UNIT - I MATRICES

- **Characteristic Polynomial:**

The determinant $|A-\lambda I|$ when expanded will give a polynomial, which we call as characteristic polynomial of matrix A.

- **Characteristic Equation:**

Let A be any square matrix of order n. The characteristic equ^s of A is $|A-\lambda I| = 0$.

- **Eigen Values:**

Let A be a square matrix, the characteristic equation of A is $|A-\lambda I| = 0$. The roots of the characteristic equation are called Eigen values of A.

- **Eigen Vector:**

Let A be a square matrix. If there exists a non-zero vector X such that $AX = \lambda X$, then the vector X is called an Eigen vector of A corresponding to the Eigen value λ.

- **Note.**

1) The characteristic equation of 2×2 matrix is

$$\lambda^2 - S_1\lambda + S_2 = 0$$

S_1 = Sum of main diagonal elements.

$S_2 = |A|$

2) The characteristic equation of 3×3 matrix is

$$\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0$$

S_1 = Sum of main diagonal elements

S_2 = Sum of the minors of main diagonal elements

$S_3 = |A|$

v) if λ is an eigen val. of ortho, then $1/\lambda$ is also an ortho eigen val.

$|A||A^T| = 1$

Properties.

i) if A is orthogonal $|A| = \pm 1$
ii) if A is ortho, then A^T is also ortho.
iii) if A is ortho, A^{-1} is also ortho.
iv) if A & B are ortho, AB is also ortho.

- **Orthogonal matrix:**

A square matrix A is said to be orthogonal if $AA^T = A^TA = I$. Since $A^{-1}A = AA^{-1} = I$, it follows that a matrix A is orthogonal if $A^T = A^{-1}$.

- **Diagonalization of the matrix:**

Step 1: To find the characteristic equation

Step 2: To find eigen values

Step 3: To find eigen vectors

Step 4: Check whether the eigen vectors are orthogonal ; $X_1^TX_2 = 0$, $X_2^TX_3 = 0$ etc.

Step 5: To form normalized matrix N ; $N = [\bar{X}_1 \;\; \bar{X}_2 \;\; \bar{X}_3]$

$$\bar{X}_1 = \begin{bmatrix} x_1/\sqrt{x_1^2+x_2^2+x_3^2} \\ x_2/\sqrt{x_1^2+x_2^2+x_3^2} \\ x_3/\sqrt{x_1^2+x_2^2+x_3^2} \end{bmatrix}$$

Step 6: Calculate N^T

Step 7: Calculate $D = N^TAN$ (Diagonal elements and eigen values of A are same).

- **Quadratic Form**

A homogenous polynomial of second degree in any number of variables is called quadratic form.

Eg: $Q = ax_1^2 + bx_2^2 + cx_3^2 + lx_1x_2 + mx_2x_3 + nx_1x_3$

Then matrix of Quadratic form is.

$$A = \begin{array}{c c} & \begin{array}{ccc} x_1 & x_2 & x_3 \end{array} \\ \begin{array}{c} x_1 \\ x_2 \\ x_3 \end{array} & \begin{bmatrix} a & l/2 & n/2 \\ l/2 & b & m/2 \\ n/2 & m/2 & c \end{bmatrix} \end{array}$$

$$Q = X^TAX = [x_1 \;\; x_2 \;\; x_3] \begin{bmatrix} a & l/2 & n/2 \\ l/2 & b & m/2 \\ n/2 & m/2 & c \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

• Nature of the Quadratic form:

Let $D_1 = a_{11}$

$$D_2 = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$$

$$D_3 = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

(p) Index → No. of +ve terms

(s) Signature → No. of +ve terms − No. of −ve terms / $s = 2p - r$

(r) Rank → No. of non-zero diagonal elements.

Nature:

1. Positive Definite — $D_n > 0$; All the eigen values are +ve.
2. Negative Definite — $D_n < 0$; All the eigen values are −ve.
3. Positive Semi-definite — $D_n > 0$; Atleast one value is zero
4. Negative Semi-definite — $D_n < 0$; Atleast one value is Zero.
5. Indefinite — All other Cases ; None of the above.

• Reduction of a Quadratic form to Canonical form:

Working Rule:

1. Construct the quadratic form to matrix form 'A'.
2. Diagonalize the matrix A.
3. Canonical form = $Y^T D Y$

$X = N\cancel{X}$

and $Q = X^T A X$

$Q = \cancel{X}^T(N^T A N)X$

$= Y^T D Y$

• Properties of Eigen Values:

1. Sum of the eigen values = Sum of the diagonal elements = Trace

$$\lambda_1 + \lambda_2 + \lambda_3 + \lambda_n \cdots = a_{11} + a_{22} + a_{33} + a_{nn} \cdots$$

