Unit - II Two dimensional of r.v's

Definition: Let S be a sample space associated with a random experiment E. Let X and Y be two r.v's defined on S. Then the pair (X, Y) is called a two dimensional r.v.

Example: The measurement of heights and weights of a students in a class results in the outcomes (H, W) having two characteristics. This is a two dimensional r.v.

Types of two dimensional r.v's

(1) Discrete random Variable

(2) Continuous random Variable

Discrete r.v: If the set of all values of (X, Y) is finite or countably infinite then (X, Y) is called a two dimensional discrete r.v. The values of (X, Y) can be represented as (x_i, y_j), $i = 1, 2, 3 \ldots$; $j = 1, 2, 3 \ldots$

Example: Consider the experiment of tossing a coin twice. The sample space is $S = \{HH, HT, TH, TT\}$.

Let X denote the number of heads obtained in the first toss and

Y denote the number of heads in the second toss.

Then	S	HH	HT	TH	TT
	X(S)	1	1	0	0
	Y(S)	1	0	1	0

(X, Y) is a two dimensional r.v.

Joint Probability Mass function of discrete r.v

The function $p(x_i, y_j) = P(X = x_i, Y = y_j)$ is the Joint prob. mass function of the discrete r.v (X, Y)

if (i) $P(x_i, y_j) \geq 0 \quad \forall\ i = 1, 2, \ldots n;\ j = 1, 2, \ldots m$

(ii) $\sum_i \sum_j P(x_i, y_j) = 1.$

We denote $P(X = x_i, Y = y_j) = P(x_i, y_j) = p_{ij}$.

The set of triples (x_i, y_j, p_{ij}), $i = 1, 2, \ldots n;\ j = 1, 2, \ldots m$ is called the joint prob. distribution of (X, Y).

Note:

The set $\{x_i, y_j, p_{ij}\}$; $i = 1, 2, \ldots n;\ j = 1, 2, \ldots m$

Can be represented in form of a table.

$x \backslash y$	y_1	y_2	y_3	$\cdots$	y_m	total
x_1	p_{11}	p_{12}	p_{13}	$\cdots$	p_{1m}	$p_{1\cdot}$
x_2	p_{21}	p_{22}	p_{23}	$\cdots$	p_{2m}	$p_{2\cdot}$
x_3	p_{31}	p_{32}	p_{33}	$\cdots$	p_{3m}	$p_{3\cdot}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\cdots$	$\cdots$	$\vdots$
x_n	p_{n1}	p_{n2}	p_{n3}	$\cdots$	p_{nm}	$p_{n\cdot}$
total	$p_{\cdot 1}$	$p_{\cdot 2}$	$p_{\cdot 3}$	$\cdots$	$p_{\cdot m}$	1

Marginal distributions

The marginal distribution ~~of~~ for X alone is given by

$$P(X=x_i) = \sum_j P(X=x_i, Y=y_j) = p_{i\cdot}$$

The marginal distribution for Y alone is given by

$$P(Y=y_j) = \sum_i P(X=x_i, Y=y_j) = p_{\cdot j}$$

Conditional probability distribution

Let $P(X=x_i, Y=y_j)$ be the ~~[illegible]~~ joint prob. function of a two dimensional r.v (X,Y). Then the Conditional prob. function of X given $Y=y_j$

$$P(X=x_i / Y=y_j) = \frac{P[X=x_i, Y=y_j)}{P(Y=y_j)} = \frac{p_{ij}}{p_{\cdot j}}$$

Cumulative distribution function

For the r.v (X, Y), the cumulative dist. function is

$$F(x, y) = P(X \leq x, Y \leq y)$$

|||^{ly}, the conditional prob. fn of Y given $X = x_i$

$$P(Y = y_j / X = x_i) = \frac{P(X = x_i, Y = y_j)}{P(X = x_i)} = \frac{p_{ij}}{p_{i.}}$$

Independence of two random Variables

If (X, Y) is a two-dimensional discrete r.v such that $P\{X = x_i / Y = y_j\} = P\{X = x_i\}$

ie, $\frac{p_{ij}}{p_{.j}} = p_{i.}$ ie, $p_{ij} = p_{i.} \times p_{.j}$ for all i, j then X and Y are said to be independent r.v's.

⊛ ⇒

1.) For the bivariate prob. distribution of (X, Y) given below, find $P(X \leq 1)$, $P(Y \leq 3)$, $P(X \leq 1, Y \geq 3)$, $P(X \leq 1 / Y \leq 3)$, $P(Y \leq 3 / X \leq 1)$ and $P(X + Y \leq 4)$.

~~Sol:~~

X \ Y	1	2	3	4	5	6
0	0	0	1/32	2/32	2/32	3/32
1	1/16	1/16	1/8	1/8	1/8	1/8
2	1/32	1/32	1/64	1/64	0	2/64

Sol:

Sol:

$X \backslash Y$	1	2	3	4	5	6	$P(X=x)$
0	0	0	$1/32$	$2/32$	$2/32$	$3/32$	$8/32$
1	$1/16$	$1/16$	$1/8$	$1/8$	$1/8$	$1/8$	$20/32$
2	$1/32$	$1/32$	$1/64$	$1/64$	0	$2/24$	$4/32$
$P(Y=y)$	$3/32$	$3/32$	$11/64$	$13/64$	$6/32$	$16/64$	1

(i) $P(X \le 1) = P(X=0) + P(X=1) = \frac{8}{32} + \frac{20}{32} = \frac{28}{32}$ or $7/8$

(ii) $P(Y \le 3) = P(Y=1) + P(Y=2) + P(Y=3)$

$= 3/32 + 3/32 + \frac{11}{64} = \frac{23}{64}$

(iii) $P(X \le 1, Y \le 3) = \sum_{j=1}^{3} P(X=0, Y=j) + \sum_{j=1}^{3} P(X=1, Y=j)$

$= P(X=0, Y=1) + P(X=0, Y=2) + P(X=0, Y=3)$

$+ P(X=1, Y=1) + P(X=1, Y=2) + P(X=1, Y=3)$

$= \left(0 + 0 + \frac{1}{32}\right) + \left(\frac{1}{16} + \frac{1}{16} + \frac{1}{8}\right) = 9/32$

(iv) $P(X \le 1 / Y \le 3) = \frac{P(X \le 1, Y \le 3)}{P(Y \le 3)} = \frac{9/32}{23/64} = \frac{18}{23}$

(iv) $P(Y \le 3 / X \le 1) = \frac{P(X \le 1, Y \le 3)}{P(X \le 1)} = \frac{9/32}{7/8} = 9/28$

(v) $P(X+Y \le 4) = P(X=0, Y=1) + P(X=0, Y=2) + P(X=0, Y=3)$

$+ P(X=0, Y=4) + P(X=1, Y=1) + P(X=1, Y=2) + P(X=1, Y=3)$

$+ P(X=2, Y=1) + P(X=2, Y=2)$

$= \sum_{j=1}^{4} P(X=0, Y=j) + \sum_{j=1}^{3} P(X=1, Y=j) + \sum_{j=1}^{2} P(X=2, Y=j)$

$$P(X+Y\le 4) = \left(0+0+\frac{1}{32}+\frac{2}{32}\right)+\left(\frac{1}{16}+\frac{1}{16}+\frac{1}{8}\right)+\left(\frac{1}{32}+\frac{1}{32}\right) = \frac{13}{32}$$

2.) The joint prob. function (X,Y) is given by $P(x,y)=K(2x+3y)$, $x=0,1,2$; $y=1,2,3$. Find all the marginal distributions, all conditional prob. distributions. Also find the prob. distribution of $(X+Y)$. and $P(X+Y>3)$

Sol:

X \ Y	1	2	3	$P(X=x)$
0	3K	6K	9K	18K
1	5K	8K	11K	24K
2	7K	10K	13K	30K
$P(Y=y)$	15K	24K	33K	72K

We know that $\sum p_{ij}=1 \Rightarrow 72K=1$

$K=1/72$

Hence the joint prob. function is given by

X \ Y	1	2	3	Total
0	$3/72$	$6/72$	$9/72$	$18/72$
1	$5/72$	$8/72$	$11/72$	$24/72$
2	$7/72$	$10/72$	$13/72$	$30/72$
Total	$15/72$	$24/72$	$33/72$	1

Marginal distribution of X

$X=x$	0	1	2
$P(X=x)$	$18/72$	$24/72$	$30/72$

Marginal distribution of Y

$Y=y$	1	2	3
$P(Y=y)$	$\frac{15}{72}$	$\frac{24}{72}$	$\frac{33}{72}$

The conditional distribution of X given Y

Conditional dist. of X given Y=1

$$P(X=0/Y=1) = \frac{P(X=0, Y=1)}{P(Y=1)} = 3/72 \Big/ \frac{15}{72} = \frac{3}{15} = \frac{1}{5}$$

$$P(X=1/Y=1) = \frac{P(X=1, Y=1)}{P(Y=1)} = 5/72 \Big/ 15/72 = \frac{5}{15} = \frac{1}{3}$$

$$P(X=2/Y=1) = \frac{P(X=2, Y=1)}{P(Y=1)} = 7/72 \Big/ 15/72 = \frac{7}{15}$$

Conditional dist. of X given Y=2

$$P(X=0/Y=2) = \frac{P(X=0, Y=2)}{P(Y=2)} = 6/72 \Big/ 24/72 = \frac{6}{24} = \frac{1}{4}$$

$$P(X=1/Y=2) = \frac{P(X=1, Y=2)}{P(Y=2)} = 8/72 \Big/ 24/72 = \frac{8}{24} = \frac{1}{3}$$

$$P(X=2/Y=2) = \frac{P(X=2, Y=2)}{P(Y=2)} = 10/72 \Big/ 24/72 = \frac{10}{24} = \frac{5}{12}$$

Conditional dist. of X given Y=3

$$P(X=0/Y=3) = \frac{P(X=0, Y=3)}{P(Y=3)} = \frac{9/72}{33/72} = \frac{9}{33}$$

$$P(X=1/Y=3) = \frac{P(X=1, Y=3)}{P(Y=3)} = \frac{11/72}{33/72} = \frac{11}{33} = \frac{1}{3}$$

$$P(X=2/Y=3) = \frac{P(X=2, Y=3)}{P(Y=3)} = \frac{13/72}{33/72} = \frac{11}{33}$$

The conditional dist. of Y given X

Conditional dist. of Y given X=0

$$P(Y=1/X=0) = \frac{P(Y=1, X=0)}{P(X=0)} = \frac{3/72}{18/72} = \frac{1}{6}$$

$$P(Y=2/X=0) = \frac{P(Y=2, X=0)}{P(X=0)} = \frac{6/72}{18/72} = \frac{6}{18} = \frac{1}{3}$$

$$P(Y=3/X=0) = \frac{P(Y=3, X=0)}{P(X=0)} = \frac{9/72}{18/72} = \frac{9}{18} = \frac{1}{2}$$

Conditional dist. of Y given X=1

$$P(Y=1/X=1) = \frac{P(Y=1, X=1)}{P(X=1)} = \frac{5/72}{24/72} = 5/24$$

$$P(Y=2/X=1) = \frac{P(Y=2, X=1)}{P(X=1)} = \frac{8/72}{24/72} = 8/24 = 1/3$$

$$P(Y=3/X=1) = \frac{P(Y=3, X=1)}{P(X=1)} = \frac{11/72}{24/72} = \frac{11}{24}$$

Conditional dist. of Y given X=2

$$P(Y=1/X=2) = \frac{P(Y=1, X=2)}{P(X=2)} = \frac{7/72}{30/72} = 7/30$$

$$P(Y=2/X=2) = \frac{P(Y=2, X=2)}{P(X=2)} = 10/72 \Big/ 30/72 = \frac{10}{30} = 1/3$$

$$P(Y=3/X=2) = \frac{P(Y=3, X=2)}{P(X=2)} = 13/72 \Big/ 30/72 = \frac{13}{30}$$

To find $P(X+Y>3)$

$P(X+Y>3) = P(X+Y=4) + P(X+Y=5)$

$= \frac{21}{72} + \frac{13}{72} = \frac{34}{72}$

Prob. dist. of $X+Y$

$X+Y$	P
1 (0, 1)	$3/32$
2 (0, 2) (1, 1)	$5/72 + \frac{6}{72} = \frac{11}{72}$
3 (0, 3), (1, 2), (2, 1)	$\frac{9}{72} + \frac{8}{72} + \frac{7}{72} = \frac{24}{72}$
4 (1, 3), (2, 2)	$\frac{11}{72} + \frac{10}{72} = \frac{21}{72}$
5 (2, 3)	$\frac{13}{72}$
Total	1

3) The joint p.d.f of (X, Y), where X and Y are discrete is given in the following table.

