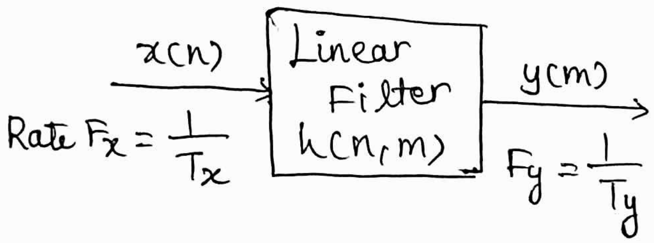
Multirate Digital Signal processing

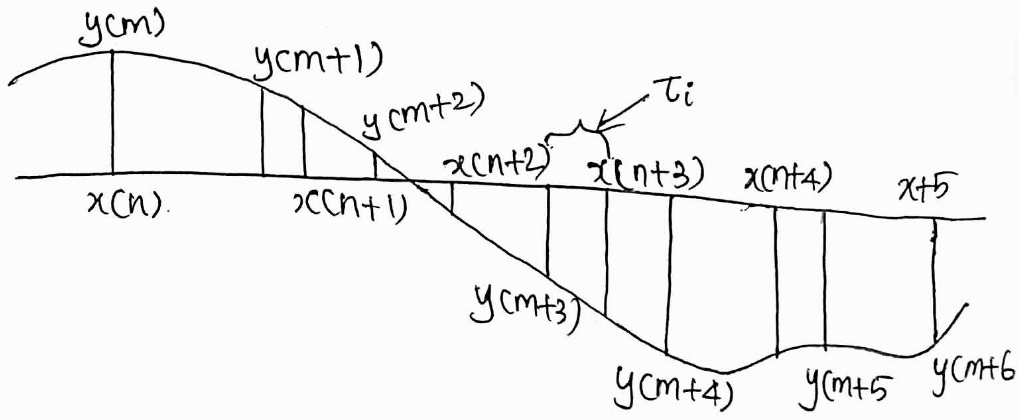
Introduction:-

The process of Sampling rate conversion in the digital domain can be viewed as a linear filtering operation. The input signal $x(n)$ is characterized by the sampling rate $F_x = \frac{1}{T_x}$ and the output signal $y(m)$ is characterized by the sampling rate $F_y = \frac{1}{T_y}$, where T_x and T_y are the corresponding sampling intervals. In the main part of our treatment, the ratio of $\frac{F_y}{F_x}$ is constrained to be rational.

$$\frac{F_y}{F_y} = \frac{I}{D}$$

where D and I are relatively prime integers.





sampling rate Conversion viewed as a linear filtering process.

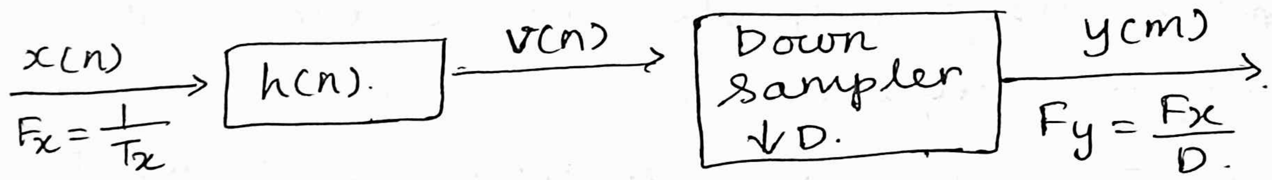
Decimation ~~of~~ by a factor

$x(n) \rightarrow$ signal

$X(\omega) \rightarrow$ Spectrum

* $x(n)$ is to be downsampled by an integer factor D.
* $X(\omega)$ is assumed to be non-zero in the frequency interval $0 \le |\omega| \le \pi$ (or) $|F| \le \frac{F_x}{2}$
* If we reduce the sampling rate by selecting every D^{th} value of $x(n)$, the resulting signal will be an aliased version of $x(n)$, with a folding frequency of $\frac{F_x}{2D}$.
* To avoid aliasing, we must first reduce the bandwidth of $x(n)$ to $F_{max} = \frac{F_x}{2D}$ (or) to $\omega_{max} = \frac{\pi}{D}$.
* Then we may downsample by D and thus avoid aliasing.

Decimation process:-



* The input sequence $x(n)$ is passed through a lowpass filter, charactorized by the impulse response $h(n)$ and a frequency response $H_D(\omega)$, which ideally satifies the condition,

$$H_D(\omega) = \begin{cases} 1, & |\omega| \le \frac{\pi}{D} \\ 0, & \text{otherwise.} \end{cases}$$

* Thus the filter eliminates the spectrum of $X(\omega)$ in the range $\frac{\pi}{D} < \omega < \pi$.

* The output of the filter is a sequence $v(n)$ given as

$$v(n) = \sum_{K=0}^{\infty} h(K)\, x(n-K)$$

* which is then downsampled

$$y(m) = v(mD$$

$$= \sum_{K=0}^{\infty} h(K)\, x(mD-K)$$

* Although the filtering operation on $x(n)$ is linear and time invariant.

* The downsampling operation in combination with the filtering results in a time-~~in~~ Variant system.

This is easily Verified.

* The overall linear operation on $x(n)$ is not time in Variant.

* The frequency domain characteristics of the output sequence ~~x~~ $y(m)$ can be obtained by the relating the spectrum of $y(m)$ to the spectrum of the input sequence $x(n)$

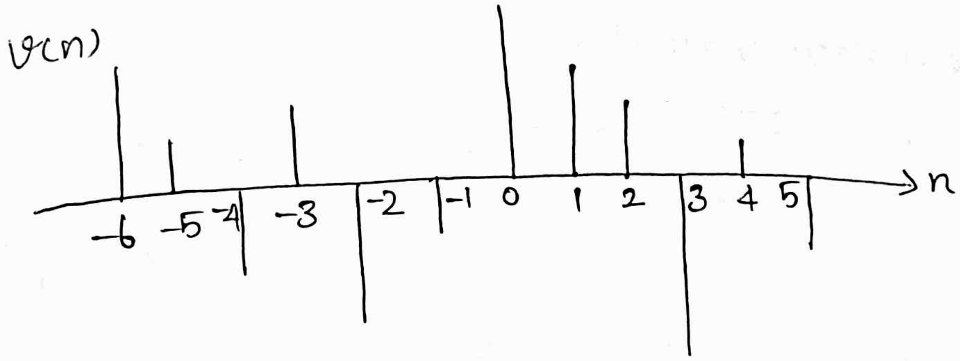
$$\bar{v}(n) = \begin{cases} v(n), & n = 0, \pm D, \pm 2D, \ldots \\ 0, & \text{otherwise} \end{cases}$$

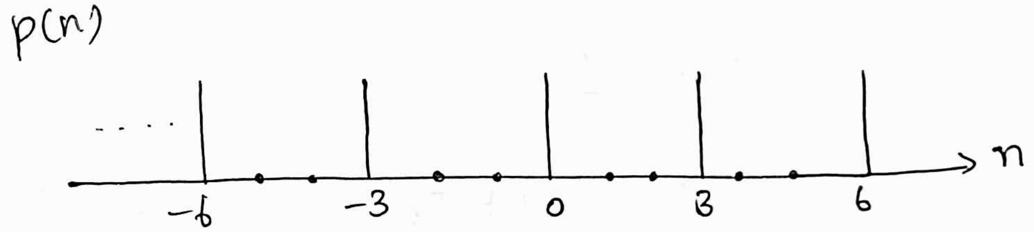
* The ~~f~~ Discrete Fourier Series representation of $p(n)$ is

$$p(n) = \frac{1}{D} \sum_{K=0}^{D-1} e^{j2\pi Kn/D}$$

$$\therefore \bar{v}(n) = v(n)\, p(n)$$

$$y(m) = \bar{v}(mD)$$
$$= v(mD)\, p(mD)$$
$$= v(mD)$$





The z-transform of the output sequence $y(m)$ is

$$Y(z) = \sum_{m=-\infty}^{\infty} y(m) z^{-m}$$

$$= \sum_{m=-\infty}^{\infty} \bar{v}(mD) z^{-m}$$

$$Y(z) = \sum_{m=-\infty}^{\infty} \bar{v}(m) z^{-m/D}$$

where the last step follows from the fact that $\bar{v}(m) = 0$, except at multiplies of D.

$$Y(z) = \sum_{m=-\infty}^{\infty} v(m) \left[\frac{1}{D} \sum_{K=0}^{D-1} e^{j2\pi mK/D} \right] z^{-m/D}$$

$$= \frac{1}{D} \sum_{K=0}^{D-1} \sum_{m=-\infty}^{\infty} v(m) \left(e^{-j2\pi K/D} z^{1/D}\right)^{-m}$$

$$= \frac{1}{D} \sum_{K=0}^{D-1} V\left(e^{-j2\pi K/D} z^{1/D}\right)$$

$$= \frac{1}{D} \sum_{K=0}^{D-1} H_D(e^{-j2\pi K/D} z^{1/D})\, X(e^{-j2\pi K/D} z^{1/D})$$

where the last step follows $V(Z) = H_D(z) X(z)$

* The frequency variable is

$$\omega_y = \frac{2\pi F}{F_y} = 2\pi F T_y \qquad (\because F_y = 1/T_y \Rightarrow T_y = 1/F_y).$$

∴ the sampling rates are

$$F_y = \frac{F_x}{D}$$

$$\omega_x = \frac{2\pi F}{F_x} = 2\pi F T_x \qquad (\because F_x = 1/T_x).$$

$$\omega_y = D\omega_x$$

Thus the frequency range $0 \leq |\omega_x| \leq \frac{\pi}{D}$ is stretched into the corresponding frequency range $0 \leq |\omega_y| \leq \pi$ by the downsampling process.

* We conclude that the spectrum $Y(\omega_y)$

$$Y(\omega_y) = \frac{1}{D} \sum_{K=0}^{D-1} H_D\left(\frac{\omega_y - 2\pi K}{D}\right) \times \left(\frac{\omega_y - 2\pi K}{D}\right)$$

with a properly designed filter $H_D(\omega)$, the aliasing is eliminated and the first term vanish.

$$\text{Hence, } Y(\omega_y) = \frac{1}{D} H_D\left(\frac{\omega_y}{D}\right) \times \left(\frac{\omega_y}{D}\right)$$

$$= \frac{1}{D} \times \left(\frac{\omega_y}{D}\right)$$

for $0 \leq |\omega_y| \leq \pi$.

Interpolation by a Factor

An increase in the sampling rate by an integer factor of I can be accomplished by interpolating $I-1$ new samples between successive values of the signal.

