5.3.2 Advantages and disadvantages of digital filters

Advantages

1. Unlike analog filter, the digital filter performance is not influenced by component ageing, temperature and power supply variations.
2. A digital filter is highly immune to noise and possess considerable parameter stability.
3. Digital filters afford a wide variety of shapes for the amplitude and phase responses.
4. There are no problems of input or output impedance matching with digital filters.
5. Digital filters can be operated over a wide range of frequencies.
6. The coefficients of digital filter can be programmed and altered any time to obtain the desired characteristics.
7. Multiple filtering is possible only in digital filter.

Disadvantage

1. The quantization error arises due to finite word length in the representation of signals and parameters.

5.4 Analog Lowpass Filter Design

The most general form of analog filter transfer function is

$$H(s) = \frac{N(s)}{D(s)} = \frac{\sum_{i=0}^{M} a_i s^i}{1 + \sum_{i=1}^{N} b_i s^i} \qquad \text{... (5.3a)}$$

where $H(s)$ is the laplace transform of the impulse response $h(t)$ that is,

$$H(s) = \int_{-\infty}^{\infty} h(t) e^{-st}\, dt \qquad \text{... (5.3b)}$$

and $N \geq M$ must be satisfied. For a stable analog filter, the poles of $H(s)$ lies in the left half of the s-plane. Now we study two types of analog filter design. They are

1. Butterworth filter 2. Chebyshev filter

5.5 Analog lowpass Butterworth Filter

The magnitude function of the Butterworth lowpass filter is given by

$$|H(j\Omega)| = \frac{1}{\left[1+\left(\Omega/\Omega_c\right)^{2N}\right]^{1/2}} \quad N = 1, 2, 3, \ldots \qquad \ldots (5.4)$$

where N is the order of the filter and Ω_c is the cutoff frequency. As shown in Fig. 5.5 the function is monotonically decreasing, where the maximum response is unity at $\Omega = 0$. The ideal response is shown by the dashed line. It can be seen that the magnitude response approaches the ideal lowpass characteristics as the order N increases. For values $\Omega < \Omega_c$ $|H(j\Omega)| \approx 1$, for values $\Omega > \Omega_c$, the value of $|H(j\Omega)|$ decreases rapidly. At $\Omega = \Omega_c$, the curves pass through 0.707, which corresponds to -3 dB point.

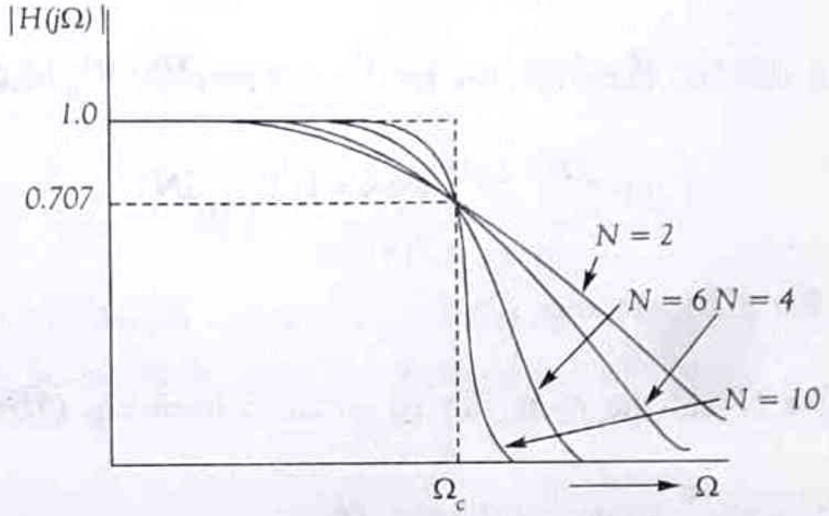

Fig. 5.5 Lowpass Butterworth magnitude response

From the Eq. (5.4), we can get magnitude square function of a normalized Butterworth filter (to 1 rad/sec cut off frequency) as

$$|H(j\Omega)|^2 = \frac{1}{1+(\Omega)^{2N}} \quad N = 1, 2, 3, \ldots \qquad \ldots (5.5)$$

Now, let us derive the transfer function of a stable filter. For this purpose, substituting $\Omega = \frac{s}{j}$, we can write Eq. (5.5) as

$$|H(j\Omega)|^2 = H(\Omega^2) = H(-s^2) = H(j\Omega)\, H(-j\Omega) = \frac{1}{1+\left(\frac{s}{j}\right)^{2N}} \qquad \ldots (5.6)$$

$$\Rightarrow \quad H(s)\,H(-s) = \frac{1}{1+(-1)^N s^{2N}} = \frac{1}{1+(-s^2)^N} \qquad \dots (5.7)$$

The above relations tell us that this function has poles in the LHP as well as in the RHP, because of the presence of two factors $H(s)$ and $H(-s)$. If $H(s)$ has roots in the LHP then $H(-s)$ has the corresponding roots in the RHP. These roots we can get by equating the denominator to zero.

i.e., $$1+(-s^2)^N = 0 \qquad \dots (5.8)$$

For N odd, the above Eq. (5.8) reduces to $s^{2N} = 1 = e^{j2\pi k}$

Now the roots of Eq. (5.8) can be found as

$$s_k = e^{j\pi k/N} \quad k = 1, 2, \dots 2N \qquad \dots (5.9)$$

For N even, the Eq. (5.8) reduces to $s^{2N} = -1 = e^{j(2k-1)\pi}$, which gives

$$s_k = e^{j(2k-1)\pi/2N} \text{ for } k = 1, 2, \dots 2N \qquad \dots (5.10)$$

For $N = 3$ Eq. (5.8) becomes $s^6 = 1$.

We know for N odd the roots can be obtained from Eq. (5.9)

i.e., $$s_1 = e^{j\pi/3} = \cos\frac{\pi}{3} + j\sin\frac{\pi}{3} = 0.5 + j\,0.866$$

$$s_2 = e^{j2\pi/3} = \cos\frac{2\pi}{3} + j\sin\frac{2\pi}{3} = -0.5 + j\,0.866$$

$$s_3 = e^{j\pi} = \cos\pi + j\sin\pi = -1$$

$$s_4 = e^{j4\pi/3} = \cos\frac{4\pi}{3} + j\sin\frac{4\pi}{3} = -0.5 - j\,0.866$$

$$s_5 = e^{j5\pi/3} = \cos\frac{5\pi}{3} + j\sin\frac{5\pi}{3} = 0.5 - j\,0.866$$

$$s_6 = e^{j2\pi} = \cos 2\pi + j\sin 2\pi = 1$$

All the above poles are located in the s-plane as shown in Fig. 5.6. It is found that the angular separation between the poles is given by $\frac{360^\circ}{2N}$, which in this case is equal to 60° and all the poles lie on a unit circle.

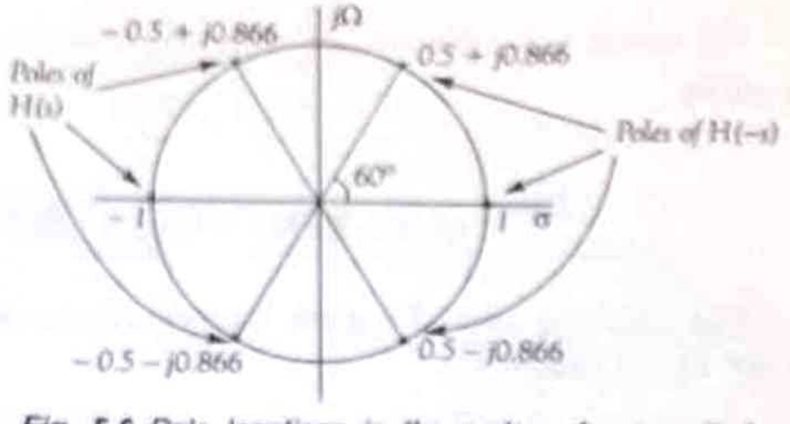

Fig. 5.6 *Pole locations in the s-plane for magnitude square function of Butterworth filter*

To ensure stability, considering only the poles that lie in the left half of the s-plane, we can write the denominator of the transfer function $H(s)$ as

$$(s+1)\{(s+0.5)^2+(0.866)^2\} = (s+1)(s^2+s+1)$$

Therefore, the transfer function of a 3rd order Butterworth filter for cutoff frequency $\Omega_c = 1$ rad/sec is

$$H(s) = \frac{1}{(s+1)(s^2+s+1)} \qquad \dots (5.11)$$

As we are interested on the poles, which lies in the left half of the s-plane, the same can be found by using the formula $s_k = e^{j\phi_k}$ where

$$\phi_k = \frac{\pi}{2} + \frac{(2k-1)\pi}{2N} \qquad k = 1 \dots N \qquad \dots (5.12)$$

For N = 4, the poles can be found from Eq. (5.12) as

$$s_1 = e^{j5\pi/8}\acute{a} = -0.3827 + j\,0.9239$$

$$s_2 = e^{j7\pi/8} = -0.9239 + j\,0.3827$$

$$s_3 = e^{j9\pi/8} = -0.9239 - j\,0.3827$$

$$s_4 = e^{j11\pi/8} = -0.3827 - j\,0.9239$$

Now the denominator of transfer function H(s) is

$$\{(s+0.3827)^2 + (0.9239)^2\}\ \{(s+0.9239)^2 + (0.3827)^2\}$$

$$= (s^2 + 1.84776\,s + 1)\ (s^2 + 0.76536s + 1)$$

For fourth order Butterworth filter the transfer function for $\Omega_c = 1\ rad/sec$ is given by

$$H(s) = \frac{1}{(s^2 + 0.76536\, s + 1)(s^2 + 1.84776\, s + 1)} \quad \text{... (5.13)}$$

The following table 5.1 gives Butterworth polynomials for various values of N for $\Omega_c = 1$ rad/sec

Table 5.1 List of Butterworth Polynomials

N	Denominator of $H(s)$
1	$s + 1$
2	$s^2 + \sqrt{2}\, s + 1$
3	$(s + 1)(s^2 + s + 1)$
4	$(s^2 + 0.76537s + 1)(s^2 + 1.8477s + 1)$
5	$(s + 1)(s^2 + 0.61803s + 1)(s^2 + 1.61803\, s + 1)$
6	$(s^2 + 1.931855s + 1)(s^2 + \sqrt{2}s + 1)(s^2 + 0.51764\, s + 1)$
7	$(s + 1)(s^2 + 1.80194\, s + 1)(s^2 + 1.247\, s + 1)(s^2 + 0.445s + 1)$

The Eq. (5.12) gives us the pole locations of Butterworth filter for $\Omega_c = 1$ rad/sec and are known as normalized poles. In general, the unnormalized poles are given by

$$s_k' = \Omega_c\, s_k \quad \text{... (5.14)}$$

The transfer function of such type of Butterworth filter can be obtained by substituting $s \rightarrow s/\Omega_c$ in the transfer function of Butterworth filter shown in table 5.1.

We can now proceed to determine the order equation given the filter specificatons. In Eq. (5.5) the filter was restricted to $-3dB$ attenuation at Ω_c. Now let the maximum passband attenuation in positive dB is $\alpha_p\ (< 3dB)$ at passband frequency Ω_p and α_s is the minimum stopband attenuation in positive dB at the stopband frequency Ω_s. Now the magnitude function can be written as

$$| H(j\Omega) | = \frac{1}{\left[1+\varepsilon^2\left(\Omega/\Omega_p\right)^{2N}\right]^{1/2}} \quad \text{... (5.15a)}$$

$$\Rightarrow | H(j\Omega) |^2 = \frac{1}{1+\varepsilon^2\left(\Omega/\Omega_p\right)^{2N}}$$

... (5.15b)

Taking logarithm on both sides we have

$$20 \log | H(j\Omega) | = 10 \log 1 - 10 \log\left[1+\varepsilon^2\left(\frac{\Omega}{\Omega_p}\right)^{2N}\right] \quad \text{... (5.16)}$$

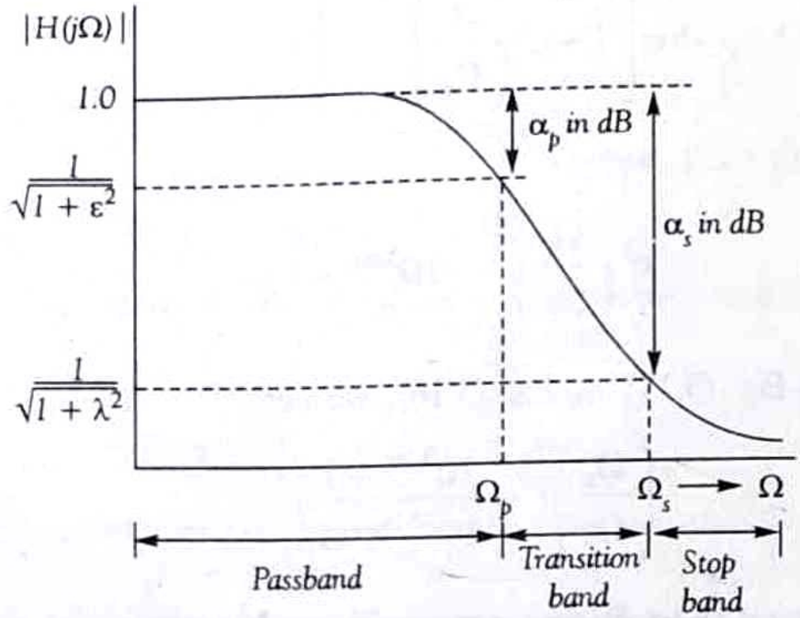

Fig. 5.7. Butterworth approximation of magnitude response

From the Fig. 5.7 we can find that at $\Omega = \Omega_p$ the attenuation is equal to α_p

Therefore, the Eq. (5.16) can be written as

$$20 \log | H(j\Omega_p) | = -\alpha_p = -10 \log(1+\varepsilon^2)$$

which gives us $\qquad \alpha_p = 10 \log(1+\varepsilon^2)$

$$0.1\,\alpha_p = \log(1+\varepsilon^2)$$

Taking antilog on both sides

$$1+\varepsilon^2 = 10^{0.1\,\alpha_p}$$

$$\Rightarrow \varepsilon = (10^{0.1\alpha_p} - 1)^{1/2} \quad \text{... (5.17)}$$

Reference to Fig. 5.7 shows that at $\Omega = \Omega_s$ the minimum stopband attenuation is equ^l to α_s. Substituting these values in Eq. (5.16)

$$20 \log |H(j\Omega_s)| = 10 \log 1 - 10 \log \left[1 + \varepsilon^2 \left(\frac{\Omega_s}{\Omega_p} \right)^{2N} \right]$$

$$-\alpha_s = -10 \log \left[1 + \varepsilon^2 \left(\frac{\Omega_s}{\Omega_p} \right)^{2N} \right]$$

$$0.1\,\alpha_s = \log \left[1 + \varepsilon^2 \left(\frac{\Omega_s}{\Omega_p} \right)^{2N} \right]$$

After simplification, we get

$$\varepsilon^2 \left(\frac{\Omega_s}{\Omega_p} \right)^{2N} = 10^{0.1\alpha_s} - 1 \qquad \text{... (5.18)}$$

Substituting Eq. (5.17) in Eq. (5.18), we get

$$\left(\frac{\Omega_s}{\Omega_p} \right)^{2N} = \frac{10^{0.1\alpha_s} - 1}{10^{0.1\,\alpha_p} - 1} \qquad \text{... (5.19)}$$

We are interested in finding expression for N, taking log of Eq. (5.19) the following expression is obtained for the order of the filter.

$$N = \frac{\log \sqrt{\dfrac{10^{0.1\,\alpha_s} - 1}{10^{0.1\,\alpha_p} - 1}}}{\log \dfrac{\Omega_s}{\Omega_p}} \qquad \text{... (5.20)}$$

Since this expression normally does not result in an integer value, we therefore, roundoff N to the next higher integer.

$$\text{i.e., } N \geq \frac{\log \sqrt{\dfrac{10^{0.1\alpha_s} - 1}{10^{0.1\alpha_p} - 1}}}{\log \dfrac{\Omega_s}{\Omega_p}} \qquad \text{... (5.21)}$$

$$\geq \frac{\log \left(\dfrac{\lambda}{\varepsilon} \right)}{\log \dfrac{\Omega_s}{\Omega_p}} \qquad \text{... (5.22)}$$

where $\varepsilon = (10^{0.1\,\alpha_p} - 1)^{0.5}$ ... (5.23)

$\lambda = (10^{0.1\,\alpha_s} - 1)^{0.5}$ ... (5.24)

For simplicity of notation, we now define the parameters A and k as follows.

$$A = \frac{\lambda}{\varepsilon} = \left(\frac{10^{0.1\,\alpha_s} - 1}{10^{0.1\,\alpha_p} - 1}\right)^{0.5} \quad \text{... (5.25)}$$

and

$$k = \frac{\Omega_p}{\Omega_s} \quad \text{... (5.26)}$$

where k is known as transition ratio.

