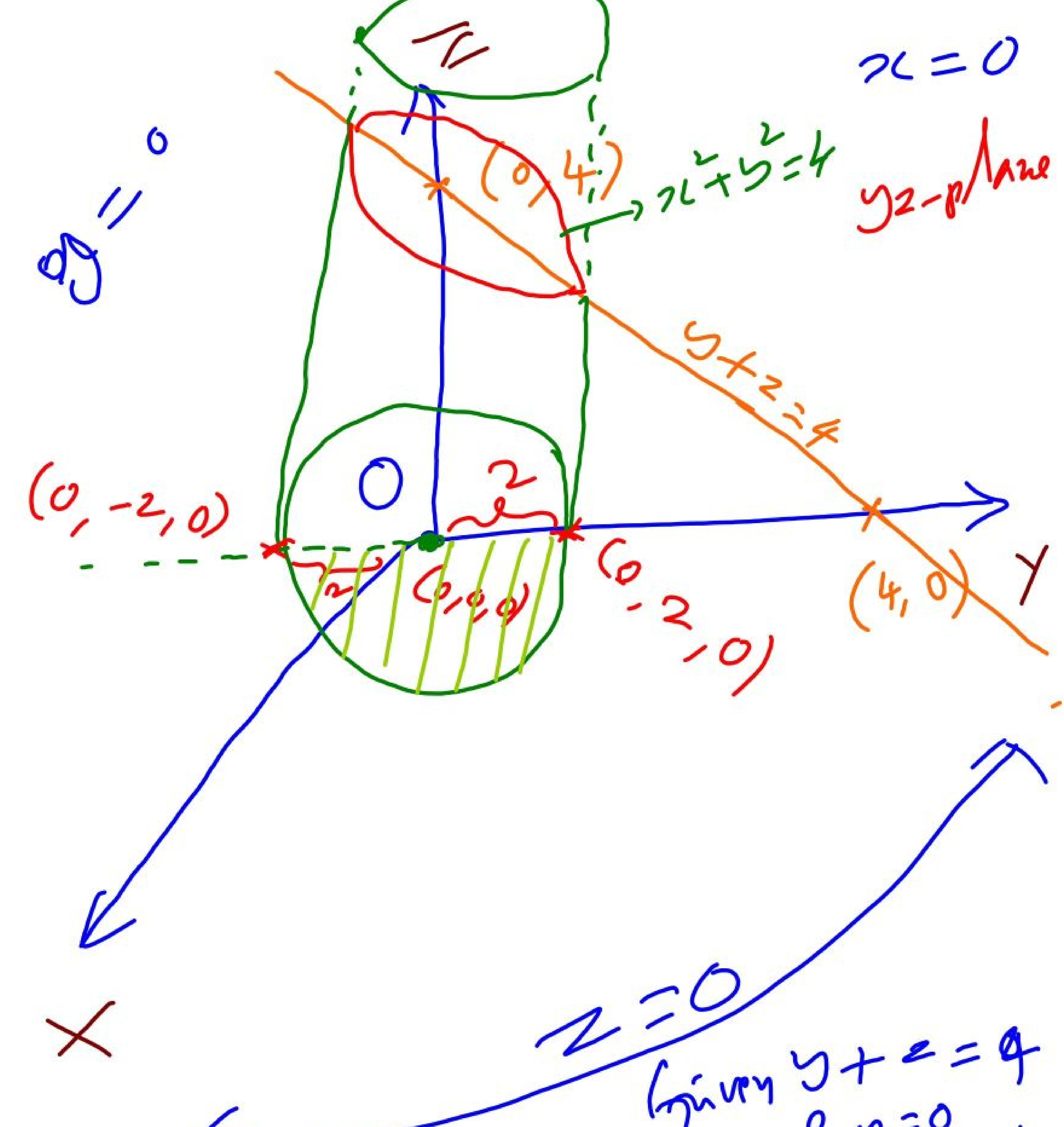
$$= \frac{3}{2} \left[0 - \frac{(-a^3)}{3} \right]$$

$$= \frac{a^3}{2} \, .$$

2) find the volume bounded by the cylinder $x^2+y^2=4$ and planes $y+z=4$ and $z=0$