* let $v(m)$ denote a sequence with a rate

$$F_y = IF_x ,$$

which is obtained from $x(n)$ by adding $I-1$ zeros between successive values of $x(n)$.

$$v(m) = \begin{cases} x(m/I), & m = 0, \pm I, \pm 2I, \ldots \\ 0, & \text{otherwise} \end{cases}$$

and its sampling rate is identical to the rate of $y(m)$.

$$V(z) = \sum_{m=-\infty}^{\infty} v(m) z^{-m}$$

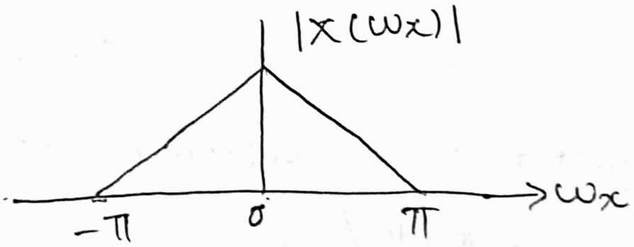
$$= \sum_{m=-\infty}^{\infty} x(m) z^{-mI}$$

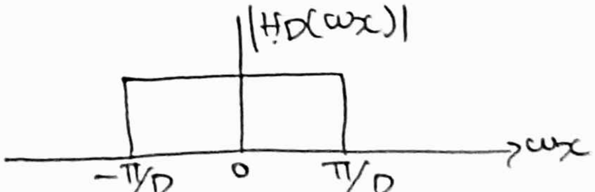
$$= X(z^I)$$

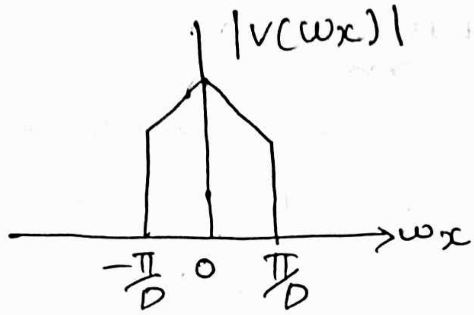
$$V(\omega_y) = X(\omega_y I)$$

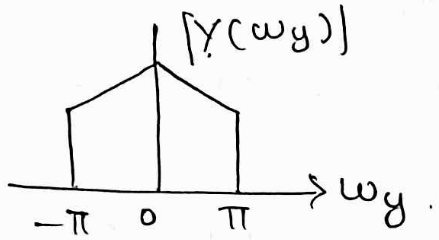
* The frequency variables ω_x and ω_y are

$$\omega_y = \frac{\omega_x}{I}$$





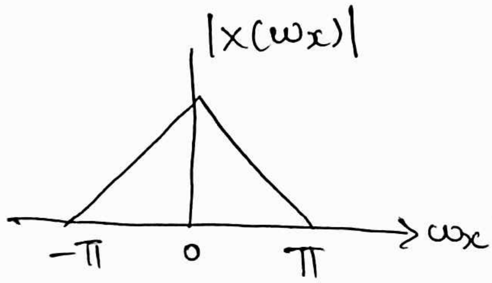


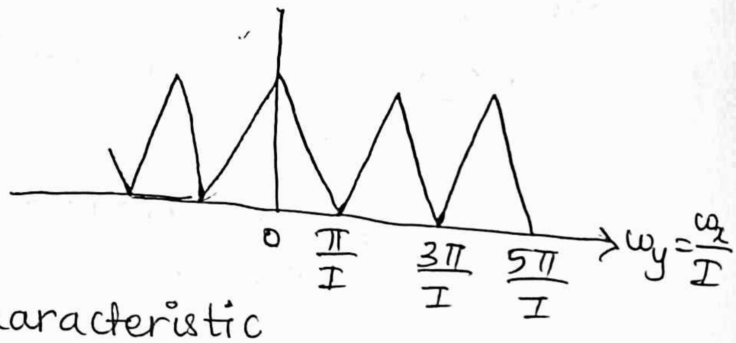


Spectra of signals in the decimation of $x(n)$ by a factor D.

Range $\Rightarrow 0 \leq \omega_y \leq \pi/I$

Spectra of $x(n)$ and $v(n)$ where $V(\omega_y) = X(\omega_y I)$.





Ideally has the characteristic

$$H_I(\omega_y) = \begin{cases} C, & 0 \leq |\omega_y| \leq \pi/I \\ 0, & \text{otherwise.} \end{cases}$$

where C is a scale factor

* The output spectrum is

$$Y(\omega_y) = \begin{cases} C X(\omega_y I), & 0 \leq |\omega_y| \leq \pi/I \\ 0, & \text{otherwise} \end{cases}$$

* The scale factor C is selected, so that the output $y(m) = x(m/I)$ for $m = 0, \pm I, \pm 2I, \ldots$

*We select the point $m=0$, Thus

$$y(0) = \frac{1}{2\pi}\int_{-\pi}^{\pi} Y(\omega_y)\, d\omega_y$$

$$= \frac{C}{2\pi}\int_{-\pi/I}^{\pi/I} X(\omega_y I)\, d\omega_y$$

$$y(0) = \frac{C}{I}\frac{1}{2\pi}\int_{-\pi}^{\pi} X(\omega_x)\, d\omega_x \qquad (\because \omega_y = \omega_x/I)$$

$$= \frac{C}{I}\, x(0)$$

Therefore, $C = I$ is the desired normalization factor.

Finally, we indicate that the output sequence $y(m)$ can be expressed as a convolution of the sequence $v(n)$ with the unit sample response $h(n)$ of the lowpass filter, Thus

$$y(m) = \sum_{K=-\infty}^{\infty} h(m-K)\, v(K)$$

Since $v(K) = 0$ except at multiplies of I

where $v(KI) = x(K)$

$$y(m) = \sum_{K=-\infty}^{\infty} h(m-KI)\, x(K)$$

Spectrum of Interpolated Signal (or) Up Sampled

The z-transform of the signal $y(n)$ is given by

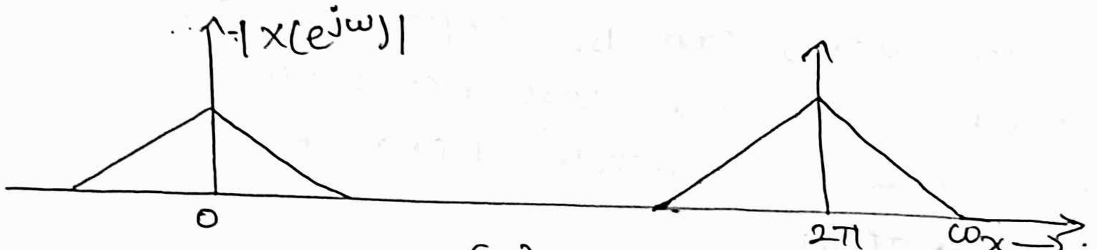
$$Y(z) = \sum_{n=-\infty}^{\infty} y(n) z^{-n}$$

$$= \sum_{n=-\infty}^{\infty} x\left(\frac{n}{L}\right) z^{-n}$$

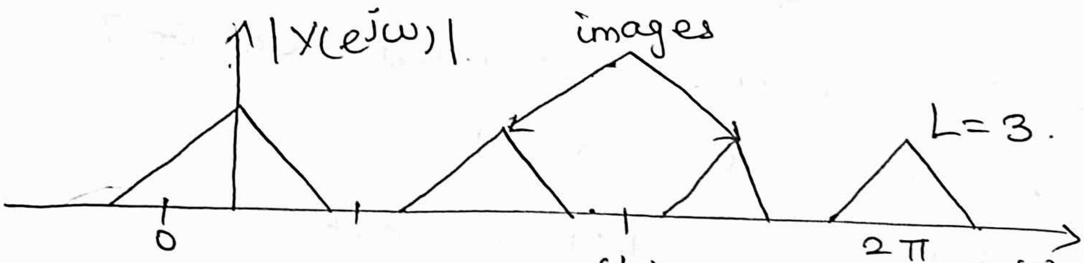
$$= \sum_{n=-\infty}^{\infty} x(n) z^{-Ln} = X(z^{L}) \longrightarrow ①$$

substituting $z = e^{j\omega}$ in ①.

$$Y(e^{j\omega}) = X(e^{j\omega L})$$



(a).



(b)

This figure shows the spectrum $X(e^{j\omega})$.

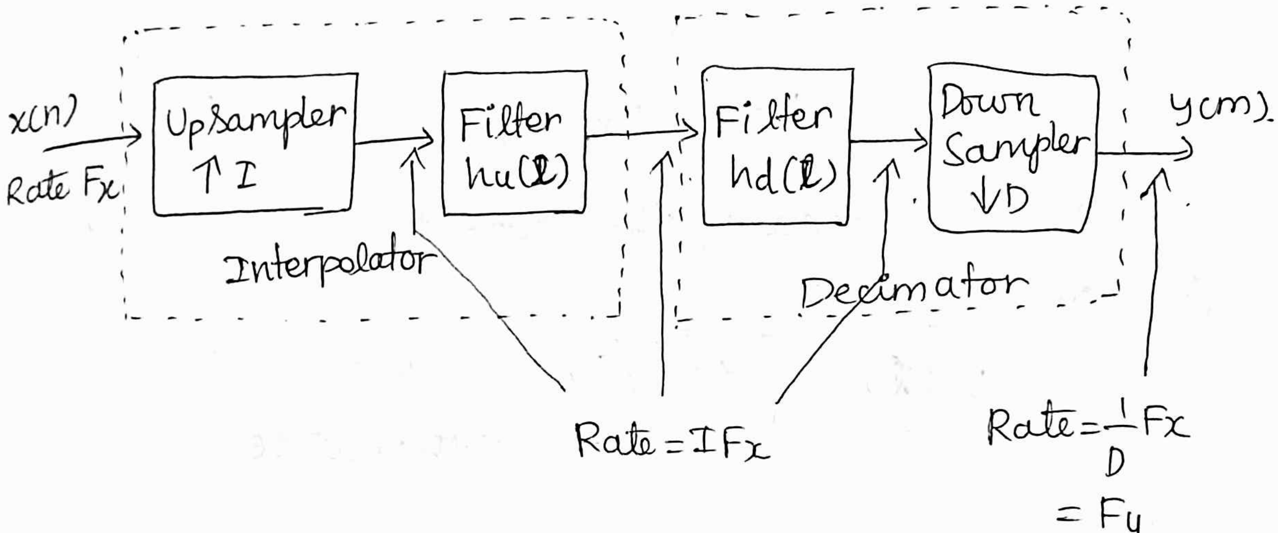
Note:-

The frequency spectrum $X(e^{j3\omega})$ is three fold repetition of $X(e^{j\omega})$.

From figure (b) $\Rightarrow$ interval $-\frac{\pi}{3} < \omega < \frac{\pi}{3}$

The signal $y(n)$ has the same spectrum as $x(n)$ apart from scaling factor. Above this range the spectrum of $y(n)$ has two spectrum of the same form after each one. These are called image spectrum and the phenomena is known as imaging.

Sample rate conversion by a rational factor I/D

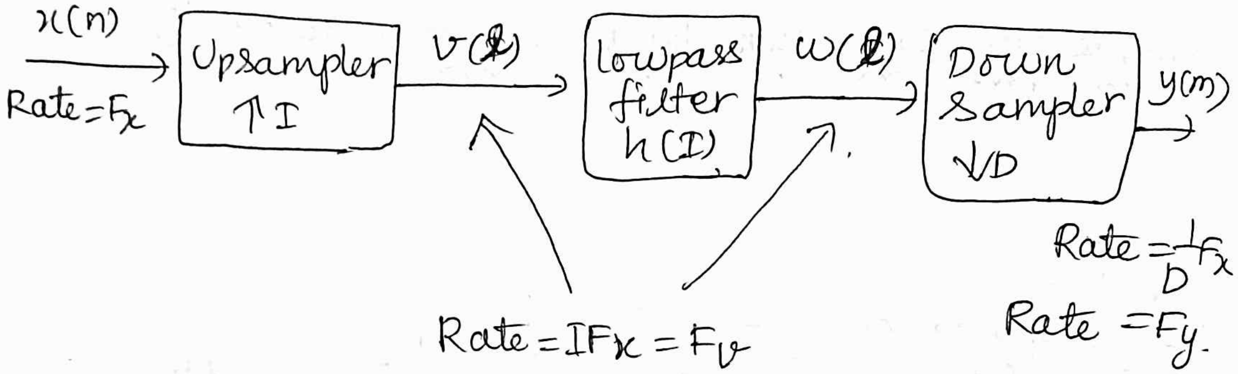


We emphasize that the importance of performing the interpolation first and the decimation second, is to preserve the desired spectral characteristics of $x(n)$. Furthermore, with the cascade configuration, the two filters with impulse response $\{hu(I)\}$ and $\{hd(I)\}$ are operated at the same rate, namely IFx and hence can be combined into a single lowpass filter with impulse response $h(I)$. The frequency response $H(\omega_v)$ of the combined filter must incorporate the filtering operations for both interpolation and decimation, and hence it

Should ideally possess the frequency response characteristics

$$H(\omega_v) = \begin{cases} I, & 0 \le |\omega_v| \le \min(\pi/D, \pi/I) \\ 0, & \text{otherwise} \end{cases}$$

where $\omega_v = \frac{2\pi F}{F_v} = \frac{2\pi F}{I F_x} = \frac{\omega_x}{I}$.