X \ Y	0	1	2
0	0.1	0.04	0.06
1	0.2	0.08	0.12
2	0.2	0.08	0.12

Verify whether X and Y are independent.

Sol:

By defn If X and Y are independent then

$P_{i.} \times P_{.j} = P_{ij}$

X \ Y	0	1	2	P(X=x)
0	0.1	0.04	0.06	0.2
1	0.2	0.08	0.12	0.4
2	0.2	0.08	0.12	0.4
P(Y=y)	0.5	0.2	0.3	1

Now $P_{0.} \times P_{.0} = 0.2 \times 0.5 = 0.1 = P_{00}$

$P_{0.} \times P_{.1} = 0.2 \times 0.2 = 0.04 = P_{01}$

$P_{0.} \times P_{.2} = 0.2 \times 0.3 = 0.06 = P_{02}$

$P_{1.} \times P_{0.} = 0.4 \times 0.5 = 0.2 = P_{10}$

$P_{1.} \times P_{.1} = 0.4 \times 0.2 = 0.08 = P_{11}$

$P_{1.} \times P_{.2} = 0.4 \times 0.3 = 0.12 = P_{12}$

$P_{2.} \times P_{.0} = 0.4 \times 0.5 = 0.2 = P_{20}$

$P_{2.} \times P_{.1} = 0.4 \times 0.2 = 0.08 = P_{21}$

$P_{2.} \times P_{.2} = 0.4 \times 0.3 = 0.12 = P_{22}$

$\therefore$ X and Y are independent.

4) The joint prob. mass function of X and Y is

$X \backslash Y$	0	1	2
0	0.1	0.04	0.02
1	0.08	0.20	0.06
2	0.06	0.14	0.3

Find the marginal prob. mass functions of X and Y, $P(X \leq 1, Y \leq 1)$ and check if X and Y independent.

Sol:

$X \backslash Y$	0	1	2	$P(X=x)$
0	0.1	0.04	0.02	0.16
1	0.08	0.20	0.06	0.34
2	0.06	0.14	0.3	0.5
$P(Y=y)$	0.24	0.38	0.38	1

The marginal dist. of X is

$X=x$	0	1	2
$P(X=x)$	0.16	0.34	0.5

The marginal dist. of Y is

$Y=y$	0	1	2
	0.24	0.38	0.38

To find $P(X \le 1, Y \le 1)$

$$P(X \le 1, Y \le 1) = P(X=0, Y \le 1) + P(X=1, Y \le 1)$$

$$= P(X=0, Y=0) + P(X=0, Y=1) + P(X=1, Y=0) + P(X=1, Y=1)$$

$$= 0.1 + 0.04 + 0.08 + 0.2 = 0.42$$

Check If X and Y independent

W.K.T $\quad p_{i.} \times p_{.j} = p_{ij}$

$$p_{0.} \times p_{.1} = 0.16 \times 0.24 = 0.038 \neq p_{01} \; (=0.1)$$

Hence X and Y are not independent.

5) The joint prob. mass function of two r.v's X and Y is given by $p(x, y) = \frac{1}{27}(2x+y)$, $x = 0, 1, 2$; $y = 0, 1, 2$.

(i) Find the conditional dist. of Y given X = 2.

(ii) Find the conditional dist. of X given Y = 1.

Sol:

Y \ X	0	1	2	P(Y=y)
0	0	2/27	4/27	6/27
1	1/27	3/27	5/27	9/27
2	2/27	4/27	6/27	12/27
P(X=x)	3/27	9/27	15/27	1

(i) Conditional distribution of Y given $X=2$

$$P(Y=0/X=2) = \frac{P(Y=0, X=2)}{P(X=2)} = \frac{4/27}{15/27} = \frac{4}{15}$$

$$P(Y=1/X=2) = \frac{P(Y=1, X=2)}{P(X=2)} = \frac{5/27}{15/27} = \frac{5}{15} = \frac{1}{3}$$

$$P(Y=2/X=2) = \frac{P(Y=2, X=2)}{P(X=2)} = \frac{6/27}{15/27} = \frac{6}{15} = \frac{2}{5}$$

(ii) Conditional distribution of X given $Y=1$

$$P(X=0/Y=1) = \frac{P(X=0, Y=1)}{P(Y=1)} = \frac{1/27}{9/27} = 1/9$$

$$P(X=1/Y=1) = \frac{P(X=1, Y=1)}{P(Y=1)} = \frac{3/27}{9/27} = 3/9 = 1/3$$

$$P(X=2/Y=1) = \frac{P(X=2, Y=1)}{P(Y=1)} = \frac{5/27}{9/27} = 5/9$$

6.) The joint prob. mass function of X and Y is given by $P(1,1)=\frac{1}{8}$, $P(1,2)=\frac{1}{4}$, $P(2,1)=\frac{1}{8}$, $P(2,2)=\frac{1}{2}$. (i) Compute the conditional mass function of X given $Y=2$ (ii) Are X and Y are independent? (iii) Compute $P(X+Y>2)$.

Sol:

$Y \backslash X$	1	2	$P(Y=y)$
1	$1/8$	$1/8$	$2/8$
2	$1/4$	$1/2$	$3/4$
$P(X=x)$	$3/8$	$5/8$	1

(i) Conditional mass function of X given Y=2

$$P(X=1/Y=2) = \frac{P(X=1, Y=2)}{P(Y=2)} = 1/4 \Big/ 3/4 = 1/3$$

$$P(X=2/Y=2) = \frac{P(X=2, Y=2)}{P(Y=2)} = \frac{1/2}{3/4} = 2/3$$

(ii) If X and Y are independent then

$$P_{i.} \times P_{.j} = P_{ij}$$

Now

$$P_{1.} \times P_{.1} = \frac{3}{8} \times \frac{2}{8} = \frac{3}{32} \neq P_{11}$$

$\therefore$ X and Y are not independent.

(iii) $P(X+Y>2) = P(X+Y=3) + P(X+Y=4)$

$$= P(X=2, Y=1) + P(X=2, Y=2) + P(X=1, Y=2)$$

$$= \frac{1}{8} + \frac{1}{2} + \frac{1}{4} = 7/8$$

7) Three balls are drawn at random without replacement from a box containing 2 white, 3 red and 4 black balls. If X denotes the number of white balls drawn and Y denotes the number of red balls drawn, find the joint prob. dist. of (X, Y)

Sol:

Given that X denotes the number of white balls drawn and Y denotes the number of red balls drawn.

So X takes the values $0, 1, 2$ and Y takes the values $0, 1, 2, 3$.

When three balls are drawn at a time without replacement.

$P(X=0, Y=0) = P$(no white and no red balls) $= P$(all are black)

$$= \frac{4C_3}{9C_3} = \frac{4\times 3\times 2}{1\times 2\times 3} \times \frac{1\times 2\times 3}{9\times 8\times 7} = \frac{1}{21}$$

$P(X=0, Y=1) = P$(drawing 1 red and 2 black balls)

$$= \frac{3C_1 \times 4C_2}{9C_3} = \frac{3}{14}$$

Similarly $P(X=0, Y=2) = \frac{3C_2 \times 4C_1}{9C_3} = \frac{1}{7}$

$$P(X=0, Y=3) = \frac{3C_3}{9C_3} = \frac{1}{84}$$

$$P(X=1, Y=0) = \frac{2C_1 \times 4C_2}{9C_3} = 1/7$$

$$P(X=1, Y=1) = \frac{2C_1 \times 3C_1 \times 4C_1}{9C_3} = 2/7$$

$$P(X=1, Y=2) = \frac{2C_1 \times 3C_2}{9C_3} = 1/14$$

$P(X=1, Y=3) = 0$ [$\because$ this event is impossible] (only 3 balls are drawn)

$$P(X=2, Y=0) = \frac{2C_2 \times 4C_1}{9C_3} = \frac{1}{21}$$

$$P(X=2, Y=1) = \frac{2C_2 \times 3C_1}{9C_3} = 1/28$$

$P(X=2, Y=2) = 0$

$P(X=2, Y=3) = 0$

The joint probability distribution is

$X \backslash Y$	0	1	2	3
0	$1/21$	$3/14$	$1/7$	$1/84$
1	$1/7$	$2/7$	$1/14$	0
2	$1/21$	$1/28$	0	0

8) Two balls are selected at random from a box that Contains 3 blue balls, 2 red balls and 3 green balls. If X is the number of blue balls and Y is the number of red balls, find (i) the joint probability function $p(x, y)$ (ii) $P(X+Y \leq 1)$.

Sol: Given X ~~der~~ denote the number of blue balls and Y denote the number of red balls.

$\therefore$ X takes the Values 0, 1, 2 and

Y takes the values 0, 1, 2.

The probabilities of selecting the balls are

$$P(x, y) = \frac{3C_x \times 2C_y \times 3C_{2-x-y}}{8C_2}$$

(i)

$X \backslash Y$	0	1	2	$P(X=x)$
0	$3/28$	$6/28$	$1/28$	$10/28$
1	$9/28$	$3/14$	0	$15/28$
2	$3/28$	0	0	$3/28$
$P(Y=y)$	$15/28$	$12/28$	$1/28$	1

(ii) $P(X+Y\le 1) = P(0,0)+P(0,1)+P(1,0) = \frac{3}{28}+\frac{3}{14}+\frac{9}{28} = \frac{9}{14}$

Joint probability density function

Let (X,Y) be a continuous two dimensional r.v.

A function f which assigns each (x,y) a real number $f(x,y)$ for all real x, y is called the joint probability density function of (X,Y) if

(i) $f(x,y)\ge 0$ for all $(x,y)\in R$, R is the range space

(ii) $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\,dx\,dy = 1$.

Note: If D is a subspace of the range space R, $P\{(X,Y)\in D\}$ is defined as

$$P\{(X,Y)\in D\} = \iint_D f(x,y)\,dx\,dy.$$

In particular $P\{a\le X\le b,\ c\le Y\le d\} = \int_c^d\int_a^b f(x,y)\,dx\,dy$

Cumulative Distribution function

If (X,Y) is a two dimensional r.v then

$F(X,Y) = P\{X\le x \text{ and } Y\le y\}$ is called the CDF of (X,Y).

In continuous case, $F(x,y) = \int_{-\infty}^{y}\int_{-\infty}^{x} f(x,y)\,dx\,dy$

Marginal probability density function:

* Marginal density function of X is denoted by $f_X(x)$ and is defined by $f_X(x) = \int_{-\infty}^{\infty} f(x, y)\, dy$.

* Marginal density function of Y is denoted by f_Y and is defined by $f_Y(y) = \int_{-\infty}^{\infty} f(x, y)\, dx$.

Conditional probability density function

* The conditional probability density function of X given Y = y is denoted by $f(x/y)$ an is defined by

$$f_{X/Y}(x/y) = \frac{f(x, y)}{f_Y(y)} \quad \text{If } f_Y(y) \neq 0$$

* The conditional probability of Y given X = x is denoted by $f(y/x)$ and is defined by

$$f_{Y/X}(y/x) = \frac{f(x, y)}{f_X(x)} \quad \text{If } f_X(x) \neq 0$$

Independent continuous random Variable

If (X, Y) is a two dimensional continuous r.v with joint p.d.f $f(x, y)$ such that $f(x, y) = f_X(x) \cdot f_Y(y)$ then X and Y are said to be independent r.v.

Marginal Distribution function

$$F_X(x) = \int_{-\infty}^{x} \left\{ \int_{-\infty}^{\infty} f_{xy}(x,y)\,dy \right\} dx$$

$$F_Y(y) = \int_{-\infty}^{y} \left\{ \int_{-\infty}^{\infty} f_{xy}(x,y)\,dx \right\} dy$$

Properties of joint distribution function

(1) $F(-\infty, y) = 0$, $F(x, -\infty) = 0$, $F(-\infty, -\infty) = 0$, $F(\infty, \infty) = 1$

(2) $P\{a < X < b,\ Y \le y\} = F(b, y) - F(a, y)$

(3) $P\{X \le x,\ c < Y < d\} = F(x, d) - F(x, c)$

(4) $P\{a < X < b,\ c < Y < d\} = F(b,d) - F(a,d) - F(b,c) + F(a,c)$

(5) If f is the joint pdf of the continuous R.V (X, Y)

then $\dfrac{\partial^2 F}{\partial x \partial y} = f(x, y)$

(6) $0 \le F(x, y) \le 1$

1) Find the marginal density function of X and Y if $f(x,y) = \frac{2}{5}(2x + 5y)$, $0 \le x \le 1$, $0 \le y \le 1$.

