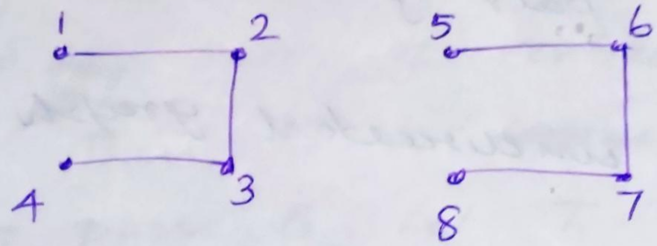
Tree :-

A Graph G(V, E) is called Tree if there is an exactly one path between every 2 vertices.



it cannot be tree from 2 to 8 there is no path.

Trees :- A connected graph without any circuit is called a Tree.

* Tree is a simple graph.
* Tree is a connected acyclic graph having no cycle.

Eg :-

collection of trees is called forest.

Note :- A tree is a simple graph as loops + parallel edges forms circuits.

Properties of Trees :-

Property 1 :-

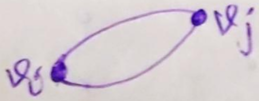
An undirected graph is a tree iff there is a unique simple path between every pair of vertices.

Proof :- Let the undirected graph T be a tree.

Then by definitions of a tree, T is connected. (There is a path between every pair of distinct vertices)

Hence, there is a simple path between any pair of vertices say v_i & v_j.

If possible let there be two paths between v_i & v_j.



i.e. one from v_i to v_j & the other from v_j to v_i.

Combination (union) of these two paths would contain circuit.

But, by definition, T cannot have a circuit.

Hence there is a unique simple path between every pair of vertices in T.

(ii) Let a unique path exists between every pair of vertices in the graph T.

Then T is connected.

If possible let T contains a circuit

This means that there is a pair of vertices v_i & v_j between two distinct path exists. which is against the data.

Hence, T cannot have a circuit & so T is a tree.

Property 2 :-

A tree with n vertices has (n-1) edges.

Proof :-

~~This~~ The property is true for n=1, 2, 3 ——

Let us now we use mathematical induction to prove the property.

Step 1 :-

single vertex is connected & has no circuit. It is true.

$|E|=0$

- connected & no circuit. It is a tree.

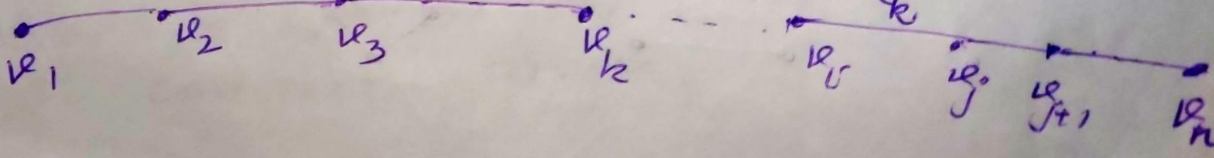
$n=2$

$|E|=1$

Step 2 :- We assume every tree with $< n$ vertices has above property.

Let us consider a tree T with n vertices we have to prove it has $n-1$ edges.

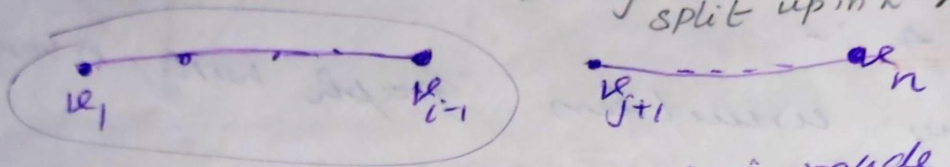
Let e_k be the edge connecting the vertices v_i & v_j of T



Then by property (1) e_k is the only path between v_i & v_j.

If we delete the edge e_k from T, T becomes disconnected. & $T - e_k$ consists of exactly two components say T_1 & T_2. which are connected

Every disconnected graph can be split up in k number of connected components.



disconnected graph is made up of connected pieces called components.