Finally the order equation for the lowpass Butterworth analog filter is given by

$$N \geq \frac{\log A}{\log (1/k)} \quad \text{... (5.27)}$$

Example 5.1: *Given the specification* $\alpha_p = 1$ dB ; $\alpha_s = 30$dB $\Omega_p = 200$ rad/sec; $\Omega_s = 600$ rad/sec. *Determine the order of the filter.*

Solution

From Eq. (5.25) $$A = \frac{\lambda}{\varepsilon} = \left(\frac{10^{0.1\,\alpha_s} - 1}{10^{0.1\,\alpha_p} - 1}\right)^{0.5}$$

$$= \left(\frac{10^3 - 1}{10^{0.1} - 1}\right)^{0.5} = 62.115$$

From Eq. (5.26) $$k = \frac{\Omega_p}{\Omega_s} = \frac{200}{600} = \frac{1}{3}$$

From Eq. (5.27) $$N \geq \frac{\log A}{\log 1/k}$$

$$\geq \frac{\log 62.115}{\log 3} = 3.758$$

Rounding off N to the next higher integer we get $N = 4$

Example 5.2: *Prove that* $\Omega_c = \dfrac{\Omega_p}{(10^{0.1\alpha_p} - 1)^{1/2N}} = \dfrac{\Omega_s}{(10^{0.1\alpha_s} - 1)^{1/2N}}$

Solution

The magnitude square function of Butterworth analog lowpass filter is given by

$$|H(j\Omega)|^2 = \frac{1}{1 + \left(\dfrac{\Omega}{\Omega_c}\right)^{2N}} \qquad \text{... (5.28)}$$

From Eq. (5.15) we know

$$|H(j\Omega)|^2 = \frac{1}{1 + \varepsilon^2 \left(\dfrac{\Omega}{\Omega_p}\right)^{2N}}$$

Comparing Eq. (5.15) and Eq. (5.28) we get

$$1 + \left(\frac{\Omega}{\Omega_c}\right)^{2N} = 1 + \varepsilon^2 \left(\frac{\Omega}{\Omega_p}\right)^{2N}$$

$$\varepsilon^2 \left(\frac{\Omega}{\Omega_p}\right)^{2N} = \left(\frac{\Omega}{\Omega_c}\right)^{2N} \qquad \text{... (5.29)}$$

Simplifying above Eq. (5.29) by substituting Eq. (5.17) we obtain

$$\left(\frac{\Omega_p}{\Omega_c}\right)^{2N} = 10^{0.1\alpha_p} - 1 \qquad \text{... (5.30)}$$

Further simplifying Eq. (5.30) we get

$$\Omega_c = \frac{\Omega_p}{(10^{0.1\alpha_p} - 1)^{1/2N}} \qquad \text{... (5.31)}$$

$$= \frac{\Omega_p}{\varepsilon^{1/N}} \qquad \text{... (5.32)}$$

From Eq. (5.19) we have

$$\left(\frac{\Omega_s}{\Omega_p}\right)^{2N} = \frac{10^{0.1\alpha_s} - 1}{10^{0.1\alpha_p} - 1}$$

$$\Omega_s = \Omega_p \left[\frac{10^{0.1\alpha_s} - 1}{10^{0.1\alpha_p} - 1} \right]^{1/2N}$$

$$= \Omega_c \, (10^{0.1\alpha_p} - 1)^{1/2N} \cdot \left[\frac{10^{0.1\alpha_s} - 1}{10^{0.1\alpha_p} - 1} \right]^{1/2N}$$

$$\Rightarrow \Omega_c = \frac{\Omega_s}{(10^{0.1\alpha_s} - 1)^{1/2N}} \qquad \text{... (5.32a)}$$

Therefore

$$\Omega_c = \frac{\Omega_p}{(10^{0.1\alpha_p} - 1)^{1/2N}} = \frac{\Omega_s}{(10^{0.1\alpha_s} - 1)^{1/2N}}$$

Example 5.3: *Design an analog Butterworth filter that has a* -2 dB *pass band attenuation at a frequency of* 20 rad/sec *an atleast* -10 dB *stopband attenuation at* 30 rad/sec.

Solution

Given $\alpha_p = 2$ dB ; $\Omega_p = 20$ rad/sec

$\alpha_s = 10$ dB ; $\Omega_s = 30$ rad/sec

$$N \geq \frac{\log \sqrt{\dfrac{10^{0.1\alpha_s} - 1}{10^{0.1\alpha_p} - 1}}}{\log \dfrac{\Omega_s}{\Omega_p}}$$

$$\geq \frac{\log \sqrt{\dfrac{10 - 1}{10^{0.2} - 1}}}{\log \dfrac{30}{20}}$$

$$\geq 3.37$$

Rounding off N to the next highest integer we get

$$N = 4$$

The normalized lowpass Butterworth filter for $N = 4$ can be found from table 5.1 as

$$H(s) = \frac{1}{(s^2 + 0.76537\,s + 1)\,(s^2 + 1.8477\,s + 1)}$$

From Eq. (5.31) we have

$$\Omega_c = \frac{\Omega_p}{(10^{0.1\,\alpha_p} - 1)^{1/2N}} = \frac{20}{(10^{0.2} - 1\,)^{1/8}} = 21.3868$$

Note:

To find the cutoff frequency Ω_c either (5.31) or (5.32a) can be used. The Eq. (5.31) satisfies passband specification at Ω_p, while the stopband specfication at Ω_s is exceeded. The Eq. (5.32a) satisfies the stopband specfication Ω_s, while the passband specification at Ω_p is exceeded. All the examples in this chapter is solved using Eq. (5.31). Students are advised to solve the exercise problems using Eq. (5.32a).

The transfer function for $\Omega_c = 21.3868$ can be obtained by substituting

$$s \rightarrow \frac{s}{21.3868} \text{ in } H(s)$$

i.e., $$H(s) = \frac{1}{\left(\frac{s}{21.3868}\right)^2 + 0.76537\left(\frac{s}{21.3868}\right) + 1} \times \frac{1}{\left(\frac{s}{21.3868}\right)^2 + 1.8477\left(\frac{s}{21.3868}\right) + 1}$$

$$= \frac{0.20921 \times 10^6}{(s^2 + 16.3686\,s + 457.394)\,(s^2 + 39.5176\;\,s + 457.394)}$$

5.6 Analog lowpass Chebyshev filters

There are two types of Chebyshev filters. Type I Chebyshev filters are all-pole filters that exhibits equiripple behaviour in the passband and a monotonic characteristics in the stopband. On the otherhand, the family of type II Chebyshev filter contains both poles and zeros and exhibits a monotonic behaviour in the passband and an equiripple behaviour in the stopband (Fig. 5.7).

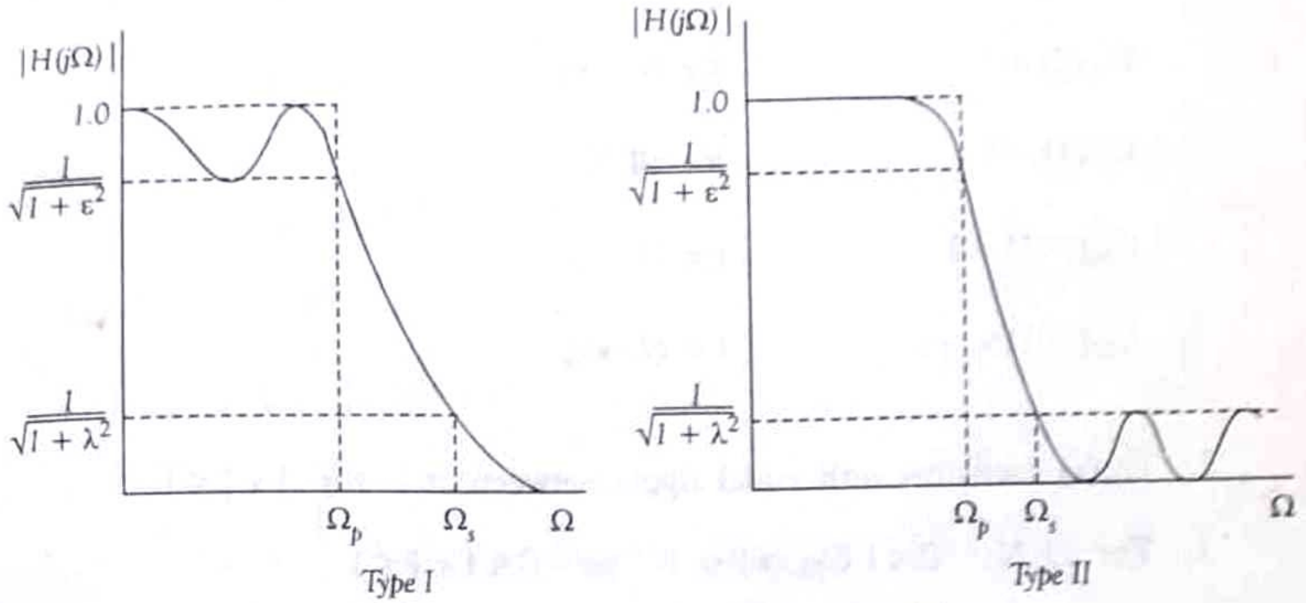

Fig. 5.7. Characteristics of Chebyshev filters

The magnitude square response of Nth order type I filter can be expressed as

$$|H(j\Omega)|^2 = \frac{1}{1+\varepsilon^2 C_N^2\left(\frac{\Omega}{\Omega_p}\right)} \quad N = 1, 2, \ldots \qquad \ldots (5.33)$$

where ε is a parameter of the filter related to the ripple in the passband and $C_N(x)$ is the Nth order Chebyshev polynomial defined as

$$C_N(x) = \cos(N \cos^{-1} x), \; |x| \le 1 \text{ (Passband)} \qquad \ldots (5.33a)$$

and

$$C_N(x) = \cosh(N \cosh^{-1} x), \; |x| > 1 \text{ (Stopband)} \qquad \ldots (5.33b)$$

The Chebyshev polynomial is defined by the recursive formula

$$C_N(x) = 2x\, C_{N-1}(x) - C_{N-2}(x), \; N > 1 \qquad \ldots (5.34)$$

where $C_0(x) = 1$ and $C_1(x) = x$

The Chebyshev polynomials described by Eq. (5.34) or Eq. (5.33a) have the following properties.

1. $C_N(x) = -C_N(-x)$ for N odd

   $C_N(x) = C_N(-x)$ for N even

   $C_N(0) = (-1)^{N/2}$ for N even

   $C_N(0) = 0$ for N odd

   $C_N(1) = 1$ for all N

   $C_N(-1) = 1$ for N even

   $C_N(-1) = -1$ for N odd

2. $C_N(x)$ oscillates with equal ripple between ± 1 for $|x| \le 1$

3. For all N $\quad 0 \le |C_N(x)| \le 1$ for $0 \le |x| \le 1$

   $$|C_N(x)| > 1 \quad \text{for} \quad |x| \ge 1$$

4. $C_N(x)$ is monotonicaly increasing for $|x| > 1$ for all N

Fig. 5.8 shows the equiripple characteristics for Chebyshev filter. For odd values of N, the oscillatory curve starts from unity and for even values of N, the oscillatory curve starts from $\dfrac{1}{\sqrt{1+\varepsilon^2}}$.

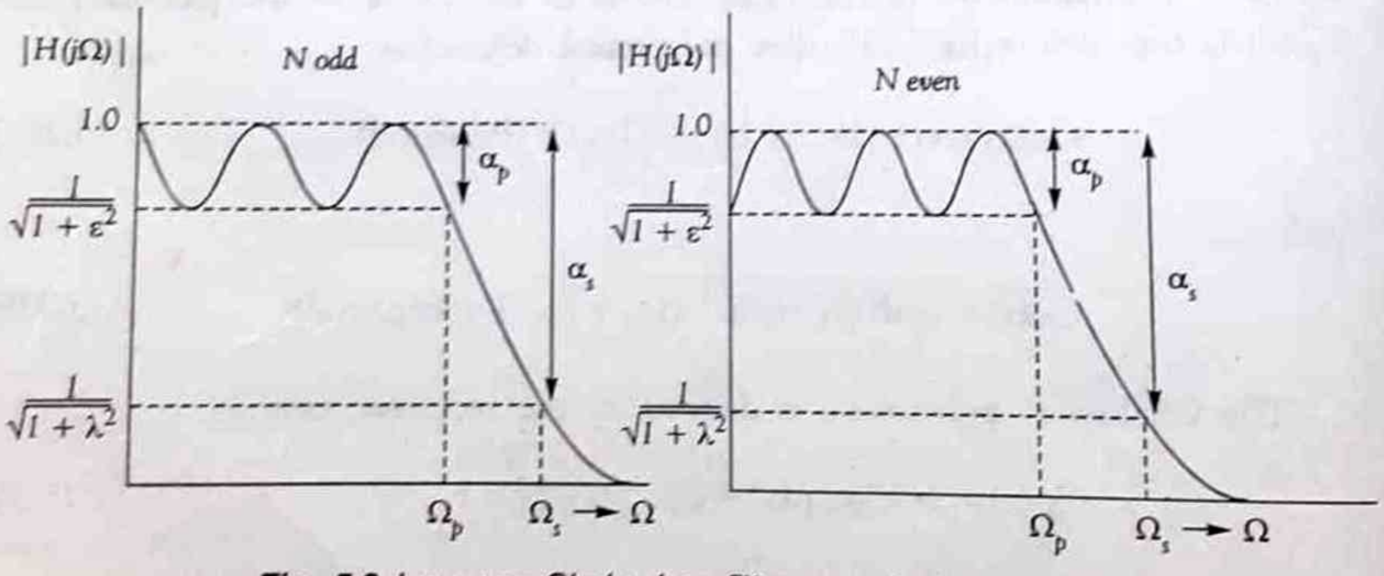

***Fig. 5.8** Lowpass Chebyshev filter magnitude response*

Taking logarithm for the Eq. (5.33) we get

$$20 \log |H(j\Omega)| = 10 \log 1 - 10 \log\left[1 + \varepsilon^2 C_N^2\left(\frac{\Omega}{\Omega_p}\right)\right] \quad \text{... (5.35)}$$

Let α_p is the attenuation in positive dB at the passband frequency Ω_p and α_s is the attenuation in positive dB at the stopband frequency Ω_s.