In the domain, the output of the upsampler is the sequence

$$v(l) = \begin{cases} x(l/I), & l = 0, \pm I, \pm 2I, \ldots \\ 0, & \text{otherwise.} \end{cases}$$

and

the output of the linear time-invariant filter is

$$w(l) = \sum_{K=-\infty}^{\infty} h(l-K)\, v(K)$$

$$= \sum_{K=-\infty}^{\infty} h(l-KI)\, x(K)$$

Finally, the output of the sampling rate converter is the sequence $[y(m)]$ obtained by downsampling the sequence $[w(l)]$ by a factor of D. Thus

$$y(m) = w(mD)$$

$$= \sum_{K=-\infty}^{\infty} h(mD - KI)\, x(K)$$

$$K = \left\lfloor \frac{mD}{I} \right\rfloor - n$$

where the notation $\lfloor r \rfloor$ denotes the largest integer contained in r.

$$y(m) = \sum_{n=-\infty}^{\infty} h\left(mD - \left\lfloor \frac{mD}{I} \right\rfloor I + nI\right) x\left(\left\lfloor \frac{mD}{I} \right\rfloor - n\right)$$

$$mD - \left\lfloor \frac{mD}{I} \right\rfloor I = mD \quad \text{modulo } I$$

$$= (mD)_I$$

$$y(m) = \sum_{n=-\infty}^{\infty} h(nI + (mD)_I)\, x\left(\left\lfloor \frac{mD}{I} \right\rfloor - n\right)$$

$$g(n,m) = h(nI + (mD)_I) \qquad -\infty < m, n < \infty$$

where $h(K)$ is the impulse response of the time-invariant lowpass filter

$$g(n, m+KI) = h(nI + (mD + KDI)_I)$$

$$= h(nI + (mD)_I)$$

$$= g(n,m)$$

Hence, $g(n,m)$ is periodic in the variable m with period I.

The frequency-domain relationships can be obtained by combining the results of the interpolation and decimation processes.

Thus,

the spectrum at the output of the linear filter with impulse response $h(l)$ is

$$V(\omega_v) = H(\omega_v) X(\omega_v I)$$

$$= \begin{cases} I X(\omega_v I), & 0 \le |\omega_v| \le \min(\pi/D, \pi/I) \\ 0, & \text{otherwise.} \end{cases}$$

The spectrum of the output sequence $y(m)$ obtained by decimating the sequence $v(n)$ by a factor of D. is

$$Y(\omega_y) = \frac{1}{D}\sum_{K=0}^{D-1} V\left(\frac{\omega_y - 2\pi K}{D}\right)$$

where $\omega_y = D\omega_v$.

Since, the linear filter prevents aliasing

$$Y(\omega_y) = \begin{cases} \frac{1}{D} X\left(\frac{\omega_y}{D}\right), & 0 \le |\omega_y| \le \min\left(\pi, \frac{\pi D}{I}\right) \\ 0, & \text{otherwise} \end{cases}$$

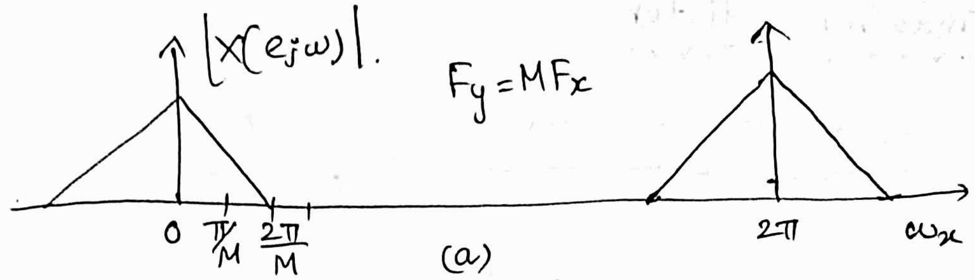
(b) Anti-aliasing.

* The Spectrum obtained after downsampling will overlap if the original spectrum is not band limited to $\omega = \frac{\pi}{M}$. This overlap causes aliasing.

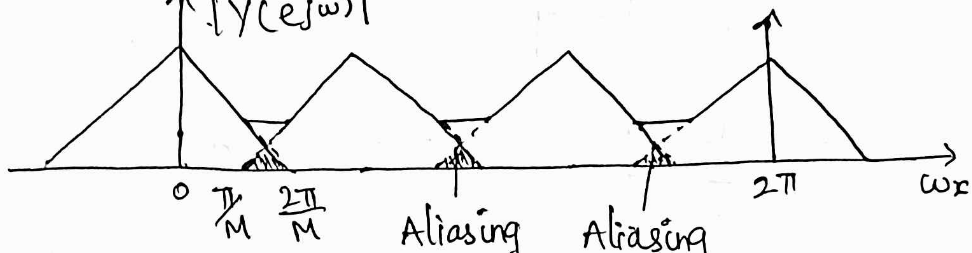
* Therfore aliasing due to downsampling a signal by a factor of M is absent if and only if the signal $x(n)$ is band limited to $\pm\frac{\pi}{M}$. If the signal $x(n)$ is not band limited to $\pm\frac{\pi}{M}$, then a lowpass filter with cut-off frequency $\frac{\pi}{M}$ is used prior to downsampling. This filter is known as anti-aliasing filter.

~~Its ideal mag~~

The complete process, i.e., filtering and then downsampling is referred to as decimation.



(a)

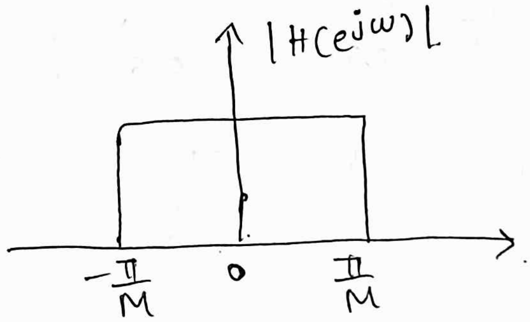


(b).

(a) Frequency spectrum of signal that is not band limited to $\omega_c = \frac{\pi}{M}$

(b) Frequency spectrum of downsampled signal with overlapping

(a).

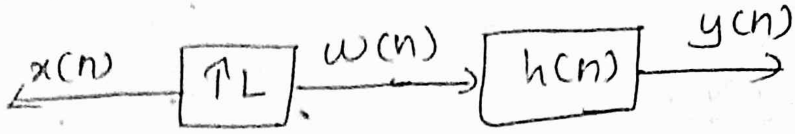


(b)

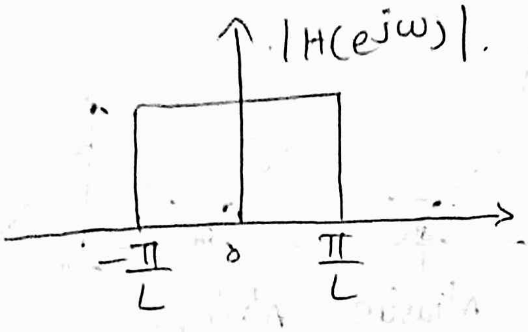
(a) Decimator with an anti-aliasing filter and a down sampler

(b) Ideal magnitude response of anti-aliasing filter.

Anti-imaging filter



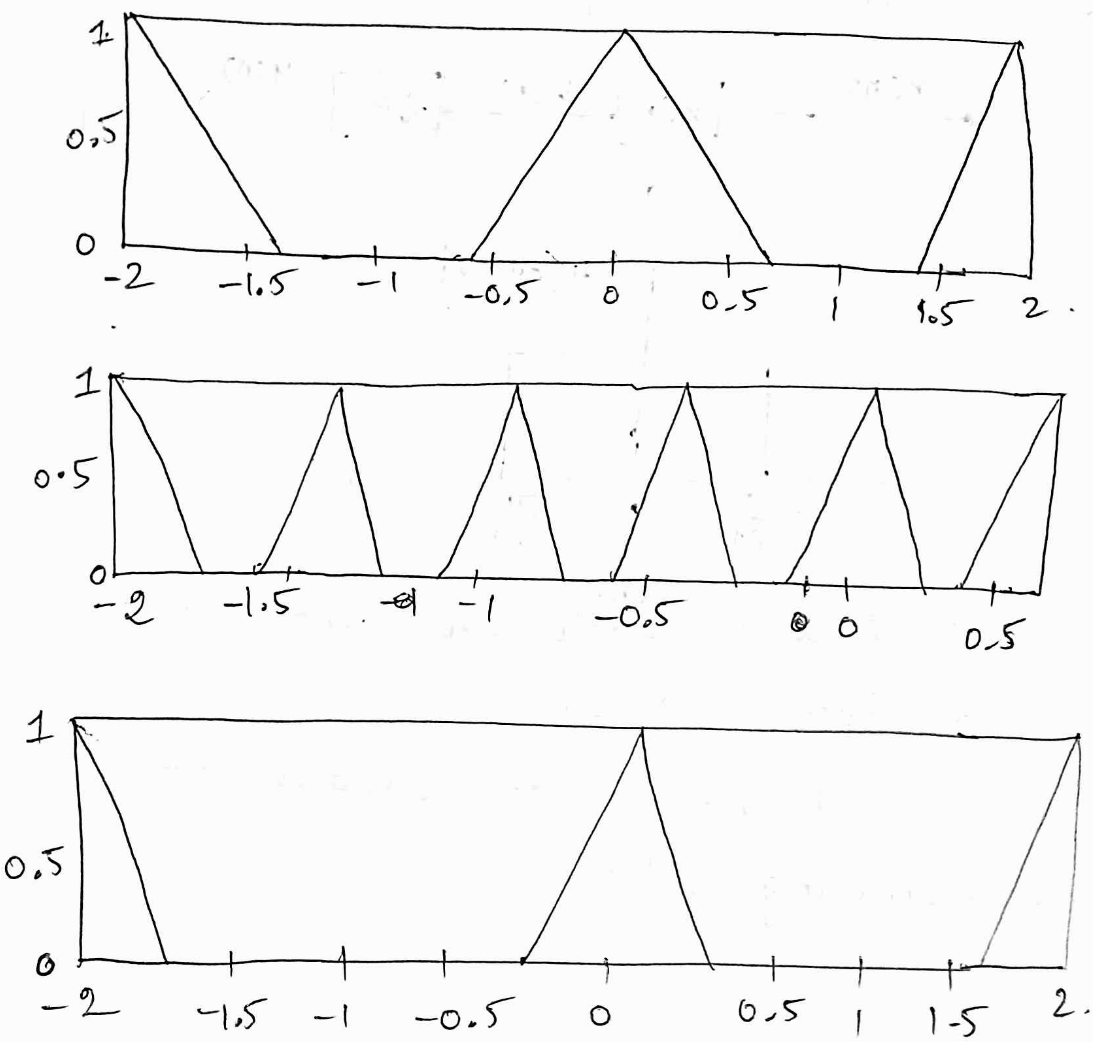
(a)



(b)

(a) Interpolator with an upsampler and anti-imaging filter.

