Unit - V   Dr. A. Saraswathi (1)

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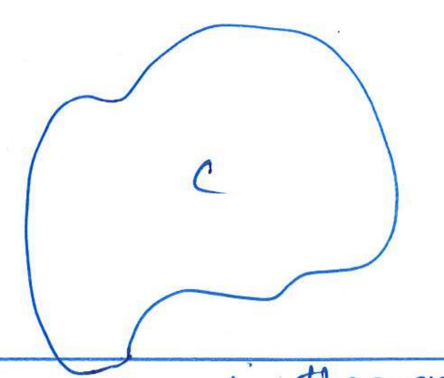
Cauchy's Integral Theorem :-

Definition :- If $f(z)$ is analytic inside and on a closed curve C then,

$$\boxed{\int_C f(z)\,.\,dz = 0}$$



Problems based on Cauchy's integral theorem:

1. evaluate :- $\displaystyle\int_C \frac{3z^2+7z-1}{z-2}\,.\,dz$ where C is the curve $|z|=1$.

given :- $|z|=1$

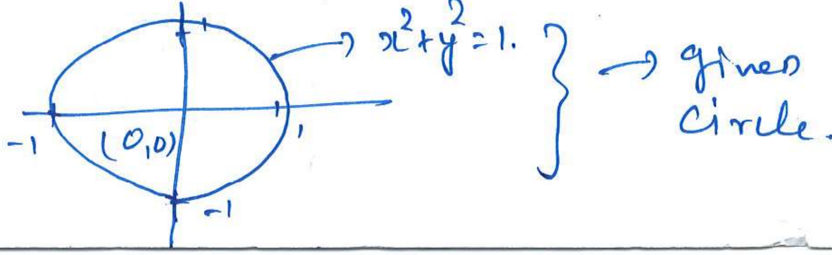
$|x+iy|=1 \Rightarrow \sqrt{x^2+y^2}=1 \quad \therefore x^2+y^2=1^2$

is a circle of radius of $r=1$ and centre $(0,0)$

Equating the Denominators in the given integral to zero.

$z-2=0 \quad \therefore \boxed{z=2} \quad \therefore$ The points $(2,0)$

and the point $(2,0)$ lies outside the circle



∴ function is not analytic when the denominator becomes zero.

∴ The function is analytic inside the circle.

By Cauchy integral theorem

$$\boxed{\int_C \frac{3z^2+7z-1}{z-2}\,dz = 0}$$

2. evaluate :- $\int_C \frac{2z+5}{(z-1)(z-2)}\,dz$ where

C is $|z| = 1/2$.

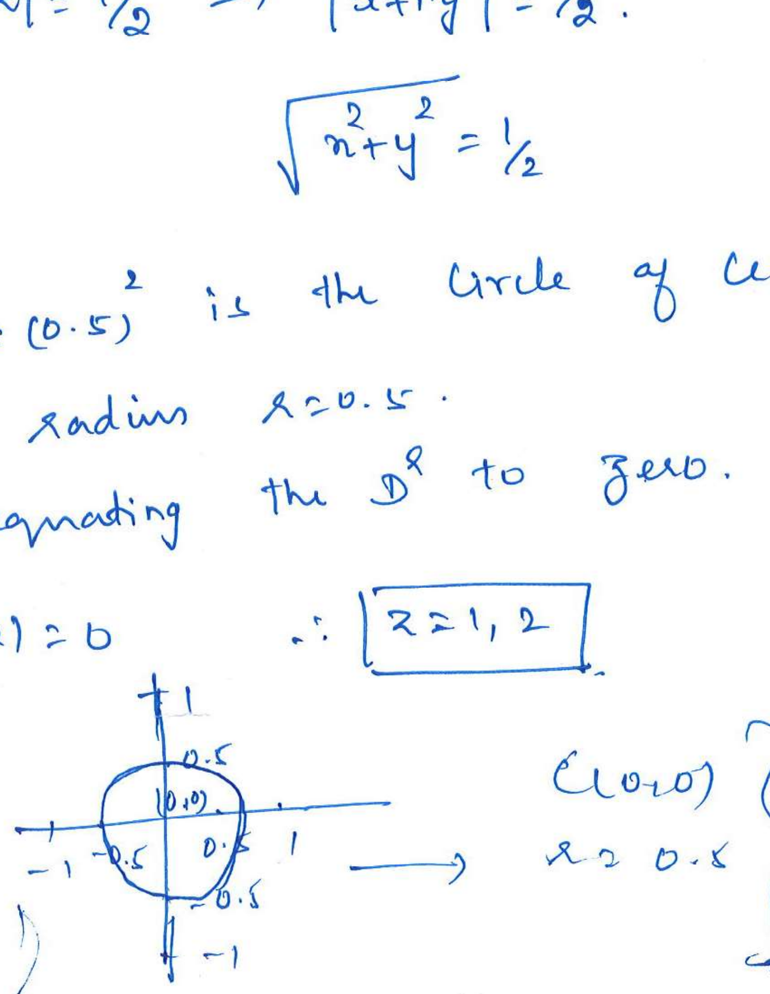
<u>Solution :-</u> $|z| = 1/2 \Rightarrow |x+iy| = 1/2$.

$$\sqrt{x^2+y^2} = 1/2$$

∴ $x^2+y^2 = (1/2)^2 = (0.5)^2$ is the circle of centre $C(0,0)$ and radius $r = 0.5$.

Now, equating the D^r to zero.

$(z-1)(z-2) = 0$ ∴ $\boxed{z = 1, 2}$



$\therefore z=1,2$ lies outside the circles

The function is not analytic when the denominator becomes zero.

$\therefore$ Both the points lie outside the circle.

The function is analytic inside the circle.

By Cauchy's integral theorem

$$\boxed{\int_C \frac{2z+5}{(z-1)(z-2)} \cdot dz = 0}$$

Q. evaluate $\int_C \frac{3z^2+7z-1}{z-2} \cdot dz$, where C is the curve $|z|=1$.

Given:- $|z|=1$

$|x+iy|=1 \Rightarrow \sqrt{x^2+y^2}=1 \Rightarrow x^2+y^2=1^2$ is the centre of circle and $r=1$.

equating the D^r to zero.

$\therefore z-2=0 \quad \therefore \boxed{z=2} \Rightarrow$ the points is $(2,0)$.

This points lies lies outside the circle

the fun is not analytic when the D^r becomes zero.

$\therefore$ By Cauchy's integral theorem $\int_C \frac{3z^2+7z-1}{z-2} \cdot dz = 0.$

4. Cauchy's integral Formula:-

If $f(z)$ is analytic inside and on a close curve 'c' and 'a' is an interior point Then,

$$\boxed{f(a) = \frac{1}{2\pi i}\int_c \frac{f(z)}{z-a}.dz.}$$

(or).

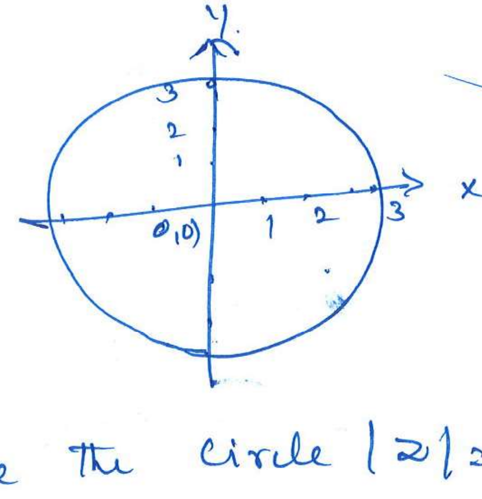
$$\boxed{\int_c \frac{f(z)}{z-a}.dz = 2\pi i . f(a).}$$

Problems based Cauchy's integral formula:.

1. Evaluate:- $\int_c \frac{z^2+2}{z-2}.dz$, where C is the circle of $|z|=3$.

Soln:- $|z|=3$

$|x+iy| = 3 \implies x^2+y^2=3^2$ is the equation of circle of radius $r=3$ and $c(0,0)$.



Now, equating the D^r to 0.

$\therefore z-2=0,\ \boxed{z=2}$.

The point lies inside the circle $|z|=3$

It's an interior point.

we have to apply the Cauchy's integral formula.

$$\int_c \frac{f(z)}{z-a} \cdot f(z) = 2\pi i\, f(a). \longrightarrow (1).$$

To find $f(a)$:-

given $f(z) = z^2 + 2.$

$f(a) = a^2 + 2.$

$f(2) = 2^2 + 2 = 4 + 2 = 6.$

Sub in (1).

$$\therefore \int_c \frac{f(z)}{z-a} \cdot f(z) = \int_C \frac{z^2+2}{z-2} \cdot dz = 2\pi i \times 6$$
$$= 12\pi i.$$

2. Evaluate: Using Cauchy's integral formula,

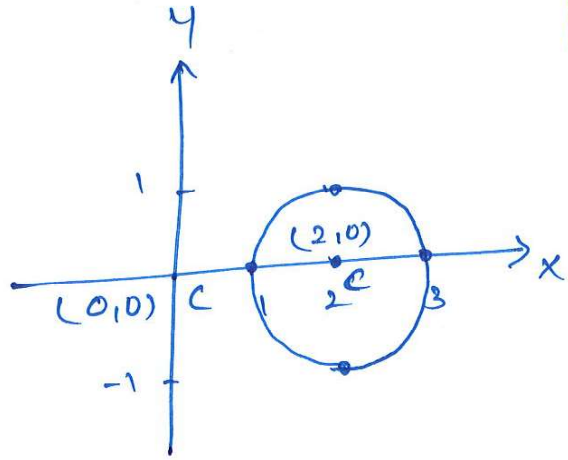
$$\int_c \frac{2z^2-4z+3}{z-2} \cdot dz, \quad \text{where } c: |z-2| = 1.$$

Soln:- $|z-2| = 1$

$\Rightarrow |x+iy-2| = 1$

$\Rightarrow |x-2+iy| = 1 \Rightarrow \sqrt{(x-2)^2 + y^2} = 1$

$\therefore (x-2)^2 + y^2 = 1^2$ is the circle of radius 1 and centre $C(2,0)$.



equating to D^r to 0.

$z-2=0 \quad \therefore \boxed{z=2}$

The point is an interior point.