Sol:

Marginal density function of X

$$f_X(x) = \int_{-\infty}^{\infty} f(x,y)\,dy = \int_0^1 \frac{2}{5}(2x + 5y)\,dy$$

$$= \frac{2}{5}\left[2xy + \frac{5y^2}{2}\right]_0^1 = \frac{2}{5}\left[2x + \frac{5}{2}\right], \quad 0 \le x \le 1.$$

Marginal density function of Y

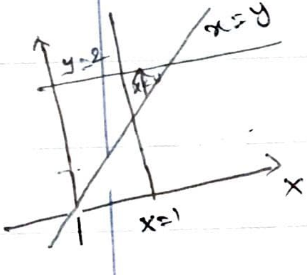
$$f_Y(y) = \int_{-\infty}^{\infty} f(x,y)\,dx = \int_0^1 \frac{2}{5}(2x+5y)\,dx$$

$$= \frac{2}{5}\left[2\frac{x^2}{2} + 5yx\right]_0^1 = \frac{2}{5}\left[1+5y\right]; \quad 0 \le y \le 1.$$

2) The joint probability density function of the two dimensional r.v is $f(x,y) = \begin{cases} 8/9\, xy, & 1 < x < y < 2 \\ 0, & \text{otherwise} \end{cases}$ (i) Find marginal density function (ii) Conditional density function of Y given X.

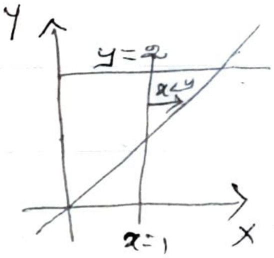
Sol:

(i) Marginal density function of X



$$f_X(x) = \int_{-\infty}^{\infty} f(x,y)\,dy = \int_x^2 \frac{8}{9}xy\,dy$$

$$= \frac{8}{9}\left(x\frac{y^2}{2}\right)_x^2 = \frac{4x}{9}\left[4-x^2\right],$$



Marginal density function of Y

$$f_Y(y) = \int_{-\infty}^{\infty} f(x,y)\,dx = \int_1^y \frac{8}{9}xy\,dx$$

$$= \frac{8}{9}\left(y\frac{x^2}{2}\right)_1^y = \frac{4}{9}y(y^2-1)$$

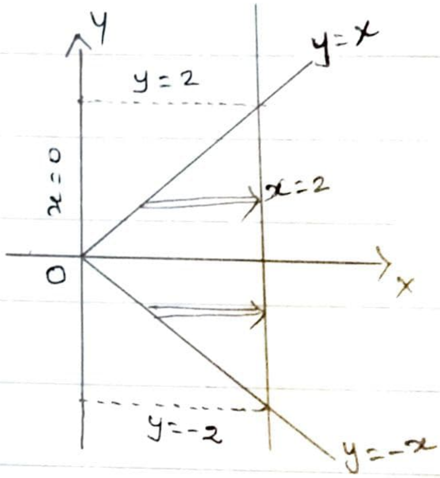
(ii) Conditional density function of Y given $X=x$ is

$$f(Y/X) = \frac{f(x,y)}{f_X(x)} = \frac{8/9\, xy}{\frac{4x}{9}(4-x^2)} = \frac{2y}{4-x^2}$$

3) Given $f_{xy}(x,y) = \begin{cases} cx(x-y), & 0<x<2,\ -x<y<x \\ 0, & \text{otherwise} \end{cases}$. (i) Evaluate c

(ii) find $f_x(x)$ (iii) $f_{y/x}(y/x)$ (iv) $f_y(y)$

$f_y(x)$ - graph



Sol:

(i) Since $f(x,y)$ is a pdf then

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\,dx\,dy = 1$$

$$\therefore \int_0^2\int_{-x}^{x} cx(x-y)\,dy\,dx = 1$$

$$\Rightarrow c\int_0^2 x\left(xy-\frac{y^2}{2}\right)_{-x}^{x} dx = 1$$

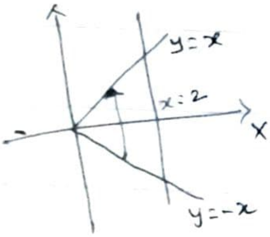
$$\Rightarrow c\int_0^2 x\left[x\cdot x-\frac{x^2}{2}+x\cdot x+\frac{x^2}{2}\right] dx = 1$$

$$\Rightarrow c\int_0^2 2x^3\,dx = 1 \Rightarrow 2c\cdot\left(\frac{x^4}{4}\right)_0^2 = 1$$

$$2c\left(\frac{16}{4}\right) = 1 \Rightarrow c = \frac{1}{8}$$

$$\therefore f_{xy}(x,y) = \frac{1}{8}x(x-y), \quad 0<x<2, \ -x<y<x.$$

(ii) To find $f_x(x)$



$$f_x(x) = \int_{-\infty}^{\infty} f(x,y)\,dy = \int_{-x}^{x}\frac{x}{8}(x-y)\,dy$$

$$= \left[\frac{x}{8}\left(xy-\frac{y^2}{2}\right)\right]_{-x}^{x} = \frac{x}{8}\left[x^2-\frac{x^2}{2}+x^2-\frac{x^2}{2}\right]$$

$$= \frac{x^3}{4}, \quad 0<x<2$$

(iii) $f_{y/x}(y/x) = \frac{f(x,y)}{f_x(x)} = \frac{\frac{1}{8}x(x-y)}{x^3/4} = \frac{x-y}{2x^2}, \quad -x<y<x, \; 0<x<2$

(iv) $f_y(y) = \int_{-\infty}^{\infty} f(x,y)\,dx$

$$= \begin{cases} \int_{y}^{2} \frac{1}{8}x(x-y)\,dx & \text{If } 0 \le y \le 2 \\ \int_{-y}^{2} \frac{1}{8}x(x-y)\,dx & \text{If } -2 \le y \le 0 \end{cases}$$

$$= \begin{cases} \frac{1}{8}\left[\frac{x^3}{3} - \frac{x^2}{2}y\right]_y^2 & \text{If } 0 \le y \le 2 \\ \frac{1}{8}\left[\frac{x^3}{3} - \frac{x^2}{2}y\right]_{-y}^2 & \text{If } -2 \le y \le 0 \end{cases}$$

$$= \begin{cases} \frac{1}{8}\left[\frac{8}{3} - \frac{4}{2}y - \frac{y^3}{3} + \frac{y^3}{2}\right] & \text{If } 0 \le y \le 2 \\ \frac{1}{8}\left[\frac{8}{3} - \frac{4}{2}y + \frac{y^3}{3} + \frac{y^3}{2}\right] & \text{If } -2 \le y \le 0 \end{cases}$$

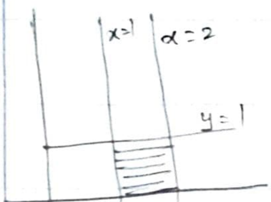
$$= \begin{cases} \frac{1}{3} - \frac{y}{4} + \frac{y^3}{48}, & 0 \le y \le 2 \\ \frac{1}{3} - \frac{y}{4} + \frac{5y^3}{48}, & -2 \le y \le 0 \end{cases}$$

4.) The joint pdf of a two dimensional r.v is given by $f(x,y) = xy^2 + \frac{x^2}{8}$, $0 \le x \le 2$ and $0 \le y \le 1$. Compute $P(X>1)$,

$P(Y<1/2)$, $P(X>1, Y<1/2)$, $P(X>1/Y<1/2)$, $P(Y<1/2 \mid X>1)$, $P(X<Y)$ and $P(X+Y \le 1)$.

Sol:

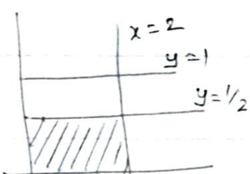
(i) $$P(X>1) = \iint\limits_{\substack{R \\ (x>1)}} f(x,y)\,dx\,dy = \int_0^1 \int_1^2 \left(xy^2 + \frac{x^2}{8}\right) dx\,dy$$



$$= \int_0^1 \left(\frac{x^2}{2}y^2 + \frac{x^3}{3\cdot 8}\right)_1^2 dy = \int_0^1 \left(2y^2 + \frac{1}{3} - \frac{y^2}{2} - \frac{1}{24}\right) dy$$

$$= \left(\frac{2y^3}{3} + \frac{1}{3}y - \frac{y^3}{6} - \frac{1}{24}y\right)_0^1 = \frac{2}{3} + \frac{1}{3} - \frac{1}{6} - \frac{1}{24} = \frac{19}{24}$$

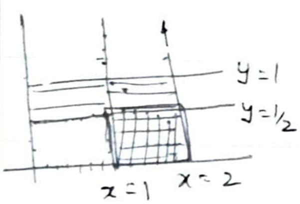
(ii) $$P(Y<1/2) = \int_0^2 \int_0^{1/2} \left(xy^2 + \frac{x^2}{8}\right) dy\,dx$$



$$= \int_0^2 \left[\frac{xy^3}{3} + \frac{x^2 y}{8}\right]_0^{1/2} dx = \int_0^2 \left(\frac{x}{24} + \frac{x^2}{16}\right) dx$$

$$= \left(\frac{x^2}{48} + \frac{x^3}{48}\right)_0^2 = \frac{1}{4}$$

(iii) $$P(X>1, Y<1/2) = \int_1^2 \int_0^{1/2} \left(xy^2 + \frac{x^2}{8}\right) dy\,dx$$



$$= \int_1^2 \left(\frac{xy^3}{3} + \frac{x^2 y}{8}\right)_0^{1/2} = \int_1^2 \left(\frac{x}{24} + \frac{x^2}{16}\right) dx$$

$$= \left(\frac{x^2}{48} + \frac{x^3}{48}\right)_1^2 = 5/24$$

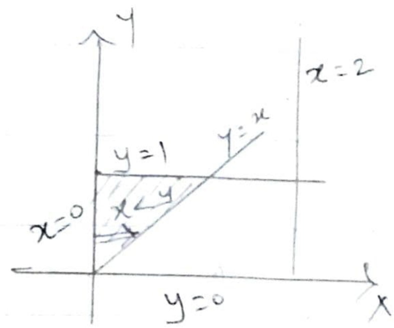
(iv) $$P(X>1/Y<1/2) = \frac{P(X>1, Y<1/2)}{P(Y<1/2)} = \frac{5/24}{1/4} = \frac{5}{6}$$

(v) $$P(Y<1/2 \mid X>1) = \frac{P(X>1, Y<1/2)}{P(X>1)} = \frac{5/24}{19/24} = \frac{5}{19}$$

(vi) $P(X<Y) = \iint_R f(x,y)\,dx\,dy$

$$= \int_0^1 \int_0^y \left(xy^2 + \frac{x^2}{8}\right) dx\,dy \quad \text{or} \quad \int_0^1 \int_x^1 \left(xy^2 + \frac{x^2}{8}\right) dy\,dx$$

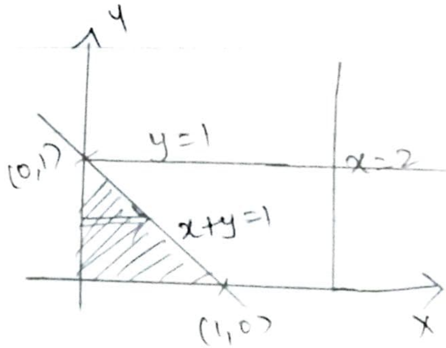
$$= \int_0^1 \left(\frac{x^2}{2} y^2 + \frac{x^3}{24}\right)_0^y dy$$



$$= \int_0^1 \left(\frac{y^4}{2} + \frac{y^3}{24}\right) dy$$

$$= \left(\frac{y^5}{10} + \frac{y^4}{96}\right)_0^1 = \frac{53}{480}$$

(vii) $P(X+Y \leq 1) = \int_0^1 \int_0^{1-y} \left(xy^2 + \frac{x^2}{8}\right) dy$



$$= \int_0^1 \left[\frac{x^2}{2} y^2 + \frac{x^3}{24}\right]_0^{1-y} dy$$

$$= \int_0^1 \left[\frac{(1-y)^2 y^2}{2} + \frac{(1-y)^3}{24}\right] dy$$

$$= \int_0^1 \left[\frac{y^2}{2}(1-2y+y^2) + \frac{1}{24}(1-3y+3y^2-y^3)\right] dy$$

$$= \left[\frac{y^3}{6} - \frac{2y^4}{8} + \frac{y^5}{10}\right]_0^1 + \frac{1}{24}\left[y - \frac{3y^2}{2} + \frac{3y^3}{3} - \frac{y^4}{4}\right]_0^1 = \frac{13}{480}$$

5) If the joint cumulative distribution function of x and y is given by $F(x,y) = \begin{cases} (1-e^{-x})(1-e^{-y}), & \text{for } x>0,\ y>0 \\ 0, & \text{elsewhere} \end{cases}$

(i) Find the marginal density function of x and y

(ii) Are X and Y independent? (iii) Find $P(1<X<2, 1<Y<2)$.

