UNIT - II : MAGNETOSTATICS AND MAXWELLS EQUATIONS

- An electrostatic field is produced by static or stationary charges.
- If the charges are moving with constant velocity, a static magnetic (or magnetostatic) field is produced.
- A magnetostatic field is produced by a constant current flow (or direct current). This current flow may be due to magnetization currents as in permanent magnets, electron - beam currents as in vaccum tubes, or conduction currents as in current - carrying wires.

*. BIOT - SAVART'S LAW :-

Biot - Savart's law states that the differential magnitude field intensity dH produced at a point 'P', by the differential current element I dl is proportional to the product I dl and the sine of the angle α between the element and the line joining P to the element and is inversely proportional to the square of the distance R between P and the element.

That is,

$$dH \propto \frac{I\,dl \sin\alpha}{R^2}$$

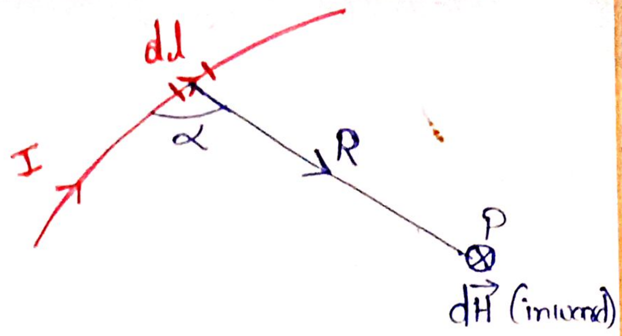


Fig:- Magnetic field $\vec{dH}$ at 'P' due to current element $I\,dl$.

or

$$dH = \frac{kI\,dl \sin\alpha}{R^2}$$

where k is the constant of proportionality. In SI units, $k = \frac{1}{4\pi}$.

$$\Rightarrow dH = \frac{I\,dl \sin\alpha}{4\pi R^2}$$

From the definition of cross product, the above equation can be written as

$$\Rightarrow \vec{dH} = \frac{I\,dl \times \vec{a}_R}{4\pi R^2} = \frac{I\,dl \times \vec{R}}{4\pi R^3}$$

where $\vec{a}_R = \frac{\vec{R}}{|\vec{R}|} = \frac{\vec{R}}{R}$.

$$\boxed{\therefore \vec{dH} = \frac{I\,dl \times \vec{a}_R}{4\pi R^2} = \frac{I\,dl \times \vec{R}}{4\pi R^3}} \longrightarrow (1)$$

NOTE :- Direction of $\vec{dH}$.

The direction of dH can be determined by the right hand rule with the right-hand thumb pointing in the direction of the current and the right-hand fingers encircling the wire in the direction of $d\vec{H}$.

(a) (b)

Fig:- Determining the direction of $\vec{dH}$ using (a) the right-hand rule or (b) the right-handed screw rule.

H or I is out $\odot$ $\otimes$ H or I is in

(a) (b)

Fig:- Conventional representation of H or (I) (a) out of the page and (b) into the page.

If we define K as the surface current density in amperes per meter and J as the volume current density in amperes per meter square, the source elements are related as

$$I\,dl \equiv K\,ds \equiv J\,dv$$

Thus in terms of the distributed current sources, the Biot-Savart law in eq. (1) becomes

$$\vec{H} = \int_L \frac{I\,d\vec{l} \times \vec{a}_R}{4\pi R^2} \quad \text{(line current)} \longrightarrow (2)$$

$$\vec{H} = \int_S \frac{K\,ds \times \vec{a}_R}{4\pi R^2} \quad \text{(surface current)} \longrightarrow (3)$$

$$\vec{H} = \int_V \frac{J\,dv \times \vec{a}_R}{4\pi R^2} \quad \text{(volume current)} \longrightarrow (4)$$

where $\vec{a}_R$ is a unit vector pointing from the differential element of current to the point of interest.

