*. MAGNETIC SCALAR AND VECTOR POTENTIALS :-

In electrostatic fields, the electric potential 'V' is related to the electric field intensity as $\vec{E} = -\nabla V$. Similarly, we can define a potential associated with magnetostatic field $\vec{B}$. In fact, the magnetic potential could be scalar V_m or vector $\vec{A}$. To define V_m and $\vec{A}$ involves recalling two important identities

$$\nabla \times (\nabla V) = 0 \longrightarrow (1)$$

$$\nabla \cdot (\nabla \times \vec{A}) = 0 \longrightarrow (2)$$

which must always hold for any scalar field 'V' and vector field $\vec{A}$.

Just as $\vec{E} = -\nabla V$, we define the magnetic scalar potential V_m (in amperes) as related to $\vec{H}$ according to

$$\boxed{\vec{H} = -\nabla V_m} \quad \text{if } \vec{J} = 0. \longrightarrow (3)$$

The condition attached to this equation is important and will be explained. we know

$$\vec{J} = \nabla \times \vec{H} = \nabla \times (-\nabla V_m) = 0 \longrightarrow (4)$$

Since 'V_m' must satisfy the condition in eq (1). Thus the magnetic scalar potential 'V_m' is only defined in a region where $\vec{J} = 0$.

we should also note that 'V_m' satisfies Laplace's equation just as 'V' does for electrostatic fields: hence

$$\nabla^2 V_m = 0, \quad (\vec{J} = 0) \longrightarrow (5)$$

we know that for a magneotostatic field

$$\nabla \cdot \vec{B} = 0 \longrightarrow (6)$$

To satisfy eq (2) and (6) simultaneously, we can define the vector magnetic potential $\vec{A}$ (in Wb/m) such that

$$\boxed{\vec{B} = \nabla \times \vec{A}} \longrightarrow (7)$$

we can define

$$\vec{A} = \int_L \frac{\mu_0 I \, \vec{dl}}{4\pi R} \quad \text{for line current}$$

$$\vec{A} = \int_S \frac{\mu_0 \vec{K} \, dS}{4\pi R} \quad \text{for surface current.}$$

$$\vec{A} = \int_V \frac{\mu_0 \vec{J} \, dv}{4\pi R} \quad \text{for volume current}$$

*. FORCES DUE TO MAGNETIC FIELDS :-

The electric force $\vec{F_e}$ on a stationary or moving electric charge Q in an electric field is given by Coulomb's experimental law and is related to the electric field intensity $\vec{E}$ as

$$\boxed{\vec{F_e} = Q\vec{E}} \longrightarrow (1)$$

This shows that if Q is positive, $\vec{F}e$ and $\vec{E}$ have the same direction.

A magnetic field can exert force only on a moving charge.

From experiments, it is found that the magnetic force $\vec{F_m}$ experienced by a charge 'Q' moving with a velocity '$\vec{u}$' in a magnetic field $\vec{B}$ is

$$\boxed{\vec{F_m} = Q\vec{u} \times \vec{B}} \longrightarrow (2)$$

This clearly shows that $\vec{F_m}$ is perpendicular to both $\vec{u}$ and $\vec{B}$.

For a moving charge 'Q' in the presence of both electric and magnetic fields, the total force on the charge is given by

$$\vec{F} = \vec{Fe} + \vec{F_m}$$

$$\boxed{\vec{F} = Q(\vec{E} + \vec{u} \times \vec{B})} \longrightarrow (3)$$

This is known as the Lorentz force equation. It relates mechanical force to electrical force.

If the mass of the charged particle moving in $\vec{E}$ and $\vec{B}$ fields is 'm', by Newton's second law of motion

$$\boxed{\vec{F} = m\frac{d\vec{u}}{dt} = Q(\vec{E} + \vec{u} \times \vec{B})} \longrightarrow (4)$$

