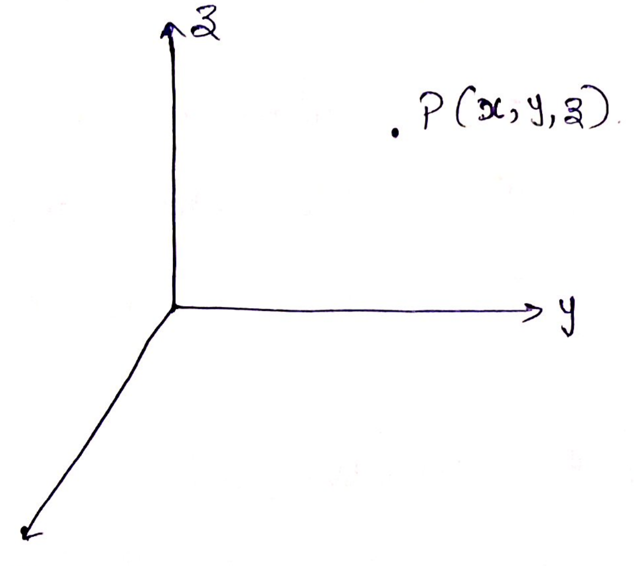
COORDINATE SYSTEMS AND TRANSFORMATION:-

- A point or vector can be represented in any curvilinear coordinate system, which may be orthogonal or non orthogonal.
- An orthogonal system is one in which the coordinates are mutually perpendicular.

  eg:- Cartesian (or rectangular), the circular cylindrical, the spherical, the elliptic cylindrical, the parabolic cylindrical.

CARTESIAN COORDINATES (X, Y, Z) :-



A point P can be represented as (x, y, z).

The ranges of the coordinate variables x, y and z are

$$-\infty < x < \infty$$
$$-\infty < y < \infty$$
$$-\infty < z < \infty$$

A vector $\vec{A}$ in Cartesian or rectangular coordinates can be written as (A_x, A_y, A_z) or $A_x\vec{a}_x + A_y\vec{a}_y + A_z\vec{a}_z$.

where $\vec{a}_x$, $\vec{a}_y$ and $\vec{a}_z$ are unit vectors along the x, y and z directions

$$\boxed{\vec{A} = A_x\vec{a}_x + A_y\vec{a}_y + A_z\vec{a}_z}$$

*. **CIRCULAR CYLINDRICAL COORDINATES :-**

The circular cylindrical coordinate system is very convenient whenever we are dealing with problems having cylindrical symmetry.

A point P in cylindrical coordinates is represented as (ρ, ϕ, z). where ρ - radius of the cylinder

ϕ - azimuthal angle.

z - same as in the Cartesian system.

The ranges of the variables are

$$0 \leq \rho < \infty$$
$$0 \leq \phi < 2\pi$$
$$-\infty < z < \infty$$

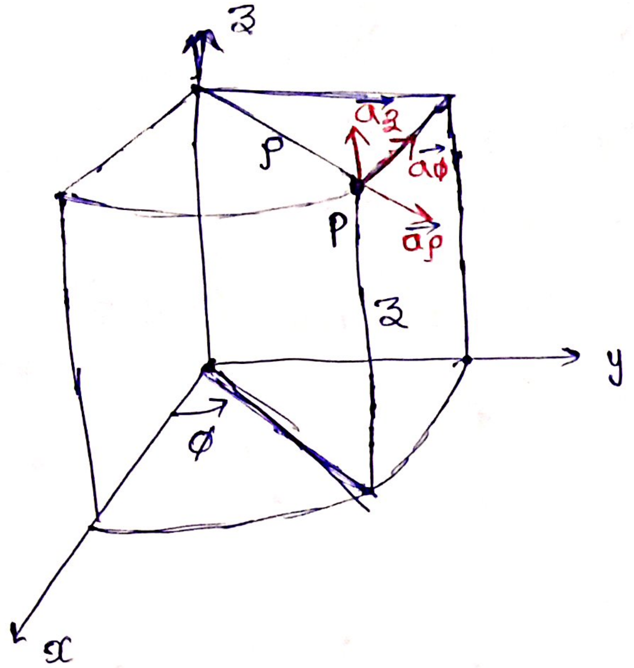


Fig:- Point P and unit vectors in the cylindrical coordinate system.

A vector '$\vec{A}$' in cylindrical coordinates can be written as (A_ρ, A_ϕ, A_z) or $A_\rho \vec{a}_\rho + A_\phi \vec{a}_\phi + A_z \vec{a}_z$. where $\vec{a}_\rho, \vec{a}_\phi$ and $\vec{a}_z$ are unit vectors in ρ, ϕ and z directions.

NOTE:- $\vec{a}_\phi$ is not in degrees; it assumes the units of $\vec{A}$. For example, if a force of 10N acts on a particle in a circular motion, the force may be represented as $\vec{F} = 10\vec{a}_\phi N$. In this case a_ϕ is in newtons.

The magnitude of '$\vec{A}$' is $|\vec{A}| = \sqrt{A_\rho^2 + A_\phi^2 + A_z^2}$.

