Function of two independent variables:

A Symbol z Which has a definite Value for every pair of values of x and y is called a function of two independent Variables x and y and we write

$$z = f(x,y) \quad or \quad \phi(x,y)$$

Limits:

The function $f(x,y)$ is said to tend to the limit l as $x \to a$ and $y \to b$ if and only if the limit l is independent of the path followed by the Point (x,y) as $x \to a$ and $y \to b$

then $\lim\limits_{\substack{x \to a \\ y \to b}} f(x,y) = l$.

Continuity:

A function $f(x,y)$ is said to be continuous at the point (a,b) if

$\lim\limits_{\substack{x \to a \\ y \to b}} f(x,y)$ exist and $= f(a,b)$

Partial derivatives

Let $z = f(x, y)$ be a function of two variables x and y.

$$\frac{\partial z}{\partial x} = p, \quad \frac{\partial z}{\partial y} = q, \quad \frac{\partial^2 z}{\partial x^2} = r, \quad \frac{\partial^2 z}{\partial x \partial y} = s$$

$$\frac{\partial^2 z}{\partial y^2} = t$$

① If $z = e^{ax+by} f(ax-by)$ prove that $b\frac{\partial z}{\partial x} + a\frac{\partial z}{\partial y} = 2abz$

Soln

$$\frac{\partial z}{\partial x} = e^{ax+by} a \, f(ax-by) + e^{ax+by} f'(ax-by) \cdot a$$

$$\frac{\partial z}{\partial y} = e^{ax+by} \cdot b \cdot f(ax-by) + e^{ax+by} f'(ax-by) \cdot (-b)$$

$$b\frac{\partial z}{\partial x} + a\frac{\partial z}{\partial y} = ab\, e^{ax+by} f(ax-by) + \cancel{ab\, e^{ax+by} f'(ax-by)} + ab\, e^{ax+by} f(ax-by) - \cancel{ab\, e^{ax+by} f'(ax-by)}$$

$$= 2ab\, e^{ax+by} f(ax-by)$$

$$= 2abz.$$

② If $u = f(y-z, z-x, x-y)$ prove that $\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = 0$.

Soln :

$$r = y - z \qquad s = z - x \qquad t = x - y$$

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial r} \cdot \frac{\partial r}{\partial x} + \frac{\partial u}{\partial s} \cdot \frac{\partial s}{\partial x} + \frac{\partial u}{\partial t} \cdot \frac{\partial t}{\partial x}$$

$$= \frac{\partial u}{\partial r} \cdot 0 + \frac{\partial u}{\partial s}(-1) + \frac{\partial u}{\partial t}(1)$$

$$\frac{\partial u}{\partial x} = -\frac{\partial u}{\partial s} + \frac{\partial u}{\partial t} \quad \text{———} \quad (1)$$

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial r} \cdot \frac{\partial r}{\partial y} + \frac{\partial u}{\partial s} \cdot \frac{\partial s}{\partial y} + \frac{\partial u}{\partial t} \cdot \frac{\partial t}{\partial y}$$

$$= \frac{\partial u}{\partial r} \cdot 1 + \frac{\partial u}{\partial s} \cdot 0 + \frac{\partial u}{\partial t} \cdot (-1)$$

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial r} - \frac{\partial u}{\partial t} \quad \text{———} \quad (2)$$

$$\frac{\partial u}{\partial z} = \frac{\partial u}{\partial r} \cdot \frac{\partial r}{\partial z} + \frac{\partial u}{\partial s} \cdot \frac{\partial s}{\partial z} + \frac{\partial u}{\partial t} \cdot \frac{\partial t}{\partial z}$$

$$= -\frac{\partial u}{\partial r} + \frac{\partial u}{\partial s}(1) + \frac{\partial u}{\partial t}(0)$$

$$\frac{\partial u}{\partial z} = -\frac{\partial u}{\partial r} + \frac{\partial u}{\partial s} \quad \text{———} \quad (3)$$

Add (1) , (2) & (3)

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = -\frac{\partial u}{\partial s} + \frac{\partial u}{\partial t} + \frac{\partial u}{\partial r} - \frac{\partial u}{\partial t} - \frac{\partial u}{\partial r} + \frac{\partial u}{\partial s}$$

$$= 0$$

③ If $u = f\left(\frac{x}{y}, \frac{y}{z}, \frac{z}{x}\right)$, Prove that :

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = 0.$$

Soln:

$$r = \frac{x}{y}, \quad s = \frac{y}{z}, \quad t = \frac{z}{x}$$

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial r}\cdot\frac{\partial r}{\partial x} + \frac{\partial u}{\partial s}\cdot\frac{\partial s}{\partial x} + \frac{\partial u}{\partial t}\cdot\frac{\partial t}{\partial x}$$

$$= \frac{\partial u}{\partial r}\cdot\frac{1}{y} + \frac{\partial u}{\partial x}(0) + \frac{\partial u}{\partial t}\left(\frac{-z}{x^2}\right)$$

$$\frac{\partial u}{\partial x} = \frac{1}{y}\frac{\partial u}{\partial r} - \frac{z}{x^2}\frac{\partial u}{\partial t} \quad \text{——— (1)}$$

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial r}\cdot\frac{\partial r}{\partial y} + \frac{\partial u}{\partial s}\cdot\frac{\partial s}{\partial y} + \frac{\partial u}{\partial t}\cdot\frac{\partial t}{\partial y}$$

$$= \frac{\partial u}{\partial r}\left(\frac{-x}{y^2}\right) + \frac{\partial u}{\partial s}\frac{1}{z} + \frac{\partial u}{\partial t}(0)$$

$$\frac{\partial u}{\partial y} = -\frac{x}{y^2}\frac{\partial u}{\partial r} + \frac{1}{z}\frac{\partial u}{\partial s} \quad \text{——— (2)}$$

$$\frac{\partial u}{\partial z} = \frac{\partial u}{\partial r}\cdot\frac{\partial r}{\partial z} + \frac{\partial u}{\partial s}\cdot\frac{\partial s}{\partial z} + \frac{\partial u}{\partial t}\cdot\frac{\partial t}{\partial z}$$

$$= \frac{\partial u}{\partial r}(0) + \frac{\partial u}{\partial s}\left(\frac{-y}{z^2}\right) + \frac{\partial u}{\partial t}\left(\frac{1}{x}\right)$$

$$\frac{\partial u}{\partial z} = -\frac{y}{z^2}\frac{\partial u}{\partial s} + \frac{1}{x}\frac{\partial u}{\partial t} \quad \text{——— (3)}$$

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z}$$

$$= \frac{x}{y}\cancel{\frac{\partial u}{\partial r}} - \frac{z}{x}\cancel{\frac{\partial u}{\partial t}} - \frac{x}{y}\cancel{\frac{\partial u}{\partial r}} + \frac{y}{z}\frac{\partial u}{\partial s}$$

$$- \frac{y}{z}\frac{\partial u}{\partial s} + \frac{z}{x}\cancel{\frac{\partial u}{\partial t}}$$

$$= 0$$

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4) Find $\frac{\partial u}{\partial x}$ & $\frac{\partial u}{\partial y}$ if $u = y^x$

Soln

$$u = y^x$$

$$\frac{\partial u}{\partial x} = y^x \log y$$

$$\frac{\partial u}{\partial y} = x\, y^{x-1}$$

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5) ~~If $u = x^2 y$ & $x^2 + xy + y^2 = 1$ then find $\frac{du}{dx}$.~~

~~soln~~

$$\frac{du}{dx} =$$

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5) If $u = \frac{x}{y} + \frac{y}{z} + \frac{z}{x}$, Show that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = 0$.

soln.

$$u = \frac{x}{y} + \frac{y}{z} + \frac{z}{x}$$

$$\frac{\partial u}{\partial x} = \frac{1}{y} - \frac{z}{x^2}$$

$$x\frac{\partial u}{\partial x} = \frac{x}{y} - \frac{z}{x}$$

$$\frac{\partial u}{\partial y} = \frac{-x}{y^2} + \frac{1}{z}$$

$$y\frac{\partial u}{\partial y} = \frac{-x}{y} + \frac{y}{z}$$

$$\frac{\partial u}{\partial z} = \frac{-y}{z^2} + \frac{1}{x}$$

$$z\frac{\partial u}{\partial z} = \frac{-y}{z} + \frac{z}{x}$$

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = \frac{x}{y} - \frac{z}{x} + \frac{y}{z} - \frac{x}{y} - \frac{y}{z} + \frac{z}{x}$$

$$= 0$$

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⑥ If $u = \log(x^2+y^2) + \tan^{-1}\left(\frac{y}{x}\right)$ prove that $u_{xx} + u_{yy} = 0$

Soln

an $u = \log(x^2+y^2) + \tan^{-1}\left(\frac{y}{x}\right)$

$$\frac{\partial u}{\partial x} = \frac{1}{x^2+y^2}(2x) + \frac{1}{1+\left(\frac{y}{x}\right)^2}\cdot\frac{-y}{x^2}$$

$$= \frac{2x}{x^2+y^2} + \frac{\cancel{x^2}}{x^2+y^2}\cdot\frac{-y}{\cancel{x^2}}$$

$$\frac{\partial u}{\partial x} = \frac{2x-y}{x^2+y^2}$$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{-2x^2 + 2y^2 + 2xy + 2x^2 - 2y^2 - 2xy}{(x^2+y^2)^2}$$

$$= 0$$

(7) If $u = x^y$, Show that $u_{xxy} - u_{xyx}$.

Soln

$$u = x^y = e^{y \log x}$$

$e^{\log x^y}$ $e^{y \log x}$

$$\frac{\partial u}{\partial y} = e^{y \log x} \cdot \log x$$

$$\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right) = \frac{\partial}{\partial x}(e^{y \log x} \cdot \log x)$$

$$= e^{y \log x} \cdot \frac{1}{x} + \log x \; e^{y \log x} \cdot y \cdot \frac{1}{x}$$

$$= \frac{x^y}{x} + x^y \cdot \frac{y}{x} \log x$$

$$u_{xy} = \frac{x^y}{x}\left[1 + y \log x\right]$$

$$\frac{\partial u}{\partial x} = e^{y \log x} \cdot \frac{y}{x}$$

$$\frac{\partial^2 u}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}\right) = \frac{\partial}{\partial y}\left(e^{y \log x} \cdot \frac{y}{x}\right)$$

$$= e^{y \log x} \cdot \frac{1}{x} + \frac{y}{x} e^{y \log x} \cdot \log x$$

$$= \frac{e^{y \log x}}{x}\left[1 + y \log x\right]$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{(x^2+y^2)(2) - (2x-y)(2x)}{(x^2+y^2)^2}$$

$$= \frac{2x^2+2y^2-4x^2+2xy}{(x^2+y^2)^2}$$

$$= \frac{-2x^2+2y^2+2xy}{(x^2+y^2)^2}$$

$$\frac{\partial u}{\partial y} = \frac{1}{x^2+y^2}(2y) + \frac{1}{1+\frac{y^2}{x^2}} \cdot \frac{1}{x}$$

$$= \frac{2y}{x^2+y^2} + \frac{x^2}{x^2+y^2} \cdot \frac{1}{x}$$

$$= \frac{2y+x}{x^2+y^2}$$

$$\frac{\partial^2 u}{\partial y^2} = \frac{(x^2+y^2)(2) - (2y+x)\cdot 2y}{(x^2+y^2)^2}$$

$$= \frac{2x^2+2y^2-4y^2-2xy}{(x^2+y^2)^2}$$

$$= \frac{2x^2-2y^2-2xy}{(x^2+y^2)^2}$$

$$\therefore u_{yx} = \frac{x^y}{x}\left[1+y\log x\right]$$

$$u_{xy} = u_{yx}$$

diff w.r.t x on both sides we get

$$u_{xxy} = u_{xyx}.$$

Euler's theorem for homogeneous function

If u be a homogenous function of degree 'n' in x and y then

$$x\cdot\frac{\partial u}{\partial x} + y\cdot\frac{\partial u}{\partial y} = n\cdot u.$$