Table :- Force on a charged Particle

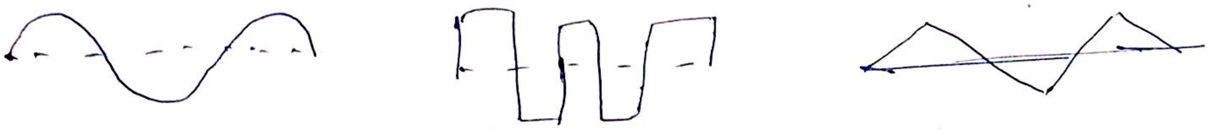
State of Particle	$\vec{E}$ field	$\vec{B}$ field	Combined $\vec{E}$ & $\vec{B}$ fields
Stationary	$Q\vec{E}$	—	$Q\vec{E}$
Moving	$Q\vec{E}$	$Q\vec{u} \times \vec{B}$	$Q(\vec{E} + \vec{u} \times \vec{B})$

- In static EM fields, electric and magnetic fields are independent of each other, whereas in dynamic EM fields, the two fields are interdependent.

- stationary charges → electrostatic fields

  steady currents → magnetostatic fields.
  [motion of electric charges with uniform constant velocity (direct current) or static magnetic charges (magnetic poles)]

  time varying currents → electromagnetic fields (or waves).



* Faradays Law:-

  steady current produces → magnetic field
  magnetism produce electricity ?

Acc. to Faraday's experiments, a static magnetic field produces no current flow, but a time varying field produces an induced voltage (called electromotive force or simply emf) in a closed circuit, which causes a flow of current.

- Faradays law:- The induced emf, V_{emf} (in volts), in any closed circuit is equal to the time rate of change of the magnetic flux linkage by the circuit. This is called Faraday's law, and it can be expressed as $V_{emf} = -\frac{d\lambda}{dt} = -N\frac{d\psi}{dt}$.

where $\lambda = N\psi$ is the flux linkage, N is the number of turns in the circuit, and ψ is the flux through each turn.

The -ve sign shows that the induced voltage acts in such a way as to oppose the flux producing it. This is known as Lenz's law, and it emphasizes that the direction of current flow in the circuit is such that the induced magnetic field produced by the induced current will oppose the change in original magnetic field.

8. Transformer & Motional electromotive forces :-

$$V_{emf} = -\frac{d\psi}{dt}$$

$$V_{emf} = \oint_L \vec{E}\cdot d\vec{l} = -\frac{d\psi}{dt} \quad \text{but} \quad \psi = \int_S \vec{B}\cdot d\vec{s}$$

$$\rightarrow \quad V_{emf} = \oint_L \vec{E}\cdot d\vec{l} = -\frac{d}{dt}\int_S \vec{B}\cdot d\vec{s}.$$

The variation of flux with time may be caused in three ways

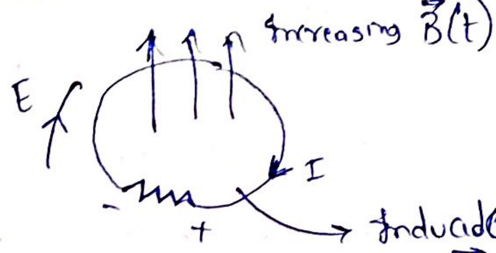
(i). By having a stationary loop in a time-varying $\vec{B}$ field.

(ii). By having a time varying loop area in a static $\vec{B}$ field.

(iii). By having a time varying loop area in a time varying $\vec{B}$ field.

(i). Stationary loop in time varying $\vec{B}$ field :- (Transformer emf).

$$V_{emf} = \oint_L \vec{E}\cdot d\vec{l} = -\frac{d}{dt}\int_S \vec{B}\cdot d\vec{s}$$



$$\rightarrow \quad \oint_L \vec{E}\cdot d\vec{l} = -\int_S \frac{\partial \vec{B}}{\partial t}\cdot d\vec{s}$$

This emf induced by the time-varying current (producing the time varying $\vec{B}$ field) in a stationary loop is often referred to as transformer emf in power analysis, since it is due to transformer action. By applying stoke's theorem to L.H.S

$$\rightarrow \quad \int_S (\nabla\times\vec{E})\cdot d\vec{s} = -\int_S \frac{\partial \vec{B}}{\partial t}\cdot d\vec{s}$$

For the two integrals to be equal, their integrands must be equal; that is

$$\boxed{\nabla\times\vec{E} = -\frac{\partial \vec{B}}{\partial t}.}$$

This is one of the Maxwell's equations for time varying fields.