Notice that the unit vectors $\vec{a}_\rho, \vec{a}_\phi$ and $\vec{a}_z$ are mutually perpendicular because it is orthogonal. then

$$\vec{a}_\rho \cdot \vec{a}_\rho = \vec{a}_\phi \cdot \vec{a}_\phi = \vec{a}_z \cdot \vec{a}_z = 1.$$

$$\vec{a}_\rho \cdot \vec{a}_\phi = \vec{a}_\phi \cdot \vec{a}_z = \vec{a}_z \cdot \vec{a}_\rho = 0.$$

$$\vec{a}_\rho \times \vec{a}_\phi = \vec{a}_z$$
$$\vec{a}_\phi \times \vec{a}_z = \vec{a}_\rho$$
$$\vec{a}_z \times \vec{a}_\rho = \vec{a}_\phi .$$

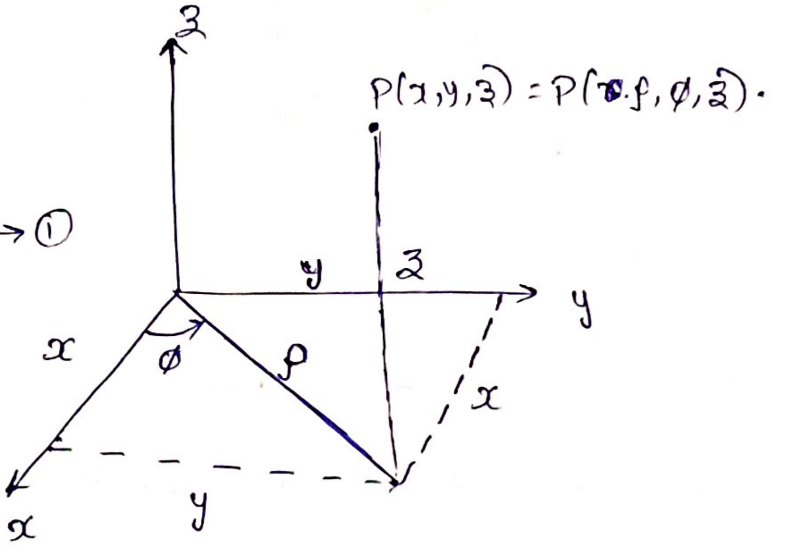
Relationship b/w (x, y, z) and (ρ, ϕ, z).

$$\sin\phi = \frac{y}{\rho}$$

$\therefore$ $\boxed{y = \rho \sin\phi} \longrightarrow (1)$

$$\cos\phi = \frac{x}{\rho}$$

$\therefore$ $\boxed{x = \rho \cos\phi} \longrightarrow (2)$



$z = z . \longrightarrow (3)$

$\boxed{\therefore\ x = \rho\cos\phi,\ y = \rho\sin\phi \text{ and } z = z} \longrightarrow (4)$ → Transforming a point from cylindrical to Cartesian coordinates.

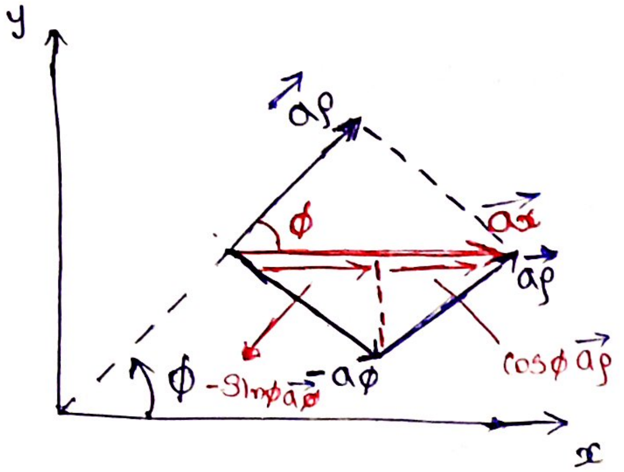
$$(1)^2 + (2)^2 \Rightarrow x^2 + y^2 = \rho^2\sin^2\phi + \rho^2\cos^2\phi$$
$$\Rightarrow x^2 + y^2 = \rho^2(\sin^2\phi + \cos^2\phi)$$
$$\Rightarrow \rho = \sqrt{x^2 + y^2} .$$

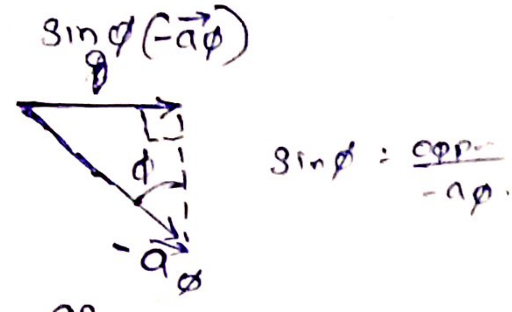
$$\frac{(1)}{(2)} \Rightarrow \frac{y}{x} = \frac{\rho\sin\phi}{\rho\cos\phi}$$
$$\Rightarrow \tan\phi = \frac{y}{x}$$
$$\Rightarrow \phi = \tan^{-1}\left(\frac{y}{x}\right).$$

