No.	Teacher's Sign / Remarks

Unit - I

1) Given point $P(-2, 6, 3)$ and vector $\vec{A} = y\,\vec{a_x} + (x+z)\,\vec{a_y}$ express P & A in cylindrical & spherical co-ordinates. Evaluate A at P in the cartesian, cylindrical and spherical system.

Gn: $P(-2, 6, 3) \Rightarrow x = -2,\ y = 6,\ z = 3.$

In CCS

$\rho = \sqrt{x^2+y^2} = \sqrt{4+32} = 6.32$

$\phi = \tan^{-1}(y/x) = \tan^{-1}(-6/2) = 108.43^\circ$

$z = 3.$

In SCS

$r = \sqrt{x^2+y^2+z^2} = \sqrt{4+36+9} = 7.$

$\theta = \tan^{-1}(\rho/z) = \tan^{-1}\left(\frac{6.32}{3}\right) = 64.62^\circ$

$\phi = \tan^{-1}(y/x) = 108.43^\circ$

$\therefore P(-2, 6, 3)$ in CCS $= P(6.32, 108.43^\circ, 3)$

$P(-2, 6, 3)$ in SCS $= P(7, 64.62^\circ, 108.43^\circ)$

In the cartesian system $\vec{A}$ at P is

$\vec{A} = y\,\vec{a_x} + (x+z)\,\vec{a_y} = 6\,\vec{a_x} + \vec{a_y}$

$\vec{A}$ in CCS

$$\begin{bmatrix} A_\rho \\ A_\phi \\ A_z \end{bmatrix} = \begin{bmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix}$$

$A_\rho = y\cos\phi + (x+z)\sin\phi$

$A_\phi = -y\sin\phi + (x+z)\cos\phi$

$A_z = 0$

But $x = \rho\cos\phi$

$y = \rho\sin\phi$

$z = z$ in terms of CCS

$\therefore A_\rho = \rho\sin\phi\cos\phi + (\rho\cos\phi + z)\sin\phi$

$A_\phi = -\rho\sin^2\phi + (\rho\cos\phi + z)\cos\phi$

$A_z = 0$

$$\therefore \vec{A} \text{ in CCS} = \left[\rho\sin\phi\cos\phi + (\rho\cos\phi + z)\sin\phi\right]\vec{a_\rho} + \left[-\rho\sin^2\phi + (\rho\cos\phi + z)\cos\phi\right]\vec{a_\phi} + [0]\cdot\vec{a_z}$$

substituting values of ρ, ϕ & z in above eqn

$$\vec{A} \text{ in CCS} = -0.9487\,\vec{a_\rho} - 6.008\,\vec{a_\phi}$$

$\vec{A}$ in SCS

$$\begin{bmatrix} A_r \\ A_\theta \\ A_\phi \end{bmatrix} = \begin{bmatrix} \sin\theta\cdot\cos\phi & \sin\theta\cdot\overset{\sin}{\cancel{\cos}}\phi & \cos\theta \\ \cos\theta\cos\phi & \cos\theta\,\sin\phi & -\sin\theta \\ -\sin\phi & \cos\phi & 0 \end{bmatrix} \begin{bmatrix} y \\ x+z \\ 0 \end{bmatrix}$$

$A_r = y\sin\theta\cos\phi + (x+z)\sin\theta\cdot\overset{\sin}{\cancel{\cos}}\phi$

$A_\theta = y\cos\theta\cdot\cos\phi + (x+z)\cos\theta\cdot\sin\phi$

$A_\phi = -y\sin\phi + (x+z)\cos\phi$

But $x = r \sin\theta \cos\phi$

$y = r \sin\theta \sin\phi$

$z = r \cos\phi$

~~$\vec{A}$ in SCS = [~~

$\therefore A_r = r\sin^2\theta \cdot \cos\phi \cdot \sin\phi + r\sin^2\theta \cdot \cos^2\phi + r\sin\theta$ $\sin\phi$ $\sin\phi \cos^2\phi$

$A_\theta = r\sin\theta \cdot \sin\phi \cos\theta \cdot \cos\phi + r\sin\theta \cdot \cos\theta \cdot \cos\phi \cdot \sin\phi + r\cos\phi \cdot \cos\theta \cdot \sin\phi$

$A_\phi = -r\sin\theta \cdot \sin^2\phi + r\sin\theta \cdot \cos^2\phi + r\cos^2\phi$

$\therefore \vec{A}$ in SCS $= \left[r\sin^2\theta \cdot \cos\phi \sin\phi + r\sin^2\theta \cdot \cos^2\phi + r\sin\theta \cdot \cos^2\phi\right]\vec{a_r} + \left[r\sin\theta \cdot \cos\theta \cdot \sin\phi \cos\phi + r\sin\theta \cdot \cos\theta \cdot \cos\phi \cdot \sin\phi + r\cos\phi \cdot \cos\theta \cdot \sin\phi\right]\vec{a_\theta} + \left[r\sin\theta \cos^2\phi + r\cos^2\phi - r\sin\theta \cdot \sin^2\phi\right]\vec{a_\phi}$

Substituting values of r, θ & ϕ in above eqn.

$\vec{A} = -0.8571\,\vec{a_r} - 0.4066\,\vec{a_\theta} - 6.008\,\vec{a_\phi}$

$|\vec{A}(x,y,z)| = |\vec{A}(\rho,\phi,z)| = |A(r,\theta,\phi)| = 6.083.$

2) Check the validity of the divergence theorem considering the field $\vec{D} = 2xy\,\vec{a_x} + x^2\,\vec{a_y}$ C/m² and the rectangular parallelopiped formed by the planes $x=0, x=1; y=0, y=2; z=0; z=3$.