The time varying $\vec{E}$ field is not conservative.

(ii). Moving loop in static $\vec{B}$ field (Motional EMF):-

When a conducting loop is moving in a static $\vec{B}$ field, an emf is induced in the loop. The force on a charge moving with uniform velocity $\vec{u}$ in a magnetic field $\vec{B}$ is $\vec{F_m} = Q\vec{u} \times \vec{B}$

we define the motional electric field $\vec{E_m}$ as $\vec{E_m} = \frac{\vec{F_m}}{Q} = \vec{u} \times \vec{B}$.

emf induced in the loop is $V_{emf} = \oint_L \vec{E_m} \cdot \vec{dl} = \oint_L (\vec{u} \times \vec{B}) \cdot \vec{dl}$

This type of emf is called motional or flux cutting emf.

[it is found in electrical machines such as motors, generators and alternators].

By applying stokes theorem

$$\int_S (\nabla \times \vec{E_m}) \cdot \vec{ds} = \int_S \nabla \times (\vec{u} \times \vec{B}) \cdot \vec{ds} \quad \text{or}$$

$$\boxed{\nabla \times \vec{E_m} = \nabla \times (\vec{u} \times \vec{B})}$$

(iii). Moving loop in time varying field:- In the general case, a moving conducting loop is in a time-varying magnetic field. Both transformer emf and motional emf are present.

$$V_{emf} = \oint_L \vec{E} \cdot \vec{dl} = -\int_S \frac{\partial \vec{B}}{\partial t} \cdot \vec{ds} + \int_S \nabla \times (\vec{u} \times \vec{B}) \cdot \vec{ds}$$

$$\Rightarrow \boxed{\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} + \nabla \times (\vec{u} \times \vec{B})}$$

*. Displacement current:-

Continuity equation:- From the principle of charge conservation, the time rate of decrease of charge with in a given volume must be equal to the net outward current flow through the surface of the volume. Thus current I_{out} coming out of the closed surface is

$$I_{out} = \oint_S \vec{J} \cdot \vec{ds} = \frac{-dQ_{in}}{dt}$$

Q_{in} is the total charge enclosed by the closed surface.

from divergence theorem $\oint_S \vec{J}\cdot d\vec{s} = \int_v \nabla\cdot\vec{J}\, dv \longrightarrow ①$

but $-\dfrac{dQ_{in}}{dt} = -\dfrac{d}{dt}\int_v \rho_v\, dv = -\int_v \dfrac{\partial \rho_v}{\partial t}\, dv \longrightarrow ②$

from ① & ② $\Rightarrow \int_v \nabla\cdot\vec{J}\, dv = -\int_v \dfrac{\partial \rho_v}{\partial t}\, dv$

$$\boxed{\therefore\ \nabla\cdot\vec{J} = -\frac{\partial \rho_v}{\partial t}}$$

continuity of current equation or just ~~current~~ continuity equation.

For static EM fields, we recall that

$$\nabla\times\vec{H} = \vec{J} \longrightarrow ①$$

but the divergence of the curl of any vector field is identically zero

$$\nabla\cdot(\nabla\times\vec{H}) = 0 = \nabla\cdot\vec{J}. \longrightarrow ②$$

The continuity of current, however, requires that $\nabla\cdot\vec{J} = -\dfrac{\partial\rho_v}{\partial t} \neq 0 \rightarrow ③$

Thus eqs ② & ③ are obviously incompatible for time-varying conditions.
we must modify eq ① to agree with eq ③. To do this, we add a term to eq ① so that it becomes

$$\nabla\times\vec{H} = \vec{J} + \vec{J}_d .$$

~~again~~ apply divergence

$$\Rightarrow \nabla\cdot(\nabla\times\vec{H}) = 0 = \nabla\cdot\vec{J} + \nabla\cdot\vec{J}_d . \longrightarrow ④$$

Inorder to eq ④ to agree with eq ③,

$$\nabla\cdot\vec{J}_d = -\nabla\cdot J = \frac{\partial\rho_v}{\partial t} = \frac{\partial}{\partial t}(\nabla\cdot\vec{D}) = \nabla\cdot\frac{\partial\vec{D}}{\partial t} .$$

or $\boxed{\vec{J}_d = \dfrac{\partial\vec{D}}{\partial t}} \longrightarrow ⑤$

The term $\vec{J}_d = \dfrac{\partial\vec{D}}{\partial t}$ is known as displacement current density and $\vec{J}$ is the conduction current density $(\vec{J} = \sigma\vec{E})$.