(b) Magnitude response of anti-imaging filter.



(a) Spectrum of the original signal
(b) Spectrum of the upsampled signal
(c) Spectrum of the signal at the output of LPF.

The frequency spectrum of upsampled signal $y(n)$ with a factor L, contains $(L-1)$ additional images of the input spectrum. These $(L-1)$ images are due to the addition of $L-1$ zero samples between successive samples of $x(n)$. Since we are not interested in image spectrum, a lowpass filter with a cutoff frequency $\omega_c = \frac{\pi}{L}$ can be used after upsampler is known as anti-imaging filter. The complete process of upsampling and filtering is known as interpolation.

Polyphase structure of decimator

$$H(z) = \sum_{m=0}^{M-1} z^{-m} P_m(z^M)$$

$$\text{where } P_m(z) = \sum_{n=0}^{[(N+1)/M]} h(Mn+m) z^{-n}$$

The z-transform of an infinite sequence is given by

$$H(z) = \sum_{n=-\infty}^{\infty} h(n) z^{-n}$$

$$H(z) = \sum_{m=0}^{M-1} z^{-m} P_m(z^m)$$

$$\text{where } P_m(z) = \sum_{r=-\infty}^{\infty} h(rM+m) z^{-r}$$

$$H(z) = \sum_{m=0}^{M-1} \sum_{r=-\infty}^{\infty} z^{-m} h(rM+m) z^{-rM}$$

$$= \sum_{m=0}^{M-1} \sum_{r=-\infty}^{\infty} h(rM+m) z^{-(rM+m)}$$

let $h(rM+m) = P_m(r)$

$$\Rightarrow H(z) = \sum_{m=0}^{M-1} \sum_{r=-\infty}^{\infty} P_m(r) z^{-(rM+m)}$$

$$Y(z) = \sum_{m=0}^{M-1} \sum_{r=-\infty}^{\infty} P_m(r) X(z) z^{-(rM+m)}$$

$$y(n) = \sum_{m=0}^{M-1} \sum_{r=-\infty}^{\infty} P_m(r) x[n-(rM+m)]$$

let $x_m(r) = x(rM-m)$ then

$$y(n) = \sum_{m=0}^{M-1} \sum_{r=-\infty}^{\infty} P_m(r) x_m(n-r)$$

$$= \sum_{m=0}^{M-1} P_m(n) * x_m(n)$$

$$= \sum_{m=0}^{M-1} y_m(n).$$

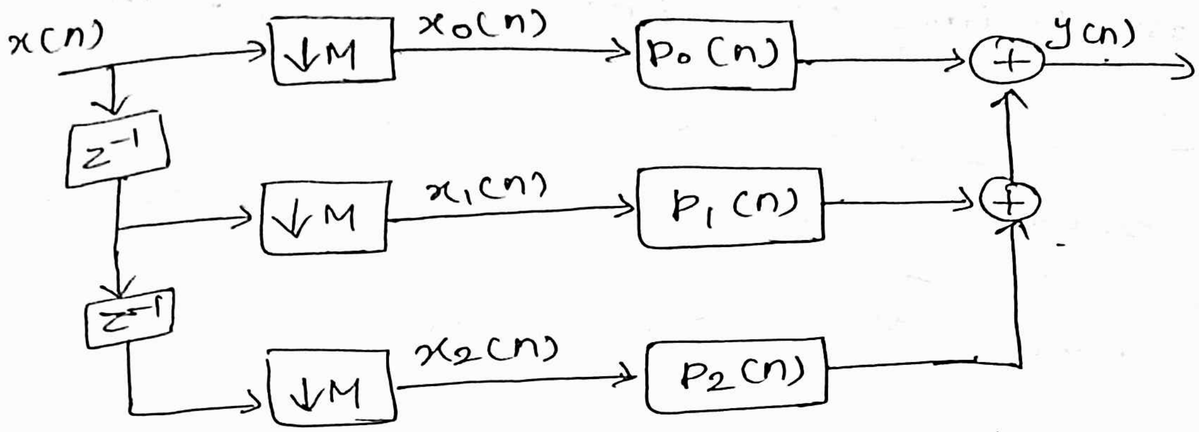
where $y_m(n) = P_m(n) * x_m(n)$

The operation $P_m(n) * x_m(n)$ is known as polyphase convolution and the overall process is polyphase filtering.

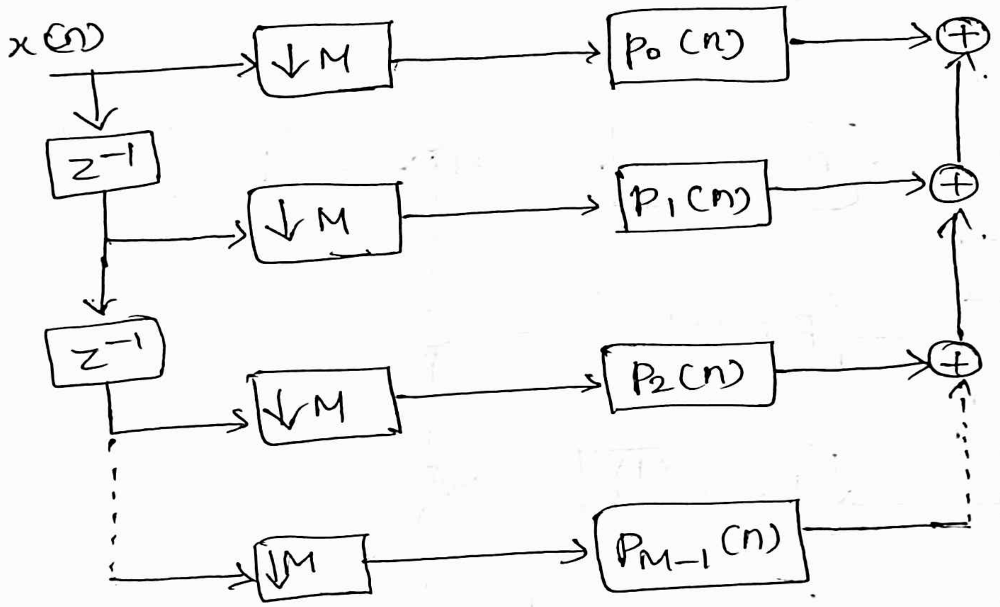
$$y(n) = \sum_{m=0}^{2} y_m(n)$$

$$= y_0(n) + y_1(n) + y_2(n)$$

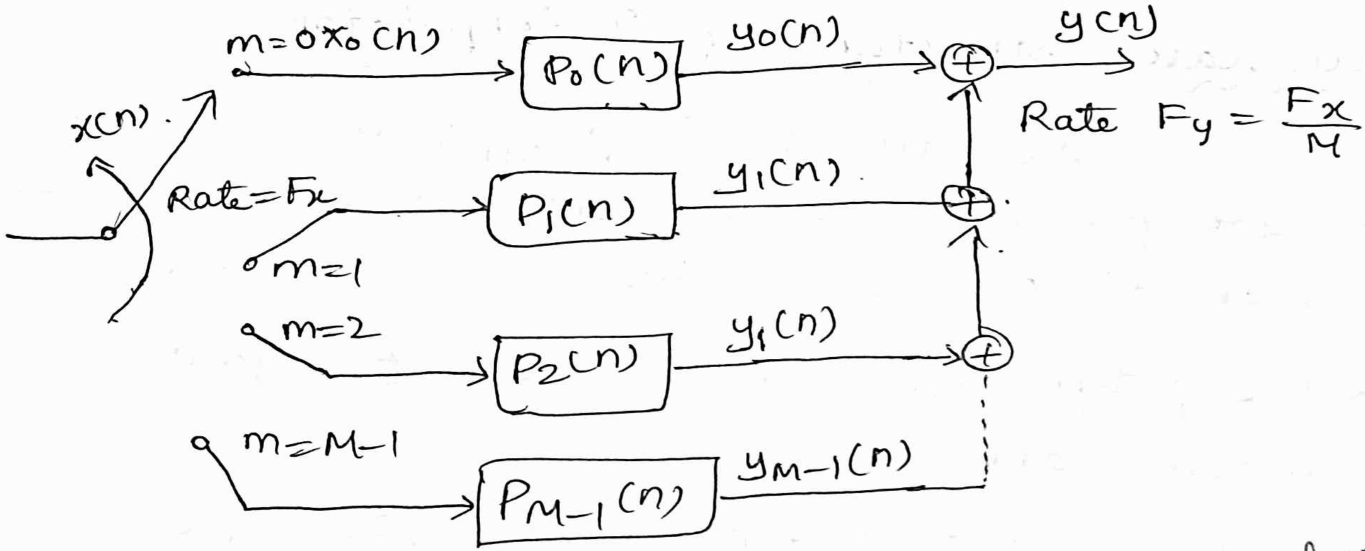
$$= P_0(n) * x_0(n) + P_1(n) * x_1(n) + P_2(n) * x_2(n)$$



polyphase structure of a 3 branch decimator



polyphase structure of a M-branch decimator



Polyphase decimator with a commutator

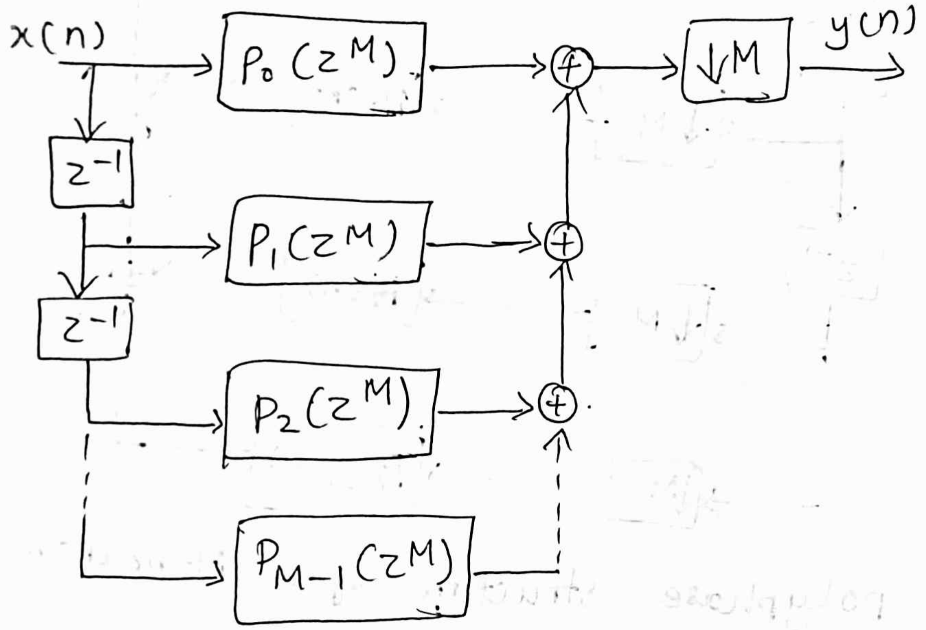
Polyphase Decimation Using the z-transform

$$X(z) \rightarrow \boxed{H(z)} \xrightarrow{V(z)} \boxed{\downarrow M} \xrightarrow{Y(z)}$$