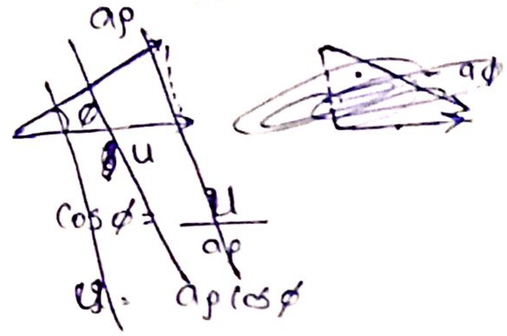
$\boxed{\therefore\ \rho = \sqrt{x^2+y^2},\ \phi = \tan^{-1}\left(\frac{y}{x}\right),\ z = z.} \longrightarrow (5)$

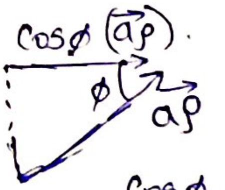
↳ Transforming a point from Cartesian to Cylindrical coordinates.

The relationships between $(\vec{a}_x, \vec{a}_y, \vec{a}_z)$ and $(\vec{a}_\rho, \vec{a}_\phi, \vec{a}_z)$ are obtained geometrically



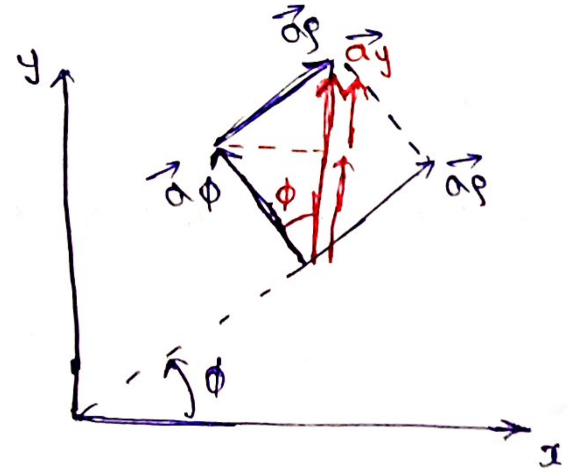


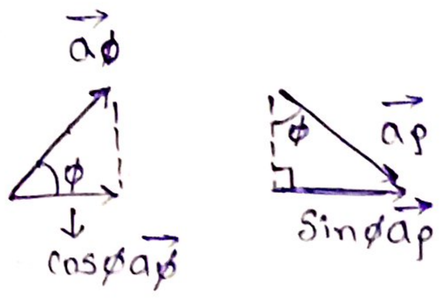




$$\cos\phi = \frac{adj}{\vec{a}_\rho}$$

$$\therefore \vec{a}_x = \cos\phi\,\vec{a}_\rho - \sin\phi\,\vec{a}_\phi.$$





$$\therefore \vec{a}_y = \sin\phi\,\vec{a}_\rho + \cos\phi\,\vec{a}_\phi.$$

$$\therefore \vec{a}_x = \cos\phi\,\vec{a}_\rho - \sin\phi\,\vec{a}_\phi \longrightarrow (6)$$

$$\vec{a}_y = \sin\phi\,\vec{a}_\rho + \cos\phi\,\vec{a}_\phi \longrightarrow (7)$$

$$\vec{a}_z = \vec{a}_z. \longrightarrow (8)$$

$$\cos\phi \times (6) \Rightarrow \cos\phi\,\vec{a}_x = \cos^2\phi\,\vec{a}_\rho - \cos\phi\sin\phi\,\vec{a}_\phi \rightarrow (9A)$$

$$\sin\phi \times (7) \Rightarrow \sin\phi\,\vec{a}_y = \sin^2\phi\,\vec{a}_\rho + \cos\phi\sin\phi\,\vec{a}_\phi. \rightarrow (9B)$$

(9A) + (9B) $\therefore \vec{a_\rho} = \cos\phi\,\vec{a_x} + \sin\phi\,\vec{a_y}$

$\sin\phi \times (6) \Rightarrow \quad \sin\phi\,\vec{a_x} = \sin\phi\cos\phi\,\vec{a_\rho} - \sin^2\phi\,\vec{a_\phi} \longrightarrow$ (10A)

$\cos\phi \times (7) \Rightarrow \quad \cos\phi\,\vec{a_y} = \sin\phi\cos\phi\,\vec{a_\rho} + \cos^2\phi\,\vec{a_\phi}. \longrightarrow$ (10B).