Using divergence theorem,

$$\oint \vec{D}\cdot\vec{ds} = \int_{vol} \nabla\cdot\vec{D}\cdot dv$$

Gn:

$$\vec{D} = 2xy\,\vec{a_x} + x^2\,\vec{a_y}$$

Evaluating R.H.S,

$$\nabla\cdot\vec{D} = \frac{\partial}{\partial x}(2xy) + \frac{\partial}{\partial y}(x^2) + 0$$

$$\nabla\cdot\vec{D} = 2y$$

$$\iiint \nabla\cdot\vec{D}\cdot dv = \int_{x=0}^{x=1}\int_{y=0}^{y=2}\int_{z=0}^{z=3} 2y\cdot dx\cdot dy\cdot dz$$

$$= \int_0^1\int_0^2 [2yz]_0^3\, dx\cdot dy$$

$$= \int_0^1\int_0^2 6y\cdot dx\cdot dy$$

$$= \int_0^1 \left[\cancel{6}^{3}\frac{y^2}{\cancel{2}}\right]_0^2 dx$$

$$= 3\times 2^2 \left[\int_0^1 dx\right]$$

divergence
2xy $\vec{a_x}$ +
plates
$=3$.
$\vec{a_x} + x^2\vec{a_y}$
$y.dz$

$$= 12\,[x]_0^1 = 12$$

Evaluating L.H.S,

$$\iint \vec{D}\cdot d\vec{S} = \iint (D)_{x=1}\, dy\, dz \cdot \vec{a_x} + \iint (D)_{x=0}\, dy\, dz(-\vec{a_x})$$
$$+ \iint (D)_{y=2}\, dx . dz(\vec{a_y}) + \iint (D)_{y=0}\, dx\, dz(-\vec{a_y})$$
$$+ \iint (D)_{z=3}\, dx . dy + \iint (D)_{z=0}\, dx . dy(-\vec{a_z})$$

$$\iint (D)_{x=1}\, dy . dz . \vec{a_x} = \int_0^2\int_0^3 2xy \,. dy . dz$$
$$= \int_0^2 [2xyz]_0^3\, dy$$
$$= \int_0^2 [6xy]\, dy$$
$$= \left[\frac{\cancel{6}^{3} x y^2}{\cancel{2}}\right]_0^2$$
$$= 12x \Rightarrow \text{at } x=1,\ = 12$$

$$\iint (D)_{x=0}\, dy . dz(-\vec{a_x}) = 0 \quad \text{at } x=0.$$

$$\iint (D)_{y=2}\, dx . dz(\vec{a_y}) = \int_0^1\int_0^3 x^2\, dx . dz$$
$$= \int_0^1 [x^2 z]_0^3\, dx$$
$$= \int_0^1 3x^2\, dx$$
$$= \left[\frac{\cancel{3}x^3}{\cancel{3}}\right]_0^1$$
$$= 1$$

$$\int_0^1\int_0^3 (D)_{y=0}(-\vec{a_y})\,dx\,dz = -\int_0^1\int_0^3 x^2\,dx\,dz$$

$$= -\cancel{3}\cdot\left[x^3/\cancel{3}\right]_0^1 = -1$$

$$\iint (\vec{D})_{z=3}\,\vec{a_z}\,dx\,dy = \int_0^1\int_0^2 (\cancel{\vec{a_z}})\cdot\vec{a_z} = 0$$

$$\iint (\vec{D})_{z=0}(-a_z)\,dx\,dy = 0$$

$$\therefore \iint \vec{D} = 12 + 0 + 1 - 1 = 12$$

$\therefore$ Hence divergence theorem is proved

$$\oiint \vec{D}\cdot\vec{ds} = \iiint_v (\nabla\cdot\vec{D})\,dv$$

3) The vector form of coulomb's law, by locating a charge of $Q_1 = 3\times10^{-4}\,C$ at M(1,2,3) and a charge of $Q_2 = -10^{-4}\,C$ at N(2,0,5) in vacuum. Derive the force exerted on Q_2 by Q_1

Gn: At M(1,2,3) charge $Q_1 = 3\times10^{-4}\,C$
At N(2,0,5) charge $Q_2 = -10^{-4}\,C$

$$\vec{R_{12}} = r_2 - r_1 = (2-1)\vec{a_x} + (0-2)\vec{a_y} + (5-3)\vec{a_z}$$

$$|\vec{R_{12}}| = \sqrt{1^2 + (-2)^2 + 2^2} = 3$$

$$\vec{a_{12}} = \frac{\vec{R_{12}}}{|R_{12}|} = \frac{\vec{a_x} - 2\vec{a_y} + 2\vec{a_z}}{3}$$

$$F_2 = \frac{Q_1 Q_2}{4\pi\varepsilon_0 R_{12}^2} \cdot \vec{a_{12}}$$

$$= \frac{-(\cancel{3}\times10^{-4})(10^{-4})}{4\times\cancel{\pi}\times\frac{1}{36\cancel{\pi}}\times10^{-9}\times\cancel{3^2}} \times \frac{\vec{a_x} - 2\vec{a_y} + 2\vec{a_z}}{\cancel{3}}$$

$$\vec{F_2} = -10\vec{a_x} + 20\vec{a_y} - 20\vec{a_z}$$

4) A point charge $Q_1 = 10\mu C$ is located at $P_1(1,2,3)$ in free space while $Q_2 = -5\mu C$ is at $P_2(1,2,10)$

a) Find the vector force exerted on Q_2 by Q_1

b) Find the co-ordinates of P_3 at which a point charge Q_3 experiences no force.

