ASSIGNMENT - 1

PART - B

① Prove that $\overline{(A-B)} = \bar{A} \cup B$ analytically.

Sol:

$\bar{A} \cup B \Rightarrow$ Let x be an element.

for $\bar{A} \cup B$, $\Rightarrow x \in \bar{A}$ and $x \in B$.

$\Rightarrow x \notin A$ and $x \in B$. $\Rightarrow x \in \bar{A}$ and $x \notin \bar{B}$

$\Rightarrow x \in \bar{A} - \bar{B}$

$(A-B)' \Rightarrow x \in A$ and $x \notin B$

$\Rightarrow x \notin \bar{A}$ and $x \in \bar{B}$

$\Rightarrow x \in (\bar{B} - \bar{A})$

$\Rightarrow (A-B)' \Rightarrow x \in (\bar{A} - \bar{B})$.

Hence, $\boxed{\bar{A} \cup B = \overline{(A-B)}}$

② If R is a relation on set $A = \{1,2,3,4,5\}$ defined by $(a,b) \in R$ if $a+b \le 6$, then list elements of R, R^{-1} & $\bar{R}$. Find the relational matrix M_R, $M_{R^{-1}}$, & $M_{\bar{R}}$.

Ⓐ Given $A = \{1,2,3,4,5\}$.

(i) $\Rightarrow R = \{(1,1), (1,2), (1,3), (1,4), (1,5), (2,1), (2,2), (2,3), (2,4), (3,1), (3,2), (3,3), (4,1), (4,2), (5,1)\}$.

(ii) elements of $R^{-1} = \{(1,1), (2,1), (3,1), (4,1), (5,1), (1,2), (2,2), (3,2), (4,2), (1,3), (2,3), (3,3), (1,4), (2,4), (1,5)\}$

(iii) elements of $R^c = \{(2,5), (3,4), (3,5), (4,3), (4,4), (4,5), (5,2), (5,3), (5,4), (5,5)\}$.

(iv) $M_R = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} \begin{array}{c} \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} \\ \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$

(v) $M_{R^{-1}} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix}$

(vi) $M_{R^c} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 \end{bmatrix}$

(3) If $f(x) = x+2$, $g(x) = x-2$. for $x \in R$, then prove that $fog = gof$

Sol: Consider $f(x) = x+2$, $g(x) = x-2$.

To show that $fog = gof$

Proof: Consider $(fog)(x) = f[g(x)]$

$= f[x-2]$ ($\because g(x) = x-2$)

$= (x-2)+2$ ($\because f(x) = x+2$)

$= x$ // —(1)

Consider $(gof)(x) = g[f(x)]$

$= g[x+2]$ ($\because f(x) = x+2$)

$= (x+2)-2$ ($\because g(x) = x-2$

$= x$ // —(2)

From (1) & (2). it is clear that $fog(x) = gof(x)$.

$\Rightarrow$ $\boxed{fog = gof}$ (Hence proved)//.

(4) If $f, g: R \to R$ where $f(x) = ax+b$; $g(x) = 1-x+x^2$ & $(gof)(x) = 9x^2-9x+3$. Find value of a, b.

Sol: Given $f: R \to R$ and $g: R \to R$

$f(x) = ax+b$; $g(x) = 1-x+x^2$; $(gof)(x) = 9x^2-9x+3$.

To find the value of a, b,

Consider $(gof)(x) = g[f(x)]$

$= g[ax+b]$

$= 1-(ax+b)+(ax+b)^2$

$= 1-ax-b+a^2x^2+2axb+b^2$

$9x^2-9x+3 = a^2x^2+(2ab-a)x+b^2-b+1$.

By comparing terms from LHS & RHS

(i) $a^2 = 9$; (ii) $2ab - a = -9$ (iii) $b^2 - b + 1 = 3$

$\downarrow$ if $a = 3$, $b^2 - b - 2 = 0$

$a = \pm 3$ $\nearrow$ $6b - 3 = -9$ $b^2 - 2b + b - 2 = 0$

$\boxed{b = -1}$ $b(b-2) + 1(b-2) = 0$

if $a = -3$, $\boxed{b = 2, \text{ (or) } b = -1}$

$-6b + 3 = -9$

$-6b = -12$

$\boxed{b = 2}$

$\therefore$ $\boxed{a = \pm 3, \ b = +2, -1}$ but consider $(gof)(x)$ also.

if $a = 3, b = -1$

$\Rightarrow$ $g[3x - 1] = 1 - (3x-1) + (3x-1)^2$

$= 1 - 3x + 1 + 9x^2 + 1 - 6x$

$= 9x^2 - 9x + 3$

if $a = -3, b = 2$

$g[-3x + 2] = 1 - (-3x+2) + (-3x+2)^2$

$= 1 + 3x - 2 + 9x^2 - 12x + 4$

$= 9x^2 - 9x + 3$ //.