$\therefore$ By Cauchy's integral formula,

$$\int_C \frac{f(z)}{z-a}\, dz = 2\pi i\,[f(a)] \longrightarrow (1)$$

<u>To find f(a):-</u>

$f(z) = 2z^2 - 4z + 3.$

$f(2) = 2(2)^2 - 4(2) + 3 = \cancel{8} - \cancel{8} + 3 = 3.$

Sub in (1).

$$\therefore \int_C \frac{f(z)}{z-a}\, dz = 2\pi i\,[3] = 6\pi i.$$

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3. evaluate using Cauchy's integral formula,

$$\int_C \frac{2z+2}{(z-1)(z-3)}\, dz, \text{ where } |z|=2.$$

3. $\int \frac{z+2}{(z-1)(z-3)} \cdot dz$, $c: |z| = 2$ $\boxed{7}$.

Soln:- $|z| = 2. \Rightarrow |x+iy| = 2.$

$\therefore \sqrt{x^2+y^2} = 2 \Rightarrow x^2+y^2 = 2^2$ is the equation of a Circle of radius $= 2$ and Centre $(0,0)$.

equating to D^r to zero.

$(z-1)(z-3) = 0 \quad \therefore \boxed{z = 1, 3}.$

$\therefore z = 1$ is lies inside the Circle.

$z = 3$ lies outside the Circle.

Since $z=1$ is an interior point and $z = 3$ is an exterior point.

$\therefore$ By cauchy's integral formula,

$$\int_c \frac{f(z)}{z-a} \cdot dz = 2\pi i \cdot f(a) \longrightarrow (1)$$

we've to apply cauchy's integral formula only the inside the Circle only {$\because$ interior point only}.

$$\therefore f(z) = \frac{z+2}{z-3}.$$

$$f(a) = \frac{a+2}{a-3} \Rightarrow f(1) = \frac{1+2}{1-3} = \frac{3}{-2}.$$

Sub: $f(a) = -3/2$ in (1).

$$\therefore \int_c \frac{f(z)}{z-a} \cdot dz = \int_c \frac{z+2}{(z-1)(z-3)} \cdot dz = 2\pi i\left[-3/2\right]$$

$$= -3\pi i.$$

4. Evaluate:- $\displaystyle\int_c \frac{\cos\pi z^2 + \sin\pi z^2}{(z-1)(z-2)} \cdot dz$, $c: |z| = 3$

given $|z| = 3 \Rightarrow |x+iy| = 3$.

$\therefore \sqrt{x^2+y^2} = 3 \Rightarrow x^2+y^2 = 3^2$ is the eqn of circle of radius $r=3$ and centre $(0,0)$.

equating the D^r to 0.

$(z-1)(z-2) = 0 \quad \therefore \boxed{z = 1, 2}$.

$z=1$ lies inside the circle.

$z=2$ lies inside the circle.

$\therefore$ Both the points analytic, it's inside the circle.

$\therefore$ Cauchy's integral formula,

$$\int_c \frac{f(z)}{z-a} \cdot dz = 2\pi i\left[f(a_1) + f(a_2)\right]. \longrightarrow (1).$$

Case ①:-

To find $f(a_1)$:-

$$f(z) = \frac{\cos\pi z^2 + \sin\pi z^2}{z-2} \quad \{\because z = a = 1\}.$$

$$a=1, \quad f(1) = \frac{\cos\pi\cdot 1^2 + \sin\pi\cdot 1^2}{1-2} = \frac{-1+0}{-1} = 1.$$

$$\therefore \boxed{f(a_1) = f(1) = 1} \qquad \left\{\begin{array}{l}\because \cos\pi = -1 \\ \sin\pi = 0\end{array}\right\}.$$

To find $f(a_2)$:-

$$f(z) = \frac{\cos\pi z^2 + \sin\pi z^2}{z-1}$$

$$a=2, \quad f(2) = \frac{\cos\pi\cdot 2^2 + \sin\pi\cdot 2^2}{2-1}$$

$$= \frac{\cos 4\pi + \sin 4\pi}{1} \qquad \left\{\begin{array}{l}\because \cos 4\pi = 1 \\ \sin 4\pi = 0\end{array}\right\}$$

$$f(2) = \frac{1+0}{1} = 1$$

$$\therefore \boxed{f(a_2) = f(2) = 1}$$

Sub in ①.

$$\left.\begin{array}{l}\therefore \displaystyle\int_c \frac{f(z)}{z-a}\,dz = \int_c \frac{\cos\pi z^2 + \sin\pi z^2}{(z-1)(z-2)}\,dz \\ \qquad = 2\pi i\,[1+1] = 4\pi i\end{array}\right\}.$$

5. **Cauchy's integral formula for Derivatives:**

$$f'(a) = \frac{1!}{2\pi i}\int_c \frac{f(z)}{(z-a)^2}\cdot dz = \frac{1}{2\pi i}\int_c \frac{f(z)}{(z-a)^2}\cdot dz$$

$$f''(a) = \frac{2!}{2\pi i}\int_c \frac{f(z)}{(z-a)^3}\cdot dz = \frac{1}{\pi i}\int_c \frac{f(z)}{(z-a)^3}\cdot dz$$

$$f'''(a) = \frac{3!}{2\pi i}\int_c \frac{f(z)}{(z-a)^4}\cdot dz = \frac{\cancel{6}\,3}{\cancel{2}\pi i}\int_c \frac{f(z)}{(z-a)^4}\cdot dz$$

$\vdots$

$$\boxed{f^{n}(a) = \frac{n!}{2\pi i}\int_c \frac{f(z)}{(z-a)^{n+1}}\cdot dz}$$

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1. evaluate :- $\int_c \frac{z^2+1}{(z-1)^2}\cdot dz$, $c : |z| = 3/2$.

Soln:- $|z| = 3/2 \Rightarrow |x+iy| = 3/2$.

$\therefore \sqrt{x^2+y^2} = 3/2 \Rightarrow x^2+y^2 = (3/2)^2$ is The radius of circle $r = 1.5$ and centre $(0,0)$.

equating the D^r to 0. $(z-1)^2 = 0$

$\boxed{z = 1, 1}$ lies twice.

$z = 1, 1$ twice.

$z = 1$ lies the circle. Since, the given integral contains denominator of the form $(z-a)^2$.

$\therefore$ we have to apply cauchy's integral formula for the derivatives,

$$f'(a) = \frac{1!}{2\pi i}\int_c \frac{f(z)}{(z-a)^2}\cdot dz.$$

We've $\int_c \frac{z^2+1}{(z-1)^2}\cdot dz \implies a = 1.$

$$\therefore f'(a) = f'(1) = \frac{1!}{2\pi i}\int_c \frac{z^2+1}{(z-1)^2}\cdot dz \longrightarrow (1).$$

$\therefore f(z) = z^2+1.$

$f'(z) = 2z$

$f'(1) = 2.(1) = 2.$

$$\therefore 2 = \frac{1}{2\pi i}\int_c \frac{z^2+1}{(z-1)^2}\cdot dz$$

$$2 \times 2\pi i = \int_c \frac{z^2+1}{(z-1)^2}\cdot dz$$

$$\therefore \boxed{\int_c \frac{z^2+1}{(z-1)^2}\cdot dz = 4\pi i}$$

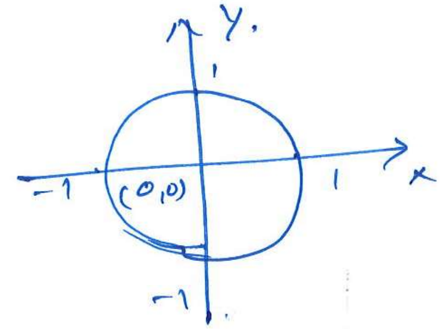
2. $\int_C \frac{3z+1}{z^2}\,dz, \quad C: |z|=1.$

Soln:- $|z|=1 \Rightarrow |x+iy|=1 \Rightarrow x^2+y^2=1^2$ is the Circle of radius $r=1$ and $C(0,0)$.

equating the D^r to 0.

$(z-0)^2=0.$

$\therefore$ $z=0$ lies inside the Circle twice.



we have to apply the derivative formula,

$$f'(a)=\frac{1!}{2\pi i}\int_C \frac{f(z)}{(z-a)^2}\,dz = \frac{1!}{2\pi i}\int_C \frac{3z+1}{z^2}\,dz \rightarrow (1).$$

here $f(z)=3z+1.$

$f'(z)=3$

$\boxed{f'(0)=3}$

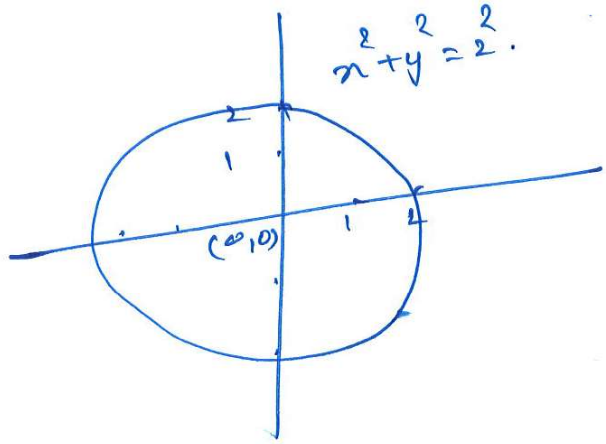
sub in (1).

$$\int_C \frac{3z+1}{z^2}\,dz = 2\pi i\left[f'(0)\right] = 2\pi i[3] = 6\pi i.$$

3. evaluate:- $\displaystyle\int_C \frac{3z^2+2z+5}{(z-1)^3}\,.dz$

where C is the circle $|z| = 2$.

<u>Soln:-</u> $|z| = 2 \Rightarrow x^2+y^2 = 2^2$ is the circle of radius $r=2$ and centre $(0,0)$.



equating the D^r to zero.

$(z-1)^3 = 0$. $\therefore \boxed{z=1,1,1}$. lies inside the circle thrice.

We have to apply Derivative's formula,

$$f''(a) = \frac{\cancel{2}!}{\cancel{2}\pi i}\int_C \frac{f(z)}{(z-a)^3}\,dz. = f''(1) = \frac{1}{\pi i}\int_C \frac{3z^2+2z+5}{(z-1)^3}\,.dz$$

here $f(z) = 3z^2+2z+5$.

$f'(z) = 6z+2$

$f''(z) = 6$.

put $z=1$, $f''(1) = 6$.

