UNIT - IV Random Process

Probablistic Signals

Random Variables   time

Sample Space

$SS \longrightarrow R$ $(X, t) \longrightarrow$ Random Process

RANDOM VARIABLE

Rule that assigns a real number to every outcome of a Random experiment.

RANDOM PROCESS

A rule that assigns a time function to every outcome of a random experiment.

Putting it collectively, a random process is a collection of random variable that are functions of real variables in time t and Sample Space S $\{X(S, t)\}$ $t \in T$

NOTE :

1. If S and t are fixed then $\{X(S, t)\}$ is a number
2. If t is fixed then the process $\{X(S, t)\}$ is Random Variable
3. If S is fixed then the process $\{X(S, t)\}$ is a single time function
4. CX

CLASSIFICATION OF Random Process.

Let $t \in T$ $S \in S$.

1. If both T & S are discrete then RP is called discrete Random sequence

ex, Outcome of the n^{th} toss of a fair dice

2. T is discrete but S is continuous, then RP is called Continuous Random Sequence.

ex, temp at the end of n^{th} hour of a day.

3. T is continuous and S is discrete, then RP is called discrete Random Process.

ex, The no. of telephone calls received in a day.

4. Both T & S is continuous, then RP is called Continuous Random process.

ex, Max temp ~~in~~ a place in the interval $(0, T)$

Stationarity PROCESS

Probability distribution or Average do not depend on T then the RP is called Stationary Process.

A RP that is not stationary is called evolutionary Process.

1^{st} ORDER STATIONARY FUNC:

If $f(x_1; t_1) = f(x_1; t_1 + h)$, where h is a positive number $h > 0$ for all $t \in T$.

then the process is called 1^{st} order stationary func

this implies the process does not change with respect to shift in time origin hence if this condition holds. The process becomes independent of time $E\{x(t)\} = \mu$ = constant

2^{nd} ORDER STATIONARY PROCESS.

If $f(x_1, x_2, t_1, t_2) = f(x_1, x_2, t_1+h, t_2+h)$

this implies the process is invariant under translation of time. $\tau = t_1 - t_2$ → time difference

N^{th} ORDER STATIONARY PROCESS (strict / strong sense stationary)

$f(x_1, x_2, \dots, x_n, t_1, t_2, \dots, t_n) = f(x_1, x_2, \dots, x_n; t_1+h, t_2+h, \dots, t_n+h)$.

AUTOCORELATION.

$R_{xx}(t)$ or $R_x(t_1, t_2) = E[X(t_1) + X(t_2)]$

$X(t_1) + X(t_2)$ are two members of $\{X(t)\}$

WIDE-SENSE OR WEAK SENSE STATIONARY PROCESS.

A random process with finite 1^{st} & 2^{nd} order moment is called WSS process if

1. $E\{x(t)\}$ is a constant.
2. Auto corelation is a func of $\tau \Rightarrow E[x(t_1) \cdot x(t_2)] = f(\tau)$

STATIONARY PROCESS.

1. Mean - Const
2. Variance - Const

Joint WSS

1. $\{X(t)\}$ must be WSS.
2. $\{Y(t)\}$ must be WSS.
3. Cross corelation $R_{XY}(t_1, t_2) = E[X(t_1) + Y(t_2)] = f(\tau)$

1. If $\{X(t)\}$ is a WSS process with auto correlation function $R_x(\tau) = A e^{-\alpha|\tau|}$ determine the 2^{nd} order moment of the RV $X(8) - X(5)$.

$E[X]$ = first order moment

$E[X^2]$ = Second order moment.

$$E[(X(8) - X(5))^2] = E[X^2(8) + X^2(5) - 2X(8)X(5)]$$

$$= E[X^2(8)] + E[X^2(5)] - 2E[X(8) \cdot X(5)]$$

$$= R_x(8,8) + R_x(5,5) - 2R_x(8,5)$$

$$= A e^{-\alpha|8-8|} + A e^{-\alpha|5-5|} + (-2Ae^{-\alpha|8-5|})$$

$$= Ae^{-\alpha(0)} + Ae^{-\alpha(0)} + (-2Ae^{-\alpha 3})$$

$$= 2A - 2Ae^{-\alpha 3}$$

$$= 2A(1 - e^{-3\alpha}).$$

2. Show That the process $X(t) = A\cos\lambda t + B\sin\lambda t$ where A & B are RV is a WSS if

i) $E(A) = E(B) = 0$

ii) $E(A^2) = E(B^2)$

iii) $E(AB) = 0$.

WSS.

