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# Root Locus :-

- Root locus technique is a tool for adjusting the location of closed-loop poles to achieve the desired system performance by varying one or more system parameters.

- Consider the open loop TF

- $G(s) = \dfrac{K}{s(s+p_1)(s+p_2)}$

- with Unity FB, the closed loop TF is given by

$$\frac{C(s)}{R(s)} = \frac{G(s)}{1+G(s)} = \frac{K}{s(s+p_1)(s+p_2)+K}$$

$\Rightarrow$ characteristic polynomial $= 0$

$$s(s+p_1)(s+p_2)+K = 0$$

Roots $\propto$ open loop gain, $K$

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⇒ Roots will take different values as the K is changed from 0 to ∞

⇒ If $K = 0$ ⇒ roots given by open loop poles

⇒ If $K = \infty$ ⇒ roots given by open-loop zeros

- The path taken by the roots of characteristic eqn when open gain K is varied from 0 to ∞ is called "root loci"
- The path taken by a root...
  ⇒ "root locus"

— other parameters can also ~~result~~ be varied to yield varied root location

$$\frac{C(s)}{R(s)} = T(s) = \frac{1}{\Delta} \sum_{K} P_K \Delta_K$$

— Using Mason's gain formula

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$\Rightarrow$ characteristic polynomial $\Rightarrow \Delta$

$\Rightarrow \Delta = 0 \Rightarrow$ char. Equation

For a single loop-system

$$\frac{C(s)}{R(s)} = \frac{G(s)}{1+G(s)H(s)}$$

$\Rightarrow$ characteristic eqn

$$1+G(s)H(s) = 0$$

$$\Rightarrow \boxed{G(s)H(s) = -1} \quad \text{— (A)}$$

$\Rightarrow$ Roots of eqn occur only for those values of s, for which $G(s)H(s) = -1$

Eqn (A) can be converted to Two Evans conditions

$$\boxed{|G(s)H(s)| = 1} \rightarrow \text{Magnitude criterion}$$

&

$$\boxed{\angle G(s)H(s) = \pm(2q+1)\pi} \text{ or } \pm(2q+1)180°$$

$$q = 0, 1, 2, 3, \ldots$$

$\downarrow$

Angle Criterion

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⇒ Magnitude Criterion: $s = s_a$ will be a point on root locus if for that value of $s$, $|G(s)H(s)| = 1$

⇒ Angle Criterion: $s = s_a$ will be a point on root locus if for that value of $s$, $\angle G(s)H(s)$ is equal to an odd multiple of $\pi$ rad ($180^\circ$)

⇒ $$G(s)H(s) = \frac{K(s+z_1)(s+z_2)(s+z_3)\cdots}{(s+p_1)(s+p_2)(s+p_3)\cdots}$$

Then $$|G(s)H(s)| = \frac{K|s+z_1| \times |s+z_2| \times |s+z_3| \times \cdots}{|s+p_1| \times |s+p_2| \times |s+p_3| \times \cdots}$$

$$= \frac{K \prod_{i=1}^{m} |s+z_i|}{\prod_{i=1}^{n} |s+p_i|}$$

∴ Magnitude Criterion:

$$K = \frac{\prod_{i=1}^{n} |s+p_i|}{\prod_{i=1}^{m} |s+z_i|}$$

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"Open-loop gain K corresponding to a point $s=s_a$ on root locus can be calculated using the above eqn.

$$K = \frac{\text{Product of length of vector from open loop poles to point } s=s_a}{\text{Product of length of vector from open-loop zeros to point } s=s_a}$$

Note : $|s+P_i|$ = length of vector drawn from $s=P_i$ to $s=s_a$

Similarly $|s+Z_i|$ = " " " from $s=z_i$ to $s=s_a$

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## Angle criterion:

$$\angle G(s)H(s) = \angle(s+z_1) + \angle(s+z_2) + \angle(s+z_3) + \ldots - \angle(s+P_1) - \angle(s+P_2) - \angle(s+P_3) - \ldots$$

$$\Rightarrow \angle G(s)H(s) = \sum_{i=1}^{m} \angle(s+z_i) - \sum_{i=1}^{n} \angle(s+P_i)$$

## Angle Criterion

$$\angle G(s)H(s) = \pm 180^\circ (2q+1)$$

$$\Rightarrow \sum_{i=1}^{m} \angle(s+z_i) - \sum_{i=1}^{n} \angle(s+p_i) = \pm 180^\circ(2q+1)$$

This equation can be used to check whether a point $s = s_a$ is a point on root locus or not.