$-$ (10A) + (10B) $\Rightarrow \quad \vec{a_\phi} = -\sin\phi\,\vec{a_x} + \cos\phi\,\vec{a_y}$

and $\vec{a_z} = \vec{a_z}$.

$$\therefore \quad \vec{a_\rho} = \cos\phi\,\vec{a_x} + \sin\phi\,\vec{a_y} \longrightarrow (9)$$

$$\vec{a_\phi} = -\sin\phi\,\vec{a_x} + \cos\phi\,\vec{a_y} \longrightarrow (10)$$

$$\vec{a_z} = \vec{a_z} \longrightarrow (11)$$

we know in Cartesian coordinates

$$\vec{A} = A_x\vec{a_x} + A_y\vec{a_y} + A_z\vec{a_z}.$$

$$\Rightarrow \quad A = A_x(\cos\phi\,\vec{a_\rho} - \sin\phi\,\vec{a_\phi}) + A_y(\sin\phi\,\vec{a_\rho} + \cos\phi\,\vec{a_\phi}) + A_z\vec{a_z}.$$

$$\Rightarrow \quad A = (A_x\cos\phi + A_y\sin\phi)\vec{a_\rho} + (-A_x\sin\phi + A_y\cos\phi)\vec{a_\phi} + A_z\vec{a_z}$$

$$\therefore \quad A_\rho = A_x\cos\phi + A_y\sin\phi$$

$$A_\phi = -A_x\sin\phi + A_y\cos\phi$$

$$A_z = A_z$$

In matrix form

$$\begin{bmatrix} A_\rho \\ A_\phi \\ A_z \end{bmatrix} = \begin{bmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix} \longrightarrow (12)$$

In circular cylindrical $A = A_\rho \vec{a}_\rho + A_\phi \vec{a}_\phi + A_z \vec{a}_z$

$$\Rightarrow A = A_\rho(\cos\phi\, \vec{a}_x + \sin\phi\, \vec{a}_y) + A_\phi(-\sin\phi\, \vec{a}_x + \cos\phi\, \vec{a}_y) + A_z \vec{a}_z$$

$$\Rightarrow A = \underbrace{(A_\rho \cos\phi - A_\phi \sin\phi)}_{A_x}\vec{a}_x + \underbrace{(A_\rho \sin\phi + A_\phi \cos\phi)}_{A_y}\vec{a}_y + A_z \vec{a}_z$$

in Matrix form

$$\begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix} = \begin{bmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} A_\rho \\ A_\phi \\ A_z \end{bmatrix} \longrightarrow (13)$$

An alternative way of obtaining eq (12) & (13) is by using dot product.

$$\begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix} = \begin{bmatrix} \vec{a}_x\cdot\vec{a}_\rho & \vec{a}_x\cdot\vec{a}_\phi & \vec{a}_x\cdot\vec{a}_z \\ \vec{a}_y\cdot\vec{a}_\rho & \vec{a}_y\cdot\vec{a}_\phi & \vec{a}_y\cdot\vec{a}_z \\ \vec{a}_z\cdot\vec{a}_\rho & \vec{a}_z\cdot\vec{a}_\phi & \vec{a}_z\cdot\vec{a}_z \end{bmatrix} \begin{bmatrix} A_\rho \\ A_\phi \\ A_z \end{bmatrix}$$

* SPHERICAL COORDINATES (r, θ, ϕ) :-

- The spherical coordinate system is most appropriate when one is dealing with problems having a degree of spherical symmetry.
- A point 'P' can be represented as (r, θ, ϕ).

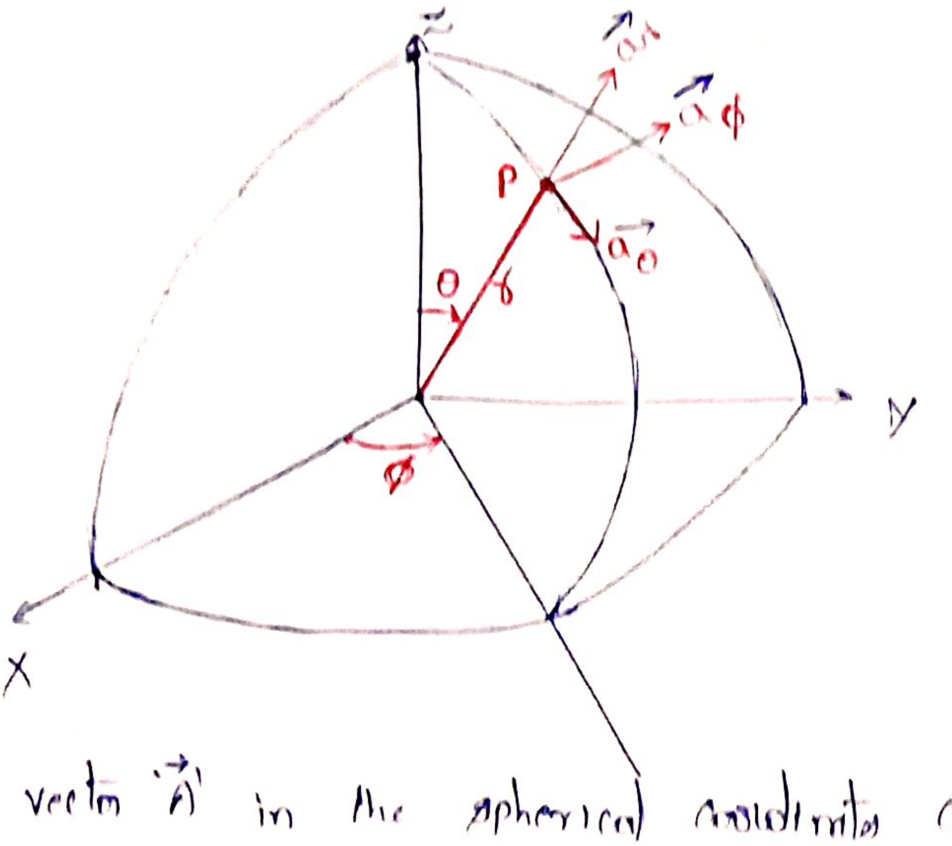
  'r' - the radius of the sphere.