1. Mean $E\{x(t)\}$ is Const

2. $R_x(t_1, t_2) = E[X(t_1) \cdot x(t_2)] = f(\tau)$

$x(t) = A\cos\lambda t + B\sin\lambda t$

$E[x(t)] = E[A\cos\lambda t + B\sin\lambda t]$

$= E[A\cos\lambda t] + E[B\sin\lambda t]$.

$= \cos\lambda t(E[A]) + \sin\lambda t\, E[B]$.

$= \cos\lambda t(0) + \sin\lambda t(0)$

$E[x(t)] = 0$. is a Constant

Hence i) is proved.

$X(t_1) = A\cos\lambda t_1 + B\sin\lambda t_1$

$X(t_2) = A\cos\lambda t_2 + B\sin\lambda t_2$.

$R_{xx}(t_1, t_2) = E[X(t_1) * X(t_2)]$.

$= E[A\cos\lambda t_1 + B\sin\lambda t_2) \cdot (A\cos\lambda t_2 + B\sin\lambda t_2)]$.

$= E[A^2\cos\lambda t_1\cos\lambda t_2 + AB\cos\lambda t_1\sin\lambda t_2 +$
$AB\cos\lambda t_2\sin\lambda t_2 + B^2\sin\lambda t_2]$.

$= E[A^2\cos\lambda t_1\cos\lambda t_2] + E[AB\cos\lambda t_1\sin\lambda t_2] +$
$E[AB\cos\lambda t_2\sin\lambda t_2] + E[B^2\sin\lambda t_2]$

$= \cos\lambda t_1 \cos\lambda t_2 \, E[A^2] + E[B^2] \sin\lambda t_1 \sin\lambda t_2$
$+ E[AB] \cos\lambda t_1 \sin\lambda t_2 + E[AB] \cos\lambda t_2 \sin\lambda t_2$

ii) $E(A^2) = E(B^2) = k$. iii) $E(AB) = 0$.

$= k[\cos\lambda t_1 \cos\lambda t_2 + \sin\lambda t_1 \sin\lambda t_2] + 0$.

$= k[\cos(\lambda t_1 - \lambda t_2)]$. $\cos A \cos B + \sin A \sin B$
$(= \cos(A-B))$

$= k[\cos\lambda(t_1 - t_2)]$

$R_{xx}(t_1,t_2) = k[\cos\lambda\tau)$ is a func of τ.

3. If $X(t) = B\cos wt + A\sin wt$ show that $X(t)$ is WSS if A & B are uncorrelated with 0 mean and same variance & w is a const.

To prove $X(t)$ is a WSS
i) Mean is a Const
ii) Autocorelation is a func of τ.

Uncorelated $\Rightarrow E(AB) = 0$.

Zero mean $\Rightarrow E(A) = E(B) = 0$

Same Variance $\Rightarrow Var(A) = Var(B)$

$E[A^2] - [E(A)]^2 = E[B^2] - [E(B)]^2$

$E(A^2) = E(B^2)$.

$E[X(t)] = E[B\cos wt] + E[A\sin wt)$

$= \cos wt \, E[B] + E[A] \sin wt$.

$= \cos wt(0) + \sin wt(0)$

$E[X(t)] = 0$

$R_{xx}(t_1, t_2) = E[x(t_1) \cdot x(t_2)]$

$x(t_1) = B\cos\omega t_1 + A\sin\omega t_1$

$x(t_2) = B\cos\omega t_2 + A\sin\omega t_2$

$R_{xx}(t_1, t_2) = E[(B\cos\omega t_1 + A\sin\omega t_1) \cdot (B\cos\omega t_2 + A\sin\omega t_2)]$

$= E[B^2\cos\omega t_1\cos\omega t_2 + AB\sin\omega t_1\cos\omega t_2 + AB\sin\omega t_2\cos\omega t_1 + A^2\sin\omega t_1\sin\omega t_2]$

$= E[B^2\cos\omega t_1\cos\omega t_2] + E[AB\sin\omega t_1\cos\omega t_2] + E[AB\sin\omega t_2\cos\omega t_1] + E[A^2\sin\omega t_1\sin\omega t_2]$

$= \cos\omega t_1\cos\omega t_2 E(B^2) + E(AB)\sin\omega t_1\cos\omega t_2 + E[AB]\sin\omega t_2\cos\omega t_1 + E(A^2)\sin\omega t_1\sin\omega t_2$

$= K[\cos\omega t_1\cos\omega t_2 + \sin\omega t_1\sin\omega t_2]$

$= K[\cos(\omega t_1 - \omega t_2)]$

$= K[\cos\omega(t_1 - t_2)]$

$\because K = E(A^2) = E(B^2)$

$R_{xx}(t_1, t_2) = K(\cos\omega\tau)$

$\therefore$ Auto corelation is a fun of τ.