Sol:

W.K.T $f(x,y) = \dfrac{\partial^2 F}{\partial x \partial y}$

$= \dfrac{\partial}{\partial x}\left(\dfrac{\partial F}{\partial y}\right) = \dfrac{\partial}{\partial x}\dfrac{\partial}{\partial y}\left\{(1-e^{-x})(1-e^{-y})\right\}$

$= \dfrac{\partial}{\partial x}(1-e^{-x})\left[-e^{-y}(-1)\right] = e^{-y}\dfrac{\partial}{\partial x}(1-e^{-x})$

$= e^{-y}(-e^{-x})(-1) = e^{-y}.e^{-x} = e^{-(x+y)}$

$\therefore f(x,y) = \begin{cases} e^{-(x+y)}, & x>0, y>0 \\ 0, & \text{otherwise} \end{cases}$

(i) The marginal density function of X is

$f_X(x) = \int_{-\infty}^{\infty} f(x,y)\,dy = \int_0^{\infty} e^{-(x+y)}\,dy$

$= e^{-x}.\left(\dfrac{e^{-y}}{-1}\right)_0^{\infty} = e^{-x}, \quad x>0$

The marginal density function of Y is

$f_Y(x) = \int_{-\infty}^{\infty} f(x,y)\,dx = \int_0^{\infty} e^{-(x+y)}\,dx$

$= e^{-y}\left(\dfrac{e^{-x}}{-1}\right)_0^{\infty} = e^{-y}, \quad y>0$

(ii) $f_X(x)\cdot f_Y(y) = e^{-x}.e^{-y} = e^{-(x+y)} = f(x,y)$

$\therefore$ X and Y are independent

(iii) $P(1<X<2,\ 1<Y<2) = P(1<X<2) \cdot P(1<Y<2)$

[as X and Y are independent]

$$= \int_1^2 f_X(x)\,dx \cdot \int_1^2 f_Y(y)\,dy$$

$$= \int_1^2 e^{-x}\,dx \int_1^2 e^{-y}\,dy = \left[\frac{e^{-x}}{-1}\right]_1^2 \left[\frac{e^{-y}}{-1}\right]_1^2$$

$$= (e^{-2} - e^{-1})^2 = \left(\frac{1-e}{e^2}\right)^2$$

6.) The joint pdf of (X, Y) is $f(x,y) = \begin{cases} \frac{1}{4}[1+xy(x^2-y^2)], & |x|\le 1, |y| \le 1 \\ 0 & , \text{otherwise} \end{cases}$

Show that X and Y are not independent.

Sol:

The marginal pdf of X is

$$f_X(x) = \int_{-\infty}^{\infty} f(x,y)\,dy = \int_{-1}^{1} \frac{1}{4}[1+xy(x^2-y^2)]\,dy$$

$$= \frac{1}{4}\left[y + \frac{x^3 y^2}{2} - \frac{x y^4}{4}\right]_{-1}^{1} = \frac{1}{2}$$

Similarly $f_Y(y) = \frac{1}{2}$

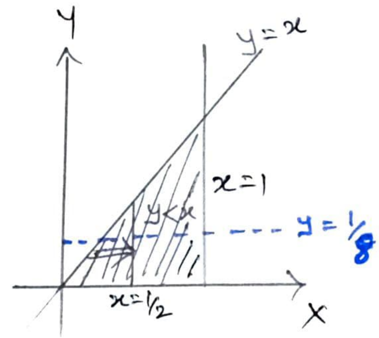
$$\therefore f_X(x) \cdot f_Y(y) = \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4} \neq f(x,y)$$

Hence X and Y are not independent.

7.) If the joint pdf of the r.v (X, Y) is $f(x,y) = \begin{cases} 8xy, & 0<x<1,\ 0<y<x \\ 0, & \text{otherwise} \end{cases}$

Find (i) $P(Y<\frac{1}{8} / X<\frac{1}{2})$ and (ii) $f(y/x)$.

Sol:



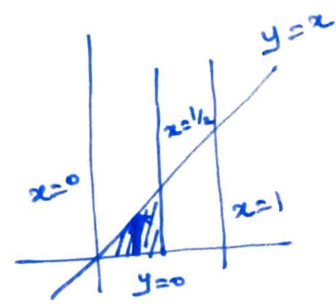
(i) $P\left(Y<\frac{1}{8} \middle| X<\frac{1}{2}\right) = \dfrac{P\left(X<\frac{1}{2}, Y<\frac{1}{8}\right)}{P(X<\frac{1}{2})}$

$$P\left(X<\tfrac{1}{2}, Y<\tfrac{1}{8}\right) = \int_0^{1/8}\int_y^{1/2} 8xy\,dx\,dy$$

$$= \int_0^{1/8} 8y\left(\frac{x^2}{2}\right)_y^{1/2}$$

$$= 4\int_0^{1/8} y\left(\frac{1}{4}-y^2\right)dy$$

$$= 4\left[\frac{1}{4}\cdot\frac{y^2}{2} - \frac{y^4}{4}\right]_0^{1/8} = \frac{31}{64^2}$$



$$P(X<\tfrac{1}{2}) = \int_0^{1/2} f_X(x)\,dx \quad \text{or} \quad \int_0^{1/2}\int_0^{x} f(x,y)\,dy\,dx$$

$$= \int_0^{1/2}\int_0^{x} 8xy\,dy\,dx$$

$$= \int_0^{1/2} 8x\left(\frac{y^2}{2}\right)_0^{x} dx = \int_0^{1/2} 4x^3\,dx$$

$$= 4\left(\frac{x^4}{4}\right)_0^{1/2} = \frac{1}{16}$$

$$\therefore P\left(Y<\tfrac{1}{8} \middle| X<\tfrac{1}{2}\right) = \frac{31/64^2}{1/16} = \frac{31}{256}$$

(ii) $f(y/x) = \dfrac{f(x,y)}{f(x)}$

Now $f(x) = \int_{-\infty}^{\infty} f(x,y)\,dy = \int_0^x 8xy\,dy$

$= 8x \cdot \left(\dfrac{y^2}{2}\right)_0^x = 4x^3 \;;\; 0 < x < 1$

$\therefore f\left(\dfrac{y}{x}\right) = \dfrac{8xy}{4x^3} = \dfrac{2y}{x^2}$

8) The Joint pdf of r.v's X and Y is given by $f(x,y) = kxy\,e^{-(x^2+y^2)}$, $x>0, y>0$. Find the value of k and also prove that X and Y are independent.

Sol:

(i) W.K.T. $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\,dx\,dy = 1$

$\Rightarrow \int_0^{\infty}\int_0^{\infty} kxy\,e^{-(x^2+y^2)}\,dx\,dy = 1$

$\Rightarrow k\int_0^{\infty} x e^{-x^2}\,dx \int_0^{\infty} y e^{-y^2}\,dy = 1$

put $x^2 = t$ | $y^2 = s$

$2x\,dx = dt$ | $2y\,dy = ds$

when $x=0 \Rightarrow t=0$
$x=\infty \Rightarrow t=\infty$
$y=0 \Rightarrow s=0$
$y=\infty \Rightarrow s=\infty$

$\Rightarrow k\int_0^{\infty} e^{-t}\dfrac{dt}{2}\int_0^{\infty} e^{-s}\dfrac{ds}{2} = 1$

$\Rightarrow \dfrac{k}{4}\left(\dfrac{e^{-t}}{-1}\right)_0^{\infty}\left(\dfrac{e^{-s}}{-1}\right)_0^{\infty} = 1$

$\Rightarrow \frac{K}{4}(0+1)(0+1)=1 \Rightarrow K=4$

$\therefore f(x,y)=4xye^{-(x^2+y^2)}, \quad x>0,\ y>0$

(ii) $f_X(x)=\int_{-\infty}^{\infty} f(x,y)\,dy$

$=\int_0^{\infty} 4xye^{-(x^2+y^2)}\,dy$

$=4xe^{-x^2}\int_0^{\infty} ye^{-y^2}\,dy$

$=4xe^{-x^2}\int_0^{\infty} \frac{e^{-s}\,ds}{2}$

$=2xe^{-x^2}\left(\frac{e^{-s}}{-1}\right)_0^{\infty}=2xe^{-x^2}, \quad x>0$

Similarly $f_Y(y)=2ye^{-y^2}, \quad y>0$

$\therefore f_X(x)\cdot f_Y(y)=2xe^{-x^2}\cdot 2ye^{-y^2}=4xye^{-(x^2+y^2)}$

$=f(x,y)$

Hence X and Y are independent.

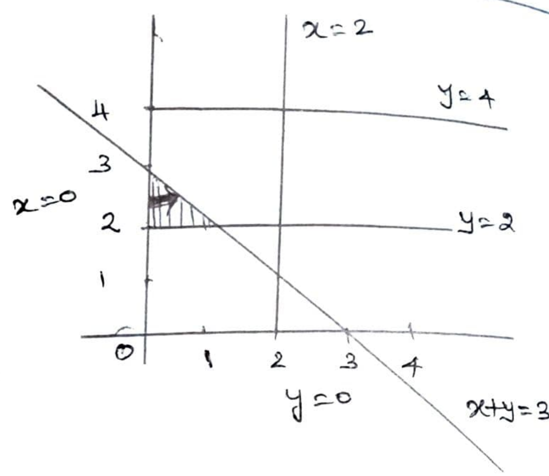
9.) If X and Y are r.v's having the joint density function

$f(x,y)=\begin{cases}\frac{1}{8}(6-x-y), & 0<x<2,\ 2<y<4\\ 0, & \text{elsewhere.}\end{cases}$ Find (i) $P(X+Y<3)$

(ii) $P(X<1, Y<3)$ (iii) $P(X<1/Y<3)$.

Sol:

$$P(X+Y<3) = \int_{y=2}^{3}\int_{x=0}^{3-y} \frac{1}{8}(6-x-y)\,dx\,dy$$



$$= \frac{1}{8}\int_2^3 \left[\frac{(6-x-y)^2}{-2}\right]_0^{3-y} dy$$

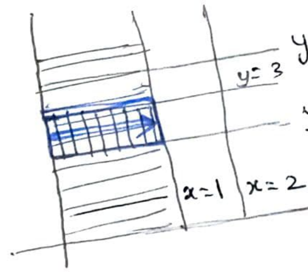
$$= \frac{-1}{8\times 2}\int_2^3 \left\{[6-3+y-y]^2-(6-y)^2\right\} dy$$

$$= -\frac{1}{16}\int_2^3 (9-36+12y-y^2)\,dy$$

$$= -\frac{1}{16}\int_2^3 (-27+12y-y^2)\,dy = -\frac{1}{16}\left[-27y+12\frac{y^2}{2}-\frac{y^3}{3}\right]_2^3$$

$$= \frac{5}{24}$$

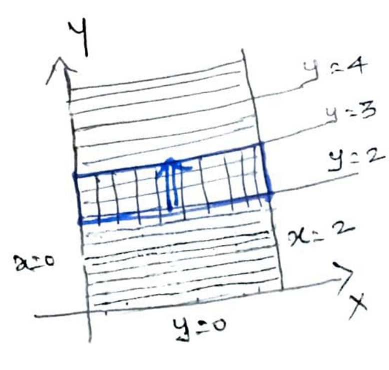
$$P(X<1, Y<3) = \int_2^3\int_0^1 \frac{1}{8}(6-x-y)\,dx\,dy$$



$$= \frac{1}{8}\int_2^3 \left[(6-y)x-\frac{x^2}{2}\right]_0^1 dy$$

$$= \frac{1}{8}\int_2^3 \left[6-y-\frac{1}{2}\right] dy = \frac{1}{8}\left[\frac{11}{2}y-\frac{y^2}{2}\right]_2^3 = 3/8$$

$$P(X<1/Y<3) = \frac{P(X<1, Y<3)}{P(Y<3)}$$



Now $P(Y<3) = \int_0^2\int_2^3 \frac{1}{8}(6-x-y)\,dy\,dx$

$$= \int_0^2 \frac{1}{8}\left[(6-x)y-\frac{y^2}{2}\right]_2^3 dx$$

$$= \frac{1}{8}\int_0^2 \left(\frac{7}{2}-x\right) dx = \frac{1}{8}\left(\frac{7}{2}x-\frac{x^2}{2}\right)_0^2 = \frac{5}{8}$$

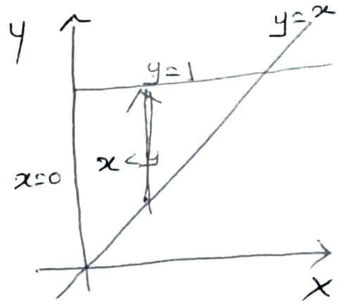
$$\therefore\ P(X<1/Y<3) = \frac{3/8}{5/8} = \frac{3}{5}$$

10) Given the joint pdf of (X, Y) as $f(x,y) = \begin{cases} 8xy, & 0<x<y<1 \\ 0, & \text{otherwise}, \end{cases}$

Find the marginal and conditional prob. functions of X and Y. Are X and Y independent? Also find $P(X+Y>1)$.