Substituting ⑤ in $\nabla\times\vec{H} = \vec{J} + \vec{J}_d$ we get

$$\nabla\times\vec{H} = \vec{J} + \frac{\partial\vec{D}}{\partial t} .$$

This is Maxwell's equation (based on Ampere's circuit law) for a time-varying field.

Based on the displacement current density, we define displacement current as

$$I_d = \oint_S \vec{J}_d \cdot \vec{ds} = \int \frac{\partial \vec{D}}{\partial t} \cdot \vec{ds}$$

Note:- The displacement current ~~density~~ is a result of time-varying electric field. A typical example of such current is the current through a capacitor when an alternating voltage source is applied to its plates.

* Maxwell's equations in final forms:-

Differential or point	Integral	Remarks
$\nabla \cdot \vec{D} = \rho_v$	$\oint_S \vec{D} \cdot \vec{ds} = \int_v \rho_v \, dv$	Gauss's law
$\nabla \cdot \vec{B} = 0$	$\oint_S \vec{B} \cdot \vec{ds} = 0$	Non existance of isolated magnetic charge
$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$	$\oint_L \vec{E} \cdot \vec{dl} = -\frac{d}{dt} \int_S \vec{B} \cdot \vec{ds}$	Faraday's law.
$\nabla \times \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t}$	$\oint_L \vec{H} \cdot \vec{dl} = \int_S \left(\vec{J} + \frac{\partial \vec{D}}{\partial t}\right) \cdot \vec{ds}$	Amperes circuit law

* Time-Varying potentials:-

For static EM fields we obtained the electric scalar potential as

$$V = \int_v \frac{\rho_v \, dv}{4\pi\epsilon R} \rightarrow (1)$$

and the magnetic vector potential as

$$\vec{A} = \int_v \frac{\mu \vec{J} \, dv}{4\pi R} \rightarrow (2)$$

We would like to examine what happens to these potentials when the fields are time varying. Recall that $\vec{A}$ was defined from the fact that $\nabla \cdot \vec{B} = 0$, which still hold for time varying fields. Hen the relation

$$\vec{B} = \nabla \times \vec{A} \longrightarrow (3)$$ holds for time varying situations

we have
$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} = -\frac{\partial}{\partial t}(\nabla \times \vec{A}) = -\nabla \times \frac{\partial \vec{A}}{\partial t}$$

$$\Rightarrow \quad \nabla \times \left(\vec{E} + \frac{\partial \vec{A}}{\partial t}\right) = 0 \longrightarrow (4)$$

Since the curl of the gradient of a scalar field is identically zero, the solution to eq (4) is

$$\vec{E} + \frac{\partial \vec{A}}{\partial t} = -\nabla V \quad \text{or}$$

$$\vec{E} = -\nabla V - \frac{\partial \vec{A}}{\partial t} \longrightarrow (5)$$

From eqs (3) & (5), we can determine the vector fields $\vec{B}$ & $\vec{E}$, provided the potentials $\vec{A}$ & V are known. Taking divergence to eq. (5) we obtain

$$\nabla \cdot \vec{E} = -\nabla^2 V - \frac{\partial}{\partial t}(\nabla \cdot \vec{A}).$$

$$\Rightarrow \quad \nabla \cdot \vec{E} = \frac{\rho_v}{\varepsilon} = -\nabla^2 V - \frac{\partial}{\partial t}(\nabla \cdot \vec{A})$$

$$\Rightarrow \quad \nabla^2 V + \frac{\partial}{\partial t}(\nabla \cdot \vec{A}) = -\frac{\rho_v}{\varepsilon} \longrightarrow (6).$$

taking curl for eq (3), we get

$$\nabla \times \vec{B} = \nabla \times \nabla \times \vec{A}$$

$$\Rightarrow \quad \nabla \times \nabla \times \vec{A} = \nabla \times \mu \vec{H} = \mu (\nabla \times \vec{H})$$

