Graph theory

A Graph G is a triple (V, E, g) where V is a non empty set called the set of vertices. E may be an empty set called set of edges g is a mapping called incidence mapping that relates to each edge $e \in E$ with ordered or unordered pair of elements of v

ie $g(e) = \{u, v\}$

$g : E \rightarrow$ collection of two element subset of V.

ie $g : E \rightarrow V \times V$.

Ex :- Draw a graph G where

$V = \{v_1, v_2\ v_3, v_4\}$

$E = \{e_1, e_2, e_3, e_4, e_5\}$

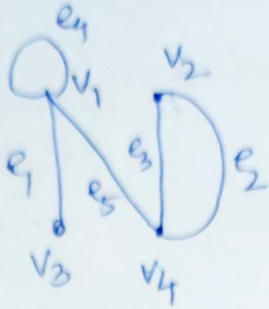
$g(e_1) = \{v_1, v_3\}$

$g(e_2) = \{v_2, v_4)$

$g(e_3) = \{v_2,$

$g(e_4) = \{$

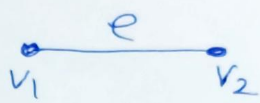
$g(e_5) = \{v_1, v_4\}$



usually vertice with single lette where as edges with pair of letters like uv to denote the edge between vertices u & v.

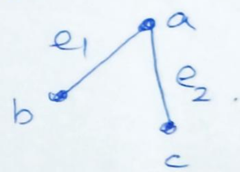
adjacent vertices :-

Two vertices are said to be adjacent that are connected by an edge.



Adjacent edge. :-

Two edge e_1 & e_2 are adjacent edges if they have a common end vertex



Incidence :-

Let e be an edge with end vertices u, v

then the vertex u is said to be incident with the edge e & edge e is said to be incident with vertex u.

An edge is said to be incident on its two endpoints. [incident → an edge connected with 2 vertices.

Self loop :-

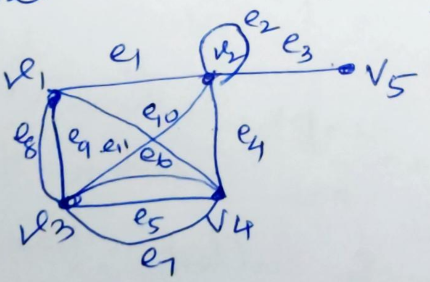
An edge whose end vertices are same or equal
edge from a vertex to itself
if the edge joins a vertex to itself.

Parallel edges :- (or) Multiple edges.

Two or more edges are called Parallel edges when they have same end vertices.

Degree of vertex :-

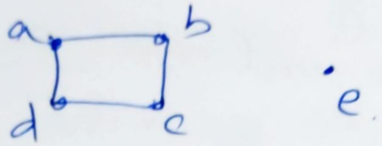
Let G be a graph & V be a vertex in G. The degree of vertex V is the no. of edges incident with V. where we consider self loop as twice. The degree of the vertex will be denoted by deg (V) or d(V)



$deg(v_1) = 4$
$deg(v_2) = 6$
$deg(v_3) = 6$
$deg(v_4) = 5$
$de(v_5) = 1$

Isolated vertex :-

In a graph, a vertex that is not adjacent to any vertex is called an Isolated vertex.

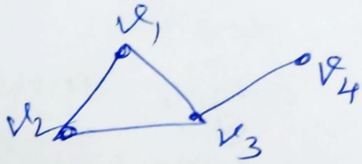


e is an isolated vertex.

Note :- deg of ~~the~~ an isolated vertex is zero

pendent vertex :-

A vertex that is adjacent to exactly one other vertex is called a pendent vertex.



Note :- A vertex in a graph G is called pendent vertex if its degree is one.

Trivial graph :-

A graph whose vertex set containing only one vertex & edge set is empty is called trivial graph.

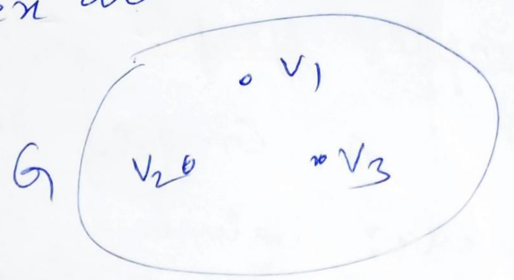
that

pendent edge :-

An edge which is incident with the pendent vertex is called pendent edge.

Null graph :-

A Graph G is called a null graph if it has no edge. For the null graph all vertex are isolated vertex.

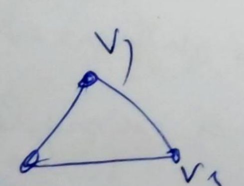


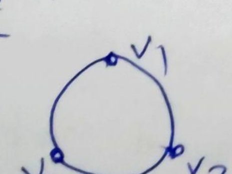
Representation of a graph :-

In a plane, vertex set of graph is represented by points & an edge is represented by a line segment or an arc not necessarily straig

$G(V, E)$ $V = \{v_1, v_2, v_3\}$

$E = \{e_1, e_2, e_3\}$





Finite & Infinite graph :-

A Graph G(V, E) is said to be finite if it has finite number of vertices & edges. Otherwise it is said to be infinite.

Types of graphs :-

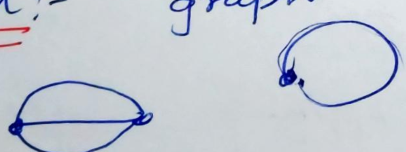
Simple graph :-

A Graph in which there is only one edge between every pair of vertices is called a simple graph.

Graph :-

A graph is a collection of vertices (atleast one) & edges. Each edge must go from a vertex to a vertex.

Ex :- graph



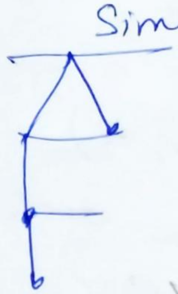
not graph

Simple Graph is a graph, whereas graph need not be a simple graph.

A simple graph is a graph in which there is no more than one edge between any two vertices & in which no edge goes from a vertex to the same vertex.

between any two vertices in the graph there can be 0 edge or 1 edge.

Simple



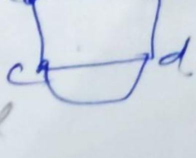
not simple

without self loop & parallel edges

Not simple.

Multigraph :-

A graph which may have loops & multiple edges or parallel edges is called a multi graph.



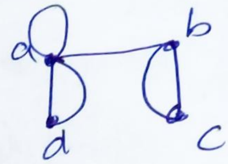
* A graph G having no self loops but containing parallel edges.

Weighted graph :-

A graph in which a number is assigned to each edge is called a Weighted graph.

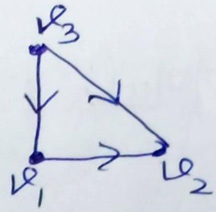
Pseudo graph :-

A graph which has loops and parallel edges is called a pseudo graph.



Directed graph :-

If in a graph G(V, E) each edge is associated with an ordered pairs of vertices, then G(V, E) is called as Directed graph or diagraph



undirected graph :-

If an edge e is associated with an unordered pairs of vertices then G is called an undirected graph.