(1) Using Euler's theorem, given $u(x,y)$ is a homogeneous function of degree n, Prove that $x^2u_{xx} + 2xyu_{xy} + y^2u_{yy} = n(n-1)u$

soln

By Euler's thm,

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = nu \quad \text{——} \quad (1)$$

diff w.r.t 'x'

$$x\frac{\partial^2 u}{\partial x^2} + \frac{\partial u}{\partial x} + y\frac{\partial^2 u}{\partial x\partial y} = n\cdot\frac{\partial u}{\partial x}$$

$$x\frac{\partial^2 u}{\partial x^2} + y\frac{\partial^2 u}{\partial x\partial y} = n\cdot\frac{\partial u}{\partial x} - \frac{\partial u}{\partial x}$$

$$x\frac{\partial^2 u}{\partial x^2} + y\frac{\partial^2 u}{\partial x\partial y} = (n-1)\frac{\partial u}{\partial x} \quad \text{——} \quad (2)$$

diff w.r.t 'y' we get

$$x \frac{\partial^2 u}{\partial y \partial x} + y \frac{\partial^2 u}{\partial y^2} + \frac{\partial u}{\partial y} = n \cdot \frac{\partial u}{\partial y}$$

$$x \frac{\partial^2 u}{\partial y \partial x} + y \frac{\partial^2 u}{\partial y^2} = (n-1) \frac{\partial u}{\partial y} \quad \text{--- (3)}$$

(2) $\times x$ + (3) $\times y$

$$x^2 \frac{\partial^2 u}{\partial x^2} + xy \frac{\partial^2 u}{\partial x \partial y} + xy \frac{\partial^2 u}{\partial y \partial x} + y^2 \frac{\partial^2 u}{\partial y^2} = (n-1)\left(x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} \right)$$

$$x^2 \frac{\partial^2 u}{\partial x^2} + 2xy \frac{\partial^2 u}{\partial x \partial y} + y^2 \frac{\partial^2 u}{\partial y^2} = n(n-1) \cdot u .$$

(2) If $u = \tan^{-1}\left(\frac{x^3+y^3}{x-y}\right)$ Prove that

$$x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = \sin 2u .$$

Soln: Gn $u = \tan^{-1}\left(\frac{x^3+y^3}{x-y}\right)$

$$f = \tan u = \frac{x^3+y^3}{x-y} = \frac{x^3(1+y^3/x^3)}{x(1-y/x)}$$

$\therefore f$ is a homogeneous function of degree 2.

$$x \frac{\partial f}{\partial x} + y \frac{\partial f}{\partial y} = 2f$$

$$x \frac{\partial}{\partial x}(\tan u) + y \frac{\partial}{\partial y}(\tan u) = 2 \tan u$$

$$x \sec^2 u \frac{\partial u}{\partial x} + y \cdot \sec^2 u \cdot \frac{\partial u}{\partial y} = 2 \tan u$$

$$\sec^2 u \left(x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} \right) = 2 \cdot \frac{\sin u}{\cos u}$$

$$x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = 2 \frac{\sin u}{\cancel{\cos} u} \cdot \cos^{\cancel{2}} u$$

$$x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = \sin 2u .$$

(3) If $u = \sin^{-1}\left(\frac{x+y}{\sqrt{x}+\sqrt{y}}\right)$ then prove that

$$x^2 \frac{\partial^2 u}{\partial x^2} + 2xy \frac{\partial^2 u}{\partial x \partial y} + y^2 \frac{\partial^2 u}{\partial y^2} = -\frac{\sin u \cos 2u}{4\cos^3 u}$$

Soln

$$f = \sin u = \frac{x+y}{\sqrt{x}+\sqrt{y}} = \frac{x}{\sqrt{x}}\left(\frac{1+y/x}{1+\sqrt{y}/\sqrt{x}}\right)$$

$$= x^{1/2}\left(\frac{1+y/x}{1+\sqrt{y}/\sqrt{x}}\right)$$

f is a homogeneous function of degree $1/2$

$$x \frac{\partial f}{\partial x} + y \frac{\partial f}{\partial y} = \frac{1}{2} f$$

$$x \frac{\partial}{\partial x}(\sin u) + y \frac{\partial}{\partial y}(\sin u) = \frac{1}{2} \sin u$$

$$x \frac{\partial}{\partial u}(\sin u) \cdot \frac{\partial u}{\partial x} + y \frac{\partial}{\partial u}(\sin u) \frac{\partial u}{\partial y} = \frac{1}{2} \sin u$$

$$x \cos u \frac{\partial u}{\partial x} + y \cos u \frac{\partial u}{\partial y} = \frac{1}{2} \sin u$$

$$\cos u \left(x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} \right) = \frac{1}{2} \sin u$$

$$x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = \frac{1}{2} \frac{\sin u}{\cos u}$$

Total derivatives

$$\frac{dy}{dx} = \frac{-\left(\frac{\partial f}{\partial x}\right)}{\left(\frac{\partial f}{\partial y}\right)}$$

$$\frac{du}{dt} = \frac{\partial u}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial u}{\partial y}\cdot\frac{dy}{dt}$$

1) Find $\frac{dy}{dx}$ if $x^3+y^3-3axy = 0$

Soln

Gn $x^3+y^3-3axy = 0$

$$f = x^3+y^3-3axy$$

$$\frac{dy}{dx} = \frac{-\left(\frac{\partial f}{\partial x}\right)}{\left(\frac{\partial f}{\partial y}\right)}$$

$$\frac{\partial f}{\partial x} = 3x^2-3ay$$

$$\frac{\partial f}{\partial y} = 3y^2-3ax$$

$$\frac{dy}{dx} = \frac{-(3x^2-3ay)}{(3y^2-3ax)} = \frac{-\cancel{3}(x^2-ay)}{\cancel{3}(y^2-ax)}$$

$$\frac{dy}{dx} = \frac{ay-x^2}{y^2-ax}$$

2) If $x^y + y^x = 1$, then find $\frac{dy}{dx}$

Soln

Gn $x^y + y^x = 1$

$f = x^y + y^x - 1$ $\qquad$ $\frac{d}{dx}(a^x) = a^x \log a.$

$\frac{\partial f}{\partial x} = y x^{y-1} + y^x \log y$

$\frac{\partial f}{\partial y} = x^y \log x + x y^{x-1}$

$\frac{dy}{dx} = \frac{-\left(\frac{\partial f}{\partial x}\right)}{(\partial f/\partial y)}$

$= \frac{-(y x^{y-1} + y^x \log y)}{x^y \log x + x y^{x-1}}$

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3) Using the definition of total derivatives find the value of $\frac{du}{dt}$ given $u = y^2 - 4ax$, $x = at^2$ $y = 2at$

Soln

$\frac{du}{dt} = \frac{\partial u}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial u}{\partial y} \cdot \frac{dy}{dt}$

$\frac{\partial u}{\partial x} = -4a$ $\qquad$ $\frac{dx}{dt} = 2at$, $\frac{dy}{dt} = 2a$

$\frac{\partial u}{\partial y} = 2y$

$$\frac{du}{dt} = (-4a)(2at) + (2y)(2a)$$
$$= 8a^2t + 4ay.$$

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4) find $\frac{du}{dt}$, if $u = \sin(x/y)$ where $x = e^t$, $y = t^2$

Soln

$$u = \sin(x/y), \quad x = e^t, \quad y = t^2$$

$$\frac{du}{dt} = \frac{\partial u}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial u}{\partial t} \cdot \frac{dy}{dt}$$

$$\frac{\partial u}{\partial x} = \cos(x/y)(1/y) = 1/y \cos(x/y)$$

$$\frac{\partial u}{\partial y} = \cos(x/y)\left(\frac{-x}{y^2}\right) = -\frac{x}{y^2}\cos(x/y)$$

$$\frac{dx}{dt} = e^t, \quad \frac{dy}{dt} = 2t$$

$$\frac{du}{dt} = \frac{1}{y}(\cos(x/y)\, e^t - \frac{x}{y^2}\cos(x/y)\, 2t$$

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5) If $u = x/y$, where $x = e^t$, $y = \log t$ find $\frac{du}{dt}$.

Soln

Gn $u = x/y$, $x = e^t$, $y = \log t$

$$\frac{du}{dt} = \frac{\partial u}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial u}{\partial y} \cdot \frac{dy}{dt}$$

$$\frac{\partial u}{\partial x} = \frac{1}{y} , \quad \frac{\partial u}{\partial y} = \frac{-x}{y^2}$$

$$\frac{dx}{dt} = e^t , \quad \frac{dy}{dt} = 1/t$$

$$\frac{du}{dt} = \frac{1}{y} \cdot e^t - \frac{x}{y^2} \cdot \frac{1}{t}$$

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6) If $u = x^2 y$ and $x^2 + xy + y^2 = 1$ then find du/dx

Soln

Gn $u = x^2 y$ $\quad x^2 + xy + y^2 = 1$

$f = x^2 + xy + y^2 - 1$

$$\frac{du}{dx} = \frac{\partial u}{\partial x} \cdot \frac{dx}{dx} + \frac{\partial u}{\partial y} \cdot \frac{dy}{dx}$$

$$= \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} \cdot \frac{dy}{dx}$$

$$\frac{\partial u}{\partial x} = 2xy \qquad \frac{\partial u}{\partial y} = x^2$$

$$\frac{dy}{dx} = \frac{-\left(\partial f/\partial x\right)}{\left(\partial f/\partial y\right)}$$

$$\frac{\partial f}{\partial x} = 2x + y \qquad \frac{\partial f}{\partial y} = x + 2y$$

$$\frac{dy}{dx} = \frac{-(2x+y)}{x+2y)}$$

$$\frac{du}{dx} = 2xy + x^2 \cdot \left(-\frac{(2x+y)}{x+2y}\right)$$

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7) If $u = x \log(xy)$ & $x^3 + y^3 + 3axy = 0$

find $\frac{du}{dx}$

Soln

$$\frac{du}{dx} = \frac{\partial u}{\partial x} \cdot \frac{dx}{dx} + \frac{\partial u}{\partial y} \cdot \frac{dy}{dx}$$

$$= \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} \cdot \frac{dy}{dx}$$

$$\left(\frac{dy}{dx}\right) = \frac{-\left(\frac{\partial f}{\partial x}\right)}{\left(\frac{\partial f}{\partial y}\right)}$$

$$f = x^3 + y^3 + 3axy$$

$$\frac{\partial f}{\partial x} = 3x^2 + 3ay = 3(x^2 + ay)$$

$$\frac{\partial f}{\partial y} = 3y^2 + 3ax = 3(y^2 + ax)$$

$$\frac{dy}{dx} = \frac{-3(x^2+ay)}{3(y^2+ax)}$$

$$u = x(\log x + \log y)$$

$$\frac{\partial u}{\partial x} = x \cdot \frac{1}{x} + \log(xy)(1)$$

$$= 1 + \log(xy)$$

$$\frac{\partial u}{\partial y} = x\left(\frac{1}{y}\right) = \frac{x}{y}$$

$$\frac{du}{dx} = 1 + \log(xy) - \frac{x}{y}\frac{(x^2+ay)}{(y^2+ax)}$$

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(8) If $z = f(x,y)$ where $x = u^2 - v^2$ $y = 2uv$