Since T did not contain any circuit, T_1 & T_2 also will not have circuit.

Hence both T_1 & T_2 are trees each having less than n vertices say r & $n-r$ resp.

$\therefore$ By the induction assumption, T_1 has $r-1$ edges. & T_2 has $n-r-1$ edges.

$\therefore$ T has $r-1+(n-r-1)+1$

$= n-1$ edges.

Thus, a tree with n vertices has $n-1$ edges. b4 =b

Property 3 :-

Any connected graph with n vertices & $(n-1)$ edges is a tree.

property 4 :-

Any circuit-less graph with n vertices & $(n-1)$ edges is a tree.

Spanning Trees :-

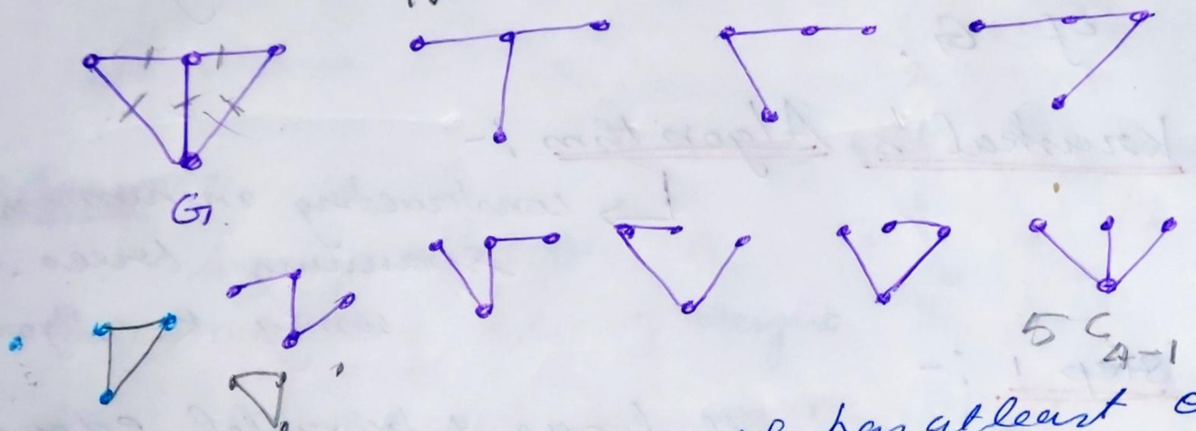
If the subgraph T of a connected graph G is a tree containing all the vertices of G, with minimum number of edges & some edges of G, then T is called a spanning tree of G.

Ex :- Let us consider the graph G.

Since G has 4 vertices, any spanning tree of G will also have 4 vertices & have 3 edges. (by property) has

Since G has 5 edges, removal of 2 edges may result in spanning tree. This can be done in $^5C_2 = 10$ ways, but 2 of these 10 ways result in disconnected graphs. All the possible spanning tree

$|E| \, C_{|V|-1}$



G

$5 \, C_{4-1}$

<u>Note</u> :- Every connected graph has atleast one spanning tree. This is obvious when G has no circuit, as G is its own spanning tree. If G has a circuit we can get a spanning tree by deleting an edge from the circuit.

Subgraph :- A subgraph for a graph whose vertices & edges are subsets of the original graph

Minimum spanning tree :-

If G is a connected weighted graph, the spanning tree of G with the smallest total weight (sum of the weights of its edges) must be minimum is called the minimum spanning tree of G.

Kruskal's Algorithm :-

→ constructing minimum spanning trees by using K. Algori

Step 1 :-

Remove all loops & parallel edges (select least cost)

(ii) The edges of the given graph G are arranged in the order of increasing weights.

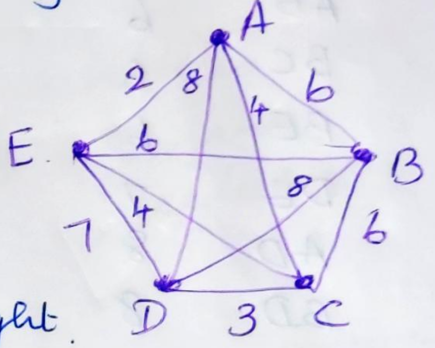
(iii) An edge G with minimum weight is selected as an edge of the required spanning tree.