1. Given a RV Y with characteristic func. and a random process RP $\{X(t)\}$, $x(t) = \cos(\lambda t + y)$

$\phi(\omega) = E[e^{i\omega y}] = E[\cos\omega y + i\sin\omega y]$. Show that $X(t)$ is a WSS if $\phi(1) = \phi(2) = 0$.

$$\phi(1) = E[\cos y + i \sin y] = 0.$$

$$\Rightarrow E[\cos Y] + i E[\sin Y] = 0 + i0$$

$$E[\cos Y] = 0 \quad E[\sin Y] = 0. \quad \text{--- (1)}$$

$$\phi(2) = E[\cos 2Y] + i E[\sin 2Y] = 0.$$

$$E[\cos 2Y] = 0 \quad E[\sin 2Y] = 0. \quad \text{--- (2)}$$

To prove $X(t)$ is WSS,

1. Mean $\rightarrow$ Const.

$$E[X(t)] = E[\cos(\lambda t + Y)]$$

$$= E[\cos\lambda t \cdot \cos Y - \sin\lambda t \sin Y]$$

$$= \cos\lambda t \, E[\cos Y] - \sin\lambda t \, E[\sin Y]$$

By Condition (1).

$$E[X(t)] = 0$$

2. $X(t_1) = \cos(\lambda t_1 + Y)$ $\quad R_{XX}(t_1, t_2) = E[X(t_1) \cdot X(t_2)]$

$X(t_2) = \cos(\lambda t_2 + Y)$. $\quad \cos A \cos B = \frac{1}{2}[\cos(A+B) + \cos(A-B)]$

$$R_{XX}(t_1, t_2) = E[\cos(\lambda t_1 + Y) \cdot \cos(\lambda t_2 + Y)]$$

$$= E\left[\frac{1}{2}[\cos(\lambda t_1 + Y + \lambda t_2 + Y) + \cos(\lambda t_1 + Y - \lambda t_2 - Y)]\right]$$

$$= \frac{1}{2} E[\cos(\lambda t_1 + \lambda t_2 + 2Y) + \cos(\lambda t_1 - \lambda t_2)]$$

$$= \frac{1}{2} E[\cos(\lambda t_1 + \lambda t_2 + 2Y)] + \frac{1}{2} E[\cos(\lambda t_1 - \lambda t_2)]$$

$= \frac{1}{2} E[\cos(\lambda t_1 + \lambda t_2)\cos(2Y) - \sin\lambda(t_1+t_2)\sin(2Y)]$

$+ \frac{1}{2} E[\cos\lambda(t_1 - t_2)]$

$= \frac{1}{2}\cos\lambda(t_1+t_2) E[\cos 2Y] - \frac{1}{2}\sin\lambda(t_1+t_2) E[\sin 2Y]$

$+ \frac{1}{2} E[\cos\lambda(t_1-t_2)]$

$= \frac{1}{2} E[\cos\lambda(t_1-t_2)]$

$= \frac{1}{2}[\cos\lambda(t_1-t_2)]$ $\qquad t_1 - t_2 = \tau.$

$= \frac{1}{2}(\cos\lambda\tau)$ = function of τ.

$(1-x)^{-1} = 1 + x + x^2 + x^3 + x^4 + \ldots$

$(1-x)^{-2} = 1 + 2x + 3x^2 + 4x^3 + \ldots$

$(1-x)^{-3} = 1 + 3x + 6x^2 + 10x^3 + 15x^4 + \ldots$

$= \frac{1}{2}[1\times 2 + 2\times 3x + 3\times 4x^2 + 4\times 5x^3 + \ldots]$

2. The process $\{X(t)\}$ whose prob. distribution under a certain condition is given by $P\{X(t)=n\} = \frac{(at)^{n-1}}{(1+at)^{n+1}}, n = 1,2,3$

$= \frac{at}{1+at}, n=0$ Verify if the process $\{X(t)\}$ is stationary

n	0	1	2	3	4	5 ...
$P\{X(t)=n\}$	$\frac{at}{1+at}$	$\frac{1}{(1+at)^2}$	$\frac{at}{(1+at)^3}$	$\frac{(at)^2}{(1+at)^4}$	$\frac{(at)^3}{(1+at)^5}$	...