At $\Omega = \Omega_p$ Eq. (5.35) can be written as

$$\alpha_p = 10 \log(1 + \varepsilon^2) \quad (\because C_N(1) = 1)$$

which gives

$$\varepsilon = (10^{0.1\alpha_p} - 1)^{1/2} \quad \text{... (5.36)}$$

at $\Omega = \Omega_s$ Eq. (5.35) can be written as

$$\alpha_s = 10 \log\left[[1 + \varepsilon^2 C_N^2\left(\frac{\Omega_s}{\Omega_p}\right)\right]$$

$$= 10 \log\left[1 + \varepsilon^2 \left\{\cosh(N \cosh^{-1}(\Omega_s/\Omega_p))\right\}^2\right] \quad \boxed{\because \frac{\Omega_s}{\Omega_p} > 1} \quad \text{... (5.37)}$$

Substituting Eq. (5.36) for ε in Eq. (5.37), solving for N and rounding it to the next higher integer, we get

$$N \geq \frac{\cosh^{-1}\sqrt{\dfrac{10^{0.1\alpha_s} - 1}{10^{0.1\alpha_p} - 1}}}{\cosh^{-1}\dfrac{\Omega_s}{\Omega_p}} \quad \text{... (5.38)}$$

Substituting Eq. (5.25) and Eq. (5.26), in Eq. (5.38) we obtain

$$N \geq \frac{\cosh^{-1} A}{\cosh^{-1}(1/k)} \quad \text{... (5.39)}$$

In Eq. (5.39) $\cosh^{-1}(x)$ can be evaluated using the identity

$$\cosh^{-1} x = \ln\left[x + \sqrt{x^2 - 1}\right]$$

5.6.1 Pole locations for Chebyshev filter

The poles for the type I filter are obtained by setting the denominator of Eq. (5.33) equal to zero.

$$\text{i.e.,}\quad 1+\varepsilon^2 C_N^2\left(-\frac{js}{\Omega_p}\right)=0 \qquad \text{... (5.40)}$$

Simplify the above equation we get

$$C_N\left(-\frac{js}{\Omega_p}\right)=\pm j/\varepsilon=\cos\left[N\cos^{-1}\left(\frac{-js}{\Omega_p}\right)\right] \qquad \text{... (5.40a)}$$

We define

$$\cos^{-1}\left(-\frac{js}{\Omega_p}\right)=\phi-j\theta \qquad \text{... (5.40b)}$$

$$\boxed{\begin{aligned}\cos\theta&=\frac{e^{j\theta}+e^{-j\theta}}{2}\\ \cos j\theta&=\frac{e^{j(j\theta)}+e^{-j(j\theta)}}{2}=\frac{e^{-\theta}+e^{\theta}}{2}\\ &=\cosh\theta\end{aligned}}$$

Now Eq. (5.40a) yields

$$\pm\frac{j}{\varepsilon}=\cos\,[N\,\phi-j\,N\theta)$$

$$=\cos(N\phi)\cos(jN\theta)+\sin(N\phi)\sin(jN\theta)$$

$$=\cos(N\phi)\cosh(N\theta)+j\sin(N\phi)\sinh(N\theta) \qquad \text{... (5.41)}$$

Equating the real and imaginary parts of both sides of Eq. (5.41) results in

$$\cos(N\phi)\cosh(N\theta)=0 \qquad \text{... (5.42a)}$$

$$\sin(N\phi)\sinh(N\theta)=\pm\frac{1}{\varepsilon} \qquad \text{... (5.42b)}$$

since $\cosh(N\theta)>0$ for θ real, then in order to satisfy Eq. (5.42a), we have

$$\phi=\frac{(2k-1)\pi}{2N}\qquad k=1,2,\ldots N \qquad \text{... (5.43)}$$

Using this result and Eq. (5.42b) we can now solve for θ, where $\sin N\phi=\pm 1$.

We then obtain

$$\theta=\pm\frac{1}{N}\sinh^{-1}\left(\frac{1}{\varepsilon}\right) \qquad \text{... (5.44)}$$

Combining Eq. (5.44), Eq. (5.43) and Eq. (5.40b), we obtain the left half plane locations given by

$$s_k=j\,\Omega_p\cos(\phi-j\theta)$$

$$=j\,\Omega_p\,[\cos\phi\cosh\theta+j\sin\phi\sinh\theta)$$

$$=\Omega_p\,[-\sin\phi\sinh\theta+j\cos\phi\cosh\theta] \qquad \text{... (5.45)}$$

The Eq. (5.45) can be simplified using the identity

$$\sinh^{-1}(\varepsilon^{-1}) = \ln(\varepsilon^{-1} + \sqrt{1+\varepsilon^{-2}}) \qquad \boxed{\because \sinh^{-1} x = \ln(x + \sqrt{1+x^2})}$$

or

$$\mu = e^{\sinh^{-1}(\varepsilon^{-1})} = \varepsilon^{-1} + \sqrt{1+\varepsilon^{-2}} \qquad \text{... (5.46)}$$

From Eq. (5.44) we can write

$$\sinh\theta = \sinh\left(\frac{1}{N}\sinh^{-1}\frac{1}{\varepsilon}\right) \qquad \boxed{\because \sinh x = \frac{e^{x} - e^{-x}}{2}}$$

$$= \frac{e^{(1/N)\sinh^{-1}(1/\varepsilon)} - e^{-(1/N)\sinh^{-1}(1/\varepsilon)}}{2}$$

$$= \frac{\left[e^{\sinh^{-1}(\varepsilon^{-1})}\right]^{1/N} - \left[e^{-\sinh^{-1}(\varepsilon^{-1})}\right]^{1/N}}{2}$$

$$= \frac{\mu^{1/N} - \mu^{-1/N}}{2} \qquad \text{... (5.47)}$$

In the same way,

$$\cosh\theta = \frac{\mu^{1/N} + \mu^{-1/N}}{2} \qquad \text{... (5.48)}$$

$$\text{Now } s_k = \Omega_p\left[-\sin\phi\left(\frac{\mu^{1/N} - \mu^{-1/N}}{2}\right) + j\cos\phi\left(\frac{\mu^{1/N} + \mu^{-1/N}}{2}\right)\right] \qquad \text{... (5.49)}$$

$$= -a\sin\phi + jb\cos\phi \qquad \text{... (5.50)}$$

$$= -a\sin\frac{(2k-1)\pi}{2N} + j\,b\cos\frac{(2k-1)\pi}{2N}$$

$$= a\cos\left[\frac{\pi}{2} + \frac{(2k-1)\pi}{2N}\right] + j\,b\sin\left[\frac{\pi}{2} + \frac{(2k-1)\pi}{2N}\right]$$

$$= a\cos\phi_k + j\,b\sin\phi_k \qquad \text{... (5.51)}$$

$$= \sigma_k + j\,\Omega_k \,,\quad k = 1, 2, \ldots N \qquad \text{... (5.51a)}$$

The poles of a Chebyshev filter can be determined by using Eq. (5.51)

$$\text{where } a = \Omega_p\left[\frac{\mu^{1/N} - \mu^{-1/N}}{2}\right] \quad \ldots (5.52)$$

$$b = \Omega_p\left[\frac{\mu^{1/N} + \mu^{-1/N}}{2}\right] \quad \ldots (5.53)$$

$$\phi_k = \frac{\pi}{2} + \frac{(2k-1)\pi}{2N} \quad k = 1, 2 \ldots N \quad \ldots (5.54)$$

and μ is given by Eq. (5.46)

The poles of the Chebyshev transfer function are located on an ellipse in the s-plane as shown in Fig. 5.9. The equation of the ellipse is given by

$$\frac{\sigma_k^2}{a^2} + \frac{\Omega_k^2}{b^2} = 1 \quad \ldots (5.55)$$

where a & b are minor and major axis of the ellipse respectively.

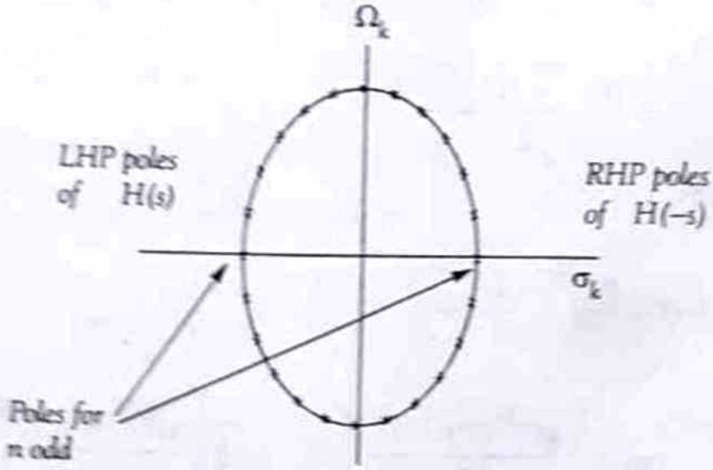

Fig. 5.9 Locus of the poles of Chebyshev filter

5.6.2 Chebyshev Type - 2 Filter

Chebyshev type-2 filter has both poles and zeros. The magnitude square response is given by

$$|H(j\Omega)|^2 = \frac{1}{1 + \varepsilon^2\left[\dfrac{C_N^2\left(\dfrac{\Omega_s}{\Omega_p}\right)}{C_N^2\left(\dfrac{\Omega_s}{\Omega}\right)}\right]}$$

where $C_N(x)$ is the Nth-order Chebyshev polynomial, Ω_s is the stopband frequency and Ω_p is the passband frequency.

The zeros are located on the imaginary axis at the points

$$s_k = j\frac{\Omega_s}{\sin\phi_k} \quad k = 1, 2, \ldots N \qquad \ldots (5.57)$$

The poles are located at the points (x_k, y_k), where

$$x_k = \frac{\Omega_s \sigma_k}{\sigma_k^2 + \Omega_k^2} \quad k = 1, 2, \ldots N \qquad \ldots (5.58a)$$

$$y_k = \frac{\Omega_s \Omega_k}{\sigma_k^2 + \Omega_k^2} \quad k = 1, 2, \ldots N \qquad \ldots (5.58b)$$

where $\sigma_k = a\cos\phi_k \quad k = 1, 2, \ldots N$ ... (5.59a)

$$\Omega_k = b\sin\phi_k \quad k = 1, 2, \ldots N \qquad \ldots (5.59b)$$

and

$$\mu = \lambda + \sqrt{1+\lambda^2} \qquad \ldots (5.60)$$

For the given specifications ε, λ, Ω_s and Ω_p, the order of the filter

$$N = \frac{\cosh^{-1}\left(\dfrac{\lambda}{\varepsilon}\right)}{\cosh^{-1}\dfrac{\Omega_s}{\Omega_p}} \qquad \ldots (5.61a)$$

$$= \frac{\cosh^{-1} A}{\cosh^{-1} 1/k} \qquad \ldots (5.61b)$$

where $A = \dfrac{\lambda}{\varepsilon}$

$$k = \frac{\Omega_p}{\Omega_s}$$

$\varepsilon = (10^{0.1\alpha_p} - 1)^{0.5}$ and

$\lambda = (10^{0.1\alpha_s} - 1)^{0.5}$

5.7 Comparison between Butterworth filter and Chebyshev filter

1. The magnitude response of Butterworth filter decreases monotonically as the frequency Ω increases from 0 to ∞, whereas the magnitude response of the Chebyshev filter exhibits ripples in the passband or stopband according to the type.

2. The transition band is more in Butterworth filter when compared to Chebyshev filter.

3. The poles of the Butterworth filter lies on a circle, whereas the poles of the Chebyshev filter lies on an ellipse.

4. For the same specifications, the number of poles in Butterworth is more when compared to the Chebyshev filter i.e., the order of the Chebyshev filter is less than that of Butterworth. This is a great advantage because less number of discrete components will be necessary to construct the filter.

5.8 Frequency Transformation in Analog Domain

So far we concentrated on designing a lowpass filter for the given specifications. In this section we discuss about the frequency transformations that can be used to design lowpass filters with different passband frequency, highpass filters, bandpass filters and bandstop filters from a normalized lowpass analog filter ($\Omega_c = 1$ rad/sec).

5.8.1 Lowpass to Lowpass filter

Given a normalized lowpass filter, it is desired to have a lowpass filter with a different cutoff frequency Ω_c (or passband frequency Ω_p). This can be accomplished by the transformation given in Eq. (5.62a)

$$s \rightarrow \frac{s}{\Omega_c} \qquad \text{... (5.62a)}$$

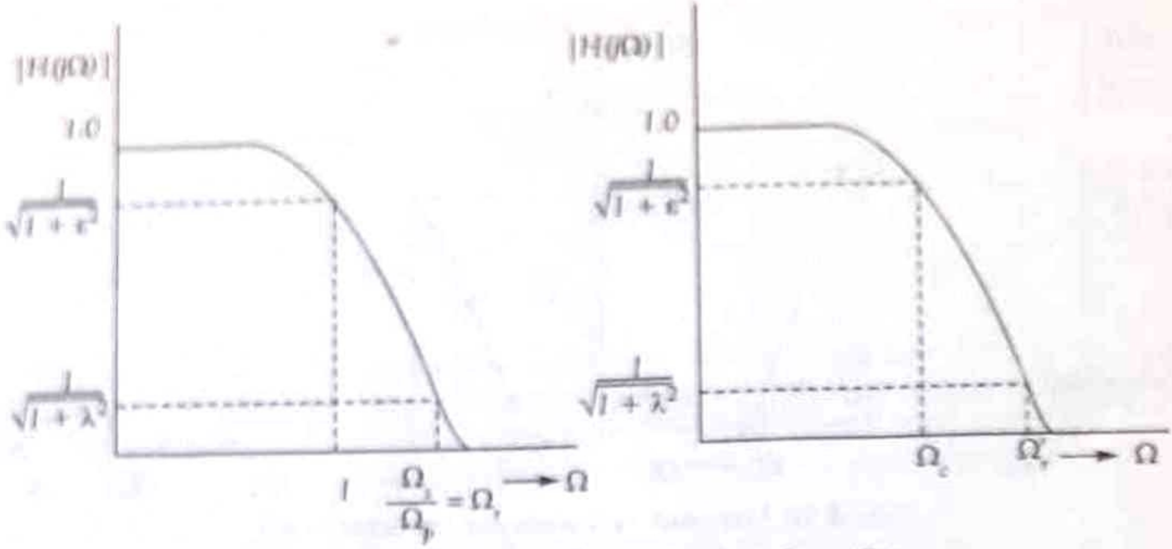

Fig. 5.10 Lowpass to Lowpass transformation

5.8.2 Lowpass to Highpass

Given a normalized lowpass filter, it is desired to have a highpass filter with cutoff frequency Ω_c. Then the transformation is

$$s \rightarrow \frac{\Omega_c}{s} \qquad \text{... (5.62b)}$$

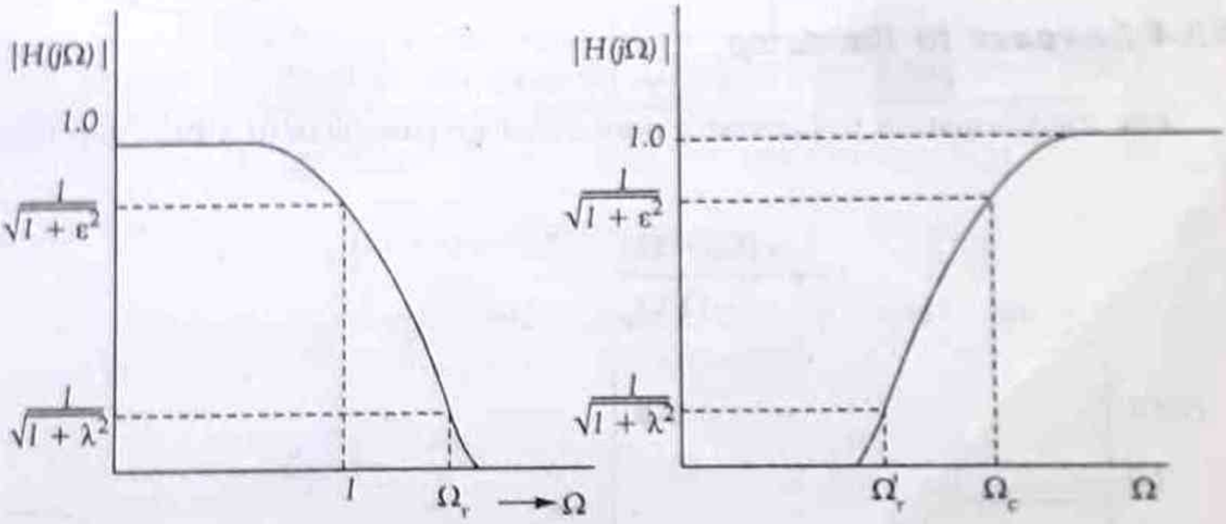

Fig. 5.11 Lowpass to Highpass transformation

5.8.3 Lowpass to Bandpass

The transformation for converting a normalized lowpass filter to a bandpass filter with cutoff frequencies Ω_l, Ω_u can be accomplished by

$$s \rightarrow \frac{s^2 + \Omega_l \Omega_u}{s(\Omega_u - \Omega_l)} \qquad (5.63a)$$