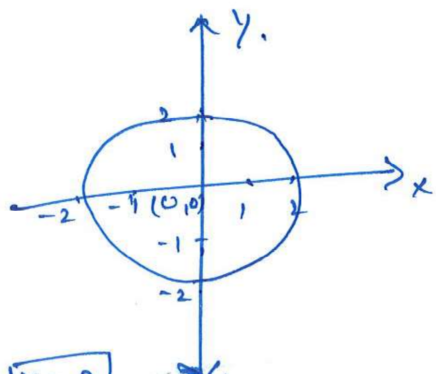
$$f''(1) = \frac{1}{\pi i}\int_C \frac{f(z)}{(z-1)^3}\cdot dz$$

$$(6) = \frac{1}{\pi i}\int_C \frac{3z^2+2z+5}{(z-1)^3}\cdot dz$$

$$\therefore \boxed{6\pi i = \int_C \frac{3z^2+2z+5}{(z-1)^3}\cdot dz}$$

4. $$\int_C \frac{(z+2)}{(z-1)^2(z-3)}\cdot dz, \quad C : |z| = 2.$$

<u>Solution</u>:



$|z| = 2 \Rightarrow |x+iy| = 2.$

$\therefore x^2+y^2 = 2^2$ is the

circle centre $(0,0)$ and $\boxed{r=2}$.

equating the denominator to zero.

$$(z-1)^2\cdot(z-3) = 0$$

$\therefore z=1$ lies inside the circle (twice).

$z=3$ lies outside the outside circle.

Applying the derivative formula,

$$f'(a) = \frac{1!}{2\pi i}\int_C \frac{f(z)}{(z-a)^2}\cdot dz \longrightarrow (1)$$

To find $f'(1)$:-

given $f(z) = \dfrac{z+2}{z-3}$.

$$f'(z) = \frac{(z-3)\cdot 1 - (z+2)(1)}{(z-3)^2}$$

$$f'(z) = \frac{\cancel{z} - 3 - \cancel{z} - 2}{(z-3)^2} = \frac{-5}{(z-3)^2}.$$

$$\therefore f'(1) = \frac{-5}{(1-3)^2} = \frac{-5}{(-2)^2} = \frac{-5}{4}.$$

sub $f'(a) = f'(1) = -5/4$ in (1).

$$-5/4 = \frac{1}{2\pi i} \int_c \frac{z+2}{(z-1)^2 (z-3)} \cdot dz$$

$$-\frac{5}{\cancel{4}_2} \times \cancel{2}\pi i = \int_c \frac{z+2}{(z-1)^2 \cdot (z-3)} \cdot dz$$

$$\therefore \boxed{\int_c \frac{z+2}{(z-1)^2 (z-3)} \cdot dz = -5/2 \pi i}$$

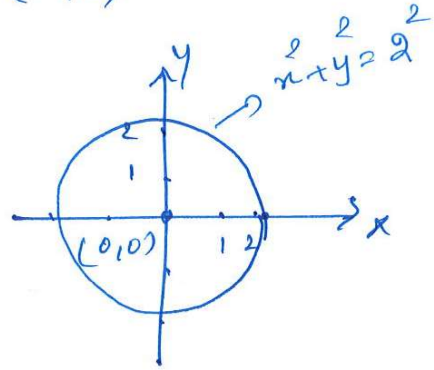
5. evaluate $\int_C \frac{e^{2z}}{(z-1)^3}\,dz$, where C is the circle of $|z|=2$.

<u>Soln:-</u> $|z|=2 \Rightarrow |x+iy|=2 \Rightarrow x^2+y^2=2^2$ is the circle of radius of centre $(0,0)$ and $r=2$.

equating the D^r to 0.

$(z-1)^3=0.$

$\boxed{z=1,1,1}$ lies inside the circle.



Applying derivative formula,

$$f''(a)=\frac{2!}{2\pi i}\int_C \frac{f(z)}{(z-a)^3}\,dz \longrightarrow (1)$$

<u>To find $f''(a)$:-</u>

$$f(z)=e^{2z}$$

$$f'(z)=2.e^{2z}$$

$$f''(z)=2\left[e^{2z}.2\right]=4e^{2z}.$$

$$\therefore f''(a)=4e^{2a}$$

$$\boxed{f''(1)=4e^{2}} \quad \{\because a=1\}.$$

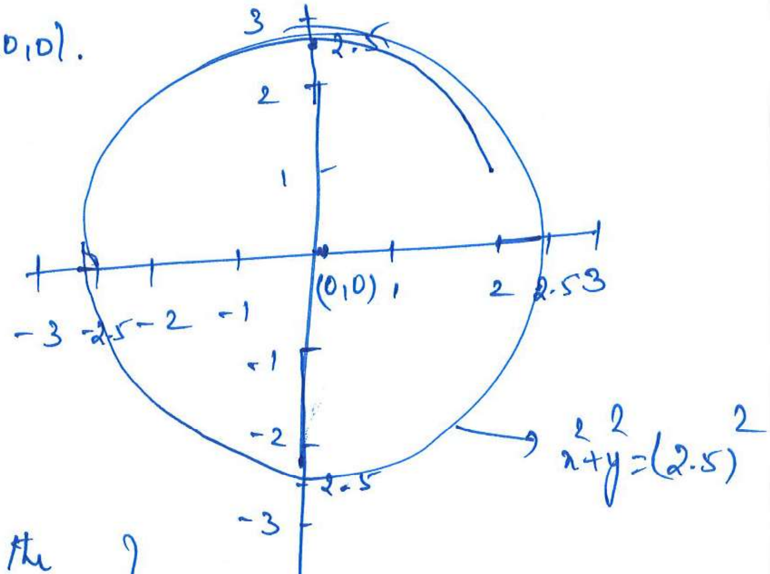
Sub in (1)

$$4e^2 = \frac{\cancel{2}!}{\cancel{2}\pi i}\int_c \frac{e^{2z}}{(z-1)^3}\,dz$$

$$\therefore \boxed{\int_c \frac{e^{2z}}{(z-1)^3}\,dz = 4e^2\,\pi i}$$

6. $\displaystyle\int_c \frac{e^{-z}}{(z+1)^4}\,dz$ where $c : |z| = 5/2$

Soln:- $|z| = 5/2 \Rightarrow x^2+y^2 = (5/2)^2$ is the radius of $r = 2.5$ and centre $(0,0)$.



equating the D^r to zero.

$$(z+1)^4 = 0$$

$\boxed{z=-1}$ lies inside the circle.

$\therefore$ By cauchy's derivative formula,

$$f'''(a) = \frac{3!}{2\pi i}\int_c \frac{f(z)}{(z-a)^4}\,dz.$$

$$f'''(a) = \frac{3!}{2\pi i}\int_C \frac{e^{-z}}{(z+1)^4}\,dz \longrightarrow (1).$$

<u>To find $f'''(a)$:-</u> $\left\{(z-a)^4 = (z+1)^4 \Rightarrow (a=-1)\right\}$.

$$f(z) = e^{-z}, \qquad \Rightarrow f(0) = e^{-z} = e^{-0} = 1$$

$$f'(z) = -e^{-z} \qquad \Rightarrow f'(-1) = -e^{+1} = -e^{+1} = -e$$

$$f''(z) = -\left[-e^{-z}\right] = e^{-z} \Rightarrow f''(-1) = e^{+1} = e.$$

$$f'''(z) = -e^{-z} \qquad \Rightarrow f'''(-1) = -e^{+1} = -e,$$

$\therefore$ sub $\boxed{-e = f'''(-1)}$ in (1).

$$\therefore -e = \frac{\cancel{6}\,3}{\cancel{2}\pi i}\int_C \frac{e^{-z}}{(z+1)^4}\,dz \qquad \left\{\because 3! = 3\times 2\times 1\right\}$$

$\therefore 3! = 6$

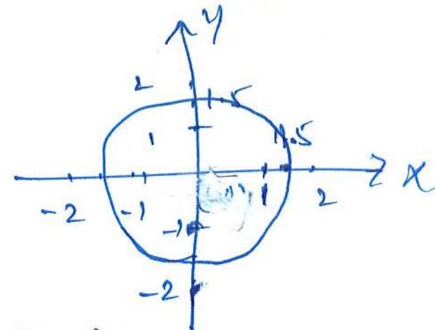
$$\boxed{\frac{-\pi i e}{3} = \int_C \frac{e^{-z}}{(z+1)^4}\,dz}$$

6. evaluate $\int_c \frac{z^2+3}{(z-1)(z+2)^2} \cdot dz$, $|z| = 3/2$.

Soln:- $|z| = 3/2 \Rightarrow |x+iy| = 3/2 \Rightarrow x^2+y^2 = (1.5)^2$.

is the eqn of circle of radius of $r = 1.5$.

equating the D^r to 0.



$(z-1)(z+2)^2 = 0$

$z = 1$ lies inside the circle

and $z = -2$ (twice) lies outside the circle.

$\therefore$ we have to apply cauchy's integral formula,

$$f(a) = \frac{1}{2\pi i} \int_c \frac{f(z)}{z-a} \cdot dz \qquad \{\because a = 1\}.$$

$$f(a) = \frac{1}{2\pi i} \int_c \frac{z^2+3}{(z-1)(z+2)^2} \cdot dz \longrightarrow (1).$$

$\longrightarrow f(z)$.

here $f(z) = \frac{z^2+3}{(z+2)^2}$

$$f(a) = \frac{a^2+3}{(a+2)^2} \Rightarrow f(1) = \frac{1^2+3}{(1+2)^2} = \frac{1+3}{9} = \frac{4}{9}.$$

$\therefore$ Sub in (1)

$$\frac{4}{9} = \frac{1}{2\pi i} \int_c \frac{z^2+3}{(z-1)(z+2)^2} \cdot dz \Rightarrow \frac{8\pi i}{9} = \int_c \frac{z^2+3}{(z-1)(z+2)^2} \cdot dz$$

Singularities :-

Singular point :-

A point at which the function $f(z)$ is not analytic is called a singular point.

Isolated Singularity :-

A singular point $z=a$ is called as an isolated singular point if there is no other singular point very close to it.

i.e., a small circle can be drawn around 'a' (which does not include other singular points).

eg :- Let $f(z) = \frac{1}{z(z+2)}$

Here, the points $z=0$ and $z=-2$ are the points at which the function $f(z)$ is not analytic.

i.e., these are isolated singular points of $f(z)$.