Sol:- Given Cylinder is

$$x^2 + y^2 = 4 \Rightarrow x^2 + y^2 = 2^2$$



Given $y + z = 4$

if $y = 0$, $z = 4$

if $z = 0$, $y = 4$

$$\iint z\,dx\,dy \Rightarrow$$

$\because\ y + z = 4$

$z = 4 - y$

$\because\ x^2 + y^2 = 4$

if x is dependent and y is independent then $x^2 = 4 - y^2$

$x = \sqrt{4 - y^2}$

$\therefore$ x is varies from 0 to $\sqrt{4 - y^2}$

y is varies from -2 to 2

$$\because \iint z\,dx\,dy = 2\int_{y=-2}^{2}\int_{x=0}^{\sqrt{4-y^2}} (4-y)\,dx\,dy$$

$$= 2\int_{y=-2}^{2} (4-y) \left\{ \int_{x=0}^{\sqrt{4-y^2}} 1 \, dx \right\} dy$$

$$= 2\int_{y=-2}^{2} (4-y) \left[x \right]_0^{\sqrt{4-y^2}} dy$$

$$= 2\int_{y=-2}^{2} (4-y) \left(\sqrt{4-y^2} - 0\right) dy$$

$$= 2\int_{y=-2}^{2} (4-y) \sqrt{4-y^2} \, dy$$

$$= 2\int_{y=-2}^{2} 4\sqrt{4-y^2} \, dy - 2\int_{y=-2}^{2} y\sqrt{4-y^2} \, dy$$

$$\int_{-a}^{a} f(x)\,dx = \begin{cases} 2f(x) \\ 0 \end{cases}$$

even func $f(-x) = f(x)$

odd func $f(-x) = -f(x)$

$f(y) = 4\sqrt{4-y^2}$

$f(-y) = 4\sqrt{4-(-y)^2}$

$= 4\sqrt{4-y^2}$

$= f(y)$

$f(y)$ is even

$f(y) = y\sqrt{1-y^2}$

$f(-y) = -y\sqrt{1-(-y)^2}$

$= \underbrace{-y\sqrt{1-y^2}}$

$= -f(y)$

$f(y)$ is odd

$$= 2\int_{y=-2}^{2} 4\sqrt{4-y^2}\,dy - 2(0)$$

$$= (2\times 4)\int_{y=-2}^{2} \sqrt{4-y^2}\,dy$$

$$\boxed{\because \int \sqrt{a^2-x^2}\,dx = \frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right)}$$

$$= 8\left[\frac{y}{2}\sqrt{2^2-y^2} + \frac{2^2}{2}\sin^{-1}\left(\frac{y}{2}\right)\right]_{y=-2}^{2}$$

$$= 8\left\{\left[\frac{2}{2}\sqrt{2^2-2^2} + \frac{2^2}{2}\sin^{-1}\left(\frac{2}{2}\right)\right.\right.$$
$$- \left[\frac{-2}{2}\sqrt{2^2-(-2)^2} + \frac{2^2}{2}\right.$$

$$= 8\left|\left\{0 + 2\sin^{-1}(1) - \left[0 + 2\sin^{-1}(-1)\right]\right\}\right|$$

$$\sin^{-1}\left(\frac{-2}{2}\right)\Big]\Big\}\Big|$$

$$= 8\left|\left\{2\sin^{-1}\left(\sin\frac{\pi}{2}\right) - 2\sin^{-1}\left(\sin\left(\pi+\frac{\pi}{2}\right)\right)\right\}\right|$$

$$\left(\because \sin\left(\pi+\frac{\pi}{2}\right) = -\sin\frac{\pi}{2} = -1\right)$$

$$= 8\left|\left(\cancel{2}\frac{\pi}{\cancel{2}} - 2\left(\pi+\frac{\pi}{2}\right)\right]\right|$$