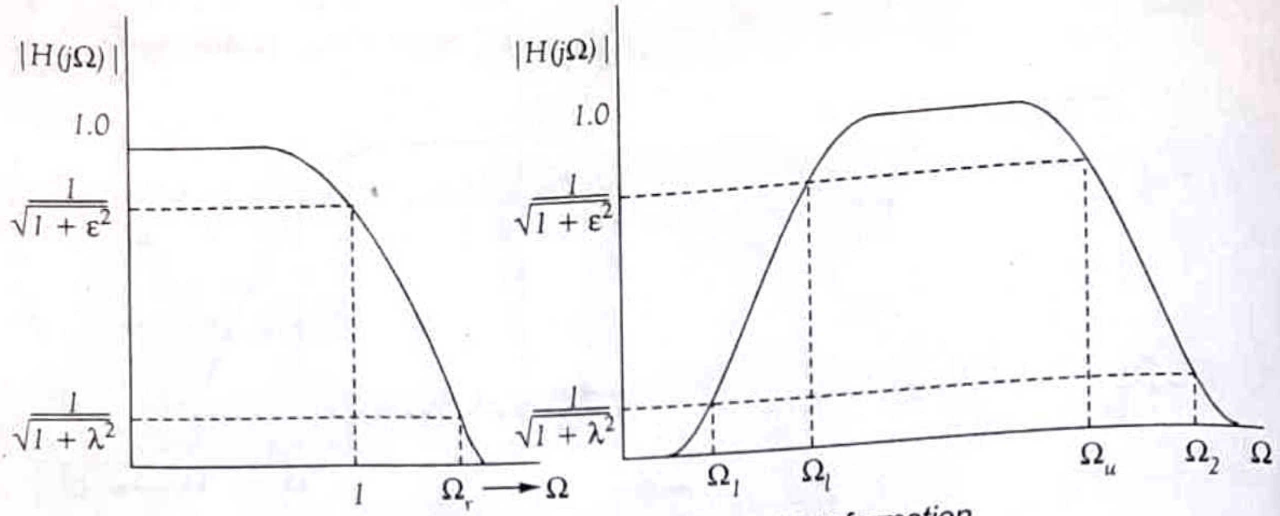

Fig. 5.12 Lowpass to bandpass transformation

$$\Omega_r = \min\{|A|, |B|\} \qquad \text{... (5.63b)}$$

$$A = \frac{-\Omega_1^2 + \Omega_1\,\Omega_u}{\Omega_1\,(\Omega_u - \Omega_l)} \qquad \text{... (5.63c)}$$

$$B = \frac{\Omega_2^2 - \Omega_l\,\Omega_u}{\Omega_2(\,\Omega_u - \Omega_l)} \qquad \text{... (5.63d)}$$

5.8.4 Lowpass to Bandstop

The transformation to convert a normalized lowpass filter to a bandstop filter is

$$s \rightarrow \frac{s\,(\Omega_u - \Omega_l)}{s^2 + \Omega_l\,\Omega_u}$$

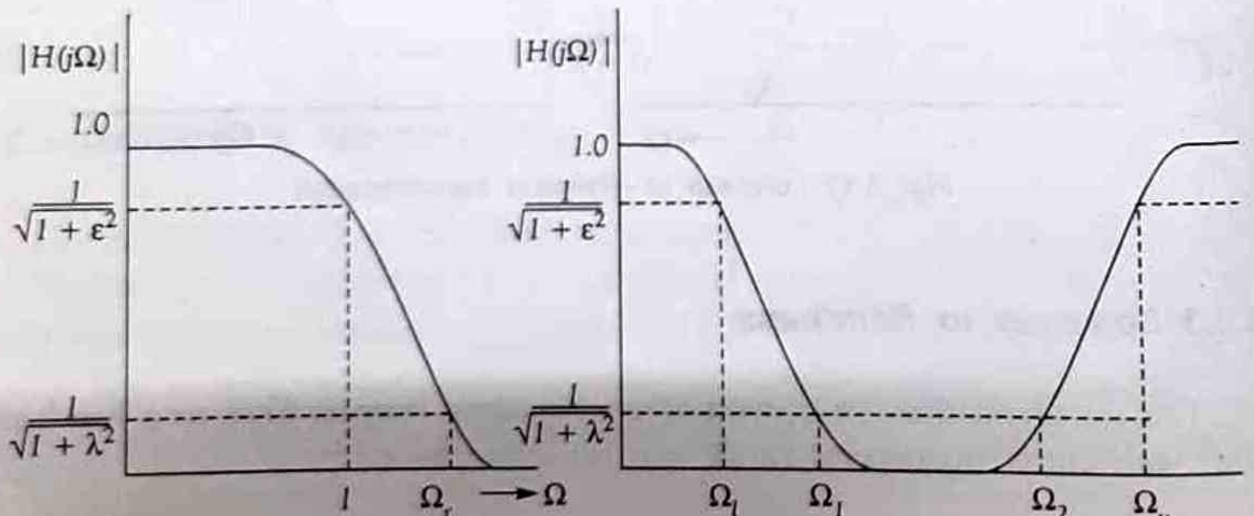

Fig. 5.13 Lowpass to bandstop transformation

$$\Omega_r = \min\{\,|A|, |B|\,\} \quad \text{... (5.63e)}$$

$$A = \frac{\Omega_1(\Omega_u - \Omega_l)}{-\Omega_1^2 + \Omega_l\Omega_u} \quad \text{... (5.63f)}$$

$$B = \frac{\Omega_2(\Omega_u - \Omega_l)}{-\Omega_2^2 + \Omega_l\Omega_u} \quad \text{... (5.63g)}$$

5.9 Steps to design an analog Butterworth lowpass filter

1. From the given specifications find the order of the filter N.
2. Round off it to the next higher integer.
3. Find the transfer function $H(s)$ for $\Omega_c = 1$ rad/sec for the value of N.
4. Calculate the value of cutoff frequency Ω_c.
5. Find the transfer function $H_a(s)$ for the above value of Ω_c by substituting $s \rightarrow \frac{s}{\Omega_c}$ in $H(s)$.

5.10 Steps to design an analog Chebyshev lowpass filter

1. From the given specifications find the order of the filter N.
2. Round off it to the next higher integer.
3. Using the following formulas find the value of a and b, which are minor and major axis of the ellipse respectively.

$$a = \Omega_p \frac{[\mu^{1/N} - \mu^{-1/N}]}{2}; \quad b = \Omega_p \left[\frac{\mu^{1/N} + \mu^{-1/N}}{2}\right]$$

where $\mu = \varepsilon^{-1} + \sqrt{\varepsilon^{-2} + 1}$

$\varepsilon = \sqrt{10^{0.1\alpha_p} - 1}$

Ω_p = Passband frequency

α_p = maximum allowable attenuation in the passband.

($\because$ For normalized Chebyshev filter $\Omega_p = 1$ rad/sec)

4. Calculate the poles of Chebyshev filter which lies on an ellipse by using the formula.

$$s_k = a\cos\phi_k + j\,b\sin\phi_k \quad k = 1, 2, \ldots N$$

$$\text{where } \phi_k = \frac{\pi}{2} + \left(\frac{2k-1}{2N}\right)\pi \quad k = 1, 2, \ldots N$$

5. Find the denominator polynomial of the transfer function using above poles.

6. The numerator of the transfer function depends on the value of N.

   (i) For N odd substitute $s = 0$ in the denominator polynomial and find the value. This value is equal to the numerator of the transfer function.

   ($\because$ For N odd the magnitude response $|H(j\Omega)|$ starts at 1.)

   (ii) For N even substitute $s = 0$ in the denominator polynomial and divide the result by $\sqrt{1+\varepsilon^2}$. This value is equal to the numerator.

To find the transfer function of highpass, bandpass and bandstop filters of any type first find the transfer function of normalized lowpass filter and use suitable transformation.

Example 5.4: *Design a Chebyshev filter with a maximum passband attenuation of 2.5 dB at* $\Omega_p = 20$ rad/sec *and the stopband attenuation of 30 dB at* $\Omega_s = 50$ rad/sec.

Solution

Given $\Omega_p = 20$ rad/sec; $\alpha_p = 2.5\ dB$

$\Omega_s = 50$ rad/sec; $\alpha_s = 30\ dB$

$$\text{We know } N = \frac{\cosh^{-1}\lambda/\varepsilon}{\cosh^{-1} 1/k}$$

$$\lambda = \sqrt{10^{0.1\alpha_s} - 1} = 31.607$$

$$\varepsilon = \sqrt{10^{0.1\,\alpha_p} - 1} = 0.882$$

$$k = \frac{\Omega_p}{\Omega_s} = 0.4$$

Now $N \geq \dfrac{\cosh^{-1}\dfrac{31.607}{0.882}}{\cosh^{-1}\dfrac{1}{0.4}} = 2.726$

i.e., $N = 3$

$$\mu = \varepsilon^{-1} + \sqrt{1+\varepsilon^{-2}} = 2.65$$

$$a = \Omega_p \frac{[\mu^{1/N} - \mu^{-1/N}]}{2} = 6.6$$

$$b = \Omega_p \left[\frac{\mu^{1/N} + \mu^{-1/N}}{2}\right] = 21.06$$

$$s_k = a\cos\phi_k + j\,b\sin\phi_k \quad k = 1, 2, 3$$

$$\phi_k = \frac{\pi}{2} + \left(\frac{2k-1}{2N}\right)\pi \quad k = 1, 2, 3$$

$$\phi_1 = 120°, \phi_2 = 180°, \phi_3 = 240°$$

$$s_1 = -3.3 + j\,18.23$$

$$s_2 = -6.6$$

$$s_3 = -3.3 - j\,18.23$$

Denominator of $H(s) = (s + 6.6)\,(s^2 + 6.6s + 343.2)$

Numerator of $H(s) = 2265.27$

Transfer function $H(s) = \dfrac{2265.27}{(s + 6.6)\,(s^2 + 6.6s + 343.2)}$

5.11 Design of IIR filters from analog filters.

There are several methods that can be used to design digital filters having an infinite duration unit sample response. The techniques described are all based on converting an analog filter into digital filter. If the conversion technique is to be effective, it should possess the following desirable properties.

1. The $j\Omega$-axis in the s-plane should map into the unit circle in the z-plane. Thus there will be a direct relationship between the two frequency variables in the two domains.
2. The left-half plane of the s-plane should map into the inside of the unit circle in the z-plane. Thus a stable analog filter will be converted to a stable digital filter.

The four most widely used methods for digitizing the analog filter into a digital filter include

1. Approximation of derivatives.
2. The impulse invariant transformation.
3. The bilinear transformation.
4. The matched z-transformation technique.

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1. Approximation of derivatives.
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3. The bilinear transformation.
4. The matched z-transformation technique.

5.11.1 Approximation of Derivatives

One of the simplest methods of digitizing an analog filter into a digital filter is to approximate the differential equation by an equivalent difference equation.

For the derivative $\frac{dy(t)}{dt}$ at time $t = nT$, we substitute the backward difference $\frac{y(nT) - y(nT - T)}{T}$

$$\text{Thus} \quad \left.\frac{dy(t)}{dt}\right|_{t=nT} = \frac{y(nT) - y(nT - T)}{T}$$

$$= \frac{y(n) - y(n-1)}{T} \qquad \text{... (5.64)}$$

where T represents the sampling interval and $y(n) \equiv y(nT)$

We know laplace transform of $\frac{dy(t)}{dt} = sY(s)$, which can be represented as shown in Fig. 5.13a

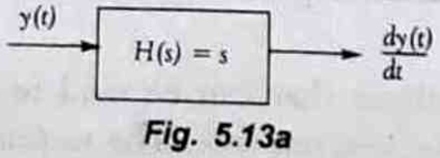

Fig. 5.13a

The z- transform of $\frac{y(n) - y(n-1)}{T}$ is $\frac{(1 - z^{-1})\, Y(z)}{T}$,

which can be represented as shown in Fig. 5.13b.

Fig. 5.13b

Comparing Fig. 5.13a and Fig. 5.13b for analog and digital domains we get

$$s = \frac{1 - z^{-1}}{T} \quad \text{... (5.65)}$$

Consequently the system function for the digital IIR filter obtained as a result of the approximations of the derivatives by finite difference is

$$H(z) = H(s)\Big|_{s = \frac{1 - z^{-1}}{T}} \quad \text{... (5.66)}$$

From the relation $s = \frac{1 - z^{-1}}{T}$ we get

$$z = \frac{1}{1 - sT} = \frac{1}{1 - j\,\Omega T} = \frac{1 + j\Omega T}{1 + \Omega^2 T^2} \quad \text{... (5.67)}$$

$$= \frac{1}{1 + \Omega^2 T^2} + \frac{j\,\Omega T}{1 + \Omega^2 T^2} \quad \text{... (5.68)}$$

$$= x + jy \quad \text{... (5.69)}$$

where $x = \frac{1}{1 + \Omega^2 T^2}$ and $y = \frac{\Omega T}{1 + \Omega^2 T^2}$

we can find that x and y are related by

$$x^2 + y^2 = x$$

which can be written as

$$\left(x - \frac{1}{2}\right)^2 + y^2 = \frac{1}{4} \quad \text{... (5.70)}$$

Thus the image in the z-plane of the $j\Omega$-axis of the s-plane is of radius 1/2 as shown in Fig. 5.14, which has the following characteristics.

1. The left half s-plane maps inside a circle of radius 1/2 centered at $z = 1/2$ in the z-plane.

2. The right half s-plane maps into the region outside the circle of radius 1/2 in the z-plane.
3. The $j\Omega$-axis maps on to the perimeter of the circle of radius 1/2 in the z-plane.

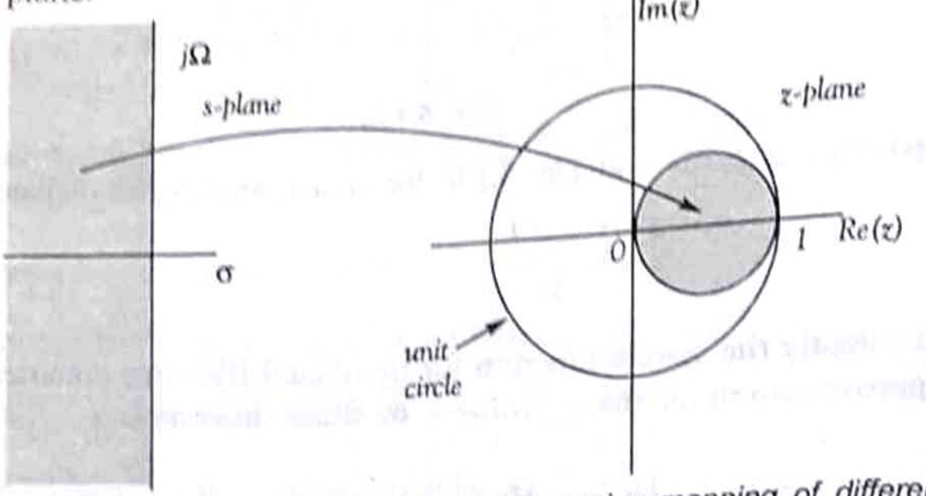

Fig 5.14 Mapping of the s-plane to the z-plane using mapping of differentials.

Thus this analog-to-digital transformation technique does map a stable analog filter into a stable digital filter, but the $j\Omega$-axis does not even map onto the $z = e^{j\omega}$ circle. Thus this transformation severely restricts the digital filter pole locations that are confined to relatively small frequencies. As a result, this mapping is restricted to the design of lowpass filters and bandpass filters having relatively small resonant frequencies. It is not possible to transform a highpass analog filter into a corresponding highpass digital filter.