1. Mean

$E[X(t)] = \sum n P(n)$

$= 0\cdot\frac{at}{1+at} + 1\cdot\frac{1}{(1+at)^2} + 2\cdot\frac{at}{(1+at)^3} + 3\cdot\frac{(at)^2}{(1+at)^4}$

$+ 4\frac{(at)^3}{(1+at)^5} + \ldots$

$$= \frac{1}{(1+at)^2} + \frac{2at}{(1+at)^3} + \frac{3(at)^2}{(1+at)^4} + \frac{4(at)^3}{(1+at)^5} + \dots$$

$$= \frac{1}{(1+at)^2}\left[1 + \frac{2at}{(1+at)} + \frac{3(at)^2}{(1+at)^2} + 4\left(\frac{at}{1+at}\right)^3 + \dots\right]$$

$$= \frac{1}{(1+at)^2}\left[1 - \frac{at}{1+at}\right]^{-2}$$

$$= \frac{1}{(1+at)^2}\left[\frac{1+at-at}{1+at}\right]^{-2}$$

$$= \frac{(1+at)^2}{(1+at)^2} = 1.$$

2. Variance = const.

$Var(x] = E[x^2] - E[x]^2.$

$n^2 = n(n+1-1)$
$= n(n+1) - n$

$E[x^2(t)] = \sum n^2 . P(n)$

$$= \sum n(n+1) P(n) - \sum n P(n).$$

$$= \left[0(1) \cdot \frac{a(t)}{1+at} + (1)(2) \frac{1}{(1+at)^2} + 2(3) \frac{at}{(1+at)^3} + 3(4) \frac{(at)^2}{(1+at)^4} + \dots\right] - 1$$

$$= \left[1 \cdot 2 \frac{1}{(1+at)^2} + 2 \cdot 3 \frac{at}{(1+at)^3} + 3 \cdot 4 \frac{(at)^2}{(1+at)^4} + \dots\right] - 1$$

$$= \frac{2}{(1+at)^2}\left[1 \cdot \cancel{2} \cdot 3 \frac{at}{(1+at)} + 6\left(\frac{at}{1+at}\right)^2 + \dots\right] - 1$$

$$= \frac{2}{(1+at)^2}\left[1 - \frac{at}{1+at}\right]^{-3} - 1$$

$$= \frac{2}{(1+at)^2}\left[\frac{1+at-at}{1+at}\right]^{-3} - 1 = \frac{2}{(1+at)^{-1}}$$

$$= 2(1+at) - 1 = 2 + 2at - 1$$

$$E[x^2(t)] = 2at + 1$$

$Var(t) = E[x^2(t)] - E[x(t)]^2 = 2at + 1 - 1$

$Var[x(t)] = 2at$

$\therefore Var[x(t)]$ is not const.

$\therefore$ P(X(t)) is not stationary.

Property of Expectation

$$E\{g(x)\} = \int_{-\infty}^{\infty} g(x) . f_x(x) . dx$$

where $f_x(x)$ is the marginal density function.

Uniform distribution

$$f(x) = \frac{1}{b-a} \quad ; a < x < b.$$

Q. Show that the Random process $\{x(t)\} = A\cos(\omega t + \theta)$ WSS A & ω are Constant θ is uniformly distributed RV in the interval $(0, \cancel{4}2\pi)$

$$f(\theta) = \frac{1}{2\pi}$$

① $E[x(t)] = E[A\cos(\omega t + \theta)]$.

$$= \int_{-\infty}^{\infty} A\cos(\omega t + \theta) . \frac{1}{2\pi} . d\theta$$

$$= \int_{0}^{2\pi} A\cos(\omega t + \theta) . \frac{1}{2\pi} . d\theta.$$

$$= \frac{A}{2\pi} \int_{0}^{2\pi} \cos(\omega t + \theta) . d\theta.$$

$$= \frac{A}{2\pi} \Big[\sin(\omega t + \theta) \Big]_{0}^{2\pi}$$

$\sin(2\pi+\theta) = \sin\theta$

$$= \frac{A}{2\pi}\left[\sin(\omega t + 2\pi) - \sin(\omega t + 0)\right]$$

$$= \frac{A}{2\pi}\left[\sin\omega t - \sin\omega t\right]$$

$$= 0.$$

Mean $= 0$.

2. $R_{xx}(t_1, t_2) = E[x(t_1) \cdot x(t_2)]$.

$$= E\left[A\cos(\omega t_1 + \theta) \cdot A\cos(\omega t_2 + \theta)\right]$$

$$= \frac{A^2}{2} E\left[\cos(\omega t_1 + \theta + \omega t_2 + \theta) + \cos(\omega t_1 + \theta - \omega t_2 - \theta)\right]$$

$$= \frac{A^2}{2} E\left[\cos(\omega t_1 + \omega t_2) + 2\theta)\right].$$

$$= \frac{A^2}{2}\int_0^{2\pi} \cos(\omega t_1 + \omega t_2 + 2\theta)\frac{1}{2\pi}\,d\theta + \frac{A^2}{2} E\left[\cos(\omega t_1 - \omega t_2)\right].$$

$$= \frac{A^2}{2}\left[\frac{\sin(\omega t_1 + \omega t_2 + 2\theta)}{4\pi}\right]_0^{2\pi} + \frac{A^2}{2}\cos(\omega t_1 - \omega t_2)$$

$$= \frac{A^2}{8\pi}\left[\sin\omega(t_1 + t_2) - \sin\omega(t_1 + t_2)\right] + \frac{A^2}{2}\cos(\omega\tau).$$

$$= 0 + \frac{A^2}{2}\cos(\omega\tau)$$

$$= \frac{A^2}{2}\cos(\omega\tau).$$

$x(t) = 20\sin(10t+\theta)$. A & w are constant, θ uniform RV in [0, 2π) is a WSS.