PT $\frac{\partial^2 z}{\partial u^2} + \frac{\partial^2 z}{\partial v^2} = 4(u^2+v^2)\left(\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2}\right)$

Soln

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial u} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial u}$$

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}(2u) + \frac{\partial z}{\partial y}(2v)$$

$$\frac{\partial}{\partial u} = 2u\frac{\partial}{\partial x} + 2v\frac{\partial}{\partial y}$$

$$\frac{\partial^2 z}{\partial u^2} = \frac{\partial}{\partial u}\left(\frac{\partial z}{\partial u}\right) = \left(2u\frac{\partial}{\partial x} + 2v\frac{\partial}{\partial y}\right)$$

$$\left(2u\frac{\partial z}{\partial x} + 2v\frac{\partial z}{\partial y}\right)$$

$$= 4u^2\frac{\partial^2 z}{\partial x^2} + 4uv\frac{\partial^2 z}{\partial x\partial y} + 4uv\frac{\partial^2 z}{\partial y\partial x} + 4v^2\frac{\partial^2 z}{\partial y^2}$$

$$\frac{\partial^2 z}{\partial u^2} = 4u^2\frac{\partial^2 z}{\partial x^2} + 8uv\frac{\partial^2 z}{\partial x\partial y} + 4v^2\frac{\partial^2 z}{\partial y^2} \quad \text{--- (1)}$$

$$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial v}$$

$$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}(-2v) + \frac{\partial z}{\partial y}(2u)$$

$$\frac{\partial}{\partial v} = -2v\frac{\partial}{\partial x} + 2u\frac{\partial}{\partial y}$$

$$\frac{\partial^2 z}{\partial v^2} = \frac{\partial}{\partial v}\left(\frac{\partial z}{\partial v}\right)$$

$$= \left(-2v\frac{\partial}{\partial x} + 2u\frac{\partial}{\partial y}\right)\left(-2v\frac{\partial z}{\partial x} + 2u\frac{\partial z}{\partial y}\right)$$

$$= 4v^2\frac{\partial^2 z}{\partial x^2} - 4uv\frac{\partial^2 z}{\partial x\partial y} - 4uv\frac{\partial^2 z}{\partial y\partial x} + 4u^2\frac{\partial^2 z}{\partial y^2}$$

$$\frac{\partial^2 z}{\partial v^2} = 4v^2\frac{\partial^2 z}{\partial x^2} - 8uv\frac{\partial^2 z}{\partial x\partial y} + 4u^2\frac{\partial^2 z}{\partial y^2} \quad \text{--- (2)}$$

① + ② =>

$$\frac{\partial^2 z}{\partial u^2} + \frac{\partial^2 z}{\partial v^2} = 4u^2 \frac{\partial^2 z}{\partial x^2} + 8uv \frac{\partial^2 z}{\partial x \partial y} + 4u^2 \frac{\partial^2 z}{\partial y^2}$$

$$+ 4v^2 \frac{\partial^2 z}{\partial x^2} - 8uv \frac{\partial^2 z}{\partial x \partial y} + 4u^2 \frac{\partial^2 z}{\partial y^2}$$

$$= 4 \frac{\partial^2 z}{\partial x^2}(u^2+v^2) + 4 \frac{\partial^2 z}{\partial y^2}(u^2+v^2)$$

$$\frac{\partial^2 z}{\partial u^2} + \frac{\partial^2 z}{\partial v^2} = 4(u^2+v^2)\left(\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2}\right)$$

Q If $x = r\cos\theta$, $y = r\sin\theta$ PT

$$\frac{\partial^2 r}{\partial x^2} + \frac{\partial^2 r}{\partial y^2} = \frac{1}{r}\left[\left(\frac{\partial r}{\partial x}\right)^2 + \left(\frac{\partial r}{\partial y}\right)^2\right]$$

Soln

Gn $x = r\cos\theta$ $\quad$ $y = r\sin\theta$

$x^2 = r^2\cos^2\theta$ $\quad$ $y^2 = r^2\sin^2\theta$

$$x^2+y^2 = r^2(\cos^2\theta + \sin^2\theta)$$

$$x^2+y^2 = r^2 \quad \text{— ①}$$

diff ① w.r.to x

$$2x + 0 = 2r\frac{\partial r}{\partial x}$$

$$\frac{\cancel{2}x}{\cancel{2}r} = \frac{\partial r}{\partial x}$$

$$\frac{\partial r}{\partial x} = \frac{x}{r}$$

$$\frac{\partial^2 r}{\partial x^2} = \frac{r(1) - x\frac{\partial r}{\partial x}}{r^2} = \frac{r - x \cdot \frac{x}{r}}{r^2}$$

$$= \frac{r^2 - x^2}{r} \times \frac{1}{r^2}$$

$$\frac{\partial^2 r}{\partial x^2} = y^2/r^3$$

diff ① w.r. to y

$$0 + 2y = 2r\frac{\partial r}{\partial y}$$

$$\frac{\cancel{2}y}{\cancel{2}r} = \frac{\partial r}{\partial y}$$

$$\Rightarrow \frac{\partial r}{\partial y} = \frac{y}{r}$$

$$\frac{\partial^2 r}{\partial y^2} = \frac{r(1) - y\frac{\partial r}{\partial y}}{r^2} = \frac{r - y \cdot \frac{y}{r}}{r^2}$$

$$= \frac{r^2 - y^2}{r} \times \frac{1}{r^2}$$

$$\frac{\partial^2 r}{\partial y^2} = x^2/r^3$$

$$\frac{\partial^2 r}{\partial x^2}+\frac{\partial^2 r}{\partial y^2} = \frac{1}{r}\left(\frac{x^2+y^2}{r^2}\right)$$

$$= \frac{1}{r}\left[\frac{x^2}{r^2}+\frac{y^2}{r^2}\right]$$

$$\frac{\partial^2 r}{\partial x^2}+\frac{\partial^2 r}{\partial y^2} = \frac{1}{r}\left[\left(\frac{\partial r}{\partial x}\right)^2+\left(\frac{\partial r}{\partial y}\right)^2\right]$$

---

10) If z is a function of x & y and u & v are two other variables such that $u = lx+my$, $v = ly-mx$

Show that $\frac{\partial^2 z}{\partial x^2}+\frac{\partial^2 z}{\partial y^2} = (l^2+m^2)\left(\frac{\partial^2 z}{\partial u^2}+\frac{\partial^2 z}{\partial v^2}\right)$

Soln

$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial x} + \frac{\partial z}{\partial v}\cdot\frac{\partial v}{\partial x}$$

$$= \frac{\partial z}{\partial u}(l) + \frac{\partial z}{\partial v}(-m)$$

$$\frac{\partial z}{\partial x} = \left(l\frac{\partial z}{\partial u} - m\frac{\partial z}{\partial v}\right)$$

$$\frac{\partial}{\partial x} = \left(l\frac{\partial}{\partial u} - m\frac{\partial}{\partial v}\right)$$

$$\frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right)$$

$$= \left(l \frac{\partial}{\partial u} - m \frac{\partial}{\partial v} \right) \left(l \frac{\partial z}{\partial u} - m \frac{\partial z}{\partial v} \right)$$

$$= l^2 \frac{\partial^2 z}{\partial u^2} - lm \frac{\partial^2 z}{\partial u \partial v} - lm \frac{\partial^2 z}{\partial v \partial u} + m^2 \frac{\partial^2 z}{\partial v^2}$$

$$\frac{\partial z}{\partial y} = \frac{\partial z}{\partial u} \cdot \frac{\partial u}{\partial y} + \frac{\partial z}{\partial v} \cdot \frac{\partial v}{\partial y}$$

$$= \frac{\partial z}{\partial u}(m) + \frac{\partial z}{\partial v}(l)$$

$$\frac{\partial z}{\partial y} = m \frac{\partial z}{\partial u} + l \frac{\partial z}{\partial v}$$

$$\frac{\partial}{\partial y} = m \frac{\partial}{\partial u} + l \frac{\partial}{\partial v}$$

$$\frac{\partial^2 z}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y} \right) = \left(m \frac{\partial}{\partial u} + l \frac{\partial}{\partial v} \right) \left(m \frac{\partial z}{\partial u} + l \frac{\partial z}{\partial v} \right)$$

$$= m^2 \frac{\partial^2 z}{\partial u^2} + lm \frac{\partial^2 z}{\partial u \partial v} + lm \frac{\partial^2 z}{\partial v \partial u} + l^2 \frac{\partial^2 z}{\partial v^2}$$

$$\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = m^2 \frac{\partial^2 z}{\partial u^2} + l^2 \frac{\partial^2 z}{\partial u^2} - lm \frac{\partial^2 z}{\partial u \partial v} - lm \frac{\partial^2 z}{\partial v \partial u} + m^2 \frac{\partial^2 z}{\partial v^2} + lm \frac{\partial^2 z}{\partial u \partial v} + lm \frac{\partial^2 z}{\partial v \partial u} + l^2 \frac{\partial^2 z}{\partial v^2}$$

$$= (l^2+m^2)\left(\frac{\partial^2 z}{\partial u^2}\right) + (l^2+m^2)\left(\frac{\partial^2 z}{\partial v^2}\right)$$

$$\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = (l^2+m^2)\left[\left(\frac{\partial^2 z}{\partial u^2}\right) + \frac{\partial^2 z}{\partial v^2}\right]$$

Jacobian:

Definition

If u_1, u_2, u_3 are functions of variable x_1, x_2, x_3 then the Jacobian of the transformation from x_1, x_2, x_3 to u_1, u_2, u_3 is defined by

$$\frac{\partial(u_1, u_2, u_3)}{\partial(x_1, x_2, x_3)} = \begin{vmatrix} \frac{\partial u_1}{\partial x_1} & \frac{\partial u_1}{\partial x_2} & \frac{\partial u_1}{\partial x_3} \\ \frac{\partial u_2}{\partial x_1} & \frac{\partial u_2}{\partial x_1} & \frac{\partial u_2}{\partial x_3} \\ \frac{\partial u_3}{\partial x_1} & \frac{\partial u_3}{\partial x_2} & \frac{\partial u_3}{\partial x_3} \end{vmatrix}$$

Properties of Jacobian

1) If u & v are the function of x & y then $\frac{\partial(u,v)}{\partial(x,y)} \times \frac{\partial(x,y)}{\partial(u,v)} = 1$

2) If u & v are functions of x & y and x & y are functions of r & s

then $$\frac{\partial(u,v)}{\partial(r,s)} = \frac{\partial(u,v)}{\partial(x,y)} \cdot \frac{\partial(x,y)}{\partial(r,s)}$$

3) If u, v, w are functionally dependent function of three independent variables x, y, z then

$$\frac{\partial(u, v, w)}{\partial(x, y, z)} = 0$$

1) If $x = u^2 - v^2$, $y = uv$ then evaluate the Jacobian of x, y w.r.to u, v

Soln

Gn $x = u^2 - v^2$ $y = uv$

$$\frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$$

$$= \begin{vmatrix} 2u & -2v \\ v & u \end{vmatrix}$$

$$= 2u^2 + 2v^2$$

$$\frac{\partial(x, y)}{\partial(u, v)} = 2(u^2 + v^2)$$

2) If $u = \frac{y^2}{2x}$, $v = \frac{x^2 + y^2}{2x}$ find $\frac{\partial(u, v)}{\partial(x, y)}$