Here the vectors originating from origin and up to these points are:

$$\vec{r_1} = \vec{a_x} + 2\vec{a_y} + 3\vec{a_z}$$

$$\vec{r_2} = \vec{a_x} + 2\vec{a_y} + 10\vec{a_z}$$

The vector form P_1 to P_2 is

$$R_{12} = r_2 - r_1$$

$$= (\vec{a_x} + 2\vec{a_y} + 10\vec{a_z}) - (\vec{a_x} + 2\vec{a_y} + 3\vec{a_z})$$

$$= 7\vec{a_z}$$

$$|\vec{R_{12}}| = 7.$$

a) The force on Q_1 is

$$F_{21} = \frac{Q_1 Q_2}{4\pi \varepsilon_0 |R_{12}|^3} \cdot R_{12} = \frac{-10\times10^{-6}\times 5\times10^{-6}\times 7\vec{a_z}}{\cancel{4\pi}\times\frac{1}{36\cancel{\pi}}\times10^{-9}\times 7^3}$$

$$\vec{F_{21}} = -9.18\, a_z \text{ mN}$$

b) Let a point charge Q_3 be situated at point $P_3(1,2,z)$ such that the force experienced by it is zero

$$r_3 = \vec{a_x} + 2\vec{a_y} + z\,\vec{a_z}$$

$$\begin{aligned} R_{13} &= r_3 - r_1 \\ &= (\vec{a_x} + 2\vec{a_y} + z\vec{a_z}) - (\vec{a_x} + 2\vec{a_y} + 3\vec{a_z}) \\ &= (z-3)\,\vec{a_z} \end{aligned}$$

$$\begin{aligned} R_{23} &= r_3 - r_2 \\ &= (\vec{a_x} + 2\vec{a_y} + z\vec{a_z}) - (\vec{a_x} + 2\vec{a_y} + 10\vec{a_z}) \\ &= (z-10)\,\vec{a_z} \end{aligned}$$

$$\frac{Q_1 Q_3}{4\pi \varepsilon_0 |R_{13}|^3} R_{13} + \frac{Q_2 Q_3 R_{23}}{4\pi \varepsilon_0 |R_{23}|^3} = 0$$

$$\frac{10}{(z-3)^2} = \frac{5}{(z-10)^2} = \frac{2}{z^2} = \frac{1}{(z-7)^2}$$

$$2(z-10)^2 = (z-3)^2$$

$$z^2 - 34z + 191 = 0$$

$$z = \frac{34 \pm \sqrt{(34)^2 - 4 \times 191}}{2}$$

$$z = 26.89 \ \& \ 7.10$$

The required point is $P_3(1, 2, 26.9)$

5) A charge of $-0.3\ \mu C$ is located at $A(25, -30, 15)$ in cm and a second charge of $0.5\ \mu C$ is at $B(-10, 8, 12)$ cm. Find $\vec{E}$ at a) the origin b) $P(15, 20, 50)$ cm

Gn: Charge $-0.3\ \mu C$ located at $A(25, -30, 15)$

Charge $0.5\ \mu C$ located at $B(-10, 8, 12)$

$$r_{OA} = r_A - r_O$$

$$r - r_1 = (0a_x + 0a_y + 0a_z) - (25a_x - 30a_y + 15a_z)$$

$$= -25\vec{a_x} + 30\vec{a_y} - 15\vec{a_z}$$

$$|r - r_1| = 41.83 \text{ cm}$$

$$r - r_2 = (0\vec{a_x} + 0\vec{a_y} + 0\vec{a_z}) - (-10\vec{a_x} + 8\vec{a_y} + 12\vec{a_z})$$

$$= +10\vec{a_x} - 8\vec{a_y} - 12\vec{a_z}$$

$$|r - r_2| = 17.549 \text{ cm}$$

$$\vec{E} = \frac{Q_1}{4\pi\varepsilon_0 |r - r_1|^2} \cdot \vec{a_1} + \frac{Q_2}{4\pi\varepsilon_0 |r - r_2|^2} \cdot \vec{a_2}$$

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$$= \frac{-0.3\times10^{-6}}{4\pi\times\frac{1}{36\pi}\times10^{-9}\times(41.833\times10^{-2})^2} \frac{(-25\vec{a_x} + 30\vec{a_y} - 15\vec{a_z})\times10^{-2}}{(41.833\times10^{-2})}$$

$$+ \frac{0.5\times10^{-6}}{4\pi\times\frac{1}{36\pi}\times10^{-9}\times(17.54\times10^{-2})^2} \times \frac{10\vec{a_x} - 8\vec{a_y} - 12\vec{a_z}}{17.54\times10^{-2}}$$

$$= -9\cancel{25}25 a_x - \cancel{77.75}\, a_y - 94.54\, a_z$$

P(15, 20, 50) cm.

$$r - r_1 = (15\vec{a_x} + 20\vec{a_y} + 50\vec{a_z}) - (25\vec{a_x} - 30\vec{a_y} + 15\vec{a_y})$$

$$= -10\vec{a_x} + 50\vec{a_y} + 35\vec{a_z}$$

$$|r - r_1| = 61.846$$

$$r - r_2 = (15\vec{a_x} + 20\vec{a_y} + 50\vec{a_z}) - (-10\vec{a_x} + 8\vec{a_y} + 12\vec{a_z})$$

$$= 25\vec{a_x} + 12\vec{a_y} + 38\vec{a_z}$$

$$|r - r_2| = 47.042$$

$$E = \frac{Q_1}{4\pi \varepsilon_0 |r - r_1|^2} \cdot \vec{a_1} + \frac{Q_2}{4\pi \varepsilon_0 |r - r_2|^2} \cdot \vec{a_2}$$

$$E = \frac{-0.3 \times 10^{-6}}{4\pi \times \frac{1}{36\pi} \times 10^{-9}} \times \frac{-10a_x + 50a_y + 35a_z}{(61.846)^3} +$$

$$\frac{0.5 \times 10^{-6}}{4\pi \times \frac{1}{36\pi} \times 10^{-9}} \times \frac{(25\vec{a_x} + 12\vec{a_y} + 38\vec{a_z})}{(47.042)^3}$$

$$\vec{E} = 1954\, \vec{a_x} + 3327\, \vec{a_y} + 5159\, \vec{a_z}$$

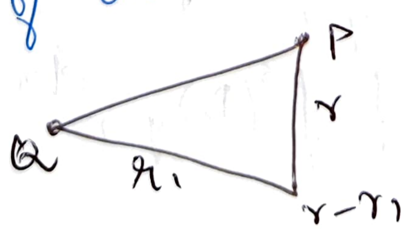
6) Find the electric field at point $P(r=2,\ \theta = 25^\circ,\ \phi = 90^\circ)$ if $\vec{D} = 0.3r^2 \vec{a_r}$ in free space.