$$= \left|8\pi - 16\left(\frac{2\pi+\pi}{2}\right)\right|$$

$$= |8\pi - 8(3\pi)|$$

$$= |8\pi - 24\pi|$$

$= |-16\pi|$

$= 16\pi$

$$\iiint dx\,dy\,dz$$

$z \quad y \quad x \quad x^2+y^2=a^2$

$y+z=4$

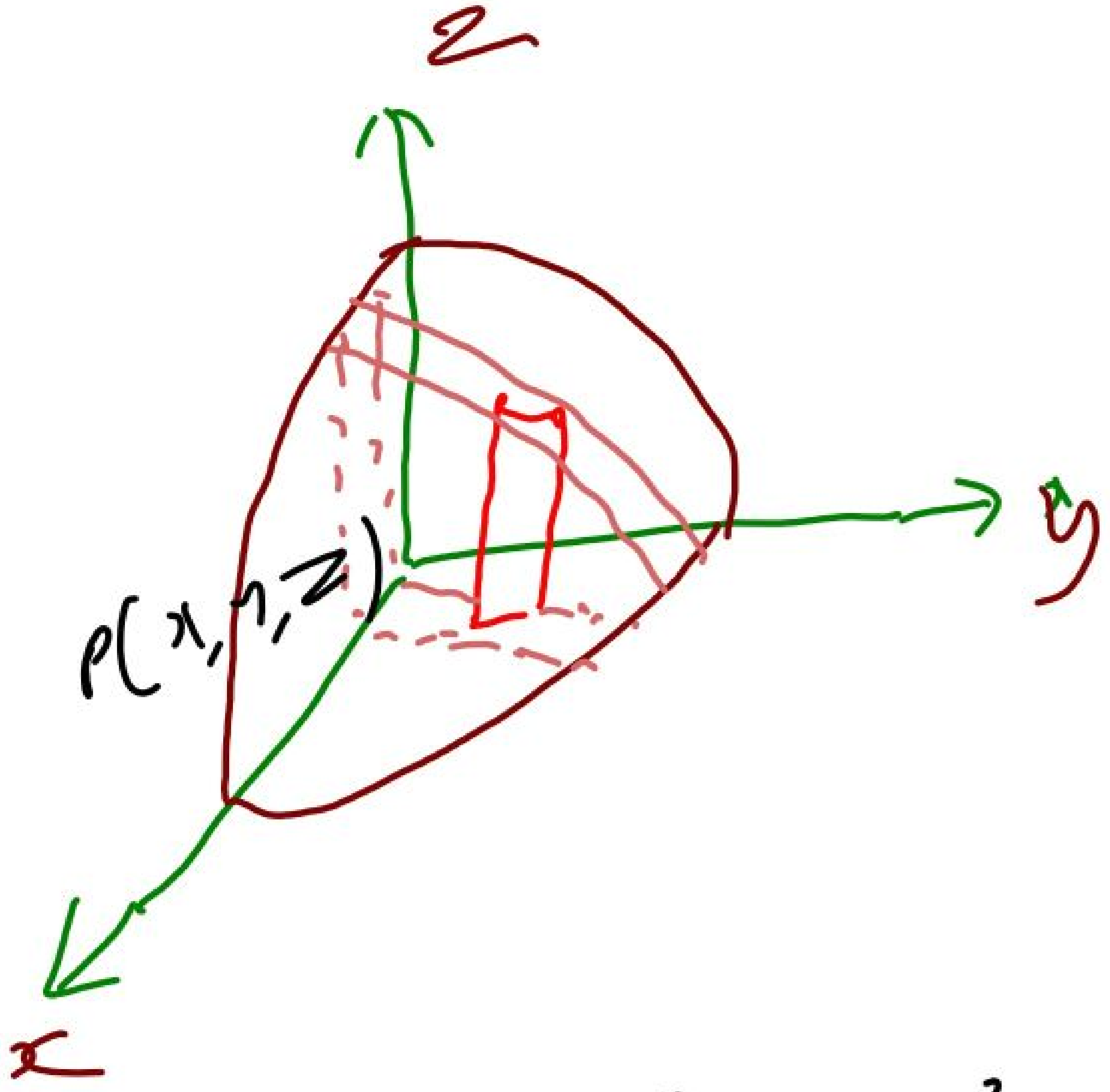
$z=0$

3) find the volume of the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$

Sol:- Given ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$

$$\iiint dx\,dy\,dz = 8\iiint dx\,dy\,dz$$

$$\boxed{\int_{-a}^{a}\int_{-b}^{b}\int_{-c}^{2} = 2\int_0^a 2\int_0^b 2\int_0^c = 8\int_0^a\int_0^b\int_0^c}$$