Soln

Given $u = \frac{y^2}{2x}$ $v = \frac{x^2 + y^2}{2x}$

$$\frac{\partial u}{\partial x} = \frac{y^2}{2}\left(\frac{-1}{x^2}\right) = \frac{-y^2}{2x^2}.$$

$$\frac{\partial u}{\partial y} = \frac{1}{2x}(2y) = \frac{y}{x}$$

$$\frac{\partial v}{\partial x} = \frac{2x(2x) - (x^2+y^2)(2)}{(2x)^2}$$

$$= \frac{4x^2 - 2x^2 - 2y^2}{4x^2}$$

$$= \frac{2x^2 - 2y^2}{4x^2}$$

$$\frac{\partial v}{\partial x} = \frac{x^2 - y^2}{2x^2}$$

$$\frac{\partial v}{\partial y} = \frac{1}{2x}(2y) = \frac{y}{x}$$

$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix}$$

$$= \begin{vmatrix} \frac{-y^2}{2x^2} & \frac{y}{x} \\ \frac{x^2-y^2}{2x^2} & \frac{y}{x} \end{vmatrix}$$

$$= \frac{-y^3}{2x^3} - \left(\frac{y}{x}\right)\left(\frac{x^2-y^2}{2x^2}\right)$$

$$= \frac{-y^3 - yx^2 + y^3}{2x^3}$$

$$= \frac{-yx^2}{2x^3}$$

$$\frac{\partial(u,v)}{\partial(x,y)} = \frac{-y}{2x}$$

---

3) If $x = r\cos\theta$ $y = r\sin\theta$ then find $\frac{\partial(r,\theta)}{\partial(x,y)}$

<u>Soln</u>

Gn $x = r\cos\theta$ $y = r\sin\theta$

$$\frac{\partial x}{\partial r} = \cos\theta \qquad \frac{\partial x}{\partial \theta} = -r\sin\theta$$

$$\frac{\partial y}{\partial r} = \sin\theta \qquad \frac{\partial y}{\partial \theta} = r\cos\theta$$

$$\frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix}$$

$$= r\cos^2\theta + r\sin^2\theta = r(\cos^2\theta + \sin^2\theta)$$

$$= r$$

Using property

$$\frac{\partial(x,y)}{\partial(r,\theta)} \cdot \frac{\partial(r,\theta)}{\partial(x,y)} = 1$$

$$\Rightarrow \frac{\partial(r,\theta)}{\partial(x,y)} = 1 \Big/ \frac{\partial(x,y)}{\partial(r,\theta)} = \frac{1}{r}$$

$$\frac{\partial(r,\theta)}{\partial(x,y)} = \frac{1}{r}$$

---

4) If $u = \frac{yz}{x}$, $v = \frac{xz}{y}$, $w = \frac{xy}{z}$

find $\frac{\partial(x,y,z)}{\partial(u,v,w)}$

(or)

Find the Jacobian of y_1, y_2, y_3 w.r.to x_1, x_2, x_2 if $y_1 = \frac{x_2 x_3}{x_1}$, $y_2 = \frac{x_3 x_1}{x_2}$

$y_3 = \frac{x_1 x_2}{x_3}$

Soln

Gn $u = \frac{yz}{x}$, $v = \frac{xz}{y}$, $w = \frac{xy}{z}$

$$\frac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix}$$

$$\frac{\partial u}{\partial x} = -\frac{yz}{x^2} \quad \frac{\partial u}{\partial y} = \frac{z}{x} \quad \frac{\partial u}{\partial z} = \frac{y}{x}$$

$$\frac{\partial v}{\partial x} = \frac{z}{y} \quad \frac{\partial v}{\partial y} = -\frac{xz}{y^2} \quad \frac{\partial v}{\partial z} = \frac{x}{y}$$

$$\frac{\partial w}{\partial x} = \frac{y}{z} \quad \frac{\partial w}{\partial y} = \frac{x}{z} \quad \frac{\partial w}{\partial z} = -\frac{xy}{z^2}$$

$$\frac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} -\frac{yz}{x^2} & \frac{z}{x} & \frac{y}{x} \\ \frac{z}{y} & -\frac{xz}{y^2} & \frac{x}{y} \\ \frac{y}{z} & \frac{x}{z} & -\frac{xy}{z^2} \end{vmatrix}$$

$$= -\frac{yz}{x^2}\left(\frac{x^2yz}{y^2z^2} - \frac{x^2}{yz}\right) - \frac{z}{x}\left(\frac{-xyz}{yz^2} - \frac{x}{y}\frac{y}{z}\right) + \frac{y}{x}\left(\frac{xz}{yz} + \frac{xyz}{y^2z}\right)$$

$$= -\frac{yz}{x^2}(0) - \frac{z}{x}\left(-2\frac{x}{z}\right) + \frac{y}{x}\left(2\frac{x}{y}\right)$$

$$= 2+2$$

$$= 4$$

Using property

$$\frac{\partial(x,y,z)}{\partial(u,v,w)} = 1 \Big/ \frac{\partial(u,v,w)}{\partial(x,y,z)} = \frac{1}{4}$$

5) find the Jacobian $\dfrac{\partial(x,y,z)}{\partial(r,\theta,\phi)}$ of the transformation $x = r\sin\theta\cos\phi$,
$y = r\sin\theta\sin\phi$ & $z = r\cos\theta$

Soln

Gn $x = r\sin\theta\cos\phi \quad y = r\sin\theta\sin\phi$
$z = r\cos\theta$

$$\frac{\partial(x,y,z)}{\partial(r,\theta,\phi)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \phi} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \phi} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial \phi} \end{vmatrix}$$

$\frac{\partial x}{\partial r} = \sin\theta\cos\phi \quad \frac{\partial x}{\partial \theta} = r\cos\phi\cos\theta \quad \frac{\partial x}{\partial \phi} = -r\sin\theta\sin\phi$

$\frac{\partial y}{\partial r} = \sin\theta\sin\phi \quad \frac{\partial y}{\partial \theta} = r\sin\phi\cos\theta \quad \frac{\partial y}{\partial \phi} = r\sin\theta\cos\phi$

$\frac{\partial z}{\partial r} = \cos\theta \quad \frac{\partial z}{\partial \theta} = -r\sin\theta \quad \frac{\partial z}{\partial \phi} = 0$

$$\frac{\partial(x,y,z)}{\partial(r,\theta,\phi)} = \begin{vmatrix} \sin\theta\cos\phi & r\cos\phi\cos\theta & -r\sin\theta\sin\phi \\ \sin\theta\sin\phi & r\sin\phi\cos\theta & r\sin\theta\cos\phi \\ \cos\theta & -r\sin\theta & 0 \end{vmatrix}$$

$= \sin\theta\cos\phi\,(r^2\sin^2\theta\cos\phi) - r\cos\phi\cos\theta\,)$
$(-r\sin\theta\cos\theta\cos\phi) - r\sin\theta\sin\phi$
$(-r\sin^2\theta\sin\phi - r\sin\phi\cos^2\theta)$

$$= r^2 \sin^3\theta \cdot \cos^2\varphi + r^2 \cos^2\theta \cos^2\varphi \sin\theta$$
$$+ r^2 \sin^3\theta \sin^2\varphi + r^2 \sin^2\varphi \sin\theta \cos^2\theta$$

$$= r^2 \sin^3\theta (\cos^2\varphi + \sin^2\varphi) +$$
$$r^2 \sin\theta \cos^2\theta (\cos^2\varphi + \sin^2\varphi)$$

$$= r^2 \sin^3\theta + r^2 \sin\theta \cos^2\theta$$

$$= r^2 \sin\theta (\sin^2\theta + \cos^2\theta)$$

$$= r^2 \sin\theta$$

6) If $x+y+z=u$, $y+z=uv$, $z=uvw$. PT $\dfrac{\partial(x,y,z)}{\partial(u,v,w)} = u^2 v$

Soln

$$x+y+z=u, \quad y+z=uv, \quad z=uvw$$

$u = x+y+z$	$y+z=uv$	$z=uvw$
$u = x+uv$	~~$y+z=u$~~	
$x = u-uv$	$y+uvw=uv$	
	$y=uv-uvw$	

$$\frac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix}$$

$\frac{\partial x}{\partial u} = 1 - v \qquad \frac{\partial x}{\partial v} = -u \qquad \frac{\partial x}{\partial w} = 0$

$\frac{\partial y}{\partial u} = v - vw \qquad \frac{\partial y}{\partial v} = u - uw \qquad \frac{\partial y}{\partial w} = -uv$

$\frac{\partial z}{\partial u} = vw \qquad \frac{\partial z}{\partial v} = uw \qquad \frac{\partial z}{\partial w} =$

$$\frac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{vmatrix} 1-v & -u & 0 \\ v-vw & u-uw & -uv \\ vw & uw & uv \end{vmatrix}$$

$$= (1-v)\left(uv(u-uw) + u^2vw\right)$$
$$+u\left(uv(v-vw) + uv^2w\right)$$
$$= (1-v)\left(u^2v - u^2vw + u^2vw\right)$$
$$+u\left(uv^2 - uv^2w + uv^2w\right)$$
$$= u^2v - u^2v^2 + u^2v^2$$

$$\frac{\partial(x,y,z)}{\partial(u,v,w)} = u^2v$$

(11) If $u = e^{xy}$. Such that $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{1}{u}\left(\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2\right)$

Soln

Given $u = e^{xy}$

$\frac{\partial u}{\partial x} = e^{xy} \cdot y$

$\frac{\partial^2 u}{\partial x^2} = y\, e^{xy} \cdot y = y^2 e^{xy}$

$\frac{\partial u}{\partial y} = e^{xy} \cdot x$

$\frac{\partial^2 u}{\partial y^2} = e^{xy} \cdot x^2 = x^2 e^{xy}$

$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = y^2 e^{xy} + x^2 e^{xy}$

$\left(\frac{\partial u}{\partial x}\right)^2 = y^2 e^{2xy}$ $\quad \left(\frac{\partial u}{\partial y}\right)^2 = x^2 e^{2xy}$

$\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 = y^2 e^{2xy} + x^2 e^{2xy}$

$= e^{xy}\left[x^2 e^{xy} + y^2 e^{xy}\right]$

$\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 = u\left[\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right]$

$\frac{1}{u}\left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2\right] = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}$

---

(12) If f is a function of x & y and if $x = e^u \sin v$ & $y = e^u \cos v$ prove that $\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = e^{-2u}\left[\frac{\partial^2 f}{\partial u^2} + \frac{\partial^2 f}{\partial v^2}\right]$

