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ECE – A

18MAB101J

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MATHEMATICS ASSIGNMENT - I (UNIT - I)

1. Find the eigenvalues and eigenvectors of $\begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}$.

Soln: Let $A = \begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}$ be a square matrix of order 3×3, such that its characteristic equation is $|A - \lambda I| = 0$.

Then,

$$\left| \begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right| = 0$$

(or)

$$\begin{vmatrix} 6-\lambda & -2 & 2 \\ -2 & 3-\lambda & -1 \\ 2 & -1 & 3 \end{vmatrix} = 0$$

Its characteristic equation is of the form,

$$\lambda^3 - S_1 \lambda^2 + S_2 \lambda - S_3 = 0 \quad \text{— (1)}$$

where,

S_1 = Sum of leading diagonal of matrix A.
$= 6 + 3 + 3 = 12$

S_2 = Sum of minor of leading diagonal of matrix A.

$$= \begin{vmatrix} 3 & -1 \\ -1 & 3 \end{vmatrix} + \begin{vmatrix} 6 & 2 \\ 2 & 3 \end{vmatrix} + \begin{vmatrix} 6 & -2 \\ -2 & 3 \end{vmatrix}$$

$= 9 - 1 + 18 - 4 + 18 - 4$
$= 36$

$S_3 = |A|$
$= 6(9-1) - (-2)(-6+2) + 2(2-6)$
$= 48 - 8 - 8$
$= 32$

Substituting the above values in (1), we get

$$\lambda^3 - 12\lambda^2 + 36\lambda - 32 = 0$$

Using synthetic division method to find root of above equation.

$(\lambda-2)(\lambda^2-10\lambda+16)=0$

$(\lambda-2)(\lambda^2-2\lambda-8\lambda+16)=0$

$(\lambda-2)((\lambda-2)\lambda-8(\lambda-2))=0$

$(\lambda-2)(\lambda-2)(\lambda-8)=0$

$\lambda=2,2,8.$

$$
\begin{array}{c|cccc}
 & 1 & -12 & 36 & -32 \\
2 & 0 & 2 & -20 & 32 \\
\hline
 & 1 & -10 & 16 & 0
\end{array}
$$

We know that,

$[A-\lambda I]X=0$

$$
\begin{bmatrix} 6-\lambda & -2 & 2 \\ -2 & 3-\lambda & -1 \\ 2 & -1 & (3-\lambda) \end{bmatrix}
\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0
$$

then,

$$
\left.\begin{aligned}
(6-\lambda)x_1 - 2x_2 + 2x_3 &= 0 \\
-2x_1 + (3-\lambda)x_2 + (-1)x_3 &= 0 \\
2x_1 - x_2 + (3-\lambda)x_3 &= 0
\end{aligned}\right\} \text{(I)}
$$

The eigenvector Corresponding to the eigen value,

For $\lambda=8$

(I) becomes.

$-2x_1-2x_2+2x_3=0$ — 1.a

$-2x_1+(-5)x_2-x_3=0$ — 1.b

$2x_1-x_2-5x_3=0$ — 1.c

From 1.a

$x_1=x_3-x_2$

Sub in 1.c

$2x_3+2x_2-x_2-5x_3=0$

$-3x_3-3x_2=0$

$x_2=-x_3$ //

Sub in 1a

$x_1=x_3+x_3$

$x_1=2x_3$ //

1b

$x_3=-2x_1-5x_2$

Sub in 1.a

$-2x_1-2x_2-2x_1-5x_2=0$

$-4x_1-7x_2=0$

$x_1=\frac{-7}{4}x_2$ (or) $4x_1=-7x_2$

Sub in 1.b

$x_3=\frac{7}{2}x_2-5x_2 \Longrightarrow x_3=-\frac{3}{2}x_2$ (or) $2x_3=-3x_2$

The eigenvector Corresponding to the eigenvalue for $\lambda = 8$ is,

$$X_1 = \cancel{\begin{bmatrix} -\frac{7k}{4} \\ k \\ -\frac{3k}{2} \end{bmatrix}} \quad \begin{bmatrix} 2k \\ -k \\ k \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}_{k=1}$$