$$\Rightarrow \quad \nabla \times \nabla \times \vec{A} = \mu\left(\vec{J} + \frac{\partial \vec{D}}{\partial t}\right) = \mu\left(\vec{J} + \varepsilon \frac{\partial E}{\partial t}\right)$$

$$\Rightarrow \quad \nabla \times \nabla \times \vec{A} = \mu \vec{J} + \mu \varepsilon \frac{\partial}{\partial t}\left(-\nabla V - \frac{\partial \vec{A}}{\partial t}\right).$$

$$\Rightarrow \quad \nabla \times \nabla \times \vec{A} = \mu \vec{J} - \mu \varepsilon \nabla\left(\frac{\partial V}{\partial t}\right) - \mu \varepsilon \frac{\partial^2 \vec{A}}{\partial t^2} \longrightarrow (7)$$

By applying the vector identity

$$\nabla \times \nabla \times \vec{A} = \nabla(\nabla \cdot \vec{A}) - \nabla^2 \vec{A}$$

(7) $\Rightarrow$ $$\nabla^2 \vec{A} - \nabla(\nabla \cdot \vec{A}) = -\mu \vec{J} + \mu \epsilon \nabla\left(\frac{\partial V}{\partial t}\right) + \mu\epsilon \frac{\partial^2 \vec{A}}{\partial t^2} \quad \longrightarrow (8)$$

(*) A vector field is uniquely defined when its curl and divergence are specified. The curl of $\vec{A}$ has been specified by eq (5); for reasons that will be obvious shortly, we may choose the divergence of $\vec{A}$ as

$$\nabla \cdot \vec{A} = -\mu\epsilon \frac{\partial V}{\partial t} \quad \longrightarrow (9)$$

This choice relates $\vec{A}$ and $\vec{V}$ and it is called the Lorentz Condition for potentials.

(6) $\Rightarrow$ $$\nabla^2 V - \mu\epsilon \frac{\partial^2 V}{\partial t^2} = -\frac{\rho_v}{\epsilon} \quad \longrightarrow (10)$$

and (8) $\Rightarrow$ $$\nabla^2 \vec{A} - \mu\epsilon \frac{\partial^2 \vec{A}}{\partial t^2} = -\mu \vec{J} \quad \longrightarrow (11)$$

Eq's (10) + (11) are called wave equations.

The solutions for the wave eq's can be written as

$$V = \int_v \frac{[\rho_v]\, dv}{4\pi\epsilon R} \quad \longrightarrow (12)$$

Retarded electric scalar pot

and $$\vec{A} = \int_v \frac{\mu [\vec{J}]\, dv}{4\pi R} \quad \longrightarrow (13)$$

Retarded magnetic vector potential.

The terms $[\rho_v]$ (or $[\vec{J}]$) means that the time 't' in $\rho_v(x,y,z,t)$ [or $\vec{J}(x,y,z,t)$] is replaced by the retarded time 't'' given by

$$t' = t - \frac{R}{u}$$, where $R = |r - r'|$ is the distance between the source point 'r'' and the observation point 'r' and $u = \frac{1}{\sqrt{\mu\epsilon}}$ is the velocity of wave propagation. (free space 3×10^8 m/s)

*. Time harmonic fields:-

A time harmonic field is one that varies periodically or sinusoidally with time.

Sinusoids are easily expressed in phasors, which are more convenient to work with.

A phasor is a complex number that contains the amplitude and the phase of a sinusoidal oscillations. As a complex number, a phasor 'z' can be represented as.

$$z = x + jy = r \angle \phi. \quad \text{(or)}$$

$$z = r e^{j\phi} = r(\cos\phi + j\sin\phi)$$

$$r = \sqrt{x^2+y^2} = |z| \quad \text{and} \quad \phi = \tan^{-1}(y/x).$$

The phasor 'z' can be represented in rectangular form as $z = x + jy$ or in polar form as $z = r\angle\phi = re^{j\phi}$.

If $z_1 = x_1 + jy_1 = r_1 \angle \phi_1$, $z_2 = x_2 + jy_2 = r_2 \angle \phi_2$

addition : $z_1 + z_2 = (x_1 + x_2) + j(y_1 + y_2)$

subtraction: $z_1 - z_2 = (x_1 - x_2) + j(y_1 - y_2)$

Multiplication: $z_1 z_2 = r_1 r_2 \angle \phi_1 + \phi_2$

division : $\frac{z_1}{z_2} = \frac{r_1}{r_2} \angle \phi_1 - \phi_2$.