  θ - angle between the z-axis and the position vector of 'P'.

  ϕ - angle measured from the x-axis.

ranges:

$$0 \le r < \infty$$
$$0 \le \theta \le \pi$$
$$0 \le \phi < 2\pi.$$



A vector '$\vec{A}$' in the spherical coordinates can be written as

(A_r, A_θ, A_ϕ) or $\vec{A} = A_r\vec{a}_r + A_\theta\vec{a}_\theta + A_\phi\vec{a}_\phi$.

where $\vec{a}_r, \vec{a}_\theta$ & $\vec{a}_\phi$ are unit vectors along the r, θ, ϕ directions.

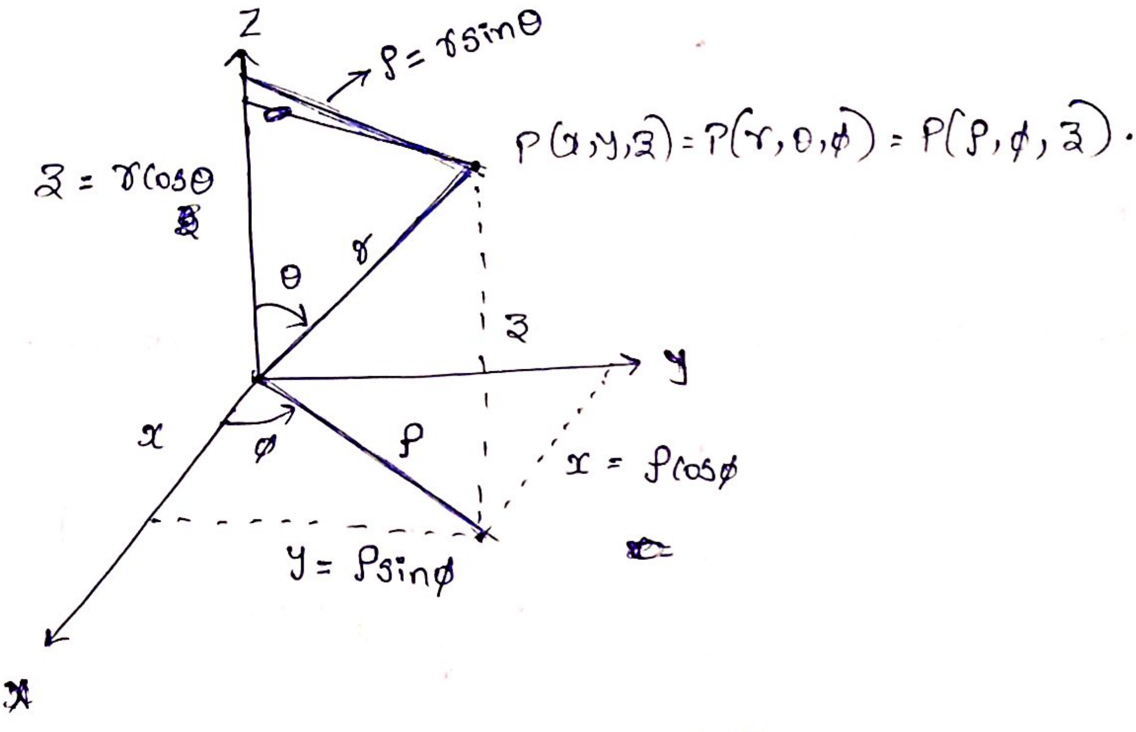
The magnitude of A is $|\vec{A}| = \sqrt{A_r^2 + A_\theta^2 + A_\phi^2}$.

$$\vec{a}_r \cdot \vec{a}_r = \vec{a}_\theta \cdot \vec{a}_\theta = \vec{a}_\phi \cdot \vec{a}_\phi = 1$$

$$\vec{a}_r \cdot \vec{a}_\theta = \vec{a}_\theta \cdot \vec{a}_\phi = \vec{a}_\phi \cdot \vec{a}_r = 0$$

$$\vec{a}_r \times \vec{a}_\theta = \vec{a}_\phi, \quad \vec{a}_\theta \times \vec{a}_\phi = \vec{a}_r, \quad \vec{a}_\phi \times \vec{a}_r = \vec{a}_\theta.$$

Transforming a point from Cartesian to spherical, cylindrical to spherical and vica versa :-



$\rho = r\sin\theta$, $\phi = \tan^{-1}\left(\frac{y}{x}\right)$, $\theta = \tan^{-1}\left(\frac{\rho}{z}\right)$.