Soln: $\vec{D} = \varepsilon_0 \vec{E}$

$$\vec{E} = \frac{\vec{D}}{\varepsilon_0} = \frac{0.3r^2}{8.854 \times 10^{-2}} \vec{a_r}$$

$$\vec{E} = \frac{0.3 \times 4}{8.854 \times 10^{-2}} \vec{a_r} = 1.355 \times 10^{11} \vec{a_r}$$

7) Find the electric field intensity at $P(-4, 6, -5)$ in free space caused by a charge of $0.1\ mC$ at $(2, -1, -3)$



$$r - r_1 = -6\vec{a_x} + 7\vec{a_y} - 2\vec{a_z}$$

$$|r - r_1| = \sqrt{6^2 + 7^2 + 2^2} = 9.433$$

$$\vec{E} = \frac{0.1 \times 10^{-3}}{4\pi \times 8.854 \times 10^{-12}} \times \frac{(-6\vec{a_x} + 7\vec{a_y} - 2\vec{a_z})}{(9.433)^3}$$

$$= 1071.3\,(-6\vec{a_x} + 7\vec{a_y} - 2\vec{a_z})$$

8) Let $\vec{D} = 6xyz^2\,\vec{a_x} + 3x^2z^2\vec{a_y} + 6x^2yz\,\vec{a_z}$ C/m². Find the total charge lying within the region bounded by $x=1$ and $x=3$, $y=0$ & 1, $z=-1$ & 1 by seperately evaluating each side of the divergence theorem.

Soln: According to Gauss's Law,

$$Q = \oint_S \vec{D}\cdot d\vec{s} = \int_{-1}^{1}\int_{0}^{1} (D)_{x=1}\,dy\,dz\,(-\vec{a_x}) + \int_{-1}^{1}\int_{0}^{1} (D)_{x=3}\,dy\,dz\,(+\vec{a_x})$$

$$+ \int_{-1}^{1}\int_{1}^{3} (D)_{y=0}\,dx\,dz\,(-\vec{a_y}) + \int_{-1}^{1}\int_{1}^{3} (D)_{y=1}\,dx.dz.(\vec{a_y})$$

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$$+\int_0^1\int_1^3 (D)_{z=-1}\, dx\, dy\, (-\vec{a_z}) +$$

$$\int_0^1\int_1^3 (D)_{z=1}\, dx . dy\, (\vec{a_z})$$

Here $(\vec{D})_{x=1} \cdot \vec{a_x} = 6xyz^2\vec{a}_x \cdot \vec{a_x}$

$$= 6xyz^2 = 6\times 1\times yz^2 = 6yz^2$$

|||ly $(\vec{D})_{x=3} \cdot \vec{a_x} = 18yz^2$

$(\vec{D})_{y=0} \cdot \vec{a_y} = 3x^2z^2$

$(\vec{D})_{y=1} \cdot \vec{a_y} = 3x^2z^2$

$(\vec{D})_{z=-1} \cdot \vec{a_z} = -6x^2y$

$(\vec{D})_{z=1} \cdot \vec{a_z} = 6x^2y$

Then the integral is

$$= -6\int_{-1}^{1}\int_0^1 yz^2\, dy\, dz + 18\int_{-1}^{1}\int_0^1 yz^2\, dy\, dz$$

$$-3\int_{-1}^{1}\int_1^3 x^2z^2\, dx\, dz + 3\int_{-1}^{1}\int_1^3 x^2z^2\, dx\, dz$$

$$+6\int_0^1\int_1^3 x^2y\, dx\, dy + 6\int_{-1}^{1}\int_1^3 x^2y\, dx . dy$$

$$= 12\times\frac{1}{2}\times 2\times\frac{1}{3} + 12\times\left(\frac{27}{3}-\frac{1}{3}\right)\times\frac{1}{2} = 56\,C$$

Method 2:

According to Gauss's Law,

$$Q = \int_{vol} (\nabla \cdot \vec{D})\, dV$$

$$\nabla \cdot \vec{D} = \frac{\partial}{\partial x}(6xyz^2) + \frac{\partial}{\partial y}(3x^2z^2) + \frac{\partial}{\partial z}(6x^2yz$$

$$= 6yz^2 + 6x^2y$$

$$\int_{-1}^{1}\int_{0}^{1}\int_{1}^{3} (6yz^2 + 6x^2y) . dx . dy . dz$$

$$= [x]_1^3 \, [y^2]_0^1 \, [z^3]_{-1}^{1} + [x^3]_1^3 \, [y^2]_0^1 \, [z]_{-1}^{1}$$

$$= 50C$$

q) Find the gradient of the following scalar fields

a) $V = e^{-z} \sin 2x \cosh y$

Soln: $\nabla V = \frac{\partial V}{\partial x} . \vec{a_x} + \frac{\partial V}{\partial y} . \vec{a_y} + \frac{\partial V}{\partial z} . \vec{a_z}$

$$= \frac{\partial}{\partial x}(e^{-z} \sin 2x . \cosh y)\vec{a_x} +$$

$$\frac{\partial}{\partial y}(e^{-z} \sin 2x . \cosh y)\vec{a_y} +$$

$$\frac{\partial}{\partial z}(e^{-z} \sin 2x . \cosh y)\vec{a_z}$$

$= 2e^{-z}\cos 2x \cosh y \cdot \vec{a_x} + e^{-z}\sin 2x \sinh y \vec{a_y}$

$- e^{-z}\sin 2x \cosh y \cdot \vec{a_z}$

b) $U = \rho^2 z \cos 2\phi$

Soln:

$$\nabla V = \frac{\partial U}{\partial \rho}\vec{a_\rho} + \frac{1}{\rho}\frac{\partial U}{\partial \phi}\vec{a_\phi} + \frac{\partial U}{\partial z}\cdot\vec{a_z}$$