$\angle(s+p_i)$ = Angle of vector drawn from $s = p_i$ to $s = s_a$ &

$\angle(s+z_i)$ = Angle of vector drawn from $s = z_i$ to $s = s_a$

$\therefore$ (sum of angles of vector from open loop zeros to point $s = s_a$) − (sum of angles of vector from open loop poles to point $s = s_a$) $= \pm 180^\circ(2q+1)$

## Construction of Root locus

- can be made like this (trial & error)
  - locate a point $s = s_a$
  - Determine angles of vectors from poles & zeros
  - Use angle criteria,
  - Use the guess to locate the trial point on root locus

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- shift the trial point
- Repeat the process

* A no. of points are determined this way
* Join the points by smooth curve
* The value of K can be determined from magnitude criterion.

Tedious process / time consuming

- A set of rules are developed to reduce the task involved in sketching the root locus

## Rules for Construction of Root Locus :-

(1) The root locus is symmetrical about the real axis

(2) Each branch of the root locus originates from an open-loop pole corresponding to $K=0$ & terminates at either on a finite open loop zero (or open loop zero at infinity) corresponding to $K=\infty$. The no. of branches of the root locus terminating on infinity $= n-m$ (no. of poles - no. of zeros)

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(3) Segments of the real axis having an odd number of real axes open-loop poles plus zeros of their right are parts of root locus

(4) The $n-m$ root locus branches that tend to infinity, do so along straight line asymptotes making angles with real axis given by

$$\phi_A = \frac{180^\circ(2q+1)}{n-m}, \quad q = 0, 1, 2, \ldots, n-m$$

(5) The point of intersection of the asymptotes with the real axis is at $s = \sigma_A$ where

$$\sigma_A = \frac{\text{Sum of poles} - \text{Sum of zeros}}{n-m}$$

(6) The breakaway and breakin points of the root locus are determined from the roots of the equation $\frac{dK}{ds} = 0$

If $r$ no. of branches of root locus meet at a point, then they will breakaway at an angle of $\pm 180^\circ/r$

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(7) The angle of departure from a complex open-loop pole is given by

$$\phi_p = \pm 180^\circ (2q+1) + \phi \quad q = 0,1,2\ldots$$

where $\phi$ = Net angle of contribution at the zero by all other open-loop poles & zeros

(8) The point of intersection of root locus branches with imaginary axis can be determined by use of the Routh Criterion. Alternatively they can be evaluated by letting $s = j\omega$ in characteristic eqn & equating the real part & imaginary part to zero, to solve for $\omega$ & K

- The value of $\omega$ = intersection points on imaginary axis
- K is value of gain at the intersection points

(9) The open-loop gain K at any point $s = s_a$ on the root locus is given by

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$$K = \frac{\prod_{i=1}^{n} |s_a + P_i|}{\prod_{i=1}^{m} |s_a + z_i|}$$

## Procedure for Constructing Root Locus

**Step-1:- Locate the poles & zeros**

- Draw s-plane on graph sheet, choose same scales for real & imaginary axes
- Mark poles by "X" & zeros by "O"
- No. of root locus branches = no. of poles of open-loop TF.
- Origin of ~~at~~ root locus is at a pole & end is at a zero
- Let $n$ = No. of poles, $m$ = No. of zeros
- $m$ root locus branches end at finite zeros
- $n-m$ branches will end at zeros at infinity

**Step-2:- Root locus on real axis**

- To determine the part of locus on real axis, take a test point on real axis. If the total no. of poles + zeros on real axis

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to the right of test point = odd

⇒ Then the test point lies on root locus

⇒ If it is ~~odd~~ even, Then the test point does not lie on root locus.

Step-3:- Angles of asymptotes & centroid

- $n-m$ root locus branches will terminate at zeros at infinity
- These branches will take the asymptotic path & meet the asymptote at $\infty$
- No. of asymptotes = no. of locus branches going to infinity

$$\text{Angle of Asymptotes} = \frac{\pm 180(2q+1)}{n-m}$$

$$q = 0, 1, 2 \ldots (n-m)$$

Centroid (meeting point of asymptotes with real axis)

$$= \frac{\text{Sum of poles} - \text{Sum of zeros}}{n-m}$$

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## Step-4:- Breakaway and Breakin Points

- Either lie at real axis or ~~real~~ exist as complex conjugate pairs.
- If there exists a root locus on real axis between two poles, there exists a breakaway point
- If there is a root locus on real axis between 2 zeros there exist a breakin point.
- Between pole & zero there may be or may not be breakaway or breakin point.