5.11.2 Design of IIR Filter using Impulse Invariance Technique

In impulse invariance method the IIR filter is designed such that the unit impulse response $h(n)$ of digital filter is the sampled version of the impulse response of analog filter.

The z-transform of an infinite impulse response is given by

$$H(z) = \sum_{n=0}^{\infty} h(n)\, z^{-n} \qquad \text{... (5.71)}$$

$$H(z) \Big|_{z=e^{sT}} = \sum_{n=0}^{\infty} h(n) e^{-sTn}$$

Let us consider the mapping of points from the s-plane to the z-plane implied by the relation

$$z = e^{sT} \qquad \text{... (5.72)}$$

If we substitute $s = \sigma + j\Omega$ and express the complex variable z in polar form as $z = re^{j\omega}$ we get

$$re^{j\omega} = e^{(\sigma + j\Omega)T}$$

$$= e^{\sigma T} e^{j\Omega T} \qquad \text{... (5.73)}$$

which gives

$$r = e^{\sigma T} \qquad \text{... (5.74a)}$$

and

$$\omega = \Omega T \qquad \text{... (5.74b)}$$

The first term in the product in Eq. (5.73), $e^{\sigma T}$, has a magnitude of $e^{\sigma T}$ and an angle of 0 – it is a real number. The second term $e^{j\Omega T}$, has unity magnitude and an angle of ΩT. Therefore, our analog pole is mapped to a place in the z-plane of magnitude $e^{\sigma T}$ and angle ΩT. The real part of the analog pole determines the radius of the z-plane pole and the imaginary part of the analog pole dictates the angle of the digital pole.

Consider any pole on the $j\Omega$-axis, where $\sigma = 0$ as shown in Fig. 5.15.

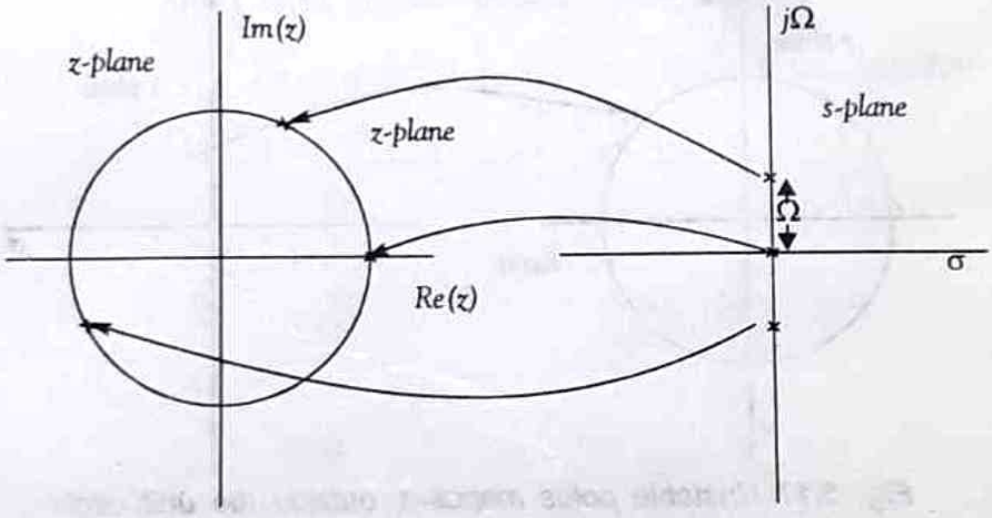

Fig. 5.15 jΩ-axis mapping to the unit circle.

These poles map to the z-plane at a radius $r = e^{0.T} = 1$. Therefore, the impulse invariant mapping map poles from the s-plane's $j\Omega$-axis to the z-plane's unit circle.

Now consider the poles in the left half of s-plane where $\sigma < 0$. These poles maps inside the unit circle as shown in Fig. 5.16, because $r = e^{\sigma T} < 1$ for $\sigma < 0$.

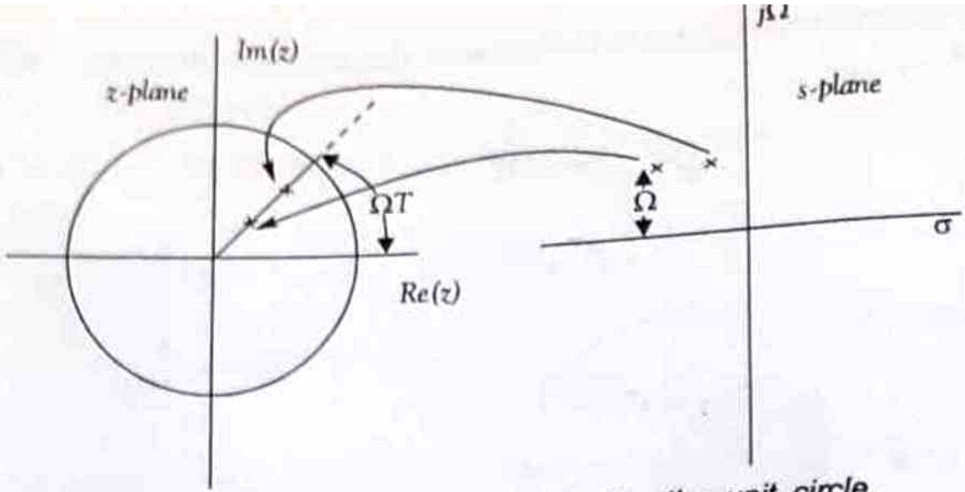

Fig. 5.16 Stable poles mapping inside the unit circle.

Therefore, all s-plane poles with negative real parts map to z-plane poles inside the unit circle - stable analog poles are mapped to stable digital poles. The impulse invariant mapping preserves the stability of the filter.

All poles in the right half of the s-plane map to digital poles outside the unit circle.

$$r = e^{\sigma T} > 1 \text{ for } \sigma > 0$$

This mapping is shown in Fig. 5.17.

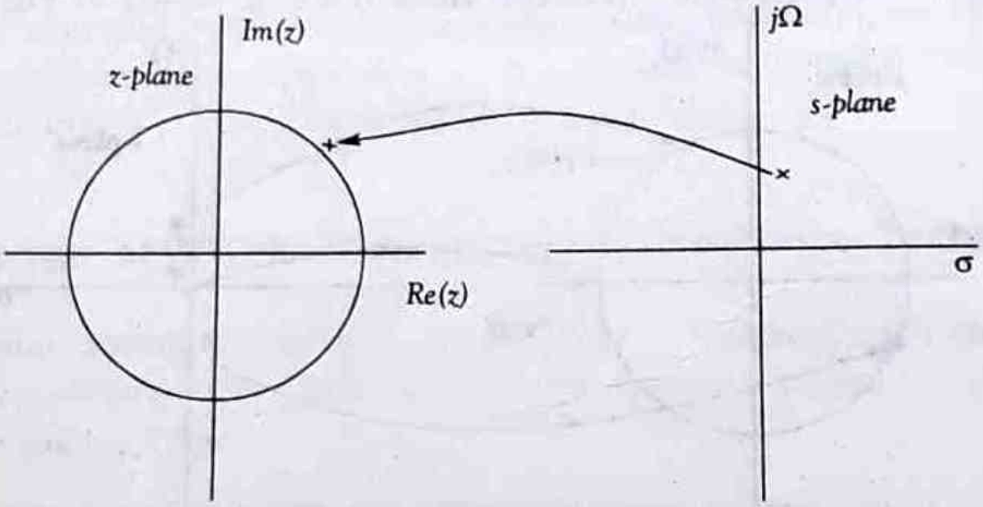

Fig. 5.17 Unstable poles mapping, outside the unit circle

Although the $j\Omega$-axis is mapped into the unit circle, it is not one-to-one mapping rather it is many-to-one mapping, where many points in s-plane are mapped to a single point in the z-plane. The easiest way to explain this is to consider two poles in the s-plane with identical real parts, but with imaginary components differing by $\frac{2\pi}{T}$ as shown in Fig. 5.18.

Let the poles are

$$s_1 = \sigma + j\,\Omega$$

$$s_2 = \sigma + j\left(\Omega + \frac{2\pi}{T}\right) \qquad \text{... (5.75)}$$

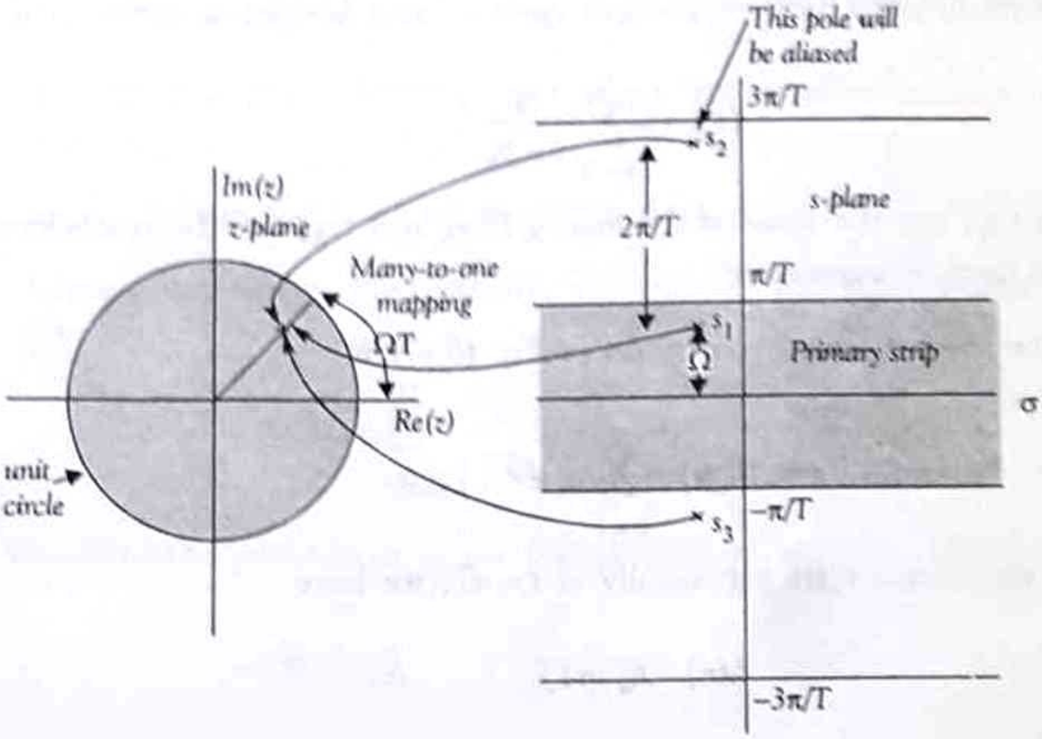

Fig. 5.18 Impulse invariant pole mapping

These poles map to z-plane poles z_1 and z_2, via impulse invariant mapping.

$$z_1 = e^{(\sigma + j\Omega)T} = e^{\sigma T} \cdot e^{j\Omega T} \qquad \text{... (5.76a)}$$

$$z_2 = e^{\left[\sigma + j\left(\Omega + \frac{2\pi}{T}T\right)\right]} = e^{\sigma T} \cdot e^{j\Omega T + j2\pi}$$

$$= e^{\sigma T} \cdot e^{j\Omega T} \quad (\because e^{j2\pi} = 1) \qquad \text{... (5.76b)}$$

From Eq. (5.76a) and Eq. (5.76b) we find that these poles map to the same location in the z-plane. There are an infinite number of s-plane poles that map to the same location in the z-plane. They must have the same real parts and imaginary parts that differ by some integer multiple of $\frac{2\pi}{T}$

This is the big disadvantage of impulse invariant mapping. The s-plane poles having imaginary parts greater than $\frac{\pi}{T}$ or less than $-\frac{\pi}{T}$ causes aliasing, when sampling analog signals.

The analog poles will not be aliased by the impulse invariant mapping if they are confined to the s-plane's "Primary strip" (within π/T of the real axis).

Let $H_a(s)$ is the system function of an analog filter. This can be expressed in partial fraction form as

$$H_a(s) = \sum_{k=1}^{N} \frac{c_k}{s - p_k} \qquad \dots (5.77)$$

where $\{p_k\}$ are the poles of the analog filter and $\{c_k\}$ are the coefficients in the partial fraction expansion

The inverse laplace transform of Eq. (5.77) is

$$h_a(t) = \sum_{k=1}^{N} c_k \, e^{p_k t} \quad t \geq 0 \qquad \dots (5.78)$$

If we sample $h_a(t)$ periodically at $t = nT$, we have

$$h(n) = h_a(nT)$$

$$= \sum_{k=1}^{N} c_k \, e^{p_k nT} \qquad \dots (5.79)$$

We know $$H(z) = \sum_{n=0}^{\infty} h(n) \, z^{-n} \qquad \dots (5.80)$$

Substituting Eq. (5.79) in Eq. (5.80) we obtain

$$H(z) = \sum_{n=0}^{\infty} \sum_{k=1}^{N} c_k \, e^{p_k nT} \, z^{-n}$$

$$= \sum_{k=1}^{N} c_k \sum_{n=0}^{\infty} \left(e^{p_k T} z^{-1} \right)^n$$

$$= \sum_{k=1}^{N} \frac{c_k}{1 - e^{p_k T} z^{-1}} \qquad \dots (5.81)$$

i.e., If $H_a(s) = \sum_{k=1}^{N} \frac{c_k}{s - p_k}$ then $H(z) = \sum_{k=1}^{N} \frac{c_k}{1 - e^{p_k T} z^{-1}}$

Due to the presence of alaising, the impulse invariant method is appropriate for the design of lowpass and bandpass filters only. The impulse invariance method is unsuccessful for implementing digital filters such as a highpass filter.

Steps to design a digital filter using Impulse Invariance method

1. For the given specifications, find $H_a(s)$, the transfer function of an analog filter.
2. Select the sampling rate of the digital filter, T seconds per sample
3. Express the analog filter transfer function as the sum of single-pole filters.

$$H_a(s) = \sum_{k=1}^{N} \frac{c_k}{s - p_k}$$

4. Compute the z-transform of the digital filter by using the formula

$$H(z) = \sum_{k=1}^{N} \frac{c_k}{1 - e^{p_k T} z^{-1}}$$

5.11.3 Design of IIR Filter Using Bilinear Transformation

The bilinear transformation is a conformal mapping that transforms the $j\Omega$ axis into the unit circle in the z-plane only once, thus avoiding aliasing of frequency components. Furthermore, all points in the LHP of 's' are mapped inside the unit circle in the z-plane and all points in the RHP of 's' are mapped into corresponding points outside the unit circle in the z-plane.

Let us consider an analog linear-filter with system function

$$H(s) = \frac{b}{s+a} \qquad \text{... (5.82)}$$

which can be written as

$$\frac{Y(s)}{X(s)} = \frac{b}{s+a}$$

so $\quad sY(s) + aY(s) = bX(s) \qquad$ ... (5.83)

This can be characterized by the differential equation

$$\frac{dy(t)}{dt} + ay(t) = bx(t) \quad \ldots (5.84)$$

$y(t)$ can be approximated by the trapezoidal formula.

Thus

$$y(t) = \int_{t_0}^{t} y'(\tau)\, d\tau + y(t_0) \quad (5.85)$$

where $y'(t)$ denotes the derivative of $y(t)$.