Removable Singularity :-

A singular point $z=a$ is called as removable singularity of $f(z)$ if $\lim_{z \to a} f(z)$ exists finitely.

eg:- $\frac{\sin z}{z} = \frac{1}{z}\left[z - \frac{z^3}{3!} + \frac{z^5}{5!} - \dots\right]$

$$= 1 - \frac{z^2}{3!} + \frac{z^4}{5!} \dots$$

{Using the series of sin ce.}

ie., there is no negative power of z.

$z=0$ is a removable singularity of $f(z)$.

Also $\lim_{z \to 0} f(z) = \lim_{z \to 0} \left(\frac{\sin z}{z}\right) = 1$ finite.

$\therefore z=0$ is a Removable singularity.

Essential singularity:-

In Laurent's series of $f(z)$, if there are infinite number of terms containing negative powers of $z-a$, then $z=a$ is an essential singularity.

Pole:- An isolated singular point 'a' is called as pole.

Example:- Let $f(z) = \frac{z+2}{(z-1)(z-3)}$.

Then the poles are $\boxed{z=1, \ z=3}$

The co-efficient of $\frac{1}{z-a}$ in the expansion of $f(z)$, about the singular point $z=a$ is defined as the residue of $f(z)$ at $z=a$.

Expanding $f(z)$ in terms of positive powers and negative powers of $z-a$ in Laurent's Series, the co-effi of $\frac{1}{z-a}$ is a_{-1}.

By definition,

$$\boxed{a_{-1} = \frac{1}{2\pi i}\int_C f(z).dz.}$$

Cauchy's Residue theorem:-

If $f(z)$ is analytic inside a closed curve except at a finite number of singular points $a_1, a_2 \ldots\ldots a_n$ then,

$$\boxed{\int_C f(z).dz = 2\pi i\,[\text{Sum of the residues}].}$$

Residue calculation:-

Residue at $z=a$ is given by

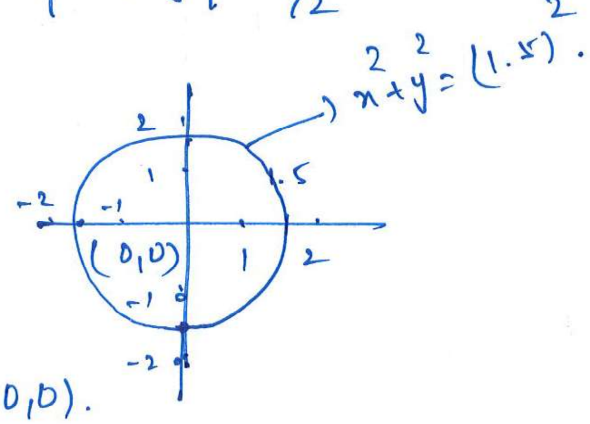
$$\boxed{\left.\begin{array}{l}\text{Resi of}\\ z=a\end{array}\right\} = \lim_{z\to a}(z-a).f(z)}$$

{∵ singular points are called as poles.}.

Problems for the cauchy's Residue's theorem:-

1. $\int_C \frac{z^2+1}{(z-1)(z-2)} \cdot dz$, where $|z| = 3/2$.

Solution :- $|z| = 3/2 \Rightarrow |x+iy| = 3/2$



$\therefore x^2+y^2 = (1.5)^2$ is the equation of circle of radius $r=1.5$ and $C(0,0)$.

equating the D^r to zero.

$(z-1)(z-2)=0$ $\therefore$ $\boxed{z=1,2}$ lies inside the circle.

But $z=2$ lies outside the circle.

Hence, we have to calculate residue of $z=1$ (only).

$\therefore$ **Residue of $z=1$.** :-

$$\lim_{z\to 1}(z-1)\cdot f(z) = \lim_{z\to 1} (z-1)\, f(z)$$

$$= \lim_{z\to 1} \cancel{(z-1)} \cdot \left[\frac{z^2+1}{\cancel{(z-1)}\cdot z-2}\right]$$

$$= \frac{1^2+1}{1-2} = \frac{2}{-1} = -2.$$

$\therefore \boxed{\text{Resi at } z=1 = -2}$.

Hence By cauchy's Residue theorem,

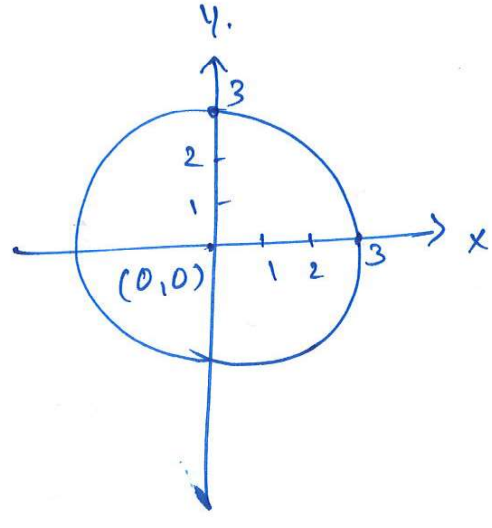
$$\int_C f(z).dz = 2\pi i \left[\text{Resi at } z=1\right].$$

$$= 2\pi i\left[-2\right] = -4\pi i$$

2. $\displaystyle\int_C \frac{\cos\pi z^2 + \sin\pi z^2}{(z-1)(z-2)}.dz$ where C is $|z|=3$.

Soln:- $|z|=3 \Rightarrow |x+iy|=3 \Rightarrow x^2+y^2=3^2$ is the equation of Circle radius $r=3$ and $C(0,0)$.

Soln:- $|z|=3$



equating the D^r to zero}.

$\therefore (z-1)(z-2)=0$

$\therefore z=1,2$ lies inside the Circle.

we have to calculate the residue at $z=1,2$.

Resi at $z=1$:-

$$\lim_{z\to1}(z-1).f(z) = \lim_{z\to1}\cancel{(z-1)}.\left[\frac{\cos\pi z^2+\sin\pi z^2}{\cancel{(z-1)}(z-2)}\right].$$

$$= \lim_{z \to 1} \left[\frac{\cos \pi z^2 + \sin \pi z^2}{(z-2)} \right]$$ $\boxed{6}$.

$$\left.\begin{array}{l}\text{Resi at} \\ z=1\end{array}\right\} = \frac{\cos \pi \cdot 1^2 + \sin \pi \cdot 1^2}{1-2} = \frac{-1+0}{-1} = 1.$$

To find the residue at $z=2$:-

$$\lim_{z \to 2} (z-2) \cdot f(z) = \lim_{z \to 2} \cancel{(z-2)} \cdot \left[\frac{\cos \pi z^2 + \sin \pi z^2}{(z-1)\cancel{(z-2)}} \right]$$

$$\left\{\begin{array}{l} \because \cos 4\pi = 1 \\ \sin 4\pi = 0 \\ \cos \pi = -1 \\ \sin \pi = 0 \end{array}\right\} = \frac{\cos \pi \cdot 2^2 + \sin \pi \cdot 2^2}{2-1}$$

$$= \frac{\cos 4\pi + \sin 4\pi}{1} = \frac{1+0}{1} = 1.$$

$$\therefore \left.\begin{array}{l}\text{Residue at} \\ z=2\end{array}\right\} = 1.$$

Hence By cauchy's Residue's theorem,

$$\int_C f(z) \cdot dz = 2\pi i \left[\text{Sum of the Residues.} \right]$$

$$= 2\pi i [1+1] = 2\pi i [2] = 4\pi i.$$

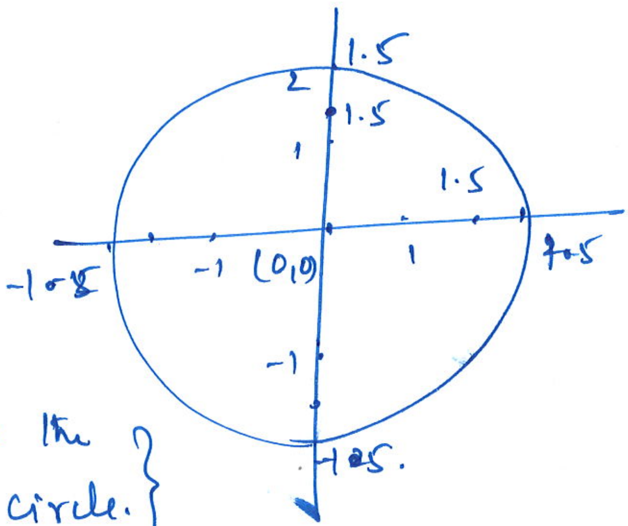
3. $\int_C \frac{7z-2}{z(z-1)(z-3)} \cdot dz$, $C : |z-1| = 3/2$

evaluate Using Cauchy Residue's theorem.

Soln:- $|z-1| = 3/2$

$$|x+iy-1| = 3/2 \Rightarrow |x-1+iy| = 3/2$$

$\therefore (x-1)^2 + y^2 = (3/2)^2$ is the equation of a circle with centre $(1,0)$ and radius $r = 1.5$.



equating the D^r to 0.

$z(z-1)(z-3) = 0$.

$z = 0, 1$ lies inside the circle.

and $z = 3$ lies inside the given Circle.

$\therefore$ we have to find Out the residue at $z=0, z=1$ only.

