logic

* Definition: Statement (proposition)
  The declarative statement which is either true or false is called proposition.

Ex (i) $2+2=5 \rightarrow$ Proposition
  (ii) Open the door - not a proposition

Connectives

→ Conjuction.

When p and q are any two statements The proposition p and q denoted by $p \wedge q$ (is called Conjunction of p and q

Truth Table:

(P.Q)

True = 1
False = 0

p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

→ Disjunction:

When p and q are two statements the proposition p or q denoted by $p \vee q$ is called disjunction of p and q

Truth Table :

(p+q)

p	q	p ∨ q
T	F	T
T	T	T
F	T	T
F	F	F

→ Negation :

The negation of a proposition p is a -ve statement of p

p	⌐p or ~p
T	F
F	T

Conditional Proposition.

"When p and q are two statements If p then q" denoted by $p \to q$ is called conditional proposition

Truth Table.

p	q	$p \to q$
T	T	T
T	F	F
F	T	T
F	F	T

Biconditional Operation

"p if and only if q" denoted by $p \leftrightarrow q$.

Truth Table

p	q	$p \leftrightarrow q$
T	T	T
T	F	F
F	T	F
F	F	T

Algebra of Propostions :

(i) Idempotent law.

(a) $p \wedge q \equiv p$

(b) $p \vee q \equiv p$

(ii) Identity law.

(a) $p \vee F = p$

(b) $p \wedge T = p$.

(iii) Dominant law.

$p \vee T = T$

$p \wedge F = F$

- Commutative law

(i) $p \vee q \equiv q \vee p$

(ii) $p \wedge q \equiv q \wedge p$

- Associative law

(i) $(p \vee q) \vee r \equiv \cancel{q}\, p \vee (q \vee r)$

(ii) $(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$

- Distributive law

(i) $p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$

(ii) $p \wedge (q \vee r) = (p \wedge q) \vee (p \wedge r)$

- Demorgan's law

(i) $\neg(p \vee q) = \neg p \wedge \neg q$

(ii) $\neg(p \wedge q) = \neg p \vee \neg q$

- Equivalences involving Conditionals.

(i) $p \rightarrow q \equiv \neg p \vee q$

(ii) $p \rightarrow q \equiv \neg q \rightarrow \neg p$

(iii) $(p \rightarrow q) \wedge (p \rightarrow r) \equiv p \rightarrow (q \wedge r)$

(iv) $(p \to q) \wedge (q \to r) \equiv (p \vee q) \to r$

- Implication formula

(i) $p \wedge q \Rightarrow p$ or q

(ii) $p \to p \vee q$

(iii) $p \to q, p \Rightarrow q$

Qu ① Construct true table for.

1) $(p \to q) \vee (q \to r)$

p	q	r	p → q (a)	q → r (b)	a v b
T	T	T	T	T	T
T	T	F	T	F	T
T	F	T	F	T	T
F	T	T	T	T	T
T	F	F	F	T	T
F	F	T	T	T	T
F	T	F	T	F	T
F	F	F	T	T	T

Totology

If it is trow for every troth value of $P_1, P_2 \ldots P_n$ then called totology

A compound proposition $(P_1, P_2, P_3 \ldots P_n)$ is called contradiction. If it is false for every truth values of P_1, P_2 etc.

$P \vee \neg P$ is a tautology

$P \wedge \neg P$ is contradiction.

Qu Determine which of the following compound propositions are tautologies and which of them are contraditions

(i) $(\neg q \wedge (p \to q)) \to \neg p$

p	q	$\neg q$ (a)	$p \to q$ (b)	$a \wedge b$ (c)	$\neg p$ (d)	$c \to d$
T	F	T	F	F	F	T
T	T	F	T	F	F	T
F	F	T	T	T	T	T
F	T	F	T	F	T	T

T

Tautology

(ii) $[(p \to q) \wedge (q \to r)] \to (p \to r)$.

p	q	r	(a) $p \to q$	(b) $q \to r$	(c) $a \wedge b$	(d) $p \to r$	$c \to d$
T	T	T	T	T	T	T	T
F	F	F	T	T	T	T	T
T	F	T	F	T	F	T	T
F	T	T	T	T	T	T	T
T	T	F	T	F	F	F	T
T	F	F	F	T	F	F	T
F	T	F	T	F	F	T	T
F	F	T	T	T	T	T	T