For $\lambda = 2$

(I) becomes

$$4x_1 - 2x_2 + 2x_3 = 0$$

$$-2x_1 + x_2 - x_3 = 0$$

$$2x_1 - x_2 + x_3 = 0$$

$$\Rightarrow \quad 2x_1 - x_2 + x_3 = 0$$

$x_3 = 0$

we get

$$2x_1 = x_2$$

$\therefore$ The eigenvector Corresponding to the eigenvalue for $\lambda = 2$ is,

$$X_2 = \begin{bmatrix} \frac{k}{2} \\ 2k \\ 0 \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} \frac{1}{2} \\ 2 \\ 0 \end{bmatrix}_{k=1}$$

Since A is a symmetric matrix, there exists another vector X_3 such that X_1, X_2 and X_3 are orthogonal to each other.

Let $X_3 = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$

then

$$X_1^T X_2 = 0$$

$$\begin{bmatrix} 2 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1/2 \\ 1 \\ 0 \end{bmatrix} = \quad 1 - 1 = 0$$

$$X_1^T X_3 = 0$$

$$\begin{bmatrix} 2 & -1 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = 0$$

$2a - b + c = 0 \quad - (1)$

and $X_2^T X_3 = 0$

$$\begin{bmatrix} 1/2 & 1 & 0 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = 0$$

$\frac{a}{2} + b = 0 \quad - (2)$

From (2)

$a = -2b$

sub in (1)

$-4b - b + c = 0$

$-5b + c = 0$

$c = 5b$

$\therefore \quad X_3 = \begin{bmatrix} -2k \\ k \\ 5k \end{bmatrix}$ or $\begin{bmatrix} -2 \\ 1 \\ 5 \end{bmatrix}_{k=1}$

$\therefore$ The eigenvalues for matrix A is $\lambda = 2, 2, 8$ and the eigenvectors corresponding to the eigenvalues are

$X_1 = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$, $X_3 = \begin{bmatrix} 1/2 \\ 1 \\ 0 \end{bmatrix}$ and $X_3 = \begin{bmatrix} -2 \\ 1 \\ 5 \end{bmatrix}$ //

2. Verify Cayley-Hamilton Theorem and find A^{-1} and A^4 when

$$A = \begin{bmatrix} 1 & 2 & -2 \\ 1 & 1 & 1 \\ 1 & 3 & -1 \end{bmatrix}.$$

Let $A = \begin{bmatrix} 1 & 2 & -2 \\ 1 & 1 & 1 \\ 1 & 3 & -1 \end{bmatrix}$ be a square matrix of order 3×3, such that its characteristic equation is,

$$|A - \lambda I| = 0$$

then,

$$\left| \begin{bmatrix} 1 & 2 & -2 \\ 1 & 1 & 1 \\ 1 & 3 & -1 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right| = 0$$

$$\begin{vmatrix} 1-\lambda & 2 & -2 \\ 1 & 1-\lambda & 1 \\ 1 & 3 & -1-\lambda \end{vmatrix} = 0$$

Its Characteristic equation is of the form,

$$\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0 \quad -(1)$$

where,

S_1 = Sum of leading diagonal of matrix A.

$= 1 + 1 + (-1)$

$= 1$

S_2 = Sum of minor of leading diagonal of matrix A.

$$= \begin{vmatrix} 1 & 1 \\ 3 & -1 \end{vmatrix} + \begin{vmatrix} 1 & -2 \\ 1 & -1 \end{vmatrix} + \begin{vmatrix} 1 & 2 \\ 1 & 1 \end{vmatrix}$$

$= -1-3 + (-1+2) + 1-2$

$= -4 + 1 - 1$

$= -4$

$S_3 = |A|$

$= 1(-1-3) - 2(-1-1) - 2(3-1)$

$= -4 + 4 - 4$

$= -4$

Substituting the above values in (1), we get.

$$\lambda^3 - 1\lambda^2 - 4\lambda + 4 = 0 \quad -(2)$$

By Cayley-Hamilton Theorem,
Every square matrix satisfies its own characteristic equation.