Parallel edges :- (or) Multiple edges

If in a graph (directed or undirected) certain pair of vertices are joined by more than one edge, such edges are called parallel edges.

Degree of a vertex in directed graph :-

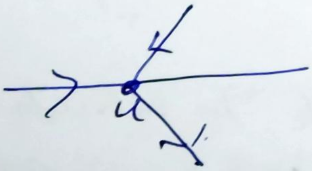
In a directed graph each vertex has an indegree & an out-degree.

no. of edges which are incident towards to u.

" " " " " incident from u.

indegree + $d^{+}(u) = 2$

out " - $d^{-}(u) = 1$



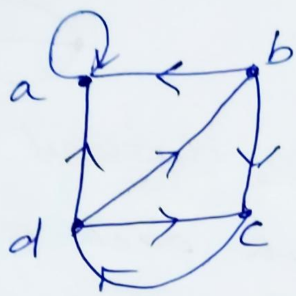
In degree of a graph :-

In degree of vertex v is the number of edges which are coming into the vertex v & is denoted by $deg^{+}(v)$

out degree of a graph :-

outdegree of vertex v is the number of edges which are going out from vertex v & is denoted by $\deg^{-}(v)$

Eg :-



$\deg^{+}(a) = 3$ $\quad d^{-}(a) = 1$

$\deg^{+} b = 1$ $\quad d^{-}(b) = 2$

$\deg^{+} c = 2$ $\quad d(c^{-}) = 1$

$\deg^{+} d = 1$ $\quad d^{-}(d) = 3$

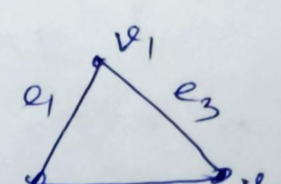
$\overline{\quad 7 \quad}$ $\quad \overline{\quad 7 \quad}$

Note :- loop at a vertex contributes 1 to both indegree & out degree

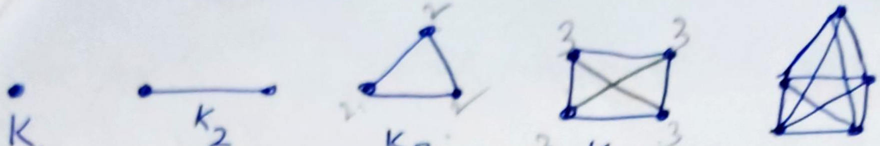
$$\sum \deg^{+}(v) = \sum \deg^{-}(v).$$

complete Graph :-

A simple Graph in which there is exactly one edge between each pair of distinct vertices is called a complete graph.



complete Graph on n vertices is denoted by K_n. every pair of vertices are adjacent



K_1 K_2 K_3 K_4 K_5

no. of Edges $= {}^nC_2 = \frac{n(n-1)}{2}$

Degree of Every vertex $= (n-1)$

Regular Graph :-

A graph G is said to be regular if every vertex of a simple graph has degree of each vertex ~~same~~ is same.

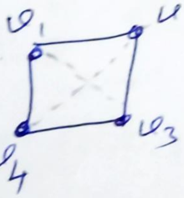
If every vertex in a regular graph has degree n, then the graph is called n-regular

i.e. $\deg(v_i) = \deg(v_j) \quad \forall i, j$

$\quad \& \; v_i, v_j \in G(V, E)$

No. of edges $E = \frac{n \times d}{2}$ $\quad d \rightarrow$ degree of a vertex

EX :-



$d(v_1) = 2$

$de(v_2) = de(v_3) = de(v_4)$

$= 2$.

$\therefore$ Every complete graph is regular graph. But the Converse is not true.

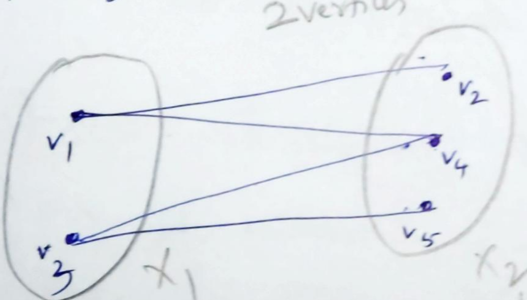
Bipartite graph :-

A Graph G is called Bipartite Graph if its vertex set V is partitioned into two disjoint non-empty subsets i.e. X_1 & X_2 in such that each edge is incident with one vertex from X_1 & other from X_2 & no edge in G connect either two vertices in X_1 or two vertices in X_2.

Eg :- $V = \{v_1, v_2 \dots v_5\}$

$X_1 = \{v_1, v_3\}$ 2 vertices $\quad X_2 = \{v_2, v_4, v_5\}$ 5-2 vertices

It has no loop

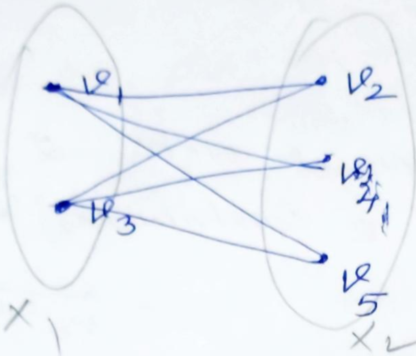


Notation :- complete bipartite graf

If $G(V, E)$ be a bipartite graph with V_1 having m vertices & V_2 having n vertices & is denoted by $K_{m,n}$.

complete bipartite graph :-

A complete bipartite graph is a graph such that there must be an edge between every vertices in X_1 ~~to~~ X_2.



each vertices of X_1 are connected to each vertices of V_2

$K_{m,n} \Rightarrow m$ - no. of ~~edges~~ vertices in X_1

n - no. of ~~edges~~ vertices in V_2

Theorem :- (The Handshaking theorem)

If $G = (V, E)$ is an undirected graph with e edges then $\sum_i \deg v_i = 2e$

i.e. The sum of degree of all the vertices is equal to twice the number of edges.

proof :-

Let G be a graph with n vertices $v_1, v_2 \dots v_n$ with m edges.

we now prove that

$$d(v_1) + d(v_2) + \dots + d(v_n) = 2m$$

If e be an edge, then e is either a loop or incident with two distinct vertices

If e is a loop at vertex v then e contributes two to the degree of v.

If e is incident with 2 vertices say u, v then e contributes one degree to each vertex.

$\therefore$ Sum of the degree of each vertex $= 2m$.

$\because$ Total no. of edges is m.

$\therefore$ total contribution is $2m$.

Theorem 2 :-

The number of vertices of odd degree in an undirected graph is even.

(OR)

In every graph there are an even number of vertices of odd degree.

Proof :-

Let G is a graph with m edges.
Let $v_1, v_2, \dots, v_k$ be the odd degree vertices in G & $u_1, u_2, u_3, \dots, u_r$ be the even degree vertices in G.
Then by the Handshaking theorem,

$$d(v_1)+d(v_2)+\dots d(v_k)+d(u_1)+d(u_2)\dots d(u_r) = 2m$$

$\because$ $u_1, u_2, \dots, u_r$ are even degree vertices

$\therefore$ $d(u_1)+d(u_2)+\dots+d(u_r) = \text{even no.}$

Also $2m$ is an even number.