(iii) $[(p \vee q) \wedge (p \to r) \wedge (p \to r)] \to r$.

p	q	r	$p \vee q$ (a)	$p \to r$ (b)	$a \wedge b$	$(a \wedge b) \wedge b$ (c)	$c \to r$
T	T	T	T	T	T	T	T
T	F	F	T	F	F	F	T
F	T	F	T	T	T	T	T
F	F	T	F	T	F	F	F
F	T	T	T	T	T	T	T
T	F	T	T	T	T	T	T
T	T	F	T	F	F	F	T
F	F	F	F	T	F	F	T

Qu Show that

$$(\neg q \wedge (p \to q)) \to \neg p$$

is a tautology without using truthtable

$$[\neg q \wedge (p \to q)] \to \neg p$$

$$\Rightarrow \neg[\neg q \wedge (\neg p \vee q)] \vee \neg p$$

$$\Rightarrow [q \vee (p \wedge \neg q)] \vee \neg p$$

$$\Rightarrow [(q \vee p) \wedge (q \vee \neg q)] \vee \neg p$$

$$\Rightarrow [(q \vee p) \wedge T] \vee \neg p$$

$$\Rightarrow (q \vee p) \vee \neg p$$

$$\Rightarrow q \vee (p \vee \neg p)$$

$$\Rightarrow q \vee T$$

$$\Rightarrow T$$

Qu Prove that

$$(q \to p) \to (\neg p \to \neg q)$$

is a tautology

$$(q \to p) \to (\neg p \to \neg q)$$

$\Rightarrow (\neg p \to \neg q) \to (\neg p \to \neg q)$

$\Rightarrow \neg(\neg p \to \neg q) \vee (\neg p \to \neg q)$

$\Rightarrow T$

Qu Show that:

$(p \wedge q) \to (p \to q)$ is a tautology

$(p \wedge q) \to (p \to q)$

$= \neg(p \wedge q) \vee (p \to q)$

$= \neg(p \wedge q) \vee (\neg p \vee q)$

$= (\neg p \vee \neg q) \vee (\neg p \vee q)$

$= \neg p \vee (\neg q \vee q)$

$= \neg p \vee T = T$

Qu Show that

$\neg(q \to r) \wedge r \wedge (p \to q)$ is

a contradiction.

$\neg(q \to r) \wedge r \wedge (p \to q)$

$= \neg(\neg q \vee r) \wedge r \wedge (\neg p \vee q)$

$\Rightarrow (q \wedge \neg r) \wedge r \wedge (\neg p \vee q)$

$= (q \wedge \neg r) \wedge r \wedge (\neg p \vee q)$

$= (q \wedge (\neg r \wedge r)) \wedge (\neg p \vee q)$

$= (q \wedge F) \wedge (\neg p \vee q)$

$= F \wedge (\neg p \vee q)$

$= F$

Dual of a proposition

The dual of a compound proposition $p(P_1, P_2 \cdots P_n)$ is a proposition obtained by replacing each $\vee$ by $\wedge$, each $\wedge$ by $\vee$, each T by F and each F by T

Ex:

The dual of $(P \vee ((q \wedge r) \rightarrow T)$ is $p \wedge ((q \vee r) \rightarrow F)$

Theory of Inference

A valid conclusion C logically follows from a set of given proposition $P_1, P_2, \ldots P_n$, if $P_1 \wedge P_2 \wedge \ldots \wedge P_n \Rightarrow C$

Direct method.

Problem :-

Qu Show that $\neg p$ follows logically from a given set of propositions

$p \rightarrow q$, $r \rightarrow \neg q$ and r.

1 $p \rightarrow q$ Rule P

2 $r \rightarrow \neg q$ Rule P

3. r Rule P.

4. $\neg q$ Rule T (2, 3)

5. $\neg q \rightarrow \neg p$ (Rule T, 1)

6 $\neg p$ (Rule T, 5, 4)

Note: $p \rightarrow q$, $q \rightarrow r \Rightarrow p \rightarrow r$.

Qu Show tha $a \vee b$ follows logically from a given set of propositions

$p \vee q$; $(p \vee q) \rightarrow \neg r$, $\neg r \rightarrow (s \wedge \neg t)$

$(s \wedge \neg t) \rightarrow (a \vee b)$.

