Maths

Note :- If R is a relation from set A to itself, R is called a relation on the set A. i.e R is the improper subset of A x A.

$$R \subseteq A \times A$$

Domain And Range of a relation:

let R be a relation from A to B i.e $R \subseteq A \times B$.
The domain D of relation R is set of all first element of ordered pairs which belong to R.

i.e $D(R) = \{a \mid a \in A \ \& \ (a,b) \in R \text{ for some } b \in B\}$

Range of a relation:

Range of relation is set of all 2nd element of ordered pairs in R. i.e

$\text{Range}(R) = \{b \mid b \in B, (a,b) \in R, \text{ for some } a \in A\}$

Clearly the domain of a relation from $A \to B$ is a subset of A and its range is a subset of B.

Eg: $A = \{1,2,3,4\}$

$B = \{r,s,t\}$

& $R = \{(1,r),(2,s),(3,r)\}$.

$D(R) = \{1,2,3\}$

$R(R) = \{r,s\}$

Eg: let R be a relation from A.

$A = \{2,3,4,5\}$

$B = \{3,6,7,10\}$

which is defined by expression ~~x~~ x divides y

① Write R as set of ordered pairs

② Find its domain and range.

∴ ① $R = \{(2,6),(2,10),(3,6),(3,3),(5,10)\}$

$D(R) = \{2,3,5\}$

$R(R) = \{6,10,3\}$

# Types of Relation

## ① Universal Relation:

A relation R on a set A is said to be universal relation if $R = A \times A$

$A = \{1, 2, 3\}$

then $R = A \times A$

$R = \{(1,1), (1,2), (1,3), (2,1), (2,2), (2,3), (3,1), (3,2), (3,3)\}$

## ② Void or Null Relation:

A relation R on set A is called void relation if R is the null set $(\emptyset)$.

Eg: $A = \{3, 4, 5\}$

then $a \in b$, if $a + b > 10$

$\therefore R = \emptyset$

## ③ Inverse Relation:

When R is any relation from set A to a set B, the inverse of R is denoted by $R^{-1}$ from $B \to A$, which consists of those ordered pairs brought by interchanging the elements of ordered pairs in R.

if $R : A \to B$

$R^{-1} : B \to A$.

$R^{-1} = \{(b, a) \mid (a, b) \in R\}$

if $aRb$, then $bR^{-1}A$

Eg. $A = \{2, 3, 5\}$

$B = \{6, 8, 10\}$

Consider R such that '<'

$R = \{(2,6), (2,8), (2,10), (3,6), (3,8), (3,10), (5,6), (5,8), (5,10)\}$

$R^{-1} = \{(6,2), (8,2), (10,2) \ldots\}$

## Operation on Relations:

**Union And Intersection of Relations**

Let R and S are two relations from a set A and set B, then $R \cup S$ and $R \cap S$ are defined as follows.

$$R \cup S = \{(a,b) \mid (a,b) \in R \text{ or } (a,b) \in S\}$$

OR

$$= \{(a,b) \mid a(R \cup S)b \Leftrightarrow aRb \vee aSb\}$$

$$R \cap S = \{(a,b) \mid a(R \cap S)b \Leftrightarrow aRb \wedge aSb\}$$

Let $A = \{1,2,3\}$, $B = \{2,b\}$.

We can define R and S by:

$R = \{(1,2), (1,b), (2,b), (3,b)\}$

$S = \{(1,2), (1,b), (2,2), (2,b), (3,2), (3,b)\}$

$R \cup S = \{(1,2), (1,b), (2,2), (2,b), (3,2), (3,b)\}$

$R \cap S = \{(1,2), (1,b), (2,b), (3,b)\}$

$R - S$ $\quad a(R-S)b = aRb \wedge a \not S b$.

$A = \{x, y, 3\}$ $\quad B = \{1, 2, 3\}$

$C = \{x, y\}$ $\quad D = \{2, 3\}$.