$\therefore$ $\boxed{a = \pm 3, \ b = -1, 2}$

(5) Verify whether the given ~~function~~ relation R on $A = \{a, b, c, d\}$ is an equivalence relation (or) not. justify your answer.

Sol: Given $A = \{a, b, c, d\}$ & R is relation on A.

$\Rightarrow R = \{(a,a), (a,c), (a,d), (b,b), (c,a), (c,c), (d,a), (d,d)\}$.

To prove that R is equivalence relation (or) not, we need to check whether R is reflexive, and symmetric and transitive.

$\Rightarrow$ (i) reflexive:- R is said to be reflexive iff $R = \{(a,a) \mid a \in A\}$.

$\Rightarrow R = \{(a,a), (b,b), (c,c), (d,d)\}$ $\Rightarrow$ R is reflexive.

(ii) Symmetric:- R is said to be symmetric if there exist $a, b \in A$ such that if aRb, then bRa also exists

$\Rightarrow$ if $(a,b) \in R$, then $(b,a) \in R$.

So, if we observe, we have (a,c) & (c,a); (a,d) & (d,a).

So, R is symmetric.

(iii) Transitive:- A relation R is said to be transitive if whenever aRb & bRc then aRc.

$\Rightarrow R = \{(a,a), (a,c), (a,d), (b,b), (c,a), (c,c), (d,a), (d,d)\}$.

$\Rightarrow (a,c) \in R$ & $(c,a) \in R \Rightarrow (a,a) \in R$ (True).

$(a,d) \in R$ & $(d,a) \in R \Rightarrow (a,a) \in R$ (True)

$(a,a) \in R$ & $(a,c) \in R \Rightarrow (a,c) \in R$ (True).

$(d,a) \in R$ & $(a,c) \in R \Rightarrow (d,c) \notin R$ (False).

So, R is not transitive.

$\therefore$ R is not an equivalence relation.

(6) Find the matrix representation of $R \cup S$ & $R \cap S$ wher $R = \{(1,1), (1,3), (2,2)\}$ & $S = \{(1,2), (1,3), (2,1), (2,2), (3,3)\}$ are relations defined on $A = \{1,2,3\}$.

Sol:

$$M_R = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad M_S = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$\Rightarrow$ (i) $M_{R \cup S} = \begin{bmatrix} 1 \vee 0 & 0 \vee 1 & 1 \vee 1 \\ 0 \vee 1 & 1 \vee 1 & 0 \vee 0 \\ 0 \vee 0 & 0 \vee 0 & 0 \vee 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$\downarrow$

$M_R \vee M_S$

(ii) $M_{R \cap S} = \begin{bmatrix} 1 \wedge 0 & 0 \wedge 1 & 1 \wedge 1 \\ 0 \wedge 1 & 1 \wedge 1 & 0 \wedge 0 \\ 0 \wedge 0 & 0 \wedge 0 & 0 \wedge 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

$\downarrow$

$M_R \wedge M_S$

PART-C

① State and prove Demorgan's law of set theory.

Ⓐ • There are 2 demorgan's law of set theory namely law of union & law of intersection.

(i) Law of union:- $(A \cup B)^c = A^c \cap B^c$.

The complement of the union of 2 sets A & B is equal to the intersection of individual complements of the 2 sets.

Proof: To prove, $(A \cup B)^c = A^c \cap B^c$.

Consider $(A \cup B)^c \Rightarrow$ Let x be an element.

$\Rightarrow (A \cup B)^c \Rightarrow \{x \notin (A \cup B)\}$

$\Rightarrow \{x \mid \sim(x \in (A \cup B))\}$

$\Rightarrow \{x \mid \sim(x \in A \lor x \in B)\}$

$\Rightarrow \{x \mid \sim(x \in A) \land \sim(x \in B)\}$

$\Rightarrow \{x \mid x \in A^c \land x \in B^c\}$

$\boxed{(A \cup B)^c \Rightarrow \boxed{A^c \cap B^c}}$ //.

(ii) Law of intersection:- $(A \cap B)^c = A^c \cup B^c$.