$$\nabla U = \frac{\partial}{\partial \rho}(\rho^2 z \cos 2\phi)\cdot\vec{a_\rho} + \frac{1}{\rho}\frac{\partial}{\partial \phi}(\rho^2 z\cdot\cos 2\phi)\cdot\vec{a_\phi}$$

$$+ \frac{\partial}{\partial z}(\rho^2 z\cdot\cos 2\phi)\vec{a_z}$$

$$= 2\rho z\cdot\cos 2\phi\cdot\vec{a_\rho} - 2\rho z\cdot\sin 2\phi\cdot\vec{a_\phi} +$$

$$\rho^2\cdot\cos\phi\cdot$$

c) $W = 10r \sin^2\theta\cdot\cos\phi$

Soln: $\nabla W = \frac{\partial W}{\partial r}\cdot\vec{a_r} + \frac{1}{r}\frac{\partial W}{\partial \theta}\cdot a_\theta + \frac{1}{r\sin\theta}\frac{\partial W}{\partial \phi}\cdot\vec{a_\phi}$

$$= \frac{\partial}{\partial r}(10r\cdot\sin^2\theta\cdot\cos\phi)\cdot\vec{a_r} + \frac{1}{r}\cdot\frac{\partial}{\partial\theta}(10r\sin^2\theta\cdot\cos\phi)\vec{a_\theta}$$

$$+ \frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}(10r\sin^2\theta\cdot\cos\phi)\cdot\vec{a_\phi}$$

x^2yz

scalar

$\nabla W = 10\sin^2\theta \cdot \cos\phi \, \vec{a_r} + 10\sin 2\theta \cdot \cos\phi \, \vec{a_\theta} -$
$10\sin\theta \cdot \sin\phi \, \vec{a_\phi}$

c)

Sol

10) Determine the divergence of these vector fields.

a) $\vec{P} = x^2yz\,\vec{a_x} + xz\vec{a_z}$

Soln: $\nabla \cdot \vec{P} = \frac{\partial}{\partial x} P_x + \frac{\partial}{\partial y} P_y + \frac{\partial}{\partial z} P_z$

$\nabla \cdot \vec{P} = \frac{\partial}{\partial x}(x^2yz) + 0 + \frac{\partial}{\partial z}(xz)$

$= 2xyz + x$

b) $\vec{Q} = \rho\sin\phi\,\vec{a_\rho} + \rho^2 z\,\vec{a_\phi} + z\cos\phi \cdot \vec{a_z}$

11)

Soln:

$\nabla \cdot \vec{Q} = \frac{1}{\rho}\frac{\partial}{\partial \rho}(\rho Q_\rho) + \frac{1}{\rho}\frac{\partial}{\partial \phi}(Q_\phi) + \frac{\partial}{\partial z} Q_z$

$= \frac{1}{\rho}\frac{\partial}{\partial \rho}[\rho^2 \sin\phi] + \frac{1}{\rho}\frac{\partial}{\partial \phi}[\rho^2 z] + \frac{\partial}{\partial z}(z\cos\phi)$

$= \frac{1}{\cancel{\rho}} \times 2\cancel{\rho}\sin\phi + 0 + \cos\phi$

$\nabla \cdot \vec{Q} = 2\sin\phi + \cos\phi$

c) $\vec{T} = \frac{1}{r^2}\cos\theta \cdot \vec{a_r} + r\sin\theta \cdot \cos\phi \, \vec{a_\theta} + \cos\theta \cdot \vec{a_\phi}$

Soln. $\nabla \cdot \vec{T} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 T_r) + \frac{1}{r\sin\theta}\frac{\partial (T_\theta \sin\theta)}{\partial\theta} +$

$\frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}(T_\phi)$

$\nabla \cdot \vec{T} = \frac{1}{r^2}\frac{\partial}{\partial r}(\cos\theta) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(r\sin^2\theta\cos\phi) +$

$\frac{1}{r\sin\theta} \cdot \frac{\partial}{\partial\phi}(\cos\theta)$

$= 0 + \frac{1}{r\sin\theta} \cdot 2r \cdot \sin\theta \cdot \cos\theta \cdot \cos\phi + 0$

$\nabla \cdot \vec{T} = 2\cos\theta \cdot \cos\phi$

11) Given that $\vec{D} = z\rho\cos^2\phi \cdot \vec{a_z}$ C/m^2, Calculate the charge density at $(1, \pi/4, 3)$ and the total charge enclosed by the cylinder of radius 1m with $-2 \leq z \leq 2m$.

Soln: $\rho_v = \nabla \cdot D = \frac{\partial D_z}{\partial z} = \rho\cos^2\phi$

At $(1, \pi/4, 3)$

$\rho_v = 1 \cdot \cos^2(\pi/4) = 0.5 \; C/m^3$.

But

$$\int_{vol} \rho_v \cdot dv = Q \text{ (total charge)}$$

$$\therefore \int_{z=-2}^{2} dz \int_{\phi=0}^{2\pi} \cos^2\phi \, d\phi \int_{\rho=0}^{1} \rho^2 d\rho$$

$$= 4 \times \pi \times \frac{1}{3} = \frac{4\pi}{3} \text{ C.}$$

12) If $\vec{D} = (2y^2 + z)\vec{a_x} + 4xy\,\vec{a_y} + x\vec{a_z}$ C/m²

Find

a) The volume charge density at $(-1,0,3)$

b) The flux through the cube defined by $0 \leq x \leq 1$, $0 \leq y \leq 1$, $0 \leq z \leq 1$