$R: A \to B \Rightarrow \{(x,1), (x,2), (y,3)\}$

$S: C \to D \Rightarrow \{(x,2), (y,3)\}$.

$R - S = \{(x,1)\}$

**Complement of R denoted by R'**

$aR'b = a \not R b$

$R' = A \times B - R$

**Identity Relation :-**

A relation R on a set A is called an identity relation on A if $R = \{(a,a) \mid a \in A$ is denoted by $I_A\}$

$A = \{1, 2, 3\}$

then $R = \{(1,1), (2,2), (3,3)\}$ is identity relation on A.

let $A=\{1,2,3\}$, $B=\{1,4\}$

Consider Relation R such that 'i<j'

Find R, R', $R^{-1}$

$A\times B=\{(1,1),(1,4),(2,1),(2,4),(3,1),(3,4)\}$

$\therefore R=\{(1,4),(2,4),(3,4)\}$

$R'=\{(1,1),(2,1),(3,1)\}$

$R^{-1}=\{(4,1),(4,2),(4,3)\}$

**Composition of Relation:**

Suppose that A, B, C are sets R is a relation from $A\to B$, S is a relation from $B\to C$, we can define a new relation the composition of R and S, written as $R\circ S$, is a relation from $A\to C$

$R\circ S=\{(a,c) \mid (a,b)\in R \ \& \ (b,c)\in S\}$

let $A=\{1,2,3\}$

$B=\{2,3,6,8,12\}$

$C=\{13,17,22\}$

$R=\{(1,2),(1,3)(1,12),(2,3),(2,6),(2,8),(2,12)\}$

$S=\{(2,13),(2,17),(3,13),(3,22),(8,22)\}$

$R\circ S=\{(1,17),(1,13),(1,22),(2,13),(2,22)\}$

**Properties of Relation:**

let A be any non-empty set and let R be a relation on A.

i) R is called a reflexive relation if $aRa$ for every $a\in A$

i.e $(a,a)\in R$ for every $a\in A$.

ii) R is called symmetric whenever $aRb$, then $bRa$

i.e $(a,b)\in R$ then $(b,a)\in R$

iii) R is transitive if $(a,b),(b,c) \in R \Rightarrow (a,c) \in R$.

iv) R is irreflexive if $(a,a) \notin R$ for all $a \in A$

v) R is anti symmetric if $aRb$ & $bRa$, then $a=b$; OR, $aRb$ then $b \not R a$.

vi) R is asymmetric if and only if it is both anti-symmetric and irreflexive

Note: Symmetric and Anti-symmetric relations are not opp. because a relation R can contain both the properties or cannot.

Equivalence Relation:

A relation R on a non empty set A is called an equivalence relation if R is reflexive, symmetric and transitive.

Q) If R is a relation on the set of positive integer, such that $(a,b) \in R$ if $a^2+b$ is even. Prove R is an equivalence relation

$(a,b) \in R$

Let's assume $(a,a) \in R$

$\rightarrow a^2+a = a(a+1)$ is even as either $a$ or $(a+1)$ is even

$\therefore (a,a) \in R$

Hence R is reflexive

$\rightarrow$ When $a^2+b$ is even, a & b must be even or both odd.

$b^2+a$ is even, a & b must be even or both odd.

$\therefore$ R is symmetric

$\rightarrow$ When a, b, c, $a^2+b$, $b^2+c$ are even. Also $a^2+c$ is even

when a, b, c are odd, $a^2+b$ and $b^2+c$ are even.

Also $a^2+c$ is even.

Then $(a,b) \in R$, $(b,c) \in R$

$\therefore (a,c) \in R$.