If the characteristic =n is in the form

$$B(s) + KA(s) = 0$$

$$\therefore \boxed{K = \frac{-B(s)}{A(s)}}$$

The roots of $\frac{dK}{ds} = 0$ are the actual breakaway or breakin points

## Step-5:- Angle of Departure/Arrival

$$\text{Angle for Departure (from Complex pole A)} = 180^\circ - \left(\begin{array}{c}\text{sum of angles of} \\ \text{vectors to the} \\ \text{complex pole A} \\ \text{from other poles}\end{array}\right) + \left(\begin{array}{c}\text{sum of angles of vectors} \\ \text{to complex pole} \\ \text{A from zeros}\end{array}\right)$$

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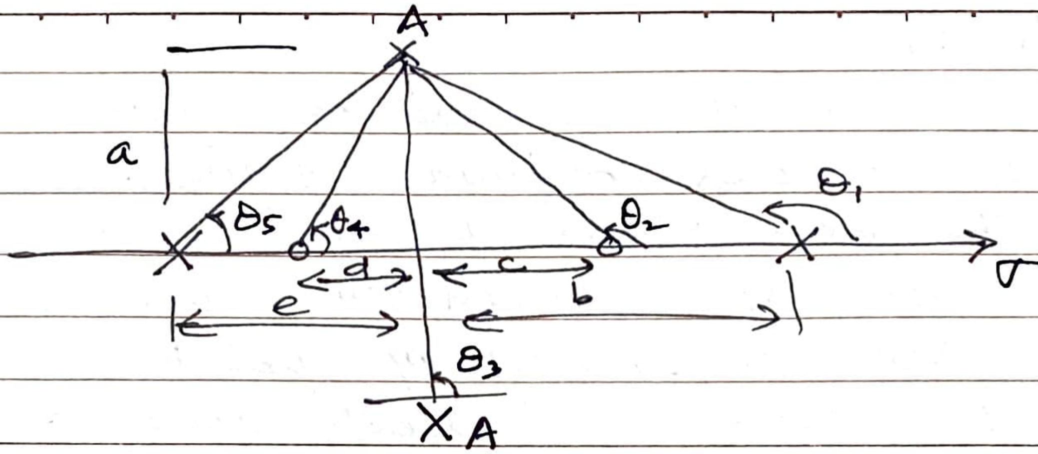



$\theta_1 = 180^\circ - \tan^{-1}\left(\frac{a}{b}\right)$

$\theta_2 = 180^\circ - \tan^{-1}\left(\frac{a}{c}\right)$

$\theta_3 = 90^\circ$

$\theta_4 = \tan^{-1}\left(\frac{a}{d}\right)$

$\theta_5 = \tan^{-1}\left(\frac{a}{e}\right)$

Angle of Departure at A

$= 180^\circ - (\theta_1 + \theta_3 + \theta_5) + (\theta_2 + \theta_4)$

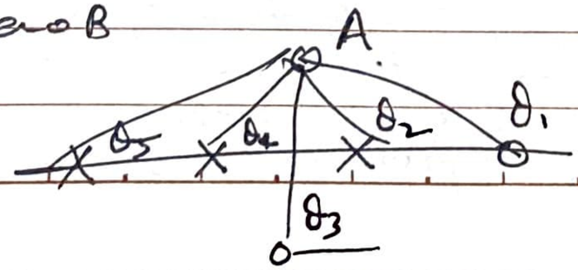
At $A^*$

$= -$ [Angle of Dep. at A]

* Angle of Arrival at a complex zero A $= 180^\circ -$ [sum of angles of vectors to the complex zero A from all other zeros] + [sum of angles of vectors to complex zero A from poles]

Ex: ~~In above fig~~
~~If poles & zeros are swapped~~

Then Angle of arrival at zero B $= 180^\circ - (\theta_1 + \theta_3) + (\theta_2 + \theta_4 + \theta_5)$



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## Step 6: Points of Intersection of Root locus with Imaginary Axis

- Can be determined by the three methods:
  1. By Routh Hurwitz array
  2. By trial & error approach
  3. letting $s = j\omega$ in the characteristic eqn & separating the real & imaginary parts
  - Two eqns are obtained
    - (a) By equating real part to zero
    - (b) By equating imaginary part to zero

- solve the two eqns for $\omega$ & K
- The value of $\omega$ gives the points where the roots locus crosses imaginary axis
- The value of k gives the value of gain at these crossing points

## Step 7: Test points & root locus

- Choose the test point.
- Using protector, roughly estimate the angles of vectors drawn to this point & Adjust the point to satisfy angle criterion

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- Repeat the procedure for more test points

- Sketch the root locus by the information gained from points 1 — 6

## Example :-

## Determination of open-loop Gain for a specified Damping of the Dominant Roots

Dominant Pole: pair of complex conjugate pole which decides the transient response of the system.