$$f(\theta) = \frac{1}{2\pi}$$

1. $E\{X(t)\} = E\{20\sin(\omega t+\theta)\}$

$$= 20\int_0^{2\pi} \sin(\omega t+\theta) \cdot \frac{1}{2\pi} \, d\theta$$

$$= \frac{20}{2\pi}\left[-\cos(\omega t+\theta)\right]_0^{2\pi}$$

$$= \frac{-10}{\pi}\left[\cos(\omega t+2\pi) - \cos(\omega t)\right]$$

$$= -\frac{10}{\pi}\left[\cos\omega t - \cos\omega t\right]$$

$E\{x(t)\} = 0$. $\omega = 10$.

Mean → 0.

$\sin A \sin B = \frac{1}{2}\left[\cos(A-B) - \cos(A+B)\right]$

2. $R_{xx}(t_1, t_2) = E[x(t_1) \cdot x(t_2)]$

$$= E\left[20\sin(10t_1+\theta) \cdot 20\sin(10t_2+\theta)\right]$$

$$= \frac{400}{2} E\left[\cos(10t_1+\theta-10t_2-\theta) - \cos(10t_1+\theta+10t_2+\theta)\right]$$

$$= 200E\left[\cos(10t_1-10t_2) - \cos(10t_1+10t_2+2\theta)\right]$$

$$= 200E\left[\cos(10t_1-10t_2)\right] - \frac{1}{2\pi}\int_0^{2\pi}\cos(10(t_1+t_2)+2\theta)\,d\theta$$

$$= 200\left[\cos(10\tau)\right] - \frac{1}{2\pi}\left[\frac{\sin(10(t_1+t_2)+2\theta)}{2}\right]_0^{2\pi}$$

$$= 200\left[\cos(10\tau)\right] - 0$$

AUTO CORRELATION FUNCTIONS:

Definition

$\{x(t)\}$ either WSS.

$R_{xx}(\tau) = E[X(t) \cdot X(t+\tau)]$. $\tau = t_2 - t_1$
$= t_1 + \tau - t_1$
$= \tau$

1. Autocorrelation is an even func of 'τ'

$R_{xx}(\tau) = R_{xx}(-\tau)$

$R_{xx}(\tau) = E[X(t) \cdot X(t+\tau)]$

$R_{xx}(-\tau) = E[X(t) \cdot X(t-\tau)]$.

$= E[X(t_1+\tau) \cdot X(t_1)] = R_{xx}(\tau)$.

2. The mean square value of the RP obtained by $\tau = 0$.

$R_{xx}(0) = E[X(t) \cdot X(t+\tau)]$.

$R_{xx}(0) = E[x(t) \cdot x(t)]$

$= E[X^2(t)]$

$E[X^2(t)] \rightarrow$ mean Square.

3. If a RP $\{x(t)\}$ has no periodic components and $E\{x(t)\}$ is denoted by $\bar{x}$ then:

Square of mean

$(\bar{X})^2 = \lim_{|\tau| \to \infty} R_{xx}(\tau)$.

mean of X

$\bar{X} = \mu_x = \sqrt{\lim_{\tau \to \infty} R_{xx}(\tau)}$

4. $\{X(t)\}$ is Periodic then its autocorrelation function is also periodic with same period.

Let $\{x(t)\}$ be periodic with period τ.

$$x(t) = x(t+\tau)$$
$$= x(\tau + n\tau) \quad n = 1, 2, 3, 4, \ldots$$
$$R_{xx}(\tau) = E[x(t) \cdot x(t+\tau)]$$
$$= E[x(t) \cdot x(t+n\tau)]$$
$$= R_{xx}(t - n\tau - t)$$
$$= R_{xx}(-n\tau)$$
$$= R_{xx}(n\tau).$$