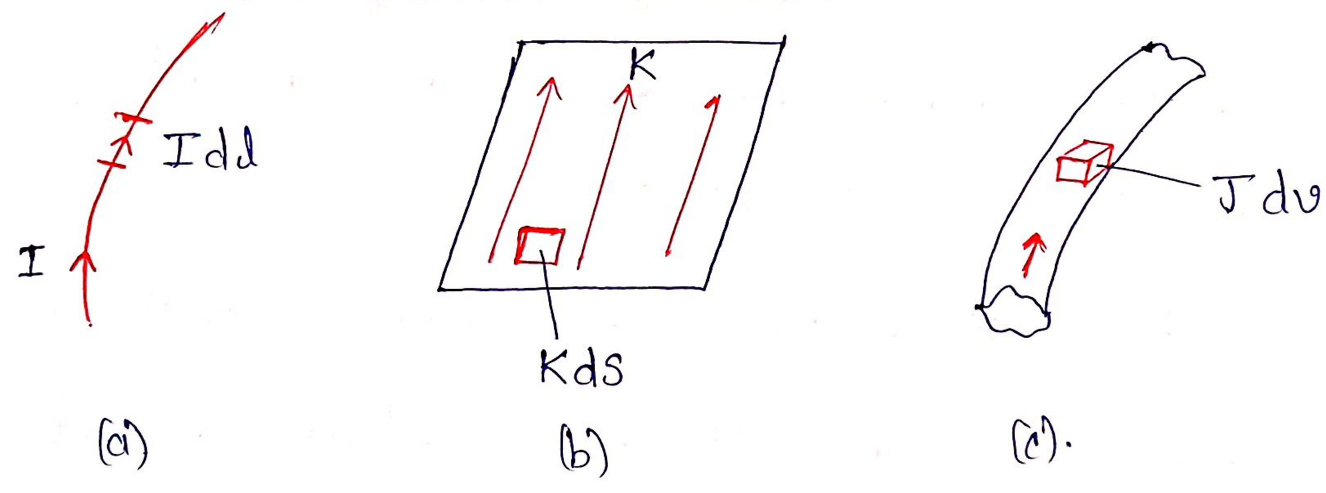
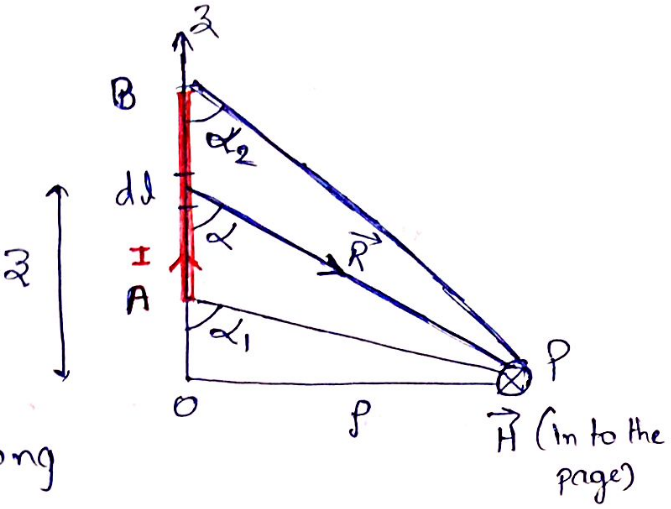


Fig:- Current distributions (a) line current (b) surface current (c) volume current.

Magnetic ~~Flux density~~ Field Intensity due to finite length current carrying conductor:- (straight conductor).

Consider a straight current carrying filamentary conductor of finite length 'AB'.



Assume that the conductor is along the z-axis with its upper and lower ends, respectively, subtending angles α_2 and α_1 at 'P', the point at which $\vec{H}$ is to be determined.

Note:- The current flows from point A, where $\alpha = \alpha_1$, to point B, where $\alpha = \alpha_2$.

If we consider the contribution $\vec{dH}$ at 'P' due to an element $\vec{dl}$ at $(0, 0, z)$,

$$\vec{dH} = \frac{I\,\vec{dl} \times \vec{R}}{4\pi R^3} \longrightarrow (1)$$

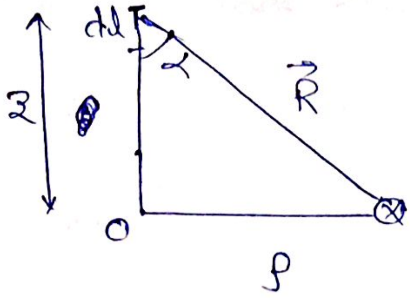
But $\vec{dl} = dz\,\vec{a_z}$ and $\vec{R} = \rho\,\vec{a_\rho} - z\,\vec{a_z}$, so

$$\vec{dl} \times \vec{R} = dz\,\vec{a_z} \times (\rho\,\vec{a_\rho} - z\,\vec{a_z})$$

$$\Rightarrow \vec{dl} \times \vec{R} = \rho\,dz\,\vec{a_z} \times \vec{a_\rho} - z\,dz\,\vec{a_z} \times \vec{a_z}$$