The approximation of the integral in Eq. (5.85) by the trapezoidal formula at $t = nT$ and $t_0 = nT - T$ yields

$$y(nT) = \frac{T}{2}\Big[\, y'(nT) + y'\,(nT - T)\,\Big] + y(nT - T) \quad \ldots (5.86)$$

From the differential Eq. (5.84) we obtain

$$y'\,(nT) = -\,ay\,(nT) + bx\,(nT) \quad \ldots (5.87)$$

Substituting Eq. (5.87) in Eq. (5.86) we get

$$y(nT) = \frac{T}{2}\,[-\,ay(nT) + bx(nT) - ay(nT - T) + bx(nT - T)] + y(nT - T)$$

which implies

$$y(nT) + \frac{aT}{2}\,y(nT) - \left(1 - \frac{aT}{2}\right)y(nT - T) = \frac{bT}{2}\,[x(nT) + x(nT - T)] \quad \ldots (5.88)$$

With $y(n) = y(nT)$ and $x(n) = x(nT)$ we obtain the result

$$\left(1 + \frac{aT}{2}\right)y(n) - \left(1 - \frac{aT}{2}\right)y(n-1) = \frac{bT}{2}\,[x(n) + x(n-1)]$$

The z-transform of this difference equation is

$$\left(1 + \frac{aT}{2}\right)Y(z) - \left(1 - \frac{aT}{2}\right)z^{-1}Y(z) = \frac{bT}{2}\,[1 + z^{-1}]X(z)$$

The system function of the digital filter is

$$H(z) = \frac{Y(z)}{X(z)} = \frac{\frac{bT}{2}(1+z^{-1})}{1+\frac{aT}{2}-\left(1-\frac{aT}{2}\right)z^{-1}}$$

$$= \frac{\frac{bT}{2}(1+z^{-1})}{(1-z^{-1})+\frac{aT}{2}(1+z^{-1})}$$

Dividing numerator and denominator by $\frac{T}{2}(1+z^{-1})$ we get

$$H(z) = \frac{b}{\frac{2}{T}\left(\frac{1-z^{-1}}{1+z^{-1}}\right)+a} \qquad \text{... (5.89)}$$

Comparing Eq. (5.82) and Eq. (5.89), the mapping from s-plane to the z-plane can be obtained as

$$s = \frac{2}{T}\left(\frac{1-z^{-1}}{1+z^{-1}}\right) \qquad \text{... (5.90)}$$

This relationship between s and z is known as bilinear transformation.

Let $z = re^{j\omega}$ and

$$s = \sigma + j\Omega \qquad \text{... (5.90a)}$$

then Eq. (5.90) can be expressed as

$$s = \frac{2(z-1)}{T(z+1)}$$

$$= \frac{2}{T}\left[\frac{re^{j\omega}-1}{re^{j\omega}+1}\right] = \frac{2}{T}\left[\frac{r\cos\omega - 1 + jr\sin\omega}{r\cos\omega + 1 + jr\sin\omega}\right]$$

$$= \frac{2}{T}\left[\frac{r\cos\omega - 1 + jr\sin\omega}{r\cos\omega + 1 + jr\sin\omega}\right]\left[\frac{r\cos\omega + 1 - jr\sin\omega}{r\cos\omega + 1 - jr\sin\omega}\right]$$

$$= \frac{2}{T}\left[\frac{r^2\cos^2\omega - 1 + r^2\sin^2\omega + j2r\sin\omega}{(r\cos\omega+1)^2 + r^2\sin^2\omega}\right]$$

$$= \frac{2}{T}\left[\frac{r^2\cos^2\omega - 1 + r^2\sin^2\omega + j\,2r\sin\omega}{1 + r^2\cos^2\omega + 2r\cos\omega + r^2\sin^2\omega}\right]$$

Separating imaginary and real parts, we have

$$s = \frac{2}{T}\left[\frac{r^2 - 1}{1 + r^2 + 2r\cos\omega} + j\frac{2r\sin\omega}{1 + r^2 + 2r\cos\omega}\right] \qquad \text{... (5.90b)}$$

Comparing Eq. (5.90a) and Eq. (5.90b), we have

$$\sigma = \frac{2}{T}\left[\frac{r^2 - 1}{1 + r^2 + 2r\cos\omega}\right]; \Omega = \frac{2}{T}\left[\frac{2r\sin\omega}{1 + r^2 + 2r\cos\omega}\right] \qquad \text{... (5.91)}$$

From Eq. (5.91), we find that

if $r \le 1$, then $\sigma < 0$ and if $r > 1$, then $\sigma > 0$. Consequently the LHP in 's' maps into the inside of the unit circle in the z-plane and the RHP in the 's' maps into the outside of the unit circle. When $r = 1$, then $\sigma = 0$ and

$$\Omega = \frac{2}{T}\,\frac{\sin\omega}{1 + \cos\omega} = \frac{2}{T}\,\frac{2\sin\frac{\omega}{2}\cos\frac{\omega}{2}}{2\cos^2\frac{\omega}{2}}$$

$$= \frac{2}{T}\tan\frac{\omega}{2} \qquad \text{... (5.92)}$$

or $$\omega = 2\tan^{-1}\frac{\Omega T}{2} \qquad \text{... (5.93)}$$

The warping effect

Let Ω and ω represents the frequency variables in the analog filter and the derived digital filter respectively. From Eq. (5.92) we have

$$\Omega = \frac{2}{T}\tan\frac{\omega}{2}$$

For small value of ω

$\because$ for small value of θ
$\tan\theta = \theta$

$$\Omega = \frac{2}{T}\cdot\frac{\omega}{2} = \frac{\omega}{T}$$

$$\omega = \Omega T \qquad \text{... (5.94)}$$

For low frequencies the relationship between Ω and ω are linear, as a result, the digital filter have the same amplitude response as the analog filter. For high frequencies, however, the relationship between ω and Ω becomes non-linear (see Fig. 5.19) and distortion is introduced in the frequency scale of the digital filter to that of the analog filter. This is known as the warping effect.

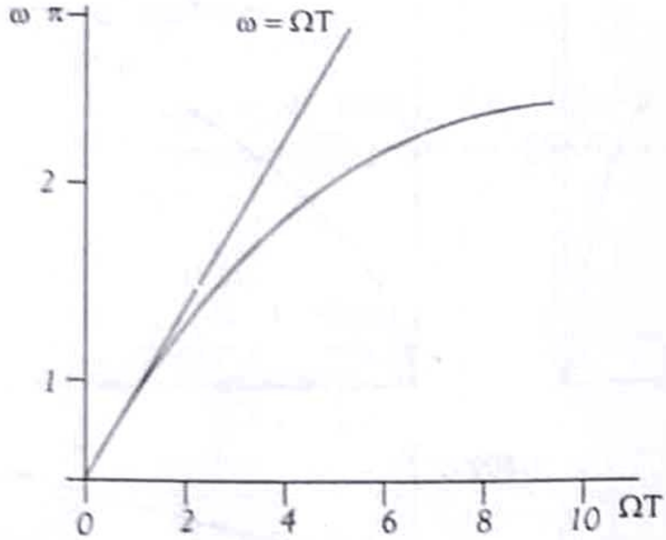

Fig. 5.19 Relationship between Ω and ω.

The influence of the warping effect on the amplitude response is shown in Fig. 5.20 by considering an analog filter with a number of passbands centered at regular intervals. The derived digital filter will have same number of pass bands. But the center frequencies and bandwidth of higher frequency passband will tend to reduce dispropotionately.

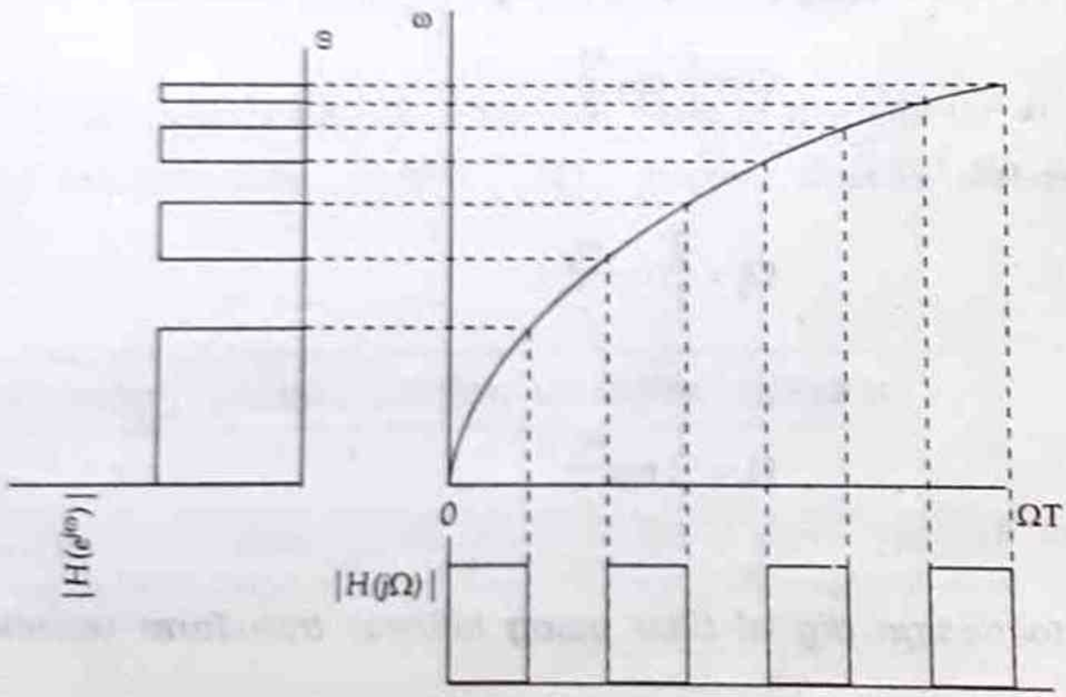

Fig. 5.20 The effect on magnitude response due to warping effect

The influence of the warping effect on the phase response is shown in Fig. 5.21. Considering an analog filter with linear phase response, the phase response of the derived digital filter will be non-linear.

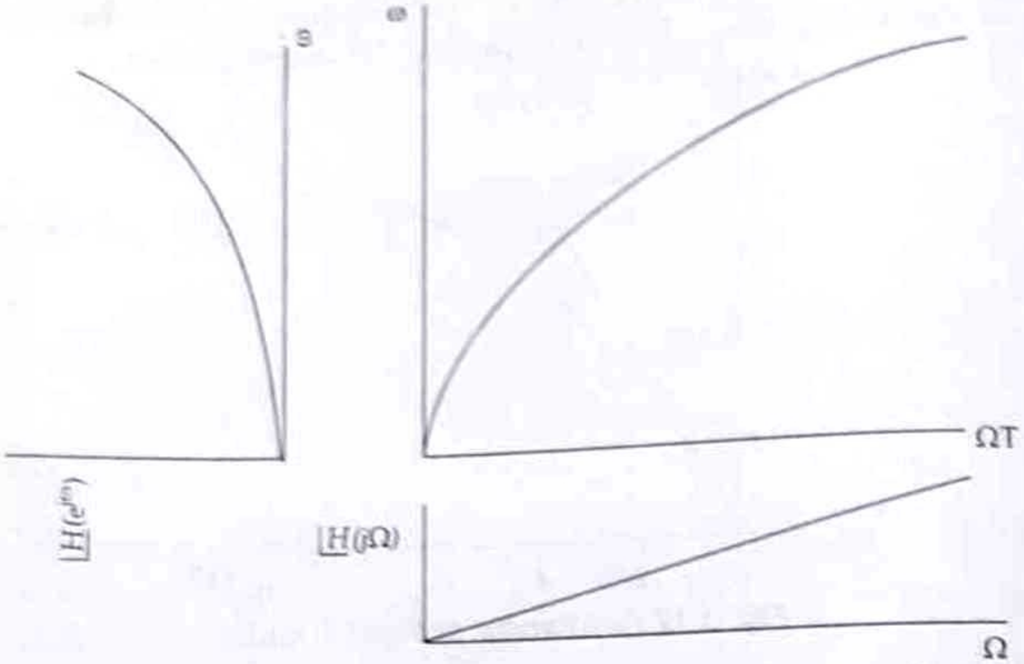

Fig. 5.21 The effect on phase response due to warping effect

Prewarping

The prewarping effect can be eliminated by prewarping the analog filter. This can be done by finding prewarping analog frequencies using the formula

$$\Omega = \frac{2}{T}\tan\frac{\omega}{2} \qquad \text{... (5.95)}$$

Therefore, we have

$$\Omega_p = \frac{2}{T}\tan\frac{\omega_p}{2} \qquad \text{... (5.96a)}$$

and

$$\Omega_s = \frac{2}{T}\tan\frac{\omega_s}{2} \qquad \text{... (5.96b)}$$

Steps to design digital filter using bilinear transform technique.

1. From the given specifications, find prewarping analog frequencies using formula $\Omega = \frac{2}{T}\tan\frac{\omega}{2}$.
2. Using the analog frequencies find H(s) of the analog filter.

3. Select the sampling rate of the digital filter, call it T seconds per sample.

4. Substitute $s = \frac{2}{T}\frac{1-z^{-1}}{1+z^{-1}}$ into the transfer function found in step 2

5.11.4 The matched *z*-transform

Another method for converting an analog filter into an equivalent digital filter is to map the poles and zeros of $H(s)$ directly into poles and zeros in the z-plane. If

$$H(s) = \frac{\prod_{k=1}^{M} (s - z_k)}{\prod_{k=1}^{N} (s - p_k)}$$

where $\{z_k\}$ are the zeros

and $\{p_k\}$ are the poles of the filter. Then the system function of the digital filter is

$$H(z) = \frac{\prod_{k=1}^{M} (1 - e^{z_k T} z^{-1})}{\prod_{k=1}^{N} (1 - e^{p_k T} z^{-1})} \qquad \text{... (5.97)}$$

where T is the sampling interval. Thus each factor of the form $(s - a)$ in $H(s)$ is mapped into the factor $1 - e^{aT} z^{-1}$. This mapping is called the matched z-transform.

5.12 Frequency transformation in digital domain

A digital lowpass filter can be converted into a digital highpass, bandstop, bandpass or another lowpass digital filter. These transformations are given below.

5.12.1 Lowpass to Lowpass

$$z^{-1} \longrightarrow \frac{z^{-1} - \alpha}{1 - \alpha z^{-1}}$$

... (5.98)

where $\alpha = \dfrac{\sin[(\omega_p - \omega'_p)/2]}{\sin[(\omega_p + \omega'_p)/2]}$

ω_p = passband frequency of lowpass filter

ω'_p = passband frequency of new filter

5.12.2 Lowpass to highpass

$$z^{-1} = -\left[\frac{z^{-1} + \alpha}{1 + \alpha z^{-1}}\right]$$

... (5.99)

where $\alpha = -\dfrac{\cos[(\omega'_p + \omega_p)/2]}{\cos[(\omega'_p - \omega_p)/2]}$

ω_p = passband frequencyof lowpass filter

ω'_p = passband frequency of highpass filter

5.12.3 Lowpass to Bandpass

$$z^{-1} \longrightarrow \frac{-\left(z^{-2} - \dfrac{2\alpha k}{1+k} z^{-1} + \dfrac{k-1}{k+1}\right)}{\dfrac{k-1}{k+1} z^{-2} - \dfrac{2\alpha k}{k+1} z^{-1} + 1}$$

where $\alpha = \dfrac{\cos[(\omega_u + \omega_l)/2]}{\cos[(\omega_u - \omega_l)/2]}$

$$k = \cot\left[\frac{\omega_u - \omega_l}{2}\right]\tan\frac{\omega_p}{2}$$ ... (5.100)

ω_u = upper cutoff frequency

ω_l = lower cutoff frequency

5.12.4 Lowpass to Bandstop

$$z^{-1} \longrightarrow \frac{z^{-2} - \frac{2\alpha}{1+k} z^{-1} + \frac{1-k}{1+k}}{\frac{1-k}{1+k} z^{-1} - \frac{2\alpha}{1+k} z^{-1} + 1}$$

where $\alpha = \dfrac{\cos\,[(\omega_u + \omega_l)/2]}{\cos\,[(\omega_u - \omega_l)/2]}$

$$k = \tan\,[(\omega_u - \omega_l)/2]\,\tan \frac{\omega_p}{2} \qquad \text{... (5.101)}$$

Examples

Example 5.5: *Convert the single pole lowpass filter with system function* $H(z) = \dfrac{0.5\,(1 + z^{-1})}{1 - 0.302 z^{-1}}$ *into bandpass filter with upper and lower cutoff frequencies* ω_u and ω_l *respectively. The lowpass filter has 3 dB bandwidth* $\omega_p = \dfrac{\pi}{6}$ and $\omega_u = \dfrac{3\pi}{4}$, $\omega_l = \dfrac{\pi}{4}$