Sol:

The marginal pdf of X is

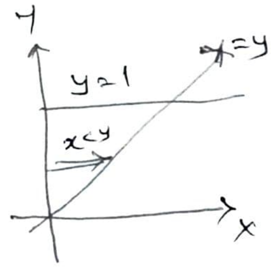


$$f_X(x) = \int_{-\infty}^{\infty} f(x,y)\,dy$$

$$= \int_{x}^{1} 8xy\,dy = 8x\left(\frac{y^2}{2}\right)_x^1$$

$$= 4x(1-x^2), \quad 0<x<1$$

The marginal pdf of Y is



$$f_Y(y) = \int_{-\infty}^{\infty} f(x,y)\,dx$$

$$= \int_{0}^{y} 8xy\,dx = 8y\left(\frac{x^2}{2}\right)_0^y$$

$$= 4y^3, \quad 0<y<1$$

The Conditional pdf of X given $Y=y$ is

$$f_{X/Y}\left(\frac{x}{y}\right) = \frac{f(x,y)}{f(y)} = \frac{8xy}{4y^3} = \frac{2x}{y^2}, \quad \begin{matrix} 0<x<y \\ 0<y<1 \end{matrix}$$

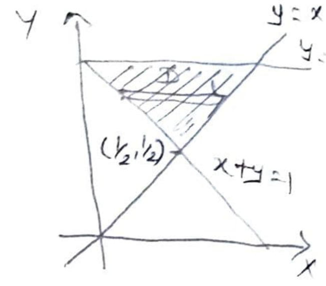
The conditional pdf of Y given $X=x$ is

$$f_{Y/X}\left(\frac{x}{y}\right) = \frac{f(x,y)}{f(x)} = \frac{8xy}{4x(1-x^2)} = \frac{2y}{1-x^2}, \quad \begin{matrix} x<y<1 \\ 0<x<1 \end{matrix}$$

Now $f_X(x) \cdot f_Y(y) = 4x(1-x^2) \cdot 4y^3 = 16xy^3(1-x^2) \neq f(x,y)$

$\therefore$ X and Y are not independent.

$$P(X+Y>1) = \iint_D f(x,y)\,dx\,dy$$



$$= \int_{1/2}^{1} \int_{1-y}^{y} 8xy\,dx\,dy$$

$$= 8\int_{1/2}^{1} y \left(\frac{x^2}{2}\right)_{1-y}^{y} dy$$

$$= 4\int_{1/2}^{1} y\left[y^2-(1-y)^2\right]dy = 4\int_{1/2}^{1} y(2y-1)\,dy$$

$$= 4\left[\frac{2y^3}{3}-\frac{y^2}{2}\right]_{1/2}^{1} = 5/6$$

11) A gun is aimed at a certain point (origin of the co-ordinate system). Because of the random factors, the actual hit point can be any point (X, Y) in a circle of radius R about the origin. Assume that the joint density of X and Y is constant in this circle given by

$$f_{XY}(x,y) = c, \quad x^2+y^2 \le R^2$$
$$= 0, \quad \text{otherwise}$$

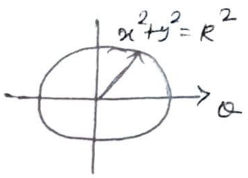
(a) Compute c and (b) show that $f_X(x) = \begin{cases} \frac{2}{\pi R}\sqrt{1-\left(\frac{x}{R}\right)^2}, & -R \le x \le R \\ 0, & \text{otherwise} \end{cases}$

Sol:

(a) By defn $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\,dx\,dy = 1$

$$\iint_{x^2+y^2 \le R} f(x,y)\,dx\,dy = 1 \Rightarrow \iint_{x^2+y^2 \le R} c\,dx\,dy = 1$$

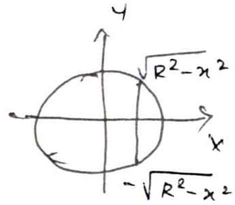
Changing over to polar co-ordinates, we have



$$\int_0^{2\pi}\int_0^{R} c\,r\,dr\,d\theta = 1 \Rightarrow c\int_0^{2\pi}\left(\frac{r^2}{2}\right)_0^R d\theta = 1$$

$$\frac{cR^2}{2}\left[\theta\right]_0^{2\pi} = 1 \Rightarrow c = \frac{1}{\pi R^2}$$

(b) Marginal density function of X



$$f_X(x) = \int_{-\infty}^{\infty} f(x,y)\,dy$$

$$= \int_{-\sqrt{R^2-x^2}}^{\sqrt{R^2-x^2}} \frac{1}{\pi R^2}\,dy$$

$$= \frac{1}{\pi R^2}\left[y\right]_{-\sqrt{R^2-x^2}}^{\sqrt{R^2-x^2}} = \frac{2\sqrt{R^2-x^2}}{\pi R^2}$$

$$= \frac{2}{\pi R}\sqrt{1-\left(\frac{x}{R}\right)^2}, \quad -R \le x \le R$$

12.) Train X arrives at a station at random in the time interval (0, T) and stops for 'a' min. Train Y arrives independently in the same interval and stops for 'b' min.

(a) Find the probability P_1 that X will arrive before Y.

(b) Find the probability P_2 that the two trains meet.

(c) Assuming that they meet, find the probability P_3

that x arrived before y.

Covariance and correlation (Introduction)

In the case of single r.v we have seen that the mean and variance are parameters of the r.v giving us information about its average behaviour. In the case of two dimensional r.v's also we would like to have a similar representation.

Covariance between two r.v's X and Y tells us about the relationship between ~~the~~ X and Y indicating how they tend to vary together.

Two variables X and Y are said to be correlated if a change in the value of one of the variables causes a change in the value of the other variable.

Though covariance is a measure of linear ~~lo~~ relationship between two variates X and Y, it does not tell us the degree of relationship or strength of the relationship. The correlation co-efficient is a measures which gives the degree of relationship between X and Y.

Covariance: Let X and Y be two r.v's defined on the same sample space. The covariance of X and Y is denoted by $Cov(X, Y)$ and is defined by

$$Cov(X, Y) = E[(X - E(X))(Y - E(Y))]$$

or $$Cov(X, Y) = E(XY) - E(X)E(Y)$$

Note: $Cov(X, X) = Var(X)$

Properties of covariance:

(1) If X and Y are independent r.v's then $Cov(X, Y) = 0$. But the converse is not true.

i.e, $Cov(X, Y) = 0$ then X and Y need not be independent.

2.) $Cov(X, Y) = Cov(Y, X)$

3) $Cov(aX, bY) = ab\,Cov(X, Y)$ where a and b are constants. This means covariance is affected by change of scale.

4.) $Cov(X + a, Y + b) = Cov(X, Y)$

This means covariance is unaffected by change of origin.

5.) $Cov(aX + b, cY + d) = ac\,Cov(X, Y)$

6) $Cov(X + Y, Z) = Cov(X, Z) + Cov(Y, Z)$

7) $Cov(X, Y + Z) = Cov(X, Y) + Cov(X, Z)$

8.) $Var(X + Y) = Var(X) + Var(Y) + 2\,Cov(X, Y)$

$Var(aX + bY) = a^2 Var(X) + b^2 Var(Y) + 2ab\,Cov(X, Y)$

9) $var(x-y) = var(x) + var(y) - 2\,cov(x,y)$

$\Rightarrow var(ax-by) = a^2 var(x) + b^2 var(y) - 2ab\,cov(x,y)$

10.) If x and y are independent, then $cov(x,y)=0$

$\therefore var(x \pm y) = var(x) + var(y)$

$\Rightarrow var(ax \pm by) = a^2 var(x) + b^2 var(y)$

Correlation: Let x and y be two r.v's. ~~defined on~~

The co-efficient of correlation between x and y is denoted by ρ_{xy} or r_{xy} and is defined by

$$\rho_{xy} = \frac{cov(x,y)}{\sigma_x \sigma_y} \quad \text{If } \sigma_x \neq 0, \sigma_y \neq 0$$

Note: (1) This is called Karl Pearson's correlation co-efficient.

2.) If x and y are independent, then $cov(x,y)=0$

$\therefore$ $\rho_{xy}=0$. Hence x and y are uncorrelated

Properties: (1) $-1 \leq \rho_{xy} \leq 1$. ie, correlation co-efficient always lies between -1 and 1.

$\Rightarrow$ If $\rho_{xy}=1$ or -1, then there is perfect positive or negative, linear relationship between x and y of the form $y = ax+b$.

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⇒ Value of ρ_{xy} is close to ± 1 indicate high degree of linear relationship between X and Y.

⇒ A Positive value of ρ_{xy} indicates Y increases as X increases, whereas a negative value of ρ_{xy} indecates Y decreases as X increases.

⇒ Value of ρ_{xy} nearer to zero indicates lack of linear relationship between X and Y.

(2) Correlation co-efficient is unaffected by change of origin and scale.

i.e, If X and Y are two r.v's and If $U = \frac{X-a}{h}$, $V = \frac{Y-b}{k}$, then $\rho_{xy} = \rho_{uv}$ where a, b, h, k are constants and $h>0$, $k>0$.

(3) $$\rho_{xy} = \frac{\sigma_x^2 + \sigma_y^2 - \sigma_{x-y}^2}{2\sigma_x \sigma_y}$$

Proof:

W.K.T $Var(X-Y) = Var(X) + Var(Y) - 2Cov(X,Y)$

or $\sigma_{x-y}^2 = \sigma_x^2 + \sigma_y^2 - 2Cov(X,Y)$

$$\therefore Cov(X,Y) = \frac{1}{2}\left[\sigma_x^2 + \sigma_y^2 - \sigma_{x-y}^2\right]$$

But $$\rho_{xy} = \frac{Cov(X,Y)}{\sigma_x \sigma_y} = \frac{\sigma_x^2 + \sigma_y^2 - \sigma_{x-y}^2}{2\sigma_x \sigma_y}$$

1) If X has mean 4 and variance 9, while Y has mean -2 and variance 5, and the two are independent, find a) $E(XY)$ b) $E(XY^2)$

Sol: ~~ans~~

Given $E(X)=4$, $\sigma_x^2=9$, $E(Y)=-2$, $\sigma_y^2=5$ and

$\Rightarrow$ X and Y are independent.

(a) $E(XY) = E(X)\cdot E(Y) = 4(-2) = -8$

(b) $E(XY^2) = E(X)\,E(Y^2)$

W.K.T $\quad Var(Y) = E(Y^2) - [E(Y)]^2$

$\Rightarrow E(Y^2) = Var(Y) + [E(Y)]^2 = 5+4 = 9$

$\therefore E(XY^2) = 4(9) = 36$

2) Let X_1 and X_2 have the joint p.m.f. $P(x_1,x_2) = \dfrac{x_1+2x_2}{18}$, $x_1 = 1,2$; $x_2 = 1,2$. Find the $cov(X_1, X_2)$.

Sol: W.K.T $\quad cov(X_1,X_2) = E(X_1X_2) - E(X_1)E(X_2)$

The joint probability dist. of X_1 and X_2 is

$X_1 \backslash X_2$	1	2	$P(X=X_1)$
1	$3/18$	$5/18$	$\frac{8}{18}$
2	$4/18$	$6/18$	$\frac{10}{18}$
$P(X=X_2)$	$\frac{7}{18}$	$\frac{11}{18}$	1

The marginal p.m.f. of X_1 is

X_1	1	2
$P(X_1)$	$\frac{8}{18}$	$\frac{10}{18}$

The marginal p.m.f. of X_2 is

X_2	1	2
$P(X_2)$	$\frac{7}{18}$	$\frac{11}{18}$

Now

$$E(X_1) = \Sigma x_1 p(x_1) = \left(1 \times \frac{8}{18}\right) + \left(2 \times \frac{10}{18}\right) = \frac{14}{9}$$

$$E(X_2) = \Sigma x_2 p(x_2) = \left(1 \times \frac{7}{18}\right) + \left(2 \times \frac{11}{18}\right) = \frac{29}{18}$$

and $E(X_1 X_2) = \Sigma\Sigma x_1 x_2 p(x_1, x_2)$

$$= \left(1 \times 1 \times \frac{3}{18}\right) + \left(1 \times 2 \times \frac{5}{18}\right) + \left(2 \times 1 \times \frac{4}{18}\right)$$

$$+ \left(2 \times 2 \times \frac{6}{18}\right) = \frac{45}{18}$$

$$\therefore \ Cov(X_1, X_2) = \frac{45}{18} - \frac{14}{9} \cdot \frac{29}{18} = -\frac{1}{162}$$

2) Let the joint probability distribution of X and Y be given by

		X		
		-1	0	1
Y	-1	1/6	1/3	1/6
	0	0	0	0
	1	1/6	0	1/6

. Find $Cov(X, Y)$.