4) Edges with minimum weight that do not form a circuit with the already selected edges are

successively added.

5) The procedure is stopped after $(n-1)$ edges have been selected.

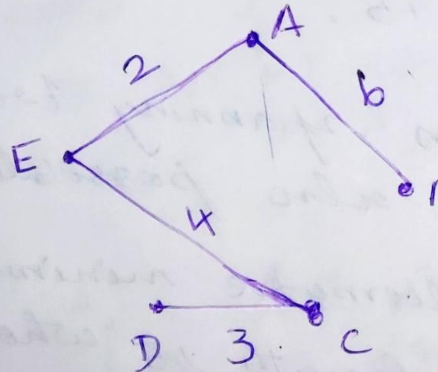
(1) Find the minimum spanning tree for the following graph using kruskal's algorithm.



Solution :-

Arrange all edges in their increasing order of Weight.

AE	DC	EC	AC	AB	EB	BC	ED	AD	BD
2	3	4	4	6	6	6	7	8	8



This graph have 5 vertices, So we have selected $(5-1)$ edges.
4 edges.
After selecting 4 edges we have to Stop the process.

Edge	Weight	included in the spanning tree or not	If not included circuit formed
AE	2	yes	–
DC	3	Yes	–
AC	4	yes	–
CE	4	NO	A-E-C-A.
AB	6	yes	–
BC	6	NO.	A-B-C-A.
BE	6	–	–
DE	7	–	–
AD	8	–	–
BD	8	–	–

The edges of the minimum spanning tree are AE, CD, AC, BC whose total length is 15.

Note :- Alternate minimum spanning tree of same length is also possible.

There are 5 other alternative minimum spanning trees of total length 15 whose edges are listed below

AE, CD, AC, BE   AE, CD, CE, AB

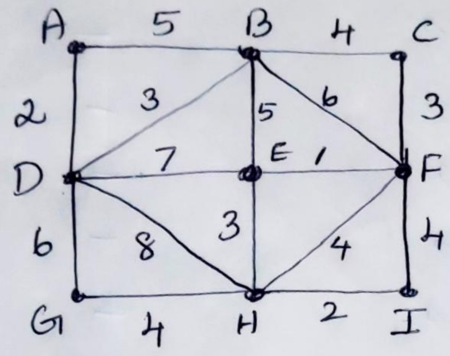
AE, CD, AC, BC

AE, CD, CE, BE.

AE, CD, CE, BC

use Kruskal's Algorithm to find a minimum spanning tree

(2) use Kuskal's algorithm to find a minimum spanning tree for the weighted graph.



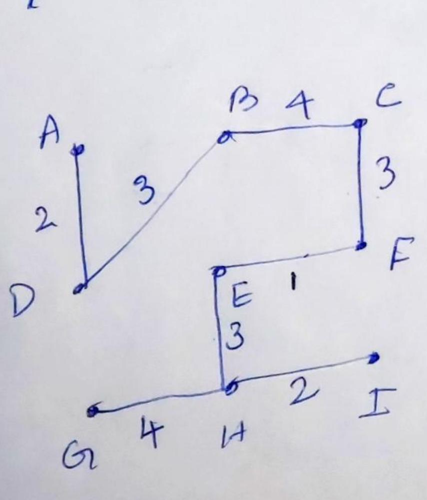
9 vertices

8 edges are selected

Solution:- Arrange all edges in their increasing order of weight.

