UNIT-I

MATRICES

Characteristic Polynomial :

The determinant $|A-\lambda I|$ when expanded will give a polynomial, which we call as characteristic polynomial of matrix A.

Characteristic Equation :

Let A be any square matrix of order n. The characteristic equation of A is $|A-\lambda I| = 0$.

Eigen values :

Let A be a square matrix, the characteristic equation of A is $|A-\lambda I| = 0$. The roots of the characteristic equation are called Eigen values of A.

Eigen vector :

Let A be a square matrix. If there exists a non-zero vector X such that $AX = \lambda X$, then the vector X is called an Eigen vector of A corresponding to the Eigen value λ.

Note :

1) The characteristic equation of 2×2 matrix is

$$\lambda^2 - S_1\lambda + S_2 = 0$$

S_1 = Sum of main diagonal elements

$S_2 = |A|$

2) The characteristic equation of 3×3 matrix is

$$\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0$$

S_1 = Sum of main diagonal elements

S_2 = Sum of the minors of main diagonal elements

$S_3 = |A|$

Problems :

1) Find the Eigen values and Eigen vectors of the matrix $\begin{bmatrix} 1 & 1 \\ 3 & -1 \end{bmatrix}$

Soln:

Let $A = \begin{bmatrix} 1 & 1 \\ 3 & -1 \end{bmatrix}$

The characteristic equation of A is $\lambda^2 - S_1\lambda + S_2 = 0$

$S_1 = 1 - 1 = 0$

$S_2 = \begin{vmatrix} 1 & 1 \\ 3 & -1 \end{vmatrix} = -1 - 3 = -4$

$\therefore \lambda^2 - 0\lambda - 4 = 0$

$\lambda^2 - 4 = 0$

$\lambda^2 = 4$

$\lambda = \pm 2$

The eigen values are $-2, 2$

To find : Eigen vectors

$(A - \lambda I) X = 0$

$$\begin{pmatrix} 1-\lambda & 1 \\ 3 & -1-\lambda \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\left.\begin{aligned} (1-\lambda) x_1 + x_2 &= 0 \\ 3x_1 + (-1-\lambda) x_2 &= 0 \end{aligned}\right\} \longrightarrow (1)$$

Case (i) : $\lambda = -2$

$3x_1 + x_2 = 0 \longrightarrow (2)$

$3x_1 + x_2 = 0 \longrightarrow (3)$

Solve (2) & (3)

$3x_1 + x_2 = 0$

$3x_1 = -x_2$

$\frac{x_1}{1} = \frac{x_2}{-3}$

$\therefore$ The eigen vector is $X_1 = \begin{pmatrix} 1 \\ -3 \end{pmatrix}$

Case (ii) : $\lambda = 2$

$-x_1 + x_2 = 0 \longrightarrow (4)$

$3x_1 - 3x_2 = 0 \longrightarrow (5)$

Solve (4) & (5),

$-x_1 + x_2 = 0$

$x_1 = x_2 \Rightarrow \frac{x_1}{1} = \frac{x_2}{1}$

$\therefore$ The eigen vector is $X_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

2) Find the Eigen values and Eigen vectors of
$\begin{bmatrix} 1 & 0 & -1 \\ 1 & 2 & 1 \\ 2 & 2 & 3 \end{bmatrix}$.

Soln :

Let $A = \begin{bmatrix} 1 & 0 & -1 \\ 1 & 2 & 1 \\ 2 & 2 & 3 \end{bmatrix}$

The characteristic equation is

$$\lambda^3 - s_1\lambda^2 + s_2\lambda - s_3 = 0$$

$s_1 = 1+2+3 = 6$

$$s_2 = \begin{vmatrix} 2 & 1 \\ 2 & 3 \end{vmatrix} + \begin{vmatrix} 1 & -1 \\ 2 & 3 \end{vmatrix} + \begin{vmatrix} 1 & 0 \\ 1 & 2 \end{vmatrix}$$

$= (6-2) + (3+2) + (2-0)$

$= 4+5+2 = 11$

$$s_3 = \begin{vmatrix} 1 & 0 & -1 \\ 1 & 2 & 1 \\ 2 & 2 & 3 \end{vmatrix}$$

$= 1(6-2) - 0 - 1(2-4)$

$= 4+2 = 6.$

$$\lambda^3 - 6\lambda^2 + 11\lambda - 6 = 0$$

To find : Eigen values

$$\begin{array}{c|cccc} 1 & 1 & -6 & 11 & -6 \\ & 0 & 1 & -5 & 6 \\ \hline & 1 & -5 & 6 & \boxed{0} \end{array}$$

$$\begin{array}{c|c} \times & + \\ \hline 6 & -5 \\ -3 & -2 \end{array}$$

$(\lambda-1)(\lambda^2 - 5\lambda + 6) = 0$

$(\lambda-1)(\lambda-2)(\lambda-3) = 0$

$\lambda = 1, 2, 3$

$\therefore$ The Eigen values are 1, 2, 3

To find : Eigen vector

$$(A - \lambda I)X = 0$$

$$\begin{bmatrix} 1-\lambda & 0 & -1 \\ 1 & 2-\lambda & 1 \\ 2 & 2 & 3-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

(4)

$$\left.\begin{aligned}(1-\lambda)x_1 + 0x_2 - x_3 &= 0\\ x_1 + (2-\lambda)x_2 + x_3 &= 0\\ 2x_1 + 2x_2 + (3-\lambda)x_3 &= 0\end{aligned}\right\} \longrightarrow (1)$$

Case (i): $\lambda = 1$

$$0x_1 + 0x_2 - x_3 = 0 \longrightarrow (2)$$

$$x_1 + x_2 + x_3 = 0 \longrightarrow (3)$$

$$2x_1 + 2x_2 + 2x_3 = 0 \longrightarrow (4).$$

Solve (2) & (3),

$$\begin{array}{cccc} & x_1 & x_2 & x_3 \\ 0 & -1 & 0 & 0 \\ 1 & 1 & 1 & 1 \end{array}$$

$$\frac{x_1}{0+1} = \frac{x_2}{-1+0} = \frac{x_3}{0-0}$$

$$\frac{x_1}{1} = \frac{x_2}{-1} = \frac{x_3}{0}$$

$$\therefore X_1 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$$

Case (ii): $\lambda = 2$

$$-x_1 + 0x_2 - x_3 = 0 \longrightarrow (5)$$

$$x_1 + 0\cdot x_2 + x_3 = 0 \longrightarrow (6)$$

$$2x_1 + 2x_2 + x_3 = 0 \longrightarrow (7)$$

Solve (6) & (7),

$$\begin{array}{cccc} & x_1 & x_2 & x_3 \\ 0 & 1 & 1 & 0 \\ 2 & 1 & 2 & 2 \end{array}$$

$$\frac{x_1}{0-2} = \frac{x_2}{2-1} = \frac{x_3}{2-0}$$

$$\frac{x_1}{-2} = \frac{x_2}{1} = \frac{x_3}{2}$$

$$\therefore X_2 = \begin{bmatrix} -2 \\ 1 \\ 2 \end{bmatrix}$$

Case (iii) : $\lambda = 3$

$$-2x_1 + 0x_2 - x_3 = 0 \longrightarrow (8)$$

$$x_1 + x_2 + x_3 = 0 \longrightarrow (9)$$

$$2x_1 + 2x_2 + 0 \cdot x_3 = 0 \longrightarrow (10)$$

Solve (9) & (10),

$$\begin{array}{ccccccc} & x_1 & & x_2 & & x_3 & \\ -1 & & 1 & & 1 & & -1 \\ 2 & & 0 & & 2 & & 2 \end{array}$$

$$\frac{x_1}{0-2} = \frac{x_2}{2-0} = \frac{x_3}{2+2}$$

$$\frac{x_1}{-2} = \frac{x_2}{2} = \frac{x_3}{4}$$

$$\frac{x_1}{-1} = \frac{x_2}{1} = \frac{x_3}{2}$$

$$\therefore X_3 = \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix}$$

$\therefore$ The eigen ~~values~~ vectors are $\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} +2 \\ -1 \\ -2 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ -2 \end{bmatrix}$

3) Find all the Eigen values and Eigen vectors of the matrix $\begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix}$

Soln :

Let $A = \begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix}$

The characteristic equation is $\lambda^3 - s_1\lambda^2 + s_2\lambda - s_3 = 0$

$$s_1 = -2 + 1 + 0 = -1$$

$$s_2 = \begin{vmatrix} 1 & -6 \\ -2 & 0 \end{vmatrix} + \begin{vmatrix} -2 & -3 \\ -1 & 0 \end{vmatrix} + \begin{vmatrix} -2 & 2 \\ 2 & 1 \end{vmatrix}$$

$$= (0 - 12) + (0 - 3) + (-2 - 4)$$

$$= -12 - 3 - 6 = -21$$

$$s_3 = \begin{vmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{vmatrix}$$

(4)

$(1-\lambda)x_1 + 0x_2 - x_3 = 0$

(6)

$= -2(0-12) - 2(0-6) - 3(-4+1)$

$= -2(-12) - 2(-6) - 3(-3)$

$= 24 + 12 + 9 = 45$

$\lambda^3 + \lambda^2 - 21\lambda - 45 = 0$

$$\begin{array}{r|rrrr} -3 & 1 & 1 & -21 & -45 \\ & 0 & -3 & 6 & 45 \\ \hline & 1 & -2 & -15 & \boxed{0} \end{array}$$

$$\begin{array}{r|r} \times & + \\ \hline -15 & -2 \\ -5 & 3 \end{array}$$

$(\lambda+3)(\lambda^2 - 2\lambda - 15) = 0$

$(\lambda+3)(\lambda+3)(\lambda-5) = 0$

$\lambda = -3, -3, 5$

$\therefore$ The Eigen values are $-3, -3, 5$.