then, in eqs (2) substituting λ with A, we get,

$$A^3 - 1A^2 - 4A + 4I = 0 \quad -(3)$$

Verifying Cayley-Hamilton Theorem,

$$-4A = -4 \begin{bmatrix} 1 & 2 & -2 \\ 1 & 1 & 1 \\ 1 & 3 & -1 \end{bmatrix} = \begin{bmatrix} -4 & -8 & +8 \\ -4 & -4 & -4 \\ -4 & -12 & +4 \end{bmatrix}$$

$$A^2 = \begin{bmatrix} 1 & 2 & -2 \\ 1 & 1 & 1 \\ 1 & 3 & -1 \end{bmatrix} \begin{bmatrix} 1 & 2 & -2 \\ 1 & 1 & 1 \\ 1 & 3 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 1+2-2 & 2+2-6 & -2+2+2 \\ 1+1+1 & 2+1+3 & -2+1-1 \\ 1+3-1 & 2+3-3 & -2+3+1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & -2 & 2 \\ 3 & 6 & -2 \\ 3 & 2 & 2 \end{bmatrix}$$

$$A^3 = A \cdot A^2$$

$$= \begin{bmatrix} 1 & 2 & -2 \\ 1 & 1 & 1 \\ 1 & 3 & -1 \end{bmatrix} \begin{bmatrix} 1 & -2 & 2 \\ 3 & 6 & -2 \\ 3 & 2 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 1+6-6 & -2+12-4 & 2-4-4 \\ 1+3+3 & -2+6+2 & 2-2+2 \\ 1+9-3 & -2+18-2 & 2-6-2 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 6 & -6 \\ 7 & 6 & 2 \\ 7 & 14 & -6 \end{bmatrix}$$

Substituting the above values in (3),
we get

$\cancel{A^3 - 2A^2 - 4A + 4I = 0}$

$$\cancel{\begin{bmatrix} 1 & 6 & -6 \\ 7 & 6 & 2 \\ 7 & 14 & -6 \end{bmatrix} + \begin{bmatrix} -2 & +4 & -4 \\ -6 & -12 & +4 \\ -6 & -4 & -4 \end{bmatrix} + \begin{bmatrix} -4 & -8 & 8 \\ -4 & -4 & -4 \\ -4 & -12 & 4 \end{bmatrix} + \begin{bmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix} = 0}$$

$$A^3 - A^2 - 4A + 4I = 0$$

$$\begin{bmatrix} 1 & 6 & -6 \\ 7 & 6 & 2 \\ 7 & 14 & -6 \end{bmatrix} + \begin{bmatrix} -1 & +2 & -2 \\ -3 & -6 & +2 \\ -3 & -2 & -2 \end{bmatrix} + \begin{bmatrix} -4 & -8 & 8 \\ -4 & -4 & -4 \\ -4 & -12 & 4 \end{bmatrix} + \begin{bmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix} = 0$$

$$\begin{bmatrix} 1-1-4+4 & 6+2-8+0 & -6-2+8+0 \\ 7-3-4+0 & 6-6-4+4 & 2+2-4+0 \\ 7-3-4+0 & 14-2-12+0 & -6-2+4+4 \end{bmatrix} = 0$$

$$0 = 0 \, //$$

$\therefore$ Cayley-Hamilton Theorem is verified.

<u>To find A^{-1}</u>

$|A| = -4 \neq 0 \quad \therefore A^{-1}$ exists.

using eqs (3)

$$A^3 - A^2 - 4A + 4I = 0$$

multipeying A^{-1} on both sides we get,

$$A^3A^{-1} - A^2A^{-1} - 4AA^{-1} + 4IA^{-1} = 0$$

$$A^2 - A - 4I + 4A^{-1} = 0$$

$$A^{-1} = \frac{1}{4}\left[-A^2 + A + 4I\right]$$

$$A^{-1} = \frac{1}{4}\left[\begin{bmatrix} -1 & 2 & -2 \\ -3 & -6 & 2 \\ -3 & -2 & -2 \end{bmatrix} + \begin{bmatrix} 1 & 2 & -2 \\ 1 & 1 & 1 \\ 1 & 3 & -1 \end{bmatrix} + \begin{bmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix}\right]$$

$$A^{-1} = \frac{1}{4}\begin{bmatrix} 4 & 4 & -4 \\ 2 & -1 & 3 \\ -2 & 1 & 1 \end{bmatrix} //$$