EF, AD, HI, BD, CF, EH, BC, GH, FH, FI, AB, BE,
1 2 2 3 3 3 4 4 4 4 5 5

DG, BF, DE, DH
6 6 7 8



Minimum weight $= 2+3+4+3+1+3+4+2$

$= 22.$

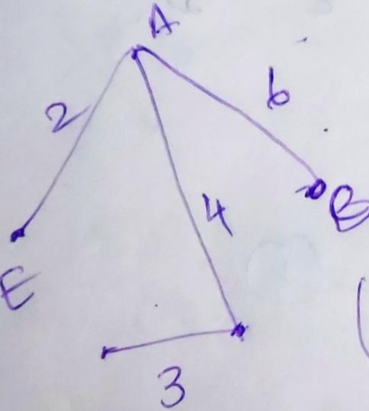
Edge	Weight	included in the spanning tree or not	If not included, circuit formed
EF	1	yes	—
AD	2	yes	—
HI	2	yes	—
BD	3	yes	—
CF	3	yes	—
EH	3	yes	—
BC	4	yes	—
GH	4	yes.	—
FH	4	—	—
FI	4	—	—
AB	5	—	—
BE	5	—	—
DG	6	—	—
BF	6	—	—
DE	7	—	—
DH.	8	—	—

Note :-

The weight of a minimum spanning tree is unique. whereas a minimum spanning tree is not unique as weights may be repeated.

The spanning tree will be unique if the weights are not repeated. i.e they are distinct.

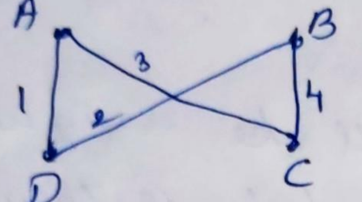
In example (1) two or more edges can have the same weights.

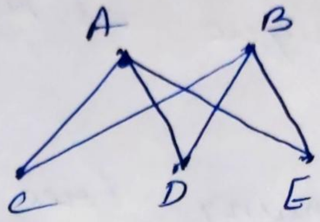


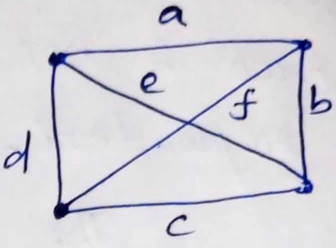
weight of the minimum spanning tree = 15 (unique)

(tree diagram is not same)

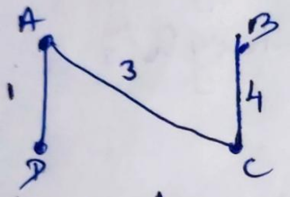
Draw all the spanning tree of the graph G_1, G_2, G_3 given in figers.

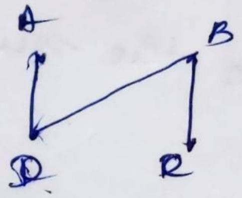


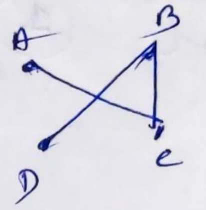


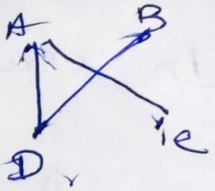


(1). number of vertices = 4
number of edges = 4





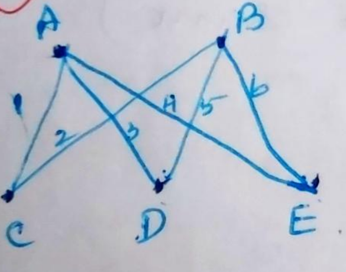


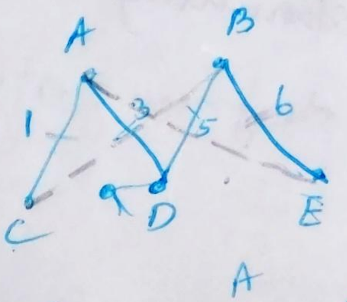


(1 3 4)

2 2.