Thus, $d(v_1)+d(v_2)+\dots d(v_k) = \text{even no.}$

$\because$ Each $d(v_i)$ is odd & their sum is even.

$\therefore$ Number of odd degree vertices must k be an even number.

Theorem 3 :-

The maximum numbers edges in simple graph with n vertices is

$$\frac{n(n-1)}{2}$$

Proof :- Since the graph is simple,

$\therefore$ It has no self loop & no parallel edges. Since it has n vertices

$\therefore$ Each vertex is adjacent to atmost

$(n-1)$ vertices.

i.e the degree of each vertex is atmost $(n-1)$

$\therefore$ Sum of degree of all vertices

is $n(n-1)$

we know that

$n(n-1) = 2m$ $\quad$ $m \rightarrow$ no. of edges

$$m = \frac{n(n-1)}{2}$$

$\therefore$ Thus the maximum no of edges in a simple graph with n vertices $\frac{n(n-1)}{2}$.

Theorem 4 :-

The number of edges in a complete graph with n vertices is $\frac{n(n-1)}{2}$.

Proof :- A complete graph is a simple graph in which there is an edge between every distinct vertex.

∵ the graph G has n vertex

∴ Each vertex must be adjacent with remaining $(n-1)$ vertices.

∴ Degree of each vertex is exactly $(n-1)$. Thus the sum of degree of all vertices is $n(n-1)$

∵ Sum of the vertices = 2 × no. of edges

$$n(n-1) = 2e$$

$$e = \frac{n(n-1)}{2}$$

Isomorphism of Graphs :-

Two graphs $G_1(V_1, E_1)$ $G_2(V_2,$

are Isomorphic if there is an one to one & onto function (a bijection) $f : V_1 \longrightarrow V_2$ such that a, b are adjacent in G_1 iff $f(a)$ & $f(b)$ are adjacent in G_2

i.e. f preserves adjacency & non adjacency.

i.e. $a, b \in V_1 \iff f(a), f(b) \in V_2$

$a, b \notin V_1 \iff f(a), f(b) \notin V_2$

$G_1 \cong G_2$ means G_1 & G_2 are isomorphic.

Note :- An isomorphic graphs must have

(i) the same number of vertices

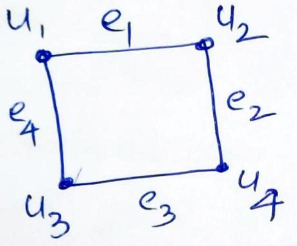
(ii) the same number of edges

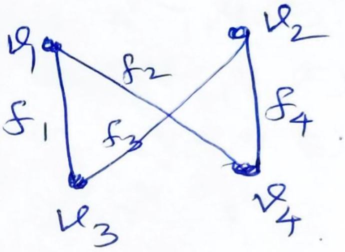
(iii) equal no. of vertices with same degree

This property is called an invariant w.r.t isomorphic Graphs. If any of these conditions is not satisfied in two groups, they cannot be isomorphic. These conditions are necessary but not sufficient

Problems :-

① Show that the following graphs are isomorphic.





$G_1(U, E_1)$ $\qquad$ $G_2(V, E_2)$

$U = \{u_1, u_2, u_3, u_4\}$ $\qquad$ $V = \{v_1, v_2, v_3, v_4\}$

$E_1 = \{e_1, e_2, e_3, e_4\}$ $\qquad$ $E_2 = \{f_1, f_2, f_3, f_4$

No. of vertices of G_1 $= 4$

No. of edges of G_1 $= 4$

$d(u_1) = 2$

$d(u_2) = 2$

$d(u_3) = 2$

$d(u_4) = 2$

No. of vertices of $G_2 = 4$

" " edges of $G_2 = 4$

$d(v_1) = 2$

$d(v_2) = 2$

$d(v_3) = 2$

$d(v_4) = 2$

No of vertices of degree 2 = 4 (G$_1$) | No of v... of degree 2 (G$_2$).

Vertex Adjacency

$$u_1 < \begin{matrix} u_2 \ 2 \\ u_3 \ 2 \end{matrix} \qquad v_1 < \begin{matrix} v_4 \ (2) \\ v_3 \ (2) \end{matrix}$$

Suppose vertex u_1 is corresponds to vertex v_1

$u_1 \Leftrightarrow v_1$
$u_2 \Leftrightarrow v_4$
$u_3 \Leftrightarrow v_3$
$u_4 \Leftrightarrow v_2$

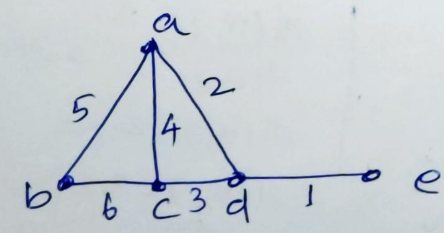
Suppose this vertex correspondence is correct or not by checking edge correspondence with the two graphs.

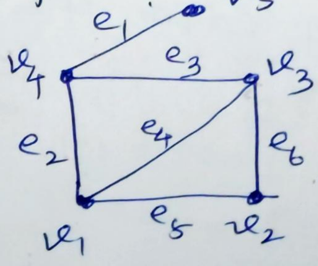
$u_1 - u_2 \Leftrightarrow v_1 - v_4$
$u_1 - u_3 \Leftrightarrow v_1 - v_3$
$u_3 - u_4 \Leftrightarrow v_3 - v_2$
$u_4 - u_2 \Leftrightarrow v_2 - v_4$

every edge in a First graph there is a correspond edge in 2nd graph then we can called edge preserving

clearly f is 1-1 & onto and adjacency is preserved. $\therefore$ $G_1(V_1, E_1)$, $G_2(V, E_2)$ are isomorphic.

(2). Check whether the following Graphs are isomorphic.





$G_1(V_1, E_1)$ $\qquad$ $G_2(V_2, E_2)$

No. of vertices of $G_1 = 5$ = No. of vertices of G_2

No. of edges of $G_1 = 6$ = No. of edges of G_2

$d(G_1)$	$d(G_2)$
$d(a) = 3$	$d(v_1) = 3$
$d(b) = 2$	$d(v_2) = 2$
$d(c) = 3$	$d(v_3) = 3$
$d(d) = 3$	$d(v_4) = 3$
$d(e) = 1$	$d(v_5) = 1$.

No. of vertices of deg 1 in G_1 = 1 = No. of vertices of deg 1 in G_2

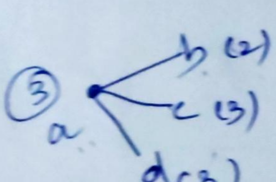
$e = v_5$.

No. of vertices of deg 2 in G_1 = 1 = No. of vertices of deg 2 in G_2

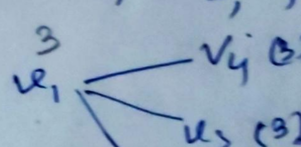
$b = v_2$.