Q) Let $A=\{1,2,3,4\}$

$R=\{(1,1),(1,2),(2,1),(1,3),(3,1),(2,2),(3,3)\}$

$S=\{(1,1),(1,2),(1,3),(2,1),(2,2),(2,3),(3,1),(3,2),(3,3)\}$

Check R & S is reflexive, symmetric, transitive

i) R is not reflexive as $(4,4) \notin R$

ii) R is symmetric

iii) R is not transitive.

i) S is not reflexive as $(4,4) \notin S$.

ii) S is symmetric as $(1,2) \in S$ and $(2,1) \in S$, $(2,3) \in S$ and $(3,2) \in S$.

iii) S is transitive

Q. Let $A=\{1,2,3,4,5,6,7\}$ and relation R defined as $R=\{(x,y) \mid |x-y|=2\}$. Is R an equivalence relation.

$R=\{(1,3),(2,4),(3,5),(4,6),(5,7),(7,5),(6,4),(5,3),(4,2),(3,1)\}$

R is not reflexive : $(a,a) \notin R \ \forall a \in A$.

R is symmetric as $(1,3),(3,1)$

R is not transitive as $(1,3) \in R$, $(3,5) \in R$, $(1,5) \notin R$.

Q) Let R be a relation on the set of integers such that $3a+4b=7n$ for some integer n. Prove that R is equivalence relation.

$(a,a) \in R$ if and only if $3a+4a = 7a$ (for some integer a)

$\therefore aRa$

$\therefore$ R is reflexive.

$(a, b) \in R \Rightarrow 3a + 4b = 7n$

$(b, a) \in R \Rightarrow 3b + 4a = 7n$

$$3b + 4a = (7a + 7b) - (3a + 4b)$$
$$= 7(a+b) - 7n = 7(a+b-n) = 7(t) \quad [t \text{ is any integer}]$$

$\therefore$ R is symmetric as $(b, a) \in R$

$aRb, bRc \Rightarrow 3a + 4b = 7n$
$3b + 4c = 7m$

$$3a + 4c = 7n - 4b + 7m - 3b$$
$$= 7(m+n) - 7b$$
$$= 7(m+n-b)$$

$\therefore (a, c) \in R$

$\therefore$ R is an equivalence relation.

**Partial order relation:**

A relation R on a set A is called a partial ordering or partial order relation if and only if R is reflexive, anti-symmetric and transitive.

**Poset:**

A set A together with partial ordering R is called partially ordered set or Poset.

Note: It is convenient to denote partial ordering as $\leq$. This symbol does not necessarily mean less than or equal to as it is used for real numbers.

**Representation of Relation by Graphs:**

Let R be a relation on set A. To represent R graphically we follow the following conditions.

(i) Each element of A is represented by a point called node or vertex.

(ii) Whenever element A is related to element B and arc or st-line is drawn from $A \to B$ called arc or edges. It starts from $A \to B$. The direction is indicated by an arrow. The resulting diagram is called directed graph.

(iii) The edge of the form $(a, a)$ represented by using an arc from vertex a back to itself is called loop.

Note :- In a digraph of R, the indegree of a vertex is number of edges terminating at the vertex but outdegree of a vertex is number of edges leaving the vertex.

→ A relation R is reflexive if and only if there is a loop at every vertex of the digraph of the relation R.

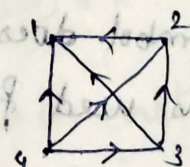
→ A relation R is symmetric if and only if for every edge between distinct vertices in its digraph, there is an edge in the opposite direction, i.e $(b, a)$ in R whenever $(a, b)$ in R.

→ Relation R is transitive if and only if whenever there is an edge from vertex A to vertex B and from B to C, there is an edge between A to C.

→ The Relation R is anti-symmetric if and only if there are never two edges in opposite direction between distinct vertices. In other words, a relation on a set A is anti-symmetric if there is atmost 1 directed edge b/w every pair of vertices.

eg: ① $A = \{1, 2, 3, 4\}$

$R = \{(x, y) \mid x > y\}$




$R = \{(2,1), (3,2), (4,1), (4,2), (4,3), (3,1)\}$

R is not reflexive, since there is no loop.

R is not symmetric, since there is no ~~opposite~~ parallel lines.