$$\boxed{\therefore\ \vec{dl} \times \vec{R} = \rho\,dz\,\vec{a_\phi}}$$

$$R = \sqrt{\rho^2 + z^2}$$



Now, eq. (1) can be written as

$$\vec{dH} = \frac{I \rho \, dz}{4\pi (\rho^2 + z^2)^{3/2}} \vec{a_\phi}$$

$$\therefore \vec{H} = \int \frac{I \rho \, dz}{4\pi (\rho^2 + z^2)^{3/2}} \vec{a_\phi} \longrightarrow (2)$$

from figure, $\tan\alpha = \frac{\rho}{z} \Rightarrow z = \rho \cot\alpha$

and $dz = -\rho \, \mathrm{cosec}^2\alpha \, d\alpha$

then eq. (2) becomes

$$\Rightarrow \vec{H} = \int \frac{I \rho \, (-\rho \, \mathrm{cosec}^2\alpha \, d\alpha)}{4\pi (\rho^2 + \rho^2 \cot^2\alpha)^{3/2}} \vec{a_\phi}$$

$$\Rightarrow \vec{H} = \frac{-I}{4\pi} \int_{\alpha_1}^{\alpha_2} \frac{\rho^2 \, \mathrm{cosec}^2\alpha \, d\alpha}{(\rho^2(1 + \cot^2\alpha))^{3/2}} \vec{a_\phi}$$

$$\Rightarrow \vec{H} = \frac{-I}{4\pi} \int_{\alpha_1}^{\alpha_2} \frac{\rho^2 \, \mathrm{cosec}^2\alpha \, d\alpha}{\rho^3 \, \mathrm{cosec}^3\alpha} \vec{a_\phi}$$

$$\Rightarrow \vec{H} = \frac{-I}{4\pi\rho} \vec{a_\phi} \int_{\alpha_1}^{\alpha_2} \sin\alpha \, d\alpha$$

$$\Rightarrow \vec{H} = \frac{-I}{4\pi\rho} \vec{a_\phi} \left(-\cos\alpha\right)\Big|_{\alpha_1}^{\alpha_2}$$

$$\boxed{\vec{H} = \frac{I}{4\pi\rho} (\cos\alpha_2 - \cos\alpha_1) \vec{a_\phi}} \longrightarrow (3)$$

Note:- From eq. ③, $\vec{H}$ is always along the unit vector a_ϕ (i.e., along concentric circular paths) irrespective of the length of the wire or the point of interest P.

Special Cases:-

When the conductor is semi-infinite (with respect to P) so that point 'A' is now at O(0,0,0) and while 'B' at $(0,0,\infty)$, $\alpha_1 = 90^\circ$, $\alpha_2 = 0^\circ$ and eq. ③ becomes

$$\boxed{\vec{H} = \frac{I}{4\pi\rho}\,\vec{a_\phi}}$$ Semi-infinite.

When the conductor is infinite in length, point 'A' is at $(0,0,-\infty)$ while 'B' is at $(0,0,\infty)$; $\alpha_1 = 180^\circ, \alpha_2 = 0^\circ$, eq. ③ reduces to

$$\boxed{\vec{H} = \frac{I}{2\pi\rho}\,\vec{a_\phi}}$$ Infinite length

To find unit vector '$\vec{a_\phi}$' in above eqs is not always easy. A simple approach is to determine $\vec{a_\phi}$ from

$$\boxed{\vec{a_\phi} = \vec{a_l} \times \vec{a_\rho}}$$

where $\vec{a_l}$ is a unit vector along the line current and $\vec{a_\rho}$ is a unit vector along the perpendicular line from the line current to the field point.

*. AMPERE'S CIRCUIT LAW - MAXWELL'S EQUATION:-

Ampere's circuit law states that the line integral of $\vec{H}$ around a closed path is the same as the net current I_{enc} enclosed by the path.

In other words, the circulation of $\vec{H}$ equals I_{enc}, that is

$$\boxed{\oint \vec{H} \cdot \vec{dl} = I_{enc}} \longrightarrow (1)$$

By applying Stokes's theorem to the LHS of eq. (1), we obtain

$$\Rightarrow \quad I_{enc} = \oint \vec{H} \cdot \vec{dl} = \int_S (\nabla \times \vec{H}) \cdot \vec{ds} \longrightarrow (2)$$

But

$$I_{enc} = \int_S \vec{J} \cdot \vec{ds} \longrightarrow (3)$$

$$\Rightarrow \quad \int_S (\nabla \times \vec{H}) \cdot \vec{ds} = \int_S \vec{J} \cdot \vec{ds}$$

$$\boxed{\therefore \int_S (\nabla \times \vec{H}) \cdot \vec{ds} = \int_S \vec{J} \cdot \vec{ds}} \quad \text{INTEGRAL FORM}$$

(OR)

$$\boxed{\nabla \times \vec{H} = \vec{J}} \quad \text{POINT FORM (OR) DIFFENETIAL FORM}$$

This is the third Maxwell's equation.