Sol:

The probability mass functions of x and y are

x :	-1	0	1
$p(x)$:	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$

y :	-1	0	1
$p(y)$:	$2/3$	0	$1/3$

Now $E(X) = \sum x\, p(x) = \left((-1)\times\frac{1}{3}\right) + \left(0\times 1/3\right) + \left(1\times 1/3\right) = 0$

$E(Y) = \sum y\, p(y) = \left((-1)\times\frac{2}{3}\right) + (0\times 0) + \left(1\times 1/3\right) = -1/3$

$E(XY) = \sum\sum xy\, p(x,y)$

$= \left((-1)(-1)\times\frac{1}{6}\right) + \left((-1)\times 0\times 1/3\right) + \left((-1)\times 1\times\frac{1}{6}\right)$

$+ 0 + \left(1\times(-1)\times 1/6\right) + 0 + \left(1\times 1\times 1/6\right) = 0$

$\therefore Cov(X,Y) = E(XY) - E(X)E(Y) = 0$

Note:

$p(-1,-1) = \frac{1}{6}$ and $p(X=-1) = 1/3$ and $p(Y=-1) = 2/3$

$\therefore p(-1,-1) \neq p(X=-1)\,P(Y=-1)$

$\Rightarrow$ X and Y are not independent.

3.) If X and Y are independent r.v with mean 2 and 3 and variance 1 and 2 respectively, find the mean and variance of $Z = 2X - 5Y$.

Sol: Given $E(X)=2$, $E(Y)=3$, $\sigma_x^2=1$, $\sigma_y^2=2$

$\bar{z}=E(Z)=E(2X-5Y)=2E(X)-5E(Y)$

$=2(2)-5(3)=-11$

Variance of Z: $Var(Z)=E(Z^2)-[E(Z)]^2$

Now $E(Z^2)=E[(2X-5Y)^2]$

$=E[4X^2-20XY+25Y^2]$

$=4E(X^2)-20E(XY)+25E(Y^2)$

W.K.T $Var(X)=E(X^2)-[E(X)]^2$

$\Rightarrow E(X^2)=Var(X)+[E(X)]^2=1+4=5$

Similarly $E(Y^2)=Var(Y)+[E(Y)]^2=2+9=11$

$\therefore$ ~~Var~~ $E(Z^2)=4(5)-20E(X)E(Y)+25(11)$ $\because$ X & Y are independent

$=20-20(2)(3)+275=175$

$\Rightarrow Var(Z)=175-(-11)^2=54.$

If $Y=-2X+3$, find $Cov(X,Y)$.

Sol:

$Cov(X,Y)=Cov(X,-2X+3)=-2Cov(X,X)=-2\sigma_x^2$

[W.K.T ~~Since~~ $Cov(aX+b, cY+d)=ac\,Cov(X,Y)$

$a=1$, $c=-2$ $\Rightarrow Cov(X,-2X+3)=-2Cov(X,X)$]

5.) If x and y are independent R.V. with variance 2 and 3, find variance of $3x+4y$.

Sol:

Given $\sigma_x^2 = 2$, $\sigma_y^2 = 3$

$\therefore Var(3x+4y) = 9 Var(x) + 16 Var(y)$

$= 9(2) + 16(3) = 66$

6.) The correlation co-efficient of two r.v x and y is $-\frac{1}{4}$ while their variances are 3 and 5. Find $Cov(X,Y)$.

Sol:

Given $\rho_{xy} = -\frac{1}{4}$, $\sigma_x^2 = 3$ and $\sigma_y^2 = 5$

W.K.T

$$\rho_{xy} = \frac{Cov(X,Y)}{\sigma_x \sigma_y} \Rightarrow -\frac{1}{4} = \frac{Cov(X,Y)}{\sqrt{3}\sqrt{5}}$$

$$\Rightarrow Cov(X,Y) = \frac{-\sqrt{3}\sqrt{5}}{4} = -0.968$$

7.) Two r.v are related by $Y = 4X+9$. Find ρ_{xy}.

Sol:

W.K.T. $\rho_{xy} = \frac{Cov(X,Y)}{\sigma_x \sigma_y}$ and Given $Y = 4X+9$

Now $Cov(X,Y) = Cov(X, 4X+9) = 4Cov(X,X)$ by property

$\Rightarrow Cov(X,Y) = 4\sigma_x^2$

and $Var(Y) = Var(4X+9) = 16 Var(X) = 16\sigma_x^2$

$$\therefore \rho_{xy} = \frac{4\sigma_x^2}{\sigma_x \cdot 4\sigma_x} = 1$$

$\Rightarrow$ There is a perfect positive correlation between x & y.

Let X be a r.v with mean 3 and variance of X is 2 and $Y = -6X + 22$. Find the mean of Y and correlation co-efficient between X and Y.

Sol:

Given $E(X) = 3$, $\sigma_x^2 = 2$ and $Y = -6X + 22$

Mean of $Y = E(Y) = E(-6X + 22)$

$= -6E(X) + 22 = -6(3) + 22 = 4$

$$r(X,Y) = \frac{Cov(X,Y)}{\sigma_x \sigma_y}$$

Now $Cov(X,Y) = Cov(X, -6X+22) = -6 Cov(X,X)$ by property

$= -6 Var(X) = -6(2) = -12$

and $\sigma_y^2 = Var(Y) = var(-6X + 22) = 36 Var(X) = 72$

$$\therefore r(X,Y) = \frac{-12}{\sqrt{2}\sqrt{72}} = -1$$

$\Rightarrow$ There is a perfect negative linear correlation between X and Y.

If the independent r.v's X and Y have variances 36 and 16 respectively, find the correlation co-efficient between $X+Y$ and $X-Y$

Sol:

Given X and Y are independent r.v.

$\therefore Cov(X,Y)=0$

Given $\sigma_X^2=36$, $\sigma_Y^2=16$

Let $U=X+Y$ and $V=X-Y$

$$\therefore \rho_{UV}=\frac{Cov(U,V)}{\sigma_U \sigma_V}$$

$$Cov(U,V)=Cov(X+Y,\ X-Y)$$

$=Cov(X+Y,\ X)-Cov(X+Y,\ Y)$ [$\because$ by property]

$=Cov(X,X)+Cov(Y,X)-Cov(X,Y)-Cov(Y,Y)$

$=\sigma_X^2-\sigma_Y^2$ $\because$ by property $Cov(X,Y)=Cov(Y,X)$

$=36-16=20$

$\sigma_U^2=Var\,U=Var(X+Y)$

$=Var(X)+Var(Y)$ $\because$ X and Y are independent

$=36+16=52$

Similarly $\sigma_V^2=Var\,V=36+16=52$

$$\therefore \rho_{UV}=\frac{20}{\sqrt{52}\sqrt{52}}=0.3$$

If X and Y are discrete r.v's with $Var(X) = Var(Y) = \sigma^2$, $Cov(X,Y) = \frac{\sigma^2}{2}$, find (i) ρ_{XY} (ii) $Var(2X-3Y)$ (iii) $Var(5X)$ (iv) If X and Y are independent find $Var(3X+9Y)$.

Sol:

Given $Var(X) = Var(Y) = \sigma^2$, $Cov(X,Y) = \sigma^2/2$

(i) $\rho_{XY} = \frac{Cov(X,Y)}{\sigma_X \sigma_Y} = \frac{\sigma^2}{2.\sigma\sigma} = \frac{1}{2}$

(ii) $Var(2X-3Y)$

W.K.T $var(aX-bY) = a^2 var(X) + b^2 var(Y) - 2ab Cov(X,Y)$

$\therefore Var(2X-3Y) = 4 var(X) + 9 var(Y) - 12 Cov(X,Y)$

$= 4\sigma^2 + 9\sigma^2 - 12.\frac{\sigma^2}{2} = 13\sigma^2 - 6\sigma^2 = 7\sigma^2$

(iii) $Var(5X) = 25 var(X) = 25\sigma^2$

(iv) $Var(3X+9Y)$

If X and Y are independent then

$var(aX+bY) = a^2 var(X) + b^2 var(Y)$

$\therefore Var(3X+9Y) = 9 Var(X) + 81 var(Y) = 9\sigma^2 + 81\sigma^2 = 90\sigma^2$

For r.v's X and Y having $\mu_X = 2$, $\mu_Y = 3$, $\sigma_X^2 = 9$, $\sigma_Y^2 = 16$ and $\rho_{XY} = 2/3$ find (i) $Cov(X,Y)$ (ii) $E(X^2)$ (iii) $E(Y^2)$.

Sol: Given $\mu_x = 2$, $\mu_y = 3$, $\sigma_x^2 = 9$, $\sigma_y^2 = 16$ and $\rho_{xy} = 2/3$

(i) $Cov(X, Y) = \rho_{xy}\sigma_x\sigma_y$ $\qquad \because \rho_{xy} = \frac{Cov(X,Y)}{\sigma_x\sigma_y}$

$= \frac{2}{3}\cdot 3\cdot 4 = 8$

(ii) $E(X^2) = Var(X) + [E(X)]^2$ $\qquad \because Var(X) = E(X^2) - [E(X)]^2$

$= 9 + 4 = 13$

(iii) $E(Y^2) = Var(Y) + [E(Y)]^2 = 16 + 9 = 25$

If $z = ax + by$ and γ is the correlation co-efficient between X and Y, show that $\sigma_z^2 = a^2\sigma_x^2 + b^2\sigma_y^2 + 2ab\gamma\sigma_x\sigma_y$.

proof:

$Var(z) = Var(ax+by)$

$= E[ax+by - E(ax+by)]^2$ $\qquad \because Var(X) = E(X - E(X))^2$

$= E[ax+by - aE(X) - bE(Y)]^2$

$= E[a(X - E(X)) + b(Y - E(Y))]^2$

$= E[a^2(X - E(X))^2 + b^2(Y - E(Y))^2$

$+ 2ab(X - E(X))(Y - E(Y))]$

$= a^2 E[(X - E(X))^2] + b^2 E[(Y - E(Y))^2]$

$+ 2ab E[(X - E(X))(Y - E(Y))]$

$= a^2 Var(X) + b^2 Var(Y) + 2ab Cov(X, Y)$

$$\Rightarrow \sigma_Z^2 = a^2\sigma_X^2 + b^2\sigma_Y^2 + 2ab\gamma\sigma_x\sigma_y \qquad \because \rho_{xy} = \frac{Cov(X,Y)}{\sigma_x\sigma_y}$$

A r.v X has mean value 3 and Variance 2. A new r.v is defined as $Y = 3X - 11$. Check whether

a) X and Y are orthogonal to each other.

b) X and Y are Correlated to each other.

Sol:

Given $E(X) = 3$ $\sigma_X^2 = 2$, $Y = 3X - 11$

(a) If X and Y are orthogonal then $E(XY) = 0$

W.K.T $Cov(X,Y) = E(XY) - E(X)E(Y)$

Now $Cov(X,Y) = Cov(X, 3X-11) = 3Cov(X,X) = 3\sigma_X^2$

$\because Cov(aX+b, cY+d) = ac\,Cov(X,Y)$

$\therefore Cov(X,Y) = 3 \cdot 2 = 6$

$E(Y) = E(3X-11) = 3E(X) - 11 = 3(3) - 11 = -2$

$\Rightarrow E(XY) = Cov(X,Y) + E(X)E(Y)$

$= 6 + (3)(-2) = 0$

$\therefore$ X and Y are orthogonal.

(b) $r_{XY} = \dfrac{Cov(X,Y)}{\sigma_X\sigma_Y}$

Now $\sigma_Y^2 = Var(Y) = Var(3X-11) = 9Var(X) = 9(2) = 18$

$\therefore r_{xy} = \frac{6}{\sqrt{2}\sqrt{9}} = \frac{6}{\sqrt{18}}$

Hence x and y are correlated.

14) Let x and y be r.v's having the following joint probability distribution

$y \backslash x$	0	1	2	$P_y(y)$
0	$3/28$	$9/28$	$3/28$	$\frac{15}{28}$
1	$3/14$	$3/14$	0	$3/7$
2	$1/28$	0	0	$1/28$
$P_x(x)$	$5/14$	$15/28$	$3/28$	1

Find the Cov(X, Y).