Soln a) $\rho_v = \nabla \cdot \vec{D} = \frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z}$

$$= \frac{\partial}{\partial x}(2y^2 + z) + \frac{\partial}{\partial y}(4xy) + \frac{\partial}{\partial z}(x)$$

$$= 4x = 4(-1) = -4 \text{ C/m}^3$$

b) From Gauss's law,

$$\psi = Q$$

$$Q = \int_{vol} \rho_v \cdot dv = \int_{vol} \rho_v \cdot dx \cdot dy \cdot dz$$

$$= \int_0^1 4x\,dx \int_0^1 dy \int_0^1 dz.$$

$$= 4\left[\frac{x^2}{2}\right]_0^1 \{y\}_0^1 \, [z]_0^1$$

$$= 4\left(1/2\right)(1)(1) = 2C$$

The total charge enclosed by the cube

$$Q = 2C.$$

13) The finite sheet $0 \le x \le 1$, $0 \le y \le 1$ on the $z=0$ plane has a charge density $\rho_s = xy(x^2+y^2+25)^{3/2}\ nC/m^2$. Find

a) The total charge on the sheet.

b) The electric field at $(0,0,5)$

c) The force experienced by a $-1\,mC$ charge located at $(0,0,5)$

<u>Soln:</u> $Q = \int_S \rho_s \cdot dS$

$$= \int_0^1 \int_0^1 xy(x^2+y^2+25)^{3/2} \times 10^{-9}\, dx \cdot dy \ nC$$

Assume $u = x^2$

$du = 2x \cdot dx$

$x = \frac{1}{2} du = \frac{1}{2} d(x^2)$

$$= \frac{1}{2}\int_0^1 y \int_0^1 (x^2+y^2+25)^{3/2}\, d(x^2)\, dy \quad nc$$

$$= \frac{1}{2}\int_0^1 y\, dy \; \frac{[x^2+y^2+25]^{3/2+1}}{3/2+1}$$

$$= \frac{1}{\cancel{2}}\int_0^1 y\, dy \; \frac{[x^2+y^2+25]^{5/2}}{5/\cancel{2}}$$

$$= \frac{1}{5}\int_0^1 y\left\{[x^2+y^2+25]^{5/2}\right\}_0^1 dy$$

$$= \frac{1}{5}\int_0^1 y\left[(1+y^2+25)^{5/2} - (0+y^2+25)^{5/2}\right] dy$$

$$= \frac{1}{5}\int_0^1 y\left[(y^2+26)^{5/2} - (y^2+25)^{5/2}\right] dy$$

Now $y\cdot dy = \frac{1}{2} d(y^2)$

$$= \frac{1}{5}\int_0^1 \frac{1}{2}\left[(y^2+26)^{5/2} - (y^2+25)^{5/2}\right] d(y^2)$$

$$= \frac{1}{10}\left[\frac{(y^2+26)^{7/2} - (y^2+25)^{7/2}}{7/2}\right] d(y^2)$$

nC

$$= \frac{1}{10}\left[\frac{(y^2+26)^{7/2} - (y^2+25)^{7/2}}{7/2}\right]_0^1$$

$$= \frac{1}{10} \times \frac{2}{7}\left[(27)^{7/2} - (26)^{7/2} - \left[(26)^{7/2} - (25)^{7/2}\right]\right]$$

$$= \frac{1}{35}\left[(27)^{7/2} - 2(26)^{7/2} - (25)^{7/2}\right]$$

$= 33.15$ nC.

$$\vec{E} = \int_s \frac{\rho_s\, ds}{4\pi\varepsilon_0 r^2} \cdot \vec{a_R} = \int_s \frac{\rho_s\, ds\,(r-r')}{4\pi\varepsilon_0 |r-r'|^3}$$

$5/2]\, dy$

where $|r-r'| = (0,0,5) - (x,y,0)$
$= (x, -y, 5)$

$$\vec{E} = \int_0^1\int_0^1 \frac{\cancel{10^{-9}} \times \cancel{xy(x^2+y^2+25)^{3/2}}\,(-x\vec{a_x} - y\vec{a_y} + 5\vec{a_z})\, dx\, dy}{\cancel{4\pi} \times \frac{\cancel{10^{-9}}}{\cancel{36\pi}_{9}}\,\cancel{(x^2+y^2+25)^{3/2}}}$$

$d(y^2)$)

$$= 9\int_0^1\int_0^1\left[-x^2y\,\vec{a_x} - xy^2\,\vec{a_y} + 5xy\,\vec{a_z}\right] \cdot dx\, dy$$

$d(y^2)$)

$$= 9\left[-\left(\int_0^1 x^2\, dx \int_0^1 y\, dy\right)\vec{a_x} - \left(\int_0^1 x\, dx \int_0^1 y^2 dy\right)\vec{a_y} + \left(5\int_0^1 x\, dx \int_0^1 y\, dy\right)\vec{a_z}\right.$$

$$= q\left[-\left\{\left[\frac{x^3}{3}\right]_0^1 \left[\frac{y^2}{2}\right]_0^1\right\}\vec{a_x} - \left\{\left[\frac{x^2}{2}\right]_0^1 \left[y^3/3\right]_0^1\right\}\vec{a_y}\right.$$

$$\left. + 5\left\{\left[\frac{x^2}{2}\right]_0^1 \left[\frac{y^2}{2}\right]_0^1\right\}\vec{a_z}\right]$$

$$= q\left[-\left(\frac{1}{3}\right)(1/2)\vec{a_x} - \left\{(1/2)(1/3)\right\}\vec{a_y} + \left[5\left(\tfrac{1}{2}\right)\left(\tfrac{1}{2}\right)\right]\vec{a_z}\right]$$

$$= q\left[-\frac{1}{6}\vec{a_x} - \frac{1}{6}\vec{a_y} + \frac{5}{4}\vec{a_z}\right]$$

$$= (-1.5, 1.5, -11.25)$$

3) $\vec{F} = q\vec{E}$

$$= (1\times10^{-3})(-1.5, -1.5, 11.25)$$

$$= (+1.5, 1.5, -11.25) \text{ mN}$$

14) The point charges $-1nC$, $4nC$ & $3nC$ are located at $(0,0,0)$, $(0,0,1)$ and $(1,0,0)$ resply. Find the energy in the system.