The complements of intersection of 2 sets A & B is equal to the ~~intersection~~ union of individual complements of 2 sets.

Proof:- To prove, $(A \cap B)^c = A^c \cup B^c$

$\Rightarrow$ Consider $(A \cap B)^c = \{x \mid x \notin A \cap B\}$

$= \{x \mid \sim(x \in A \cap B)\}$

$= \{x \mid \sim[(x \in A) \land (x \in B)]\}$

$= \{x \mid \sim(x \in A) \lor \sim(x \in B)\}$

$= \{x \mid x \in \bar{A} \lor x \in \bar{B}\} = \bar{A} \cup \bar{B}$

(2) If A, B and C are sets then prove the statement $(A-B)-C = A-(B\cup C)$ analytically.

Sol: To prove $(A-B)-C = A-(B\cup C)$.

Consider $A-(B\cup C) = \{x \mid x\in A \text{ and } x\notin (B\cup C)\}$

$= \{x \mid x\in A \text{ and } (x\notin B \text{ and } x\notin C)$

$= \{x \mid (x\in A \text{ and } x\notin B) \text{ and } x\notin C\}$

$= \{x \mid x\in (A-B) \text{ and } x\notin C\}$

$= \{x \mid x\in [(A-B)-C]\}$.

$\boxed{A-(B\cup C) = (A-B)-C}$

Hence, proved.

(3) If R is a relation on set of integers such that $(a,b)\in R$ iff $3a+4b=7n$ for some integer n, prove that R is an equivalence relation.

(A) To prove R is equivalence relation, we must prove that R is reflexive, Symmetric & Transitive.

(i) Reflexive:- To prove, reflexive, $(a,a)\in R$.

$\Rightarrow$ Given, $3a+4b = 7n$

$a=b$

$\Rightarrow$ $3a+4a = 7n$

$7a = 7n$.

$a=n$, $n=$ integer $\Rightarrow$ $a=$ integer

$\Rightarrow$ $\boxed{R \text{ is reflexive}}$

(ii) Symmetric:- if $(a,b)\in R$ then $(b,a)\in R$.

$\Rightarrow \quad 3a+4b = 7n \quad -①$

$3b+4a = 7a+7b-(3a+4b)$

$= 7a+7b-7n$

$= 7(\underbrace{a}_{int}+\underbrace{b}_{int}-\underbrace{n}_{int}) = 7m$ where $m=(a+b-n)$ is also an integer.

$\Rightarrow$ $3b+4a$ is also an integer.

$\Rightarrow$ R is symmetric.

(iii) Transitive:- if $(a,b) \in R$ & $(b,c) \in R \Rightarrow (a,c) \in R$.

$\Rightarrow \quad 3a+4b = 7n \quad -①$

$3b+4c = 7p \quad -②$

$\Rightarrow \quad 3a+7b+4c = 7(n+p)$

$\Rightarrow \quad 3a+4c = 7(n+p-b) = 7q$ where $q=(n+p-b)$ is also an integer.

$\Rightarrow \boxed{3a+4c = 7q}$ $\therefore$ R is transitive.

$\therefore$ Since, R is symmetric, reflexive & transitive, R is an equivalence relation. Hence, proved

④ If R is a relation on set $A=\{1,2,4,6,8\}$, defined by aRb iff b/a is an integer. Show that R is a partial ordering on A

Sol: R is said to be a partial ordering on A iff R is reflexive, anti-symmetric & transitive

$\Rightarrow$ Consider $A=\{1,2,4,6,8\}$, Condition: $(b/a \in \mathbb{Z})$

$\Rightarrow \quad R=\{(1,1),(1,2),(1,4),(1,6),(1,8),(2,2),(2,4),(2,6),(2,8),$
$(4,4),(4,8),(6,6),(8,8)\}.$

(i) Reflexive:- R is reflexive if $R=\{(a,a) \mid a \in A\}$.

$\Rightarrow$ R contains $\{(1,1), (2,2), (4,4), (6,6), (8,8)\}$

$\Rightarrow$ R is reflexive.

(ii) Anti - symmetric:- R is said to be antisymmetric,

if aRb & $bRa \Rightarrow \boxed{a=b}$.

$\Rightarrow \frac{a}{b}$ is an integer and $\frac{b}{a}$ is also an integer

$\Rightarrow$ $a=b$ Hence, R is anti-symmetric.

(iii) Transitive:- R is said to be transitive if aRb & bRc

$\Rightarrow$ aRc.