The M-branch polyphase decomposition of H(z) is given by

$$H(z) = \sum_{m=0}^{M-1} z^{-m} P_m(z^{-m}).$$

The subfilters $P_0(z), P_1(z), \dots P_{M-1}(z)$ are FIR filters and when combined in the right phase sequence produce the original filter H(z):

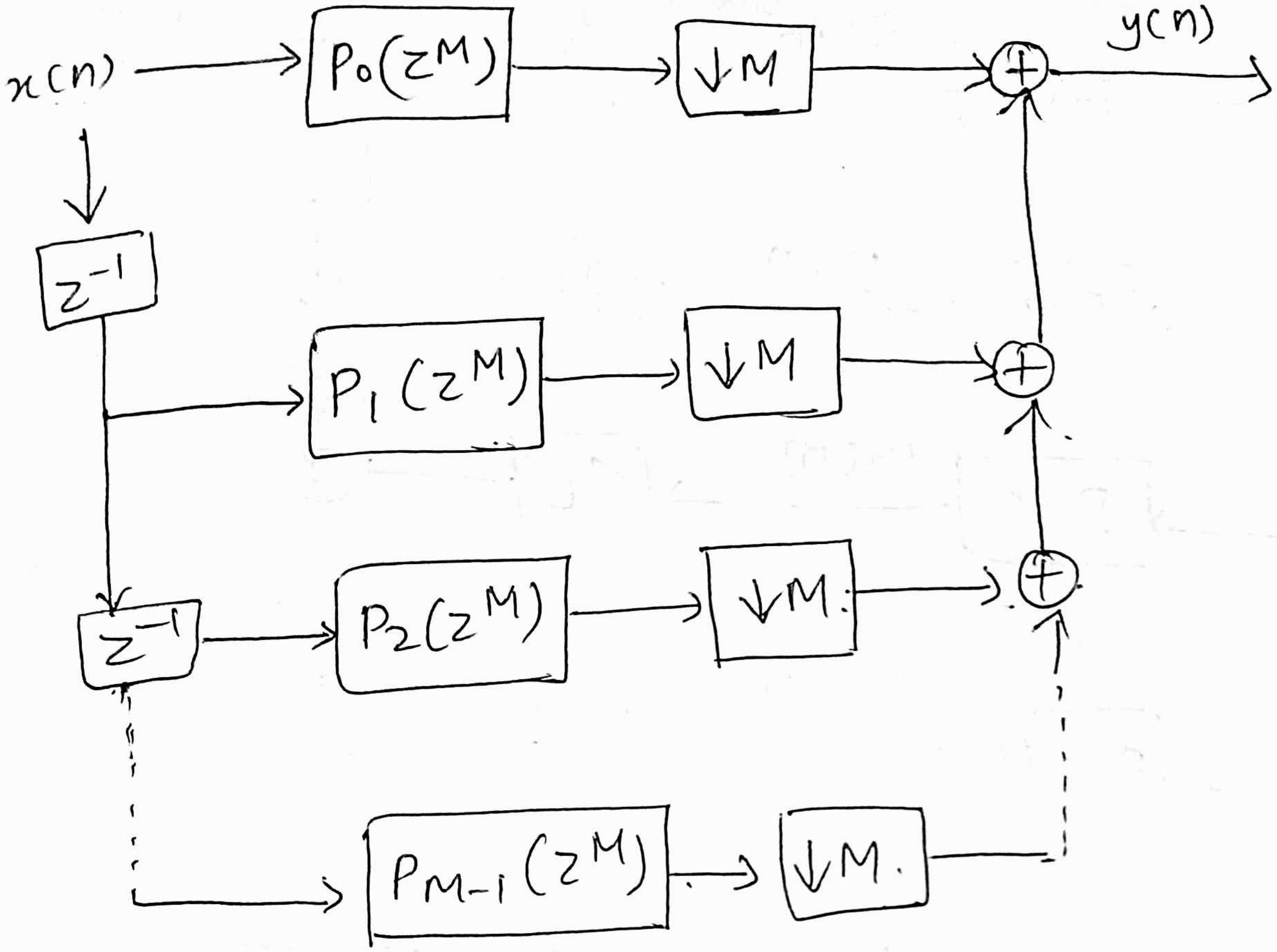


Polyphase structure of Interpolator

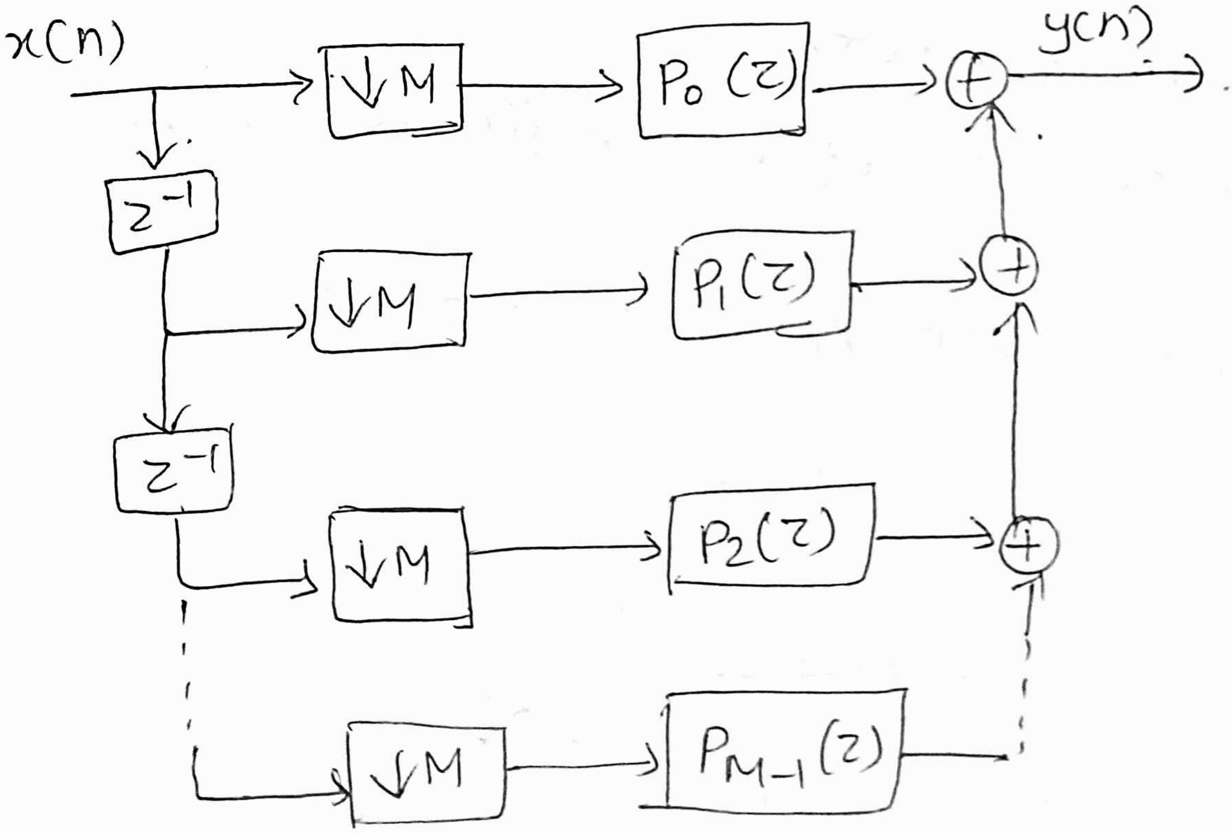
We can obtain polyphase structure of interpolator, which consists of a set of L sub-filters connected in parallel.

* The polyphase components of impulse response are

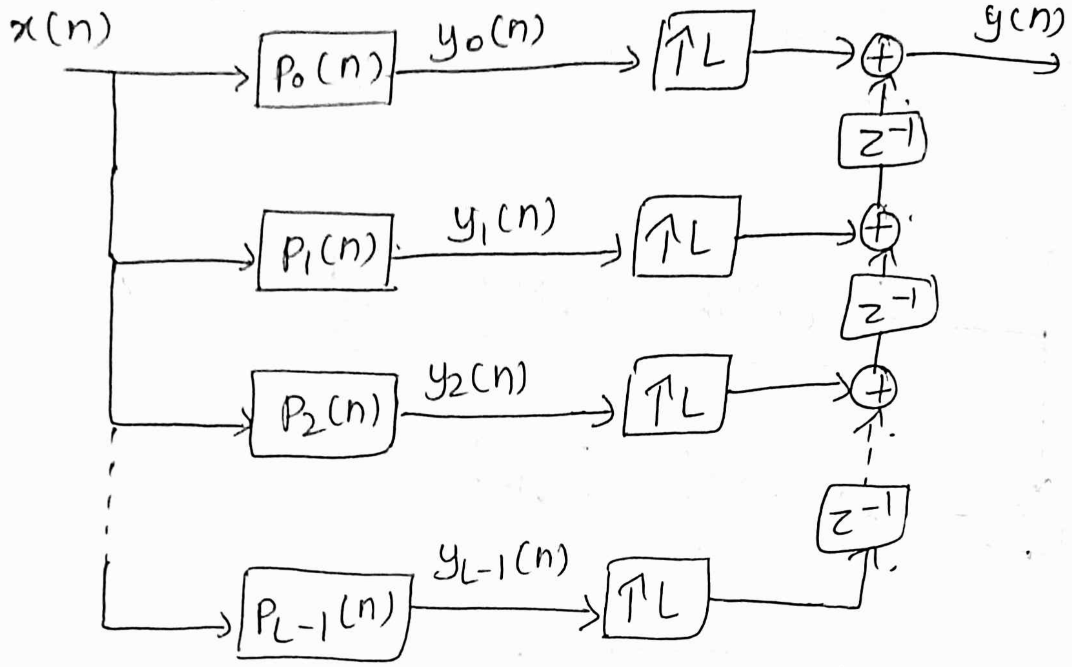
$$P_m(n) = h(nL+m), \quad m = 0, 1, 2, \dots, L-1$$


$x(n)$
$P_0(z^M)$
$\downarrow M$
$y(n)$
z^{-1}
$P_1(z^M)$
$\downarrow M$
z^{-1}
$P_2(z^M)$
$\downarrow M$
$P_{M-1}(z^M)$
$\downarrow M$

(a)


$x(n)$
$\downarrow M$
$P_0(z)$
$y(n)$
z^{-1}
$\downarrow M$
$P_1(z)$
z^{-1}
$\downarrow M$
$P_2(z)$
$\downarrow M$
$P_{M-1}(z)$

(b)

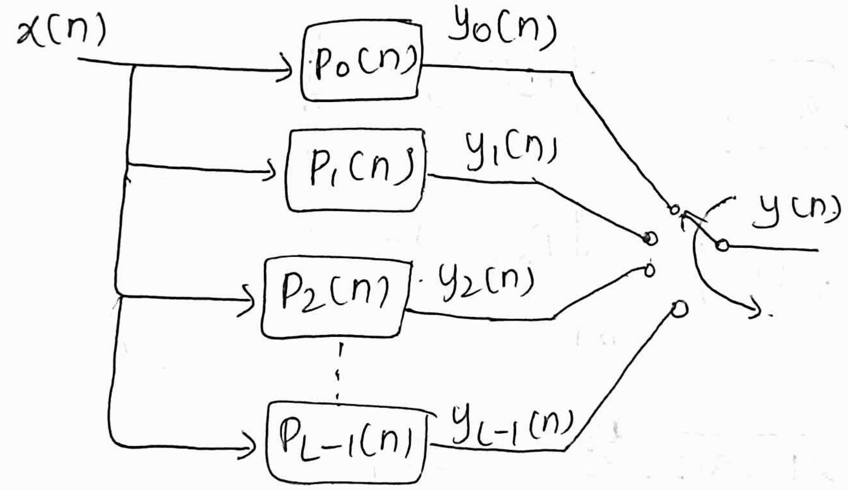


where

h(n) is the impulse response of anti-imaging filter.