Case (1):- Resi at $z=0$

$$\left.\begin{matrix}\text{Resi at}\\ z=0\end{matrix}\right\} = \lim_{z\to 0} (z-0)\cdot f(z) = \lim_{z\to 0} (\cancel{z}) \cdot \left[\frac{7z-2}{\cancel{z}(z-1)(z-3)}\right]$$

$$= \lim_{z \to 0} \left[\frac{7z-2}{(z-1)(z-3)} \right]$$

$$= \frac{0-2}{(0-1)(0-3)} = \frac{-2}{3}.$$

<u>Case ②</u> :- Resi at $z=1$.

$$\lim_{z \to 1} (z-1) \cdot f(z) = \lim_{z \to 1} (\cancel{z-1}) \cdot \left[\frac{7z-2}{z(\cancel{z-1})(z-3)} \right].$$

$$= \lim_{z \to 1} \left[\frac{7z-2}{z(z-3)} \right].$$

$$\left.\begin{array}{l}\text{Resi at} \\ z=1\end{array}\right\} = \frac{7(1)-2}{1(1-3)} = \frac{7-2}{-2} = \frac{5}{-2}.$$

$$\int_C \frac{7z-2}{z(z-1)(z-3)} \cdot dz = 2\pi i \left[\text{Sum of the Residues} \right].$$

$$= 2\pi i \left[\text{Resi at } z=0 + \text{Resi at } z=1 \right]$$

$$= 2\pi i \left[-2/3 - 5/2 \right]$$

$$= -2\pi i \left[2/3 + 5/2 \right]$$

$$= -\cancel{2}\pi i \left[\frac{4+15}{\cancel{6}_3} \right] = -\frac{\cancel{2}\pi i}{\cancel{6}_3} [19]$$

$$\int_C \frac{7z-2}{z(z-1)(z-3)} \cdot dz = -19\pi i / 3$$

Residue's for Multiple poles:-

Case ①:- For a pole of order 2, the residue is given by

$$\lim_{z \to a} \frac{1}{1!} \frac{d}{dz} (z-a)^2 \cdot f(z)$$

Case ②:- For a pole of 3, the residue is given by

$$\lim_{z \to a} \frac{1}{2!} \frac{d^2}{dz^2} \cdot (z-a)^3 \cdot f(z)$$

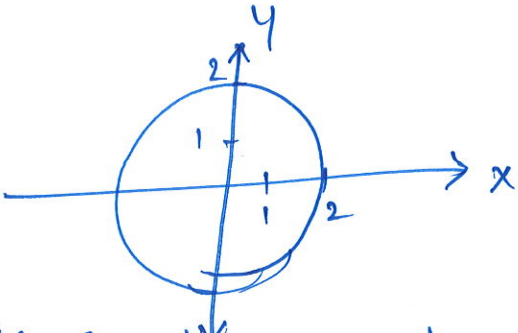
Case ③:- For a pole of order 4, the residue is given by

$$\lim_{z \to a} \frac{1}{3!} \frac{d^3}{dz^3} (z-a)^4 \cdot f(z)$$

Problems for Residue's at Multiple poles:-

1. Evaluate:- $\int_C \frac{3z^2+5z-1}{(z-1)^2} \cdot dz$ where $C :- |z| = 2$

Soln:- $|z| = 2 \Rightarrow x^2 + y^2 = 2^2$ is the eqn of a circle of radius $r = 2$ and $C(0,0)$.



equating the D^r to 0.

$(z-1)^2 = 0$.

$\boxed{z = 1, 1}$ lies twice, is a pole of order 2.

since, it's a pole of order 2.

$\therefore$ we have to apply the cauchy's residue's theorem.

$\therefore$ Resi at $z=1 = \lim_{z\to 1} \frac{1}{1!} \frac{d}{dz}\left[(z-1)^2 \cdot f(z)\right]$

$$= \lim_{z\to 1} \cdot \frac{d}{dz}\left[\cancel{(z-1)^2} \cdot \left(\frac{(3z^2+5z-1)}{\cancel{(z-1)^2}}\right)\right]$$

$$= \lim_{z\to 1} \cdot \frac{d}{dz}\left[(3z^2+5z-1)\right]$$

$$= \lim_{z\to 1}\left[6z+5\right] = 6(1)+5 = 11.$$

By cauchy's theorem,

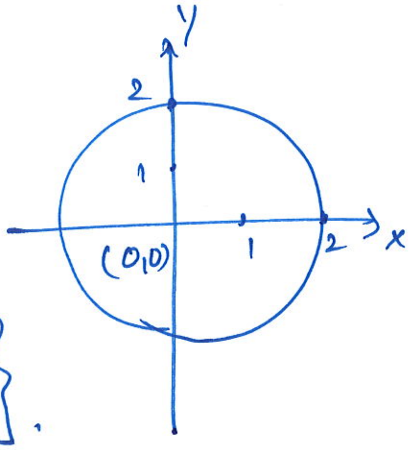
$$\int_C \frac{3z^2+5z-1}{(z-1)^2} \cdot dz = 2\pi i\left[\text{Resi at } z=1,1\right].$$

$$= 2\pi i\left[11\right]$$

$$= 22\pi i$$

2. $\int_C \frac{z^2+1}{z^2(z-1)} \cdot dz$ at $|z|=2$. $\boxed{11}$

Soln:- $|z|=2 \Rightarrow x^2+y^2=2^2$ is the eqn of a Circle of radius of $r=2$, Centre $(0,0)$.



equating the D^r to zero.

$z^2(z-1)=0$.

$z=0,0$ lies inside the Circle (twice).

$z=1$ lies inside the Circle $|z|=2$.

Hence we've to find the Residue's at $z=0, z=1$.

Case ①:- Resi at $z=0$ $= \lim\limits_{z\to 0} \frac{1}{1!}\frac{d}{dz}\left[(z-0)^2 \cdot f(z).\right]$

$$= \lim_{z\to 0} \frac{d}{dz}\left[\cancel{z^2}\cdot\frac{(z^2+1)}{\cancel{z^2}(z-1)}\right]$$

$$= \lim_{z\to 0} \cdot \frac{d}{dz}\left[\frac{z^2+1}{z-1}\right].$$

$$= \lim_{z\to 0}\left[\frac{(z-1)(2z)-(z^2+1)(1)}{(z-1)^2}\right]$$

$$= \left[\frac{(0-1)(\cdot 0)-(0+1)}{(0-1)^2}\right]$$

$$\frac{0-1}{1} = -1.$$

$\therefore$ $\boxed{\text{Resi at } z=0 = -1.}$

Case (2) :- Resi at $z=1 = \lim_{z\to 1} (z-1).f(z)$

$$= \lim_{z\to 1} \cancel{(z-1)}\left[\frac{z^2+1}{z^2\cancel{(z-1)}}\right].$$

$$= \lim_{z\to 1}\left[\frac{z^2+1}{z^2}\right]$$

$$= \frac{1^2+1}{1^2} = 2.$$

$\therefore$ By Cauchy's residue's theorem,

$$\int_c f(z).dz = 2\pi i\,[\text{Sum of the Residues}].$$

$$= 2\pi i\,[\text{Resi of } z=0 + \text{Resi at } z=1]$$

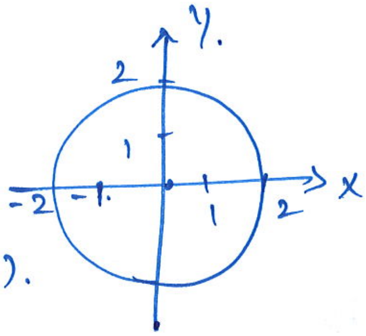
$$= 2\pi i\,[-1+2]$$

$$= 2\pi i\,[1]$$

$$\boxed{\int_c \frac{z^2+1}{z^2(z-1)}\,dz = 2\pi i}$$

8. $\int_C \frac{e^{2z}}{(z+1)^3} \cdot dz$, $C: |z| = 2$ $\boxed{13}$

Soln:- $|z| = 2 \Rightarrow x^2+y^2 = 2^2$ is the equation of circle of radius of $r = 2$ and centre $(0,0)$.



equating the D^r to zero.

$(z+1)^3 = 0 \Rightarrow z = -1$ (thrice).

$\therefore z = -1$ lies inside the circle and $z = -1$ is a pole order 3.

$\therefore$ By cauchy's Residue Calculation:-

$$\left.\begin{matrix}\text{Resi}\\ \text{at } z=a\end{matrix}\right\} = \lim_{z \to a} \cdot \frac{1}{2!} \frac{d^2}{dz^2}\left[(z-a)^3 \cdot f(z)\right] \quad \left\{\because \text{pole of order } 3\right\}.$$

$$\therefore \left.\begin{matrix}\text{Resi at}\\ z=-1\end{matrix}\right\} = \lim_{z \to -1} \frac{1}{2!} \frac{d^2}{dz^2}\left[\cancel{(z+1)^3} \cdot \frac{e^{2z}}{\cancel{(z+1)^3}}\right]$$

$$= \lim_{z \to -1} \frac{1}{2!} \frac{d^2}{dz^2} \left\{e^{2z}\right\}.$$

$$= \lim_{z \to -1} \frac{1}{2} \cdot \frac{d}{dz}\left[\frac{d}{dz} \cdot e^{2z}\right]$$

$$= \lim_{z \to -1}\left(\frac{1}{2} \cdot \frac{d}{dz}\left\{2.e^{2z}\right\}\right) = \lim_{z \to -1} \frac{1}{2}\left\{2^2 . e^{2z}\right\}$$

$$\lim_{z \to -1} 2e^{2z} = 2e^{-2}.$$

∴ By Cauchy's residue's theorem,

$$\int_C f(z)\cdot dz = \int_C \frac{e^{2z}}{(z+1)^2}\,dz = 2\pi i\left[\text{sum of the Residues at } z=-1\right]$$

$$= 2\pi i\left[2e^{-2}\right].$$

$$= 4\pi i\, e^{-2}.$$

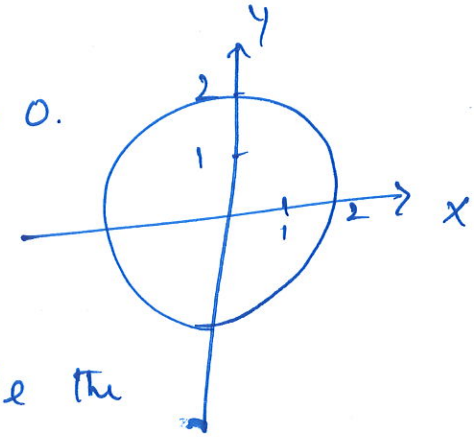
---

4. $\int_C \frac{dz}{z^3(z+4)}$, $C: |z|=2$.

<u>Given:</u> $|z|=2 \Rightarrow |x+iy|=2 \Rightarrow x^2+y^2=2^2$ is the equation of radius $r=2$, and centre $(0,0)$.

<u>Soln:</u> equating the D^r to 0.

$z^3(z+4)=0$.



∴ $z=0,0,0$ lies inside the circle (thrice).

∴ $z=-4$ lies outside the circle.

∴ We have to find the Residue at $z=0$ is a pole of order 3.