- In higher order systems dominant poles are given by the poles which are very close to origin, provided all other poles are lying far away from the dominant poles

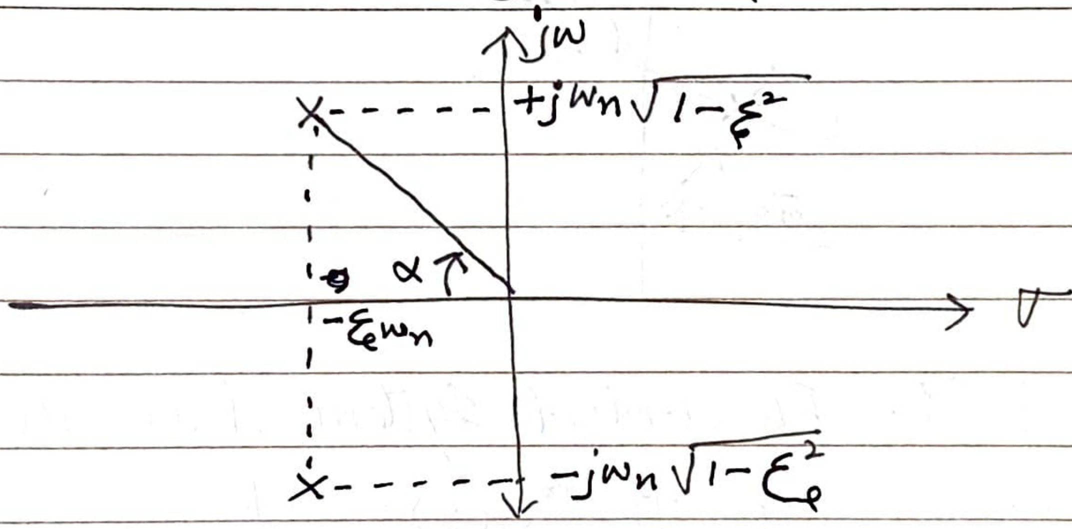
- The far away poles will have less effect on transient response second order, closed loop TF

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\xi\omega_n s + \omega_n^2}$$

- The dominant poles $s_d$ & $s_d^*$ are given by

$$s_d = \frac{-2\xi\omega_n \pm \sqrt{4\xi^2\omega_n^2 - 4\omega_n^2}}{2}$$

$$= -\xi\omega_n \pm j\omega_n\sqrt{1-\xi^2}$$




$$\cos\alpha = \frac{\xi\omega_n}{\omega_n} = \xi$$

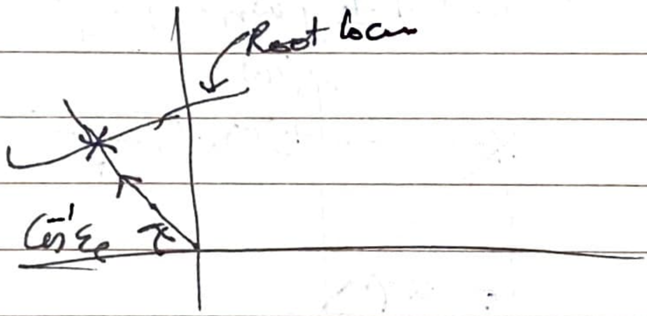
$$\therefore \alpha = \cos^{-1}\xi$$

- To fix dominant pole on root locus, draw line at angle $\cos^{-1}\xi$ wrt negative real axis

- The meeting point of this line with root locus ⇒ Dominant pole

- The value of K

$$K_{sd} = \frac{\text{Product of lengths of vectors from open loop poles to dominant pole}}{\text{Prod. of lengths of vectors from open loop zeros to dominant pole}}$$




(1) Unity FB Control System has an open-loop TF $G(s) = \dfrac{K}{s(s^2+4s+13)}$

Sketch Root locus

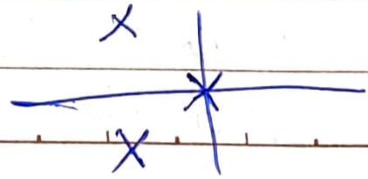
Soln:

Step 1: Locate poles & zeros of OLTF

$$s(s^2+4s+13)=0$$

Poles $P_1=0$, $P_2=-2+j3$, $P_3=-2-j3$

Mark poles by X



No zeros

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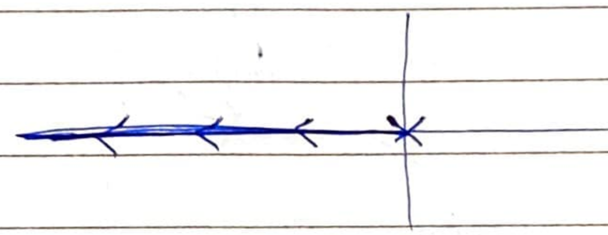
Step-2: To find root locus on real axis

- Only one (pole + zero) at origin
$$\underset{1}{\text{pole}} + \underset{0}{\text{zero}}$$