Square root: $\sqrt{z} = \sqrt{r} \angle \phi/2$

Complex conjugate: $z^* = x - jy = r\angle -\phi = re^{-j\phi}$

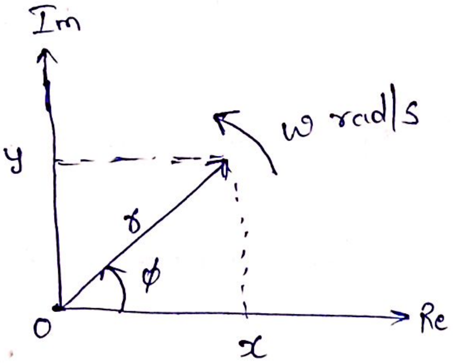


Fig:- Representation of a phasor $z = x + jy = r\angle\phi$.

To introduce the time element, we let $\phi = \omega t + \theta$

then $$re^{j\phi} = re^{j\theta} e^{j\omega t}$$

$$\mathrm{Re}\,(re^{j\phi}) = r\cos(\omega t + \theta), \quad \mathrm{Im}\,(re^{j\phi}) = r\sin(\omega t + \theta).$$

Thus a sinusoidal current $I(t) = I_0 \cos(\omega t + \theta)$, for example, equals the real part of $I_0 e^{j\theta} e^{j\omega t}$. The current $I'(t) = I_0 \sin(\omega t + \theta)$, which is imaginary part of $I_0 e^{j\theta} e^{j\omega t}$, can also be represented as the real part of $I_0 e^{j\theta} e^{j\omega t} e^{-j90^\circ}$ because $\sin\alpha = \cos(\alpha - 90^\circ)$. However, in performing our mathematical operations, we must be consistent in our use of either the real part or the imaginary part but not both at the same time.

The complex term $I_o e^{j\theta}$, which results from dropping the time factor $e^{j\omega t}$ in $I(t)$, is called 'phasor current', denoted by I_s. that is

$$I_s = I_o e^{j\theta} = I_o \angle\theta.$$

where the subscript 's' denotes the phasor form of $I(t)$. Thus $I(t) = I_o \cos(\omega t + \theta)$, the instantaneous form, can be expressed as

$$I(t) = Re\left(I_s e^{j\omega t}\right)$$

In general, a phasor could be a scalar or vector. If a vector $\vec{A}(x,y,z,t)$ is a time harmonic field, the phasor form of $\vec{A}$ is $\vec{A}_s(x,y,z)$; the two quantities are related as

$$\vec{A} = Re\left(\vec{A}_s e^{j\omega t}\right)$$

for example, if $\vec{A} = A_o \cos(\omega t - \beta x) a_y$, we can write $\vec{A}$ as

$$\vec{A} = Re\left(A_o e^{-j\beta x} a_y e^{j\omega t}\right).$$

the phasor form of $\vec{A}$ is $\vec{A}_s = A_o e^{-j\beta x} a_y$.

$$\frac{\partial \vec{A}}{\partial t} = \frac{\partial}{\partial t} Re\left(\vec{A}_s e^{j\omega t}\right) = Re\left(j\omega A_s e^{j\omega t}\right).$$

$$\frac{\partial \vec{A}}{\partial t} \rightarrow j\omega \vec{A}_s.$$

$$\int A\, \partial t \rightarrow \frac{A_s}{j\omega}.$$

instantaneous form $\vec{A}(x,y,z,t)$ and its phasor $\vec{A}_s(x,y,z)$; the former is time dependent and real, whereas the latter is time invariant and generally complex. It is easy to work with $\vec{A}_s$ and get $\vec{A}$ from $\vec{A}_s$.

$\vec{E}(x,y,z,t)$, $\vec{D}(x,y,z,t)$, $\vec{H}(x,y,z,t)$, $\vec{B}(x,y,z,t)$, $\vec{J}(x,y,z,t)$ and $\rho_v(x,y,z,t)$.