∴ Resi at $z=0$:-

$$= \lim_{z\to 0} \frac{1}{2!} \frac{d^2}{dz^2}\left\{(z-0)^3 \cdot f(z)\right\} = \left.\text{Resi at}\right\}_{z=0}.$$

$$= \lim_{z\to 0} \frac{1}{2} \cdot \frac{d^2}{dz^2}\left[\cancel{z^3} \cdot \frac{1}{\cancel{z^3}(z+4)}\right]$$

$$= \lim_{z\to 0} \cdot \frac{1}{2} \frac{d^2}{dz^2}\left[\frac{1}{z+4}\right].$$

$$= \lim_{z\to 0}\left[\frac{1}{2} \cdot \frac{d}{d}\left[\cdot \frac{d}{dz}\left(\frac{1}{z+4}\right)\right]\right]$$

$$= \lim_{z\to 0}\left[\frac{1}{2} \cdot \frac{d}{dz}\left[\frac{-1}{(z+4)^2}\right]\right].$$

$$= \lim_{z\to 0}\left[\frac{1}{\cancel{2}} \cdot \left[\frac{\cancel{2}}{(z+4)^3}\right]\right] \quad \left\{\because \frac{d}{dx}(1/x) = -1/x^2\right\}$$

$$\left\{\frac{d}{dx}\left(-1/x^2\right) = 2/x^3\right\}.$$

$$= \lim_{z\to 0}\left[\frac{1}{(z+4)^3}\right].$$

$$= \frac{1}{(0+4)^3} = \frac{1}{64}.$$

∴ By Cauchy's Residue theorem,

$$\int_C \frac{dz}{z^3(z+4)} \cdot dz = 2\pi i\,[\text{Resi at } z=0] = 2\pi i\left[\frac{1}{64}\right] = \frac{\pi i}{32}.$$

Contour Integration :-

Introduction:- Certain real integrals can be evaluated using Complex integration technique. The closed curves considered for this purpose are called as Contours.

Type - ① :- Integral of the form $\int_{-\infty}^{\infty} f(x).dx$.

For this type of integrals, consider a large semi-circle Γ and above x-axis with radius R.

Let f(z) be the function by replacing x by z. Find the poles as usual.

All the points, i.e., positive poles will be inside and negative poles will be outside. Calculate the residue for the poles, which are inside an apply Cauchy's residue theorem.

$$\therefore \boxed{\int_C f(z).dz = \int_{-R}^{R} f(x).dx + \int_{\Gamma} f(z).dz.}$$

Type-I. An integral of the form $\int_{-\infty}^{\infty} f(x).dx$

1. evaluate : $\int_{-\infty}^{\infty} \frac{1}{(x^2+1)(x^2+4)} . dx$

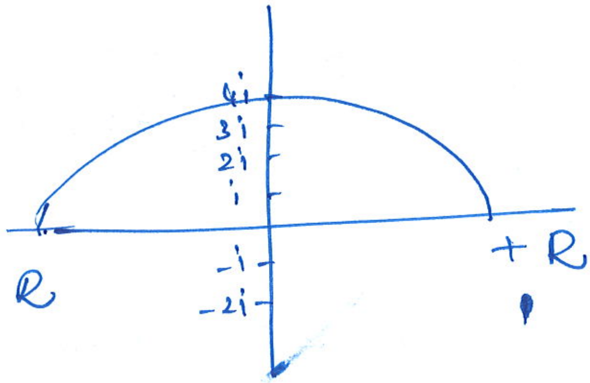
Soln:- $f(z) = \frac{1}{(z^2+1).(z^2+4)}$

poles are at

$z^2+1=0$ | $z^2+4=0$

$z^2=-1$ | $z^2=-4$

$\boxed{z=\pm i}$ | $\boxed{z=\pm 2i}$



$\therefore$ The poles are $i, -i, 2i$ and $-2i$.

Here the poles $i, 2i$ inside the semi-circle.

and $z = -i, -2i$ lies outside the semi-circle.

$\Rightarrow$ we have to calculate Residue at $z = i, 2i$ only.

Case ①:- Resi at $z=i$ $\Big\} = \lim_{z \to i} (z-i) . f(z)$

$$= \lim_{z \to i} (z-i) \left[\frac{1}{(z^2+1)(z^2+4)} \right]$$

$$= \lim_{z \to i} (z-\cancel{i}) \cdot \left[\frac{1}{(z+i)(z-\cancel{i})(z^2+4)} \right].$$

$$\left\{ \because \ z^2+1 = z^2 - -1 = (z+i)(z-i) \right\}$$

$$= \frac{1}{(i+i)(i^2+4)} = \frac{1}{2i(-1+4)} = \frac{1}{3 \times 2i}$$

$$= \frac{1}{6i}$$

Case ②

$$\left.\begin{array}{l} \text{Resi at} \\ z = 2i \end{array}\right\} = \lim_{z \to 2i} (z-2i) \cdot f(z)$$

$$= \lim_{z \to 2i} (z-\cancel{2i}) \left[\frac{1}{(z^2+1)(z+2i)(z-\cancel{2i})} \right]$$

$$= \lim_{z \to 2i} \left[\frac{1}{(z^2+1)(z+2i)} \right]$$

$$\left.\begin{array}{l} (2i)^2 = 4 \times -1 \\ = -4 \end{array}\right\} = \frac{1}{[(2i)^2+1][2i+2i]}.$$

$$= \frac{1}{(-4+1)(4i)} = \frac{-1}{12i}.$$

$\therefore$ By Cauchy's residue's theorem,

$$\int_C f(z).\,dz = \int_{-\infty}^{\infty} f(x).\,dx = 2\pi i\,[\text{sum of the residues}].$$

$$= 2\pi i\left[\frac{1}{6i} + \frac{-1}{12i}\right]$$

$$= 2\pi \cancel{i}\left[\frac{2-1}{12\cancel{i}}\right]$$

$$= 2\pi\cdot\left(\frac{1}{12}\right)$$

$$= \pi/6.$$

2. Evaluate:- $\displaystyle\int_0^{\infty} \frac{1}{1+x^2}.\,dx$

Soln:- $\displaystyle\int_0^{\infty} \frac{1}{1+x^2}.\,dx = \frac{1}{2}\int_{-\infty}^{\infty}\frac{1}{1+x^2}.\,dx$

here $f(z) = \dfrac{1}{1+z^2}$.

poles at $1+z^2=0 \Rightarrow z^2=-1 \quad \therefore \boxed{z=\pm i}$.

The poles are at $i, -i$. Here the pole i lies inside the semi-circle and $z=-i$ lies outside the circle.

Residue at $z=i$:-

$$\lim_{z\to i}(z-i)\,f(z) = \lim_{z\to i}(z-i)\left[\frac{1}{z^2+1}\right].$$

$$= \lim_{z\to i}\cancel{(z-i)}\left[\frac{1}{(z+i)\cancel{(z-i)}}\right]$$

$$= \frac{1}{i+i} = \frac{1}{2i}$$

∴ By cauchy's Residue's theorem,

$$\int_0^\infty\left(\frac{1}{1+x^2}\right).dx = \frac{1}{\cancel{2}}\left[\cancel{2}\pi i\right].\left(\frac{1}{2i}\right)$$

$$= \frac{\pi i}{2i} = \pi/2.$$

$$\therefore \boxed{\int_0^\infty\left(\frac{1}{1+x^2}\right).dx = \pi/2.}$$

3. $\displaystyle\int_0^\infty \frac{1}{(1+x^2)^2}.dx = \frac{1}{2}\int_{-\infty}^\infty \frac{1}{(1+x^2)^2}.dx$

Let $f(z) = \dfrac{1}{(1+z^2)^2}.$

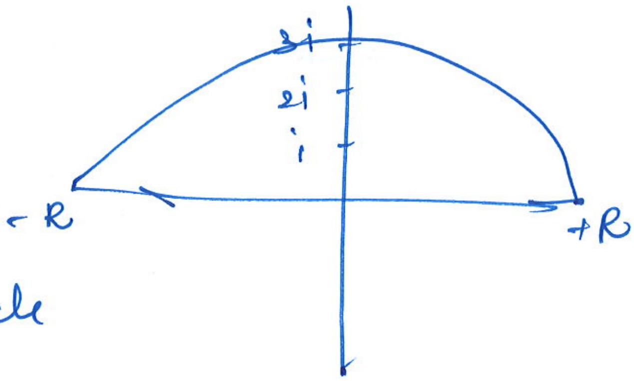
The poles are at $z^2+1=0$.

$z^2=-1 \Rightarrow \boxed{z=\pm i}$

$z^2+1=0$

$\boxed{z=\pm i}$

$z = \pm i \;,\; z = \pm i$



$\therefore z = i, i$ lies inside the semi-circle and is a pole of order 2.

$\therefore$ Residue at $z = i, i$:-

$$\left.\begin{matrix}\text{Resi at} \\ z = i\end{matrix}\right\} = \lim_{z \to i} \frac{d}{dz}\left[(z-i)^2 \cdot f(z)\right]$$

$$= \lim_{z \to i} \frac{d}{dz}\left[(z-i)^2 \cdot \frac{1}{(z^2+1)^2}\right]$$

$$\because \left\{z^2+1 = (z+i)(z-i)\right\} = \lim_{z \to i} \frac{d}{dz}\left[(z-i)^2 \cdot \frac{1}{\left[(z+i)(z-i)\right]^2}\right]$$

$$= \lim_{z \to i} \frac{d}{dz}\left[\cancel{(z-i)^2} \cdot \frac{1}{\cancel{(z-i)^2} \cdot (z+i)^2}\right]$$

$$= \lim_{z \to i}\left[\frac{-2}{(z+i)^3}\right] = \frac{-2}{(i+i)^3}$$

$$\left\{\frac{1}{i^3} = \frac{1}{-i} \;\; \cancel{+i} \quad \text{and } i^3 = -i\right\}$$

$$= \frac{-2}{(2i)^3} = \frac{-2}{-8i} = \frac{1}{4i}$$

$\therefore$ By Cauchy's Residue's theorem,

$$\int_0^\infty \frac{1}{(1+x^2)^2}\,dx = \frac{1}{\cancel{2}}\cdot\cancel{2}\pi i\left[\frac{1}{4i}\right] = \pi/4.$$

<u>Type ②</u> :- Integral of the form

$$\int_{-\infty}^{\infty} f(x, \sin x, \cos x)\cdot dx.$$

Sin x and cos x are present only in the Numerator. For this type of integrals, we have to write cos ax as real part of e^{iax} and sin ax as imaginary part of e^{iax} and apply the method as type 1.