To find : Eigen vector

$(A - \lambda I)X = 0$

$$\begin{bmatrix} -2-\lambda & 2 & -3 \\ 2 & 1-\lambda & -6 \\ -1 & -2 & 0-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\left.\begin{array}{l} (-2-\lambda)x_1 + 2x_2 - 3x_3 = 0 \\ 2x_1 + (1-\lambda)x_2 - 6x_3 = 0 \\ -x_1 - 2x_2 + (-\lambda)x_3 = 0 \end{array}\right\} \longrightarrow (1)$$

Case (i) : $\lambda = 5$

$-7x_1 + 2x_2 - 3x_3 = 0 \longrightarrow (2)$

$2x_1 - 4x_2 - 6x_3 = 0 \longrightarrow (3)$

$-x_1 - 2x_2 - 7x_3 = 0 \longrightarrow (4)$

Solve (2) & (3),

$$\begin{array}{cccccc} & x_1 & & x_2 & & x_3 \\ 2 & & -3 & & -7 & & 2 \\ -4 & & -6 & & 2 & & -4 \end{array}$$

$$\frac{x_1}{-12-12} = \frac{x_2}{-6-42} = \frac{x_3}{28-4}$$

$$\frac{x_1}{-24} = \frac{x_2}{-48} = \frac{x_3}{24}$$

$$\frac{x_1}{1} = \frac{x_2}{2} = \frac{x_3}{-1}$$

$$\therefore X_1 = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$$

Case (ii) : $\lambda = -3$

$x_1 + 2x_2 - 3x_3 = 0 \longrightarrow (5)$

$2x_1 + 4x_2 - 6x_3 = 0 \longrightarrow (6)$

$-x_1 - 2x_2 + 3x_3 = 0 \longrightarrow (7)$

Equations are same

$(5) \Rightarrow x_1 + 2x_2 - 3x_3 = 0$

Put $x_1 = 0$.

$(5) \Rightarrow 2x_2 - 3x_3 = 0$

$$2x_2 = 3x_3$$

$$\frac{x_2}{3} = \frac{x_3}{2}$$

$$\therefore X_2 = \begin{bmatrix} 0 \\ 3 \\ 2 \end{bmatrix}$$

Case (iii) : $\lambda = -3$

double $X_1 = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$ $\quad X_2 = \begin{bmatrix} 0 \\ 3 \\ 2 \end{bmatrix}$

$x_1 + 2x_2 - x_3 = 0 \longrightarrow (8)$

$0x_1 + 3x_2 + 2x_3 = 0 \longrightarrow (9)$.

Solve (8) & (9)

$$\begin{array}{cccccccc} & x_1 & & x_2 & & x_3 & \\ 2 & & -1 & & 1 & & 2 \\ 3 & \times & 2 & \times & 0 & \times & 3 \end{array}$$

$$\frac{x_1}{4+3} = \frac{x_2}{0-2} = \frac{x_3}{3-0} \qquad \therefore X_3 = \begin{bmatrix} 7 \\ -2 \\ 3 \end{bmatrix}$$

$\therefore$ The Eigen vectors are $\begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 3 \\ 2 \end{bmatrix}, \begin{bmatrix} 7 \\ -2 \\ 3 \end{bmatrix}$

Last.

(8) 4) Find the Eigen values and Eigen vectors of

$$\begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

Soln:

Let $A = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 0 & 0 & 1 \end{bmatrix}$

The characteristic equation is $\lambda^3 - s_1\lambda^2 + s_2\lambda$

$s_1 = 2 + 2 + 1 = 5$

$s_2 = \begin{vmatrix} 2 & 1 \\ 0 & 1 \end{vmatrix} + \begin{vmatrix} 2 & 1 \\ 0 & 1 \end{vmatrix} + \begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix}$

$= (2-0) + (2-0) + (4-1)$

$= 2 + 2 + 3 = 7.$

$s_3 = \begin{vmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 0 & 0 & 1 \end{vmatrix}$

$= 2(2-0) - 1(1-0) + 1(0-0)$

$= 4 - 1 = 3.$

$\lambda^3 - 5\lambda^2 + 7\lambda - 3 = 0.$

$\therefore \lambda = 3, 1, 1.$

To find: Eigen vectors

$(A - \lambda I)X = 0$

$$\begin{bmatrix} 2-\lambda & 1 & 1 \\ 1 & 2-\lambda & 1 \\ 0 & 0 & 1-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\left.\begin{aligned} (2-\lambda)x_1 + x_2 + x_3 &= 0 \\ x_1 + (2-\lambda)x_2 + x_3 &= 0 \\ 0x_1 + 0x_2 + (1-\lambda)x_3 &= 0 \end{aligned}\right\} \rightarrow (1)$$

Case (i): $\lambda = 3$

$-x_1 + x_2 + x_3 = 0 \longrightarrow (2)$

$x_1 - x_2 + x_3 = 0 \longrightarrow (3)$

$0x_1 + 0x_2 - 2x_3 = 0 \longrightarrow (4)$

Solve (2) & (3)

$$\begin{array}{cccccc} & x_1 & & x_2 & & x_3 \\ 1 & & 1 & & -1 & 1 \\ -1 & & 1 & & 1 & -1 \end{array}$$

$$\frac{x_1}{1+1} = \frac{x_2}{1+1} = \frac{x_3}{1-1}$$

$$\frac{x_1}{2} = \frac{x_2}{2} = \frac{x_3}{0} \Rightarrow \frac{x_1}{1} = \frac{x_2}{1} = \frac{x_3}{0}$$

$$\therefore X_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

Case (ii) : $\lambda = 1$

$$x_1 + x_2 + x_3 = 0 \longrightarrow (5)$$
$$x_1 + x_2 + x_3 = 0 \longrightarrow (6)$$
$$0x_1 + 0 \cdot x_2 + 0 \cdot x_3 = 0 \longrightarrow (7)$$

Put $x_1 = 0$ in (5)

$$0 + x_2 + x_3 = 0$$
$$x_2 = -x_3$$
$$\frac{x_2}{-1} = \frac{x_3}{1}$$
$$X_2 = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$$

Case (iii) : $\lambda = 1$

$$X_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \quad X_2 = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$$

$$x_1 + x_2 + 0x_3 = 0 \longrightarrow (8)$$
$$0x_1 - x_2 + x_3 = 0 \longrightarrow (9)$$

Solve (8) & (9),

$$\begin{array}{cccccc} & x_1 & & x_2 & & x_3 \\ 1 & & 0 & & 1 & 1 \\ -1 & & 1 & & 0 & -1 \end{array}$$

$$\frac{x_1}{1-0} = \frac{x_2}{0-1} = \frac{x_3}{-1-0}$$

$$\frac{x_1}{1} = \frac{x_2}{-1} = \frac{x_3}{-1}$$

$$\begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix} = \begin{bmatrix} 2-1-1 \\ 1-2-1 \\ 0+0-1 \end{bmatrix} = \begin{bmatrix} 0 \\ -2 \\ -1 \end{bmatrix}$$

$\therefore X_3 = \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix}$

$\therefore$ The Eigen vectors are $\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix}$.

5) Find the Eigen values and Eigen vectors of $A = \begin{bmatrix} 1 & 1 & 3 \\ 1 & 5 & 1 \\ 3 & 1 & 1 \end{bmatrix}$

Soln :

Let $A = \begin{bmatrix} 1 & 1 & 3 \\ 1 & 5 & 1 \\ 3 & 1 & 1 \end{bmatrix}$

The characteristic equation is $\lambda^3 - s_1\lambda^2 + s_2\lambda - s_3 = 0$

$s_1 = 1 + 5 + 1 = 7$

$s_2 = \begin{vmatrix} 5 & 1 \\ 1 & 1 \end{vmatrix} + \begin{vmatrix} 1 & 3 \\ 3 & 1 \end{vmatrix} + \begin{vmatrix} 1 & 1 \\ 1 & 5 \end{vmatrix}$

$= (5-1) + (1-9) + (5-1)$

$= 4 - 8 + 4 = 0$.

$s_3 = \begin{vmatrix} 1 & 1 & 3 \\ 1 & 5 & 1 \\ 3 & 1 & 1 \end{vmatrix}$

$= 1(5-1) - 1(1-3) + 3(1-15)$

$= 1(4) - 1(-2) + 3(-14)$

$= 4 + 2 - 42 = -36$

$\frac{14}{3}$
$\overline{42}$

$\therefore \lambda^3 - 7\lambda^2 + 0\lambda + 36 = 0$

$$\begin{array}{r|rrrr} -2 & 1 & -7 & 0 & 36 \\ & 0 & -2 & 18 & -36 \\ \hline & 1 & -9 & 18 & \boxed{0} \end{array}$$

$$\begin{array}{c|c} \times & + \\ \hline 18 & -9 \\ -6 & -3 \end{array}$$

$(\lambda + 2)(\lambda^2 - 9\lambda + 18) = 0$

$\lambda = -2, \ \lambda = 3, 6$.

$\therefore \lambda = -2, 3, 6$.

To find : Eigen vectors

$(A - \lambda I) X = 0$

$$\begin{bmatrix} 1-\lambda & 1 & 3 \\ 1 & 5-\lambda & 1 \\ 3 & 1 & 1-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\left.\begin{aligned} (1-\lambda)x_1 + x_2 + 3x_3 &= 0 \\ x_1 + (5-\lambda)x_2 + x_3 &= 0 \\ 3x_1 + x_2 + (1-\lambda)x_3 &= 0 \end{aligned}\right\} \longrightarrow (1)$$

Case (i) : $\lambda = -2$

$$3x_1 + x_2 + 3x_3 = 0 \longrightarrow (2)$$
$$x_1 + 7x_2 + x_3 = 0 \longrightarrow (3)$$
$$3x_1 + x_2 + 3x_3 = 0 \longrightarrow (4)$$

Solve (2) & (3),

$$\begin{array}{cccccc} & x_1 & & x_2 & & x_3 \\ 1 & & 3 & & 3 & & 1 \\ 7 & & 1 & & 1 & & 7 \end{array}$$

$$\frac{x_1}{1-21} = \frac{x_2}{3-3} = \frac{x_3}{21-1}$$

$$\frac{x_1}{-20} = \frac{x_2}{0} = \frac{x_3}{20}$$

$$\frac{x_1}{1} = \frac{x_2}{0} = \frac{x_3}{-1}$$

$$\therefore X_1 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

Case (ii) : $\lambda = 3$

$$-2x_1 + x_2 + 3x_3 = 0 \longrightarrow (5)$$
$$x_1 + 2x_2 + x_3 = 0 \longrightarrow (6)$$
$$3x_1 + x_2 - 2x_3 = 0 \longrightarrow (7).$$

Solve (5) & (6),

$$\begin{array}{cccccc} & x_1 & & x_2 & & x_3 \\ 1 & & 3 & & -2 & & 1 \\ 2 & & 1 & & 1 & & 2 \end{array}$$

$$\frac{x_1}{1-6} = \frac{x_2}{3+2} = \frac{x_3}{-4-1}$$

$$\frac{x_1}{-5} = \frac{x_2}{5} = \frac{x_3}{-5}$$

$$\frac{x_1}{1} = \frac{x_2}{-1} = \frac{x_3}{1}$$

$$\therefore X_2 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

Case (iii) : $\lambda = 6$

$$-5x_1 + x_2 + 3x_3 = 0 \longrightarrow (8)$$

$$x_1 - x_2 + x_3 = 0 \longrightarrow (9)$$

$$3x_1 + x_2 - 5x_3 = 0 \longrightarrow (10).$$

Solve (8) & (9),

$$\begin{array}{cccccccc} & x_1 & & x_2 & & x_3 & \\ 1 & & 3 & & -5 & & 1 \\ -1 & & 1 & & 1 & & -1 \end{array}$$

$$\frac{x_1}{1+3} = \frac{x_2}{3+5} = \frac{x_3}{5-1}$$

$$\frac{x_1}{4} = \frac{x_2}{8} = \frac{x_3}{4}$$

$$\frac{x_1}{1} = \frac{x_2}{2} = \frac{x_3}{1}$$

$$\therefore X_3 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$$

$\therefore$ The Eigen vectors are $\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$

Orthogonal matrix :

A square matrix A is said to be orthogonal if $AA^T = A^TA = I$ since $A^{-1}A = AA^{-1} = I$, it follows that a matrix A is orthogonal if $A^T = A^{-1}$.