Soln Given $x = e^u \sin v$ $\quad y = e^u \cos v$

$$\frac{\partial f}{\partial u} = \frac{\partial f}{\partial x} \cdot \frac{\partial x}{\partial u} + \frac{\partial f}{\partial y} \cdot \frac{\partial y}{\partial u}$$

$$\frac{\partial f}{\partial u} = \frac{\partial f}{\partial x} e^u \sin v + \frac{\partial f}{\partial y} e^u \cos v$$

$$\frac{\partial}{\partial u} = \frac{\partial}{\partial x} \cdot e^u \sin v + \frac{\partial}{\partial y} e^u \cos v$$

$$\frac{\partial^2 f}{\partial u^2} = \frac{\partial}{\partial u}\left(\frac{\partial f}{\partial u}\right)$$

$$= \left(e^u \sin v \frac{\partial}{\partial x} + e^u \cos v \frac{\partial}{\partial y}\right)\left(e^u \sin v \frac{\partial f}{\partial x} + e^u \cos v \frac{\partial f}{\partial y}\right)$$

$$= e^{2u} \sin^2 v \frac{\partial^2 f}{\partial x^2} + e^{2u} \sin v \cos v \frac{\partial^2 f}{\partial x \partial y} + e^{2u} \sin v \cos v \frac{\partial^2 f}{\partial y \partial x} + e^{2u} \cos^2 v \frac{\partial^2 f}{\partial y^2}$$

$$\frac{\partial^2 f}{\partial u^2} = e^{2u} \sin^2 v \frac{\partial^2 f}{\partial x^2} + 2e^{2u} \sin v \cos v \frac{\partial^2 f}{\partial x \partial y} + e^{2u} \cos^2 v \frac{\partial^2 f}{\partial y^2}$$

$$\frac{\partial f}{\partial v} = \frac{\partial f}{\partial x} \cdot \frac{\partial x}{\partial v} + \frac{\partial f}{\partial y} \cdot \frac{\partial y}{\partial v}$$

$$= \frac{\partial f}{\partial x} \cdot e^u \cos v + \frac{\partial f}{\partial y} \left(-e^u \sin v\right)$$

$$\frac{\partial}{\partial v} = e^u \cos v \frac{\partial}{\partial x} - e^u \sin v \frac{\partial}{\partial y}$$

$$\frac{\partial^2 f}{\partial v^2} = \frac{\partial}{\partial v}\left(\frac{\partial f}{\partial x}\right)$$

$$= \left(e^u \cos v \frac{\partial}{\partial x} - e^u \sin v \frac{\partial}{\partial y}\right)\left(e^u \cos v \frac{\partial f}{\partial x} - e^u \sin v \frac{\partial f}{\partial y}\right)$$

$$= e^{2u} \cos^2 v \frac{\partial^2 f}{\partial x^2} - e^{2x} \cos v \sin v \frac{\partial^2 f}{\partial x \partial y} - e^{2u} \sin v \cos v \frac{\partial^2 f}{\partial y \partial x} + e^{2u} \sin^2 v \frac{\partial^2 f}{\partial y^2}$$

$$\frac{\partial^2 f}{\partial v^2} = e^{2u} \cos^2 v \frac{\partial^2 f}{\partial x^2} - 2e^{2u} \cos v \sin v \frac{\partial^2 f}{\partial x \partial y} + e^{2u} \sin^2 v \frac{\partial^2 f}{\partial y^2}$$

$$\frac{\partial^2 f}{\partial u^2} + \frac{\partial^2 f}{\partial v^2} = e^{2u} \sin^2 v \frac{\partial^2 f}{\partial x^2} + 2e^{2u} \sin v \cos v \frac{\partial^2 f}{\partial y \partial x} + e^{2u} \cos^2 v \frac{\partial^2 f}{\partial y^2} + e^{2u} \cos^2 v \frac{\partial^2 f}{\partial x^2} - 2e^{2u} \cos v \sin v \frac{\partial^2 f}{\partial x \partial y} + e^{2u} \sin^2 v \frac{\partial^2 f}{\partial y^2}$$

$$= e^{2u} \frac{\partial^2 f}{\partial x^2} (\sin^2 v + \cos^2 v) + e^{2u} \frac{\partial^2 f}{\partial y^2} (\cos^2 v + \sin^2 v)$$

$$= e^{2u} \frac{\partial^2 f}{\partial x^2} + e^{2u} \frac{\partial^2 f}{\partial y^2}$$

$$= e^{2u} \left[\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} \right]$$

$$\frac{1}{e^{2u}} \left[\frac{\partial^2 f}{\partial u^2} + \frac{\partial^2 f}{\partial v^2} \right] = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}$$

$$e^{-2u}\left[\frac{\partial^2 f}{\partial u^2}+\frac{\partial^2 f}{\partial v^2}\right]=\frac{\partial^2 f}{\partial x^2}+\frac{\partial^2 f}{\partial y^2}$$

Taylor's Series.

$$f(x,y) = f(a,b) + \frac{1}{1!}\left[h f_x(a,b) + k f_y(a,b)\right]$$
$$+ \frac{1}{2!}\left[h^2 f_{xx}(a,b) + 2hk f_{xy}(a,b) + k^2 f_{yy}(a,b)\right]$$
$$+ \frac{1}{3!}\left[h^3 f_{xxx}(a,b) + 3h^2k f_{xxy}(a,b) + 3hk^2 f_{xyy}(a,b)\right.$$
$$\left. + h^3 f_{yyy}(a,b)\right] + \cdots$$

where $h = x-a$ $\quad k = y-b$

① Expand $x^2y+3y-2$ in powers of $x-1$ & $y+2$ upto 3^{rd} degree terms.

Soln

$f(x,y) = x^2y+3y-2 \Rightarrow f(1,-2) = -10$

$h = x-1 \quad k = y+2$

$\boxed{a=1} \quad \boxed{b=-2}$

$f_x = 2xy \qquad f_x = -4$

$f_y = x^2+3 \qquad f_y = 4$

$f_{xx} = 2y \qquad f_{xx} = -4$

$f_{xy} = +2x \qquad f_{xy} = 2$

$f_{yy} = 0 \qquad f_{yy} = 0$

$f_{xxx} = 0$ $\qquad$ $f_{xxx} = 0$

$f_{xxy} = 2$ $\qquad$ $f_{xxy} = 2$

$f_{xyy} = 0$ $\qquad$ $f_{xyy} = 0$

$f_{yyy} = 0$ $\qquad$ $f_{yyy} = 0$

$$f(x,y) = f(a,b) + \frac{1}{1!}\left[h f_x(a,b) + k f_y(a,b)\right] + \frac{1}{2!}\left[h^2 f_{xx}(a,b) + 2hk f_{xy}(a,b) + k^2 f_{yy}(a,b)\right] + \frac{1}{3!}\left[h^3 f_{xxx}(a,b) + 3h^2k f_{xxy}(a,b) + 3hk^2 f_{xyy}(a,b) + k^3 f_{yyy}(a,b)\right] + \cdots$$

$$= -10 + \frac{1}{1!}\left[(x-1)(-4) + (y+2)4\right] + \frac{1}{2!}\left[(x-1)^2(-4) + 2(x-1)(y+2)(2) + (y+2)^2(0)\right] + \frac{1}{3!}\left[(x-1)^3(0) + 3(x-1)^2(y+2)(2)\right] + \cdots$$

$$= -10 + \left[-4(x-1) + 4(y+2)\right] + \left[-2(x-1)^2 + 2(x-1)(y+2)\right] + (x-1)^2(y+2) + \cdots$$

2) Expand $e^x \sin y$ in powers of x and y as far as the terms of the 3rd degree using Taylor's expansion

$\sin \pi/4 = 1/\sqrt{2}$
$\cos \pi/4 = 1/\sqrt{2}$

$e^0 = 1 \quad \sin 0 = 0$
$\cos 0 = 1$

$(0,0) \qquad\qquad (-1, \pi/4)$

Soln

$f(x,y) = e^x \sin y \qquad f(0,0) = 0 \qquad f(-1,\pi/4) = \frac{1}{\sqrt{2}e}$

$f_x(x,y) = e^x \sin y \qquad f_x(0,0) = 0 \qquad f_x(-1,\pi/4) = \frac{1}{\sqrt{2}e}$

$f_y(x,y) = e^x \cos y \qquad f_y(0,0) = 1 \qquad f_y(-1,\pi/4) = \frac{1}{\sqrt{2}e}$

$f_{xx}(x,y) = e^x \sin y \qquad f_{xx}(0,0) = 0 \qquad f_{xx}(-1,\pi/4) = \frac{1}{\sqrt{2}e}$

$f_{xy}(x,y) = e^x \cos y \qquad f_{xy}(0,0) = 1 \qquad f_{xy}(-1,\pi/4) = \frac{-1}{\sqrt{2}e}$

$f_{yy}(x,y) = -e^x \sin y \qquad f_{yy}(0,0) = 0 \qquad f_{yy}(-1,\pi/4) = \frac{-1}{\sqrt{2}e}$

$f_{xxx}(x,y) = e^x \sin y \qquad f_{xxx}(0,0) = 0 \qquad f_{xxx}(-1,\pi/4) = \frac{1}{\sqrt{2}e}$

$f_{xxy}(x,y) = e^x \cos y \qquad f_{xxy}(0,0) = 1 \qquad f_{xxy}(-1,\pi/4) = \frac{1}{\sqrt{2}e}$

$f_{xyy}(x,y) = -e^x \sin y \qquad f_{xyy}(0,0) = 0 \qquad f_{xyy}(-1,\pi/4) = \frac{-1}{\sqrt{2}e}$

$f_{yyy}(x,y) = -e^x \cos y \qquad f_{yyy}(0,0) = -1 \qquad f_{yyy}(-1,\pi/4) = \frac{-1}{\sqrt{2}e}$

$h = x - a = x - 0 = x$
$k = y - a = y - 0 = y$

$$f(x,y) = f(a,b) + \frac{1}{1!}\left[h f_x(a,b) + k f_y(a,b)\right] + \frac{1}{2!}\left[h^2 f_{xx}(a,b) + 2hk f_{xy}(a,b) + k^2 f_{yy}(a,b)\right] + \frac{1}{3!}\left[h^3 f_{xxx}(a,b) + 3h^2 k f_{xxy}(a,b) + 3hk^2 f_{xyy}(a,b) + k^3 f_{yyy}(a,b)\right] + \cdots$$

$$= 0 + \frac{1}{1!}\left[x \cdot 0 + y \cdot 1\right] + \frac{1}{2!}\left[x^2 \cdot 0 + 2x \cdot y(1) + y^2 0\right] + \frac{1}{3!}\left[x^3 \cdot 0 + 3x^2 y \cdot 1 + 3xy^2(0) + y^3(-1)\right] + \cdots$$

$$= \frac{1}{1!}[y] + \frac{1}{2!}[2xy] + \frac{1}{3!}[3x^2y - y^3] + \cdots$$

$$= y + xy + \frac{1}{3!}[3x^2y - y^3] + \cdots$$

(ii) where $h = x - a = x + 1$ $\boxed{a = -1}$

$k = y - b = y - \pi/4$ $\boxed{b = \pi/4}$

$$f(x,y) = \frac{1}{\sqrt{2}e} + \frac{1}{1!}\left[(x+1)\frac{1}{\sqrt{2}e} + (y-\pi/4)\frac{1}{\sqrt{2}e}\right]$$

$$+ \frac{1}{2!}\left[(x+1)^2\frac{1}{\sqrt{2}e} + 2(x+1)(y-\pi/4)\frac{1}{\sqrt{2}e} + (y-\pi/4)^2 \cdot \frac{-1}{\sqrt{2}e}\right] +$$

$$\frac{1}{3!}\left[(x+1)^3\frac{1}{\sqrt{2}e} + 3(x+1)^2(y-\pi/4)\frac{1}{\sqrt{2}e} + 3(x+1)(y-\pi/4)\frac{-1}{\sqrt{2}e} + (y-\pi/4)^3\frac{-1}{\sqrt{2}e}\right] + \cdots$$

$$= \frac{1}{\sqrt{2}e}\left[1 + \frac{1}{1!}[(x+1) + (y-\pi/4)] + \frac{1}{2!}\left[(x+1)^2 + 2(x+1)(y-\pi/4) - (y-\pi/4)^2\right] + \frac{1}{3!}\left[(x+1)^3 + 3(x+1)^2(y-\pi/4) - 3(x+1)(y-\pi/4)^2 - (y-\pi/4)^3\right] + \cdots\right]$$

(3) Expand $e^x \log(1+y)$ in powers of x and y upto terms of 3^{rd} degree.