1- evaluate :- $\displaystyle\int_{-\infty}^{\infty} \frac{\cos x}{1+x^2}\,dx.$

<u>Soln</u>:- $$\int_{-\infty}^{\infty} \frac{\cos x}{1+x^2}\,dx = \int_{-\infty}^{\infty} \frac{R.P \text{ of } (e^{ix})}{1+x^2}\,dx$$

$$\{\because e^{i\theta} = \cos\theta + i\sin\theta\}.$$

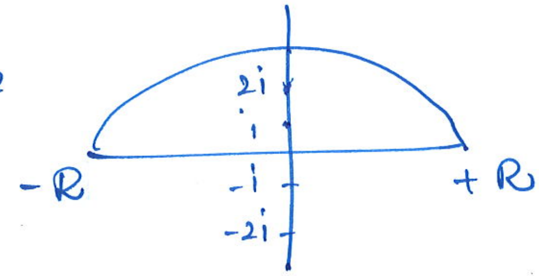
$$= R.P \int_{-\infty}^{\infty}\left(\frac{e^{ix}}{1+x^2}\right).dx$$

Now, $f(z) = \dfrac{e^{iz}}{1+z^2}.$

The poles are at $1+z^2=0$.

$$z^2=-1. \quad \therefore \boxed{z=\pm i}$$

$z=i$ lies inside the Circle and $z=-i$ lies outside the Circle.



The point $z=i$ lies inside the Semi-Circle and the point $z=-i$ lies outside the semi-Circle.

$\therefore$ we've to calculate the residue at $z=i$ only.

Resi at $z=i$:-

$$\left.\begin{array}{l}\text{Resi at}\\ z=i\end{array}\right\} = \lim_{z\to i}(z-i)\,f(z)$$

$$= \lim_{z\to i}(z-i)\left[\frac{e^{iz}}{1+z^2}\right].$$

$$= \lim_{z\to i}\cancel{(z-i)}\cdot\left[\frac{e^{iz}}{(z+i)\cancel{(z-i)}}\right]$$

$$= \frac{e^{i.i}}{i+i} = \frac{e^{i^2}}{2i} = \frac{e^{-1}}{2i}.$$

$\therefore$ By Cauchy's Residue's theorem,

$$\int_{-\infty}^{\infty} \frac{\cos x}{1+x^2} \cdot dx = R.P \text{ of } 2\pi i \left[\begin{array}{c} \text{Sum of the} \\ \text{Residues} \end{array} \right]$$

$$= R.P \text{ of } 2\pi i \left[\frac{e^{-1}}{2i} \right]$$

$$= R.P \left[\pi e^{-1} \right] = \pi e^{-1}.$$

---

2. Evaluate :- $\displaystyle\int_{-\infty}^{\infty} \frac{\sin x}{1+x^2} \cdot dx$

<u>Soln</u>:- $$\int_{-\infty}^{\infty} \frac{\sin x}{1+x^2} \cdot dx = \int_{-\infty}^{\infty} \frac{I.P \text{ of } e^{ix}}{1+x^2} \cdot dx$$

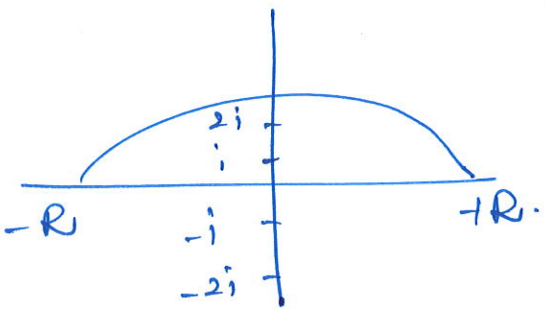
$$= I.P \int_{-\infty}^{\infty} \frac{e^{ix}}{1+x^2} \cdot dx$$

Consider $f(z) = \dfrac{e^{iz}}{1+z^2}$.

The poles are $1+z^2 = 0$.

$\therefore z^2 = -1$. $\boxed{z = \pm i}$.

$\therefore$ The point $z = i$ lies outside the Semi-Circle and the point $z = -i$ lies outside the Semi-Circle.



$\therefore$ Residue at $z=i$ $\Big\} = \lim_{z\to i} (z-i)\, f(z)$

$$= \lim_{z\to i} \cancel{(z-i)} \left[\frac{e^{iz}}{(z+i)\cancel{(z-i)}} \right]$$

$$= \frac{e^{i.i}}{i+i} = \frac{e^{i^2}}{2i} = \frac{e^{-1}}{2i} \quad \{\because i^2=-1\}.$$

$\therefore$ By cauchy's Residue's theorem,

$$\int_{-\infty}^{\infty} \frac{\sin x}{1+x^2}\,.dx = \text{I.P of } 2\pi i \left[\text{Resi at } z=i \right]$$

$$= \text{I.P} \left[2\pi i \left[\frac{e^{-1}}{2i} \right] \right]$$

$$= \text{I.P.} \left[\pi e^{-1} \right] = \text{I.P} \left[\pi e^{-1} + 0i \right]$$

$\because \left\{ \begin{array}{l} \text{I.P of } a+i0 = 0 \\ \text{R.p of } 0+ib = 0 \end{array} \right\} \therefore$ $\boxed{\int_{-\infty}^{\infty} \left(\frac{\sin x}{1+x^2} \right) dx = 0.}$

3. evaluate :- $\int_0^\infty \left(\frac{x\sin x}{x^2+4}\right).dx$

Soln:- $\int_0^\infty \frac{x\sin x}{x^2+4}).dx = \frac{1}{2}\int_{-\infty}^{\infty}\left(\frac{I.P\ x[e^{ix}]}{x^2+4}\right).dx$

$\left\{ \because \begin{array}{l} \sin x = I.P \text{ of } e^{ix} \\ e^{ix} = \cos x + i\sin x \end{array} \right\}$

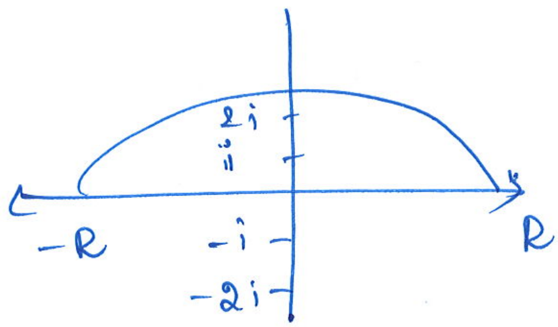
consider $f(z) = \frac{ze^{iz}}{z^2+4}$.

The poles are $z^2+4=0$, $z^2=-4$.

$$\boxed{z = \pm 2i}$$

$\therefore$ The point $z=2i$ lies inside the semi-circle.

and the point $z=-2i$ lies outside the semi-circle.



Resi at $z=2i$:-

$$\lim_{z\to 2i}(z-2i)\,f(z) = \lim_{z\to 2i}(z-2i)\cdot\left[\frac{ze^{iz}}{z^2+4}\right]$$

$$= \lim_{z\to 2i}\cancel{(z-2i)}\cdot\left[\frac{ze^{iz}}{(z+2i)\cancel{(z-2i)}}\right]$$

$$= \frac{2i\left(e^{i(2i)}\right)}{(2i+2i)} = \frac{\cancel{2i}\,e^{2i^2}}{\cancel{4i}_{2}}$$

$$= \frac{1}{2}e^{2(-1)}$$

$$= \frac{e^{-2}}{2}.$$

$\therefore$ By cauchy's Residue's theorem,

$$\int_{-\infty}^{\infty}\left(\frac{x\sin x}{x^2+4}\right).dx = \text{I.P of } \tfrac{1}{\cancel{2}}\cdot\cancel{2}\pi i\left(\begin{array}{l}\text{Sum of the}\\\text{Residue}\\\text{at } z=2i\end{array}\right)$$

$$= \text{I.P}\left[\pi i\left[\frac{e^{-2}}{2}\right)\right]$$

$$\int_{-\infty}^{\infty}\left(\frac{x\sin x}{x^2+4}\right)dx = \pi\frac{e^{-2}}{2}.$$

type-3 :- Integrals of the form.

$$\int_0^{2\pi} f(\sin\theta, \cos\theta) \cdot d\theta.$$

For this type of integral, Consider a unit Circle with centre Origin.

All the poles less than 1 (numerically) will be inside the circle and poles greater than 1 (numerically) will be outside the Circle.

Calculate the residue for poles inside the Circle. Apply cauchy's residue theorem, to find the value of the integral

put $z = e^{i\theta}$.

$$dz = e^{i\theta} [i d\theta].$$

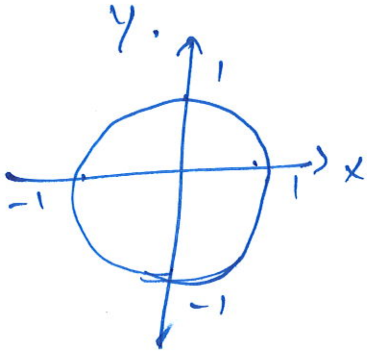
$$\frac{dz}{ie^{i\theta}} \Rightarrow d\theta = \frac{dz}{iz} \quad \{\because z = e^{i\theta}\}.$$

Also, $\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2} = \frac{z + \frac{1}{z}}{2} = \frac{z^2+1}{2z}.$

$$\sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2} = \frac{z - \frac{1}{z}}{2i} = \frac{z^2-1}{2zi}.$$

$\therefore \cos\theta = \frac{z^2+1}{2z}, \quad \sin\theta = \frac{z^2-1}{2zi}.$

1. evaluate :- $\int_0^{2\pi} \frac{d\theta}{5-3\cos\theta} \rightarrow (1)$



Soln :- put $z = e^{i\theta}$

$$d\theta = \frac{dz}{iz}.$$

and $\cos\theta = \frac{z^2+1}{2z}.$

$\therefore$ from (1)

$$\int_0^{2\pi} \frac{d\theta}{5-3\cos\theta} = \int_c \frac{dz/iz}{5-3\left[\frac{z^2+1}{2z}\right]}.$$

$$= \int_c \frac{dz \times 2\cancel{z}}{i\cancel{z}\left[10z-3z^2-3\right]}$$

$$= \frac{2}{i}\int_c \frac{dz}{-3z^2+10z-3}$$

$$= \frac{2}{i}\int_c \frac{dz}{-3\left[z^2+\frac{10z}{-3}+1\right]}$$

$$= \frac{-2}{3i}\int_c \frac{dz}{\left(z^2-\frac{10}{3}z+1\right)}.$$

$\boxed{30}$

The poles are at $z^2 - \frac{10}{3} z + 1 = 0$.

$$z = \frac{-b \pm \sqrt{b^2 - 4ac}}{2 \times a} \quad \left\{ \because a = 1,\ b = \frac{-10}{3},\ c = 1 \right\}.$$

$$= \frac{\frac{10}{3} \pm \sqrt{\frac{100}{9} - 4 \times 1 \times 1}}{2 \times 1}.$$

$$= \frac{\frac{10}{3} \pm \sqrt{\frac{100 - 36}{9}}}{2} = \frac{\frac{10}{3} \pm \sqrt{\frac{64}{9}}}{2}$$

$$= \frac{\frac{10}{3} \pm 8/3}{2}$$

$$= \frac{\frac{10}{3} + 8/3}{2},\ \frac{\frac{10}{3} - 8/3}{2}.$$

$$= \frac{18/3}{2},\ \frac{2/3}{2}$$

$$= 6/2,\ 2/6$$

$$\boxed{z = 3, 1/3.}$$

$\therefore$ The pole $z = 1/3$ lies inside the circle.

and the pole $z = 3$ lies outside the circle.