To Find A^4,

using eqn ③

$$A^3 - A^2 - 4A + 4I = 0$$

Multiplying A on both sides,

$$A^4 - A^3 - 4A^2 + 4I = 0$$

$$A^4 = A^3 + 4A^2 - 4I = 0$$

$$= \begin{bmatrix} 1 & 6 & -6 \\ 7 & 6 & 2 \\ 7 & 14 & -6 \end{bmatrix} + \begin{bmatrix} 4 & -8 & 8 \\ 12 & 24 & -8 \\ 12 & 8 & 8 \end{bmatrix} + \begin{bmatrix} -4 & 0 & 0 \\ 0 & -4 & 0 \\ 0 & 0 & -4 \end{bmatrix}$$

$$A^4 = \begin{bmatrix} 1 & -2 & 2 \\ 19 & 26 & -6 \\ 19 & 22 & -2 \end{bmatrix}$$

3\. Using Cayley-Hamilton Theorem find the value of polynomial

$$A^8 - 5A^7 + 7A^6 - 3A^5 + A^4 - 5A^3 + 8A^2 - 2A + I \text{ - for } A = \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix}$$

Soln

Let $A = \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix}$ be a square matrix of order 3×3, such that its characteristics equation is,

$$|A - \lambda I| = 0$$

then,

$$\begin{vmatrix} 2-\lambda & 1 & 1 \\ 0 & 1-\lambda & 0 \\ 1 & 1 & 2-\lambda \end{vmatrix} = 0$$

Its characteristic polynomial is,

$$\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0 \quad — ①$$

where,

S_1 = Sum of leading diagonal of matrix A.

$= 2 + 1 + 2$

$= 5$

S_2 = Sum of minors of leading diagonal of matrix A.

$$= \begin{vmatrix} 1 & 0 \\ 1 & 2 \end{vmatrix} + \begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} + \begin{vmatrix} 2 & 1 \\ 0 & 1 \end{vmatrix}$$

$= 2 - 0 + 4 - 1 + 2 - 0$

$= 7$

$S_3 = |A|$

$= 2(2) - 1(0) + 1(0 - 1)$

$= 3$

Substituting the above values in ①,

$$\lambda^3 - 5\lambda^2 + 7\lambda - 3 = 0 \quad — ②$$

By Cayley-Hamilton Theorem,
every square matrix staisfies its own characteristic equation.

Replacing λ with A in eqn ②, we get

$$A^3 - 5A^2 + 7A - 3I = 0$$

Given polynomial,

$$A^8-5A^7+7A^6-3A^5+A^4-5A^3+8A^2-2A+I$$

$$\Rightarrow A^5(A^3-5A^2+7A-3I)+A(A^3-5A^2+8A-2I)+I$$

$$A^5(A^3-5A^2+7A-3I)+A(A^3-5A^2+7A+A-3I+I)+I$$

$$A^5(A^3-5A^2+7A-3I)+A(A^3-5A^2+7A-3I)+A^2+A+I$$

According to Cayley-Hamilton theorem,

$$A^3-5A^2+7A-3I=0$$

we get,

$$\Rightarrow 0+0+A^2+A+I \quad \text{—(3)}$$

$$A^2=AA=\begin{bmatrix}2&1&1\\0&1&0\\1&1&2\end{bmatrix}\begin{bmatrix}2&1&1\\0&1&0\\1&1&2\end{bmatrix}$$

$$=\begin{bmatrix}4+0+1 & \cancel{2}+1+1 & 2+0+2\\ 0+0+0 & 0+1+0 & 0+0+0\\ 2+0+2 & 1+\cancel{1}+2 & 1+0+4\end{bmatrix}$$

$$=\begin{bmatrix}5&4&4\\0&1&0\\4&4&5\end{bmatrix}$$

Substituting in (3), we get

$$A^2+A+I=$$

$$\begin{bmatrix}5&4&4\\0&1&0\\4&4&5\end{bmatrix}+\begin{bmatrix}2&1&1\\0&1&0\\1&1&2\end{bmatrix}+\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}$$

$$=\begin{bmatrix}8&5&5\\0&3&0\\5&5&8\end{bmatrix}$$

4. Reduce the quadratic form $Q = 3x^2 + 5y^2 + 3z^2 - 2xy - 2yz + 2xz$ to canonical form and hence find its nature, rank, index and signature.