R is transitive

R is anti symmetric, since no two edges in opp. direction

② $A = \{1, 2, 3, 4\}$

$R = \{(1,1), (1,2), (1,3), (2,2), (3,3), (3,4), (4,1), (4,4)\}$

R is reflexive, there is loop

R is not-symmetric, since no parallel lines

R is not-transitive

Matrix <u>representation</u> of relation :

If R is a relation from the set $A = \{a_1, a_2 \cdots a_n\}$ to the set $B = \{b_1, b_2, b_3 \cdots b_n\}$ where the elements of A and B are assumed to be in a specific order, the relation R can be represented by the matrix $M_R = [m_{ij}]$

$$m_{ij} = \begin{cases} 1 & \text{if } (a_i, a_j) \in R \\ 0 & \text{if } (a_i, a_j) \notin R \end{cases}$$

eg: If $A = \{a_1, a_2, a_3\}$, $B = \{b_1, b_2, b_4\}$

$R = \{(a_1, b_2), (a_2, b_1), (a_2, b_3), (a_2, b_4), (a_3, b_2), (a_3, b_4)\}$

$$M_R = \begin{array}{c|cccc} & b_1 & b_2 & b_3 & b_4 \\ \hline a_1 & 0 & 1 & 0 & 0 \\ a_2 & 1 & 0 & 1 & 1 \\ a_3 & 0 & 1 & 0 & 1 \end{array}$$

Note: If R and S be two relations on A, then

$M_{R\cup S} = M_R \vee M_S$

$M_{R\cap S} = M_R \wedge M_S$

$M_{R^{-1}} = (M_R)^T$ T → Transpose

$M_R = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{pmatrix}$ & $M_S = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$

$M_{R\cup S} = M_R \vee M_S$

$= \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{pmatrix} \vee \begin{pmatrix} 1 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$

$M_{R\cap S} = M_R \wedge M_S = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ \cdots \end{pmatrix} \wedge \begin{pmatrix} 1 & 0 & 1 \\ 1 & 0 & 0 \\ \cdots \end{pmatrix} = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ \cdots \end{pmatrix}$

$M_{R\circ S} = M_R \odot M_S$

$$= \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 1\vee0\vee0 & 0\vee0\vee1 & 1\vee0\vee0 \\ 0\vee0\vee1 & 0\vee0\vee1 & 0\vee0\vee0 \\ 1\vee0\vee0 & 0\vee0\vee0 & 1\vee0\vee0 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$$

**Note:** If R is reflexive if and only if the ordered pair $(a_i, a_i) \in R$ for $i = 1, 2, \dots n$, $m_{ii} = 1$ for $i = 1, 2 \dots n$. In other words, R is reflexive if all the elements in the principal diagonal of $M_R$ are equal to 1.

→ The relation R on the set $A = \{a_1, a_2 \dots a_n\}$ is symmetric if and only if $(a_i, a_j) \in R$ whenever $(a_j, a_i) \in R$, i.e. $(a_i, a_j) \in R$ we will have $m_{ji} = 1$ when $m_{ij} = 1$.

If $M_R = (M_R)^T$

→ The matrix of an anti-symmetric relation has a property that when $m_{ij} = 1$ $(i \neq j)$ then $m_{ji} = 0$.

→ If R is transitive if and only if $R^n$ is a subset of R i.e. $R^n \subseteq R$ for $n \geq 1$.

If $M_{R^2} = M_R$, then R is transitive. The converse is not true

R is transitive if and only if $R^2 \subseteq R$

① If R is a relation on $A = \{1, 2, 3\}$ such that $(a, b) \in R$ if and only if $a + b =$ even. Find $M_R$ and also find relation matrices $R^{-1}, \bar{R}, R^2$.