2. Product of the eigen values = $|A| = \lambda_1 \cdot \lambda_2 \cdot \lambda_3 \cdot \lambda_n$

3. The eigen values of diagonal matrix (or) Upper triangular matrix (or) lower triangular matrix are the diagonal elements

$$\begin{bmatrix} a_{11} & 0 & 0 \\ 0 & a_{22} & 0 \\ 0 & 0 & a_{33} \end{bmatrix} \quad \begin{bmatrix} a_{11} & 0 & 0 \\ a_{21} & a_{22} & 0 \\ a_{31} & a_{32} & a_{23} \end{bmatrix} \quad \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{bmatrix}$$

Diagonal matrix — Upper triangular matrix — Lower Triangular matrix

$$\lambda_1 = a_{11}, \quad \lambda_2 = a_{22}, \quad \lambda_3 = a_{33}$$

4. A and A^T have the same eigen values. $|A - \lambda I| = |A^T - \lambda I|$

5. If λ is an eigen value of A, then $k\lambda$ is an eigen value of kA. $(kA)X = (k\lambda)X$

6. If λ is an eigen value of A, then λ^k is an eigen value of A^k $\quad A(AX) = A(\lambda X) \Rightarrow A^2X = \lambda(AX) \Rightarrow A^2X = \lambda^2 X$

7. If λ is an eigen value of A, the $1/\lambda$ is an eigen value of A^{-1}, provided A is an non-singular matrix.

8. $\frac{|A|}{\lambda} \rightarrow$ is an eigen value of adj A.

$$A^{-1} = \frac{1}{|A|} \operatorname{adj} A$$

$$A^{-1}X = \frac{1}{\lambda} X$$

$$\frac{1}{|A|} \operatorname{adj} A\, X = \frac{1}{\lambda} X$$

$$\operatorname{adj} A\, X = \frac{|A|}{\lambda} X$$

Cayley-Hamilton Theorem: (CHT)

Every Square matrix Satisfies its own characteristic equation.

W.k.t Chst eqn of a matrix is

$$P(\lambda) = |A - \lambda I|$$

By CHT,

$$P(A) = 0.$$

MATRICES

Quadratic Form (QF) & Canonical Form (C.F)

QF = Homogeneous Exp of degree two

$QF = ax_1^2 + bx_2^2 + cx_3^2 + l x_1x_2 + m x_2x_3 + n x_1x_3$

Its matrix form

$Q.F = X^T A X$

$$X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \quad A = \begin{array}{c c} & \begin{array}{ccc} x_1 & x_2 & x_3 \end{array} \\ \begin{array}{c} x_1 \\ x_2 \\ x_3 \end{array} & \begin{bmatrix} a & 1/2\, l & n/2 \\ 1/2\, l & b & m/2 \\ n/2 & m/2 & c \end{bmatrix} \end{array}$$

$C.F = Y^T D Y$;

$X = NY \Rightarrow Q.F.\ X^T A X$

$\therefore C.F = Y^T(N^T A N)Y$

$= Y^T D Y$

Eigenvalue and Eigenvector Problems

$\rightarrow AX = \lambda X$

$|A - \lambda I| = 0$ Chs eqn

i) $A^n X = \lambda^n X$

ii) $(k A)X = (k\lambda)X$

iii) $\lambda_1 + \lambda_2 + \lambda_3 =$ Trace of A

iv) $\lambda_1 \cdot \lambda_2 \cdot \lambda_3 = |A|$

v) λ of Diago, upper Δ and lower Δ matrix = Diagonal elements

$\text{adj}A \cdot X = \frac{|A|}{\lambda} X$

$A^{-1} X = \frac{1}{\lambda} X$

Orthogonal Matrix

$\rightarrow AA^T = I = A^T A$, ie $A^T = A^{-1}$

i) $|A||A^T| = 1$ it gives $|A| = \pm 1$

ii) A is ortho, then A^T is also ortho

iii) A is ortho, then A^{-1} is also ortho

iv) A, B is ortho, AB is also ortho

v) If λ is eigenvalue of ortho, then $1/\lambda$ is also an eigen value.

Diagonalization of a matrix by orthogonal Transformation

$D = N^T A N$; N is Normalised matrix

$N^T = [\bar{X}_1 \ \bar{X}_2 \ \bar{X}_3]$

$$\bar{X}_1 = \begin{bmatrix} \dfrac{x_1}{\sqrt{x_1^2 + x_2^2 + x_3^2}} \\ \dfrac{x_2}{\sqrt{x_1^2 + x_2^2 + x_3^2}} \\ \dfrac{x_3}{\sqrt{x_1^2 + x_2^2 + x_3^2}} \end{bmatrix}; \quad \begin{array}{l} X_1^T X_2 = 0 \\ X_2^T X_3 = 0 \\ X_1^T X_3 = 0 \end{array}$$

The diagonal elements of D are the eigenvalues of A.

h Index Sum of terms

s Signature = Diff of no of +ve - no of -ve no

r Rank = No of non-zero Diagonal elements of D

Nature

- +ve Definite = $\lambda_1 \lambda_2 \lambda_3 > 0$
- −ve " = $\lambda_1 \lambda_2 \lambda_3 < 0$
- +ve Semi " = $\lambda_1 \lambda_2\ 0 > 0$
- −ve Semi " = $\lambda_1 \lambda_2\ 0 < 0$
- Indefinite = None of the above

Cayley-Hamilton Theorem

$\rightarrow P(\lambda) = |A - \lambda I|$ Charac eqt

By CHT

Every Square matx, Satisfies its own Charac eqn.