1) show that $A = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$ is orthogonal.

Soln :

Let $A = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$

$$A^T = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

$$AA^T = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

$$= \begin{bmatrix} \cos^2\theta + \sin^2\theta & -\sin\theta\cos\theta + \sin\theta\cos\theta \\ -\sin\theta\cos\theta + \sin\theta\cos\theta & \sin^2\theta + \cos^2\theta \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

$\therefore A$ is orthogonal.

Diagonalization of the matrix :

Working Rule :

Step : 1

To find characteristic equation

Step : 2

To find Eigen values

Step : 3

To find Eigen vectors

Step : 4

check whether the Eigen vectors are orthogonal.

Step : 5

To form normalized matrix N

Step : 6

To calculate N^T

Step : 7

calculate $D = N^T A N$

[Diagonal elements and Eigen values are same].

Problems :

1) Diagonalize the matrix $\begin{bmatrix} 2 & 1 & -1 \\ 1 & 1 & -2 \\ -1 & -2 & 1 \end{bmatrix}$ by means of orthogonal transformation.

Soln :

Let $A = \begin{bmatrix} 2 & 1 & -1 \\ 1 & 1 & -2 \\ -1 & -2 & 1 \end{bmatrix}$

The characteristic equation is $\lambda^3 - s_1\lambda^2 + s_2\lambda - s_3$

$s_1 = 2+1+1 = 4$

$s_2 = \begin{vmatrix} 1 & -2 \\ -2 & 1 \end{vmatrix} + \begin{vmatrix} 2 & -1 \\ -1 & 1 \end{vmatrix} + \begin{vmatrix} 2 & 1 \\ 1 & 1 \end{vmatrix}$

$= (1-4) + (2-1) + (2-1)$

$= -3+1+1 = -1.$

$s_3 = \begin{vmatrix} 2 & 1 & -1 \\ 1 & 1 & -2 \\ -1 & -2 & 1 \end{vmatrix}$

$= 2(1-4) - 1(1-2) - 1(-2+1)$

$= 2(-3) - 1(-1) - 1(-1)$

$= -6+1+1 = -4$

$\therefore \lambda^3 - 4\lambda^2 - \lambda + 4 = 0.$

$\lambda = 4, 1, -1.$

To find : Eigen vectors

$(A - \lambda I)X = 0$

$$\begin{bmatrix} 2-\lambda & 1 & -1 \\ 1 & 1-\lambda & -2 \\ -1 & -2 & 1-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\left.\begin{aligned} (2-\lambda)x_1 + x_2 - x_3 &= 0 \\ x_1 + (1-\lambda)x_2 - 2x_3 &= 0 \\ -x_1 - 2x_2 + (1-\lambda)x_3 &= 0 \end{aligned}\right\} \rightarrow (1)$$

Case (i) : $\lambda = 4$

$-2x_1 + x_2 - x_3 = 0 \longrightarrow (2)$

$x_1 - 3x_2 - 2x_3 = 0 \longrightarrow (3)$

$-x_1 - 2x_2 - 3x_3 = 0 \longrightarrow (4)$

Solve (2) & (3),

$$\begin{array}{cccccc} & x_1 & & x_2 & & x_3 \\ 1 & & -1 & & -2 & & 1 \\ -3 & & -2 & & 1 & & -3 \end{array}$$

$$\frac{x_1}{-2-3} = \frac{x_2}{-1-4} = \frac{x_3}{6-1}$$

$$\frac{x_1}{-5} = \frac{x_2}{-5} = \frac{x_3}{5}$$

$$\frac{x_1}{1} = \frac{x_2}{1} = \frac{x_3}{-1}$$

$$\therefore X_1 = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$$

Case (ii): $[\lambda = 1]$

$$+2x_1 + x_2 - x_3 = 0 \longrightarrow (5)$$

$$x_1 + 0x_2 - 2x_3 = 0 \, . \longrightarrow (6)$$

$$-x_1 - 2x_2 + 0x_3 = 0 \longrightarrow (7)$$

Solve (5) & (6)

$$\begin{array}{cccc} & x_1 & x_2 & x_3 \\ 1 & -1 & 1 & 1 \\ 0 & -2 & 1 & 0 \end{array}$$

$$\frac{x_1}{-2-0} = \frac{x_2}{-1+2} = \frac{x_3}{0-1}$$

$$\frac{x_1}{-2} = \frac{x_2}{1} = \frac{x_3}{-1}$$

$$\therefore X_2 = \begin{bmatrix} -2 \\ 1 \\ -1 \end{bmatrix}$$

Case (iii): $\lambda = -1$

$$3x_1 + x_2 - x_3 = 0 \longrightarrow (8)$$

$$x_1 + 2x_2 - 2x_3 = 0 \longrightarrow (9)$$

$$-x_1 - 2x_2 + 2x_3 = 0 \longrightarrow (10)$$

Solve (8) & (9),

$$\begin{array}{cccc} & x_1 & x_2 & x_3 \\ 1 & -1 & 3 & 1 \\ 2 & -2 & 1 & 2 \end{array}$$

$$\frac{x_1}{-2+2} = \frac{x_2}{-1+6} = \frac{x_3}{6-1}$$

$$\frac{x_1}{0} = \frac{x_2}{5} = \frac{x_3}{5}$$

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$$\frac{x_1}{0} = \frac{x_2}{1} = \frac{x_3}{1}$$

$$\therefore X_3 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

$\therefore$ The Eigen vectors are $X_1 = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}, X_2 = \begin{bmatrix} -2 \\ 1 \\ -1 \end{bmatrix}, X_3 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$

To find : orthogonal

$$X_1^T X_2 = (-1 \; -1 \;\; 1) \begin{pmatrix} -2 \\ 1 \\ -1 \end{pmatrix}$$

$$= 2 - 1 - 1 = 0$$

$$X_2^T X_3 = (-2 \;\; 1 \; -1) \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$$

$$= 0 + 1 - 1 = 0$$

$$X_3^T X_1 = (0 \;\; 1 \;\; 1) \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix} = 0 - 1 + 1 = 0.$$

To form Normalized matrix, $N = \begin{bmatrix} -\frac{1}{\sqrt{3}} & -\frac{2}{\sqrt{6}} & 0 \\ -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} \end{bmatrix}$

$$N^T = \begin{bmatrix} -\frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ -\frac{2}{\sqrt{6}} & \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{6}} \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$$

$$D = N^T A N$$

$$= \begin{bmatrix} -\frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ -\frac{2}{\sqrt{6}} & \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{6}} \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} 2 & 1 & -1 \\ 1 & 1 & -2 \\ -1 & -2 & 1 \end{bmatrix} \begin{bmatrix} -\frac{1}{\sqrt{3}} & -\frac{2}{\sqrt{6}} & 0 \\ -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} \\ +\frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} \end{bmatrix}$$

$$= \begin{bmatrix} 4 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

2) Diagonalize the matrix $A = \begin{bmatrix} 3 & 1 & 1 \\ 1 & 3 & -1 \\ 1 & -1 & 3 \end{bmatrix}$ by means of orthogonal transformation. (17)

Soln :

Let $A = \begin{bmatrix} 3 & 1 & 1 \\ 1 & 3 & -1 \\ 1 & -1 & 3 \end{bmatrix}$

The characteristic equation is $\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0$

$S_1 = 3+3+3 = 9$

$S_2 = \begin{vmatrix} 3 & -1 \\ -1 & 3 \end{vmatrix} + \begin{vmatrix} 3 & 1 \\ 1 & 3 \end{vmatrix} + \begin{vmatrix} 3 & 1 \\ 1 & 3 \end{vmatrix}$

$= (9-1) + (9-1) + (9-1)$

$= 8 + 8 + 8 = 24$

$S_3 = \begin{vmatrix} 3 & 1 & 1 \\ 1 & 3 & -1 \\ 1 & -1 & 3 \end{vmatrix}$

$= 3(9-1) - (3+1) + 1(-1-3)$

$= 24 - 4 - 4 = 16$.

$\therefore \lambda^3 - 9\lambda^2 + 24\lambda - 16 = 0$.

$\lambda = 1, 4, 4$.