<u>Note:</u>- $\nabla \times \vec{H} = \vec{J} \neq 0$; that is a magnetostatic field is not Conservative.

Note:-

→ Ampere's law is similar to Gauss's law, since Ampere's law is easily applied to determine $\vec{H}$ when the current distribution is symmetrical.

→ Ampere's law is a special case of Biot-Savart's law.

*. APPLICATIONS OF AMPERE'S LAW:-

Note:- For symmetrical current distribution, $\vec{H}$ is either parallel or perpendicular to $\vec{dl}$. When '$\vec{H}$' is parallel to $\vec{dl}$, $|H|$ = Constant.

A. Infinite line current :-

Consider an infinitely long filamentary current 'I' along the z-axis.

To determine $\vec{H}$ at an observation point 'P', we allow a closed path to pass through 'P'. This path on which Ampere's law is to be applied, is known as an Amperian path (analogous to the term "Gaussian Surface").

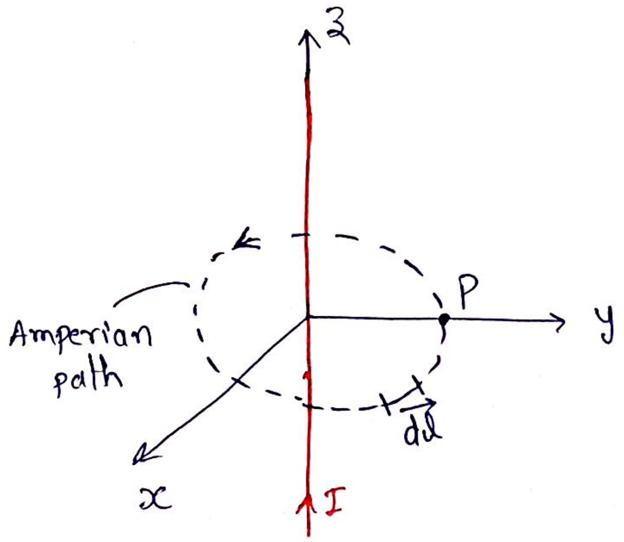


Fig:- Ampere's law applied to an infinite filamentary line current.

We choose a concentric circle as the Amperian path in view of equation $\vec{H} = \frac{I}{2\pi\rho}\vec{a}_\phi$, which shows that $\vec{H}$ is

Constant provided 'ρ' is constant.

Since the path encloses the whole current 'I', according to Ampere's law,

$$I = \oint \vec{H} \cdot \vec{dl} = \oint H_\phi \vec{a_\phi} \cdot \rho d\phi \, \vec{a_\phi}$$

$$\Rightarrow I = H_\phi \oint \rho \, d\phi = H_\phi \, 2\pi\rho$$

$$\Rightarrow H_\phi = \frac{I}{2\pi\rho} \quad \text{or} \quad \vec{H} = H_\phi \vec{a_\phi}$$

$$\boxed{\vec{H} = \frac{I}{2\pi\rho} \vec{a_\phi}}$$

B. Infinite sheet of current :-

Consider an infinite current sheet in the $z=0$ plane. If the sheet has a uniform current density $\vec{K} = K_y a_y$ A/m