- Any point on negative real axis has odd no. of poles + zeros on right
∴ A part of root locus

⇒ starts from pole, ends on zero at infinity



Step 3: To find Angles of Asymptotes & Centroid

- No finite zero
- Three poles ⇒ Three branches } All three branches end at infinity

∴ Three Asymptotes are reqd

Angle of Asymptotes: $\dfrac{\pm 180^\circ(2q+1)}{n-m}$ $q = 0, 1, \ldots n-m$

$q = 0 \Rightarrow$ Angles $= \dfrac{\pm 180^\circ}{3} = \pm 60^\circ$

$q = 1 \Rightarrow$ Angles $= \dfrac{\pm 180^\circ \times 3}{3} = \pm 180^\circ$

$q = 2 \Rightarrow$ Angles $= \dfrac{\pm 180^\circ \times 5}{3}$ $\pm 300^\circ = \mp 60^\circ$

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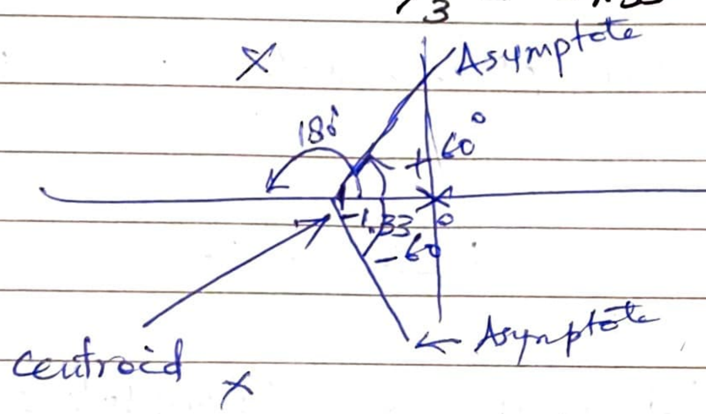
$q = 3$, Angles $= \dfrac{\pm 180^\circ \times 7}{3} = \pm 420^\circ = \pm 60^\circ$

Here Three value (distinct) : $60^\circ, 180^\circ, -60^\circ$

Centroid $= \dfrac{\text{sum of poles} - \text{Sum of zeros}}{n - m}$

$= -4/3 = -1.33$




Step-4: To find Breakaway & breakin points

CLTF $\dfrac{C(s)}{R(s)} = \dfrac{K}{s(s^2+4s+13)+K}$

characteristic eq

$s^3 + 4s^2 + 13s + K = 0$

$K = -s^3 - 4s^2 - 13s$

$\dfrac{dK}{ds} = -3s^2 - 8s - 13$



put $dK/ds = 0$

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$\Rightarrow s = -1.33 \pm j1.6$

Check for K, when $s = -1.33 + j1.6$

$$K = -(s^3 + 4s^2 + 13s) = -\left[(-1.33+j1.6)^3 + 4(-1.33+j1.66)^2 + 13(-1.33+j1.6)\right]$$

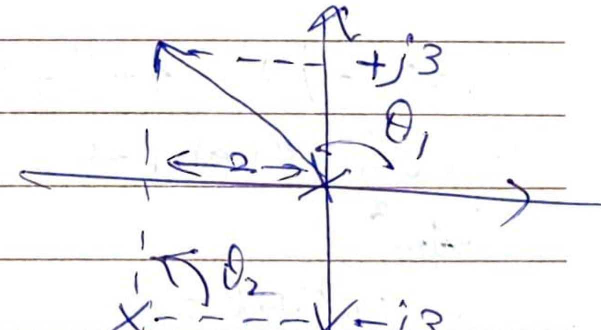
$\neq$ positive & real

Also for $s = -1.33 - j1.6$, K is not positive & real

$\Rightarrow$ Points are not actual breakaway or breakin points

Step 5: Angles of Departure

Draw vectors from all others poles to $P_2$




$\theta_1 = 180^\circ - \tan^{-1}(3/2)$

$= 123.7$

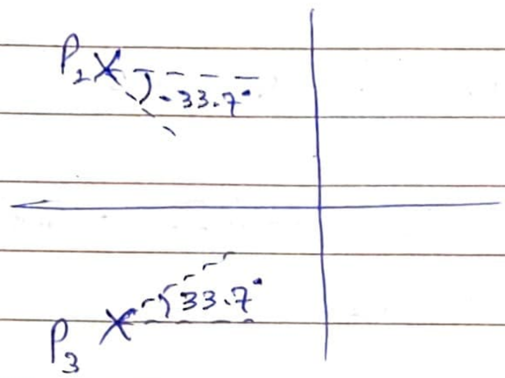
$\theta_2 = 90^\circ$

Angle of Departure from complex pole $P_2 = 180 - (\theta_1 + \theta_2)$

$\Rightarrow$ Ang of Dep ($P_2$) $= -33.7^\circ$

- Angle of Departure at complex pole $P_3$ = − Angle of Dep ...