$r^2 = \rho^2 + z^2 = x^2 + y^2 + z^2$,

$(\because \rho^2 = x^2 + y^2)$

$$\boxed{\begin{aligned} \therefore r &= \sqrt{x^2+y^2+z^2} \\ \theta &= \tan^{-1}\left(\frac{\sqrt{x^2+y^2}}{z}\right) \\ \phi &= \tan^{-1}(y/x). \end{aligned}}$$

Cartesian to Spherical.

$$\boxed{\begin{aligned} x &= \rho\cos\phi = r\sin\theta\cos\phi \\ y &= \rho\sin\phi = r\sin\theta\sin\phi \\ z &= r\cos\theta . \end{aligned}}$$

Spherical to Cartesian.

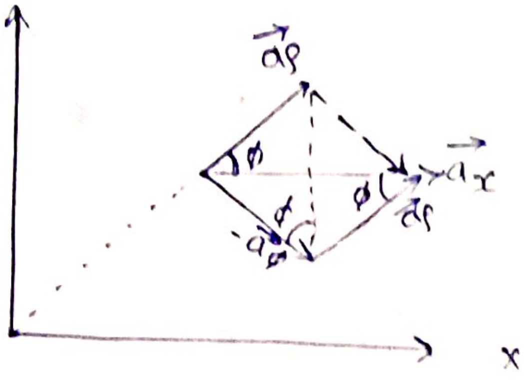
$$\boxed{\begin{aligned} r &= \sqrt{\rho^2 + z^2} \\ \theta &= \tan^{-1}\left(\frac{\rho}{z}\right) \\ \phi &= \phi \end{aligned}}$$

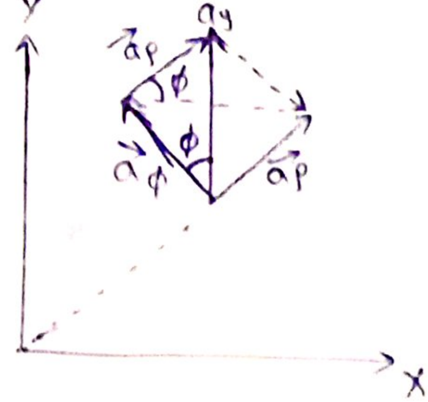
Cylindrical to Spherical

$$\boxed{\begin{aligned} \rho &= r\sin\theta \\ \phi &= \phi \\ z &= r\cos\theta \end{aligned}}$$

Spherical to cylindrical.

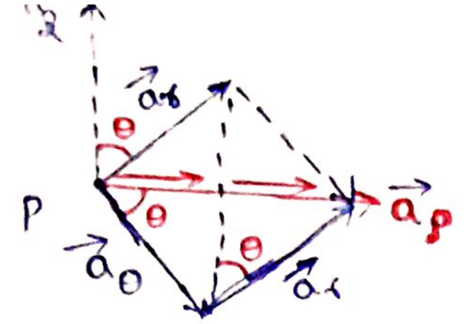
The relationships between unit vectors a_x, a_y, a_z and a_r, a_θ, a_ϕ can be obtained geometrically.

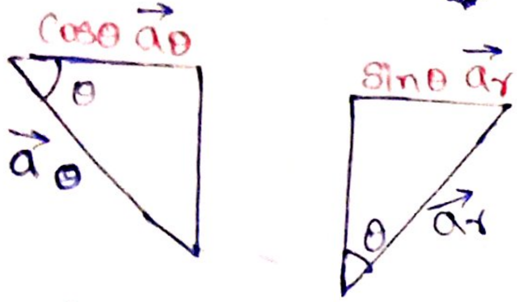




$$\vec{a_x} = \cos\phi\,\vec{a_\rho} - \sin\phi\,\vec{a_\phi}$$

$$\vec{a_y} = \sin\phi\,\vec{a_\rho} + \cos\phi\,\vec{a_\phi}$$

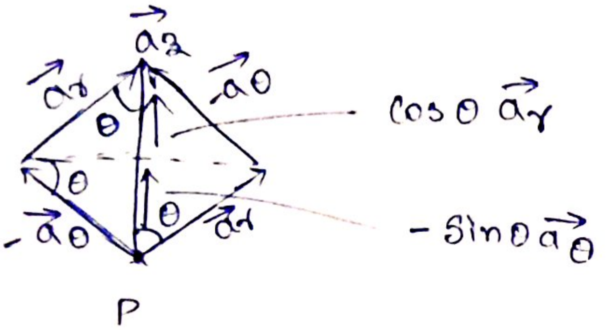


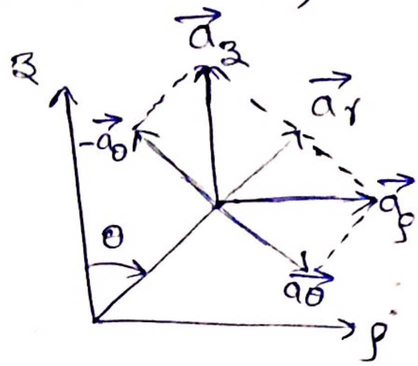


$\therefore \vec{a}_\rho = \sin\theta\,\vec{a}_r + \cos\theta\,\vec{a}_\theta.$

$\vec{a}_x = \sin\theta\cos\phi\,\vec{a}_r + \cos\theta\cos\phi\,\vec{a}_\theta - \sin\phi\,\vec{a}_\phi$