No. of vertices of deg 3 in G_1 = 3 = No. of vertices of deg 3 in G_2

(a c d) $\qquad$ v_1, v_3, v_4



$a \Leftrightarrow v_1$
$b \leftrightarrow v_2$
$c \rightarrow v_3$ | v_4
$d - v_4$ | v_3.
$e \rightarrow v_5$



$a-b \Leftrightarrow u_1-u_2$ $\quad u_1-u_2$

$b-c \Leftrightarrow u_2-u_3$ $\quad u_2-u_4$

$c-d \Leftrightarrow u_3-u_4$ $\quad u_4-u_3$

$c \to a \Leftrightarrow u_3-u_1$ $\quad u_4-u_1$

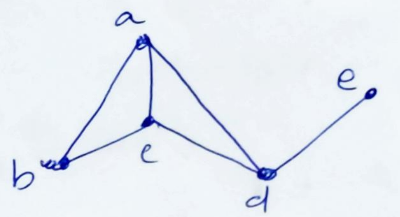
$d \to a \Leftrightarrow u_4-u_1$ $\quad u_3-u_1$

$d-e \Leftrightarrow u_4-u_5$. $\quad u_3-u_5$

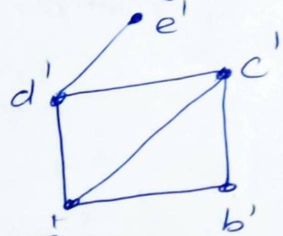
f is a bijection & preserves adjacency

$\therefore$ f is an isomorphism.

③ Check whether the following graphs are isomorphic. If not give reasons



$G_1(V_1, E_1)$



$G_2(V_2, E_2)$

No. of vertices of G_1 = 5 = No. of vertices of G_2

No. of edges of G_1 = 6 = " " edges of G_2

$d(G_1)$	$d(G_2)$
$d(a) = 3$	$d(a') = 3$
$d(b) = 2$	$d(b') = 2$
$d(c) = 3$	$d(c') = 3$
$d(d) = 3$	$d(d') = 3$
$d(e) = 1$	$d(e') = 1$

No. of vertices of deg 1 = 1 = No. of vertices of deg 1
in G_1

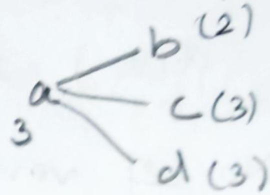
$e = e'$.

No. of vertices of deg 2 = 1 = No. of vertices of deg 2 in G_2

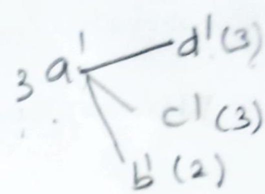
$b = b'$

No. of vertices of deg 3 = 3 = No. of vertices of deg 3 in G_2
in G_1

a, c, d (a', c', d')

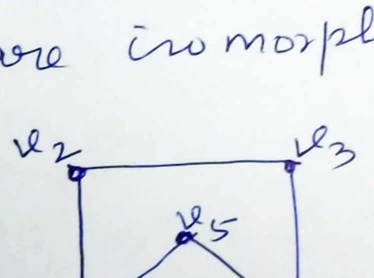


$a \to a'$ $a-b = a'-b'$
$b \leftrightarrow b'$ $a-c = a'-c'$
$c \to c'$ | d' | $a-d = a'-d'$
$d \to d'$ | c' | $b-c = b'-c'$ d'
$e \leftrightarrow e'$ $c-d - c'-d'$ $d'-c'$
$d-e - d'-e'$

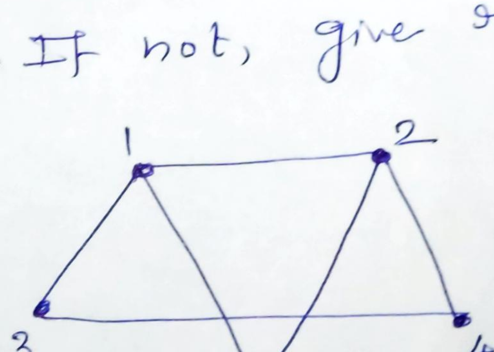


∴ f is a bijection & preserves adjacency ∴ f is an isomorphism.

④. Check whether the following graphs are isomorphic. If not, give reason.



$G_1(V_1, E_1)$



$G_2(V_2, E_2)$

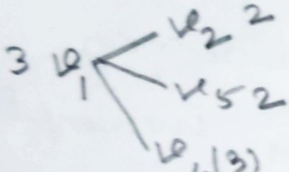
Soln. No. of vertices in $G_1 = 5 =$ No. of vertices in G_2

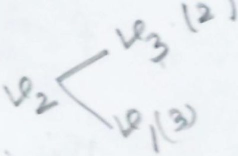
No. of edges in $G_1 = 6 =$ No. of edges in G_2

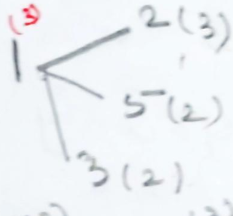
$d(G_1)$	$d(G_2)$
$d(v_1) = 3$	$d(1) = 3$
$d(v_2) = 2$	$d(2) = 3$
$d(v_3) = 2$	$d(3) = 2$
$d(v_4) = 3$	$d(4) = 2$
$d(v_5) = 2$	$d(5) = 2$

No. of vertices of deg 2 in G_1 $= 3 =$ No. of vertices of deg 2 in G_2

No. of vertices of deg 3 in G_1 $= 2 =$ No. of vertices of deg 3 in G_2







$v_1 \to 1$

$v_2 - 3$

$v_3 - 4$

$v_4 - 2$

$v_5 - 5$

$v_1 - v_2 = 1 - 3$

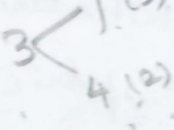
$v_1 - v_4 \Rightarrow 1 - 2$

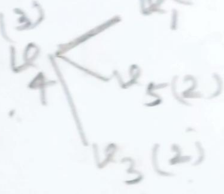
$v_1 - v_5 - 1 - 5$

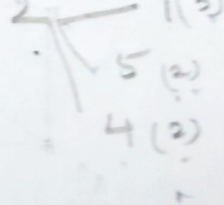
$v_2 - v_3 \leftrightarrow 3 - 4$

$v_3 - v_4 \leftrightarrow 4 - 2$

$v_4 - v_5 \leftrightarrow 2 - 5$

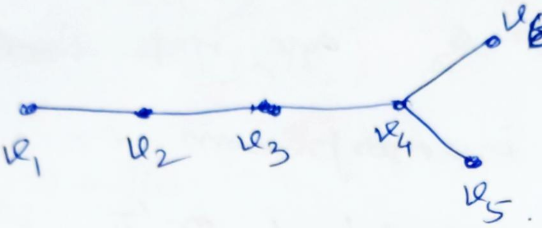




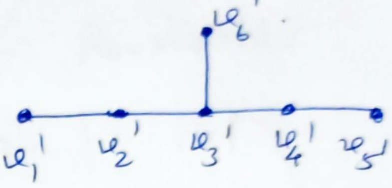


f is a bijection & preserves adjacency
$\therefore f$ is an isomorphism.