$P_3 \rightarrow 33.7^\circ$




Step6: Find crossing point on imaginary axis

characteristic eqn

$$s^3 + 4s^2 + 13s + K = 0$$

put $s = j\omega$

$$(j\omega)^3 + 4(j\omega)^2 + 13(j\omega) + K = 0$$

$$\Rightarrow -j\omega^3 - 4\omega^2 + 13j\omega + K = 0$$

Equating Real parts

$$-4\omega^2 + K = 0$$

$$\Rightarrow K = 4\omega^2$$

$$\boxed{K = 52}$$

Equating imaginary parts

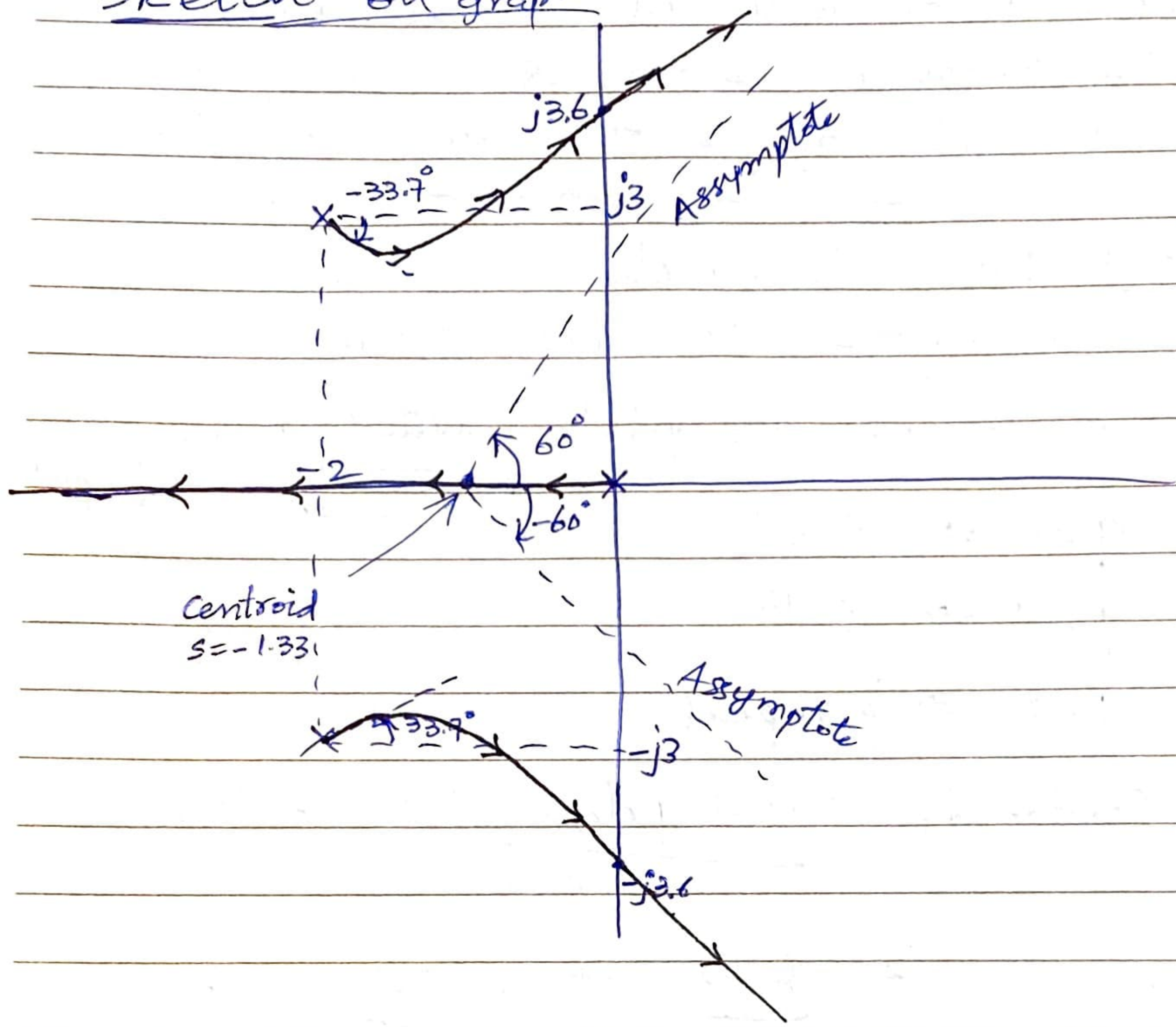
$$-\omega^3 + 13\omega = 0$$

$$\Rightarrow \omega(-\omega^2 + 13) = 0$$

$$\Rightarrow \omega^2 = 13 \Rightarrow \boxed{\omega = \pm 3.6}$$

∴ Crossing point of root locus $= \pm 3.6$, value of $K$ at those points $= 52$

sketch on graph




② Sketch the root locus of the system whose open-loop TF is $G(s) = \dfrac{K}{s(s+2)(s+4)}$