Sol: $Cov(X,Y) = E(XY) - E(X)E(Y)$

Now $E(X) = \sum x p(x) =$ Marginal p.m.f. of X is

X	0	1	2
$P_x(x)$	$5/14$	$15/28$	$3/28$

Marginal p.m.f. of Y is

Y	0	1	2
$P_y(y)$	$15/28$	$3/7$	$1/28$

Now $E(X) = \sum x\, p(x) = 0 + \left(1 \times \frac{15}{28}\right) + \left(2 \times \frac{3}{28}\right) = 3/4$

$E(Y) = \Sigma y\, p(y) = 0 + \left(1 \cdot \times \frac{3}{7}\right) + \left(2 \times \frac{1}{28}\right) = \frac{1}{2}$

$E(XY) = \Sigma\Sigma xy\, p(x, y)$

$= 0+0+0+\left(1\times 0\times \frac{9}{28}\right)+\left(1\times 1\times \frac{3}{14}\right)+(1\times 2\times 0)$

$+\left(2\times 0\times \frac{3}{28}\right)+(2\times 1\times 0)+(2\times 2\times 0)$

$= \frac{3}{14}$

$\therefore \text{Cov}(X, Y) = E(XY) - E(X)E(Y) = \frac{3}{14} - \frac{3}{4}\times\frac{1}{2} = -\frac{9}{56}$

Let (X, Y) be the two dimensional r.v. described by the joint p.d.f $f(x, y) = \begin{cases} 8xy, & 0 \le x \le 1, 0 \le y \le x \\ 0, & \text{otherwise} \end{cases}$

Find the $\text{Cov}(X, Y)$.

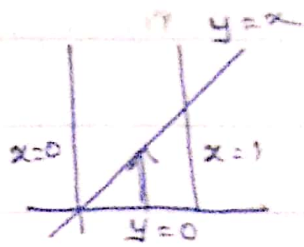
Sol:

$\text{Cov}(X, Y) = E(XY) - E(X)E(Y)$ where $E(X) = \int_{-\infty}^{\infty} x f(x)\,dx$

$E(Y) = \int_{-\infty}^{\infty} y f(y)\,dy$, $E(XY) = \int_{-\infty}^{+\infty}\int_{-\infty}^{\infty} xy\, f(x, y)\,dx\,dy$

First we compute the marginal density functions of X and Y.

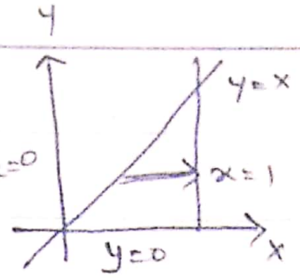
$f_X(x) = \int_0^x f(x, y)\,dy$



$= \int_0^x 8xy\,dy = 8x\left(\frac{y^2}{2}\right)_0^x$

$= 4x^3$

$$f_Y(y) = \int_y^1 f(x,y)\,dx$$



$$= \int_y^1 8xy\,dx = 8y\left(\frac{x^2}{2}\right)_y^1$$

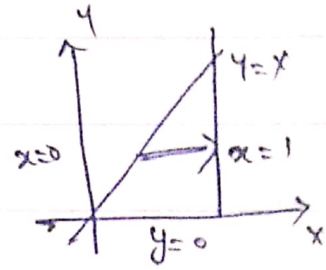
$$= 4y(1-y^2)$$

$$E(X) = \int_0^1 x f(x)\,dx = \int_0^1 4x^4\,dx = 4\left(\frac{x^5}{5}\right)_0^1 = \frac{4}{5}$$

$$E(Y) = \int_0^1 y f(y)\,dy = 4\int_0^1 y^2(1-y^2)\,dy.$$

$$= 4\left[\frac{y^3}{3} - \frac{y^5}{5}\right]_0^1 = \frac{8}{15}$$

$$E(XY) = \int_0^1 \int_y^1 xy f(x,y)\,dx\,dy$$



$$= \int_0^1 \int_y^1 8x^2y^2\,dx\,dy = 8\int_0^1 y^2\left(\frac{x^3}{3}\right)_y^1 dy$$

$$= \frac{8}{3}\int_0^1 y^2(1-y^3)\,dy = \frac{8}{3}\left[\frac{y^3}{3} - \frac{y^6}{6}\right]_0^1 = \frac{4}{9}$$

$$\therefore \operatorname{Cov}(X,Y) = E(XY) - E(X)E(Y) = \frac{4}{9} - \left(\frac{4}{5} \times \frac{8}{15}\right) = \frac{4}{225}$$

16) Two r.v's X and Y have joint density function

$$f(x,y) = \begin{cases} 2-x-y, & 0<x<1,\ 0<y<1 \\ 0, & \text{otherwise} \end{cases}$$

Find Cov(X,Y) and Correlation Co-efficient of X and Y.

Sol: (i) $Cov(X, Y) = E(XY) - E(X)E(Y)$

To find the mean of X and Y, we shall find their marginal distributions.

$$f_X(x) = \int_{-\infty}^{\infty} f(x,y)\,dy = \int_0^1 (2-x-y)\,dy$$

$$= \left[(2-x)y - \frac{y^2}{2}\right]_0^1 = \cancel{2x}\ 2 - x - 1/2$$

$$= \frac{3}{2} - x, \quad 0 < x < 1$$

$$f_Y(y) = \int_0^1 (2-x-y)\,dx$$

$$= \left[(2-y)x - \frac{x^2}{2}\right]_0^1 = 2 - y - \frac{1}{2}$$

$$= \frac{3}{2} - y, \quad 0 < y < 1$$

$$\therefore E(X) = \int_0^1 x f_X(x)\,dx = \int_0^1 x\left(\frac{3}{2} - x\right)dx$$

$$= \left[\frac{3}{2}\cdot\frac{x^2}{2} - \frac{x^3}{3}\right]_0^1 = \frac{5}{12}$$

$$E(Y) = \int_0^1 y f_Y(y)\,dy = \int_0^1 y(3/2 - y)\,dy$$

$$= \left(\frac{3}{2}\frac{y^2}{2} - \frac{y^3}{3}\right)_0^1 = \frac{5}{12}$$

$$E(XY) = \int_0^1\int_0^1 xy\, f(x,y)\,dx\,dy$$

$$= \int_0^1\int_0^1 xy(2-x-y)\,dx\,dy$$

$$= \int_0^1 y\left[(2-y)\frac{x^2}{2} - \frac{x^3}{3}\right]_0^1 dy$$

$$= \int_0^1 y\left[(2-y)\cdot\frac{1}{2} - \frac{1}{3}\right] dy = \int_0^1 \left(\frac{2}{3}y - \frac{y^2}{2}\right) dy$$

$$= \left[\frac{2}{3}\cdot\frac{y^2}{2} - \frac{y^3}{6}\right]_0^1 = \frac{1}{3} - \frac{1}{6} = \frac{1}{6}$$

$$Cov(X,Y) = E(XY) - E(X)E(Y) = \frac{1}{6} - \frac{5}{12}\cdot\frac{5}{12} = \frac{-1}{144}$$

(ii) Correlation co-efficient

$$r_{XY} = \frac{Cov(X,Y)}{\sigma_X \sigma_Y}$$

$$\sigma_X^2 = var(X) = E(X^2) - (E(X))^2$$

where $E(X^2) = \int_0^1 x^2 f_X(x)\,dx = \int_0^1 x^2\left(\frac{3-2x}{2}\right) dx$

$$= \frac{1}{2}\int_0^1 (3x^2 - 2x^3)\,dx = \frac{1}{2}\left[3\cdot\frac{x^3}{3} - \frac{2x^4}{4}\right]_0^1$$

$$= \frac{1}{2}\left(1 - \frac{1}{2}\right) = \frac{1}{4}$$

Similarly

$$E(Y^2) = \int_0^1 y^2 f_Y(y)\,dy = \int_0^1 y^2\left(\frac{3-2y}{2}\right) dy = \frac{1}{4}$$

$$\therefore var(X) = \frac{1}{4} - \left(\frac{5}{12}\right)^2 = \frac{11}{144} \Rightarrow \sigma_X = \sqrt{\frac{11}{144}} = \frac{\sqrt{11}}{12}$$

|||ly $var(Y) = \frac{\sqrt{11}}{12}$

$$\Rightarrow r_{XY} = \frac{-1/144}{\frac{\sqrt{11}}{12}\cdot\frac{\sqrt{11}}{12}} = \frac{-1}{11}$$

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Let X and Y be r.v's with joint mass function given below find r_{XY}.

Y \ X	1	3
−3	0.1	0.3
2	0.2	0.1
4	0.2	0.1

Sol: $r_{XY} = \frac{Cov(X,Y)}{\sigma_X \sigma_Y}$ where

$Cov(X,Y) = E(XY) - E(X)E(Y)$, $\sigma_X^2 = E(X^2) - [E(X)]^2$

and $\sigma_Y^2 = E(Y^2) - [E(Y)]^2$

First we find the marginal distribution of X and Y

Y \ X	1	3	$P_Y(y)$
−3	0.1	0.3	0.4
2	0.2	0.1	0.3
4	0.2	0.1	0.3
$P_X(x)$	0.5	0.5	1

x	1	3
$P_X(x)$	0.5	0.5

y	−3	2	4
$P_Y(y)$	0.4	0.3	0.3

$E(X) = (1\times0.5) + (3\times0.5) = 2$

$E(Y) = (-3\times0.4) + (2\times0.3) + (4\times0.3) = 0.6$

$E(X^2) = (1\times0.5) + (9\times5) = 5$

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$$E(y^2) = (9\times0.4) + (4\times0.3) + (16\times0.3) = 9.6$$

$$E(XY) = (1\times(-3)\times(0.1)) + (1\times2\times0.2) + (1\times4\times0.2) + (3\times(-3)\times(0.3))$$
$$+ (3\times2\times0.1) + (3\times4\times0.1) = 0$$

$$Cov(X,Y) = 0 - 2\times0.6 = -1.2$$

Now $\sigma_X^2 = 5-4 = 1$, $\sigma_Y^2 = 9.6 - 0.36 = 9.24$

$$\therefore r_{XY} = \frac{-1.2}{3.04} = -0.39$$

The joint p.d.f. of x and y is given by

$$f(x,y) = \begin{cases} x+y, & 0\le x\le 1,\ 0\le y\le 1 \\ 0, & \text{otherwise} \end{cases}$$ obtain the correlation

Co-efficient between X and Y.

Sol:
$$r_{XY} = \frac{Cov(X,Y)}{\sigma_X \sigma_Y}$$

$$f(x) = \int_0^1 f(x,y)\,dy = \int_0^1 (x+y)\,dy = \left(xy + \frac{y^2}{2}\right)_0^1 = x + \frac{1}{2}$$

Similarly $f(y) = y + \frac{1}{2}$

$$E(X) = \int_0^1 x f(x)\,dx = \int_0^1 x\left(x+\tfrac{1}{2}\right)dx = \left(\frac{x^3}{3} + \frac{x^2}{4}\right)_0^1 = \frac{7}{12}$$

Similarly $E(Y) = \frac{7}{12}$

$$E(XY) = \int_0^1\int_0^1 xy\, f(x,y)\,dx\,dy = \int_0^1\int_0^1 xy(x+y)\,dx\,dy$$
$$= \int_0^1 y\left(\frac{x^3}{3} + \frac{x^2}{2}y\right)_0^1 dy = \int_0^1 y\left(\frac{1}{3} + \frac{y}{2}\right)dy$$

$$E(XY) = \int \left[\frac{y^2}{6} + \frac{y^3}{6} \right]_0^1 = 1/3$$

$$E(X^2) = \int_0^1 x^2 f(x)\,dx = \int_0^1 x^2\left(x+\tfrac{1}{2}\right)dx$$

$$= \left[\frac{x^4}{4} + \frac{x^3}{6}\right]_0^1 = \frac{5}{12}$$

$|||^{ly}$ $E(Y^2) = \frac{5}{12}$

$\therefore Cov(X,Y) = E(XY) - E(X)E(Y) = \frac{1}{3} - \frac{7}{12}\cdot\frac{7}{12} = \frac{-1}{144}$

$$\sigma_x^2 = \frac{5}{12} - \frac{49}{144} = \frac{11}{144}, \quad |||^{ly}\ \sigma_y^2 = \frac{11}{144}$$

$$\therefore r_{xy} = \frac{-1/144}{\sqrt{\frac{11}{144}}\sqrt{\frac{11}{144}}} = \frac{-1}{11}$$

19 The following table gives the joint prob. distribution of two r.v's X and Y. Find E(X), E(Y) and E(XY) verify whether X and Y are correlated.