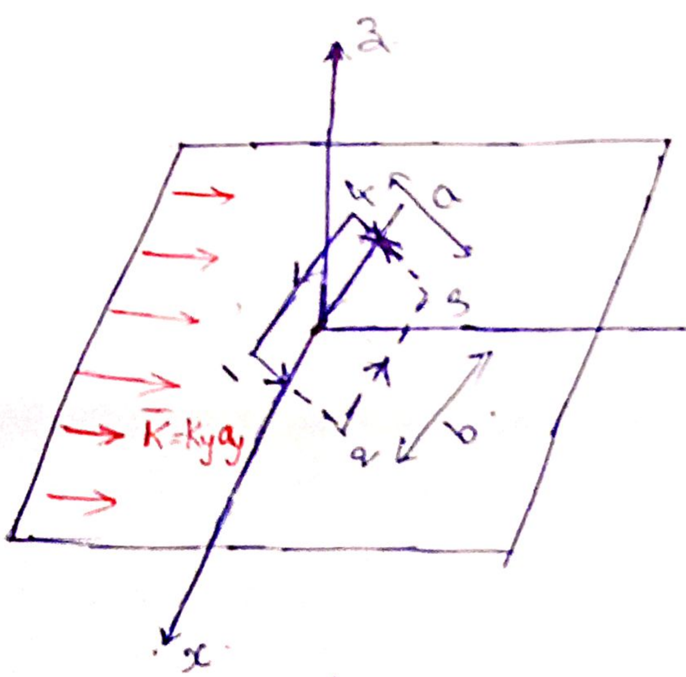


Fig :- Application of Ampere's law to an infinite sheet

applying Ampere's law to the rectangular closed path 1-2-3-4-1 (Amperian path) gives

$$\oint \vec{H} \cdot \vec{dl} = I_{enc} = K_y b \longrightarrow (1)$$

Assume that the infinite sheet comprises of filaments, $\vec{dH}$ (6)
above or below the sheet due to a pair of filamentary currents can be found by eqn $\vec{H} = \frac{I}{2\pi\rho}\vec{a_\phi}$.

The resultant $\vec{dH}$ has only x-component. Also, $\vec{H}$ on one side of the sheet is the negative of that on the other side. Owing to the infinite extent of the sheet, the sheet can be regarded as consisting of such filamentary pairs so that the characteristics of $\vec{H}$ for a pair are the same for the infinite current sheets, that is,

$$\vec{H} = \begin{cases} H_0 \vec{a_x} & z > 0 \\ -H_0 \vec{a_x} & z < 0 \end{cases}$$

where H_0 is yet to be determined.

$$\oint \vec{H}\cdot\vec{dl} = \left(\int_1^2 + \int_2^3 + \int_3^4 + \int_4^1\right)\vec{H}\cdot\vec{dl}.$$

$$\Rightarrow \oint \vec{H}\cdot\vec{dl} = 0(-a) + (-H_0)(-b) + 0(a) + H_0(b)$$

$$\Rightarrow \oint \vec{H}\cdot\vec{dl} = 2H_0 b \longrightarrow (2)$$

From eqns (1) & (2), we obtain $H_0 = \frac{1}{2}K_y$.

Then $$\vec{H} = \begin{cases} \frac{1}{2}K_y a_x & z > 0 \\ -\frac{1}{2}K_y a_x & z < 0 \end{cases} \longrightarrow (3)$$

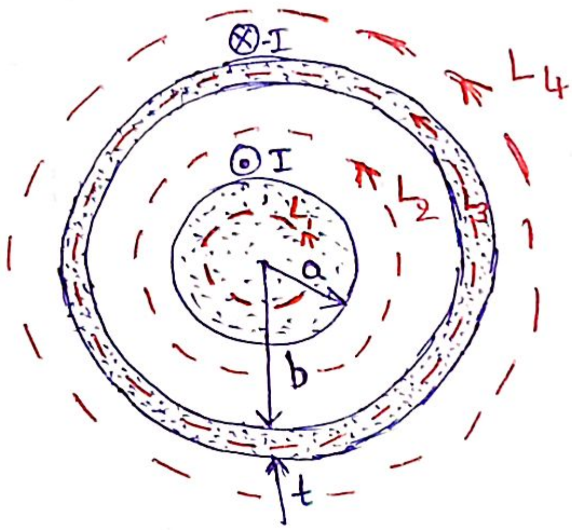
In general, for an infinite sheet of current density $\vec{K}$ A/m,

$$\boxed{\vec{H} = \frac{1}{2}\vec{K} \times \vec{a_n}} \longrightarrow (i)$$

where $\vec{a_n}$ is a unit vector directed from the current sheet to the point of interest.

c. Infinitely long Coaxial transmission line:-

Consider an infinitely long transmission line consisting of two concentric cylinders having their axis along the z-axis.



The inner conductor has radius 'a' and carries current 'I', while the outer conductor has inner radius 'b' and thickness 't' and carries return current '-I'.