To find : Eigen vectors

$(A - \lambda I)X = 0$

$$\begin{bmatrix} 3-\lambda & 1 & 1 \\ 1 & 3-\lambda & -1 \\ 1 & -1 & 3-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\left.\begin{aligned} (3-\lambda)x_1 + x_2 + x_3 &= 0 \\ x_1 + (3-\lambda)x_2 - x_3 &= 0 \\ x_1 - x_2 + (3-\lambda)x_3 &= 0 \end{aligned}\right\} \longrightarrow (1)$$

Case (i) : $\lambda = 1$

$2x_1 + x_2 + x_3 = 0 \longrightarrow (2)$

$x_1 + 2x_2 - x_3 = 0 \longrightarrow (3)$

$x_1 - x_2 + 2x_3 = 0 \longrightarrow (4)$

Solve (2) & (3)

$$\begin{array}{cccccc} x_1 & & x_2 & & x_3 & \\ 1 & 1 & & 2 & & 1 \\ 2 & -1 & & 1 & & 2 \end{array}$$

$$\frac{x_1}{-1-2} = \frac{x_2}{1+2} = \frac{x_3}{4-1}$$

$$\frac{x_1}{-3} = \frac{x_2}{3} = \frac{x_3}{3}$$

$$\frac{x_1}{-1} = \frac{x_2}{1} = \frac{x_3}{1}$$

$$\therefore X_1 = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$$

Case (ii) : $\lambda = 4$

$$-x_1 + x_2 + x_3 = 0 \longrightarrow (5)$$

$$x_1 - x_2 - x_3 = 0 \longrightarrow (6)$$

$$x_1 - x_2 - x_3 = 0 \longrightarrow (7)$$

Put $x_1 = 0$ in (6)

$$0 - x_2 - x_3 = 0$$

$$-x_2 = x_3$$

$$\frac{x_2}{1} = \frac{x_3}{-1}$$

$$\therefore X_2 = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}.$$

Case (iii) : $\lambda = 4$

$$X_1 = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} \quad X_2 = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$$

$$-x_1 + x_2 + x_3 = 0$$

$$0x_1 + x_2 - x_3 = 0$$

$$\begin{array}{cccccc} x_1 & & x_2 & & x_3 & \\ 1 & 1 & & -1 & & 1 \\ 1 & -1 & & 0 & & 1 \end{array}$$

$$\frac{x_1}{-1-1} = \frac{x_2}{0+1} = \frac{x_3}{-1-0}$$

$$\frac{x_1}{-2} = \frac{x_2}{-1} = \frac{x_3}{-1}$$

$$\therefore X_3 = \begin{bmatrix} -2 \\ -1 \\ -1 \end{bmatrix}$$

To find : Orthogonal

$$X_1 = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} \quad X_2 = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} \quad X_3 = \begin{bmatrix} -2 \\ -1 \\ -1 \end{bmatrix}$$

$$X_1^T X_2 = \begin{pmatrix} -1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} = 0 + 1 - 1 = 0$$

$$X_2^T X_3 = \begin{pmatrix} 0 & 1 & -1 \end{pmatrix} \begin{pmatrix} -2 \\ -1 \\ -1 \end{pmatrix} = 0 - 1 + 1 = 0$$

$$X_3^T X_1 = \begin{pmatrix} -2 & -1 & -1 \end{pmatrix} \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} = 2 - 1 - 1 = 0$$

$\therefore$ Eigen vectors are orthogonal.

To form : Normalized vector

$$N = \begin{bmatrix} \frac{-1}{\sqrt{3}} & 0 & \frac{-2}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{-1}{\sqrt{2}} & \frac{-1}{\sqrt{6}} \end{bmatrix}$$

$$N^T = \begin{bmatrix} -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{-2}{\sqrt{6}} & \frac{-1}{\sqrt{6}} & -\frac{1}{\sqrt{6}} \end{bmatrix}$$

$$D = N^T A N$$

$$= \begin{bmatrix} \frac{-1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ 0 & \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \\ \frac{-2}{\sqrt{6}} & \frac{-1}{\sqrt{6}} & \frac{-1}{\sqrt{6}} \end{bmatrix} \begin{bmatrix} 3 & 1 & 1 \\ 1 & 3 & -1 \\ 1 & -1 & 3 \end{bmatrix} \begin{bmatrix} \frac{-1}{\sqrt{3}} & 0 & -\frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{-1}{\sqrt{2}} & \frac{-1}{\sqrt{6}} \end{bmatrix}$$

$$= \begin{bmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

Quadratic form

A homogeneous polynomial of second degree in any number of variables is called quadratic form.

Problems :

1) Write the matrix of the quadratic form $2x_1^2 - 2x_2^2 + 4x_3^2 + 2x_1x_2 - 6x_1x_3 + 6x_2x_3$.

Soln :

$$A = \begin{array}{c} \\ x_1 \\ x_2 \\ x_3 \end{array} \begin{array}{c} \begin{array}{ccc} x_1 & x_2 & x_3 \end{array} \\ \begin{bmatrix} 2 & 1 & -3 \\ 1 & -2 & 3 \\ -3 & 3 & 4 \end{bmatrix} \end{array}$$

2) Write the matrix of the quadratic form $2x^2 + 8z^2 + 4xy + 10xz - 2yz$.

Soln :

$$A = \begin{array}{c} \\ x \\ y \\ z \end{array} \begin{array}{c} \begin{array}{ccc} x & y & z \end{array} \\ \begin{bmatrix} 2 & 2 & 5 \\ 2 & 0 & -1 \\ 5 & -1 & 8 \end{bmatrix} \end{array}$$

3) Write down the quadratic form corresponding to the matrix $\begin{bmatrix} 0 & 5 & -1 \\ 5 & 1 & 6 \\ -1 & 6 & 2 \end{bmatrix}$

Soln :

$$A = \begin{bmatrix} 0 & 5 & -1 \\ 5 & 1 & 6 \\ -1 & 6 & 2 \end{bmatrix}$$

$$Q = X^T A X$$

$$= (x_1 \ x_2 \ x_3) \begin{bmatrix} 0 & 5 & -1 \\ 5 & 1 & 6 \\ -1 & 6 & 2 \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

$$= 0x_1^2 + x_2^2 + 2x_3^2 + 10x_1x_2 + 12x_2x_3 - 2x_1x_3$$

4) Write down the quadratic form corresponding to the matrix $\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 1 & -1 & 0 \end{bmatrix}$

Soln :

Let $A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 1 & -1 & 0 \end{bmatrix}$

$$Q = X^T A X$$

$$= [x_1 \quad x_2 \quad x_3] \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 1 & -1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$= x_1^2 + x_2^2 + 0x_3^2 + 0x_1x_2 + 2x_1x_3 - 2x_2x_3$$

Nature of the Quadratic form :

Let $D_1 = a_{11}$

$$D_2 = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$$

$$D_3 = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

Note :

1) Index → No. of +ve terms
2) Signature → No. of +ve terms − No. of −ve terms
3) Rank → No. of non-zero diagonal elements

S.No	Nature	Condition
1)	Positive definite	$D_n > 0$ (+ve) (or) All the eigen values are +ve
2)	Negative definite	$D_n < 0$ (−ve) (or) All the eigen values are −ve
3)	Positive semi-definite	$D_n > 0$ & Atleast one value is zero. (or)

		All the eigen values ≥ 0 & atleast one value is zero
4)	Negative semi-definite	$D_n \leq 0$ & atleast one value is zero. (or) All the eigen values ≤ 0 & atleast one value is zero.
5)	Indefinite	All other cases.

Problems:

1) Prove that the quadratic form $x_1^2 + 2x_2^2 + 3x_3^2 + 2x_1x_2 + 2x_2x_3 - 2x_1x_3$ is indefinite.

Soln:

$$A = \begin{bmatrix} 1 & 1 & -1 \\ 1 & 2 & 1 \\ -1 & 1 & 3 \end{bmatrix}$$

$D_1 = 1$

$D_2 = \begin{vmatrix} 1 & 1 \\ 1 & 2 \end{vmatrix} = 2 - 1 = 1$

$$D_3 = \begin{vmatrix} 1 & 1 & -1 \\ 1 & 2 & 1 \\ -1 & 1 & 3 \end{vmatrix} = 1(6-1) - 1(3+1) - 1(1+2)$$

$= 1(5) - 1(4) - 1(3)$

$= 5 - 4 - 3 = -2$

$\therefore$ The nature is indefinite.

2) Discuss the nature of the quadratic form $2x_1x_2 + 2x_2x_3 - 2x_1x_3$ without reducing to the canonical form.

Soln:

$$A = \begin{bmatrix} 0 & 1 & -1 \\ 1 & 0 & 1 \\ -1 & 1 & 0 \end{bmatrix}$$

$D_1 = 0$

$D_2 = \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} = 0 - 1 = -1$

$$D_3 = \begin{vmatrix} 0 & 1 & -1 \\ 1 & 0 & 1 \\ -1 & 1 & 0 \end{vmatrix}$$

$$= 0 - 1(0+1) - 1(1-0)$$

$$= -1-1 = -2.$$

$\therefore$ The nature is negative semi-definite.

3) Find the index, signature and the nature of the quadratic form $x_1^2 + 2x_2^2 - 3x_3^2$.

Soln:

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -3 \end{bmatrix}$$

$D_1 = 1$

$$D_2 = \begin{vmatrix} 1 & 0 \\ 0 & 2 \end{vmatrix} = 2$$

$$D_3 = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -3 \end{vmatrix}$$

$$= 1(-6-0) + 0 + 0 = -6.$$

$\therefore$ The nature is indefinite.

Index $= 2$

Signature $= 2 - 1 = 1$

Reduction of a Quadratic form to canonical form

Working Rule:

1) Construct the quadratic form to matrix form 'A'.
2) Diagonalize the matrix A
3) Canonical form $= Y^T D Y$

Problems:

1) Reduce the Quadratic form $10x_1^2 + 2x_2^2 + 5x_3^2 + 6x_2x_3 - 10x_1x_3 - 4x_1x_2$ to a canonical form through an orthogonal trans

-formation. Also find index, signature, Rank and nature.