Y\X	0	1	2	3
2	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$
3	$\frac{1}{16}$	$\frac{1}{8}$	0	$\frac{1}{16}$
4	$\frac{1}{16}$	0	$\frac{1}{8}$	$\frac{1}{16}$

Sol:

$y \backslash x$	0	1	2	3	$P_y(y)$
2	$1/8$	$1/8$	$1/8$	$1/8$	$1/2$
3	$\frac{1}{16}$	$\frac{1}{8}$	0	$\frac{1}{16}$	$\frac{1}{4}$
4	$\frac{1}{16}$	0	$\frac{1}{8}$	$\frac{1}{16}$	$1/4$
$P_x(x)$	$1/4$	$1/4$	$1/4$	$1/4$	1

x	0	1	2	3
$P_x(x)$	$1/4$	$1/4$	$1/4$	$1/4$

y	2	3	4
$P_y(y)$	$1/2$	$1/4$	$1/4$

Now $E(x) = \Sigma x P_x(x) = (0 \times \frac{1}{4}) + (1 \times \frac{1}{4}) + (2 \times \frac{1}{4}) + (3 \times 1/4) = 3/2$

$E(y) = \Sigma y P_y(y) = (2 \times \frac{1}{2}) + (3 \times \frac{1}{4}) + (4 \times \frac{1}{4}) = \frac{11}{4}$

$E(xy) = \Sigma\Sigma xy\, p(x,y)$

$= (0 \times 2 \times \frac{1}{8}) + (0 \times 3 \times \frac{1}{16}) + (0 \times 4 \times \frac{1}{16}) + (1 \times 2 \times \frac{1}{8}) + (1 \times 3 \times \frac{1}{8})$

$+ (1 \times 4 \times 0) + (2 \times 2 \times 1/8) + (2 \times 3 \times 0) + (2 \times 4 \times \frac{1}{8})$

$+ (3 \times 2 \times \frac{1}{8}) + (3 \times 3 \times \frac{1}{16}) + (3 \times 4 \times \frac{1}{16})$

$= \frac{2}{8} + \frac{3}{8} + \frac{4}{8} + \frac{8}{8} + \frac{6}{8} + \frac{9}{16} + \frac{12}{16} = \frac{67}{16}$

$\therefore Cov(x,y) = E(xy) - E(x)E(y) = \frac{67}{16} - \frac{3}{2} \cdot \frac{11}{4} = \frac{1}{16}$

$\Rightarrow r_{xy} = \frac{Cov(x,y)}{\sigma_x \sigma_y} = \frac{\frac{1}{16}}{\sigma_x \sigma_y} \neq 0 \because \sigma_x > 0, \sigma > 0$

$\therefore$ x and y are Correlated.

20.) Let X and Y be jointly distri. r.v's with correlation co-efficient $r_{XY} = \frac{1}{2}$, $\sigma_X = 2$, $\sigma_Y = 3$. Find $Var(2X - 4Y + 3)$.

Sol:

W.K.T. $Var(aX+bY) = a^2 var(X) + b^2 var(Y) + 2ab\,Cov(X,Y)$

$\Rightarrow Var(2X - 4Y + 3) = 4\,Var(X) + 16\,Var(Y) - 16\,Cov(X,Y)$

Now $\rho_{XY} = \frac{Cov(X,Y)}{\sigma_X \sigma_Y} \Rightarrow Cov(X,Y) = \sigma_X \sigma_Y \rho_{XY}$

$= 2 \times 3 \times \frac{1}{2} = 3$

$\therefore Var(2X - 4Y + 3) = 4(4) + 16(9) - 16(3) = 112$

21.) If X and Y are independent r.v. with density fn's $f(x) = \frac{8}{x^3}$, $x > 2$ and $f(y) = 2y$, $0 < y < 1$ respectively and $Z = XY$ find $E(Z)$.

Sol:

$E(Z) = E(XY) = E(X) \cdot E(Y)$ [$\because X$ and Y are independent]

Now $E(X) = \int_2^\infty x \frac{8}{x^3}\, dx = 8 \left(\frac{-1}{x} \right)_2^\infty = 4$ and

$E(Y) = \int_0^1 y \cdot 2y\, dy = \left(\frac{2y^3}{3} \right)_0^1 = \frac{2}{3}$

$\therefore E(Z) = 4 \times \frac{2}{3} = 8/3$

Conditional Mean and Conditional Covariance

Discrete:

$$E(X/Y=y_j) = \sum_{i=1}^{n} x_i \, P(X=x_i/Y=y_j)$$

$$V(X/Y=y_j) = E[X^2/Y=y_j] - [E(X/Y=y_j)]^2$$

Continuous:

$$E(X/Y) = \int_{-\infty}^{\infty} x f\left(\frac{x}{y}\right) dx$$

$$V(X/Y) = E(X^2/Y) - [E(X/Y)]^2$$

— x —

Transformation of r.v

If (X, Y) is a two dimensional r.v with joint pdf $f_{XY}(x,y)$ and If $Z = g(X,Y)$ and $W = h(X,Y)$ are two other R.v's, then the joint pdf of (Z, W) is given by

$$f_{ZW}(z,w) = |J| f_{XY}(x,y) \quad \text{where } J = \frac{\partial(x,y)}{\partial(z,w)}$$

is called the Jacobian of the transformation and is given by

$$J = \begin{vmatrix} \frac{\partial x}{\partial z} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial z} & \frac{\partial y}{\partial w} \end{vmatrix}$$

Standard transformations of this types are

(i) $Z = X+Y$ (ii) $Z = XY$ (iii) $Z = \frac{X}{Y}$ (iv) $Z = \sqrt{X^2+Y^2}$

Note:

(1) Many problems of type $Z = g(X,Y)$ can be solved by introducing an auxiliary variable $W = h(X,Y)$ and obtain the joint pdf of (Z, W).

(2) The range space of (Z, W) is obtained from the range space of (X, Y).

(3) The marginal density function of Z is

$$f_Z(z) = \int_{-\infty}^{\infty} f_{ZW}(z,w)\,dw$$

Remark: (i) $W = Y$ If $= X+Y$, $= \frac{X}{Y}$, $= X - Y$

(ii) $W = X + Y$ If $= \frac{X}{X+Y}$

(iii) $W = \tan^{-1}\left(\frac{Y}{X}\right)$ If $= \sqrt{X^2+Y^2}$.

1.) The joint pdf of a two dimensional r.v (X, Y) is given by $f(x, y) = x + y$, $0 \leq x, y \leq 1$. Find the pdf of $U = XY$.

Sol:

Given $U = XY$.

Assume $V = Y \Rightarrow u = xy$, $v = y$

$\Rightarrow x = \frac{u}{y}$, $v = y$

ie, $x = \frac{u}{v}$, $y = v$

The joint p.d.f of (U, V) is given by

$$f(u, v) = |J| f(x, y)$$

where J is the Jacobian and $J = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$

Now

$$\Rightarrow J = \begin{vmatrix} \frac{1}{v} & \frac{-u}{v^2} \\ 0 & 1 \end{vmatrix} = \frac{1}{v}$$

$$\therefore f(u, v) = \frac{1}{|v|}\left(\frac{u}{v} + v\right)$$

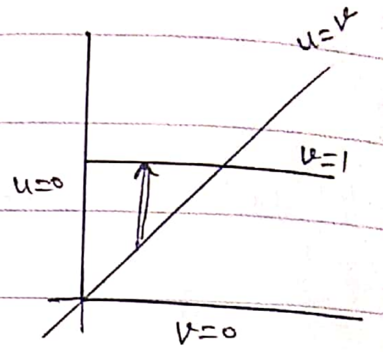
Range space: Given limits for x, y:

$$0 \leq x \leq 1, \quad 0 \leq y \leq 1$$

$\Rightarrow 0 \leq \frac{u}{v} \leq 1, \quad 0 \leq v \leq 1$

ie, $0 \leq u \leq v, \quad 0 \leq v \leq 1$

The p.d.f of U is



$$f_U(u) = \int_{-\infty}^{\infty} f(u,v)\,dv$$

$$= \int_u^1 \frac{1}{v}\left(\frac{u}{v} + v\right) dv$$

$$= \int_u^1 \left(\frac{u}{v^2} + 1\right) dv$$

$$= \left[\frac{-u}{v} + v\right]_u^1 = 2(1-u), \quad 0 \leq u \leq 1$$

2) If X and Y are independent r.v's with $f_X(x) = e^{-x} u(x)$ and $f_Y(y) = 3e^{-3y} u(y)$, find $f_Z(z)$ if $Z = \frac{X}{Y}$.

Sol:

Given the pdf of X is $f_X(x) = e^{-x} u(x)$

Here $u(x)$ is the unit step function defined by

$$u(x) = \begin{cases} 1, & x \geq 0 \\ 0, & x < 0 \end{cases}$$

$$\therefore f_X(x) = e^{-x}, \quad x \geq 0$$

Similarly $f_Y(y) = 3e^{-3y}, \quad y \geq 0$

Since X and Y are independent r.v's then joint pdf $f_{XY}(x,y) = f_X(x) \cdot f_Y(y)$

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$= 3e^{-(x+3y)}$, $x \geq 0,\ y \geq 0$

Given $z = \frac{x}{y}$. Assume ~~z~~ $w = y$

find x, y interms of z, w

$$\Rightarrow z = \frac{x}{w} \Rightarrow x = zw$$

ie, $x = zw$, $y = w$

Find the Jacobian

$$J = \frac{\partial(x, y)}{\partial(z, w)} = \begin{vmatrix} w & z \\ 0 & 1 \end{vmatrix} = w$$

Find the Joint pdf of z, w

$$f(z, w) = |J|\, f(x, y)$$

$$= w \cdot 3e^{-(zw + 3w)} = 3we^{-w(z+3)}$$

<u>Range</u> Space: Given limits for x, y: $x \geq 0,\ y \geq 0$

$\Rightarrow zw \geq 0$, $w \geq 0$

$\Rightarrow z \geq 0$ as $w \geq 0$

The marginal density function of z

$$f_z(z) = \int_0^\infty 3we^{-w(z+3)}\,dw$$

$$= 3\left[\frac{we^{-w(z+3)}}{-(z+3)} - \frac{1 \cdot e^{-w(z+3)}}{(z+3)^2}\right]_0^\infty$$

$$= \frac{3}{(z+3)^2}, \quad z \geq 0$$

3.) If X and Y each follow an exponential distributed with parameter 1 and are independent, find the pdf of $U = X - Y$.

Sol: Given X and Y are exponential distributed.

ie) $f_X(x) = e^{-x}, \; x > 0$ | $\because f(x) = \lambda e^{-\lambda x}$

and $f_Y(y) = e^{-y}, \; y > 0$ | $\lambda = 1$

$\Rightarrow f(x) = e^{-x}$

The joint pdf of (X, Y) is given by

$$f(x, y) = e^{-(x+y)}, \; x > 0, \; y > 0$$

[Since X and Y are independent $f(x,y) = f(x) f(y)$]

Given $U = X - Y$

Assume $V = Y$

$\Rightarrow U = X - V$

ie, $X = U + V, \; Y = V$

Now

$$J = \begin{vmatrix} x_u & x_v \\ y_u & y_v \end{vmatrix} = \begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix} = 1$$

The joint pdf of (U, V) is given by

$$f(u, v) = 1 \cdot e^{-(u+v+v)} = e^{-(u+2v)}$$

Given limits for x, y: $x \geq 0, \; y > 0$

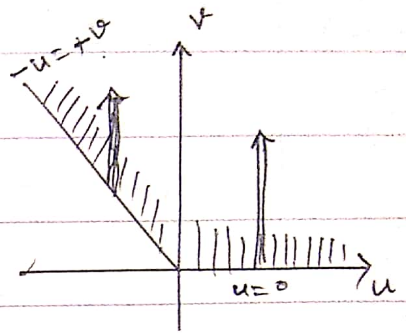
$\Rightarrow u + v > 0$ and $v > 0$

Case (i): $u < 0$ then $v > -u$

Case (ii): $u > 0$ then $v > 0$

The pdf of U is given by

$$f_U(u) = \int_{-\infty}^{\infty} f(u,v)\,dv$$



$$= \begin{cases} \int_{v=-u}^{\infty} e^{-(u+2v)}\,dv, & u<0 \\ \int_{0}^{\infty} e^{-(u+2v)}\,dv, & u>0 \end{cases}$$

$$= \begin{cases} \dfrac{e^{u}}{2}, & u<0 \\ \dfrac{e^{-u}}{2}, & u>0 \end{cases}$$