Since the current distribution is symmetrical, we apply Ampere's law along the Amperian path for each of the four possible regions;

$$0 \le \rho \le a$$
$$a \le \rho \le b$$
$$b \le \rho \le b+t \quad \text{and} \quad \rho \ge b+t.$$

For region $0 \le \rho \le a$, we apply Ampere's law to path L_1, (7)

giving $$\oint_{L_1} \vec{H} \cdot \vec{dl} = I_{enc} = \int \vec{J} \cdot \vec{ds} \longrightarrow (1)$$

Since the current is uniformly distributed over the cross section,

$$\vec{J} = \frac{I}{\pi a^2} \vec{a_z} \quad \text{and} \quad \vec{ds} = \rho \, d\phi \, d\rho \, \vec{a_z}$$

$$\Rightarrow \quad I_{enc} = \int \vec{J} \cdot \vec{ds} = \int \frac{I}{\pi a^2} \vec{a_z} \cdot \rho \, d\phi \, d\rho \, \vec{a_z}$$

$$= \int_{\phi=0}^{2\pi} \int_{\rho=0}^{a} \frac{I \rho}{\pi a^2} \, d\phi \, d\rho$$

$$= \frac{I}{\pi a^2} \times 2\pi \times \int_{\rho=0}^{a} \rho \, d\rho$$

$$= \frac{I}{\pi a^2} \times 2\pi \times \left. \frac{\rho^2}{2} \right|_0^a$$

$$= \frac{I \rho^2}{a^2}$$

$\boxed{I_{enc} = I}$

eq (1) becoms $\Rightarrow$ $$\oint_{L_1} \vec{H} \cdot \vec{dl} = \frac{I \rho^2}{a^2}$$

$$\Rightarrow \quad H_\phi \int_{L_1} dl = \frac{I \rho^2}{a^2}$$

$$\Rightarrow \quad H_\phi \, 2\pi\rho = \frac{I \rho^2}{a^2}$$

$$\boxed{\therefore \; H_\phi = \frac{I \rho}{2\pi a^2}} \longrightarrow (2)$$

For region $a \le \rho \le b$, we use path L_2 as the Amperian path.

$$\oint_{L_2} \vec{H} \cdot \vec{dl} = I_{enc} = I$$

$$\Rightarrow \quad H_\phi \, 2\pi\rho = I$$

$$\boxed{\therefore \; H_\phi = \frac{I}{2\pi\rho}} \longrightarrow (3)$$

For region $b \le \rho \le b+t$, we use path L_3, getting

$$\oint_{L_3} \vec{H} \cdot \vec{dl} = H_\phi \, 2\pi\rho = I_{enc} \longrightarrow (4)$$

where $\quad I_{enc} = I + \int \vec{J} \cdot \vec{ds}$

$\vec{J}$ in this case is the current density (current per unit area) of the outer conductor and is along $-\vec{a_z}$, that is,

$$\vec{J} = -\frac{I}{\pi\left[(b+t)^2 - b^2\right]} \vec{a_z}$$

Thus $\quad I_{enc} = I + \int \vec{J} \cdot \vec{ds}$

$$= I + \int \frac{-I}{\pi\left[(b+t)^2 - b^2\right]} \vec{a_z} \cdot \rho \, d\phi \, d\rho \, \vec{a_z}$$

$$= I - \frac{I}{\pi\left[(b+t)^2 - b^2\right]} \int_{\phi=0}^{2\pi} \int_{\rho=b}^{\rho=b+t} \rho \, d\rho \, d\phi$$

$$I_{enc} = I\left[1 - \frac{\rho^2 - b^2}{t^2 + 2bt}\right]$$

eq (4), becomes

$$\Rightarrow H_\phi \, 2\pi\rho = I\left[1 - \frac{\rho^2 - b^2}{t^2 + 2bt}\right]$$

$$\Rightarrow H_\phi = \frac{I}{2\pi\rho}\left[1 - \frac{\rho^2 - b^2}{t^2 + 2bt}\right]$$

$$\boxed{H_\phi = \frac{I}{2\pi\rho}\left[1 - \frac{\rho^2 - b^2}{t^2 + 2bt}\right]} \longrightarrow (5)$$

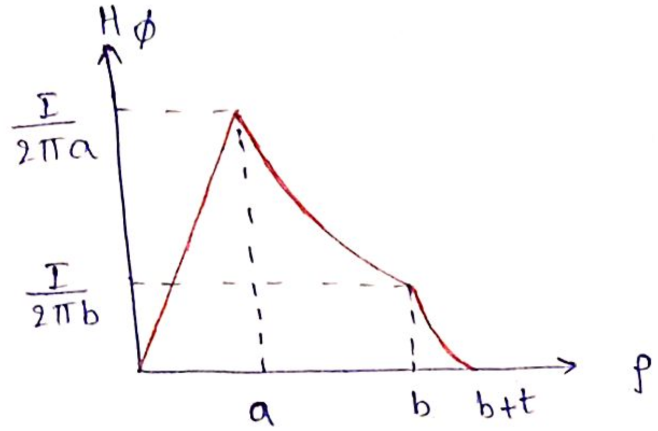
For region $\rho \geq b+t$, we use path L_4, getting