Soln. The Quadratic form can be expressed in terms of product of matrices,

$$Q = X^T A X \text{ , where } X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

and

$$A = \begin{bmatrix} 3 & -1 & 1 \\ -1 & 5 & -1 \\ 1 & -1 & 3 \end{bmatrix}$$

A is a symmetric matrix, such that its characteristic equation is

$$|A - \lambda I| = 0$$

Then its characteristic polynomial is,

$$\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0 \quad - (1)$$

where,

S_1 = Sum of diagonal

$= 3+5+3 = 11$

S_2 = Sum of minor of diagonal

$$= \begin{vmatrix} 5 & -1 \\ -1 & 3 \end{vmatrix} + \begin{vmatrix} 3 & 1 \\ 1 & 3 \end{vmatrix} + \begin{vmatrix} 3 & -1 \\ -1 & 5 \end{vmatrix}$$

$= 15 - 1 + 9 - 1 + 15 - 1$

$= 36$

$S_3 = |A|$

$= 3(14) + 1(-2) + 1(-4)$

$= 36$

Sub the above values in (1), we get

$$\lambda^3 - 11\lambda^2 + 36\lambda - 36 = 0$$

Using synthetic division,

$$\begin{array}{r|rrrr} & 1 & -11 & 36 & -36 \\ 2 & 0 & 2 & -18 & 36 \\ \hline & 1 & -9 & 18 & 0 \end{array}$$

$$\Rightarrow (\lambda-2)(\lambda^2 - 9\lambda + 18) \Rightarrow (\lambda-2)((\lambda-3)(\lambda-6)) = 0 \quad (11)$$

$\lambda = 2, 3, 6$

To find eigenvector

we know that,

$$(A - \lambda I) X = 0$$

$$\begin{bmatrix} 3-\lambda & -1 & 1 \\ -1 & 5-\lambda & -1 \\ 1 & -1 & 3-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

For $\lambda = 2$

$$\begin{bmatrix} 1 & -1 & 1 \\ -1 & 3 & -1 \\ 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

Cross multiplying,

$$\begin{matrix} -1 & 1 & 1 & -1 \\ 3 & -1 & -1 & 3 \end{matrix}$$

$$\frac{x_1}{1-3} = \frac{x_2}{-1+1} = \frac{x_3}{3-1}$$

$$\Rightarrow \frac{x_1}{-2} = \frac{x_2}{0} = \frac{x_3}{2}$$

$$X_1 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

For $\lambda = 3$

$$\begin{bmatrix} 0 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

Cross multiplying

$$\begin{matrix} -1 & 1 & 0 & -1 \\ 2 & -1 & -1 & 2 \end{matrix}$$

$$\frac{x_1}{1-2} = \frac{x_2}{-1} = \frac{x_3}{-1} \qquad X_2 = \begin{bmatrix} -1 \\ -1 \\ -1 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

For $\lambda = 6$

$$\begin{bmatrix} -3 & -1 & 1 \\ -1 & -1 & -1 \\ 1 & -1 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

Cross multiplying

$$\begin{matrix} -1 & -1 & -1 & -1 \\ -1 & -3 & 1 & -1 \end{matrix}$$

$$\frac{x_1}{3-1} = \frac{x_2}{-1-3} = \frac{x_3}{1+1}$$

$$X_3 = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$$

To find diagonal of A by orthogonal transformation we know that

$$D = N^T A N$$

N is the Normalised Modal Matrix,

$$N = [\bar{X_1} \quad \bar{X_2} \quad \bar{X_3}]$$

$$N = \begin{bmatrix} -1/\sqrt{2} & 1/\sqrt{3} & 1/\sqrt{6} \\ 0 & 1/\sqrt{3} & -2/\sqrt{6} \\ 1/\sqrt{2} & 1/\sqrt{3} & 1/\sqrt{6} \end{bmatrix}, \quad N^T = \begin{bmatrix} -1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 1/\sqrt{3} & 1/\sqrt{3} & 1/\sqrt{3} \\ 1/\sqrt{6} & -2/\sqrt{6} & 1/\sqrt{6} \end{bmatrix}$$