5. Max Value of $R_{xx}(\tau)$ is attained at $\tau = 0$.

$$|R_{xx}(\tau)| \leq R_{xx}(0) \longrightarrow E[x^2(t)].$$

Proof, By Cauchy schwartz inequality,

$$E[XY]^2 \leq E[X^2] \cdot E[Y^2].$$

$$Y = x(t+\tau) \qquad X = x(t).$$

$$E[x(t) \cdot x(t+\tau)]^2 \leq E[x^2(t)] E[x^2(t+\tau)].$$

$$R_{xx}(\tau)^2 \leq E[x^2(t)] E[x^2(t+\tau)]$$

Since we know that mean & Variance are constant for a stationary process.

$$R_{xx}(\tau)^2 \leq E[x^2(t)] E[x^2(t)].$$

$$R_{xx}(\tau)^2 \leq E[x^2(t)]^2.$$

$$R_{xx}(\tau) \leq E[x^2(t)]. = R_{xx}(0).$$

6 Auto correlation of R function cannot have an arbitary shape.

1. Check whether the following are valid autocorrelated function.

1. $R_{XX}(\tau) = \dfrac{25\tau^2}{4+5\tau^2}$

2. $R_{XX}(\tau) = \cos(\tau) + \dfrac{|\tau|}{T}$

3. $R_{XX}(\tau) = \tau^3 + \tau^2$.

1. $R_{XX}(\tau) = \dfrac{25\tau^2}{4+5\tau^2}$.

$R_{XX}(-\tau) = \dfrac{25(-\tau)^2}{4+5(-\tau)^2} = \dfrac{25\tau^2}{4+5\tau^2} = R_{XX}(\tau)$.

$\therefore$ It is a valid autocorrelation func.

2. $R_{XX}(\tau) = \cos(\tau) + \dfrac{|\tau|}{T}$.

$R_{XX}(-\tau) = \cos(-\tau) + \dfrac{|(-\tau)|}{T}$

$= \cos\tau + \dfrac{|\tau|}{T} = R_{XX}(\tau)$

$\therefore$ It is valid.

3. $R_{XX}(\tau) = \tau^3 + \tau^2$.

$R_{XX}(-\tau) = (-\tau^3) + \tau^2$

$= -\tau^3 + \tau^2 \neq R_{XX}(\tau)$

2. A stationary process has an $R_{xx}(\tau) = \frac{25\tau^2+36}{6.25\tau^2+4}$ find the mean & variance of the process.

mean, $\mu_x = \sqrt{\lim_{|\tau| \to \infty} R_{xx}(\tau)}$

$$\mu_x^2 = \lim_{|\tau| \to \infty} \frac{\tau^2\left[25 + \frac{36}{\tau^2}\right]}{\tau^2\left[6.25 + \frac{4}{\tau^2}\right]}$$

$$= \lim_{|\tau| \to \infty} \frac{25 + 36/\tau^2}{6.25 + 4/\tau^2}$$

$$= \frac{25}{6.25} = 4$$

$E[x^2(\tau)] =$

$\mu_x = 2.$

$$\mathrm{Var}(x) = E[x^2(\tau)] - E[x(\tau)]^2$$

$$= R_{xx}(0) - (2)^2$$

$R_{xx}(0) = \frac{36}{4} = 9$

$$= 9 - 4$$

$$= 5.$$

3. Find the mean & variance of the stationary Process, whose $R_{xx}(\tau) = 18 + \frac{2}{6+\tau^2}$.

$$\mu_x = \sqrt{\lim_{|\tau| \to \infty} R_{xx}(\tau)}$$

$$\mu_x^2 = \lim_{|\tau| \to \infty} 18 + \frac{2}{6+\tau^2}$$

$\sqrt{2 \times 9} = 3\sqrt{2}.$

$$\mu_x^2 = 18$$

$$\mu_x = \sqrt{18} = 3\sqrt{2}$$

$$Var(x) = E[x^2(t)] - E[x(t)]^2$$
$$= R_{xx}(0) - (18)$$
$$= \left(18 + \frac{2}{6}\right) - 18$$
$$= 1/3$$

3. Autocorrelation func of stationary process $R_{xx}(\tau) = 9 + 2e^{-|\tau|}$ find mean of the random variable $y = \int_0^2 x(t).dt$ & $Var[x(t)]$.

$$E[y] = E\left[\int_0^2 x(t).dt\right].$$

Expectation & Integral can be interchangeable

$$= \int_0^2 E[x(t)].dt$$

$$E[x(t)] = \mu_x = \sqrt{\lim_{\tau \to \infty} R_{xx}(\tau)}$$
$$= \sqrt{9} = 3 \qquad E[y] = \int_0^2 3.dt = [3t]_0^2 = 6$$

$$Var[x(t)] = E[x^2(t)] - E[x(t)]^2$$
$$= R_{xx}(0) - 9.$$
$$= 11 - 9 = 2.$$

4. If $\{X(t)\}$ is a process with mean $\eta(t) = 3$ and autocorrelation is $R_{XX}(t, t+\tau) = 9 + 4e^{-0.2|\tau|}$ determine mean, Var, Covar of $Z = X(5)$ & $W = X(8)$

Mean of Z : $E(Z) = E[X(5)] = \eta(5) = 3$,

$Var(Z) = E[Z^2(t)] - E[Z(t)]^2$

$E[Z^2(t)] = E[X^2(5)] = E[X(5) \cdot X(5)]$

$= 9 + 4e^{-(0.2(0))}$

$= 13$

$Var(Z) = 13 - (3)^2 = 13 - 9 = 4$.