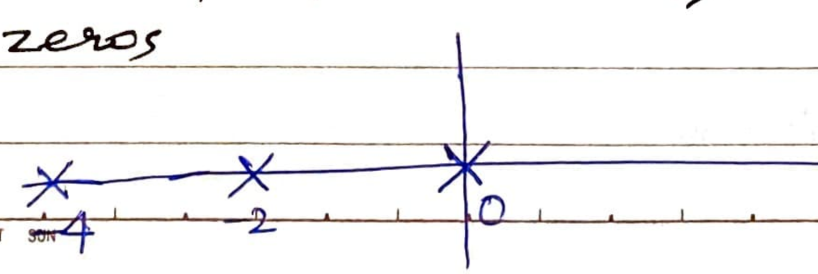
Find value of K, so that Damping ratio of closed system = 0.5

Step1: Locate the Poles & zeros of open-loop

$s(s+2)(s+4)=0$

$\Rightarrow$ Poles $P_1 = 0$, $P_2 = -2$, $P_3 = -4$

No zeros



Step 2: Root locus on Real axis

* Between $s=0$ & $s=-2$ } Root locus
* left of $s=-4$ upto $-\infty$ }


Step 3: Find Asymptotes & Centroid

$n-m=3 \Rightarrow$ Three asymptotes

Angles of Asymptotes $= \dfrac{\pm 180(2q+1)}{n-m}$ $\quad q=0,1,\ldots n-m$

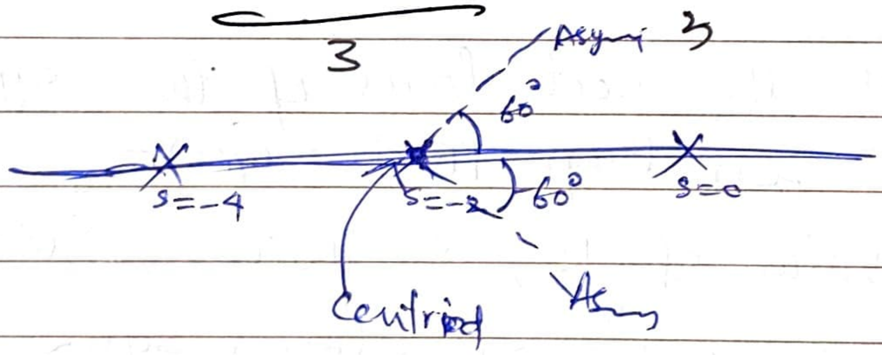
$q=0 \Rightarrow \pm 60^\circ$

$q=1 \Rightarrow \pm 180^\circ$

$q=2 \Rightarrow$ } repeat

Centroid: $\dfrac{\text{Sum of poles} - \text{Sum of zeros}}{n-m}$

$= \dfrac{0-2-4}{3} = \dfrac{-6}{3} = -2$




Step 4: Angle of Departure

- No complex pole or zero

$\Rightarrow$ Need not find

Step 5: Breakaway & Breakin Points

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CLTF $\frac{C(s)}{R(s)} = \frac{K}{s(s+2)(s+4)+K}$

$s(s+2)(s+4)+K=0$

$\Rightarrow K = -s^3 - 6s^2 - 8s$

$\frac{dK}{ds} = -3s^2 - 12s - 8$

Put $\frac{dk}{ds} = 0 \Rightarrow s = -0.845$ or $-3.154$

when $s = -0.845$, $\boxed{K = 3.08}$

~~when $s = -3.154$, $\boxed{K = -3.08}$~~

- Since K is -ve for $s = -3.154$
  This is not breakaway point
- Since K is positive & real for $s = -0.845 \Rightarrow$ This point is actual breakaway point