$$D = N^T A N$$

$$= \begin{bmatrix} -1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 1/\sqrt{3} & 1/\sqrt{3} & 1/\sqrt{3} \\ 1/\sqrt{6} & -2/\sqrt{6} & 1/\sqrt{6} \end{bmatrix} \begin{bmatrix} 3 & -1 & 1 \\ -1 & 5 & -1 \\ 1 & -1 & 3 \end{bmatrix} N$$

$$= \begin{bmatrix} -3/\sqrt{2} + 1/\sqrt{2} & 1/\sqrt{2} - 1/\sqrt{2} & -1/\sqrt{2} + 3/\sqrt{2} \\ 3/\sqrt{3} - 1/\sqrt{3} + 1/\sqrt{3} & -1/\sqrt{3} + 5/\sqrt{3} - 1/\sqrt{3} & 1/\sqrt{3} - 1/\sqrt{3} + 3/\sqrt{3} \\ 3/\sqrt{6} + 2/\sqrt{6} + 1/\sqrt{6} & -1/\sqrt{6} - 10/\sqrt{6} - 1/\sqrt{6} & 1/\sqrt{6} + 2/\sqrt{6} + 3/\sqrt{6} \end{bmatrix} N$$

$$= \begin{bmatrix} -2/\sqrt{2} & 0 & 2/\sqrt{2} \\ 3/\sqrt{3} & 3/\sqrt{3} & 3/\sqrt{3} \\ 6/\sqrt{6} & -12/\sqrt{6} & 6/\sqrt{6} \end{bmatrix} \begin{bmatrix} -1/\sqrt{2} & 1/\sqrt{3} & 1/\sqrt{6} \\ 0 & 1/\sqrt{3} & -2/\sqrt{6} \\ 1/\sqrt{2} & 1/\sqrt{3} & 1/\sqrt{6} \end{bmatrix}$$

$$D = \begin{bmatrix} 2/2 + 2/2 & 0 & 0 \\ 0 & \frac{3}{3}+\frac{3}{3}+\frac{3}{3} & 0 \\ 0 & 0 & \frac{6}{6}+\frac{24}{6}+\frac{6}{6} \end{bmatrix}$$

$$D = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 6 \end{bmatrix}$$

<u>To find Canonical form</u>

We know that

$X = NY$ and $QF = X^T A X$

then,

$Q.F = (NY)^T A (NY)$

$= Y^T N^T A N Y$, $(D = N^T A N)$

$= Y^T D Y$

Canonical form $= [x \ y \ z] \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 6 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$

$CF = 2x^2 + 3y^2 + 6z^2$

Index $(p) = 3$

Signature $(s) = 3$

Rank $(r) = 3$

Nature = Positive Definite.

5. Reduce the quadratic form $Q = x_1^2 + 2x_2x_3$ to Canonical form and hence find its nature, rank, index and signature.

Soln

We know that QF is the product of matrices.

$Q = X^T A X$ where $X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ and

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}.$$

A is a symmetric matrix, such that its characteristic equation is

$$|A - \lambda I| = 0$$

Then its characteristic polynomial is

$$\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0 \quad -(1)$$

where,

$S_1 = 1$ (Sum of diagonal)

$S_2 = \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} + \begin{vmatrix} 1 & 0 \\ 0 & 0 \end{vmatrix} + \begin{vmatrix} 1 & 0 \\ 0 & 0 \end{vmatrix}$ (Sum of minor of diagonal)

$= -1$

$S_3 = |A|$

$= -1$

Sub. the above values in (1), we get

$$\lambda^3 - \lambda^2 - \lambda + 1 = 0$$

Using Synthetic division

$$\begin{array}{c|cccc} & 1 & -1 & -1 & 1 \\ 1 & 0 & 1 & 0 & -1 \\ \hline & 1 & 0 & -1 & 0 \end{array}$$