(2)

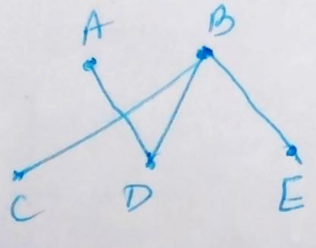


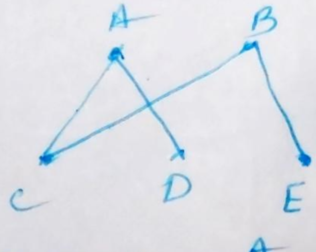


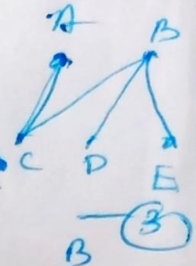
— (1)

5 vertices spanning tree should have 4 edges.

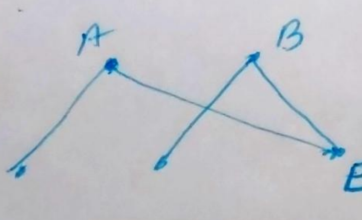
2
4
(2,4)

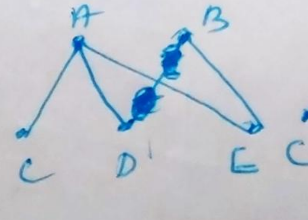


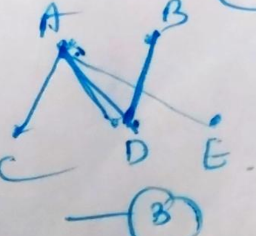




— (3)



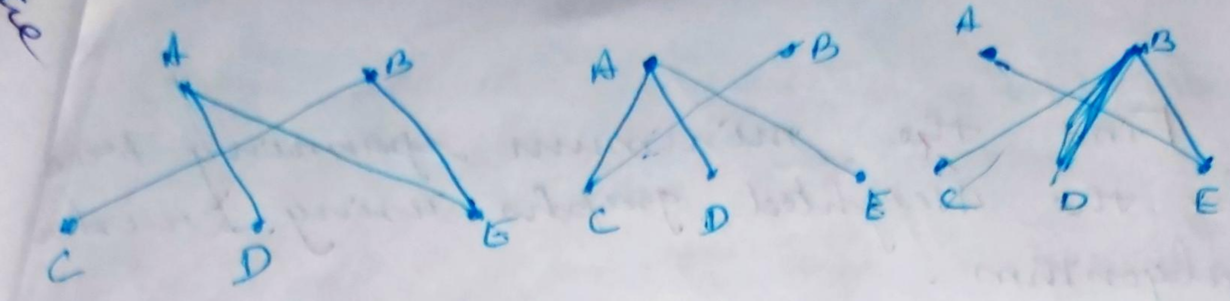


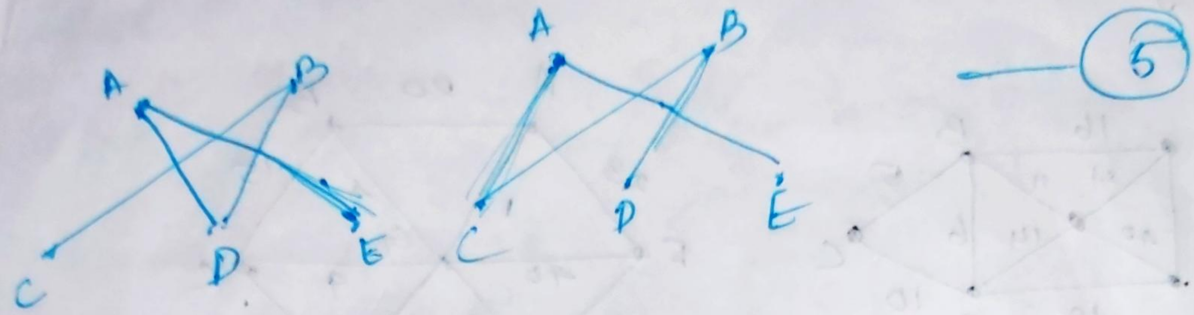


— (3)

U.vertices (1 3 5 6)

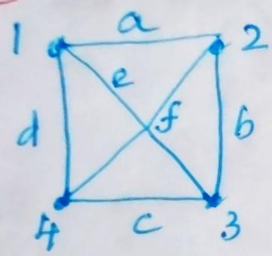
unused " (2,4)