(5) Check whether the following graphs are isomorphic. If not, give reason



$G_1(V_1, E_1)$



$G_2(V_1', E_2)$

Soln

No. of vertices in G_1 = 6 = No of vertices of G_2

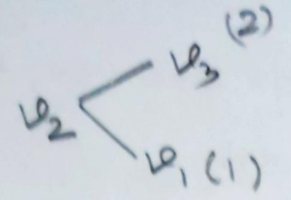
No. of edges in G_1 = 5 = No of edges in G_2

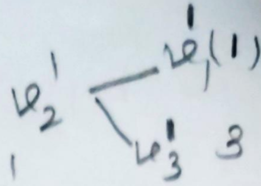
$d(G_1)$	$d(G_2)$
$d(v_1) = 1$	$d(v_1') = 1$
$d(v_2) = 2$	$d(v_2') = 2$
$d(v_3) = 2$	$d(v_3') = 3$
$d(v_4) = 3$	$d(v_4') = 2$
$d(v_5) = 1$	$d(v_5') = 1$
$d(v_6) = 1$	$d(v_6') = 1$

No of vertices of deg 1 in G_1 = 3 = No of vertices of deg 1 in G_2

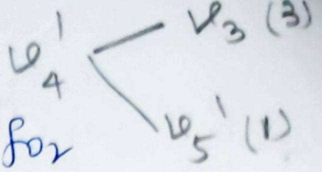
No of vertices of deg 2 in G_1 = 2 = No " " deg 2

No. of vertices of deg 3 in G_1 = 1 = No. of
v_4 v_3'





$v_2 \neq v_2'$



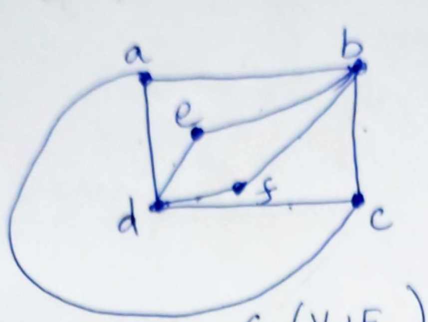
∴ Necessary conditions for graph isomorphism are satisfied.

However G_1 & G_2 are not isomorphic. For if any isomorphism exists then v_4 must be mapped onto v_3'. But v_3' is adjacent to only one pendent vertex namely v_6' whereas v_4 is adjacent to two pendent vertices namely v_5 & v_6.

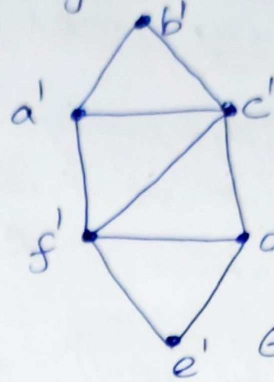
Two Graphs G & G' are said to be isomorphic. If there is a 1-1 correspondence between their vertices & between their edges s.t the adjacency of vertices is preserved. Such graphs will have the same structure differing only in the way their vertices & edges are labelled or only in the way they are represented geometrically.

(3)

6) check whether the following graphs are isomorphic. If not give reason.



$G_1(V_1, E_1)$



$G_2(V_2, E_2)$

soln

No. of vertices in G_1 = 6 = No. of vertices in G_2

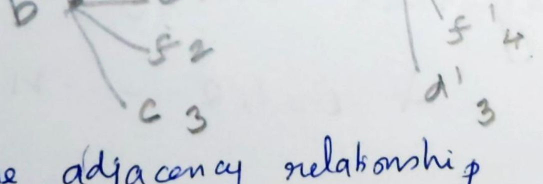
No. of edges in G_1 = 9 = No. of edges in G_2

$d(G_1)$	$d(G_2)$
$d(a) = 3$	$d(a') = 3$
$d(b) = 4$	$d(b') = 2$
$d(c) = 3$	$d(c') = 4$
$d(d) = 4$	$d(d') = 3$
$d(e) = 2$	$d(e') = 2$
$d(f) = 2$	$d(f') = 4$

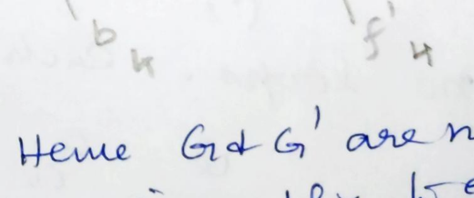
No. of vertices of deg 4 in G_1 = 2 = No. of vertices of deg 4 in G_2

No. of vertices of deg 3 in G_1 = 2 = No. of vertices of deg 3 in G_2

No. of vertices of deg 2 in G_1 = 2 = " " "



The adjacency relationship is violate in vertices having degree 4



Hence G & G' are not isomorphic to each other.

Matrix representation of graphs

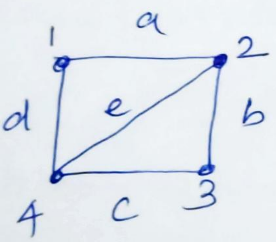
Adjacency matrix :-

Adjacency matrix of a simple graph G with n vertices $v_1, v_2 \dots v_n$ is an $n \times n$ matrix A or

$$A_G = (a_{ij}) \quad \text{where}$$

$$a_{ij} = \begin{cases} 1 & \text{if } v_i, v_j \text{ are adjacent} \\ 0 & \text{otherwise.} \end{cases}$$

Eg :-



Row & column matrix represents vertices

$$\begin{array}{c|c} & \begin{matrix} 1 & 2 & 3 & 4 \end{matrix} \\ \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \end{matrix} & \begin{pmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix} \end{array}$$

row sum = degree of vertex that the row represents.

Properties :-

(i) Since a simple graph has no loops. each diagonal entry of A

ie $a_{ii} = 0 \quad \forall \, i = 1, 2 \dots n$

(ii) Adjacency matrix of simple graph is symmetric

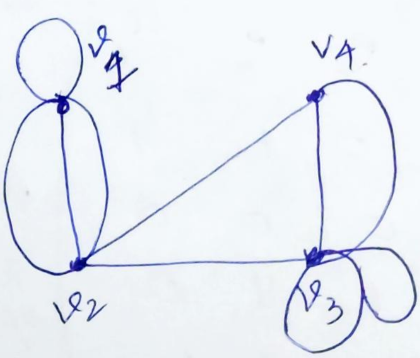
i.e. $a_{ij} = a_{ji}$ both of these entries are 1 where v_i & v_j are adjacent & both zero otherwise.