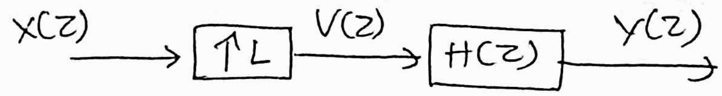
The output of L sub-filters can be represented as

$y_m(n) = x(n)\, p_m(n)$; $m = 0, 1, 2, \ldots L-1$



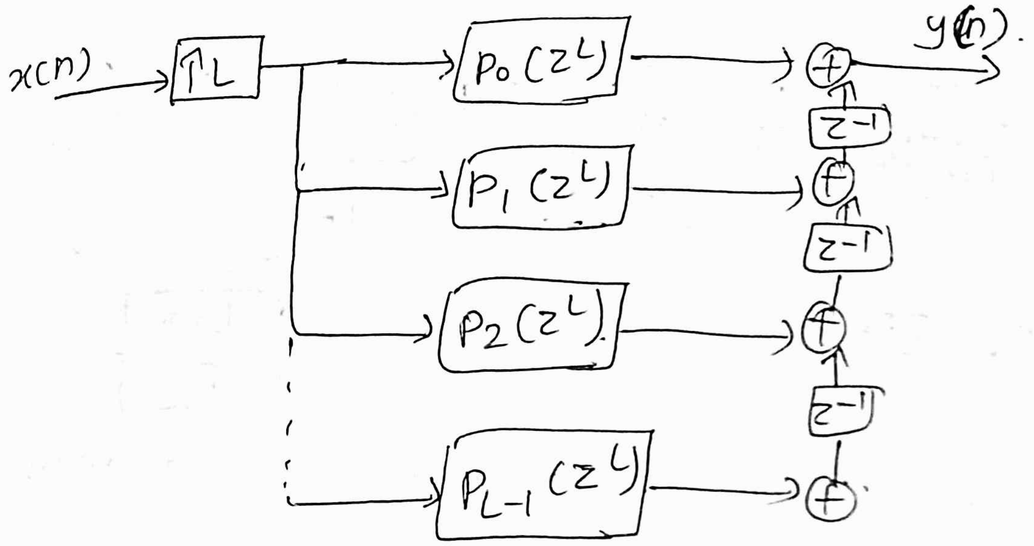
polyphase interpolation using the z-transform.

The block diagram of an interpolator

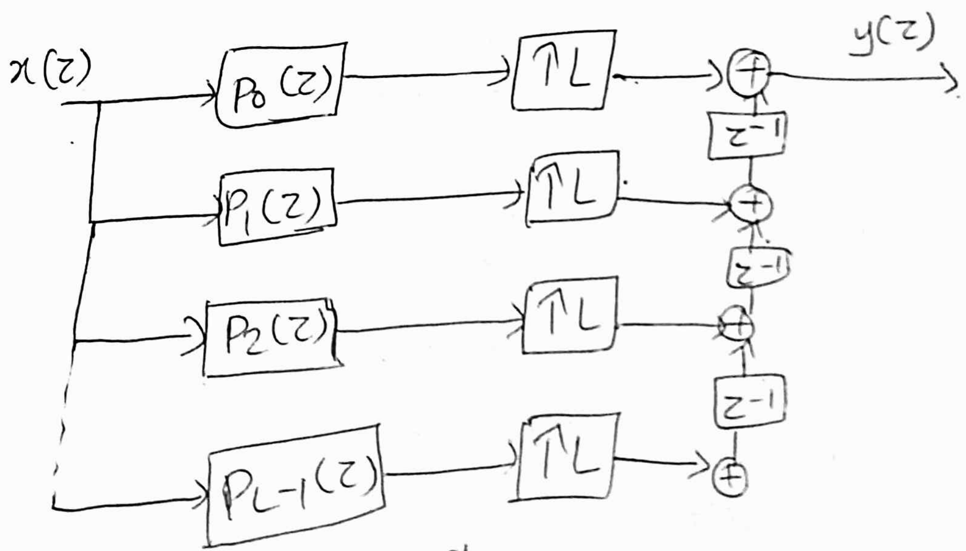


The transfer function H(Z) of the interpolator is given by

$$H(Z) = \sum_{m=0}^{L-1} z^{-m} P_m(z^L).$$



(a)



(b).

Advantages of multirate DSP.

* Reduced computational workload.
* Lower coefficients sensitivity.
* Less memory requirement.
* Low noise.

Practical Applications of multirate DSP.

1) Interfacing of digital systems with different sampling Rates.

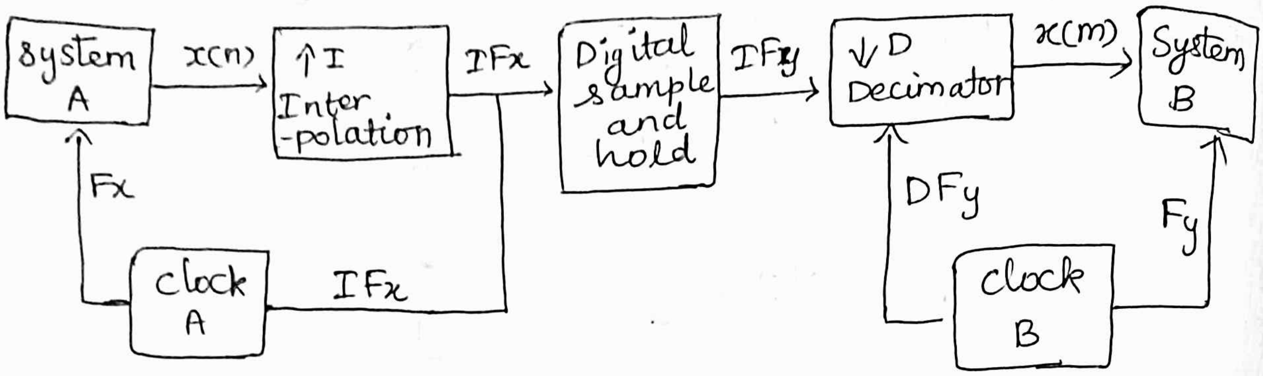


Figure:- Interfacing of two digital systems with different sampling rates.

* In practice we frequently encounter the problem of Interfacing two digital systems that are controlled by independently operating clocks.

* An analog solution to this problem is to convert the signal from the first system to analog form and then resample it at the input to the second system using the clock in this system.

* However, a simpler approach is one where the interfacing is done by a digital method using the basic sample-rate conversion methods.

* let us consider interfacing the two systems with independent ~~wt~~ clocks.

* The output of System A at rate F_x is fed to an interpolator which increases the sampling rate by I.

* The output of the interpolator ~~sy~~ is fed at the rate IF_x to a digital sample-and-hold which serves as the interface to system B at the high rate sampling IF_x.

* Signals from the digital sample-and hold are read out into system B at the clock rate DF_y of system B.

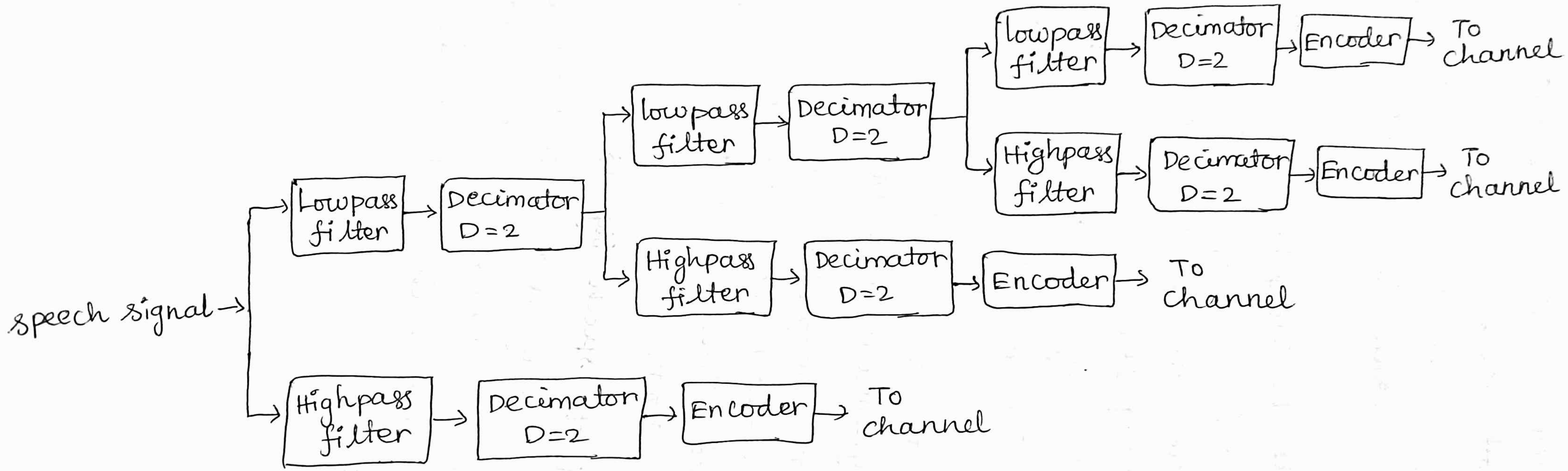
* Thus the output rate from the sample and hold is not synchronized with the input rate.

* By using linear interpolation in place of the digital sample-and-hold, we can reduce the distortion and thus reduce the size of the interpolator factor.

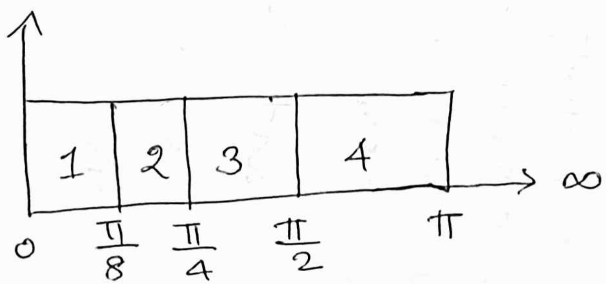
2) Subband coding of speech signals.

* Subband coding is a method, where the speech signal is subdivided into several frequency bands and each band is digitally encoded separately.

Block diagram of a Subband speech coder.