$\Rightarrow (1,2) \in R$ and $(2,2) \in R, (2,4) \in R, (2,6) \in R, (2,8) \in R$

$\Rightarrow (1,2), (1,4), (1,6), (1,8) \in R.$ — ①

$\Rightarrow (2,4) \in R$ and $(4,4), (4,8) \in R \Rightarrow (2,4), (2,8) \in R$ — ②

$\therefore$ R is transitive.

$\Rightarrow$ R is a partial order on set A, since R is reflexive, anti-symmetric & transitive.

⑤ Let $R = \{(1,1), (1,3), (1,5), (2,3), (2,4), (3,3), (3,5), (4,2), (4,4), (5,4)\}$. be relation on $A = \{1,2,3,4,5\}$. Find transitive closure using Warshall's algorithm.

Sol:

$$R = \begin{bmatrix} 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} = W_0 = \begin{bmatrix} 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

$\Rightarrow$

k	position of 1's in column on k	position of 1's in row of k	(col, row)	W_k
1	1	1, 3, 5	(1,1), (1,3), (1,5)	$\begin{bmatrix} 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 \end{bmatrix}$

2	4	(3, 4)	(4,3), (4,4)	$\begin{bmatrix} 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 \end{bmatrix}$
3	(1, 2, 3)	(1, 3, 5)	(1,1) (1,3), (1,5), (2,1), (2,3), (2,5) (3,1) , (3,3) (3,5)	$\begin{bmatrix} 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 \end{bmatrix}$
4	(2, 4, 5)	(2, 3, 4)	(2,2), (2,3) ,(2,4) (4,2), (4,3), (4,4) (5,2), (5,3), (5,4)	$\begin{bmatrix} 1 & 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 0 \end{bmatrix}$
5	(1, 2, 3)	(1, 2, 3, 4)	(1,1), (1,2), (1,3), (1,4) (2,1), (2,2), (2,3) (2,4), (3,1), (3,2), (3,3), (3,4)	$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 0 \end{bmatrix}$

$\therefore$ Transitive closure $R^{\infty} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 0 \end{bmatrix}$

6. Show that the composition of invertible function is invertible.

Sol: Consider $f: A \to B$, $g: B \to C$ are 2 invertible functions [$\Rightarrow$ They are bijective]. To show that $g \circ f$ is invertible.

- We have to show that $g \circ f$ is 1-1 & onto.

$\Rightarrow$ (i) $g \circ f$ is 1-1:-

We know that $f: \to 1-1$ & $g: 1-1$

$\Rightarrow$ Let $(g \circ f)(a_1) = (g \circ f)(a_2)$

$\Rightarrow g[f(a_1)] = g[f(a_2)]$

$\Rightarrow f(a_1) = f(a_2)$ ($\because g$ is 1-1)

$\Rightarrow \boxed{a_1 = a_2}$ ($\because f$ is 1-1)

$\because a_1 = a_2$
$g \circ f$ is 1-1

(ii) gof is onto:-

Let $c \in C$; sin g is onto, there exists $b \in B$ such that $\boxed{c = g(b)}$; Since f is onto, there will be an element $a \in A$ such that $b = f(a)$.

Hence, for every element $c = g(b) = g(f(a)) = gof(a)$

$\Rightarrow$ $\forall c \in A$ $\exists a \in A$ such that $(gof)(a) = c$.

$\therefore$ Hence $gof : A \to C$ is onto.

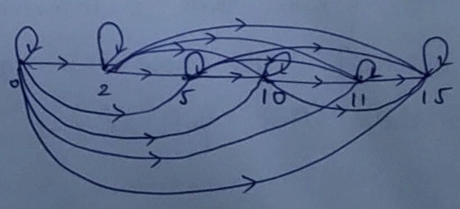
$\therefore$ Hence, gof is bijection & thus invertible.

(7) Draw the hasse-diagram for the "less than (or) equal to" relation on $\{0, 2, 5, 10, 11, 15\}$ starting from digraph.

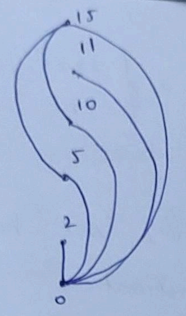
(A) Given $A = \{0, 2, 5, 10, 11, 15\}$

$R = a \leq b$ $\Rightarrow$ $R = \{(0,0), (0,2), (0,5), (0,10), (0,11), (0,15),$
$(2,2), (2,5), (2,10), (2,11), (2,15),$
$(5,5), (5,10), (5,11), (5,15), (10,10), (10,11), (10,15),$
$(11,11), (11,15), (5,15)\}$.