$R = \{(1,1), (1,3), (2,2), (3,1), (3,3)\}$

$$M_R = \begin{array}{c|c} & \begin{matrix} 1 & 2 & 3 \end{matrix} \\ \begin{matrix} 1 \\ 2 \\ 3 \end{matrix} & \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \end{array} \quad M_{R^{-1}} = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \quad M_{\bar{R}} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

$M_{R^2} = M_{R \circ R}$

$$= \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$$

Q) If R and S be the relation on set A represented by the matrices.

$$M_R = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 1 & 1 \\ 1 & 0 & 0 \end{pmatrix} \quad M_S = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$$

Find matrices that represents $R \cup S$, $R \cap S$, $R \circ S$, $S \circ R$, $R \oplus S$

Q) Examine if relation R represented by $M_R = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$ is an equivalence relation using the properties of $M_R$.

A. Since all the elements in the main diagonal of $M_R$ is 1 each, R is equivalence relation.

R is symmetric as | where

For transitive,

$$M_{R^2} = M_R \circ M_R = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} = M_R$$

R is transitive

$\therefore$ R is equivalence relation

## Hassey diagram for partial ordering :

Simplified form of a diagraph of a partial ordering on a finite set that contains sufficient info about partial ordering is called Hasse diagram.

Steps to be followed :-

1) Create a vertex for elements of the given domain. If A R B, then draw an edge from a to b that means we can draw line only in upward direction.

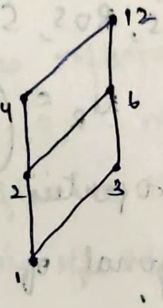
2) Remove self loops and transitive edges

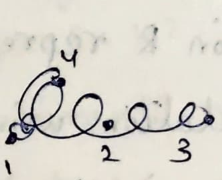
For eg: let's considering Hasse diagram for $\{(a,b) \mid a \le b\}$ on set $\{1,2,3,4\}$ starting from its digraph.

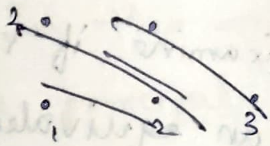
Q) Draw the Hasse diagram for $\{3, 4, 12, 24, 48, 72\}$ using $|$.

Q) Draw Hasse diagram for $(D_{12}, |)$, $D_{12}$ means set of positive integer divisor of 12.

$D_{12} = \{1, 2, 3, 4, 6, 12\}$



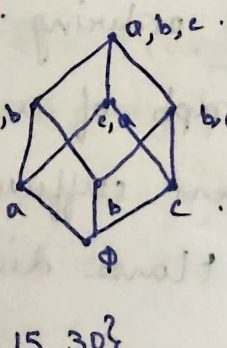

Q) Consider the divisibility relation on set $A = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$.

Q) Draw the Hasse for $(D_{30}, |)$ where $|$ is divisibility relation

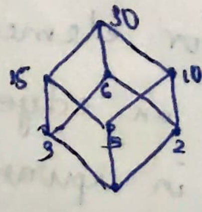
Q) Draw the Hasse representing subset relation on $P(A)$ where $A = \{a, b, c\}$

Q) Let $A = \{a, b, c\}$, draw the Hasse diagram representing the subset relation on the power set $P(A)$.

$P(A) = \{\phi, \{a\}, \{b\}, \{c\}, \{a,b\}, \{b,c\}, \{c,a\}, \{a,b,c\}\}$.




Q) $D_{30} = \{1, 2, 3, 5, 6, 10, 15, 30\}$



## Closure of Relations:

### ① Reflexive closure:

Let R be a relation on a set A. The reflexive closure of R is denoted by $R^r$ and is defined by $R \cup \{(a,a) \mid a \in A\}$.
It is the smallest reflexive relation on a set A containing R. eg: let $A = \{1,2,3,4\}$.
Let R be a relation on A $R = \{(1,1), (1,2), (2,3), (3,3), (3,4)\}$