If $z=0$ then $\frac{z^2}{c^2} = 1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}$

$$z^2 = c^2\left(1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}\right)$$

$$z = \sqrt{c^2\left(1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}\right)}$$

$$= c\sqrt{1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}}$$

z is varying from 0 to $c\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}$

if $z=0$ $\quad c\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}=0$

$$1-\frac{x^2}{a^2}-\frac{y^2}{b^2}=0$$

$$\frac{y^2}{b^2}=1-\frac{x^2}{a^2}$$

$$y^2=b^2\left(1-\frac{x^2}{a^2}\right)$$

$$y=b\sqrt{1-\frac{x^2}{a^2}}$$

y is varies from 0 to $b\sqrt{1-\frac{x^2}{a^2}}$

If $y=0 \Rightarrow b\sqrt{1-\frac{x^2}{a^2}}=0$

$$1-\frac{x^2}{a^2}=0$$

$$1=\frac{x^2}{a^2}$$

$$x^2=a^2$$

$$x=a$$

$\therefore$ x is varies from 0 to a

$$\text{Volume} \iiint dx\,dy\,dz = 8 \int_{x=0}^{a} \int_{y=0}^{b\sqrt{1-\frac{x^2}{a^2}}} \int_{z=0}^{c\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}} dx\,dy\,dz$$

$$= 8 \int_{x=0}^{a} \int_{y=0}^{b\sqrt{1-\frac{x^2}{a^2}}} \left\{ \int_{z=0}^{c\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}} (1)\,dx\,dy\,dz \right\}$$

$$= 8 \int_{x=0}^{a} \int_{y=0}^{b\sqrt{1-\frac{x^2}{a^2}}} \left(z\right)_{0}^{c\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}} dy\,dx$$

$$= 8 \int_{x=0}^{a} \int_{y=0}^{b\sqrt{1-\frac{x^2}{a^2}}} \left(c\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}-0\right)$$

$$= 8c \int_{x=0}^{a} \left\{ \int_{y=0}^{b\sqrt{1-\frac{x^2}{a^2}}} \sqrt{\left(1-\frac{x^2}{a^2}\right)-\frac{y^2}{b^2}}\,dy \right\} dx \qquad dy\,dx$$

$$= 8c \int_{x=0}^{a} \left\{ \int_{y=0}^{b\sqrt{1-\frac{x^2}{a^2}}} \sqrt{\left(1-\frac{x^2}{a^2}\right) - \frac{y^2}{b^2}}\, dy \right\} dx$$

$$= 8c \int_{x=0}^{a} \int_{y=0}^{b\sqrt{1-\frac{x^2}{a^2}}} \sqrt{\frac{1}{b^2}\left(b^2\left(1-\frac{x^2}{a^2}\right) - y^2\right)}\, dy\, dx$$

$$= \frac{8c}{b} \int_{x=0}^{a} \int_{y=0}^{b\sqrt{1-\frac{x^2}{a^2}}} \sqrt{\left(b\sqrt{1-\frac{x^2}{a^2}}\right)^2 - y^2}\, dy\, dx \quad \text{--- (1)}$$

Let $t = b\sqrt{1-\frac{x^2}{a^2}}$

$$\int_{y=0}^{t} \sqrt{t^2 - y^2}\, dy$$

$$\left[\frac{y}{2}\sqrt{t^2-y^2}+\frac{t^2}{2}\sin^{-1}\left(\frac{y}{t}\right)\right]_0^t$$

$$\left[\frac{t}{2}\sqrt{t^2-t^2}+\frac{t^2}{2}\sin^{-1}\left(\frac{t}{t}\right)\right.$$

$$\left.-\frac{0}{2}\sqrt{t^2-0^2}-\frac{t^2}{2}\sin^{-1}\left(\frac{0}{t}\right)\right]$$

$$0+\frac{t^2}{2}\sin^{-1}(1)-0-0$$

$$\frac{b^2\left(1-\frac{x^2}{a^2}\right)}{2}\frac{\pi}{2}$$

$\therefore$ from (1)

$$= \frac{8c}{b} \int_{x=0}^{a} \frac{b^2\left(1-\frac{x^2}{a^2}\right)}{2} \frac{\pi}{2}\, dx$$

$$= \frac{8c}{b} \frac{\pi}{4} b^2 \int_{x=0}^{a} \left(1-\frac{x^2}{a^2}\right) dx$$

$$= 2\pi bc \left(x - \frac{1}{a^2}\frac{x^3}{3}\right)_0^a$$

$$= 2\pi bc \left[a - \frac{1}{a^2}\frac{a^3}{3} - 0\right]$$

$$= 2\pi bc \left[a - \frac{a}{3}\right]$$

$$= 2\pi bc \left[\frac{3a-a}{3}\right]$$

$$= 2\pi bc\left(\frac{2a}{3}\right) = \frac{4\pi abc}{3}$$