(a)



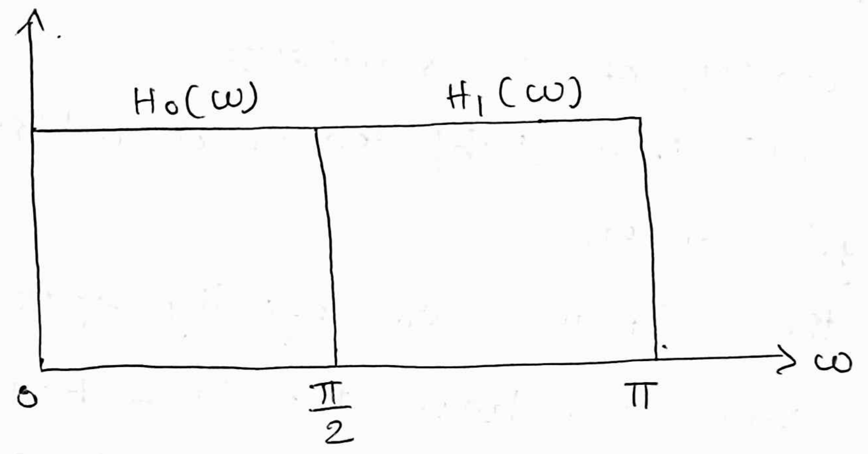
(b)

* The Synthesis Method for the Subband encoded speech signal is basically the reverse of the encoding process.

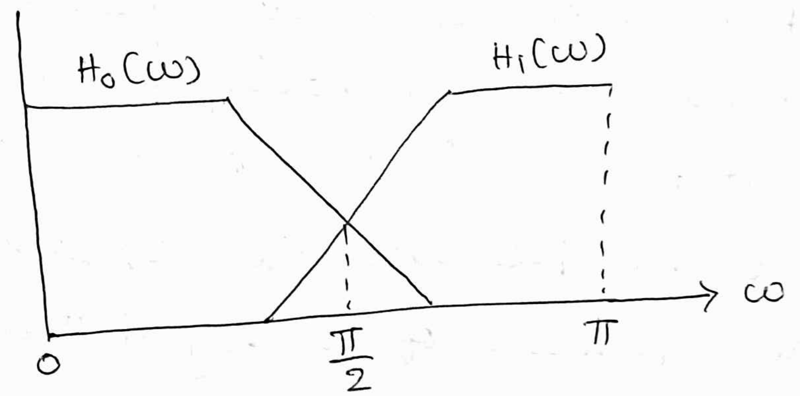
* The signals in adjacent lowpass and high pass frequency bands are interpolated, filtered and combined

* A pair of QMF (Quadrature Mirror Filters) is used in the signal synthesis of each octave of the signal.

* Subband coding is also an effective method to achieve data compression in image signal processing.



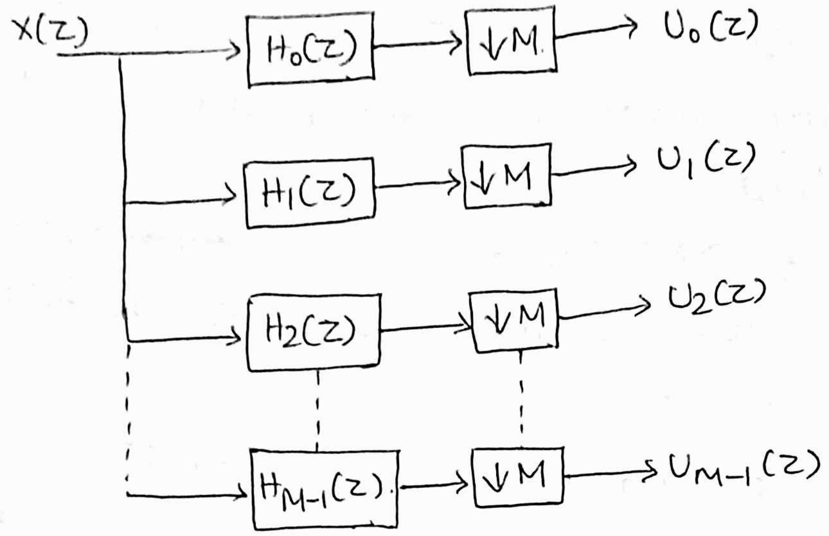
(a) Brick wall Filters.



(b) QMF

Filter Banks

1) Analysis Filter Bank



★ The M-channel analysis filter bank in shown in Figure.

★ It consists of M subfilters

★ The individual subfilter $H_K(z)$ is known as analysis filter.

★ All the subfilters are equally spaced in frequency and each have the same band width.

★ The Spectrum of the input signal $X(e^{j\omega})$ lies in the range $0 \leq \omega \leq \pi$.

★ The Filter bank splits the signal into number of subbands each having a bandwidth $\frac{\pi}{M}$.

★ The filter $H_0(z)$ is lowpass, $H_1(z)$ to $H_{M-2}(z)$ are bandpass and $H_{M-1}(z)$ is highpass

★ As the spectrum of the signal is band limited to $\frac{\pi}{M}$, the sampling rate can be reduced by a factor M.

★ The downsampling moves all the subband signals into the base band $0 \leq \omega \leq \frac{\pi}{M}$.

2) Synthesis Filter Bank

* The M-Channel synthesis filter Bank is dual of M-Channel analysis filter bank.

* In this case each $U_m(z)$ is fed to an Upsampler.

* The Upsampling process produce the signal $U_m(z^M)$.

* These signals are applied to filters $G_m(z)$ and finally added to get the output signal $\hat{X}(z)$.

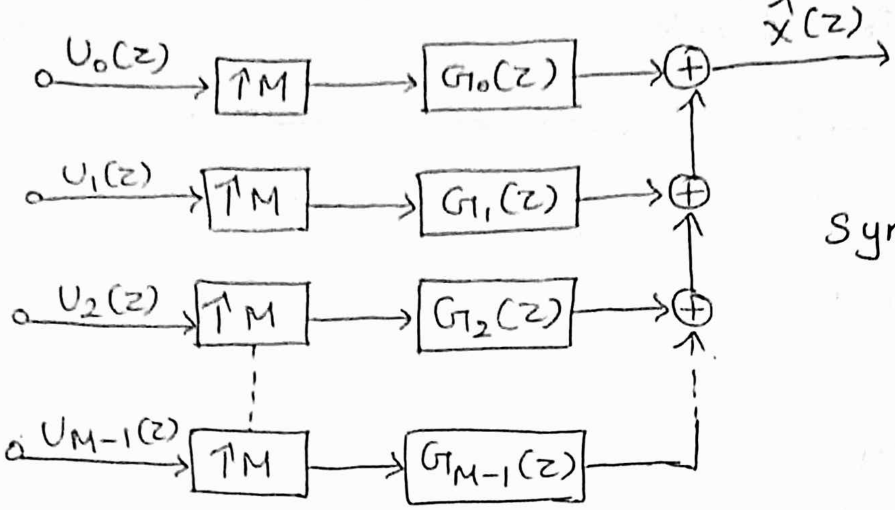
* The filter $G_0(z)$ to $G_{M-1}(z)$ have the same characteristics as the analysis filters $H_0(z)$ to $H_{M-1}(z)$.

3) Sub-band coding filter bank

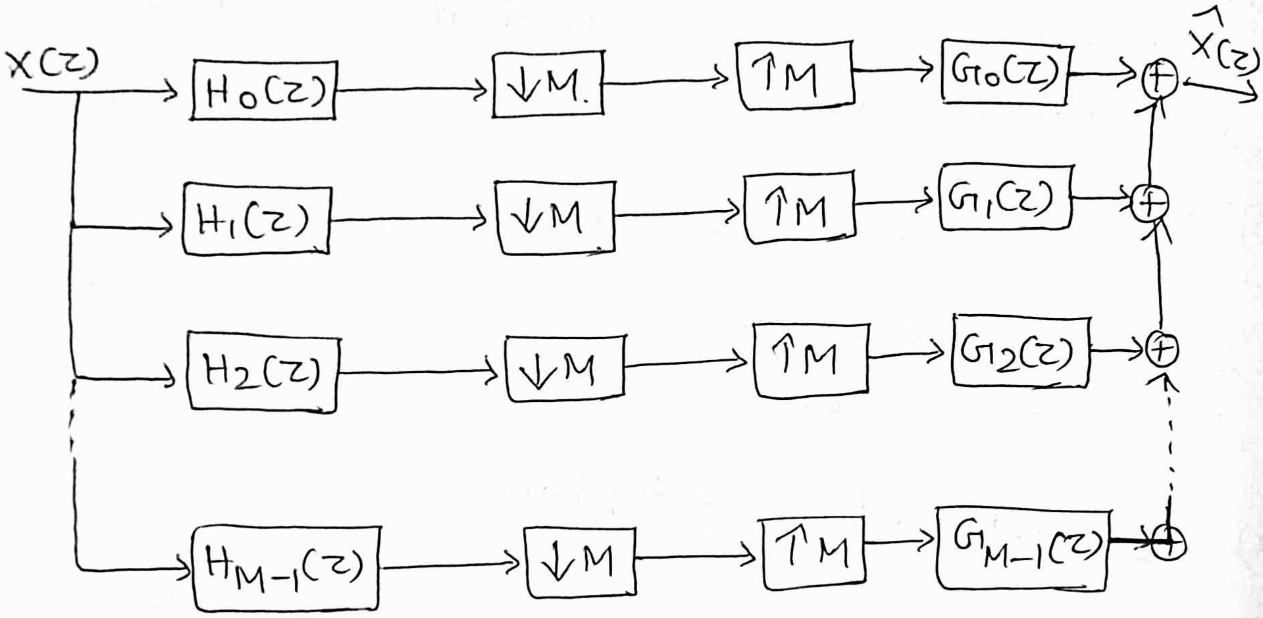
⇒ The analysis filter bank splits the broadband input signal $x(n)$ into M non-overlapping frequency band signals $X_0(z), X_1(z), \ldots X_{M-1}(z)$ of equal bandwidth.

⇒ These outputs are coded and transmitted.

⇒ The synthesis filterbank is used to reconstruct output signal $\hat{x}(n)$ which should approximate the original signal.



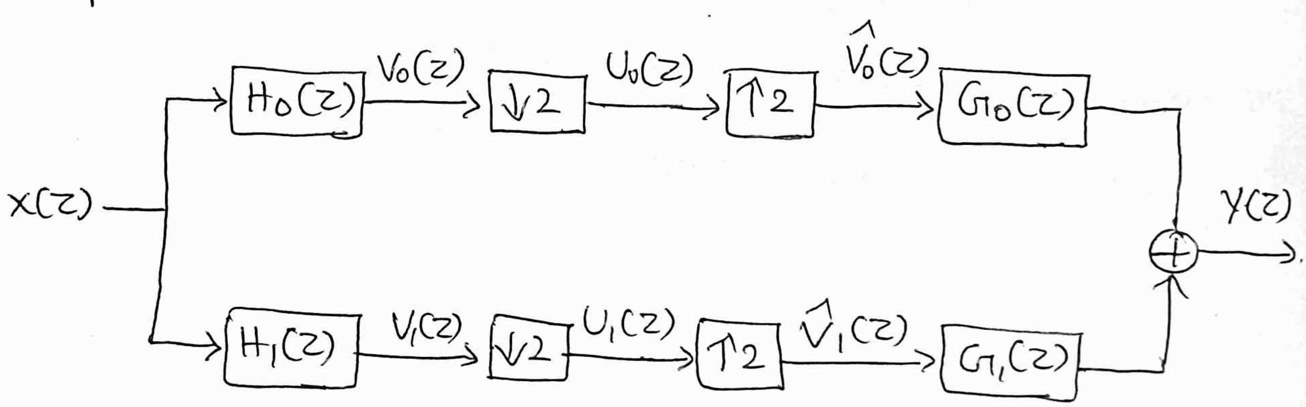
Synthesis Filter Bank



Subband Coding filter Bank.