1. $p \vee q$ (Rule P)

2. $p \vee q \rightarrow \neg r$ (Rule P)

3. $\neg r \to (s \land \neg t)$ Rule P

4. $(s \land \neg t) \to (a \lor b)$ Rule P

5. $\neg r$ (Rule T, 1, 2)

6. $(s \land \neg t)$ (Rule T, 5, 3)

7. $(a \lor b)$ (Rule T, 6, 4)

Qu Show that

$$(a \to b) \land (a \to c),\ \neg(b \land c),\ d \lor a \Rightarrow d$$

(1) $(a \to b) \land (a \to c)$ Rule P

(2) $\neg(b \land c)$ Rule P

(3) $a \to b$ Rule T (1)

(4) $d \lor a$ Rule P.

5 $a \to c$ (Rule T, 1)

6 $\neg b \lor \neg c$ (Rule T, 2)

7 $b \to \neg c$ (Rule T, 6)

8 $\neg b \to \neg a$ (Rule T, 4)

9 $\neg c \to \neg a$ (Rule T, 5)

10 $(\neg b \lor \neg c) \to \neg a$ (Rule T, 8, 9)

11 $\neg(b \lor c) \to \neg a$ (Rule T, 10)

12 $\neg a$ (Rule T, 10, 6)

13 $(d \lor a) \land \neg a$ (Rule T, 12, 13)

14 $(d \land \neg a) \lor (a \land \neg a)$ (Rule T, 13)

15 $(d \land \neg a) \lor F$ (Rule T, 14)

16 $d \land \neg a$ (Rule T, 15)

17 $\quad q \quad$ (Rule T, 16)

Qu Show that

$$p \vee q, \; p \to r, \; q \to s \Rightarrow s \vee r$$

(1) $p \vee q$ $\quad$ Rule P $\quad$ ~~$\neg p \to q$~~

(2) $p \to r$ $\quad$ Rule P)

(3) $q \to s$ $\quad$ Rule P

(4) $\neg p \to q$ $\quad$ Rule T(1)

(5) $\neg p \to s$ $\quad$ Rule T(4,3)

(6) $\neg r \to \neg p$ $\quad$ Rule T(2)

(7) $\neg r \to s$ $\quad$ Rule T(5,6)

(8) $r \vee s$

(9) $s \vee r$.

Hence proved.

Qu Prove that

$$p \vee q, \; q \to r, \; p \to s; \; \neg s \Rightarrow r \wedge (p \vee q)$$

(1) $p \vee q$ $\quad$ Rule P

2 $\quad q \to r$ $\quad$ Rule P

3 $\quad p \to s$ $\quad$ Rule P

4 $\quad \neg s$ $\quad$ Rule P

5 $\quad \neg s \to \neg p$ $\quad$ Rule T(3)

(6) $\neg p$ $\quad$ Rule T(4,5)

(7) $\neg(\neg p) \vee q$ $\quad$ Rule T(1)

(8) $\neg p \to q$ $\quad$ Rule T(1)

(9) $\neg p \to r$ $\quad$ Rule T(6,1)

(10) r $\quad$ Rule T (6,9)

(11) $r \wedge (p \vee q)$ $\quad$ Rule T(10,1)

Qn Prove that $(p \to q) \to r, p \wedge s, q \wedge t \Rightarrow r$.

(1)	$(p \to q) \to r$	Rule P
(2)	$p \wedge s$	Rule P
3	$q \wedge t$	Rule P
4	$\neg(p \to q) \vee r$	Rule T(1)
5	p	Rule T(2)
6	q	Rule T(3)
7	$\neg(\neg p \vee q) \vee r$	Rule T(4)
8	$(p \vee \neg q) \vee r$	Rule T(7)
9	$(p \vee r) \wedge (\neg q \vee r)$	Rule T(8)
10	$\neg q \vee r$	Rule T(9)
11	$q \to r$	Rule T(10)
12	r	Rule T(11,6)

Qn Prove that

$$p \to q \quad (\neg q \vee r) \wedge \neg r, \neg(\neg p \wedge s) \Rightarrow \neg s.$$