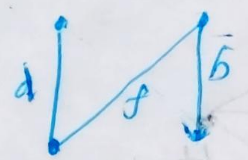




— 5

3





— f

used d, f, b

unused (a, c, e)

a

c

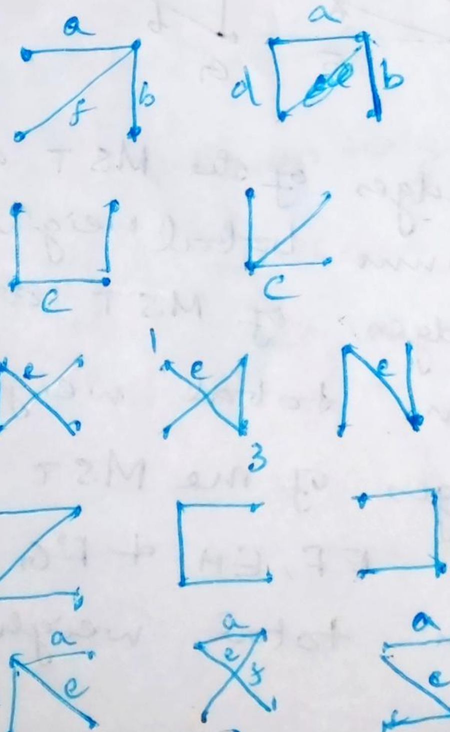
e

ac

c, e

a, e

a, c, e.

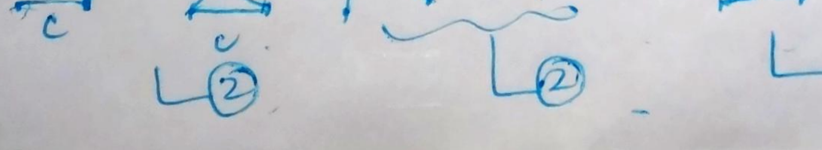


— 2

— 2

— 3

— 3

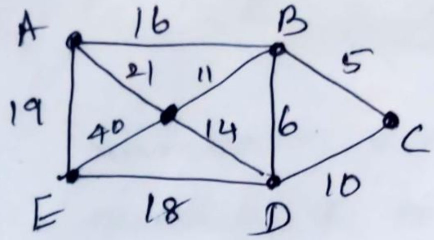


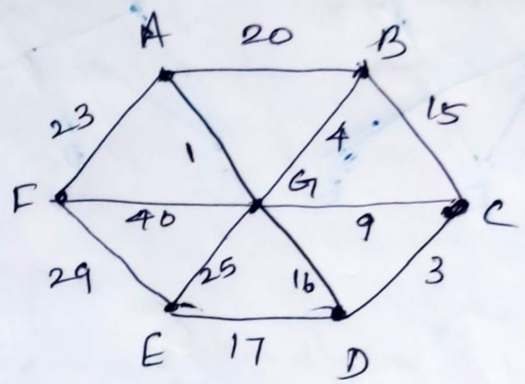
— 2

— 2

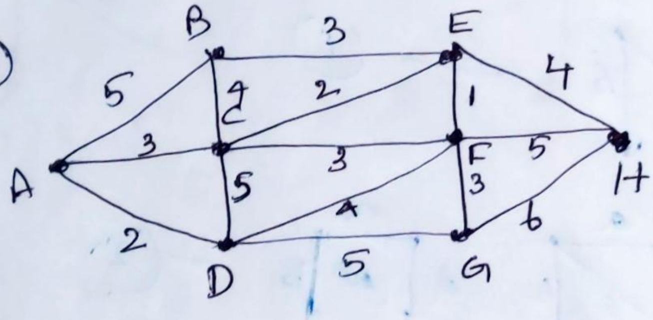
— 1

Find the minimum spanning tree for the weighted graphs using Kruskal's algorithm.





(3)



Ans (1) The edges of the MST are AB, BC, BD, BF, DE

minimum total weight = 56.