Hence, we have to calculate the residue at $z = 1/3$ only.

$\therefore$ Resi at $z = 1/3$:-

$$\lim_{z \to 1/3} (z - 1/3) f(z) = \lim_{z \to 1/3} (z - 1/3) \left[f(z) \right]$$

$$= \lim_{z \to 1/3} \cancel{(z - 1/3)} \left[\frac{1}{(z-3)\cancel{(z - 1/3)}} \right]$$

$$= \lim_{z \to 1/3} \left[\frac{1}{z-3} \right].$$

$$= \frac{1}{1/3 - 3} = \frac{1}{\frac{1-9}{3}} = \frac{1}{\frac{-8}{3}} = \frac{-3}{8}.$$

$\therefore$ By Cauchy's Residue theorem,

$$\int_0^{2\pi} \frac{d\theta}{5 - 3\cos\theta} = \frac{-2}{3i} 2\pi i \left[-3/8 \right] = \pi/2.$$

2. evaluate :- $\int_0^{2\pi} \frac{d\theta}{5+4\sin\theta}$ $\rightarrow$ (1).

$\boxed{32}$

Let $z = e^{i\theta}$, $d\theta = \frac{dz}{iz}$.

$\sin\theta = \frac{z^2-1}{2zi}$.

$$\int_0^{2\pi} \frac{d\theta}{5+4\sin\theta} = \int_c \frac{dz/iz}{5+\cancel{4}\left[\frac{z^2-1}{\cancel{2}2zi}\right]}$$

$$= \int_c \frac{dz/iz}{\frac{5\cancel{z}i+[2z^2-2]}{2i}}$$

$$= \int_c \frac{dz \times \cancel{z}i}{i\cancel{z}\left[5zi+2z^2-2\right]}$$

$$= \int_c \frac{dz}{2z^2+5zi-2}$$

$$= \int_c \frac{dz}{2(z^2+\frac{5}{2}zi-1)}.$$

$$= \frac{1}{2}\int_c \frac{dz}{(z^2+\frac{5}{2}zi-1)}$$

$\therefore$ The poles at $z^2 + \frac{5}{2} zi - 1 = 0$.

$$z = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \quad \left\{ a = 1,\ b = \frac{5i}{2},\ c = -1 \right\}.$$

$$= \frac{-\frac{5i}{2} \pm \sqrt{\frac{25}{4} i^2 - 4 \times 1 \times -1}}{2 \times 1}$$

$$= \frac{-\frac{5i}{2} \pm \sqrt{-\frac{25}{4} + 4}}{2}$$

$$= \frac{-\frac{5i}{2} \pm \sqrt{\frac{-25 + 16}{4}}}{2}$$

$$= \frac{-\frac{5i}{2} \pm \sqrt{\frac{-9}{4}}}{2}$$

$$= \frac{-\frac{5i}{2} \pm \frac{3}{2} i}{2}$$

$$= \frac{-\frac{5i}{2} + \frac{3i}{2}}{2}, \quad \frac{-\frac{5i}{2} - \frac{3i}{2}}{2}$$

$$= \frac{-2i/2}{2}, \quad \frac{-8i/2}{2} = \frac{-2i}{4}, \quad \frac{-8i}{4}$$

$$= \frac{-i}{2}, \quad -2i$$

$\therefore$ The poles are $z = -\frac{i}{2}$, $z = -2i$

Here, the pole $z = -\frac{i}{2}$ lies inside the circle and the pole $z = -2i$ lies outside the circle.

We're to calculate the residue at $z = -\frac{i}{2}$ (only).

$\therefore$ Residue at $z = -\frac{i}{2}$:-

$$\therefore \lim_{z \to -i/2} (z + i/2) f(z) = \lim_{z \to -i/2} \cancel{(z+i/2)} \cdot \left[\frac{1}{\cancel{(z+i/2)}(z+2i)}\right]$$

$$= \lim_{z \to -i/2} \left[\frac{1}{z+2i}\right]$$

$$= \frac{1}{-i/2 + 2i}$$

$$= \frac{1}{\frac{-i+4i}{2}} = \frac{2}{3i}$$

$\therefore$ By Cauchy's Residue's theorem,

$$\int_0^{2\pi} \frac{d\theta}{5+4\sin\theta} = \frac{1}{\cancel{2}} \cancel{2}\pi i \left[\text{Residue at } z = -\frac{i}{2}\right].$$

$$= \pi \cancel{i} \left[\frac{2}{3\cancel{i}}\right] = 2\pi/3.$$

3. $\int_0^{2\pi} \frac{\cos 3\theta}{5-4\cos\theta} \cdot d\theta$, evaluate using contour integral.

Soln:- $z = e^{i\theta}$.

$d\theta = \frac{dz}{iz} \Rightarrow \cos\theta = \frac{z^2+1}{2z}$.

$\cos 3\theta = R.P \text{ of } e^{i3\theta}$

Since $e^{i3\theta} = \cos 3\theta + i\sin 3\theta$

$$\int_0^{2\pi} \frac{\cos 3\theta}{5-4\cos\theta} \cdot d\theta = \int_0^{2\pi} \frac{R.P.e^{i3\theta}}{5-4\cos\theta} \cdot d\theta$$

$$= R.P \int_c \frac{(e^{i\theta})^3}{5-4\left[\frac{z^2+1}{2z}\right]} \cdot \frac{dz}{iz}$$

$$= R.P \int_c \frac{z^3}{\frac{10z-4z^2-4}{2z}} \cdot \frac{dz}{iz}$$

$$= R.P \int_c z^{\cancel{3}\,2}\left[\frac{2z}{10z-4z^2-4}\right] \frac{dz}{i\cancel{z}}$$

$$= R.P.\frac{2}{i}\int_c \frac{z^3 . dz}{-4z^2+10z-4}.$$

$$= R.P.\ \frac{2}{i}\int_C \frac{z^3 . dz}{-4\left[z^2 - \frac{10}{4}z + 1\right]}$$

$$= R.P.\ -\frac{\cancel{2}}{\cancel{4}_2 i}\int_C \frac{z^3 . dz}{z^2 - \frac{10}{4}z + 1}.$$

$$= R.P\ \frac{-1}{2i}\int_C \frac{z^3 . dz}{z^2 - 5/2\, z + 1}.$$

The poles are at $z^2 - 5/2\, z + 1 = 0$.

$$z = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \quad \{\because\ a = 1,\ b = 5/2,\ c = 1\}.$$

$$= \frac{5/2 \pm \sqrt{\frac{25}{4} - 4\times1\times1}}{2\times1}$$

$$= \frac{5/2 \pm \sqrt{\frac{25-16}{4}}}{2}$$

$$= \frac{5/2 \pm \sqrt{9/4}}{2} = \frac{5/2 \pm 3/2}{2} = \frac{\frac{3+5}{2}}{2}, \frac{\frac{5-3}{2}}{2}$$

$$= \left(\frac{8/2}{2}, \frac{2/2}{2}\right) = \left(8/4, 2/4\right) = (2, 1/2)$$

$\therefore \boxed{z = 2, \tfrac{1}{2}}$.

$\therefore$ The pole $z = \frac{1}{2}$ lies inside the circle and the pole $z = 2$ lies outside the circle.

$\therefore$ we've to calculate the residue at $z = \frac{1}{2}$ only.

Residue at $z = \frac{1}{2}$:-

$$\lim_{z \to \frac{1}{2}} (z - \tfrac{1}{2}) \cdot f(z) = \lim_{z \to \frac{1}{2}} (z - \tfrac{1}{2}) \cdot \left[\frac{z^3}{z^2 - \frac{5}{2}z + 1} \right].$$

$$= \lim_{z \to \frac{1}{2}} \cancel{(z - \tfrac{1}{2})} \cdot \left[\frac{z^3}{\cancel{(z - \frac{1}{2})}(z - 2)} \right]$$

$$= \lim_{z \to \frac{1}{2}} \left[\frac{z^3}{z - 2} \right].$$

$$= \frac{(\frac{1}{2})^3}{\frac{1}{2} - 2} = \frac{\frac{1}{8}}{\frac{1-4}{2}} = \frac{\frac{1}{8}}{\frac{-3}{2}}.$$

$$= \frac{-1}{\cancel{8}_4} \times \frac{\cancel{2}}{3} = \frac{-1}{12}.$$

By Cauchy's residue theorem,

$$\int_0^{2\pi} \frac{\cos 3\theta}{5-4\cos\theta} \cdot d\theta = R.P\left[\frac{-1}{2i}(2\pi i)\left[\begin{array}{c}\text{Residue at} \\ z=1/2\end{array}\right]\right]$$

$$= R.P.\left[-\pi \times \frac{-1}{12}\right].$$

$$= R.P\left[\pi/12\right].$$

$$\boxed{\int_0^{2\pi} \frac{\cos 3\theta}{5-4\cos\theta} \cdot d\theta = \pi/12.}$$