(1)	$p \to q$	Rule P
(2)	$(\neg q \vee r) \wedge \neg r$	Rule P
3	$\neg(\neg p \wedge s)$	Rule P
4	$\neg q \vee r$	Rule T(2)
5	$\neg r$	Rule T(2)
6	$p \vee \neg s$	Rule T(3)
7	$\neg p \to \neg s$	Rule T(6)

8 $q \to r$ Rule T(4)

9 $\neg r \to \neg q$ Rule T(8)

10 $\neg q$ Rule T(9, 5)

11 $\neg q \to \neg p$ Rule T(1)

12 $\neg p$ Rule T(10, 11)

13 $\neg s$ Rule T(12, 7)

INDIRECT METHOD :-

Prove that $p \to q,\ q \to r,\ \neg(p \wedge r),\ p \vee r \Rightarrow r$

1). $p \to q$ Rule P

2) $q \to r$ Rule P

3 $\neg(p \wedge r)$ Rule P.

4 $p \vee r$ Rule P

5 $\neg r$ Rule P (additional)

6. $p \to r$ Rule T (1, 2)

7 $\neg p \vee \neg r$ Rule T (3)

8 $\neg r \to p$ Rule T (4)

9 p Rule T (8, 5)

10 r Rule T (6, 9)

11 $r \wedge \neg r$ Rule T (10, 5)

12 F Rule (T) 11

Qu Prove that

$\neg q, p \to q, p \vee r \Rightarrow r$ using

indirect method.

(1) $\neg q$ Rule P

(2) $p \to q$ Rule P

3 $p \vee r$ Rule P

4 $\neg r$ Rule P (additional)

5 $\neg q \to \neg p$ Rule T(2)

6 $\neg p$ Rule T(5,1)

7 $\neg p \wedge \neg r$ Rule T(4,6)

8 $\neg(p \vee r)$ Rule T(7)

9 $(p \vee r) \wedge \neg(p \vee r)$ Rule T(8,3)

10. F Rule T(9)

C-P rule

If the conclusion is of the form $r \to s$, we will take r as an additional proposition and derive s from give propositions and r

Proove that

$\neg p \vee q, \neg q \vee r, r \to s \Rightarrow p \to s$ using CP rule

1	$\neg p \vee q$	Rule P
2	$\neg q \vee r$	Rule P
3	$r \to s$	Rule P
4	p	Rule p (additional)
5	$p \to q$	Rule T(1)
6	$q \to r$	Rule T(2)
7	$p \to r$	Rule T(5,6)
8	$p \to s$	Rule T(4,7)
9	s	Rule (8,4)
10	$p \to s$	Rule P

using CP rule P.T $p, p \to (q \to (r \wedge s)) \Rightarrow q \to s$

1	p	Rule P
2	$p \to (q \to (r \wedge s))$	Rule P
3	q	Rule p (additional)
4	$q \to (r \wedge s)$	Rule T (1,2)
5	$r \wedge s$	Rule T (3,4)
6	s	Rule T (5)
7	$q \to s$	Rule CP

Consistent and Inconsistent Propositions

A set of proposition $P_1, P_2, \ldots P_n$ is said to be consistent if $P_1 \wedge P_2 \wedge \ldots \wedge P_n \Rightarrow T$ and is said to be in consistent if

$$P_1 \wedge P_2 \wedge \ldots \wedge P_n \Rightarrow F$$

Qu Show that the following set of Propositions is consistent.

(i) If Rama gets his degree, he will go for a Job

(ii) If he goes for a Job, he will get married

(iii) If he goes for higher studies, he will not get married

(iv) Rama get his degree and goes for higher study.

Soln: p: Rama gets his degree

q: he will go for Job

r: he will go for higher studies.

s: he will get married.

(i) $p \rightarrow q$ Rule P

(ii) $q \rightarrow s$ Rule P

(iii) $r \to \neg s$ Rule P

(iv) $p \wedge r$ Rule P.

(v) $p \to s$ Rule T(1,2)

(vi) $s \to \neg r$ Rule T(3)

(vii) r Rule T(4)

(viii) $\neg s$ Rule T(7,3)

(ix) p Rule T(4)

(x) s Rule T(9,5)

(xi) $\neg s \wedge s$ Rule T(10,8)

(xii) F Rule T(11).

Qu Prove that each of the following set of propositions is inconsistent.