Soln:

$$W = W_1 + W_2 + W_3$$

$$= 0 + Q_2V_{21} + Q_3(V_{31} + V_{32})$$

$$= Q_2 \cdot \frac{Q_1}{4\pi\varepsilon_0 |(0,0,1)-(0,0,0)|} +$$

$$\frac{Q_3}{4\pi\varepsilon_0} \cdot \left[\frac{Q_1}{|(1,0,10)-(0,0,0)|} + \frac{Q_2}{|(1,0,0)-(0,0,1)|}\right]$$

$$= \frac{1}{4\pi\varepsilon_0}\left[Q_1 Q_2 + Q_1 Q_3 + \frac{Q_2 Q_3}{\sqrt{2}}\right]$$

$$= \frac{1}{\frac{4\pi \times 10^{-9}}{36\pi}}\left(-4 - 3 + \frac{12}{\sqrt{2}}\right) \times 10^{-18} = 13.37 \text{ nJ}$$

15) Given the potential $V = \frac{10}{r^2} \sin\theta \cos\phi$

a) Find the $\vec{D}$ at $(2, \pi/2, 0)$

b) Calculate the workdone in moving a $10\mu C$ charge from point $A(1, 30°, 120°)$ to $B(4, 90°, 60°)$

Soln:

$$\vec{D} = \varepsilon_0 \vec{E}$$

$$\vec{E} = -\nabla V = -\left[\frac{\partial V}{\partial r} \cdot \vec{a_r} + \frac{1}{r}\frac{\partial V}{\partial \theta} \cdot a_\theta + \frac{1}{r\sin\theta}\frac{\partial V}{\partial \phi} a_\phi\right]$$

$$= \frac{20}{r^3} \sin\theta \cdot \cos\phi \cdot \vec{a_r} - \frac{10}{r^3} \cos\theta \cdot \cos\phi \cdot a_\theta +$$

$$\frac{10}{r^3} \cdot \sin\phi \, \vec{a_\phi}$$

$$= -\frac{\partial}{\partial r}\left[\frac{10}{r^2} \sin\theta \cdot \cos\phi\right]\vec{a_r} - \frac{1}{r}\frac{\partial}{\partial\theta}\left[\frac{10}{r^2} \sin\theta \cdot \cos\phi\right]\vec{a_\theta}$$

$$- \frac{1}{r\sin\theta} \cdot \frac{\partial}{\partial\phi}\left[\frac{10}{r^2} \cdot \sin\theta \cdot \cos\phi\right]\vec{a_\phi}$$

$$= -\left[\frac{-2}{r^3}\right) \sin\theta \cdot \cos\phi \cdot \vec{a_r} - \frac{10}{r^3} \cos\theta \cdot \cos\phi \, \vec{a_\theta}$$

$$+ \frac{1}{r\cancel{\sin\theta}} \cdot \frac{10}{r^2} \cdot \cancel{\sin\theta} \cdot \sin\phi \cdot a_\phi$$

$$= \frac{20}{r^3} \cdot \sin\theta \cdot \cos\phi \cdot \vec{a_r} - \frac{10}{r^3} \cos\theta \cdot \cos\phi \cdot \vec{a_\theta}$$

$$+ \frac{10}{r^3} \cdot \sin\phi \cdot \vec{a_\phi}$$

At $(2, \pi/2, 0)$

$$\vec{D} = \varepsilon_0 E\left[r = 2, \theta = \frac{\pi}{2}, \phi = 0\right]$$

$$= \varepsilon_0 \left[\frac{20}{8} \vec{a_r} - 0 a_\theta + 0 a_\phi\right]$$

$$= 2.5 \varepsilon_0 \cdot \vec{a_r} \ \mathrm{C/m^2}$$

$$\vec{E} = 22.1 \vec{a_r} \ \mathrm{pC/m^2}$$

Work done:

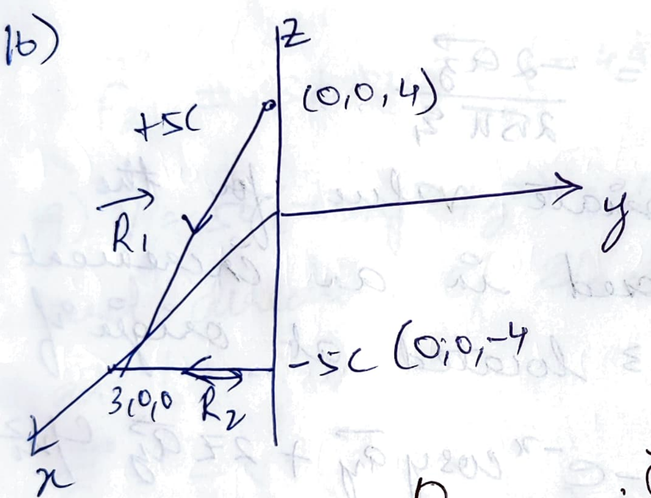
$$W = -Q\int_A^B \vec{E}\cdot dl = Q\,V_{AB}.$$

$$= Q\,[V_B - V_A]$$

$$= 10\times10^{-6}\left[\frac{10}{4^2}\sin 90^\circ \cos 60^\circ - \frac{10}{1}\sin 30^\circ \cos 120^\circ\right]$$

$$= 10\times10^{-6}\left[\frac{10}{32} - \left[\frac{-5}{2}\right]\right] = 28.125\,\mu J.$$

1b) Find the total Electric field intensity at (3,0,0)



Soln: $E_T = \dfrac{Q_1}{4\pi\varepsilon R_1^2}\cdot\vec{a}_{R_1} + \dfrac{Q_2}{4\pi\varepsilon R_2^2}\cdot\vec{a}_{R_2}$