$$\oint_{L_4} \vec{H} \cdot \vec{dl} = I - I = 0$$

$$\boxed{H_\phi = 0}$$

Putting eq ns together gives

$$\vec{H} = \begin{cases} \dfrac{I\rho}{2\pi a^2} a_\phi & 0 \leq \rho \leq a \\ \dfrac{I}{2\pi\rho} a_\phi & a \leq \rho \leq b \\ \dfrac{I}{2\pi\rho}\left[1 - \dfrac{\rho^2 - b^2}{t^2 + 2bt}\right] a_\phi & b \leq \rho \leq b+t \\ 0 & \rho \geq b+t. \end{cases}$$



<u>Note:-</u> From these examples, it can be observed that the ability to take $\vec{H}$ from under the integral sign is the key to using Ampere's law to determine $\vec{H}$.

In other words, Ampere's law can be used to find $\vec{H}$ only due to symmetric current distributions for which it is possible to find a closed path over which $\vec{H}$ is constant in magnitude.

*. MAGNETIC FLUX DENSITY - MAXWELL'S EQUATIONS :-

The magnetic flux density $\vec{B}$ is similar to the electric flux density $\vec{D}$. As $\vec{D} = \varepsilon_0 \vec{E}$ in free space, the magnetic flux density $\vec{B}$ is related to the magnetic field intensity $\vec{H}$ according to

$$\boxed{\vec{B} = \mu_0 \vec{H}} \longrightarrow (1)$$

where μ_0 is a constant known as the permeability of free space.

$$\boxed{\mu_0 = 4\pi \times 10^{-7} \; H/m}$$

Definition:- A measure of the strength of a magnetic field at a given point, expressed by the force per unit length on a conductor carrying unit current at that point.

(OR)

It is defined as the force per unit current element.

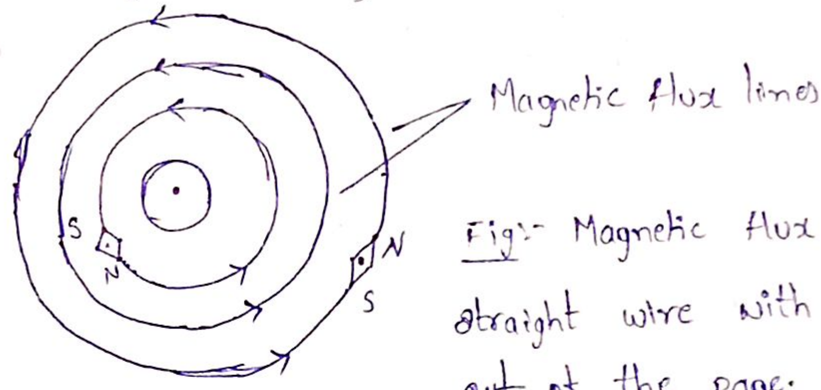
Unit:- Newton - meters per ampere (Nm/A) or Tesla (T) or Wb/m^2.

The magnetic flux through a surface 'S' is given by

$$\boxed{\psi = \int_S \vec{B} \cdot d\vec{S}} \longrightarrow (2)$$

where the magnetic flux Ψ is in Webers (Wb) and the magnetic flux density $\vec{B}$ is in Webers per square meter (Wb/m^2) or teslas (T).