$R^r = \{(1,1), (1,2) (2,3) (3,3) (3,4) (2,2) (4,4)\}$

### ② Symmetric closure:

Let R be a relation on a set A. The symmetric closure of R denoted by $R^S$ and defined by $R \cup \{(b,a) \mid (a,b) \in R\}$

eg: let $A = \{1,2,3\}$

$R = \{(1,1), (1,2), (2,1) (2,3) (1,3)\}$

$R^S = \{(1,1) (1,2) (2,1) (2,3) (1,3) (3,2) (3,1)\}$

### ③ Transitive closure:

The transitive closure of the relation R on a set A denoted by $R^t$ and defined by $R^t = R \cup \{(a,c) \mid (a,b) \in R \ \& \ (b,c) \in R\}$
If R is a relation on a set A with $|A| = n$ then

$R^t = R \cup R^2 \cup R^3 \cup \dots R^n$

Eg: Let $A = \{1,2,3\}$

$R = \{(1,1) (2,3) (3,1)\}$

$R^t = \{(1,1) (2,3) (3,1) (2,1)\}$

### ④ Closure of Relation in Matrix form:

Let R be a relation on a set A with n elements. Let $M_R$ be the matrix of R

i) $R^r$ as matrix is given by $M_{R^r} = M_R + I_n$ (where $I_n$ is $n \times n$ unit matrix)

ii) $R^S$ in matrix form $M_{R^S} = M_R + (M_R)^T$

iii) $R^t$ in matrix form $M_{R^t} = M_R + M_{R^2} + M_{R^3} + \dots + M_{R^n}$.

where $M_{R^2} = M_R \odot M_R$

eg: Let $A = \{1,2,3\}$, Relation of set A is given by

$$R = \{(1,2)(2,3)(3,1)\}$$

$$M_R = \begin{array}{c c} & \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \end{array} & \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \end{array}$$

$$M_{R^r} = M_R + I_3.$$

$$= \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix}$$

$$M_{R^s} = M_R + (M_R)^T$$

$$= \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$$

$$M_{R^t} = M_R + M_{R^2} + M_{R^3}$$

$$M_{R^2} = M_R \odot M_R = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

$$M_{R^3} = M_{R^2} \odot M_R = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$M_{R^t} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} + \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$$

→ Warshell's Algorithm:

Q) Using Warshell's Algorithm find the transitive closure of relation R on a set $A = \{1, 2, 3, 4\}$ where the relation matrix is given by

$$M_R = \begin{pmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

A:

| C | R | C × R |
|---|---|---|
| $\{1, 4\}$ | $\{1, 3, 4\}$ | (1,1) (1,3) (1,4) (4,1) (4,3) (4,4) |
| $\{2\}$ | $\{2\}$ | (2,2) |
| $\{1, 4\}$ | $\{4\}$ | (1,4) (4,4) |
| $\{1, 3, 4\}$ | $\{1, 3, 4\}$ | (1,1) (1,3) (1,4) (3,1) (3,3) (3,4) (4,1) (4,3) (4,4) |

$W_0$

$$W_1 = \begin{pmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 \end{pmatrix}$$

$$W_2 = \begin{pmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 \end{pmatrix}$$

$$W_3 = \begin{pmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 \end{pmatrix}$$

$$W_4 = \begin{pmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 \end{pmatrix}$$

Q) Let $A = \{1, 2, 3, 4\}$, a relation is defined as $R = \{(1,2)(2,1)(2,3)(3,4)\}$. Using Warshell's Algorithm find the transitive closure of R.

Q) Using Warshell's algorithm, find the transitive closure of R where $R = \{(1,1), (1,2) (2,3) (3,1) (3,2)\}$

→ Equivalence classes:

If R is an equivalence relation on a set A, then set of all elements of A that R related to an element A of A is called equivalence class of A and is denoted by $[a] = \{x : x \in A, xRa\}$

Let R be an equivalence relation on set A, then the equivalence class of any element A in A is the set of all elements that are related to A through R, i.e the subset of A containing all the elements related to A through R.

eg: let $A = \{1, 2, 3, 4, 5\}$

$$R = \{(x, y) \mid (x - y) \text{ is even}, x, y \in A\}$$

$$R = \{(1,1)(2,2)(3,3)(4,4)(5,5)(1,3)(1,5)(2,4)(3,1)(5,1)(4,2)(5,3)(3,5)\}$$

$[1] = \{1, 3, 5\}$

$[2] = \{2, 4\}$

$[3] = \{1, 3, 5\}$

$[4] = \{2, 4\}$

$[5] = \{1, 3, 5\}$

Note: i) $a \in [a]$

ii) Either $[a] \wedge [b] = \phi$

or $[a] = [b]$

i.e any two equivalent classes are either disjoint or identical.