(2) The edges of MST are AF, AG, BG, CD, CG & DE

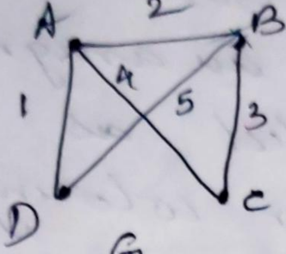
Minimum total weight = 57.

(3) The edges of the MST are AC, AD, BE, CE

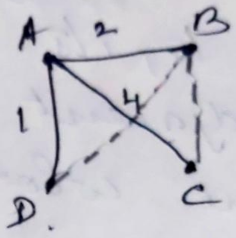
EF, EH & FG.

Minimum total weight = 18.

Draw all the spanning trees of the graph G shown in diagram



G.



number of vertices = 4

n " " edges = $n-1$

$= 3.$

edges used = 1 2, 4   3

" " 3, 5.   5

3, 5

The given graph G has 4 vertices. Hence, any spanning tree of G will also have 4 vertices & so 3 edges. Since G has 5 edges. We have to delete 2 of the edges of G to get a spanning tree.

Graph colouring :-

Graph coloring is an assignment of colors to each its vertices such that no two adjacent vertices are assigned to the same color is called as Graph coloring. or proper coloring.

Chromatic number :-

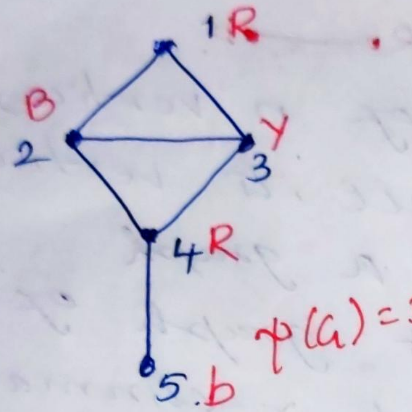
The minimum number of colours (or least number of colours) required to colours all the vertices of a given graph is called a chromatic number of a given graph.

* The chromatic number of a graph G is usually denoted by $\chi(G)$.

A group G is said to be K colourable if we can properly colour it with K colour.

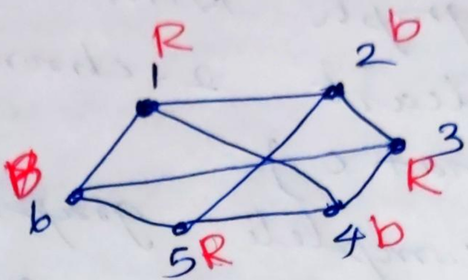
Eg :-

- Find the chromatic number for the following graph.



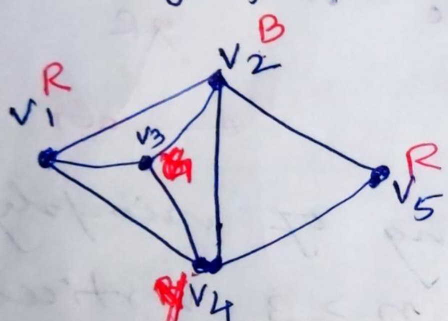
$\chi(G) = 3$

3 - colorable graph.



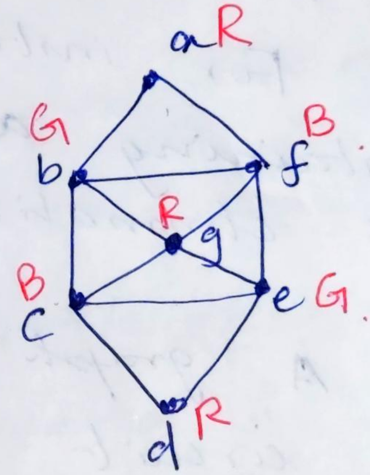
$\chi(G) = 2$

2 - colorable graph.



4 - colorable graph.

$\chi(G) = 4$



$\chi(G) = 3$.