$E[W(t)] = E[X(8)] = \eta(8) = 3$.

$Var(W) = E[W^2(t)] - E[W(t)]^2$ $\tau = 0$.

$E[W^2(t)] = E[W(t) \cdot W(t)] = E[X(8) \cdot X(8)] = 9 + 4 = 13$.

$Covar(XY) = E[XY] - E[X]E[Y]$.

Covar of $(ZW) = E[WZ] - E[Z]E[W]$

$\tau = 3 \rightarrow$ sub in R_{XX}

$= E[X(8) \cdot X(5)] - 3(3)$

$= (9 + 4e^{-0.2(3)}) - 9$.

$= 4e^{-0.6}$.

$R_{xx}(\tau) = E[x(t) \cdot x(t+\tau)]$

$\{x(t)\} \cdot \{y(t)\}$.

Cross correlation.

$R_{xy}(t, t+\tau) = E[x(t) \cdot y(t+\tau)]$.

Properties

1. $R_{xy}(\tau) = R_{yx}(-\tau)$.

Proof, By definition

$R_{xy}(\tau) = E[x(\tau)\ y(t+\tau)]$.

$R_{xy}(-\tau) = E[y(t) \cdot x(t-\tau)]$. $\quad t-\tau = t_1$

$t = t_1 + \tau$.

$= E[y(t_1+\tau) \cdot x(t_1)]$

$= E[x(t_1) \cdot y(t_1+\tau)]$

$= R_{xy}(\tau)$

Cross correlation is not an even function.

2. $|R_{xy}(\tau)| \leq \sqrt{R_{xx}(0) \cdot R_{yy}(0)}$ $\quad R_{xx}(0) \rightarrow$ mean sq value of x

Proof

$R_{xx}(0) = E[x^2(t)] \cdot R_{yy}(0) = E[y^2(t)]$.

for any real no. k

$E[(x(t) + k y(t+\tau))^2] \geq 0$.

$E[x^2(t) + k^2 y^2(t+\tau) + 2k\, x(t)\, y(t+\tau)] \geq 0$

$= E[x^2(t)] + k^2 E[x^2(t+\tau)] + 2kE[x(t).y(t+\tau)] \geq 0.$

$= b^2 - 4ac \leq 0$

$2^2 E[x(t).y(t+\tau)]^2 - 4 E[y^2(t+\tau)] E[x^2(t)] \leq 0.$

$(R_{xy}(\tau))^2 - R_{xx}(0) R_{yy}(0) \leq 0.$

$(R_{xy}(\tau))^2 \leq R_{xx}(0) R_{yy}(0).$

$|R_{xy}(\tau)| \leq \sqrt{R_{xx}(0) R_{yy}(0)}$

3. $|R_{xy}(\tau)| \leq \frac{1}{2}[R_{xx}(0) + R_{yy}(0)].$

$R_{xx}(0) = E[x^2(t)] > 0 \quad R_{yy}(0) = E[y^2(t)] > 0.$

$\therefore$ GM $\leq$ AM

$\sqrt{R_{xx}(0) R_{yy}(0)} \leq \frac{R_{xx}(0) + R_{yy}(0)}{2}$ — ①

GM $\leq$ AM

$\sqrt{ab} \leq \frac{a+b}{2}$

$|R_{xy}(\tau)| \leq \sqrt{R_{xx}(0) \cdot R_{yy}(0)}$ — ②

$|R_{xy}(\tau)| \leq \frac{R_{xx}(0) + R_{yy}(0)}{2}$

4. If $\{x(t)\}$ & $\{y(t)\}$ are orthogonal

$R_{xy}(\tau) = 0.$

5. If $\{x(t)\}$ & $\{y(t)\}$ are independent then

$R_{xy}(\tau) = \mu_x \cdot \mu_y.$

$R_{xy}(\tau) = E[x(t)\, y(t+\tau)]$

$E[xy] = E[x] \cdot E[y]$

$= E[x(t)] \cdot E[y(t)]$

$= \mu_x \cdot \mu_y.$

1. Two random process $x(t)$ & $y(t)$ are given by

$x(t) = A\cos(\omega t+\theta)$

$y(t) = A\sin(\omega t+\theta)$. Where A & ω are constant

and θ is uniformly distributed in $(0, 2\pi)$.