Adjacency matrix of a pseudograph :-

Adjacency matrix of a pseudograph with n vertices is defined by

$$A_G = (a_{ij}) \text{ where } a_{ij} = \begin{cases} m & \text{if there are } m \text{ edges between } v_i \text{ \& } v_j \\ 1 & \text{if there is a loop at } v_i,\ i = 1, 2 \ldots n \\ 0 & \text{otherwise} \end{cases}$$

Eg :-



$$\begin{array}{c c} & \begin{array}{cccc} v_1 & v_2 & v_3 & v_4 \end{array} \\ \begin{array}{c} v_1 \\ v_2 \\ v_3 \\ v_4 \end{array} & \begin{pmatrix} 1 & 3 & 0 & 0 \\ 3 & 0 & 1 & 1 \\ 0 & 1 & 2 & 2 \\ 0 & 1 & 2 & 0 \end{pmatrix} \end{array}$$

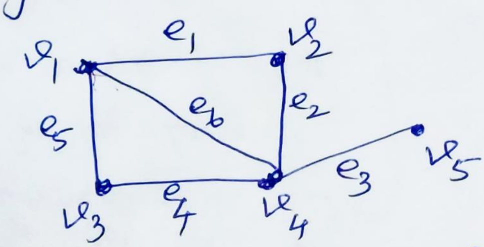
Incidence matrix :-

If $G(V, E)$ is an undire[illegible] graph with n vertices $v_1, v_2 \cdots v_n$ & m edges $e_1, e_2 \cdots e_m$ then the $n \times m$ matrix $B = [b_{ij}]$

$$b_{ij} = \begin{cases} 1 & \text{if edge } e_j \text{ is incident on } v_i \\ 0 & \text{otherwise} \end{cases}$$

is called an incidence matrix of G.

Eg :-



Rows represents vertices & columns represents edges.

$$\begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \\ v_6 \end{array} \begin{array}{c} \begin{array}{cccccc} e_1 & e_2 & e_3 & e_4 & e_5 & e_6 \end{array} \\ \begin{pmatrix} 1 & 0 & 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \end{array}$$

row sum = degree of the vertex

column = 2

every edge has 2 ends

properties :-

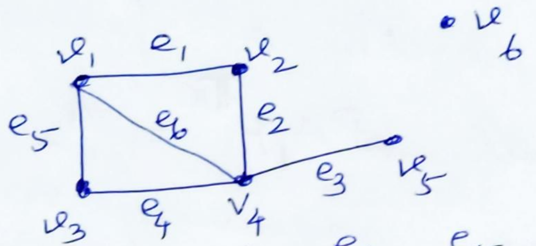
(i) Each column entries contains exactly two unit entries.

(2) A row with all 0 entries corresponds to an isolated vertex

(3) A row with single unit entry corresponds to a pendent vertex

4) $deg(v_j)$ = to the number of 1's in the i^{th} row.

Example :-



$$
\begin{array}{c}
\\ v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \\ v_6
\end{array}
\begin{array}{c}
\begin{array}{cccccc} e_1 & e_2 & e_3 & e_4 & e_5 & e_6 \end{array} \\
\begin{pmatrix}
1 & 0 & 0 & 0 & 1 & 1 \\
1 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 1 & 0 \\
0 & 1 & 1 & 1 & 0 & 1 \\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}
\end{array}
$$

Isomorphism & Adjacency Matrix

Result :- (without proof)

1. Two graphs are isomorphic iff their vertices can be labelled in such a way that the corresponding adjacency matrices are equal.

2. Two graphs G_1 & G_2 are isomorphic iff there exists a permutation matrix P such that $PA_1P^T = A_2$ where A_1 & A_2 are adjacency matrices of G_1 & G_2 resp.

Permutation matrix :-

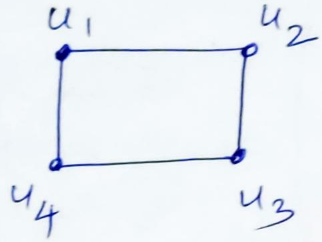
A matrix whose rows are the rows of the unit matrix, but not necessarily in the same order, is called a permutation matrix.

Ex :-
$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = I_3$$

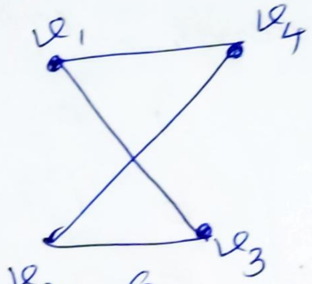
$$\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$ is a permutation matrix.

Example :-

(1) Prove that the following graphs are isomorphic $G_1(V_1, E_1)$, $G_2(V_2, E_2)$ using adjacency matrices.



G_1



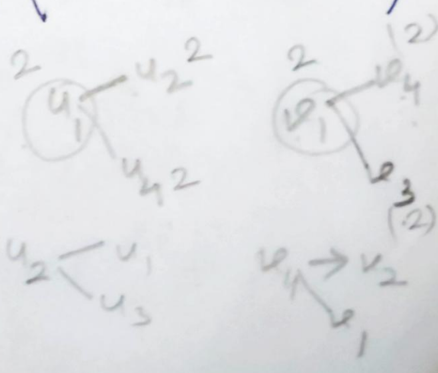
G_2

Adjacency matrix of G_1, G_2

$$\begin{array}{c} \\ u_1 \\ u_2 \\ u_3 \\ u_4 \end{array} \begin{array}{c} \begin{array}{cccc} u_1 & u_2 & u_3 & u_4 \end{array} \\ \begin{pmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{pmatrix} \end{array} \qquad \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \end{array} \begin{array}{c} \begin{array}{cccc} v_1 & v_2 & v_3 & v_4 \end{array} \\ \begin{pmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{pmatrix} \end{array}$$

$d(u_1) = 2$
$d(u_2) = 2$
$d(u_3) = 2$
$d(u_4) = 2$

$d(v_1) = 2$
$d(v_2) = 2$
$d(v_3) = 2$
$d(v_4) = 2$



To find a permutation matrix P such that $PA_1P^T = A_2$

A_1 & A_2 are 4×4 matrices.

$\therefore$ P is also a 4×4 matrix whose rows are the rows of unit matrix I_4 not necessarily in the same order.

$$I_4 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$u_1 — v_1$
$u_2 — v_4$
$u_3 — v_2$
$u_4 — v_3$

$\deg(u_1) = 2$; $\deg(v_1) = 2$

Hence first row of P can be taken as the first row of I_4

$u_1 - u_2 \Leftrightarrow v_1 - v_4$
$u_1 - u_4 \Leftrightarrow v_1 - v_3$
$u_4 - u_3 \Leftrightarrow v_3 - v_2$
$u_2 - u_3 \Leftrightarrow v_4 - v_2$

$\deg(u_2) = 2 = \deg(v_4)$

ie ie second vertex of G_1 corresponds to 4^{th} vertex of G_2. $\therefore$ 2^{nd} row of I_4 can be taken as the 4^{th} row of P.

$\deg(u_3) = \deg(v_2) = 2$

3^{rd} vertex of G_1 corresponds to 2^{nd} vertex of G_2.

$\therefore$ 3^{rd} row of I_4 can be taken as the 2^{nd} row of P.

$\deg(u_4) = \deg(v_3)$.