Soln:

$$A = \begin{bmatrix} 10 & -2 & -5 \\ -2 & 2 & 3 \\ -5 & 3 & 5 \end{bmatrix}$$

The characteristic equation is $\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0$

$S_1 = 10 + 2 + 5 = 17$

$$S_2 = \begin{vmatrix} 2 & 3 \\ 3 & 5 \end{vmatrix} + \begin{vmatrix} 10 & -5 \\ -5 & 5 \end{vmatrix} + \begin{vmatrix} 10 & -2 \\ -2 & 2 \end{vmatrix}$$

$= (10 - 9) + (50 - 25) + (20 - 4)$

$= 1 + 25 + 16 = 42$

$$S_3 = \begin{vmatrix} 10 & -2 & -5 \\ -2 & 2 & 3 \\ -5 & 3 & 5 \end{vmatrix}$$

$= 10(10 - 9) + 2(-10 + 15) - 5(-6 + 10)$

$= 10(1) + 2(5) - 5(4)$

$= 10 + 10 - 20 = 0.$

$\therefore \lambda^3 - 17\lambda^2 + 42\lambda = 0$

$\lambda(\lambda^2 - 17\lambda + 42) = 0$

$\lambda = 0, 14, 3.$

To find: Eigen vectors

$(A - \lambda I)X = 0$

$$\begin{bmatrix} 10-\lambda & -2 & -5 \\ -2 & 2-\lambda & 3 \\ -5 & 3 & 5-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\left.\begin{aligned} (10-\lambda)x_1 - 2x_2 - 5x_3 &= 0 \\ -2x_1 + (2-\lambda)x_2 + 3x_3 &= 0 \\ -5x_1 + 3x_2 + (5-\lambda)x_3 &= 0 \end{aligned}\right\} \longrightarrow (1)$$

Case (i) : $\lambda = 0$

$$10x_1 - 2x_2 - 5x_3 = 0 \longrightarrow (2)$$
$$-2x_1 + 2x_2 + 3x_3 = 0 \longrightarrow (3)$$
$$-5x_1 + 3x_2 + 5x_3 = 0 \longrightarrow (4)$$

Solve (2) & (3)

$$\begin{array}{cccc} & x_1 & x_2 & x_3 \\ -2 & -5 & 10 & -2 \\ 2 & 3 & -2 & 2 \end{array}$$

$$\frac{x_1}{-6+10} = \frac{x_2}{10-30} = \frac{x_3}{20-4}$$

$$\frac{x_1}{4} = \frac{x_2}{-20} = \frac{x_3}{16}$$

$$\frac{x_1}{1} = \frac{x_2}{-5} = \frac{x_3}{4}$$

$$X_1 = \begin{bmatrix} 1 \\ -5 \\ 4 \end{bmatrix}$$

Case (ii) : $\lambda = 14$

$$-4x_1 - 2x_2 - 5x_3 = 0 \longrightarrow (5)$$
$$-2x_1 - 12x_2 + 3x_3 = 0 \longrightarrow (6)$$
$$-5x_1 + 3x_2 - 9x_3 = 0 \longrightarrow (7)$$

Solve (5) & (6)

$$\begin{array}{cccc} & x_1 & x_2 & x_3 \\ -2 & -5 & -4 & -2 \\ -12 & 3 & -2 & -12 \end{array}$$

$$\frac{x_1}{-6-60} = \frac{x_2}{10+12} = \frac{x_3}{48-4}$$

$$\frac{x_1}{-66} = \frac{x_2}{22} = \frac{x_3}{44}$$

$$\frac{x_1}{-6} = \frac{x_2}{2} = \frac{x_3}{4}$$

$$\frac{x_1}{-3} = \frac{x_2}{1} = \frac{x_3}{2}$$

$$\therefore X_2 = \begin{bmatrix} -3 \\ 1 \\ 2 \end{bmatrix}$$

Case (iii) : $\lambda = 3$

$$7x_1 - 2x_2 - 5x_3 = 0 \longrightarrow (8)$$

$$-2x_1 - x_2 + 3x_3 = 0 \longrightarrow (9)$$

$$-5x_1 + 3x_2 + 2x_3 = 0 \longrightarrow (10)$$

Solve (8) & (9)

$$\begin{array}{cccccc} & x_1 & & x_2 & & x_3 \\ -2 & & -5 & & 7 & & -2 \\ -1 & & 3 & & -2 & & -1 \end{array}$$

$$\frac{x_1}{-6-5} = \frac{x_2}{10-21} = \frac{x_3}{-7-4}$$

$$\frac{x_1}{-11} = \frac{x_2}{-11} = \frac{x_3}{-11}$$

$$\frac{x_1}{-1} = \frac{x_2}{-1} = \frac{x_3}{-1}$$

$$\therefore X_3 = \begin{bmatrix} -1 \\ -1 \\ -1 \end{bmatrix}$$

To verify : The Eigen vectors are orthogonal.

$$X_1 = \begin{bmatrix} 1 \\ -5 \\ 4 \end{bmatrix} \quad X_2 = \begin{bmatrix} -3 \\ 1 \\ 2 \end{bmatrix} \quad X_3 = \begin{bmatrix} -1 \\ -1 \\ -1 \end{bmatrix}$$

$$X_1^T X_2 = (1 \ -5 \ 4)\begin{pmatrix} -3 \\ 1 \\ 2 \end{pmatrix} = -3-5+8 = 0$$

$$X_2^T X_3 = (-3 \ 1 \ 2)\begin{pmatrix} -1 \\ -1 \\ -1 \end{pmatrix} = 3-1-2 = 0$$

$$X_3^T X_1 = (-1 \ -1 \ -1)\begin{pmatrix} 1 \\ -5 \\ 4 \end{pmatrix} = -1+5-4 = 0$$

$\therefore$ Eigen vectors are orthogonal.

To form : Normalized vector

$$N = \begin{bmatrix} \frac{1}{\sqrt{42}} & \frac{-3}{\sqrt{14}} & \frac{-1}{\sqrt{3}} \\ \frac{-5}{\sqrt{42}} & \frac{1}{\sqrt{14}} & \frac{-1}{\sqrt{3}} \\ \frac{4}{\sqrt{42}} & \frac{2}{\sqrt{14}} & \frac{-1}{\sqrt{3}} \end{bmatrix}$$

$$N^T = \begin{bmatrix} \frac{1}{\sqrt{42}} & \frac{-5}{\sqrt{42}} & \frac{4}{\sqrt{42}} \\ \frac{-3}{\sqrt{14}} & \frac{1}{\sqrt{14}} & \frac{2}{\sqrt{14}} \\ \frac{-1}{\sqrt{3}} & \frac{-1}{\sqrt{3}} & \frac{-1}{\sqrt{3}} \end{bmatrix}$$

$$D = N^T A N$$

$$= \begin{bmatrix} \frac{1}{\sqrt{42}} & \frac{-3}{\sqrt{14}} & \frac{-1}{\sqrt{3}} \\ \frac{-5}{\sqrt{42}} & \frac{1}{\sqrt{14}} & \frac{-1}{\sqrt{3}} \\ \frac{4}{\sqrt{42}} & \frac{2}{\sqrt{14}} & \frac{-1}{\sqrt{3}} \end{bmatrix} \begin{bmatrix} 10 & -2 & -5 \\ -2 & 2 & 3 \\ -5 & 3 & 5 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{42}} & \frac{-5}{\sqrt{42}} & \frac{4}{\sqrt{42}} \\ \frac{-3}{\sqrt{14}} & \frac{1}{\sqrt{14}} & \frac{2}{\sqrt{14}} \\ \frac{-1}{\sqrt{3}} & \frac{-1}{\sqrt{3}} & \frac{-1}{\sqrt{3}} \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 14 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

To form : Canonical form.

Canonical form $= Y^T D Y$

$$= [y_1 \ y_2 \ y_3] \begin{bmatrix} 0 & 0 & 0 \\ 0 & 14 & 0 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

$$= 0y_1^2 + 14y_2^2 + 3y_3^2$$

Index = 2

Signature = 2 − 0 = 2

Rank = 2.

∴ The nature is positive semi-definite.

2) Reduce the quadratic form $2x^2+5y^2+3z^2+4xy$ to canonical form by orthogonal reduction and state its nature.

Soln:

$$A = \begin{bmatrix} 2 & 2 & 0 \\ 2 & 5 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

The characteristic equation is $\lambda^3 - s_1\lambda^2 + s_2\lambda - s_3 = 0$

$S_1 = 2+5+3 = 10$

$$S_2 = \begin{vmatrix} 5 & 0 \\ 0 & 3 \end{vmatrix} + \begin{vmatrix} 2 & 0 \\ 0 & 3 \end{vmatrix} + \begin{vmatrix} 2 & 2 \\ 2 & 5 \end{vmatrix}$$

$= (15-0)+(6-0)+(10-4)$

$= 15+6+6 = 27$

$$S_3 = \begin{vmatrix} 2 & 2 & 0 \\ 2 & 5 & 0 \\ 0 & 0 & 3 \end{vmatrix}$$

$= 2(15-0)-2(6-0)+0$

$= 30-12 = 18.$

$\therefore \lambda^3 - 10\lambda^2 + 27\lambda - 18 = 0$

$\lambda = 1, 3, 6.$

To find : Eigen vector

$(A-\lambda I)X = 0$

$$\begin{bmatrix} 2-\lambda & 2 & 0 \\ 2 & 5-\lambda & 0 \\ 0 & 0 & 3-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\left.\begin{aligned} (2-\lambda)x_1 + 2x_2 + 0\cdot x_3 &= 0 \\ 2x_1 + (5-\lambda)x_2 + 0x_3 &= 0 \\ 0x_1 + 0x_2 + (3-\lambda)x_3 &= 0 \end{aligned}\right\} \longrightarrow (1)$$

Case (i): $\lambda = 3$

$-x_1 + 2x_2 + 0x_3 = 0 \longrightarrow (2)$

$2x_1 + 2x_2 + 0x_3 = 0 \longrightarrow (3)$

$0x_1 + 0x_2 + 0x_3 = 0 \longrightarrow (4)$

Solve (2) & (3)

$$\begin{array}{cccccc} & x_1 & & x_2 & & x_3 \\ 2 & & 0 & & -1 & & 2 \\ 2 & & 0 & & 2 & & 2 \end{array}$$

$$\frac{x_1}{0-0} = \frac{x_2}{0+0} = \frac{x_3}{-2-4}$$

$$\frac{x_1}{0} = \frac{x_2}{0} = \frac{x_3}{-6}$$

$$\frac{x_1}{0} = \frac{x_2}{0} = \frac{x_3}{-1}$$

$$\therefore \; X_1 = \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix}$$

Case (ii) : $\lambda = 6$

$$-4x_1 + 2x_2 + 0 \cdot x_3 = 0 \longrightarrow (5)$$

$$2x_1 - x_2 + 0x_3 = 0 \longrightarrow (6)$$

$$0x_1 + 0x_2 - 3x_3 = 0 \longrightarrow (7)$$

Solve (6) & (7)

$$\begin{array}{cccccc} & x_1 & & x_2 & & x_3 \\ -1 & & 0 & & 2 & & -1 \\ 0 & & -3 & & 0 & & 0 \end{array}$$

$$\frac{x_1}{3-0} = \frac{x_2}{0+6} = \frac{x_3}{0+0}$$

$$\frac{x_1}{3} = \frac{x_2}{6} = \frac{x_3}{0}$$

$$\frac{x_1}{1} = \frac{x_2}{2} = \frac{x_3}{0}$$

$$\therefore \; X_2 = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$$

Case (iii) : $\lambda = 1$

$$x_1 + 2x_2 + 0x_3 = 0 \longrightarrow (8)$$

$$2x_1 + 4x_2 + 0x_3 = 0 \longrightarrow (9)$$

$$0x_1 + 0x_2 + 2x_3 = 0 \longrightarrow (10)$$

Solve (8) & (10)

$$\begin{array}{cccccc} & x_1 & & x_2 & & x_3 \\ 2 & & 0 & & 1 & & 2 \\ 0 & & 2 & & 0 & & 0 \end{array}$$

$$\frac{x_1}{4-0} = \frac{x_2}{0-2} = \frac{x_3}{0-0}$$

$$\frac{x_1}{4} = \frac{x_2}{-2} = \frac{x_3}{0}$$

$$\frac{x_1}{2} = \frac{x_2}{-1} = \frac{x_3}{0}$$

$$\therefore X_3 = \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}$$

To find : The vectors are orthogonal.

$$X_1 = \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix} \quad \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} = X_2 \quad X_3 = \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}$$

$$X_1^T X_2 = (0 \quad 0 \quad -1) \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} = 0 + 0 + 0 = 0$$

$$X_2^T X_3 = (1 \quad 2 \quad 0) \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix} = 2 - 2 + 0 = 0$$

$$X_3^T X_1 = (2 \quad -1 \quad 0) \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix} = 0 + 0 + 0 = 0$$

$\therefore$ The Eigen vectors are orthogonal.