$\vec{a}_y = \sin\theta\sin\phi\,\vec{a}_r + \cos\theta\sin\phi\,\vec{a}_\theta + \cos\phi\,\vec{a}_\phi$





$\vec{a}_z = \cos\theta\,\vec{a}_r - \sin\theta\,\vec{a}_\theta.$

$\therefore \vec{a}_x = \sin\theta\cos\phi\,\vec{a}_r + \cos\theta\cos\phi\,\vec{a}_\theta - \sin\phi\,\vec{a}_\phi \longrightarrow (1)$

$\vec{a}_y = \sin\theta\sin\phi\,\vec{a}_r + \cos\theta\sin\phi\,\vec{a}_\theta + \cos\phi\,\vec{a}_\phi \longrightarrow (2)$

$\vec{a}_z = \cos\theta\,\vec{a}_r - \sin\theta\,\vec{a}_\theta \longrightarrow (3)$

$\sin\theta\cos\phi \times (1) \Rightarrow \sin\theta\cos\phi\,\vec{a}_x = \sin^2\theta\cos^2\phi\,\vec{a}_r + \sin\theta\cos\theta\cos^2\phi\,\vec{a}_\theta - \sin\theta\sin\phi\cos\phi\,\vec{a}_\phi$ $\hookrightarrow (4a)$

$\sin\theta\sin\phi \times (2) \Rightarrow \sin\theta\sin\phi\,\vec{a}_y = \sin^2\theta\sin^2\phi\,\vec{a}_r + \sin\theta\cos\theta\sin^2\phi\,\vec{a}_\theta + \sin\theta\sin\phi\cos\phi\,\vec{a}_\phi$ $\hookrightarrow (4b)$

$\cos\theta \times (3) \Rightarrow \cos\theta\,\vec{a}_z = \cos^2\theta\,\vec{a}_r - \sin\theta\cos\theta\,\vec{a}_\theta$ $\hookrightarrow (4c)$

$(4a) + (4b) + (4c) \Rightarrow \vec{a}_r = \sin\theta\cos\phi\,\vec{a}_x + \sin\theta\sin\phi\,\vec{a}_y + \cos\theta\,\vec{a}_z.$

$$① \times \cos\theta\cos\phi \Rightarrow \cos\theta\cos\phi\,\vec{a}_x = \sin\theta\cos\theta\cos^2\phi\,\vec{a}_r + \cos^2\theta\cos^2\phi\,\vec{a}_\theta - \sin\phi\cos\theta\cos\phi\,\vec{a}_\phi \rightarrow (5a)$$

$$② \times \cos\theta\sin\phi \Rightarrow \cos\theta\sin\phi\,\vec{a}_y = \sin\theta\cos\theta\sin^2\phi\,\vec{a}_r + \cos^2\theta\sin^2\phi\,\vec{a}_\theta + \sin\phi\cos\theta\cos\phi\,\vec{a}_\phi \rightarrow (5b)$$

$$③ \times -\sin\theta \Rightarrow -\sin\theta\,\vec{a}_z = -\sin\theta\cos\theta\,\vec{a}_r + \sin^2\theta\,\vec{a}_\theta \rightarrow (5c)$$

$$(5a) + (5b) + (5c) \quad \vec{a}_\theta = \cos\theta\cos\phi\,\vec{a}_x + \cos\theta\sin\phi\,\vec{a}_y - \sin\theta\,\vec{a}_z .$$

$$① \times -\sin\phi \Rightarrow -\sin\phi\,\vec{a}_x = -\sin\theta\sin\phi\cos\phi\,\vec{a}_r - \sin\phi\cos\theta\cos\phi\,\vec{a}_\theta + \sin^2\phi\,\vec{a}_\phi \rightarrow (6a)$$

$$② \times \cos\phi \Rightarrow \cos\phi\,\vec{a}_y = \sin\theta\sin\phi\cos\phi\,\vec{a}_r + \sin\phi\cos\theta\cos\phi\,\vec{a}_\theta + \cos^2\phi\,\vec{a}_\phi \rightarrow (6b)$$

$$(6a) + (6b) \Rightarrow \vec{a}_\phi = -\sin\phi\,a_x + \cos\phi\,a_y .$$

we know $\quad A = A_r\vec{a}_r + A_\theta\vec{a}_\theta + A_\phi\vec{a}_\phi$

$$\Rightarrow A = A_r(\sin\theta\cos\phi\,\vec{a}_x + \sin\theta\sin\phi\,\vec{a}_y + \cos\theta\,\vec{a}_z) +$$
$$A_\theta(\cos\theta\cos\phi\,\vec{a}_x + \cos\theta\sin\phi\,\vec{a}_y - \sin\theta\,\vec{a}_z) +$$
$$A_\phi(-\sin\phi\,\vec{a}_x + \cos\phi\,\vec{a}_y)$$

$$\Rightarrow A = (A_r\sin\theta\cos\phi + A_\theta\cos\theta\cos\phi - A_\phi\sin\phi)\,\vec{a}_x +$$
$$(A_r\sin\theta\sin\phi + A_\theta\cos\theta\sin\phi + A_\phi\cos\phi)\,\vec{a}_y$$
$$+ (A_r\cos\theta - A_\theta\sin\theta)\,\vec{a}_z .$$