Solution

The digital-to-digital transformation from lowpass filter to a bandpass filter is

$$z^{-1} \rightarrow \frac{-\left(z^{-2} - \frac{2\,\alpha\,k}{k+1} + \frac{k-1}{k+1}\right)}{\frac{k-1}{k+1} z^{-2} - \frac{2\,\alpha\,k}{k+1} z^{-1} + 1}$$

where

$$k = \cot\left[\frac{\omega_u - \omega_l}{2}\right] \tan \frac{\omega_p}{2} = \cot\left(\frac{\frac{3\pi}{4} - \frac{\pi}{4}}{2}\right) \tan \frac{\pi}{12}$$

$$= \cot\left(\frac{\pi}{4}\right) \tan \frac{\pi}{12}$$

$$= 0.268$$

$$\alpha = \frac{\cos\frac{\omega_u + \omega_l}{2}}{\cos\frac{\omega_u - \omega_l}{2}} = \frac{\cos\left(\frac{\frac{3\pi}{4} + \frac{\pi}{4}}{2}\right)}{\cos\left(\frac{\frac{3\pi}{4} - \frac{\pi}{4}}{2}\right)} = \frac{\cos\frac{\pi}{2}}{\cos\frac{\pi}{4}} = 0$$

Substituting the values of α and k in the transformation

$$z^{-1} \longrightarrow \frac{-\left(z^{-2} + \frac{0.268 - 1}{0.268 + 1}\right)}{\frac{0.268 - 1}{0.268 + 1} z^{-2} + 1}$$

i.e.,

$$z^{-1} \longrightarrow \frac{-(z^{-2} - 0.577)}{-0.577z^{-2} + 1}$$

Now the transfer function of bandpass filter can be obtained by substituting the above transformation in $H(z)$

i.e.,

$$H(z) = 0.5 \frac{\left[1 + \frac{-z^{-2} + 0.577}{1 - 0.577z^{-2}}\right]}{1 - 0.302\left(\frac{-z^{-2} + 0.577}{1 - 0.577z^{-2}}\right)}$$

$$= 0.5\left[\frac{1.577(1 - z^{-2})}{0.82575 - 0.275z^{-2}}\right]$$

$$= \frac{0.955(1 - z^{-2})}{(1 - 0.333z^{-2})}$$

Example 5.6: *Obtain an analog Chebyshev filter transfer function that satisfies the constraints* $\frac{1}{\sqrt{2}} \le |H(j\Omega)| \le 1 \quad 0 \le \Omega \le 2$

$|H(j\Omega)| < 0.1 \quad \Omega \ge 4$

Solution

Step 1: From the given data we can find that

$$\frac{1}{\sqrt{1+\varepsilon^2}} = \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{1+\lambda^2}} = 0.1,$$

$\Omega_p = 2$ and $\Omega_s = 4$

from which we can obtain $\varepsilon = 1$ and $\lambda = 9.95$.

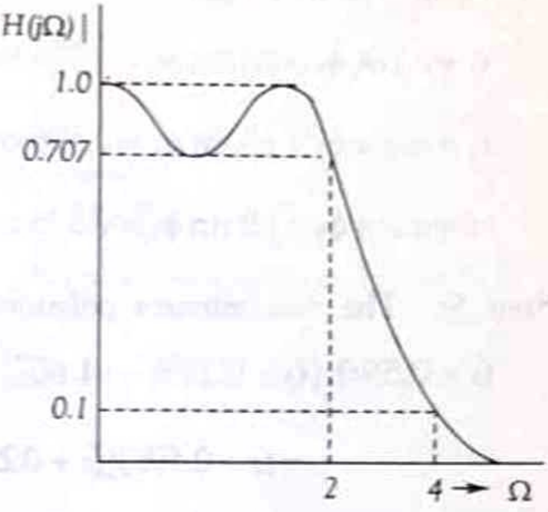

Fig. 5.22. Magnitude response of example 5.6

We know

$$N \geq \frac{\cosh^{-1}\dfrac{\lambda}{\varepsilon}}{\cosh^{-1}\dfrac{\Omega_s}{\Omega_p}} = \frac{\cosh^{-1} 9.95}{\cosh^{-1} 2} = 2.269$$

Step 2: Rounding N to next higher value we get $N = 3$. For N odd, the oscillatory curve starts from unity as shown in Fig. 5.22.

Step 3: Finding the values of a and b

$$\mu = \varepsilon^{-1} + \sqrt{1+\varepsilon^{-2}} = 2.414$$

$$a = \Omega_p\left[\frac{\mu^{1/N} - \mu^{-1/N}}{2}\right] = 2\left[\frac{(2.414)^{1/3} - (2.414)^{-1/3}}{2}\right]$$

$$= 0.596$$

$$b = \Omega_p\left[\frac{\mu^{1/N} + \mu^{-1/N}}{2}\right] = 2\left[\frac{(2.414)^{1/3} + (2.414)^{-1/3}}{2}\right]$$

$$= 2.087$$

Step 4: To calculate the poles of Chebyshev filter

$$\phi_k = \frac{\pi}{2} + \frac{(2k-1)\pi}{2N} \quad k = 1, 2, 3$$

$$\phi_1 = 120^\circ, \phi_2 = 180^\circ, \phi_3 = 240^\circ$$

We know $s_k = a\cos\phi_k + jb\sin\phi_k \quad k = 1, 2, 3$

From which we get

$$s_1 = a\cos\phi_1 + j\,b\sin\phi_1 = 0.596\cos 120° + j\,2.087\sin 120° = -0.298 + j\,1.807$$

$$s_2 = a\cos\phi_2 + j\,b\sin\phi_2 = 0.596\cos 180° + j\,2.087\sin 180° = -0.596$$

$$s = a\cos\phi_3 + j\,b\sin\phi_3 = 0.596\cos 240° + j\,2.087\sin 240° = -0.298 - j\,1.807$$

Step 5: The denominator polynomial is given by

$$(s+0.596)\{(s+0.298) - j\,1.807\}\{(s+0.298) + j\,1.807\}$$

$$= (s+0.596)[(s+0.298)^2 + (1.807)^2]$$

$$= (s+0.596)(s^2 + 0.596s + 3.354)$$

Step 6: The numerator of $H(s)$ can be obtained by substituting $s = 0$ (for N odd) in the denominator.

i.e., The numerator of $H(s) = 2$

The transfer function of Chebyshev filter for the given specifications is given by $H(s) = \dfrac{2}{(s+0.596)\,(s^2 + 0.596\,s + 3.354)}$

Example 5.7: *Given the specifications* $\alpha_p = 3dB$; $\alpha_s = 16\,dB$ $f_p = 1KHz$ *and* $f_s = 2KHz$. *Determine the order of the filter using Chebyshev approximation. Find H(s).*

Solution

From the given data we can find

$$\Omega_p = 2\pi \times 1000\ Hz = 2000\pi \text{ rad/sec}$$

$$\Omega_s = 2\pi \times 2000\ Hz = 4000\pi \text{ rad/sec}$$

and $\alpha_p = 3dB;\ \alpha_s = 16dB$

Step 1: $N \geq \dfrac{\cosh^{-1}\sqrt{\dfrac{10^{0.1\alpha_s}-1}{10^{0.1\alpha_p}-1}}}{\cosh^{-1}\dfrac{\Omega_s}{\Omega_p}} = \cosh^{-1}\dfrac{\sqrt{\dfrac{10^{1.6}-1}{10^{0.3}-1}}}{\cosh^{-1}\dfrac{4000\,\pi}{2000\pi}}$

$$= 1.91$$

Step 2: Rounding N to next higher value

we get $N = 2$

For N even, the oscillatory curve starts from $\frac{1}{\sqrt{1+\varepsilon^2}}$

Step 3: The values of minor axis and major axis can be found as below

$$\varepsilon = (10^{0.1\alpha_p} - 1)^{0.5} = (10^{0.3} - 1)^{0.5} = 1$$

$$\mu = \varepsilon^{-1} + \sqrt{1+\varepsilon^{-2}} = 2.414$$

$$a = \Omega_p \frac{[\mu^{1/N} - \mu^{-1/N}]}{2} = 2000\,\pi \frac{[(2.414)^{1/2} - (2.414)^{-1/2}]}{2} = 910\pi$$

$$b = \Omega_p \left[\frac{\mu^{1/N} + \mu^{-1/N}}{2}\right] = 2000\pi \left[\frac{(2.414)^{1/2} + (2.414)^{-1/2}}{2}\right] = 2197\pi$$

Step 4: The poles are given by

$$s_k = a\cos\phi_k + j\,b\sin\phi_k, \quad k = 1, 2$$

$$\phi_k = \frac{\pi}{2} + \frac{(2k-1)\pi}{2N} \quad k = 1, 2$$

$$\phi_1 = \frac{\pi}{2} + \frac{\pi}{4} = 135°$$

$$\phi_2 = \frac{\pi}{2} + \frac{3\pi}{4} = 225°$$

$$s_1 = a\cos\phi_1 + j\,b\sin\phi_1 = -643.46\pi + j\,1554\pi$$

$$s_2 = a\cos\phi_2 + j\,b\sin\phi_2 = -643.46\pi - j\,1554\pi$$

Step 5: The denominator of $H(s) = (s + 643.46\pi)^2 + (1554\pi)^2$

Step 6: The numerator of $H(s) = \dfrac{(643.46\pi)^2 + (1554\pi)^2}{\sqrt{1+\varepsilon^2}}$

$$= (1414.38)^2\pi^2$$

The transfer function $H(s) = \dfrac{(1414.38)^2\pi^2}{s^2 + 1287\pi s + (1682)^2\pi^2}$

Example 5.8: *For the given specifications design an analog Butterworth filter.*

$0.9 \le |H(j\Omega)| \le 1 \quad \text{for } 0 \le \Omega \le 0.2\pi$

$|H(j\Omega)| \le 0.2 \quad \text{for} \quad 0.4\pi \le \Omega \le \pi$

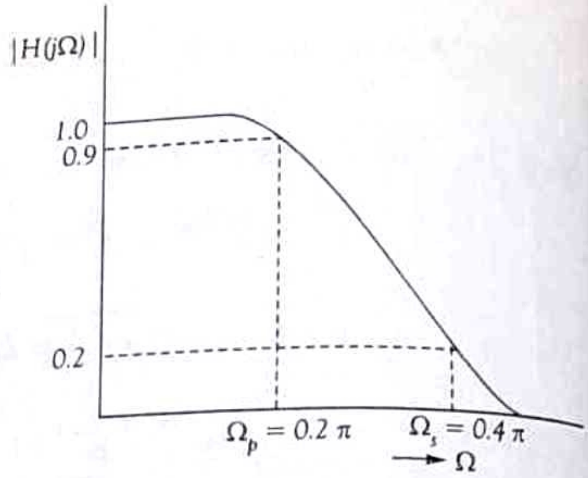

Fig. 5.23 Magnitude response of example 5.8

Solution

From the data we find $\Omega_p = 0.2\pi$;

$$\Omega_s = 0.4\pi; \frac{1}{\sqrt{1+\varepsilon^2}} = 0.9 \text{ and } \frac{1}{\sqrt{1+\lambda^2}} = 0.2$$

from which we obtain

$$\varepsilon = 0.484 \text{ and } \lambda = 4.898$$

$$N \ge \frac{\log\left(\frac{\lambda}{\varepsilon}\right)}{\log\frac{\Omega_s}{\Omega_p}} = \frac{\log\frac{4.898}{0.484}}{\log\left(\frac{0.4\pi}{0.2\pi}\right)} = 3.34$$

i.e., $N = 4$

From the table 5.1, for $N = 4$, the transfer function of normalised Butterworth filter is

$$H(s) = \frac{1}{(s^2+0.76537s+1)(s^2+1.8477s+1)}$$

we know $\Omega_c = \dfrac{\Omega_p}{(10^{0.1\alpha_p}-1)^{1/2N}} = \dfrac{\Omega_p}{\varepsilon^{1/N}} = \dfrac{0.2\pi}{(0.484)^{1/4}} = 0.24\pi$

$H(s)$ for $\Omega_c = 0.24\pi$ can be obtained by substituting $s \to \dfrac{s}{0.24\pi}$ in $H(s)$

i.e.,

$$H(s) = \frac{1}{\left\{\left(\frac{s}{0.24\pi}\right)^2 + 0.76537\left(\frac{s}{0.24\pi}\right)+1\right\}} \times \frac{1}{\left(\frac{s}{0.24\pi}\right)^2 + 1.8477\left(\frac{s}{0.24\pi}\right)+1}$$

$$= \frac{0.323}{(s^2+0.577s+0.0576\pi^2)(s^2+1.393s+0.0576\pi^2)}$$

Example 5.9: *Determine the order and the poles of a lowpass Butterworth filter that has a 3 dB attenuation at 500 Hz and an attenuation of 40 dB at 1000 Hz.*

Solution

Given data $\alpha_p = 3\text{dB}$; $\alpha_s = 40\text{dB}$; $\Omega_p = 2 \times \pi \times 500 = 1000\pi$ rad/sec

$$\Omega_s = 2 \times \pi \times 1000 = 2000\pi \text{ rad/sec}$$

The order of the filter $N \geq \dfrac{\log \sqrt{\dfrac{10^{0.1\alpha_s} - 1}{10^{0.1\alpha_p} - 1}}}{\log \dfrac{\Omega_s}{\Omega_p}}$

$$\geq \frac{\log \sqrt{\dfrac{10^4 - 1}{10^{0.3} - 1}}}{\log \dfrac{2000\pi}{1000\pi}} = 6.6$$

rounding 'N' to nearest higher value

we get $N = 7$.