Soln

$$f(x,y) = e^x \log(1+y)$$

functions | Values at $(0,0)$

$f = e^x \log(1+y)$ | $f(0,0) = 0$

$f_x = e^x \log(1+y)$ | $f_x = 0$

$f_y = e^x \cdot \frac{1}{1+y}$ | $f_y = 1$

$f_{xx} = e^x \cdot \frac{1}{1+y}$ | $f_{xx} = 0$

$f_{xy} = e^x \cdot \frac{1}{1+y}$ | $f_{xy} = 1$

$f_{yy} = e^x \left(\frac{-1}{(1+y)^2}\right)$ | $f_{yy} = -1$

$f_{xxx} = e^x \cancel{\log(1+y)} \frac{1}{1+y}$ | $f_{xxx} = \cancel{0}\ 1$

$f_{xxy} = e^x \frac{1}{1+y}$ | $f_{xxy} = 1$

$f_{xyy} = e^x \left(\frac{-1}{(1+y)^2}\right)$ | $f_{xyy} = -1$

$f_{yyy} = e^x \frac{2}{(1+y)^3}$ | $f_{yyy} = 2$

4) Obtain the Taylor's series expansion upto second degree of $e^x \cos y$ in powers of $x+1$ and $y-\pi/4$

Sol'n

$h = x-a = x+1$ $\boxed{a=-1}$

$k = y-b = y-\pi/4$ $\boxed{b=\pi/4}$

function	Value at $(-1, \pi/4)$
$f(x,y) = e^x \cos y$	$f(-1, \pi/4) = \frac{1}{\sqrt{2}e}$
$f_x = e^x \cos y$	$f_x = \frac{1}{\sqrt{2}e}$
$f_y = -e^x \sin y$	$f_y = -\frac{1}{\sqrt{2}e}$
$f_{xx} = e^x \cos y$	$f_{xx} = \frac{1}{\sqrt{2}e}$
$f_{xy} = -e^x \sin y$	$f_{xy} = \frac{-1}{\sqrt{2}e}$
$f_{yy} = -e^x \cos y$	$f_{yy} = -\frac{1}{\sqrt{2}e}$

$$f(x,y) = f(a,b) + \frac{1}{1!}\left[h\, f_x(a,b) + k\, f_y(a,b)\right] + \frac{1}{2!}\left[h^2 f_{xx}(a,b) + 2hk\, f_{xy}(a,b) + k^2 f_{yy}(a,b)\right] + \cdots$$

$$= \frac{1}{\sqrt{2}e} + \frac{1}{1!}\left[(x+1)\frac{1}{\sqrt{2}e} + (y-\pi/4)\left(\frac{-1}{\sqrt{2}e}\right)\right]$$

$$+ \frac{1}{2!}\left[(x+1)^2 \frac{1}{\sqrt{2}e} + 2(x+1)(y-\pi/4)\left(\frac{-1}{\sqrt{2}e}\right) \right.$$

$$= \frac{1}{\sqrt{2}e} + \frac{1}{\sqrt{2}e}\left[(x+1) - (y - \pi/4)\right]$$

$$+ \frac{1}{\sqrt{2}e}\left[(x+1)^2 \mp 2(x+1)(y-\pi/4) \mp (y-\pi/4)^2\right] + \cdots$$

5) Expand $e^x \cos y$ at $(0, \pi/2)$ upto the 3^{rd} term using Taylor's Series.

<u>Soln</u>

functions	Value at $(0, \pi/2)$
$f(x,y) = e^x \cos y$	$f(0, \pi/2) = 0$
$f_x = e^x \cos y$	$f_x = 0$
$f_y = -e^x \sin y$	$f_y = -1$
$f_{xx} = e^x \cos y$	$f_{xx} = 0$
$f_{xy} = -e^x \sin y$	$f_{xy} = -1$
$f_{yy} = -e^x \cos y$	$f_{yy} = 0$
$f_{xxx} = e^x \cos y$	$f_{xxx} = 0$
$f_{xxy} = -e^x \sin y$	$f_{xxy} = -1$
$f_{xyy} = -e^x \cos y$	$f_{xyy} = 0$
$f_{yyy} = e^x \sin y$	$f_{yyy} = 1$

$h = x - a = x - 0 \quad k = y - b = y - \pi/2$

$h = x \quad k = y - \pi/2$

$$f(x,y) = f(a,b) + \frac{1}{1!}\left[h f_x(a,b) + k f_y(a,b)\right]$$
$$+ \frac{1}{2!}\left[h^2 f_{xx}(a,b) + 2hk f_{xy}(a,b) + k^2 f_{yy}(a,b)\right]$$
$$+ \frac{1}{3!}\left[h^3 f_{xxx}(a,b) + 3h^2 k f_{xxy}(a,b) + 3hk^2 f_{xyy}(a,b) + k^3 f_{yyy}(a,b)\right] + \cdots$$

$$= 0 + \frac{1}{1!}\left[x(0) + (y - \pi/2)(-1)\right]$$
$$+ \frac{1}{2!}\left[x^2(0) + 2x(y-\pi/2)(-1) + (y-\pi/2)^2(0)\right]$$
$$+ \frac{1}{3!}\left[x^3(0) + 3x^2(y-\pi/2)(-1) + 3x(y-\pi/2)^2(0) + (y-\pi/2)^3(1)\right] + \cdots$$

$$= -(y-\pi/2) + \frac{1}{2}\left[-2x(y-\pi/2)\right]$$
$$+ \frac{1}{6}\left[-3x^2(y-\pi/2) + (y-\pi/2)^3\right] + \cdots$$

Maxima and Minima for functions of two variables.

If $f_x = 0$, $f_y = 0$ & $f_{xx} = A$

$f_{xy} = B$, $f_{yy} = C$ then

(i) $f(a,b)$ is Maximum value if $AC - B^2 > 0$ and $A < 0$ (or $B < 0$)

(ii) $f(a,b)$ is minimum Value if $AC - B^2 > 0$ and $A > 0$ (or $B > 0$)

(iii) $f(a,b)$ is not an extremum (Saddle) if $AC - B^2 < 0$

(iv) $AC - B^2 \geq 0$ then the test is inconclusive

① Find the Maximum & Minimum Value of $x^2 - xy + y^2 - 2x + y$

Soln

$$f(x,y) = x^2 - xy + y^2 - 2x + y$$

$$f_x = 2x - y - 2 \qquad f_{xy} = -1 = B$$

$$f_y = -x + 2y + 1 \qquad f_{yy} = 2 = C$$

$$f_{xx} = 2 = A$$

$f_x = 0$ $\qquad$ $f_y = 0$

$2x-y-2=0$ $\qquad$ $-x+2y+1=0$

$2x-y=2$ — ① $\qquad$ $x-2y=1$ — ②

Solve ① & ②

The Stationary Points are

$x=1 \quad y=0$

	A	B	C	$AC-B^2$	Conclusion
(1, 0)	$2>0$	-1	2	$3>0$	Minimum

$\therefore$ The Minimum value of $f(x,y)$

is $f(1,0) = 1-2 = -1$

2) Examine $f(x,y) = x^3+y^3-12x-3y+20$ for its extreme values

Soln

$f(x,y) = x^3+y^3-12x-3y+20$

$f_x = 3x^2-12$

$f_y = 3y^2-3$

$$f_{xx} = 6x = A$$
$$f_{xy} = 0 = B$$
$$f_{yy} = 6y = C$$

$f_x = 0$	$f_y = 0$
$3x^2 - 12 = 0$	$3y^2 - 3 = 0$
$3x^2 = 12$	$3y^2 = 3$
$x^2 = 4$	$y^2 = 1$
$x = \pm 2$	$y = \pm 1$

The Stationary Points are
$(2,1)$ $(2,-1)$, $(-2,1)$ $(-2,-1)$

Points	A	B	C	$AC-B^2$	Con
$(2,1)$	$12 > 0$	0	6	$72 > 0$	Mini
$(2,-1)$	12	0	-6	-72	Saddle pts
$(-2,1)$	-12	0	6	-72	Saddle pts
$(-2,-1)$	$-12 < 0$	0	-6	$72 > 0$	Maxi

The Minimum Value of $f(x,y)$ is

$$f(2,1) = (2)^3 + (1)^3 - 12(2) - 3(1) + 20$$

$$= 8 + 1 - 24 - 3 + 20 = 2$$

The Maximum Value of $f(x,y)$ is

$$f(-2,-1) = (-2)^3 + (-1)^3 - 12(-2) - 3(-1) + 20$$

$$= -8 - 1 + 24 + 3 + 20$$

$$= 38.$$

(3) Examine the function $f(x,y) = x^3y^2(12-x-y)$ for Extreme values.

soln

Gn $f(x,y) = 12x^3y^2 - x^4y^2 - x^3y^3$

$$f_x = 36x^2y^2 - 4x^3y^2 - 3x^2y^3$$

$$f_y = 24x^3y - 2x^4y - 3x^3y^2$$

$$f_{xx} = 72xy^2 - 12x^2y^2 - 6xy^3 = A$$

$$f_{xy} = 72x^2y - 8x^3y - 9x^2y^2 = B$$

$$f_{yy} = 24x^3 - 2x^4 - 6x^3y = C$$

$f_x = 0$

$$36x^2y^2 - 4x^3y^2 - 3x^2y^3 = 0$$

$$x^2y^2(36 - 4x - 3y) = 0$$

$$\Rightarrow x = 0,\ y = 0$$

$$4x + 3y = 36 \quad \text{—(1)}$$

$f_y = 0$

$$24x^3y - 2x^4y - 3x^3y^2 = 0$$

$$x^3y(24 - 2x - 3y) = 0$$

$$\Rightarrow x = 0,\ y = 0.$$

$$2x + 3y = 24 \quad \text{—(2)}$$

$$4x+3y = 36$$
$$2x+3y = 24$$
$$(-)\quad (-)\quad (-)$$
$$2x = 12$$

$$\boxed{x=6}$$

$$12+3y = 24$$
$$3y = 24-12$$
$$3y = 12$$
$$\boxed{y=4} \qquad (6,4)$$

① $\Rightarrow$

$x=0 \Rightarrow y=12 \quad (0,12)$

$y=0 \Rightarrow x=9 \quad (9,0)$

② $\Rightarrow$

$x=0 \Rightarrow y=+8 \quad (0,+8)$

$y=0 \Rightarrow x=12 \quad (12,0)$

$\therefore$ The Stationary pts are $(0,0)$

$(0,12)$, $(9,0)$, $(0,+8)$, $(12,0)$, $(6,4)$

Points	A	B	C	$AC-B^2$	Conclusion
$(0,0)$	0	0	0	0	inconclusive
$(0,12)$	0	0	0	0	inconclusive
$(9,0)$	0	0	-13122	0	inconclusive
$(0,+8)$	0	0	0	0	"
$(12,0)$	6	0	-41472	0	"
$(6,4)$	$-2304<0$	-1782	-2592	2796447[illegible]	Maximum

Maximum Value of $f(x,y)$ is

$$f(6,4) = 12(6)^3(4)^2 - (6)^4(4)^2 - (6)^3(4)^3$$

$$= 41472 - 20736 - 13824$$

$$= 6912.$$

4) Find the extreme Value of $f(x,y) = x^3y^2(1-x-y)$

Soln.