To form : Normalized matrix

$$N = \begin{bmatrix} 0 & \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \\ 0 & \frac{2}{\sqrt{5}} & \frac{-1}{\sqrt{5}} \\ -1 & 0 & 0 \end{bmatrix}$$

$$N^T = \begin{bmatrix} 0 & 0 & -1 \\ \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} & 0 \\ \frac{2}{\sqrt{5}} & \frac{-1}{\sqrt{5}} & 0 \end{bmatrix}$$

$$D = N^T A N$$

$$= \begin{bmatrix} 0 & 0 & -1 \\ \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} & 0 \\ \frac{2}{\sqrt{5}} & \frac{-1}{\sqrt{5}} & 0 \end{bmatrix} \begin{bmatrix} 2 & 2 & 0 \\ 2 & 5 & 0 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} 0 & \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \\ 0 & \frac{2}{\sqrt{5}} & \frac{-1}{\sqrt{5}} \\ -1 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

To find : Canonical form

Canonical form $= Y^T D Y$

$$= [y_1 \; y_2 \; y_3] \begin{bmatrix} 3 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

$$= 3y_1^2 + 6y_2^2 + y_3^2$$

Index $= 3$

Signature $= 3$

Rank $= 3$.

$\therefore$ The Nature is positive definite.

3) Reduce the quadratic form $6x^2 + 3y^2 + 3z^2 - 4xy - 2yz + 4xz$ into a canonical form by an orthogonal reduction. Hence find its rank and nature.

Soln :

$$A = \begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}$$

The characteristic equation is $\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0$

$S_1 = 6 + 3 + 3 = 12$

$$S_2 = \begin{vmatrix} 3 & -1 \\ -1 & 3 \end{vmatrix} + \begin{vmatrix} 6 & 2 \\ 2 & 3 \end{vmatrix} + \begin{vmatrix} 6 & -2 \\ -2 & 3 \end{vmatrix}$$

$$= (9-1) + (18-4) + (18-4)$$

$$= 8 + 14 + 14 = 36$$

$$S_3 = \begin{vmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{vmatrix}$$

$$= 6(9-1) + 2(-6+2) + 2(2-6)$$

$$= 6(8) + 2(-4) + 2(-4)$$

$$= 48 - 8 - 8 = 32.$$

(32)

$\therefore \lambda^3 - 12\lambda^2 + 36\lambda - 32 = 0$

$\lambda = 8, 2, 2.$

To find : Eigen vectors

$(A - \lambda I) X = 0$

$$\begin{bmatrix} 6-\lambda & -2 & 2 \\ -2 & 3-\lambda & -1 \\ 2 & -1 & 3-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\left.\begin{aligned} (6-\lambda)x_1 - 2y_2 + 2z_3 &= 0 \\ -2x_1 + (3-\lambda)y_2 - z_3 &= 0 \\ 2x_1 - y_2 + (3-\lambda)z_3 &= 0 \end{aligned}\right\} \longrightarrow (1).$$

Case (i) : $\lambda = 8$

$-2x_1 - 2y_2 + 2z_3 = 0 \longrightarrow (2)$

$-2x_1 - 5y_2 - z_3 = 0 \longrightarrow (3)$

$2x_1 - y_2 - 5z_3 = 0 \longrightarrow (4)$

Solve (2) & (3)

$$\begin{array}{cccccccc} & x_1 & & y_2 & & z_3 & & \\ -2 & & 2 & & -2 & & -2 & \\ -5 & & -1 & & -2 & & -5 & \end{array}$$

$$\frac{x_1}{2+10} = \frac{y_2}{-4-2} = \frac{z_3}{10-4}$$

$$\frac{x_1}{12} = \frac{y_2}{-6} = \frac{z_3}{6}$$

$$\frac{x_1}{2} = \frac{y_2}{-1} = \frac{z_3}{1}$$

$$\therefore X_1 = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$$

Case (ii) : $\lambda = 2$

$4x_1 - 2y_2 + 2z_3 = 0 \longrightarrow (5)$

$-2x_1 + y_2 - z_3 = 0 \longrightarrow (6)$

$2x_1 - y_2 + z_3 = 0 \longrightarrow (7)$

Put $x = 0$ in (7)

$$0 - y + z = 0$$

$$-y = -z$$

$$\frac{y}{-1} = \frac{z}{-1}$$

$$Y = \begin{bmatrix} 0 \\ -1 \\ -1 \end{bmatrix}$$

Case (iii) : $\lambda = 2$

$$X = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} \quad Y = \begin{bmatrix} 0 \\ -1 \\ -1 \end{bmatrix}$$

$$2x - y + z = 0 \rightarrow (8)$$

$$0x - y - z = 0 \rightarrow (9)$$

Solve (8) & (9)

$$\begin{array}{cccccccc} & x & & y & & z & \\ -1 & & 1 & & 2 & & -1 \\ -1 & & -1 & & 0 & & -1 \end{array}$$

$$\frac{x}{1+1} = \frac{y}{0+2} = \frac{z}{-2+0}$$

$$\frac{x}{2} = \frac{y}{2} = \frac{z}{-2}$$

$$\frac{x}{1} = \frac{y}{1} = \frac{z}{-1}$$

$$\therefore Z = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$$

To verify : Eigen vectors are orthogonal.

$$X = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} \quad Y = \begin{bmatrix} 0 \\ -1 \\ -1 \end{bmatrix} \quad Z = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$$

$$X^T Y = (2 \ -1 \ 1) \begin{pmatrix} 0 \\ -1 \\ -1 \end{pmatrix} = 0 + 1 - 1 = 0$$

$$Y^T Z = (0 \ -1 \ -1) \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} = 0 - 1 + 1 = 0$$

(3)

$$z^T X = (1 \quad 1 \quad -1)\begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} = 2-1-1 = 0.$$

$\therefore$ Eigen vectors are orthogonal.

To find : Normalized matrix.

$$N = \begin{bmatrix} \frac{2}{\sqrt{6}} & 0 & \frac{1}{\sqrt{3}} \\ \frac{-1}{\sqrt{6}} & \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{6}} & \frac{-1}{\sqrt{2}} & \frac{-1}{\sqrt{3}} \end{bmatrix}$$

$$N^T = \begin{bmatrix} \frac{2}{\sqrt{6}} & \frac{-1}{\sqrt{6}} & \frac{1}{\sqrt{6}} \\ 0 & \frac{-1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{-1}{\sqrt{3}} \end{bmatrix}$$

$$D = N^T A N$$

$$= \begin{bmatrix} \frac{2}{\sqrt{6}} & \frac{-1}{\sqrt{6}} & \frac{1}{\sqrt{6}} \\ 0 & \frac{-1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{-1}{\sqrt{3}} \end{bmatrix} \begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix} \begin{bmatrix} \frac{2}{\sqrt{6}} & 0 & \frac{1}{\sqrt{3}} \\ \frac{-1}{\sqrt{6}} & \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{6}} & \frac{-1}{\sqrt{2}} & \frac{-1}{\sqrt{3}} \end{bmatrix}$$

$$= \begin{bmatrix} 8 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

To find : Canonical form.

Canonical form $= Y^T D Y$

$$= [y_1 \; y_2 \; y_3] \begin{bmatrix} 8 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

$$= 8y_1^2 + 2y_2^2 + 3y_3^2$$

Rank = 3

Nature is positive definite.

of determinant of A.

Properties of Eigen values :

i) Sum of the eigen values = Sum of the diagonal elements = Trace

ii) Product of the eigen values = $|A|$

iii) The eigen values of diagonal matrix (or) upper triangular matrix (or) lower triangular matrix are the diagonal elements.

iv) A and A^T have the same eigen values.

Proof :

Let λ be an eigen value of A then

$$|A-\lambda I| = 0$$

$$(A-\lambda I)^T = A^T - (\lambda I)^T$$
$$= A^T - \lambda I^T$$
$$= A^T - \lambda I$$

$$|A-\lambda I|^T = |A^T - \lambda I|$$

$$|A^T - \lambda I| = 0$$

$\therefore \lambda$ is an eigen value of A^T.

v) If λ is an eigen value of A, then $k\lambda$ is an eigen value of kA.

Proof :

Let λ be an eigen value of A then

$$AX = \lambda X$$

$$k(AX) = k(\lambda X)$$

$$(kA)X = (k\lambda)X$$

$\therefore k\lambda$ is an eigen value of kA.

vi) If λ is an eigen value of A, then λ^k is an eigen value of A^k.

Proof :

Let λ be an eigen value of A then

$AX = \lambda X$

$A(AX) = A(\lambda X)$

$A^2 X = (A\lambda) X$

$= (\lambda A) X$

$A^2 X = \lambda (AX)$

$= \lambda(\lambda X)$

$A^2 X = \lambda^2 X$

Similarly, λ^k is an eigen value of A^k.

vii) If λ is an eigen value of A then $\frac{1}{\lambda}$ is an eigen value of A^{-1} provided A is an non-singular

Proof:

Let λ be an eigen value of A.

$AX = \lambda X$

$A^{-1}(AX) = A^{-1}(\lambda X)$

$A^{-1}AX = A^{-1}\lambda X$

$IX = \lambda A^{-1} X$

$X = \lambda A^{-1} X$

$\frac{1}{\lambda} X = A^{-1} X$.