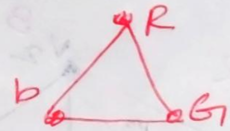
Note :-

A graph consisting of only isolated vertices is 1-chromatic.

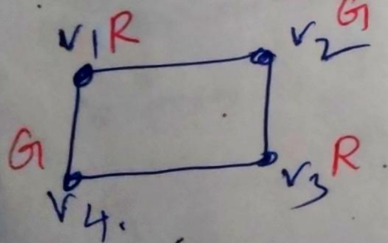
2. A graph with one or more edges is atleast 2-chromatic (also called bichromatic)

3. A complete graph of n vertices is n-chromatic as all its vertices are adjacent. Hence a graph containing a complete graph of r vertices is atleast r-chromatic

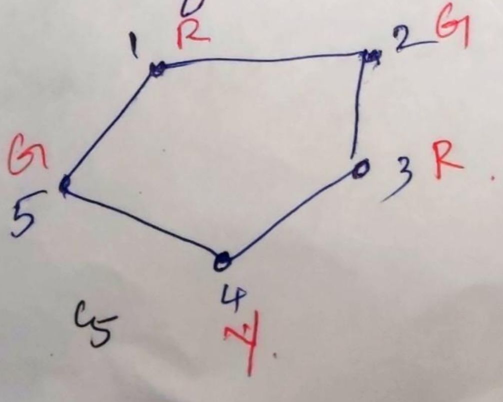
For instance every graph containing a triangle is atleast 3-chromatic.



A graph consisting of simply One circuit with $n \geq 3$. vertices is 2-chromatic if n is even & 3-chromatic if n is odd.



C_4

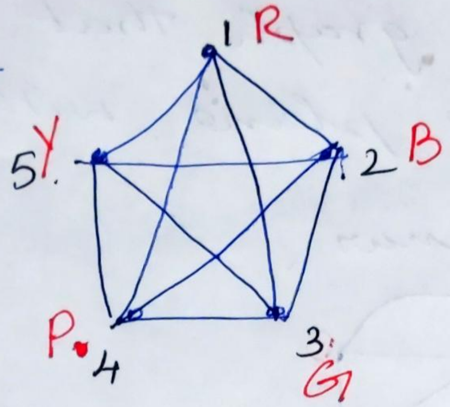


C_5

Chromatic number of K_n

No two vertices can be assigned the same color because every two vertices of this graph are adjacent. ∴ The chromatic number of K_n is n.

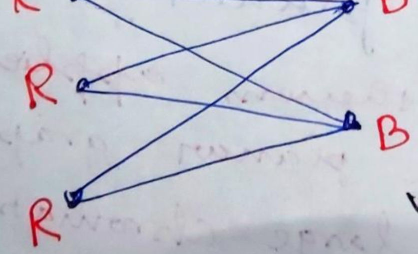
Eg:- K_5



Here five distinct colors are needed so chromatic number is 5.

(2) coloring of $K_{3,2}$ bipartite graph :-

only two colors are needed to color a complete bipartite graph



This means that we can color the set of m vertices with one color & the set of n vertices with a second color. because edges connect only a vertex from the set of m vertices & a vertex from the set of n vertices, no two adjacent vertices have the same colour.

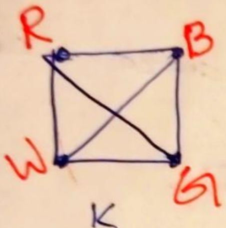
The Four color theorem :-

The chromatic number of a planar graph is no greater than four.

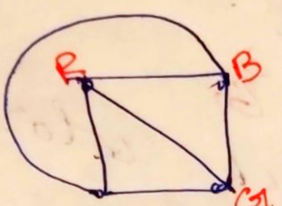
Planar graph :-

A graph that can be drawn in the plane with no crossings.

Eg:- K_4 is planar.



K_4



K_4.

A graph may be planar even if it is usually drawn with crossings, since we can draw it differently without crossing.

Note :- K_5 is non planar.

Note :- The four colour theorem applies only to planar graphs. Non planar graphs can have arbitrarily large chromatic numbers.