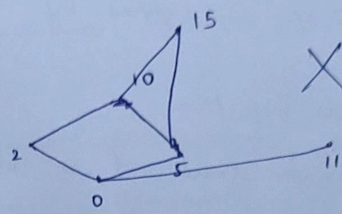
$\Rightarrow$ diagraph:-



Hasse diagram:-



$\Rightarrow$



(8) If $f: Z \times Z \to Z$ defined by $f(m,n) = 2m+3n$, then determine whether it is one - one &(or) onto

(A) Given $f: Z \times Z \to Z$ given by $f(m,n) = 2m+3n$.

(i) To prove 1-1:-

$$f(m_1, n_1) = f(m_2, n_2)$$

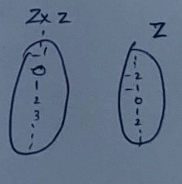
$$\Rightarrow 2m_1 + 3n_1 = 2m_2 + 3n_2$$

$$\Rightarrow 2(m_1 - m_2) \neq 3(n_2 - n_1) \quad \text{clearly } 2(m_1 - m_2) \neq 3(n_2 - n_1)$$

$\therefore$ Thus $f(m,n) = 2m+3n$ is not one-one.

(ii) To ~~prove~~ check onto:-

$$f(m,n) = 2m+3n.$$



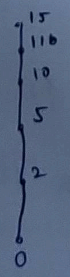
$\Rightarrow$ Let $x \in Z \times Z$; $y \in Z$.

Let $m = -1$, $n = 1$

$$\Rightarrow f(-1, 1) = 2(-1) + 3(1) = 1 \Rightarrow \in Z$$

$\therefore$ $f(m,n) = 2m+3n$ is an onto function.

(7) Hasse diagram:-



(for the digraph of R.)

⑨ If $f: Z \to N$ is defined by $f(x) = \begin{cases} 2x-1, & \text{if } x>0 \\ -2x, & \text{if } x \le 0. \end{cases}$ prove that f is bijective.

Sol:- • To prove that f is bijective, we have to prove that f is 1-1 + onto functions.

$\Rightarrow$ (i) f is 1-1:-

Case 1: when $x > 0$

$$2x_1 - 1 = 2x_2 - 1$$

$$\Rightarrow x_1 = x_2$$

Case 2: when $x \le 0$

$$-2x_1 = -2x_2$$

$$\Rightarrow x_1 = x_2$$

$\therefore$ f is 1-1

(ii) To show f is onto:-

Let y be a nat. numb ($y \in N$)

Case 1:- $x > 0$

$\Rightarrow y = 2x - 1$

$\Rightarrow x = \frac{y+1}{2}$

$\Rightarrow f\left(\frac{y+1}{2}\right) = 2\left(\frac{y+1}{2}\right) - 1 = y.$

Case: 2 $x \le 0$

$y = -2x$

$x = -\frac{y}{2}$ $\therefore$ for every y in $N \cup \{0\}$, there exists x in Z such that $x = -\frac{y}{2}$.

$\therefore$ f is onto

$\Rightarrow$ f is bijective.

⑩ Prove that R is an equivalence relation where aRb iff 3a+b is multiple of 4.

Sol: To prove R is an equivalence relation, we need to prove that R is reflexive, symmetric and transitive.

(i) Reflexive:- $R = \{(a,a) / a \in A\}$

$\Rightarrow$ $3a + b = 4n$.

$a = b$

$\Rightarrow$ $4a = 4n$

$4a = 4n$. $\Rightarrow$ divisibly by 4.

$\therefore$ R is reflexive.

(ii) symmetric:- $(a,b) \in R \Rightarrow (b,a) \in R$.

$\Rightarrow$ $3a+b = 4n$. — ①

$3b+a = 4k$ — ②

$\Rightarrow$ $4(a+b) = 4(n+k)$ $\Rightarrow$ divisible by 4.

$\Rightarrow$ R is symmetric.

(iii) Transitive:- $(a,b) \in R$ & $(b,c) \in R \Rightarrow (a,c) \in R$.

$\Rightarrow$ $3a+b = 4n$ — ①

$3b+c = 4k$ — ②

Add ①, ②

$\Rightarrow$ $3a + 4b + c = 4(n+k)$

$\Rightarrow$ $3a + c = 4(n+k-b)$ $\Rightarrow$ divisible by 4

$\Rightarrow$ R is transitive.

$\therefore$ R is an equivalence relation.