Gn $f(x,y) = x^3y^2 - x^4y^2 - x^3y^3$

$$f_x = 3x^2y^2 - 4x^3y^2 - 3x^2y^3$$

$$f_y = 2x^3y - 2x^4y - 3x^3y^2$$

$$f_{xx} = 6xy^2 - 12x^2y^2 - 6xy^3 = A$$

$$\cancel{f_{xy} = 6x^2y - 12x^3y - 9x^2y^2}$$

$f_{xy}(x,y) = 6x^2y - 8x^3y - 9x^2y^2 = B$

$f_{yy}(x,y) = 2x^3 - 2x^4 - 6x^3y$

$f_x = 0$	$f_y = 0$
$3x^2y^2 - 4x^3y^2 - 3x^2y^3 = 0$	$2x^3y - 2x^4y - 3x^3y^2 = 0$
$x^2y^2[3 - 4x - 3y] = 0$	$x^3y(2 - 2x - 3y) = 0$
$\Rightarrow x = 0,\ y = 0$	$x = 0 \quad y = 0$
$4x + 3y = 3$ — ①	$2x + 3y = 2$ — ②

Solve ① & ② $\quad x = \frac{1}{2} \quad y = \frac{1}{3}$

$\therefore$ ~~The Stationary points are $(0,0)$~~

① $\Rightarrow x = 0$	$y = 0$
$3y = 3$	$4x = 3$
$\boxed{y = 1}$	$\boxed{x = 3/4}$

$(0,1)$ ~~$(3/4, 0)$~~ $(3/4, 0)$

② $\Rightarrow x = 0$	$y = 0$
$3y = 2$	$2x = 2$
$y = 2/3$	$x = 1$

$(0, 2/3)$ ~~$(0, 2/3)$~~ $(1, 0)$

$\therefore$ The Stationary Points are

$(0,0), (\frac{1}{2}, \frac{1}{3}), (0,1), (3/4, 0)$

$(0, 2/3), (1, 0)$

Points	A	B	C	$AC-B^2$	Conclusion
$(0,0)$	0	0	0	0	Inconclusive
$(\frac{1}{2},\frac{1}{3})$	$-\frac{1}{9}<0$	$-\frac{1}{12}$	$-\frac{1}{8}$	$\frac{1}{144}>0$	Maximum
$(0,1)$	0	0	0	0	inconclusive
$(0,\frac{2}{3})$	0	0	0	0	inconclusive
$(\frac{3}{4},0)$	0	0	$\frac{27}{128}$	0	inconclusive
$(1,0)$	0	0	0	0	inconclusive

Maximum value of $f(x,y)$ is

$$f\left(\tfrac{1}{2},\tfrac{1}{3}\right) = \left(\tfrac{1}{2}\right)^3\left(\tfrac{1}{3}\right)^2\left[1-\tfrac{1}{2}-\tfrac{1}{3}\right] = \frac{1}{432}.$$

5. Discuss the Maximum & Minimum of $f(x,y) = x^2+xy+y^2+\frac{1}{x}+\frac{1}{y}$

Soln

$$f_x = 2x+y-\frac{1}{x^2}$$

$$f_y = x+2y-\frac{1}{y^2}$$

$$f_{xx} = 2+2/x^3 = A$$

$$f_{xy} = 1 = B$$

$$f_{yy} = 2 + \frac{2}{y^3} = C.$$

$f_x = 0$ $\qquad$ $f_y = 0$

$2x + y - \frac{1}{x^2} = 0$ — ① $\qquad$ $x + 2y - \frac{1}{y^2} = 0$ — ②

① − ②

$$2x + y - \frac{1}{x^2} - x - 2y + \frac{1}{y^2} = 0$$

$$x - y + \frac{1}{y^2} - \frac{1}{x^2} = 0$$

$$x - y + \frac{x^2 - y^2}{x^2 y^2} = 0$$

$$x - y + \frac{(x+y)(x-y)}{x^2 y^2} = 0$$

$$(x-y)\left[x^2 y^3 + x + y\right] = 0$$

$$x - y = 0$$

$$x = y$$

Put $x = y$ in ①

$$2y + y - \frac{1}{y^2} = 0$$

$$\frac{2y^3 + y^3 - 1}{y^2} = 0$$

$$y^3 + 1 = 0$$

$$3y^3 = 1$$

$$y^3 = 1/3$$

$$y = (1/3)^{1/3} = x.$$

The Stationary points are $\left(\left(\frac{1}{3}\right)^{1/3}, \left(\frac{1}{3}\right)^{1/3}\right)$

Points	A	B	C	$AC-B^2$	conclusion
$\left(\left(\frac{1}{3}\right)^{1/3}, \left(\frac{1}{3}\right)^{1/3}\right)$	$8>0$	1	8	$\cancel{15}$ 64 >0	Minimum

The Minimum value of $f(x,y)$ is

$$f\left(\left(\tfrac{1}{3}\right)^{1/3}, \left(\tfrac{1}{3}\right)^{1/3}\right) = \left(\tfrac{1}{3}\right)^{2/3} + \left(\tfrac{1}{3}\right)^{1/3}\left(\tfrac{1}{3}\right)^{1/3} + \left(\tfrac{1}{3}\right)^{2/3} + \frac{1}{\left(\frac{1}{3}\right)^{1/3}} + \frac{1}{\left(\frac{1}{3}\right)^{1/3}}$$

$$= \left(\tfrac{1}{3}\right)^{2/3} + \left(\tfrac{1}{3}\right)^{2/3} + \left(\tfrac{1}{3}\right)^{2/3} + 2\,3^{1/3}$$

$$= 3\left(\frac{1}{3}\right)^{2/3} + 2\,3^{1/3}$$

$$\cancel{= 3 \cdot \frac{1}{3^{2/3}} + 2 \cdot 3^{1/3}}$$

$$\cancel{= 3 \cdot \frac{1}{3^{1/3}}} \quad 3 \cdot 3^{-2/3} + 2 \cdot 3^{1/3}$$

$$= 3^{1/3} + 2 \cdot 3^{1/3} = 3 \cdot 3^{1/3}$$

$$f\left(\left(\tfrac{1}{3}\right)^{1/3}, \left(\tfrac{1}{3}\right)^{1/3}\right) = 3^{4/3}.$$

Discuss the Maximum & Minimum value of the function $f(x,y) = x^4+y^4-2x^2+4xy-2y^2$

Soln

$f(x,y) = x^4+y^4-2x^2+4xy-2y^2$

$f_x = 4x^3-4x+4y$

$f_y = 4y^3+4x-4y$

$f_{xx} = 12x^2-4 = A$

$f_{xy} = 4 = B$

$f_{yy} = 12y^2-4 = C$

$f_x = 0$

$4x^3-4x+4y=0$

$4(x^3-x+y)=0$

$x^3-x+y=0$ ——— (1)

$f_y = 0$

$4y^3+4x-4y=0$

$4(y^3+x-y)=0$

$y^3+x-y=0$ ——— (2)

(1) + (2) =>
$$\begin{aligned} x^3 - \cancel{x} + \cancel{y} &= 0 \\ y^3 + \cancel{x} - \cancel{y} &= 0 \\ \hline x^3+y^3 &= 0 \end{aligned}$$

$x^3 = -y^3$

$x = -y$ (or) $y = -x$

$$x^3 - x - x = 0$$
$$x^3 - 2x = 0$$
$$x(x^2 - 2) = 0$$
$$x = 0 \quad (\text{or}) \quad x^2 = 2$$
$$x = \pm\sqrt{2}$$

Put $x = 0 \Rightarrow y = 0$

$x = \sqrt{2} \Rightarrow y = -\sqrt{2}$

$x = -\sqrt{2} \Rightarrow y = \sqrt{2}$

$\therefore$ The stationary pts are

$(0, 0)$ $(\sqrt{2}, -\sqrt{2})$ $(-\sqrt{2}, \sqrt{2})$.

Points	A	B	C	$AC-B^2$	conclusion
$(0,0)$	-4	4	-4	0	inconclusive
$(\sqrt{2}, -\sqrt{2})$	20	4	20	$384>0$	Minimum
$(-\sqrt{2}, \sqrt{2})$	20	4	20	$384>0$	Minimum

Minimum value:

$$f(\sqrt{2}, -\sqrt{2}) = (\sqrt{2})^4 + (-\sqrt{2})^4 - 2(\sqrt{2})^2 + 4\sqrt{2}(-\sqrt{2}) - 2(-\sqrt{2})^2$$
$$= -8$$

$$f(-\sqrt{2}, \sqrt{2}) = (-\sqrt{2})^4 + (\sqrt{2})^4 - 2(-\sqrt{2})^2 + 4(-\sqrt{2})(\sqrt{2}) - 2(\sqrt{2})^2$$
$$= -8.$$

(8) ~~Fin~~

Method of Lagrange Multipliers:

To find the Maximum & Minimum Value of the function $f(x,y,z)$ Subject to $g(x,y,z)=0$

Consider $F = f + \lambda g$

Form the eqns $\frac{\partial F}{\partial x} = 0$, $\frac{\partial F}{\partial y} = 0$ $\frac{\partial F}{\partial z} = 0$

& $\frac{\partial F}{\partial \lambda} = 0$

On solving these eqns we get the Maximum and Minimum points.

(1) Find the Minimum Value of $x^2+y^2+z^2$ Subject to $x+y+z=3a$

Let $f = x^2+y^2+z^2$, $g = x+y+z-3a$

$$F = (x^2+y^2+z^2) + \lambda(x+y+z-3a)$$

$\frac{\partial F}{\partial x} = 0 \Rightarrow 2x + \lambda = 0 \Rightarrow \lambda = -2x$ — (1)

$\frac{\partial F}{\partial y} = 0 \Rightarrow 2y + \lambda = 0 \Rightarrow \lambda = -2y$ — (2)

$\frac{\partial F}{\partial z} = 0 \Rightarrow 2z + \lambda = 0 \Rightarrow \lambda = -2z$ — (3)

$\frac{\partial F}{\partial \lambda} = 0 \Rightarrow x+y+z = 3a$ —— (4)

$$2x = 2y = 2z$$

$$\Rightarrow x = y = z$$

$$x + x + x = 3a$$

$$3x = 3a \Rightarrow x = a$$

$x = a \Rightarrow y = a \quad , z = a$

Minimum point (a, a, a) is

$$f = a^2 + a^2 + a^2 = 3a^2$$

(2) A rectangular box open at the top is to have a volume of 32 cc
(3) Find the dimensions of the box (108) that required the least Material for its conclusion.

Soln

Let x, y, z be the length breadth & height of the box.

Surface area $= xy + 2yz + 2zx = f(x,y,z)$

Volume $= xyz = 32$

$$g(x,y,z) = xyz - 32$$

$$F(x,y,z) = f(x,y,z) + \lambda\, g(x,y,z)$$

$$= xy + 2yz + 2zx + \lambda(xyz - 32)$$

where λ is Lagrange Multiplier.

$F_x = 0 \Rightarrow (y+2z) + \lambda(yz) = 0$

$F_y = 0 \Rightarrow (x+2z) + \lambda(xz) = 0$

$F_z = 0 \Rightarrow (2y+2x) + \lambda(xy) = 0$

$$y + 2z = -\lambda yz$$

$$\frac{y+2z}{yz} = -\lambda$$

$$\frac{1}{z} + \frac{2}{y} = -\lambda \quad \text{——— (1)}$$

$$x + 2z = -\lambda(xz)$$

$$\frac{x+2z}{xz} = -\lambda$$

$$\frac{1}{z} + \frac{2}{x} = -\lambda \quad \text{——— (2)}$$

$$2y + 2x = -\lambda xy$$

$$\frac{2y+2x}{xy} = -\lambda$$

$$\frac{2}{x} + \frac{2}{y} = -\lambda \quad \text{——— (3)}$$

from ①, ② & ③

$$\frac{1}{z} + \frac{2}{y} = \frac{1}{z} + \frac{2}{x} = \frac{2}{x} + \frac{2}{y}$$

$$\cancel{\frac{1}{z}} + \frac{2}{y} = \cancel{\frac{1}{z}} + \frac{2}{x}$$

$$\frac{2}{y} = \frac{2}{x}$$

$$x = y \quad \text{——— } ④$$

$$\cancel{\frac{2}{x}} + \frac{2}{y} = \frac{1}{z} + \cancel{\frac{2}{x}}$$

$$\frac{2}{y} = \frac{1}{z}$$

$$2z = y \quad \text{——— } ③$$

from ③ & ④

$$x = y = 2z$$

Volume $xyz = 32$

$$(2z)(2z)z = 32$$

$$4z^3 = 32$$

$$z^3 = 32/4$$

$$z^3 = 8$$

$$\boxed{z = 2}$$

The dimension of the boxes are

$x = 4,\ y = 4\ z = 2$ $(4, 4, 2)$

(3) Obtain the dimensions of a rectangular box with out top of Maximum capacity
(A) given the surface area as 432 square meters.