$(\lambda - 1)(\lambda^2 - 1) = 0$

$\lambda = 1, 1, -1.$

To find eigenvector, we know that

$$(A - \lambda I)X = 0$$

$$\begin{bmatrix} 1-\lambda & 0 & 0 \\ 0 & 0-\lambda & 1 \\ 0 & 1 & 0-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

For $\lambda = -1$

$$\begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

Cross multiplying,

$$\begin{matrix} 0 & 0 & 2 & 0 \\ 1 & 1 & 0 & 1 \end{matrix}$$

$$\frac{x_1}{0} = \frac{x_2}{-2} = \frac{x_3}{2}$$

$$X_1 = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$$

For $\lambda = 1$

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & -1 & 1 \\ 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$$0x_1 - x_2 + x_3 = 0$$

$$\Rightarrow x \neq 0$$

$$x_3 = x_1$$

$$X_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

Since A is a symmetric matrix, the eigen vectors are orthogonal to each other.

$X_1^T X_3 = 0$ and $X_2^T X_3 = 0$

where

$$X_3 = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

then,

$$X_1^T X_3 = \begin{bmatrix} 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = 0$$

$$= -b + c = 0 \quad - \text{(I)}$$

and

$$X_2^T X_3 = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = 0$$

$$= a + b + c = 0 \quad - \text{(II)}$$

Cross multiplying (I) and (II), we get,

$$\begin{array}{cccc} -1 & 1 & 0 & -1 \\ 1 & 1 & 1 & 1 \end{array}$$

$$\frac{a}{-2} = \frac{b}{1} = \frac{c}{1}$$

$$\therefore \quad X_3 = \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}$$

To find diagonal matrix of A by orthogonal transformation, we know that,

$$D = N^T A N$$

where N is the normalised modal matrix

$$N = [\bar{X}_1 \quad \bar{X}_2 \quad \bar{X}_3]$$

$$N = \begin{bmatrix} 0 & 1/\sqrt{3} & -2/\sqrt{6} \\ -1/\sqrt{2} & 1/\sqrt{3} & 1/\sqrt{6} \\ 1/\sqrt{2} & 1/\sqrt{3} & 1/\sqrt{6} \end{bmatrix} \text{ and } N^T = \begin{bmatrix} 0 & -1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{3} & 1/\sqrt{3} & 1/\sqrt{3} \\ -2/\sqrt{6} & 1/\sqrt{6} & 1/\sqrt{6} \end{bmatrix}$$

$D = N^T A N$

$$= \begin{bmatrix} 0 & -1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{3} & 1/\sqrt{3} & 1/\sqrt{3} \\ -2/\sqrt{6} & 1/\sqrt{6} & 1/\sqrt{6} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} N$$

$$= \begin{bmatrix} 0 & 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{3} & 1/\sqrt{3} & 1/\sqrt{3} \\ -2/\sqrt{6} & 1/\sqrt{6} & 1/\sqrt{6} \end{bmatrix} \begin{bmatrix} 0 & 1/\sqrt{3} & -2/\sqrt{6} \\ -1/\sqrt{2} & 1/\sqrt{3} & 1/\sqrt{6} \\ 1/\sqrt{2} & 1/\sqrt{3} & 1/\sqrt{6} \end{bmatrix}$$

$$= \begin{bmatrix} -\frac{1}{2}-\frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{3}+\frac{1}{3}+\frac{1}{3} & 0 \\ 0 & 0 & \frac{4}{6}+\frac{1}{6}+\frac{1}{6} \end{bmatrix}$$

$$D = \begin{bmatrix} -1 & 0 & 0 \\ 0 & +1 & 0 \\ 0 & 0 & +1 \end{bmatrix}$$

<u>To find Canonical form</u>

$X = NY$ and $QF = X^T A X$

then

$QF = (NY)^T A (NY)$

$= Y^T (N^T A N) Y$

$= Y^T D Y \quad (\because D = N^T A N)$

$$= [y_1 \ y_2 \ y_3] \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

$= -y_1^2 + y_2^2 + y_3^2$ is the Canonical form.

Its Signature $(S) = 2 - 1 = 1$

Index $(p) = 2$

Rank $= 3$

Nature = Indefinite