<u>Functions</u>

Types of functions

① One to one or injective function : let f maps $A \to B$ be a function f if every distinct elements of A is assigned to distinct element of B i.e $a \neq b$ implies $f(a) \neq f(b)$ ~~or~~ for $(a, b) \in A$ or equivalently if $f(a) = f(b)$ then $a = b$.

Eg: f maps $N \to N$, $f(x) = 5x$. Check whether it ~~fun~~ satisfies one to one Yes

Eg: $f(x) = x^2$, where $f: R \to R$

$f(1) = 1$, $f(-1) = 1$

But $a \neq b$

② On to function or Subjective: let F maps $A \to B$ is a function, F is called an onto function if for every b in B there is an a in A such that $f(a) = b$, i.e Range = Codomain.

(11) One to one correspondance or bijection: A function f maps A → B is a bijection if F is both one to one and onto.

① Let $f: R \to R$, $f(x) = 2x + 1$

② Let $f: Z \to Z$, $f(x) = x^2$, verify whether $x$ is bijection.

$$f(x_1) = f(x_2)$$

$$x_1^2 = x_2^2$$

$$\Rightarrow x_1^2 - x_2^2 = 0$$

$$\Rightarrow (x_1 + x_2)(x_1 - x_2) = 0$$

$$x_1 = x_2, \quad x_1 = -x_2$$

∴ It is not one to one.

Let $f(x) = y$

$$\therefore y = x^2$$

$$x \neq \pm\sqrt{y}$$

∴ It is not onto.

Composition of Function

Let f maps $A \to B$ and g maps $B \to C$, then the composition f and g denoted by $g \circ f: A \to C$ and defined by $g \circ f(x) = g[f(x)]$ $\forall x \in A$.

$g \circ f$ is also called relative product of function f and g or left composition of g with f.

Note: $(g \circ f)(x) = g[f(x)]$

From this the range of f is the domain of g.

Not commutative i.e $f \circ g \neq g \circ f$.

Note: $g \circ f$ is defined only when Range f is subset of domain g

Equality of function

The function f and g are set to be equal if dom f = dom g

codom f = codom g

① $f(x) = g(x)$, $\forall x \in$ dom f.

Identity function

A function f maps $A \to A$ defined by $f(x) = x$ for all $x \in A$ is called the identity function on A, is denoted by $I_A$

## Inverse function

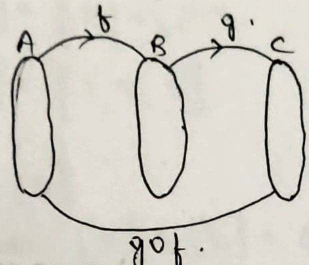
Let $f: A \to B$ and $g: B \to A$, then the function $g$ is called the inverse of function $f$ i.e $g = f^{-1}$ if $g \circ f = I_A$, and $f \circ g = I_B$

### Theorem-1

Composition of function is associative.

**Proof:** $f: A \to B$.
$g: B \to C$
$h: C \to D$
be the three function defined on $A, B, C$.

To prove $h \circ (g \circ f) = (h \circ g) \circ f$.