Soln

Let x, y, z be the dimensions of the boxes.

$$\text{Surface area} = xy + 2yz + 2zx \overset{=432}{=} f(x,y,z)$$

$$\text{Volume} = xyz = g(x,y,z)$$

$$F(x,y,z) = xyz + \lambda(xy + 2yz + 2zx - 432)$$

$$F_x = 0 \Rightarrow yz + \lambda(y + 2z) = 0$$

$$\cancel{yz \text{ or } \lambda(y+2z) = -yz}$$

$$yz = -\lambda(y + 2z)$$

$$-\frac{1}{\lambda} = \frac{y + 2z}{yz}$$

$$-\frac{1}{\lambda} = \frac{1}{z} + \frac{2}{y} \quad \text{--- (1)}$$

$$F_y = 0 \Rightarrow xz + \lambda(x + 2z) = 0$$

$$xz = -\lambda(x + 2z)$$

$$-\frac{1}{\lambda} = \frac{x + 2z}{xz}$$

$$-1/\lambda = \frac{1}{z} + \frac{2}{x} \quad \text{--- (2)}$$

$F_z = 0 \Rightarrow xy + 2\lambda(x+y) = 0$

$xy = -2\lambda(x+y)$

$-\frac{1}{\lambda} = +\frac{2(x+y)}{xy}$

$-\frac{1}{\lambda} = +\frac{2}{y} + \frac{2}{x}$ —— ③.

$\frac{\partial F}{\partial \lambda} = 0 \Rightarrow xy + 2yz + 2zx = 432$

equate ① & ②

$\frac{1}{2} + \frac{2}{y} = \frac{1}{z} + \frac{2}{x} \Rightarrow \frac{2}{y} = \frac{2}{x} \Rightarrow x = y$

equate ② & ③

$\frac{1}{z} + \frac{2}{x} = \frac{2}{x} + \frac{2}{y} \Rightarrow y = 2z$

$x = y = 2z$

$xy + 2yz + 2zx = 432$

$(2z)(2z) + 2(2z)(z) + 2z(2z) = 432$

$4z^2 + 4z^2 + 4z^2 = 432$

$12z^2 = 432$

$z^2 = 36$

$z = (36)^{1/2}$

$z = 6$

$x = 12, \quad y = 12, \quad z = 6$

The dimension of the rectangular box is $(12, 12, 6)$

(1)

(4) Find the Maximum Value of $x^m y^n z^p$ when $x+y+z=a$

(7) Soln

Gn $f = x^m y^n z^p$ $g = x+y+z-a$

$F = f + \lambda g$

$= x^m y^n z^p + \lambda (x+y+z-a)$

$\frac{\partial F}{\partial x} = 0 \Rightarrow m x^{m-1} y^n z^p + \lambda = 0$

$\Rightarrow \lambda = -m x^{m-1} y^n z^p$ — (1)

$\frac{\partial F}{\partial y} = 0 \Rightarrow n x^m y^{n-1} z^p + \lambda = 0$

$\Rightarrow \lambda = -n x^m y^{n-1} z^p$ — (2)

$\frac{\partial F}{\partial z} = 0 \Rightarrow p x^m y^n z^{p-1} + \lambda = 0$

$\lambda = -p x^m y^n z^{p-1}$ — (3)

$\frac{\partial F}{\partial \lambda} = 0 \Rightarrow x+y+z = a$

$\frac{m x^m y^n z^p}{x} = \frac{n x^m y^n z^p}{y} = \frac{p x^m y^n z^p}{z}$

$\frac{m}{x} = \frac{n}{y} = \frac{p}{z} = \frac{m+n+p}{x+y+z} = \frac{m+n+p}{a}$

$\frac{m}{x} = \frac{m+n+p}{a} \Rightarrow x = \frac{am}{m+n+p}$

$$\frac{n}{y} = \frac{m+n+p}{a} \Rightarrow y = \frac{an}{m+n+p}$$

$$\frac{p}{z} = \frac{m+n+p}{a} \Rightarrow z = \frac{ap}{m+n+p}$$

Minimum point $\left(\frac{am}{m+n+p}, \frac{an}{m+n+p}, \frac{ap}{m+n+p}\right)$

is $f = x^m y^n z^p$

$$= \left(\frac{am}{m+n+p}\right)^m \left(\frac{an}{m+n+p}\right)^n \left(\frac{ap}{m+n+p}\right)^p$$

(5) Find the volume of the parallelopiped inscribed in the ellipsoid whose equation
(6) is $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$.

Soln Let the vertex of the parallelopiped be (x, y, z)

Sides of the solids be $(2x, 2y, 2z)$

Volume $= (2x)(2y)(2z)$

$= 8xyz$

$f(x,y,z) = 8xyz$

$g(x,y,z) = \frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}-1$

$F = f + \lambda g$

$= 8xyz + \lambda\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}-1\right)$

$\frac{\partial F}{\partial x}=0 \Rightarrow 8yz+\lambda\frac{2x}{a^2}=0$

$\lambda = -8yz\cdot\frac{a^2}{2x} = -\frac{4a^2yz}{x}$ — (1)

$\frac{\partial F}{\partial y}=0 \Rightarrow 8xz+\lambda\cdot\frac{2y}{b^2}=0$

$\lambda = -8xz\frac{b^2}{2y} = -\frac{4b^2xz}{y}$ — (2)

$\frac{\partial F}{\partial z}=0 \Rightarrow 8xy+\lambda\left(\frac{2z}{c^2}\right)=0$

$\Rightarrow \lambda = -8xy\frac{c^2}{2z} = -\frac{4xyc^2}{z}$ — (3)

$\frac{\partial F}{\partial \lambda}=0 \Rightarrow \frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ — (4)

$\cancel{-}\frac{4a^2yz}{x} = \cancel{-}\frac{4b^2xz}{y} = \cancel{-}\frac{4xyc^2}{z^2}$

$$\frac{x^2}{a^2} = \frac{y^2}{b^2} = \frac{z^2}{c^2} \Rightarrow \frac{\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}}{3}$$

~~Volume~~ =

$$\frac{x^2}{a^2} = \frac{1}{3} \Rightarrow x^2 = \frac{a^2}{3} \Rightarrow x = \frac{a}{\sqrt{3}}$$

$$\frac{y^2}{b^2} = \frac{1}{3} \Rightarrow y^2 = \frac{b^2}{3} \Rightarrow y = \frac{b}{\sqrt{3}}$$

$$\frac{z^2}{c^2} = \frac{1}{3} \Rightarrow z^2 = \frac{c^2}{3} \Rightarrow z = \frac{c}{\sqrt{3}}$$

$$\text{Volume} = 8xyz$$

$$= 8 \frac{a}{\sqrt{3}} \frac{b}{\sqrt{3}} \frac{c}{\sqrt{3}}$$

$$= \frac{8abc}{3\sqrt{3}} \text{ cu units}$$

6) Find the Shortest and the Longest distance from the points $(1, 2, -1)$ to the sphere $x^2+y^2+z^2 = 24$ using Lagrange's Method of Constraint Maxima & Minima.

Soln

Let (x, y, z) be any point on the sphere

Distance from (x,y,z) & $(1,2,-1)$ is gn by

$$d = \sqrt{(x-1)^2 + (y-2)^2 + (z+1)^2}$$

$$d^2 = (x-1)^2 + (y-2)^2 + (z+1)^2$$

$$f = (x-1)^2 + (y-2)^2 + (z+1)^2$$

$$g = x^2 + y^2 + z^2 - 24$$

The Auxillary eqn is

$$F = f + \lambda g$$

$$F = (x-1)^2 + (y-2)^2 + (z+1)^2 + \lambda(x^2+y^2+z^2-24)$$

$$\frac{\partial F}{\partial x} = 0 \Rightarrow 2(x-1) + 2x\lambda = 0$$

$$\Rightarrow x - 1 + x\lambda = 0$$

$$\Rightarrow x(1+\lambda) = 1$$

$$\Rightarrow x = \frac{1}{1+\lambda} \quad \text{——} (1)$$

$$\frac{\partial F}{\partial y} = 0 \Rightarrow 2(y-2) + 2y\lambda = 0$$

$$\Rightarrow y - 2 + y\lambda = 0$$

$$\Rightarrow y(1+\lambda) = 2$$

$$\Rightarrow y = \frac{2}{1+\lambda} \quad \text{--- } (2)$$

$$\frac{\partial F}{\partial z} = 0 \Rightarrow 2(z+1) + \lambda(2z) = 0$$

$$\Rightarrow z + 1 + z\lambda = 0$$

$$\Rightarrow z(1+\lambda) = -1$$

$$\Rightarrow z = \frac{-1}{1+\lambda} \quad \text{--- } (3)$$

$$\frac{\partial F}{\partial \lambda} = 0 \Rightarrow x^2 + y^2 + z^2 = 24 \quad \text{--- } (4)$$

Sub in eqn (4)

$$\left(\frac{1}{1+\lambda}\right)^2 + \left(\frac{2}{1+\lambda}\right)^2 + \left(\frac{-1}{1+\lambda}\right)^2 = 24$$

$$\frac{6}{(1+\lambda)^2} = 24$$

$$\frac{6}{24} = (1+\lambda)^2$$

$$\Rightarrow (1+\lambda) = \pm \tfrac{1}{2}$$

$$\Rightarrow \lambda = +\tfrac{1}{2} - 1 = -\tfrac{1}{2}$$

$$\lambda = -\tfrac{1}{2} - 1 = -\tfrac{3}{2}$$

$\lambda = -\tfrac{1}{2}$

$$x = \frac{1}{1+\lambda} \Rightarrow x = \frac{1}{1/2} \Rightarrow x = 2$$

$$y = \frac{2}{1+\lambda} \Rightarrow y = \frac{2}{1/2} \Rightarrow y = 4$$

$$z = \frac{-1}{1+\lambda} \Rightarrow z = \frac{-1}{1-\tfrac{1}{2}} \Rightarrow z = -2$$

The pt's on the sphere is $(2, 4, -2)$

$\lambda = -\tfrac{3}{2}$

$$x = \frac{1}{1+\lambda} = \frac{1}{1-\tfrac{3}{2}} = \frac{1}{-\tfrac{1}{2}} = -2$$

$$y = \frac{2}{1+\lambda} = \frac{2}{1-\tfrac{3}{2}} = -4$$

$$z = \frac{-1}{1+\lambda} = \frac{-1}{1-\tfrac{3}{2}} = 2$$

The pt's on the Sphere is $(-2,-4,2)$

Distance :

(i) $(1,2,-1)$ & $(2,4,-2)$

$$d = \sqrt{(2-1)^2 + (4-2)^2 + (-2+1)^2}$$

$$= \sqrt{1+4+1} = \sqrt{6}$$

(ii) p.t's $(1,2,-1)$ & $(-2,-4,2)$

$$d = \sqrt{(-2-1)^2 + (-4-2)^2 + (2+1)^2}$$

$$= \sqrt{9+36+9}$$

$$= \sqrt{54} = 3\sqrt{6}$$

Shortest distance $\sqrt{6}$

Largest distance $3\sqrt{6}$.