(i) $p \to q$, $p \to r$, $q \to \neg r$, p

(ii) $p \to q$, $(q \vee r) \to s$, $s \to \neg p$, $p \wedge \neg r$

(i) 1) $p \to q$ Rule P

2) $p \to r$ Rule P

3) $q \to \neg r$ Rule P

4) p Rule P

5) $p \to \neg r$ Rule T(1,3)

6) $\neg r$ Rule T(4,5)

7) r Rule T(2,4)

8) $\neg r \wedge r$ Rule T(6,7)

9) F Rule T(8)

(ii) 1) $p \to q$ Rule P

2) $q \vee r \to s$ Rule P

3) $s \to \neg p$ Rule P

4) $p \wedge \neg r$ Rule P

5) $(q \vee r) \to \neg p$ Rule T(2, 3)

6) $\neg(q \vee r) \vee \neg p$ (Rule T, 5)

7) $(\neg q \wedge \neg r) \vee (\neg p)$ (Rule T, 6)

8) $(\neg q \vee \neg p) \wedge (\neg r \vee \neg p)$ Rule T(7)

9) p Rule T(4)

10) q Rule T(9, 1)

11) $p \to \neg s$ Rule T(3)

12) $\neg s$ Rule T(11, 9)

13) $\neg s \to \neg(q \vee r)$ Rule T

14) $\neg(q \vee r)$ Rule T(12, 13)

15) $\neg q \wedge \neg r$ Rule T(14)

16) $\neg q$ Rule T(15)

17) $\neg q \wedge q$ Rule T(16, 10)

18) F Rule T(17)

Predicate and Quantifier

Now consider the following statement

x is greater than 1

Here " is greater than 1" is called Predicate;

x: a number.

G(x) : x is greater than 1

Ex:- x^2 is positive for any x.

Here " for all x " is called quantifier (universal quantifier)

It is denoted by $\forall x$.

$x^2 - 2$ is positive for some x.

Here for some x is called existential quantifier.

It is denoted by $\exists x$

Ex - (1) All mean are good.

m(x) : x is a man

G(x) : x is good.

$\forall x \ (m(x) \rightarrow G(x))$.
Ever

(2) Everybody loves z

$L(x, z)$: x loves z

$\forall x \ L(x, z)$.

(3) Everybody loves Somebody

$L(x, y)$: x loves y
$\forall x \ \exists y \ L(x, y)$.

Note :- (1) $\neg \forall = \exists$
(2) $\neg \exists = \forall$

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Rule US

It is a rule which states that one can conclude that P(c) is true whenever $\forall x \ p(x)$

Rule ES

It is a rule which states that on can conclude that P(c) is true whenever $\exists x \ p(x)$.

Rule UG

The generalisation of any proposition using ~~metho~~ universal quantifier is called UG Rule.

Rule EG

The generalisation of any proposition using existential quantifier is EG Rule.

Qu. Prove that

$$\exists x\,(C(x) \wedge J(x)),\ \forall x\,(J(x) \rightarrow H(x)).$$
$$\Rightarrow \exists x\,(C(x) \wedge H(x)).$$

(1) $\exists x\,(C(x) \wedge J(x))$.

(2) $\forall x\,(J(x) \rightarrow H(x))$.

(3) $C(a) \wedge J(a)$ Rule ES(1)

(4) $J(a) \rightarrow H(a)$ Rule T(2).

(5) $J(a)$ Rule T(3)

(6) $C(a)$ Rule T(3)

(7) $H(a)$ Rule T(4,5)

(8) $C(a) \wedge H(a)$ Rule T(7,6)

(9) $\exists x\,(C(x) \wedge H(x))$ Rule EG(8).

Qu Prove that

$$\forall x\,(p(x) \to q(x))$$

$$\forall x\,(r(x) \to \neg q(x))$$

$$\Rightarrow \forall x\,[r(x) \to \neg p(x)]$$

1 $\forall x\,[p(x) \to q(x)]$ Rule P.

2 $\forall x\,[r(x) \to \neg q(x)]$ Rule P.

3 $p(a) \to q(a)$ Rule US(1).

4 $r(a) \to \neg q(a)$ Rule US(2).

5 $\neg q(a) \to \neg p(a)$ Rule T(3).

6 $r(a) \to \neg p(a)$ Rule T(4,5).

7 $\forall x\,[r(x) \to \neg p(x)]$ Rule UG (6).