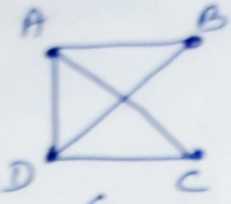
4^{th} vertex of G_1 corresponds to 3^{rd} vertex of G_2. $\therefore$ 4^{th} row of I_4 corresponds to 3^{rd} row of P.

$$P = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{pmatrix} \qquad \begin{pmatrix} 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \end{pmatrix}$$

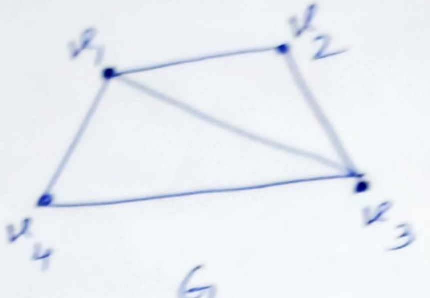
$$PA_1P^T = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{pmatrix} = A_2$$

(2) Establish the isomorphism of two graphs given in the fig by considering their adjacency matrices



G_1



G_2.

Proof:-

Adjacency matrices A_1 & A_2 of G_1 & G_2 resp are given below.

$$A_1 = \begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix} \qquad A_2 = \begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{pmatrix}$$

The matrices A_1 & A_2 are not the same

To establish isomorphism between G_1 & G_2, we have to find a permutation matrix P such that $PA_1P^T = A_2$.

Since A_1 & A_2 are fourth order matrices. P is a 4^{th} order matrix got by permuting the rows of the unit matrix I_4.

number of vertices & edges of the two graphs G_1 & G_2 are same.

G_1 | G_2

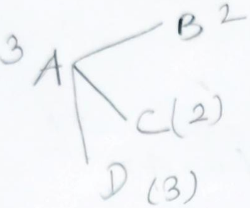
$d(A) = 3$ | $d(V_1) = 3$

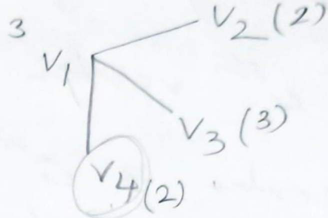
$d(B) = 2$ | $d(V_2) = 2$

$d(C) = 2$ | $d(V_3) = 3$

$d(D) = 3$ | $d(V_4) = 2$

equal no of vertices with same degree





A	V_1	
B	V_2	V_4
C	V_4	V_2
D	V_3	

$A - B \iff V_1 - V_2 \; V_4$

$A - D \iff V_1 - V_3$

$A - C \iff V_1 - V_4 \; V_2$

$B - D \iff V_2 \; V_4 - V_3$

$D - C \iff V_3 - V_4 \; V_2$

$$I_4 \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$P \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \text{ or } \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{pmatrix} ✓$$

P is also a 4×4 matrix whose rows are the rows of unit matrix I_4 not necessarily in the same order

$$I_4 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$d(A) = 3 = \deg(v_1)$

Hence First row of P can be taken as 1^{st} row of I_4.

$d(D) = d(V_3)$

ie 4^{th} vertex of G_1 corresponds to 3^{rd} of G_2

ie 4^{th} row of I_4 may can be taken as 3^{rd} r[illegible] of P.

$\deg(B) = \deg(C) = 2$ & $\deg(v_2) = \deg(v_4) = 2$

ie 2^{nd} vertex of G_1 may corresponds to 2^{nd} or 4^{th} vertex of G_2.

IIIly 3^{rd} vertex of G_1 may be corresponds to 4^{th} or 2^{nd} vertex of G_2.

Hence, 2^{nd} & 3^{rd} row of I_4 may be taken ~~as~~ either as the 2^{nd} & 4^{th} rows of P or as the 4^{th} & 2^{nd} row of P.

Thus there are 2 possible forms of P namely

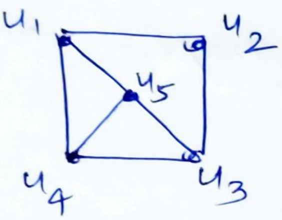
$$P = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \text{ or } \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

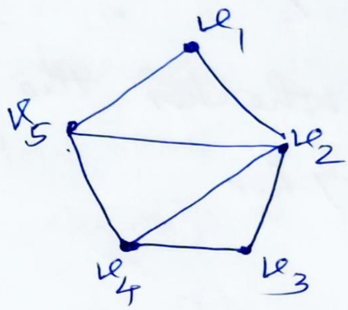
for b[illegible] easily v[illegible] Hence the[illegible]

For both the forms of P, it is easily verified that $PA_1P^T = A_2$

Hence the two graphs G_1 & G_2 are isomorphic.

(3) Determine whether the graphs are isomorphic.





5 vertices
7 edges

5 vertices
7 - edges

$d(u_1) = 3$
$d(u_2) = 2$
$d(u_3) = 3$
$d(u_4) = 3$
$d(u_5) = 3$

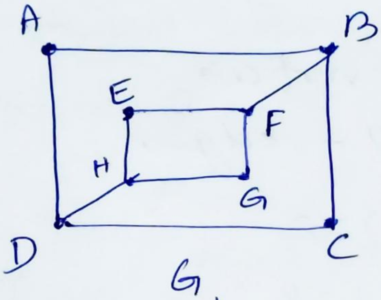
$d(v_1) = 2$
$d(v_2) = 4$
$d(v_3) = 2$
$d(v_4) = 3$
$d(v_5) = 3$

In G_1, the vertex u_2 is of degree 2 & all the other vertices are of degree 3 each. In other graph 2 vertices v_1 & v_3 are of degree 2

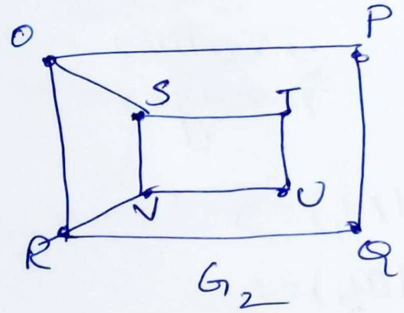
2 vertices v_4 & v_5 are of deg[ree]
3 & the vertex v_2 is of degree

Though there are equal number of vertices & equal number of edges in the two graphs, the degree of vertices are not invariant. Hence the two graphs are not isomorphic.

(4) Check whether the two graphs are isomorphic.



G_1.



G_2

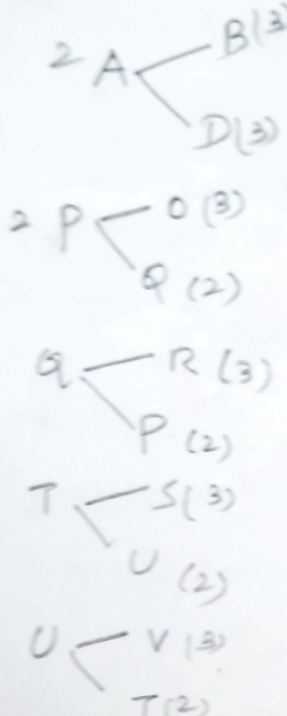
8 vertices
10 edges.