$\therefore \frac{1}{\lambda}$ is an eigen value of A^{-1}.

Note:

$\lambda \rightarrow$ Eigen value of A

$\frac{1}{\lambda} \rightarrow$ Eigen value of A^{-1}

$\frac{|A|}{\lambda} \rightarrow$ Eigen value of adj A.

Problems:

1) If the sum of the eigen values and trace of a 3×3 matrix A are equal then find the value

of determinant of A.

Soln:

Given A is a 3×3 matrix.

Let $\lambda_1, \lambda_2, \lambda_3$ be an eigen values.

WKT, Sum of the eigen values = Trace

$$\lambda_1+\lambda_2+\lambda_3 = \text{Trace}$$

$$\text{Trace}+\lambda_3 = \text{Trace}$$

$$\lambda_3 = 0$$

$|A|$ = Product of eigen values $= \lambda_1\lambda_2\lambda_3$

$|A| = 0 \quad [\because \lambda_3 = 0]$

2) Find the sum and product of the eigen values of the matrix $A = \begin{bmatrix} 2 & 1 & 2 \\ 1 & 3 & 1 \\ 2 & 1 & -6 \end{bmatrix}$

Soln:

Sum of the eigen values = Sum of the diagonal elements

$$= 2+3-6 = -1$$

Product of the eigen value $= |A|$

$$= 2(-18-1)-1(-6-2)+2(1-6)$$

$$= 2(-19)-1(-8)+2(-5)$$

$$= -38+8-10 = -40.$$

3) Find the eigen value of $-6A$, A^3 and A^{-1} where $A = \begin{bmatrix} 3 & 1 & 4 \\ 0 & 2 & 6 \\ 0 & 0 & 5 \end{bmatrix}$

Soln:

Given, A is an upper triangular matrix.

The eigen values of A is $3, 2, 5$

The eigen values of $-6A$ is $-18, -12, -30$

The eigen values of A^3 is $27, 8, 125$

The eigen values of A^{-1} is $\frac{1}{\lambda} \Rightarrow \frac{1}{3}, \frac{1}{2}, \frac{1}{5}$

4) If $2, -1, -3$ are the eigen values of the matrix A then find the eigen values of $A^2 - 2I$.

Soln:

The eigen values of A is $2, -1, -3$

The eigen values of A^2 is $4, 1, 9$

The eigen values of I is $1, 1, 1$.

The eigen values of $2I$ is $2, 2, 2$

The eigen values of $A^2 - 2I$ is $2, -1, 7$.

5) If the eigen values of matrix A of order 3×3 are $2, 3, 1$ then the eigen values of adj A.

Soln:

The eigen values of A are $2, 3, 1$.

$|A|$ = Product of eigen values $= 6$

The eigen vector of adj $A = \frac{|A|}{\lambda}$

$= \frac{6}{2}, \frac{6}{3}, \frac{6}{1}$

$= 3, 2, 6$.

6) If 3 and 6 are two eigen values of $A = \begin{bmatrix} 1 & 1 & 3 \\ 1 & 5 & 1 \\ 3 & 1 & 1 \end{bmatrix}$ write down all the eigen values of A in rows.

Soln:

Let $\lambda_1, \lambda_2, \lambda_3$ be an eigen values of A.

Given, $\lambda_1 = 3$, $\lambda_2 = 6$.

WKT, Sum of eigen values = Sum of the diagonal elements

$\lambda_1 + \lambda_2 + \lambda_3 = 1 + 5 + 1$

$3 + 6 + \lambda_3 = 7$

$\lambda_3 = -2$

The eigen values of A is $3, 6, -2$

The eigen values of A^{-1} is $\frac{1}{3}, \frac{1}{6}, -\frac{1}{2}$.

7) The product of two eigen values of matrix $A = \begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}$ is 16. find the 3rd eigen value.

Soln :

Let $\lambda_1, \lambda_2, \lambda_3$ be an eigen value of A.

Given, $\lambda_1 \lambda_2 = 16$.

WKT, Product of eigen value = |A|

$$\lambda_1 \lambda_2 \lambda_3 = 6(9-1) + 2(-6+2) + 2(2-6)$$
$$16\lambda_3 = 6(8) + 2(-4) + 2(-4)$$
$$16\lambda_3 = 48 - 8 - 8$$
$$16\lambda_3 = 32$$
$$\lambda_3 = 2.$$

8) One of the eigen value of $A = \begin{bmatrix} 7 & 4 & -4 \\ 4 & -8 & -1 \\ 4 & -1 & -8 \end{bmatrix}$ is -9 find the other two eigen values.

Soln :

Let $\lambda_1, \lambda_2, \lambda_3$ be an eigen values of A.

Given, $\lambda_1 = -9$

WKT, Sum of the eigen values = Sum of the diagonal elements

$$\lambda_1 + \lambda_2 + \lambda_3 = 7 - 8 - 8$$
$$-9 + \lambda_2 + \lambda_3 = -9$$
$$\lambda_2 + \lambda_3 = 0$$
$$\lambda_3 = -\lambda_2 \rightarrow (1)$$

WKT, Product of the eigen values = |A|

$$\lambda_1 \lambda_2 \lambda_3 = 7(64-1) - 4(-32+4) - 4(-4+32)$$
$$-9\lambda_2(-\lambda_2) = 7(63) - 4(-28) - 4(28)$$
$$9\lambda_2^2 = 441$$
$$\lambda_2^2 = 49$$
$$\lambda_2 = \pm 7$$

$\therefore \lambda_3 = \pm 7$

$\therefore$ The eigen values of A are $-9, \pm 7, \pm 7$.

Cayley - Hamilton Theorem

Every square matrix satisfies its own characteristic equation.

Problems:

1) Verify Cayley - Hamilton theorem for the matrix $A = \begin{bmatrix} 2 & 0 & -1 \\ 0 & 2 & 0 \\ -1 & 0 & 2 \end{bmatrix}$ and hence find A^{-1} & A^4.

Soln:

Given, $A = \begin{bmatrix} 2 & 0 & -1 \\ 0 & 2 & 0 \\ -1 & 0 & 2 \end{bmatrix}$

The characteristic equation is $\lambda^3 - s_1\lambda^2 + s_2\lambda - s_3 = 0$

$s_1 = 2 + 2 + 2 = 6$

$|A| = s_3 = 2\begin{vmatrix} 2 & 0 \\ 0 & 2 \end{vmatrix} + 0 - 1\begin{vmatrix} 0 & 2 \\ -1 & 0 \end{vmatrix}$

$= 2(4-0) + 0 - 1(0+2)$

$= 8 - 2 = 6$.

$s_2 = \begin{vmatrix} 2 & 0 \\ 0 & 2 \end{vmatrix} + \begin{vmatrix} 2 & -1 \\ -1 & 2 \end{vmatrix} + \begin{vmatrix} 2 & 0 \\ 0 & 2 \end{vmatrix}$

$= (4-0) + (4-1) + (4-0)$

$= 4 + 3 + 4 = 11$

$\therefore \lambda^3 - 6\lambda^2 + 11\lambda - 6 = 0$

By Cayley - Hamilton theorem,

$A^3 - 6A^2 + 11A - 6 = 0$.

$A^2 = \begin{bmatrix} 5 & 0 & -4 \\ 0 & 4 & 0 \\ -4 & 0 & 5 \end{bmatrix}$ $\therefore A^3 = \begin{bmatrix} 14 & 0 & -13 \\ 0 & 8 & 0 \\ -13 & 0 & 14 \end{bmatrix}$

$A^3 - 6A^2 + 11A - 6I$

$$= \begin{bmatrix} 14 & 0 & -13 \\ 0 & 8 & 0 \\ -13 & 0 & 14 \end{bmatrix} - \begin{bmatrix} 30 & 0 & -24 \\ 0 & 24 & 0 \\ -24 & 0 & 30 \end{bmatrix} + \begin{bmatrix} 22 & 0 & -11 \\ 0 & 22 & 0 \\ -11 & 0 & 22 \end{bmatrix} - \begin{bmatrix} 6 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 6 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$\therefore A^3 - 6A^2 + 11A - 6I = 0$

To find : A^{-1}

$A^3 - 6A^2 + 11A - 6I = 0$

$\times$ by A^{-1}

$A^2 - 6A + 11I - 6A^{-1} = 0$

$6A^{-1} = A^2 - 6A + 11I$

$A^{-1} = \frac{1}{6}[A^2 - 6A + 11I]$

$$= \frac{1}{6}\left\{\begin{bmatrix} 5 & 0 & -4 \\ 0 & 4 & 0 \\ -4 & 0 & 5 \end{bmatrix} - \begin{bmatrix} 12 & 0 & -6 \\ 0 & 12 & 0 \\ -6 & 0 & 12 \end{bmatrix} + \begin{bmatrix} 11 & 0 & 0 \\ 0 & 11 & 0 \\ 0 & 0 & 11 \end{bmatrix}\right\}$$

$$= \frac{1}{6}\begin{bmatrix} 4 & 0 & 2 \\ 0 & 3 & 0 \\ 2 & 0 & 4 \end{bmatrix}$$

To find : A^4

$A^3 - 6A^2 + 11A - 6I = 0$

$\times$ by A

$A^4 - 6A^3 + 11A^2 - 6A = 0$

$A^4 = 6A^3 - 11A^2 + 6A$

$$= \begin{bmatrix} 84 & 0 & -78 \\ 0 & 48 & 0 \\ -78 & 0 & 84 \end{bmatrix} - \begin{bmatrix} 55 & 0 & -44 \\ 0 & 44 & 0 \\ -44 & 0 & 55 \end{bmatrix} + \begin{bmatrix} 12 & 0 & -6 \\ 0 & 12 & 0 \\ -6 & 0 & 12 \end{bmatrix}$$

$$= \begin{bmatrix} 41 & 0 & -40 \\ 0 & 16 & 0 \\ -40 & 0 & 41 \end{bmatrix}$$

2) Using Cayley Hamilton theorem find A^{-1} & A^4 if

$$A = \begin{bmatrix} 1 & 2 & -2 \\ -1 & 3 & 0 \\ 0 & -2 & 1 \end{bmatrix}$$

Soln :

Given, $A = \begin{bmatrix} 1 & 2 & -2 \\ -1 & 3 & 0 \\ 0 & -2 & 1 \end{bmatrix}$

The characteristic equation is $\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0$

$S_1 = 1+3+1 = 5$.