Find the Cross correlation.

$x(t) = A\cos(\omega t+\theta)$

$y(t) = A\sin(\omega t+\theta)$

$E[g(x)] = \int_{-\infty}^{\infty} g(x) f(x).dx$

$f(\theta) = \frac{1}{2\pi}$

$\sin A \cos B = \frac{1}{2}\{\sin(A+B) + \sin(A-B)\}$

$R_{XX}(t, t+\tau) = E[x(t).y(t+\tau)]$

$= E[A\cos(\omega t+\theta). A\sin(\omega t+\omega\tau+\theta)]$

$= A^2 \int_0^{2\pi} \cos(\omega t+\theta)\sin(\omega t+\omega\tau+\theta)\frac{1}{2\pi}.d\theta$

$= \frac{A^2}{2\pi} \times \frac{1}{2}\int_0^{2\pi} \sin(\omega t+\theta+\omega t+\omega\tau+\theta) + \sin(\omega t+\theta-\omega t-\omega\tau-\theta)\, d\theta$

$= \frac{A^2}{2\pi}\left[\int_0^{2\pi} \sin(2\omega t+2\theta+\omega\tau) + \int_0^{2\pi} \sin(\omega\tau).d\theta\right.$

$= \frac{A^2}{2\pi}\left[\left[\frac{\cos(2\omega t+2\theta+\omega\tau)}{2}\right]_0^{2\pi} + \sin\omega\tau(\theta)_0^{2\pi}\right]$

$= \frac{A^2}{2\pi}\sin\omega t(2\pi - 0)$

$= \frac{A^2}{2}\sin\omega t.$

When RV belong to same family auto correlation

2. Find the cross correlation function of $Z(t) = A(t) - B(t)$ $W(t) = A(t) + B(t)$. When $A(t)$ & $B(t)$ are statistically independent Random variable with zero mean and auto correlation of

$$R_{AA}(\tau) = e^{-|\tau|}, \quad -\infty < \tau < \infty$$

$$R_{BB}(\tau) = 3e^{-|\tau|}, \quad -\infty < \tau < \infty$$

$$R_{WZ}(t, t+\tau) = E[W(t) \cdot Z(t+\tau)]$$

$$= E[[A(t) + B(t)] \cdot [A(t+\tau) - B(t+\tau)]]$$

$$= E[A(t)\, A(t-\tau)] - E[B(t) \cdot B(t-\tau)] +$$

$$E[B(t)\, A(t+\tau)] - E[A(t) \cdot B(t+\tau)]$$

A & B are independent.

$E[AB] = E(A)\, E(B)$

$E(A) = 0 \quad E(B) = 0$. $\quad E[A(t)\, A(t-\tau)] = R_{AA}(\tau)$

$E(AB) = 0$.

$$= R_{AA}(\tau) - R_{BB}(\tau)$$

$$= e^{-|\tau|} - 3e^{-|\tau|}$$

$$R_{WZ} = -2e^{-|\tau|}$$

3. Are independent with zero mean $x(t)$ & $y(t)$

Find the auto-correlation of

$z(t) = a + b\,x(t) + c\,y(t)$.

$R_{zz}(\tau) = E[z(t) \cdot z(t+\tau)]$.

$= E[[a + b\,x(t) + c\,y(t)] \cdot [a + b\,x(t+\tau) + c\,y(t+\tau)]]$

$= E[a^2] + E[a\,b\,x(t+\tau)] + E[a\,c\,y(t+\tau)] + E[a\,b\,x(t)]$

$+ E[b^2 x(t) \cdot x(t+\tau)] + E[b\,c\,x(t) \cdot y(t+\tau)] +$

$E[a\,c\,y(t)] + E[b\,c\,y(t)\,x(t+\tau)] + E[c^2 y(t)\,y(t+\tau)]$

$= a^2 + a\,b\,\underbrace{E[x(t+\tau)]}_{\to 0} + 0 + 0 + 0 + E[b^2 x(t) \cdot x(t+\tau)]$

$+ E[c^2 y(t)\,y(t+\tau)]$.

$R_{zz}(\tau) = a^2 + b^2 R_{xx}(\tau) + c^2 R_{yy}(\tau)$

$E[x(t)] = 0$

WSS

1. Mean → Const

2. $R_{xx}(t_1, t_2) = f_n(\tau)$

JOINT WSS.

$\{x(t)\}$ $\{y(t)\}$.

1. $\{x(t)\}$ is WSS.

2. $\{y(t)\}$ is WSS.

3. $R_{xy}(t_1, t_2) = f_n(\tau)$.