8-vertices
10-edges.

$d(A) = 2$
$d(B) = 3$.
$d(C) = 2$
$d(D) = 3$
$d(E) = 2$
$d(F) = 3$
$d(G) = 2$
$d(H) = 3$.

$d(O) = 3$
$d(P) = 2$ ✓
$d(Q) = 2$ ✓
$d(R) = 3$
$d(S) = 3$
$d(T) = 2$ ✓
$d(U) = 2$ ✓
$d(V) = 3$.

Also ~~they~~ both the Graph have 4 vertices each of degree 2 & 4 vertices each of degree 3.



Vertex A must corresponds to either P, Q, T or U which are of degree 2 each in G_2.

Now each of the vertices P, Q, T, U is adjacent to another vertex of degree 2.

But A is not adjacent to any vertex of degree 2 in G_1.

~~Hence~~ the two graphs G_1 & G_2 are not isomorphic.

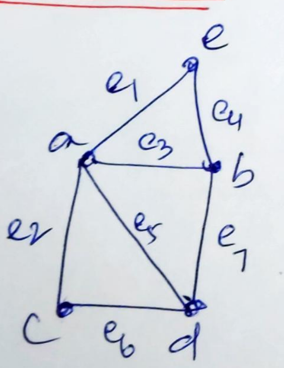
Subgraphs :-

A graph $H(V_2, E_2)$ is said to be a subgraph of $G(V_1, E_1)$ if $V_2 \subseteq V_1$ & $E_2 \subseteq E_1$.

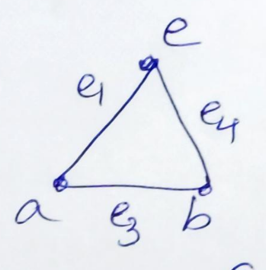
If $V_2 \subsetneq V_1$ & $E_2 \subsetneq E_1$ then H is called proper subgraph of G.

ie $H(V_2, E_2)$ & $G(V_1, E_1)$
if V_2 is a subset of V_1
& E_2 is a " " E_1.
then H is a subgraph of G.

For Ex :-



G_1



G_2

Set of vertices of G_2 is a subset of set of " " G_1
& set of edges of G_2 is a subset of set of edges of G_1.

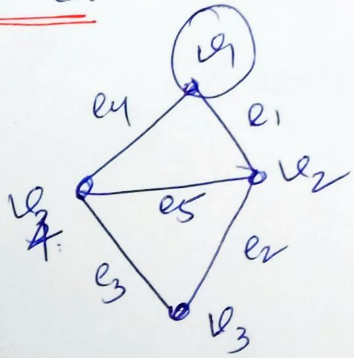
Hence G_2 is a subgraph of G_1.

<u>Note</u> :-

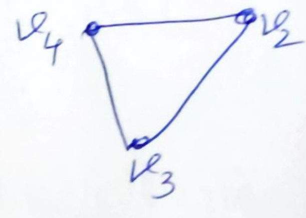
Any subgroup of a graph G can be obtained by removing certain vertices & edges from G.

Removal of an edge does not go with the removal of its adjacent vertices whereas removal of a vertex goes with the removal of any edge incident on it.

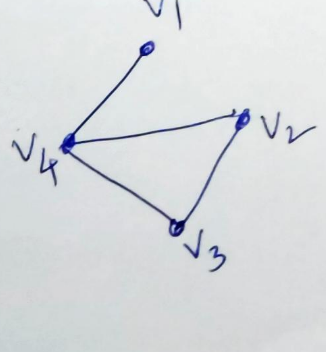
<u>For ex</u> :-



Vertex deletion (remove all edges incident with that vertex as well)



edge deletion.

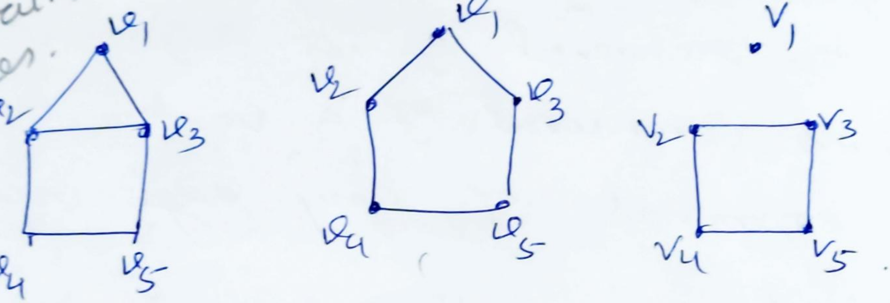


$G|e_1$

Spanning subgraph :-

A spanning subgraph is a subgraph obtained only by edge deletion. In otherswords the vertex set of the subgraph is the entire vertex set of the original graph.

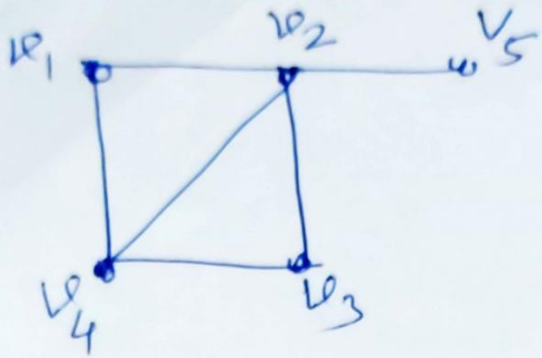
It contains all vertices.



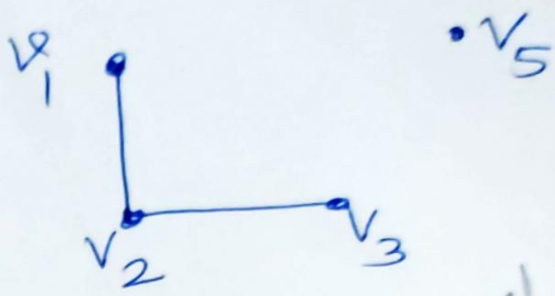
Induced subgraph :-

A subgraph obtained only by vertex deletion is called an induced subgraph.

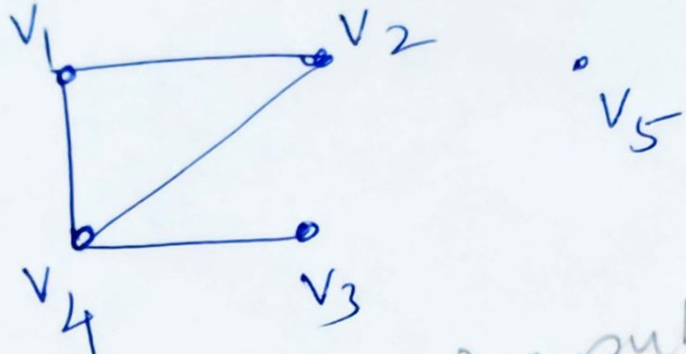
Graph G + X is the set of vertices that are deleted the resulting subgraph is $G - X$. Remove vertex + incident edges.



$G - v_2$



induced subgraph



spanning subgraph