$S_2 = \begin{vmatrix} 3 & 0 \\ -2 & 1 \end{vmatrix} + \begin{vmatrix} 1 & -2 \\ 0 & 1 \end{vmatrix} + \begin{vmatrix} 1 & 2 \\ -1 & 3 \end{vmatrix}$

$= (3-0) + (1-0) + (3+2)$

$= 3+1+5 = 9$

$S_3 = |A| = \begin{vmatrix} 1 & 2 & -2 \\ -1 & 3 & 0 \\ 0 & -2 & 1 \end{vmatrix}$

$= 1(3-0) - 2(-1-0) - 2(2-0)$

$= 1(3) - 2(-1) - 2(2) = 3+2-4 = 1$.

$\therefore \lambda^3 - 5\lambda^2 + 9\lambda - 1 = 0$.

By Cayley Hamilton theorem,

$A^3 - 5A^2 + 9A - I = 0$

$A^2 = \begin{bmatrix} -1 & 12 & -4 \\ -4 & 7 & 2 \\ 2 & -8 & 1 \end{bmatrix}$ $A^3 = \begin{bmatrix} -13 & 42 & -2 \\ -11 & 9 & 10 \\ 10 & -22 & -3 \end{bmatrix}$

To find : A^{-1}.

$A^3 - 5A^2 + 9A - I = 0$

$\times$ by A^{-1}

$A^2 - 5A + 9I - A^{-1} = 0$

$A^{-1} = A^2 - 5A + 9I$

$= \begin{bmatrix} -1 & 12 & -4 \\ -4 & 7 & 2 \\ 2 & -8 & 1 \end{bmatrix} - \begin{bmatrix} 5 & 10 & -10 \\ -5 & 15 & 0 \\ 0 & -10 & 5 \end{bmatrix} + \begin{bmatrix} 9 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 9 \end{bmatrix}$

$$= \begin{bmatrix} 3 & 2 & 6 \\ 1 & 1 & 2 \\ 2 & 2 & 5 \end{bmatrix}$$

To find : A^4

$$A^3 - 5A^2 + 9A - I = 0$$

$\times$ by A

$$A^4 - 5A^3 + 9A^2 - A = 0$$

$$A^4 = 5A^3 - 9A^2 + A$$

$$= \begin{bmatrix} 65 & 210 & -10 \\ -55 & 45 & 50 \\ 50 & -110 & -15 \end{bmatrix} - \begin{bmatrix} -9 & 108 & -36 \\ -36 & 63 & 18 \\ 18 & -72 & 9 \end{bmatrix} + \begin{bmatrix} 1 & 2 & -2 \\ -1 & 3 & 0 \\ 0 & -2 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -55 & 104 & 24 \\ -20 & -15 & 32 \\ 82 & -40 & -23 \end{bmatrix}$$

3) Using Cayley Hamilton theorem to find the value of the matrix $\begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix}$

i) $A^8 - 5A^7 + 7A^6 - 3A^5 + 8A^4 - 5A^3 + 8A^2 - 2A + I$

ii) $A^8 - 5A^7 + 7A^6 - 3A^5 + A^4 - 5A^3 + 8A^2 - 2A + I$

Soln :

$$A = \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix}$$

The characteristic equation is $\lambda^3 - s_1\lambda^2 + s_2\lambda - s_3 = 0$

$$s_1 = 2 + 1 + 2 = 5$$

$$s_2 = \begin{vmatrix} 1 & 0 \\ 1 & 2 \end{vmatrix} + \begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} + \begin{vmatrix} 2 & 1 \\ 0 & 1 \end{vmatrix}$$

$$= (2-0) + (4-1) + (2-0)$$

$$= 2 + 3 + 2 = 7$$

$$s_3 = \begin{vmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{vmatrix}$$

$$= 2(2-0) - 1(0-0) + 1(0-1)$$

$$= 4 - 1 = 3$$

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$\therefore \lambda^3 - 5\lambda^2 + 7\lambda - 3 = 0$

By Cayley Hamilton theorem,

$A^3 - 5A^2 + 7A - 3I = 0.$

i)

$$\begin{array}{r|l} & A^5 + 8A + 35I \\ \hline A^3 - 5A^2 + 7A - 3I & A^8 - 5A^7 + 7A^6 - 3A^5 + 8A^4 - 5A^3 + 8A^2 - 2A + I \\ & \underset{(-)}{A^8} \underset{(+)}{- 5A^7} \underset{(-)}{+ 7A^6} \underset{(+)}{- 3A^5} \\ \hline & 8A^4 - 5A^3 + 8A^2 - 2A + I \\ & \underset{(-)}{8A^4} \underset{(+)}{- 40A^3} \underset{(-)}{+ 56A^2} \underset{(+)}{- 24A} \\ \hline & 35A^3 - 48A^2 + 22A + I \\ & \underset{(-)}{35A^3} \underset{(+)}{- 175A^2} \underset{(-)}{+ 245A} \underset{(+)}{- 105I} \\ \hline & 127A^2 - 223A + 106I \end{array}$$

$A^8 - 5A^7 + 7A^6 - 3A^5 + 8A^4 - 5A^3 + 8A^2 - 2A + I$

$= (A^3 - 5A^2 + 7A - 3I)(A^5 + 8A + 35) + (127A^2 - 223A + 106I)$

$= 127A^2 - 223A + 106I$

$= 127 \begin{bmatrix} 5 & 4 & 4 \\ 0 & 1 & 0 \\ 4 & 4 & 5 \end{bmatrix} - 223 \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix} + 106 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$= \begin{bmatrix} 635 & 508 & 508 \\ 0 & 127 & 0 \\ 508 & 508 & 635 \end{bmatrix} - \begin{bmatrix} 446 & 223 & 223 \\ 0 & 223 & 0 \\ 223 & 223 & 446 \end{bmatrix} + \begin{bmatrix} 106 & 0 & 0 \\ 0 & 106 & 0 \\ 0 & 0 & 106 \end{bmatrix}$

$= \begin{bmatrix} 295 & 285 & 285 \\ 0 & 10 & 0 \\ 285 & 285 & 295 \end{bmatrix}$

ii)

$$\begin{array}{r|l} & A^5 + A \\ \hline A^3 - 5A^2 + 7A - 3I & A^8 - 5A^7 + 7A^6 - 3A^5 + A^4 - 5A^3 + 8A^2 - 2A + I \\ & \underset{(-)}{A^8} \underset{(+)}{- 5A^7} \underset{(-)}{+ 7A^6} \underset{(+)}{- 3A^5} \\ \hline & A^4 - 5A^3 + 8A^2 - 2A + I \\ & \underset{(-)}{A^4} \underset{(+)}{- 5A^3} \underset{(-)}{+ 7A^2} \underset{(+)}{- 3A} \\ \hline & A^2 + A + I \end{array}$$

$A^8 - 5A^7 + 7A^6 - 3A^5 + A^4 - 5A^3 + 8A^2 - 2A + I$

$= (A^3 - 5A^2 + 7A - 3I)(A^5 + A) + (A^2 + A + I)$

$= A^2 + A + I$

$$= \begin{bmatrix} 5 & 4 & 4 \\ 0 & 1 & 0 \\ 4 & 4 & 5 \end{bmatrix} + \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 8 & 5 & 5 \\ 0 & 3 & 0 \\ 5 & 5 & 8 \end{bmatrix}$$

4) Find A^n using cayley Hamilton theorem taking $A = \begin{pmatrix} 1 & 4 \\ 2 & 3 \end{pmatrix}$ hence find A^3.

Soln:

$$A = \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix}$$

The characteristic equation is $\lambda^2 - s_1\lambda + s_2 = 0$

$s_1 = 1 + 3 = 4$

$s_2 = \begin{vmatrix} 1 & 4 \\ 2 & 3 \end{vmatrix} = 3 - 8 = -5$

$\therefore \lambda^2 - 4\lambda - 5 = 0$

$\lambda = -1, 5$

To find : A^n

When λ^n is divided by $\lambda^2 - 4\lambda - 5$

Let the quotient be $Q(\lambda)$ & remainder be $a\lambda + b$.

$\lambda^n = (\lambda^2 - 4\lambda - 5)\,Q(\lambda) + (a\lambda + b)$

Put $\lambda = -1$

$(-1)^n = [(-1)^2 + 4(-1) - 5]\,Q(-1) + a(-1) + b$

$(-1)^n = -a + b \longrightarrow (1)$

(4b) Put $\lambda = 5$

$$5^n = [(5)^2 - 4(5) - 5]\, Q(5) + a(5) + b$$

$$5^n = 5a + b \rightarrow (2)$$

Solve (1) & (2),

$$(-1)^n = -a + b$$

$$\underline{\overset{(-)}{}5^n = \overset{(-)}{}5a \overset{(-)}{+} b}$$

$$(-1)^n - 5^n = -6a$$

$$a = \frac{(-1)^n - 5^n}{-6}$$

Sub in (1),

$$(-1)^n = \frac{(-1)^n - 5^n}{6} + b.$$

$$b = (-1)^n - \frac{(-1)^n - 5^n}{6}$$

$$= (-1)^n - \frac{(-1)^n}{6} + \frac{5^n}{6}$$

$$= \frac{5(-1)^n}{6} + \frac{5^n}{6} = \frac{5(-1)^n + 5^n}{6}$$

$$A^n = (A^2 - 4A - 5)\, Q(A) + aA + b$$

$$A^n = aA + b$$

$$A^n = \frac{(-1)^n - 5^n}{-6} A + \frac{5(-1)^n + 5^n}{6}$$

$$= \frac{[5^n - (-1)^n]}{6} A + \frac{5(-1)^n + 5^n}{6}$$

$$A^3 = \frac{5^3 - (-1)^3}{6} A + \frac{5(-1)^3 + 5^3}{6}$$

$$= \frac{125 + 1}{6} A + \frac{-5 + 125}{6}$$

$$= \frac{126A}{6} + \frac{120}{6} = 21A + 20I.$$

$$\therefore A^3 = 21A + 20I$$

$$A^3 = 21\begin{bmatrix}1 & 4\\2 & 3\end{bmatrix} + 20\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}$$

$$= \begin{bmatrix}21 & 84\\42 & 63\end{bmatrix} + \begin{bmatrix}20 & 0\\0 & 20\end{bmatrix}$$

$$A^3 = \begin{bmatrix}41 & 84\\42 & 83\end{bmatrix}$$