$$\begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix} = \begin{bmatrix} \sin\theta\cos\phi & \cos\theta\cos\phi & -\sin\phi \\ \sin\theta\sin\phi & \cos\theta\sin\phi & \cos\phi \\ \cos\theta & -\sin\theta & 0 \end{bmatrix} \begin{bmatrix} A_r \\ A_\theta \\ A_\phi \end{bmatrix} .$$

$$\begin{bmatrix} A_r \\ A_\theta \\ A_\phi \end{bmatrix} = \begin{bmatrix} \sin\theta\cos\phi & \sin\theta\sin\phi & \cos\theta \\ \cos\theta\cos\phi & \cos\theta\sin\phi & -\sin\theta \\ -\sin\phi & \cos\phi & 0 \end{bmatrix} \begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix} .$$

Transformation	Coordinate variables	Unit vector	Vector components
Cartesian to cylindrical	$\rho = \sqrt{x^2+y^2}$, $\phi = \tan^{-1}(y/x)$ $z = z$	$a_\rho = a_x \cos\phi + a_y \sin\phi$ $a_\phi = -a_x \sin\phi + a_y \cos\phi$ $a_z = a_z$	$A_\rho = A_x \cos\phi + A_y \sin\phi$ $A_\phi = -A_x \sin\phi + A_y \cos\phi$ $A_z = A_z$
Cylindrical to Cartesian	$x = \rho\cos\phi$, $y = \rho\sin\phi$ $z = z$	$a_x = a_\rho \cos\phi - a_\phi \sin\phi$ $a_y = a_\rho \sin\phi + a_\phi \cos\phi$ $a_z = a_z$	$A_x = A_\rho \cos\phi - A_\phi \sin\phi$ $A_y = A_\rho \sin\phi + A_\phi \cos\phi$ $A_z = A_z$
Cartesian to spherical	$r = \sqrt{x^2+y^2+z^2}$, $\theta = \tan^{-1}\left(\frac{\sqrt{x^2+y^2}}{z}\right)$, $\phi = \tan^{-1}\left(\frac{y}{x}\right)$	$a_r = a_x \sin\theta\cos\phi + a_y \sin\theta\sin\phi$ $+ a_z \cos\theta$ $a_\theta = a_x \cos\theta\cos\phi + a_y \cos\theta\sin\phi$ $- a_z \sin\theta$ $a_\phi = -a_x \sin\phi + a_y \cos\phi.$	$A_r = A_x \sin\theta\cos\phi + A_y \sin\theta\sin\phi + A_z \cos\theta$ $A_\theta = A_x \cos\theta\cos\phi + A_y \cos\theta\sin\phi - A_z \sin\theta$ $A_\phi = -A_x \sin\phi + A_y \cos\phi.$
Spherical to Cartesian	$x = r\sin\theta\cos\phi$ $y = r\sin\theta\sin\phi$ $z = r\cos\theta$	$a_x = a_r \sin\theta\cos\phi + a_\theta \cos\theta\cos\phi - a_\phi \sin\phi$ $a_y = a_r \sin\theta\sin\phi + a_\theta \cos\theta\sin\phi + a_\phi \cos\phi$ $a_z = a_r \cos\theta - a_\theta \sin\theta.$	$A_x = A_r \sin\theta\cos\phi + A_\theta \cos\theta\cos\phi - A_\phi \sin\phi$ $A_y = A_r \sin\theta\sin\phi + A_\theta \cos\theta\sin\phi + A_\phi \cos\phi$ $A_z = A_r \cos\theta - A_\theta \sin\theta.$
Cylindrical to spherical	$r = \sqrt{\rho^2+z^2}$, $\theta = \tan^{-1}\left(\frac{\rho}{z}\right)$ $\phi = \phi$	$a_r = a_\rho \sin\theta + a_z \cos\theta$ $a_\theta = a_\rho \cos\theta - a_z \sin\theta$ $a_\phi = a_\phi$	$A_r = A_\rho \sin\theta + A_z \cos\theta$ $A_\theta = A_\rho \cos\theta - A_z \sin\theta$ $A_\phi = A_\phi$
Spherical to cylindrical	$\rho = r\sin\theta$ $\phi = \phi$ $z = r\cos\theta$	$a_\rho = a_r \sin\theta + a_\theta \cos\theta$ $a_\phi = a_\phi$ $a_z = a_r \cos\theta - a_\theta \sin\theta$	$A_\rho = A_r \sin\theta + A_\theta \cos\theta$ $A_\phi = A_\phi$ $A_z = A_r \cos\theta - A_\theta \sin\theta$

$$\begin{bmatrix} \text{[illegible]} \\ A_\theta \\ A_\phi \end{bmatrix} = \begin{bmatrix} a_r \cdot a_x & a_r \cdot a_y & a_r \cdot a_z \\ a_\theta \cdot a_x & a_\theta \cdot a_y & a_\theta \cdot a_z \\ a_\phi \cdot a_x & a_\phi \cdot a_y & a_\phi \cdot a_z \end{bmatrix} \begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix}.$$