The poles of Butterworth filter is given by

$$s_k = \Omega_c e^{j\phi_k} = 1000\pi e^{j\phi_k} \quad k = 1, 2, \ldots 7$$

where $\phi_k = \dfrac{\pi}{2} + \dfrac{(2k-1)\pi}{2N} \quad k = 1, 2, \ldots 7$

Example 5.10: *Determine the order and the poles of a type I lowpass Chebyshev filter that has a 1 dB ripple in the passband and passband frequency $\Omega_p = 1000\pi$, a stopband frequency of 2000π and an attenuation of 40 dB or more.*

Solution

Given data $\alpha_p = 1$ dB; $\Omega_p = 1000\,\pi$ rad / sec; $\alpha_s = 40$ dB

$\Omega_s = 2000\pi$ rad/sec

$$N \geq \frac{\cosh^{-1}\sqrt{\frac{10^{0.1\alpha_s}-1}{10^{0.1\alpha_p}-1}}}{\cosh^{-1}\frac{\Omega_s}{\Omega_p}} \geq \frac{\cosh^{-1}\sqrt{\frac{10^4-1}{10^{0.1}-1}}}{\cosh^{-1}\frac{2000\pi}{1000\pi}} = 4.536$$

i.e., $N = 5$

$$\varepsilon = \sqrt{10^{0.1\alpha_p}-1} = 0.508\ ;\ \mu = \varepsilon^{-1} + \sqrt{1+\varepsilon^{-2}} = 4.17$$

$$a = \Omega_p\left[\frac{\mu^{1/N} - \mu^{-1/N}}{2}\right] = 289.5\pi;\ b = \Omega_p\left[\frac{\mu^{1/2} + \mu^{-1/N}}{2}\right] = 1041\pi$$

$$\phi_k = \frac{\pi}{2} + \frac{(2k-1)\pi}{2N} \quad k = 1, 2, \ldots 5$$

$$\phi_1 = 108°;\ \phi_2 = 144°;\ \phi_3 = 180°;\ \phi_4 = 216°;\ \phi_5 = 252°$$

$$s_k = a\cos\phi_k + j\,b\sin\phi_k \quad k = 1, 2, \ldots 5$$

$$s_1 = -89.5\pi + j\,989\pi;\quad s_2 = -234.2\pi + j\,612\pi;\quad s_3 = -289.5\pi$$

$$s_4 = -234.2\pi - j\,612\pi;\quad s_5 = -89.5\pi - j\,989\pi$$

Example 5.11: *Apply bilinear transformation to* $H(s) = \frac{2}{(s+1)(s+2)}$ *with* $T = 1$ *sec and find* $H(z)$

Solution

Given $H(s) = \frac{2}{(s+1)(s+2)}$

Substitute $s = \frac{2}{T}\left[\frac{1-z^{-1}}{1+z^{-1}}\right]$ in $H(s)$ to get $H(z)$

$$H(z) = H(s)\Big|_{s=\frac{2}{T}\left(\frac{1-z^{-1}}{1+z^{-1}}\right)}$$

$$= \frac{2}{(s+1)(s+2)}\Bigg|_{s=\frac{2}{T}\left(\frac{1-z^{-1}}{1+z^{-1}}\right)}$$

Given $T = 1$ sec

$$H(z) = \frac{2}{\left\{2\left(\frac{1-z^{-1}}{1+z^{-1}}\right)+1\right\}\left\{2\left(\frac{1-z^{-1}}{1+z^{-1}}\right)+2\right\}}$$

$$= \frac{2(1+z^{-1})^2}{(3-z^{-1})(4)}$$

$$= \frac{(1+z^{-1})^2}{6-2z^{-1}}$$

$$= \frac{0.166(1+z^{-1})^2}{(1-0.33z^{-1})}$$

Example 5.12: *For the analog transfer function* $H(s) = \dfrac{2}{(s+1)(s+2)}$ *determine* $H(z)$ *using impulse invariance method. Assume* $T = 1$ *sec.*

Solution

Given $H(s) = \dfrac{2}{(s+1)(s+2)}$

Using partial fraction we can write

$$H(s) = \frac{A}{s+1} + \frac{B}{s+2}$$

$$A = (s+1)\frac{2}{(s+1)(s+2)}\Bigg|_{s=-1}$$

$$= 2$$

$$B = (s+2)\frac{2}{(s+1)(s+2)}\Bigg|_{s=-2}$$

$$= -2$$

i.e.,

$$H(s) = \frac{2}{s+1} - \frac{2}{s+2}$$

$$= \frac{2}{s-(-1)} - \frac{2}{s-(-2)}$$

using impulse invariance technique we have, if

$$H(s) = \sum_{k=1}^{N} \frac{c_k}{s - p_k} \text{ then } H(z) = \sum_{k=1}^{N} \frac{c_k}{1 - e^{p_k T} z^{-1}}$$

i.e., $(s - p_k)$ is transformed to $1 - e^{p_k T} z^{-1}$

There are two poles $p_1 = -1$ and $p_2 = -2$

So

$$H(z) = \frac{2}{1 - e^{-T} z^{-1}} - \frac{2}{1 - e^{-2T} z^{-1}}$$

for $T = 1$ sec

$$H(z) = \frac{2}{1 - e^{-1} z^{-1}} - \frac{2}{1 - e^{-2} z^{-1}}$$

$$= \frac{2}{1 - 0.3678 z^{-1}} - \frac{2}{1 - 0.1353 z^{-1}}$$

$$H(z) = \frac{0.465\, z^{-1}}{1 - 0.503\, z^{-1} + 0.04976\, z^{-2}}$$

Example 5.13: Using impulse invariance with $T = 1$ sec determine $H(z)$ if $H(s) = \frac{1}{s^2 + \sqrt{2}s + 1}$

Solution

Given $H(s) = \dfrac{1}{s^2 + \sqrt{2}s + 1}$

$$h(t) = L^{-1}[H(s)] = L^{-1}\left[\frac{1}{s^2+\sqrt{2}s+1}\right]$$

$$= L^{-1}\left[\frac{1}{\left(s+\frac{1}{\sqrt{2}}\right)^2+\left(\frac{1}{\sqrt{2}}\right)^2}\right]$$

$$= L^{-1}\left[\sqrt{2}\cdot\frac{\frac{1}{\sqrt{2}}}{\left(s+\frac{1}{\sqrt{2}}\right)^2+\left(\frac{1}{\sqrt{2}}\right)^2}\right]$$

$$= \sqrt{2}\,L^{-1}\left[\frac{\frac{1}{\sqrt{2}}}{\left(s+\frac{1}{\sqrt{2}}\right)^2+\left(\frac{1}{\sqrt{2}}\right)^2}\right] = \sqrt{2}\,e^{-t/\sqrt{2}}\sin t/\sqrt{2}$$

Let $t = nT$

$$h(nT) = \sqrt{2}\,e^{-nT/\sqrt{2}}\sin\frac{nT}{\sqrt{2}}$$

If $T = 1$ sec

$$h(n) = \sqrt{2}\,e^{-n/\sqrt{2}}\sin\frac{n}{\sqrt{2}}$$

$$H(z) = z[h(n)] = \sqrt{2}\left[\frac{e^{-1/\sqrt{2}}z^{-1}\sin 1/\sqrt{2}}{1-2e^{-1/\sqrt{2}}z^{-1}\cos\frac{1}{\sqrt{2}}+e^{-\sqrt{2}}z^{-2}}\right]$$

$$= \frac{0.453\,z^{-1}}{1-0.7497\,z^{-1}+0.2432\,z^{-2}}$$

Example 5.14: *Using the bilinear transform, design a highpass filter, monotonic in passband with cutoff frequency of 1000 Hz and down 10 dB at 350 Hz. The sampling frequency is 5000 Hz.*

Solution

Given $\alpha_p = 3\ dB$; $\omega_c = \omega_p = 2\times\pi\times 1000 = 2000\pi$ rad/sec

$$\alpha_s = 10 \text{ dB}; \ \omega_s = 2 \times \pi \times 350 = 700\pi \text{ rad/sec}$$

$$T = \frac{1}{f} = \frac{1}{5000} = 2 \times 10^{-4} \text{ sec}$$

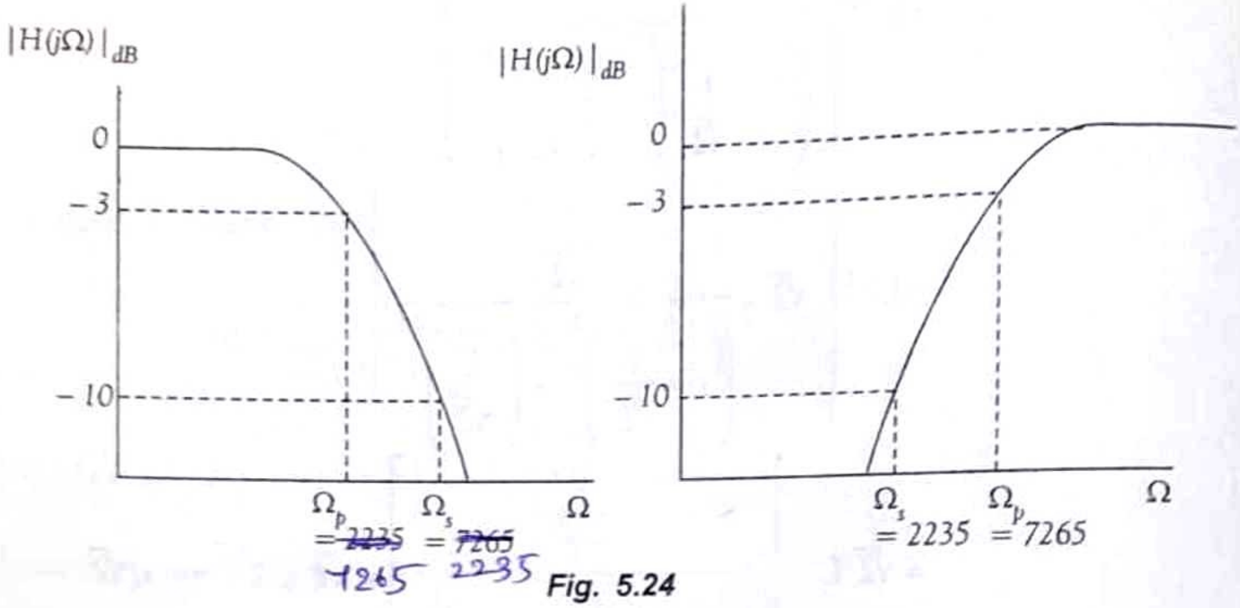

Fig. 5.24

The characteristics are monotonic in both passband and stopband. Therefore, the filter is Butterworth filter.

Prewarping the digital frequencies we have

$$\Omega_p = \frac{2}{T} \tan \frac{\omega_p T}{2} = \frac{2}{2 \times 10^{-4}} \tan \frac{(2000\pi \times 2 \times 10^{-4})}{2}$$

$$= 10^4 \tan (0.2\pi) = 7265 \text{ rad/sec}$$

$$\Omega_s = \frac{2}{T} \tan \frac{\omega_s T}{2} = \frac{2}{2 \times 10^{-4}} \tan \frac{(700\pi \times 2 \times 10^{-4})}{2}$$

$$= 10^4 \tan (0.07\pi) = 2235 \text{ rad/sec}$$

First we design a lowpass filter for the given specifications and use suitable transformation to obtain transfer function of highpass filter.

The order of the filter

$$N = \frac{\log \sqrt{\dfrac{10^{0.1\alpha_s} - 1}{10^{0.1\alpha_p} - 1}}}{\log \dfrac{\Omega_s}{\Omega_p}} = \frac{\log \sqrt{\dfrac{10^{0.1(10)} - 1}{10^{0.1(3)} - 1}}}{\log \dfrac{7265}{2235}} = \frac{\log 3}{\log 3.25} = \frac{0.4771}{0.5118} = 0.932$$

Therefore, we take $N = 1$.

The first-order Butterworth filter for $\Omega_c = 1$ rad/sec is $H(s) = \dfrac{1}{1+s}$

The highpass filter for $\Omega_c = \Omega_p = 7265$ rad/sec can be obtained by using the transformation

$$s \rightarrow \frac{\Omega_c}{s}$$

$$\text{i.e., } s \rightarrow \frac{(7265)}{s},$$

The transfer function of highpass filter

$$H(s) = \left.\frac{1}{s+1}\right|_{s=\frac{7265}{s}}$$

$$= \frac{s}{s+7265}$$

Using bilinear transformation

$$H(z) = H(s)\Big|_{s=\frac{2}{T}\left(\frac{1-z^{-1}}{1+z^{-1}}\right)}$$

$$= \left.\frac{s}{s+7265}\right|_{s=\frac{2}{2\times 10^{-4}}\left(\frac{1-z^{-1}}{1+z^{-1}}\right)}$$

$$= \frac{10000\left(\dfrac{1-z^{-1}}{1+z^{-1}}\right)}{10000\left(\dfrac{1-z^{-1}}{1+z^{-1}}\right)+7265}$$

$$= \frac{0.5792(1-z^{-1})}{1-0.1584z^{-1}}$$

Example 5.15: *Design a third order Butterworth digital filter using impulse invariant technique. Assume sampling period $T = 1\ sec$.*

Solution

From the table 5.1, for $N = 3$, the transfer function of a normalised Butterworth filter is given by

$$H(s) = \frac{1}{(s+1)(s^2+s+1)}$$

$$= \frac{A}{s+1} + \frac{B}{s+0.5+j\,0.866} + \frac{C}{s+0.5-j\,0.866}$$

Solving for A, B, C we get

$$H(s) = \frac{1}{s+1} + \frac{-0.5+0.288j}{s+0.5+j\,0.866} + \frac{-0.5-0.288j}{s+0.5-j\,0.866}$$

$$= \frac{1}{s-(-1)} + \frac{-0.5+0.288j}{s-(-0.5-j\,0.866)} + \frac{-0.5-0.288\,j}{s-(-0.5+j\,0.866)}$$

In impulse invariant technique

$$\text{If } H(s) = \sum_{k=1}^{N} \frac{c_k}{s-p_k}, \text{ then } H(z) = \sum_{k=1}^{N} \frac{c_k}{1-e^{p_k T}z^{-1}}$$

Therefore,

$$H(z) = \frac{1}{1-e^{-1}z^{-1}} + \frac{-0.5+j\,0.288}{1-e^{-0.5}e^{-j\,0.866}z^{-1}} + \frac{-0.5-j0.288}{1-e^{-0.5}e^{j\,0.866}z^{-1}}$$

$$= \frac{1}{1-0.368z^{-1}} + \frac{-1+0.66z^{-1}}{1-0.786z^{-1}+0.368z^{-2}}$$

Example 5.16: *Apply impulse invariant method and find H(z) for* $H(s) = \dfrac{s+a}{(s+a)^2+b^2}$

Solution

The inverse laplace transform of given function is

$$h(t) = \begin{cases} e^{-at} & \cos(bt) \text{ for } t \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

Sampling the function produces

$$h(nT) = \begin{cases} e^{-anT} \cos(bnT) & \text{for } n \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

$$H(z) = \sum_{n=0}^{\infty} e^{-anT} \cos(bnT) z^{-n}$$

$$= \sum_{n=0}^{\infty} \left[e^{-anT} z^{-n} \left(\frac{e^{jbnT} + e^{-jbnT}}{2} \right) \right]$$

$$= \frac{1}{2} \sum_{n=0}^{\infty} \left[(e^{-(a-jb)T} z^{-1})^n + (e^{-(a+jb)T} z^{-1})^n \right]$$

$$= \frac{1}{2} \left[\frac{1}{1 - e^{-(a-jb)T} z^{-1}} + \frac{1}{1 - e^{-(a+jb)T} z^{-1}} \right]$$

$$= \frac{1 - e^{-aT} \cos(bT) z^{-1}}{1 - 2e^{-aT} \cos(bT) z^{-1} + e^{-2aT} z^{-2}}$$

Example 5.17: *An analog filter has a transfer function* $H(s) = \dfrac{10}{s^2 + 7s + 10}$. *Design a digital filter equivalent to this using impulse invariant method.*

Solution

Given $H(s) = \dfrac{10}{s^2 + 7s + 10}$

$$= \frac{-3.33}{s+5} + \frac{3.33}{s+2} = \frac{3.33}{s-(-5)} + \frac{3.33}{s-(-2)}$$

Using impulse invariant technique we have

$$H(z) = \frac{-3.33}{1 - e^{-5T} z^{-1}} + \frac{3.33}{1 - e^{-2T} z^{-1}}$$

Assume $T = 1$ sec

$$= \frac{0.4287\, z^{-1}}{1 - 0.1421\, z^{-1} + 0.0009 z^{-2}} \qquad (\because T = 1)$$

Example 5.18: *Determine H(z) that results when the bilinear transformation is applied to* $H_a(s) = \dfrac{s^2 + 4.525}{s^2 + 0.692s + 0.504}$

Solution

In bilinear transformation

$$H(z) = H(s)\Big|_{s = \frac{2}{T}\left[\frac{1-z^{-1}}{1+z^{-1}}\right]}$$

Assume $T = 1$ sec

Then

$$H(z) = \frac{\left[2\left(\dfrac{1-z^{-1}}{1+z^{-1}}\right)\right]^2 + 4.525}{4\left[\dfrac{1-z^{-1}}{1+z^{-1}}\right]^2 + 0.692 \times 2 \times \left[\dfrac{1-z^{-1}}{1+z^{-1}}\right] + 0.504}$$

$$= \frac{1.4479 + 0.1783\,z^{-1} + 1.4479z^{-2}}{1 - 1.18752\,z^{-1} + 0.5299z^{-2}}$$

5.13 Realization of Digital filters

A digital filter transfer function can be realized in a variety of ways. There are two types of realizations 1. Recursive, 2. Non recursive

1. For recursive realization the current output $y(n)$ is a function of past outputs, past and present inputs. This form corresponds to an Infinite Impulse response (IIR) digital filter. In this section we discuss about this type of realization.

2. For non-recursive realization current output sample $y(n)$ is a function of only past and present inputs. This form corresponds to a Finite Impulse Response (FIR) digital filter.

IIR filter can be realized in many forms. They are

1. Direct form - I realization
2. Direct form - II realization
3. Transposed direct form realization