4) Quadrature - Mirror Filter (QMF) Bank

* It is a two channel Subband coding filter bank with complementary frequency responses.



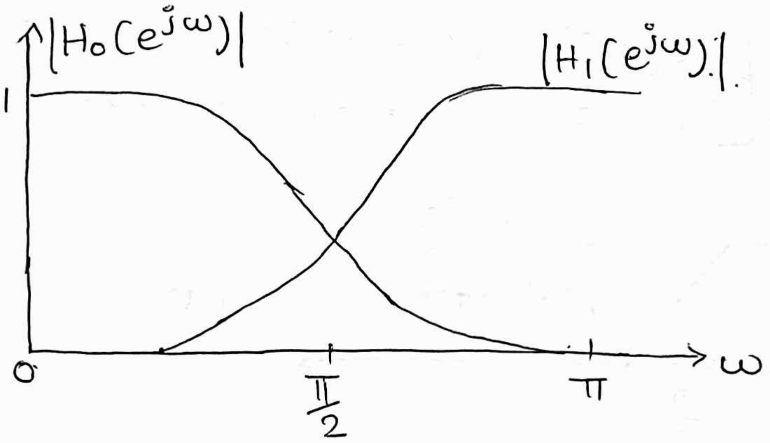
A Two channel QMF bank.

It consists of two sections.

1. Analysis section
2. Synthesis section

Analysis section

* The analysis section is a two-channel analysis filter bank.



* The signal $x(n)$ is fed to a lowpass filter $H_0(z)$ and a highpass filter $H_1(z)$ simultaneously.

* Since the normalized frequency range is $\omega=0$ to $\omega=\pi$, the cutoff frequency of highpass and lowpass filters are chosen as $\frac{\pi}{2}$

* The output of lowpass and highpass filters are

$$V_0(z) = X(z) H_0(z) \longrightarrow ①$$

$$V_1(z) = X(z)\, H_1(z) \longrightarrow ②$$

Down sampling with $M=2$, yields the subband signals

$$U_0(z) = \frac{1}{2}\left[V_0(z^{1/2}) + V_0(-z^{1/2})\right] \longrightarrow ③$$

$$U_1(z) = \frac{1}{2}\left[V_1(z^{1/2}) + V_1(-z^{1/2})\right] \longrightarrow ④$$

Sub eqn ① in ③

$$U_0(z) = \frac{1}{2}\left[X(z^{1/2})\,H_0(z^{1/2}) + X(-z^{1/2})\,H_0(-z^{1/2})\right]$$

Sub eqn ② in ④

$$U_1(z) = \frac{1}{2}\left[X(z^{1/2})\,H_1(z^{1/2}) + X(-z^{1/2})\,H_1(-z^{1/2})\right]$$

In matrix form

$$\begin{bmatrix} U_0(z) \\ U_1(z) \end{bmatrix} = \frac{1}{2}\begin{bmatrix} H_0(z^{1/2}) & H_0(-z^{1/2}) \\ H_1(z^{1/2}) & H_1(-z^{1/2}) \end{bmatrix}\begin{bmatrix} X(z^{1/2}) \\ X(-z^{1/2}) \end{bmatrix} \rightarrow ⑤$$

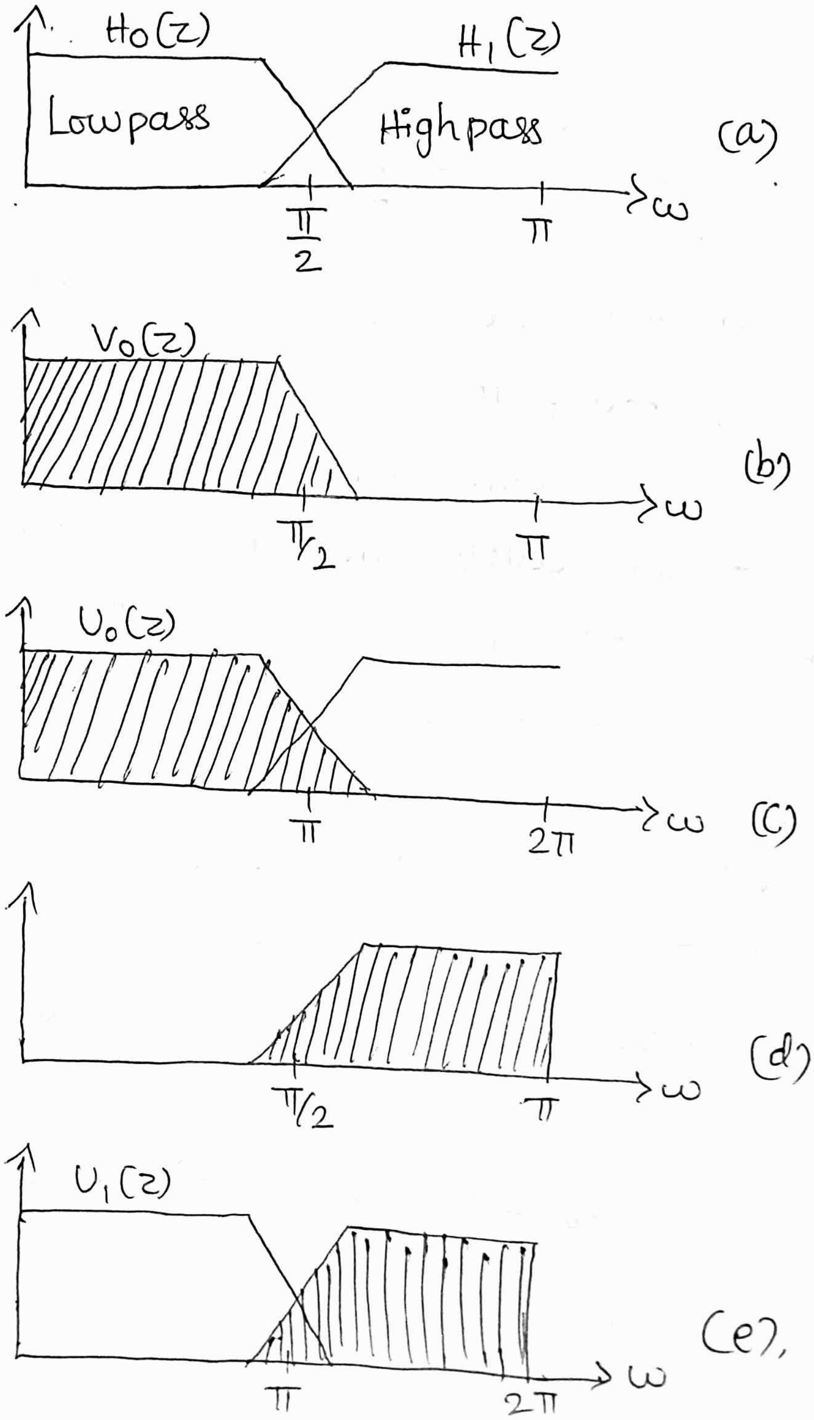


Fig :- Frequency Response characteristics of the signals.

Synthesis Section

* The signals $U_0(z)$ and $U_1(z)$ are fed to the synthesis filter bank.

* Here the signals $U_0(z)$ and $U_1(z)$ are upsampled and then passed through two filters $G_0(z)$ and $G_1(z)$ respectively.

* The filter $G_0(z)$ is a lowpass filter and eliminates the image spectrum of $U_0(z)$ in the range $\frac{\pi}{2} \leq \omega \leq \pi$.

* Meanwhile the highpass filter $G_1(z)$ eliminates most of the image spectra of $U_1(z)$ in the range $0 \leq \omega \leq \frac{\pi}{2}$.

* The image spectra is not completely eliminated.

The reconstructed output of the filter bank is

$$Y(z) = G_0(z)\hat{V}_0(z) + G_1(z)\hat{V}_1(z) \rightarrow (1)$$

$$= G_0(z)U_0(z^2) + G_1(z)U_1(z^2) \rightarrow (2)$$

where
$$\boxed{\begin{aligned}\hat{V}_0(z) &= U_1(z^2)\\ \hat{V}_1(z) &= U_1(z^2)\end{aligned}}$$

From (2)

$$Y(z) = \begin{bmatrix} G_0(z) & G_1(z) \end{bmatrix} \begin{bmatrix} U_0(z^2) \\ U_1(z^2 \end{bmatrix}$$

$$\begin{bmatrix} U_0(z^2) \\ U_1(z^2) \end{bmatrix} = \frac{1}{2} \begin{bmatrix} H_0(z) & H_0(-z) \\ H_1(z) & H_1(-z) \end{bmatrix} \begin{bmatrix} X(z) \\ X(-z) \end{bmatrix}$$

$$Y(z) = \begin{bmatrix} G_0(z) & G_1(z) \end{bmatrix} \frac{1}{2} \begin{bmatrix} H_0(z) & H_0(-z) \\ H_1(z) & H_1(-z) \end{bmatrix} \begin{bmatrix} X(z) \\ X(-z) \end{bmatrix}$$

$$Y(z) = \frac{1}{2}\left[G_0(z) H_0(z) + G_1(z) H_1(z)\right] X(z)$$

$$+ \frac{1}{2}\left[G_0(z) H_0(-z) + G_1(z) H_1(-z)\right] X(-z)$$

$$= T(z) X(z) + A(z) X(-z)$$

where

$$T(z) = \frac{1}{2}\left[G_0(z) H_0(z) + G_1(z) H_1(z)\right]$$

$$A(z) = \frac{1}{2}\left[G_0(z) H_0(-z) + G_1(z) H_1(-z)\right]$$

$T(z) \rightarrow$ describes the transfer function of the filter and is called discrete transfer function.

$A(z) \rightarrow$ due to aliasing component.

5) Alias free filter Bank.

To obtain alias-free filter bank, we can choose the synthesis filter such that

$$A(z) = 0$$

i.e.,

$$G_0(z) H_0(-z) + G_1(z) H_1(-z) = 0$$

A simple ~~so~~ sufficient condition for alias cancellation is

$$G_0(z) = H_1(-z)$$

$$G_1(z) = -H_0(z)$$

$$Y(z) = T(z) X(z)$$

sub $z=e^{j\omega}$ yields

$$Y(e^{j\omega}) = T(e^{j\omega})\, X(e^{j\omega})$$

$$= |T(e^{j\omega})|\, e^{j\theta(\omega)} X(e^{j\omega})$$

* If $|T(e^{j\omega})|$ is constant for all ω, then we do not have any amplitude distortion.
  This condition is satisfied, when $T(e^{j\omega})$ is an allpass filter.

* If $T(e^{j\omega})$ have linear phase there is no phase distortion, this condition satisfied when $\theta(\omega) = \alpha\omega + \beta$, for constant α and β.

* Therefore, we need $T(e^{j\omega})$ to be a linear phase all-pass filter in order to avoid any magnitude or phase distortion.

* If an alias-free QMF bank has no amplitude and phase distortion, then it is called a perfect reconstruction (PR) QMF bank.

$$Y(z) = K z^{-l} X(z)$$

The output

$$y(n) = Kx(n-l)$$

That is the reconstructed output of a PRQMF bank is a scaled, delayed replica of the output.