Step 6: Crossing point on imaginary axis

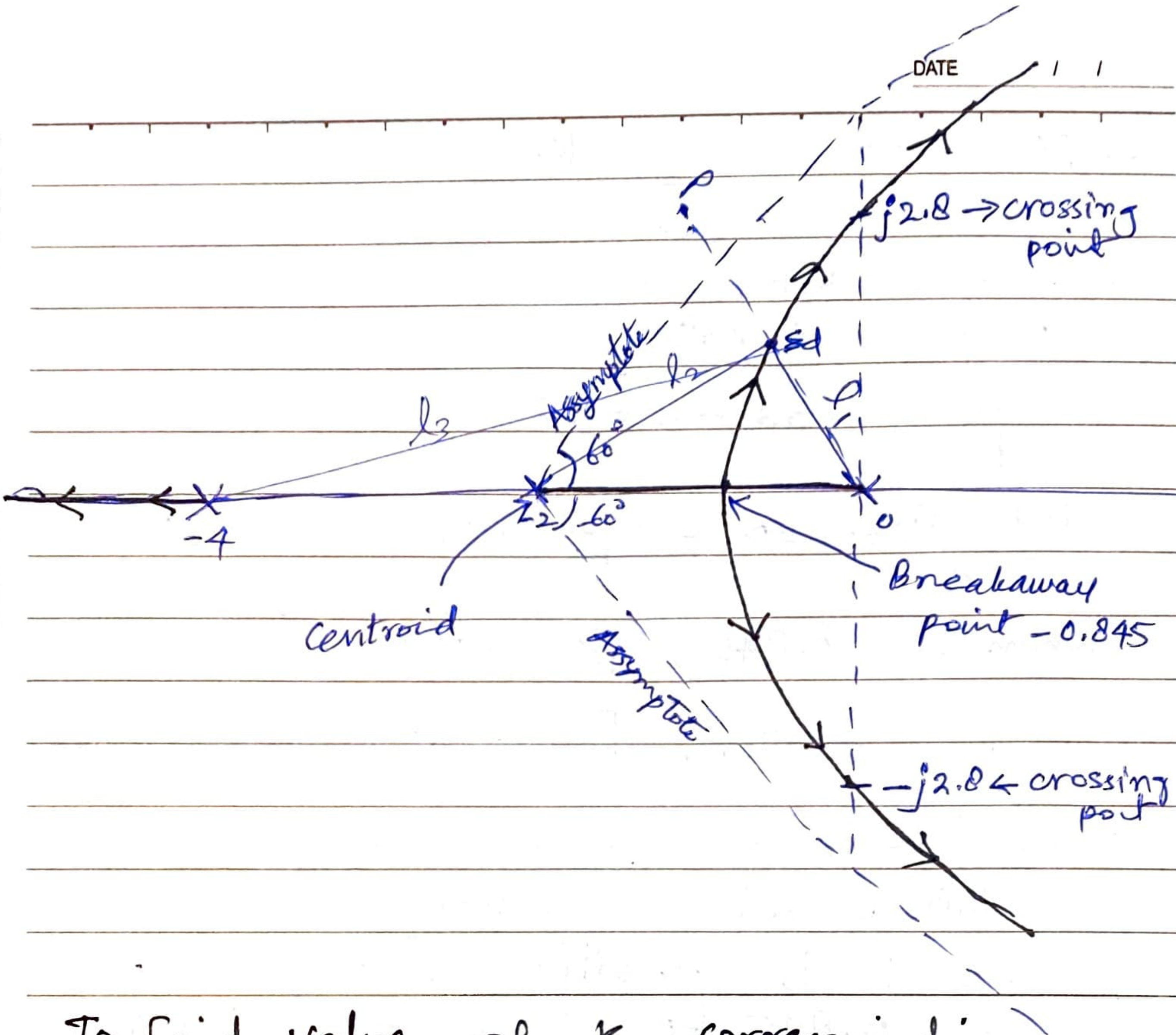
$s^3 + 6s^2 + 8s + K = 0$

Put $s = j\omega$

$\Rightarrow -j\omega^3 - 6\omega^2 + 8j\omega + K = 0$

| | |
|---|---|
| imaginary part = 0 | real part = 0 |
| $8\omega = \omega^3$ | $K = 6\omega^2$ |
| $\Rightarrow \omega^2 = 8 \Rightarrow \omega = \pm 2\sqrt{2}$ | $= 48$ |
| $\omega = \pm 2.8$ | |

crossing points $= \pm j2.8$




To find value of K corresponding to ~~θ~~ $\zeta = 0.5$

$$\alpha = \cos^{-1}\zeta = \cos^{-1}0.5 = 60^\circ$$

- Draw line op such that angle b/w op & negative real axis = 60°
- The meeting point gives dominant Pole, Sd

- $K_{sd}$ = value of K corresponding to dominant pole $s = s_d$

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$K_{sd} = \dfrac{\text{Product of lengths of vectors from all poles to pole at } s = s_d}{\text{Product of lengths of vectors from all zeros to pole at } s = s_d}$

$K_{sd} = \dfrac{l_1 \times l_2 \times l_3}{1} = 1.3 \times 1.75 \times 3.5$

$= 7.96 \approx 8$

(3) Sketch the root locus for unity FB system with $G(s) = \dfrac{K}{s(s^2+6s+10)}$

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