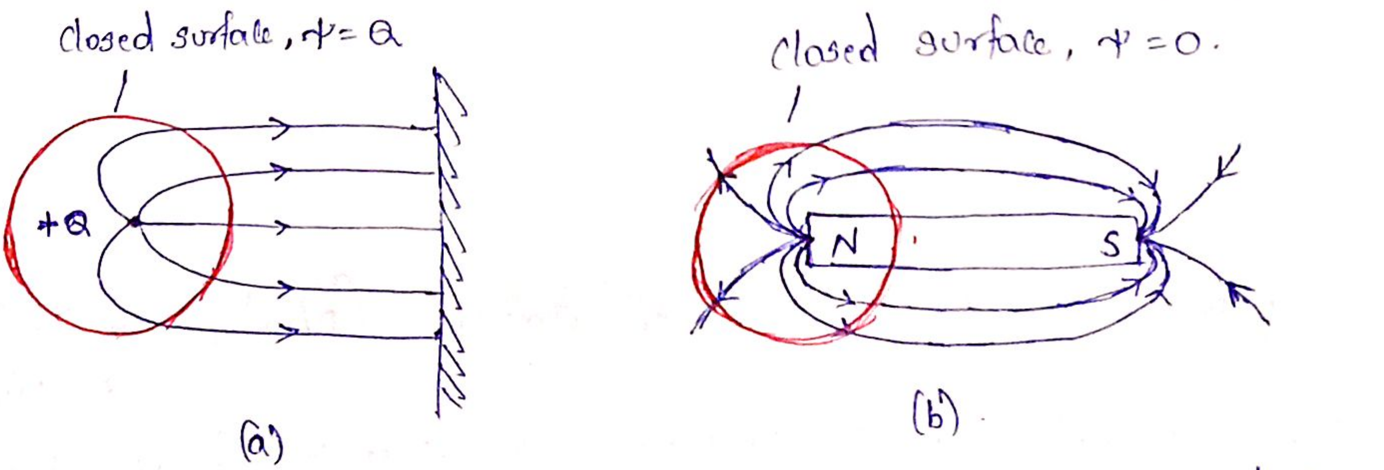
<u>Note</u>:- → A magnetic flux line is a path to which $\vec{B}$ is tangential at every point on the line. It is a line along which the needle of a magnetic compass will orient itself if placed in the presence of a magnetic field.



<u>Fig</u>:- Magnetic flux lines due to a straight wire with current coming out of the page.

→ The direction of $\vec{B}$ is taken as that indicated as "north" by the needle of the magnetic compass.

→ Each flux line is closed and has no beginning or end.



(a) (b)

<u>Fig</u>:- Flux leaving a closed surface due to (a) isolated electric charge $\Psi = \oint_S \vec{D} \cdot d\vec{S} = Q$, (b) magnetic charge, $\Psi = \oint_S \vec{B} \cdot d\vec{S} = 0$.

In an electrostatic field, the flux passing through a closed surface is the same as the charge ~~encl~~ enclosed; that is $\psi = \oint \vec{D} \cdot \vec{dS} = Q$. Thus it is possible to have an isolated electric charge.

Unlike electric flux lines, magnetic flux lines always close upon themselves. This is because it is not possible to have isolated magnetic poles (or magnetic charges).

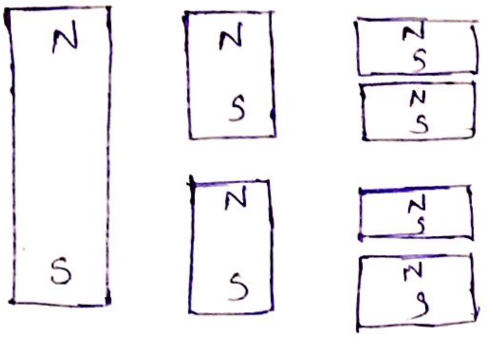


Fig:- Successive division of a bar magnet results in pieces with north and south poles, showing that magnetic poles cannot be isolated.

An isolated magnetic charge does not exist.

Thus the total flux through a closed surface in a magnetic field must be zero; that is

$$\boxed{\oint \vec{B} \cdot \vec{dS} = 0} \longrightarrow (3)$$

This equation is referred to as the law of conservation of magnetic flux or Gauss's law for magnetostatic fields.

Note:- Although the magnetostatic field is not conservative, magnetic flux is ~~conservative~~ conserved.

By applying the divergence theorem to eq ③, we obtain

$$\oint_S \vec{B} \cdot d\vec{s} = \int_v \nabla \cdot \vec{B} \, dv = 0$$

or

$$\boxed{\nabla \cdot \vec{B} = 0}$$

This equation is the fourth Maxwell's equation.

*. MAXWELL'S EQUATIONS FOR STATIC FIELDS :-

Table :- Maxwell's equations for static electric and magnetic fields.

Differential (or) Point form	Integral form	Remarks
$\nabla \cdot \vec{D} = \rho_v$	$\oint_S \vec{D} \cdot d\vec{s} = \int_v \rho_v \, dv$	Gauss's law
$\nabla \cdot \vec{B} = 0$	$\oint_S \vec{B} \cdot d\vec{s} = 0$	Nonexistence of magnetic monopole, Gauss's law for magnetostatic fields, law of Conservation of magnetic flux.
$\nabla \times \vec{E} = 0$	$\oint_L \vec{E} \cdot d\vec{l} = 0$	Conservative nature of electrostatic field.
$\nabla \times \vec{H} = \vec{J}$	$\oint_L \vec{H} \cdot d\vec{l} = \int_S \vec{J} \cdot d\vec{s}$	Ampere's law.