$g \circ f : A \to C$

Let $x \in A$, $y \in B$ and $z \in C$, so that $f(y) = x$ & $z = g(y)$.

$$g \circ f(x) = g[f(x)] = g(y) = z.$$

$$h \circ (g \circ f)x = h[g \circ f(x)] = h(z) \quad \text{— (i)}.$$

$$(h \circ g) \circ f(x) = h \circ g(f(x)) = h \circ g(y) = h[g(y)] = h(z) \quad \text{— (ii)}$$

### Theorem-2

Let $f: A \to B$, $g: B \to C$ be a function, then i) if $f$ and $g$ are one to one, then the composition is also one to one i.e. $g \circ f$ is 1-1.

ii) if $f$ and $g$ are onto then $g \circ f$ are onto.

iii) $f$ and $g$ are bijection, then $g \circ f$ is also bijection.

Since $f$ and $g$ are one to one, $g \circ f$ maps $A \to C$,

let $g \circ f(a_1) = g \circ f(a_2)$ $\forall a_1, a_2 \in A$

$\Rightarrow g[f(a_1)] = g[f(a_2)]$ $\Rightarrow g(a_1) = g(a_2)$ [$\because f$ is 1-1]

$\therefore a_1 = a_2$ [$\because g$ is 1-1] let $f$ and $g$ are onto

let $c \in C$, as $g$ is onto, there is $b$ in $B$ such that

$$g(b) = c$$

As $b \in B$ and $f$ is onto, there is $a \in A$ such that

$$f(a) = b.$$

$$g \circ f(a) = g[f(a)] = g(b) = c$$

$\therefore$ for element $c \in C$, $a \in A$

$\therefore$ $g \circ f$ is onto

As $g \circ f$ is both 1-1 and onto

$\therefore$ $g \circ f$ is bijection.

Theorem-3

let $f: A \to B$ is a function, if $f^{-1}$ exist, then it is unique.

Uniqueness of inverse function.

soln: If possible $g$ & $h$ be inverse of $f$.

$g: B \to A$ and $h: B \to A$.

then by the definition of inverse function

$\therefore g \circ f = I_A$, $f \circ g = I_B$.

$h \circ f = I_A$, $f \circ h = I_B$

$$\begin{aligned} h &= h \circ I_B \\ &= h \circ (f \circ g) \\ &= (h \circ f) \circ g. \\ &= I_A \circ g = g. \end{aligned}$$

Note: By uniqueness property, we have $(f^{-1})^{-1} = (g^{-1}) = f$.

Theorem-4

Existence of inverse.

let $f: A \to B$ be any function, then $f^{-1}$ exist, if and only if $f$ is one to one and onto.

In other words, necessary and sufficient condition for function $f: A \to B$ to be invertible is that $f$ is one to one & onto.

Proof: Assume $f$ is invertible.

To prove that $f$ is one to one and onto.

Let $f^{-1}$ exist, then exists a unique function $f^{-1} = g : B \to A$ such that $g \circ f = I_A$ and $f \circ g = I_B$

Let $a_1, a_2 \in A$, such that $f(a_1) = f(a_2)$.

where $f(a_1), f(a_2) \in B$.

Since $g : B \to A$.

$$g(f(a_1)) = g(f(a_2))$$

$$\Rightarrow (g \circ f)(a_1) = (g \circ f)(a_2)$$

$$\Rightarrow I_A(a_1) = I_A(a_2)$$

$$\Rightarrow a_1 = a_2.$$

Let $b \in B$, then $g(b) \in A$

$$b = I_B(b) = (f \circ g)(b).$$
$$= f[g(b)]$$

Thus for $b \in B$, there exist $g(b) \in A$ such that $f[g(b)] = b$.

$\therefore f$ is onto.

$\therefore f$ is 1-1 & onto, $f$ is bijection.

Conversely, let $f : A \to B$ be a bijection

$\therefore f$ is invertible (to prove)

$\because f$ is onto, for each $b \in B$ there exist an element $a \in A$, such that $f(a) = b$

We define $g : B \to A$.

$g(b) = a$ where $f(a) = b$

$\therefore g$ is well defined.

If possible $g(b) = a_1$ & $g(b) = a_2$

Paper Pattern

Part A = $5 \times 1 = 5$

Part B = $2 \times 4 = 8$.

Part C = 12 marks