$\vec{R_1} = (3,0,0) - (0,0,4) = 3\vec{a_x} - 4\vec{a_z}$

$\vec{R_2} = (3,0,0) - (0,0,-4) = 3\vec{a_x} + 4\vec{a_y}$

$\vec{a_{R_1}} = \dfrac{3\vec{a_x} - 4\vec{a_z}}{\sqrt{3^2+4^2}} = \dfrac{3\vec{a_x} - 4\vec{a_z}}{5}$

$$\vec{a}_{R_2} = \frac{3\vec{a_x} + 4\vec{a_y}}{\sqrt{3^2+4^2}} = \frac{3\vec{a_x} + 4\vec{a_y}}{5}$$

$$E_T = \frac{Q_1}{4\pi\epsilon|R_1|^3} \cdot \vec{R_1} + \frac{Q_2}{4\pi\epsilon|R_2|^3} \cdot \vec{R_2}$$

$$E_T = \frac{\cancel{5}}{4\pi\epsilon(5)^{\cancel{3}2}}(3a_x - 4a_z) - \frac{\cancel{5}}{4\pi\epsilon(5)^{\cancel{3}2}}(3a_x + 4a_z)$$

$$= \frac{3\cancel{a_x} - 4a_z - \cancel{3a_x} - 4a_z}{4\pi\epsilon(25)}$$

$$= \frac{-\overset{2}{\cancel{8}}a_y}{\cancel{4}\pi\epsilon(25)} = \frac{-2\vec{a_z}}{25\pi\epsilon}$$

17) Find an appropriate value for the total charge enclosed in an increment volume of $10^{-9}\,m^3$ located at origin of

$$\vec{D} = e^{-x}\sin y\,\vec{a_x} - e^{-x}\cos y\,\vec{a_y} + 2z\,\vec{a_z}\ C/m^2$$

$$Q = \oint\!\!\oint \vec{D}\cdot ds = \iiint_V (\nabla\cdot\vec{D})\,dV$$

$$\nabla\cdot\vec{D} = \frac{\partial}{\partial x}(e^{-x}\sin y) + \frac{\partial}{\partial y}(-e^{-x}\cos y) + \frac{\partial}{\partial z}(2z)$$

$$= \cancel{-1e^{-x}\sin y + (-e^{-x})}(-\sin y) + 2 = 2\,nC$$

18) Two
are loc
Find th
soln:
R_{21}
$|\vec{R}_{21}|$
F_{21}
$\vec{F}_{21}$

19) V
and
at
$\vec{E}$

18) Two point charges $Q_1 = 50\mu C$ and $Q_2 = 10\mu C$ are located $P_1(-1,1,3)$, $P_2(3,1,0)$ respectively. Find the force ~~on~~ Q_1 due to Q_2

Soln/:

$$R_{21} = R_1 - R_2 = -4a_x + 0\vec{a_y} + 3\vec{a_z}$$

$$|\vec{R_{21}}| = \sqrt{4^2+3^2} = \sqrt{25} = 5$$

$$F_{21} = \frac{Q_1 Q_2}{4\pi\varepsilon_0 R_{21}^2} = \frac{(\cancel{50}^{10^2}\times10^{-6})(\cancel{10}^{2}\times10^{-6})}{\cancel{4}\times\cancel{\pi}\times\frac{10^{-9}\times\cancel{5}^{2}}{\cancel{36\pi}\,9}}\frac{(-4a_x+3\vec{a_z})}{\cancel{5}}$$

$$\vec{F_{21}} = 36\times10^{-3}(-4\vec{a_x}+3\vec{a_z})\ N$$

19) $V = 50(x^2+y^2+z^2)$ volts. Find the magnitude and direction of Electric field intensity at $(1,-1,1)$

$$\vec{E} = -\nabla V$$

$$= -\left[\frac{\partial}{\partial x}(50x^2)\vec{a_x} + \frac{\partial}{\partial y}(50y^2)\vec{a_y} + \frac{\partial}{\partial z}(50z^2)\cdot\vec{a_z}\right]$$

$$= -50\left[2x\vec{a_x} + 2y\vec{a_y} + 2z\vec{a_z}\right]$$

$$\vec{E} = -100(\vec{a_x} - \vec{a_y} + \vec{a_z}) \text{ at } (1,-1,1)$$

$$|\vec{E}| = \sqrt{100^2+100^2+100^2} = \pm 100\sqrt{3}$$

$$\vec{a_E} = \frac{\vec{E}}{|\vec{E}|} = \frac{-100\vec{a_x} + 100\vec{a_y} - 100\vec{a_z}}{100\sqrt{3}}$$

$$= \frac{-100(\vec{a_x} - \vec{a_y} + \vec{a_z})}{100\sqrt{3}}$$

$$= \frac{\mp(\vec{a_x} - \vec{a_y} + \vec{a_z})}{\sqrt{3}}$$

20. Two point charges $-4\,\mu C$ and $5\,\mu C$ are located at $(2,-1,3)$ and $(0,4,-2)$ respectively. Find the potential at $(1,0,1)$. Assuming zero potential at infinity.

<u>Soln:</u> $Q_1 = -4\,\mu C$ $Q_2 = 5\,\mu C$

$$V(r) = \frac{Q_1}{4\pi\varepsilon_0|r-r_1|} + \frac{Q_2}{4\pi\varepsilon_0|r-r_2|} + C_0$$

If $V(\infty) = 0$, $C_0 = 0$.

$$\therefore |r-r_1| = (1,0,1) - (2,-1,3)$$

$$= |(-1,1,2)| = \sqrt{1+1+4}$$

$$= \sqrt{6}$$

$$\cancel{|r - r_2| = \frac{10^{-6}}{4\pi \times 10^{-9}/9\pi}\left[\frac{-4}{\sqrt{6}} + \frac{5}{r}\right]}$$

$$|r - r_2| = |(1, 0, +1) - (2, -1, 3)|$$

$$= |(1, -4, 3)| = \sqrt{1^2 + (-4)^2 + 3^2}$$

$$= \sqrt{26}.$$

Hence, $V(1,0,1) = \dfrac{10^{-6}}{4 \times \pi \times \dfrac{10^{-9}}{36\pi}}\left[\dfrac{-4}{\sqrt{6}} + \dfrac{5}{\sqrt{26}}\right]$

$$= 9 \times 10^{3}\,[-1.6337 \;\; 0.9806]$$

$$= -5.872 \text{ KV}$$