$P(A) = 0$

Eg: $\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0$

By CHT

$A^3 - S_1 A^2 + S_2 A - S_3 = 0$

UNIT-II : FUNCTION OF SEVERAL VARIABLES.

- **Function of two independent Variables:**

A Symbol z, which has a definite Value for every pair of Values of x any y is called a function of two independent Variables x any y and we write

$$z = f(x,y) \text{ or } \phi(x,y)$$

- **Limits:**

The function $f(x,y)$ is said to tend to the limit L as $x \to a$ and $y \to b$, if and only if the limit L is independent of the path followed by the point (x,y) as $x \to a$ and $y \to b$ then $\lim_{\substack{x\to a\\ y\to b}} f(x,y) = L$

$$\lim_{(x,y)\to(x_0,y_0)} [f(x,y) \pm g(x,y)] = L \pm M$$

$$\times \qquad = L \times M$$

$$\lim f(x,y)^{m/n} = L^{m/n} \qquad k f(x,y) = kL \qquad f/g = L/M$$

Path independent

- **Continuity:**

A function $f(x,y)$ is said to be Continuous at the point (a,b) if $\lim_{\substack{x\to a\\ y\to b}} f(x,y)$ exists and $= f(a,b)$.

- **Partial derivatives:**

Let $z = f(x,y)$ be a function of two variables x and y.

$$\frac{\partial z}{\partial x} = p \; ; \; \frac{\partial z}{\partial y} = q \; ; \; \frac{\partial^2 z}{\partial x^2} = r \; ; \; \frac{\partial^2 z}{\partial x \partial y} = s \; ; \; \frac{\partial^2 z}{\partial y^2} = t.$$

→ To solve, show that the limit approaches from different paths, if value of L is same, it is path independent and limit exist.

If L is path dependent, limit does not exist.

$$\lim_{\substack{x=a\\ y\to y_0}} \text{ (or) } \lim_{\substack{x\to x_0\\ y=b}} \text{ (or) } \lim_{\substack{x=my \text{ or } y^m\\ y\to y_0}} \text{ or } \lim_{\substack{x\to x_0\\ y=mx \text{ or } x^m}} f(x,y) = L$$

Different paths

- ## Total Differentiation:

For a multivariable function f, such that

$$\begin{tikzcd}[column sep=small] & & f \arrow[dl] \arrow[dr] & & \\ & x \arrow[dl] \arrow[d] & & y \arrow[dl] \arrow[d] & \\ t & r & t & r & \end{tikzcd}$$

f is a fn of (x, y)
x and y are a fun of (r, t)

then,

$$\frac{\partial f}{dt} = \frac{\partial f}{dx} \cdot \frac{dx}{dt} + \frac{\partial f}{dy} \cdot \frac{dy}{dt} \quad \text{and}$$

$$\frac{\partial f}{dr} = \frac{\partial f}{dx} \frac{dx}{dr} + \frac{\partial f}{dy} \cdot \frac{dy}{dr}.$$

- ## Eulers Theorem for homogenous function:

If u be a homogenous function of degree 'n' in x and y, then.

$$x \cdot \frac{du}{dx} + y \frac{du}{dy} = n \cdot u.$$

where u is a fn of (x, y).
$u = f(x, y)$

- ## Total derivatives: (Implicit Differentiation)

$$\frac{dy}{dx} = \frac{-\frac{\partial f}{dx}}{\frac{\partial f}{dy}} \quad \text{or} \quad \frac{-f_x}{f_y}.$$

- Jacobian:

→ Definition:

If U_1, U_2, U_3 are functions of Variable x_1, x_2, x_3 then the Jacobian of the transformation from x_1, x_2 and x_3 to U_1, U_2, U_3 is defined by.

$$\frac{\partial(U_1, U_2, U_3)}{\partial(x_1, x_2, x_3)} = \begin{vmatrix} \frac{\partial U_1}{\partial x_1} & \frac{\partial U_1}{\partial x_2} & \frac{\partial U_1}{\partial x_3} \\ \frac{\partial U_2}{\partial x_1} & \frac{\partial U_2}{\partial x_2} & \frac{\partial U_2}{\partial x_3} \\ \frac{\partial U_3}{\partial x_1} & \frac{\partial U_3}{\partial x_2} & \frac{\partial U_3}{\partial x_3} \end{vmatrix}$$

- Properties of Jacobian:

1. If U and V are the function of x and y, then

$$\frac{\partial(U,V)}{\partial(x,y)} \times \frac{\partial(x,y)}{\partial(U,V)} = 1$$

2. If U and V are the functions of x and y and x and y are functions of t and s, then

$$\frac{\partial(U,V)}{\partial(x,y)} \times \frac{\partial(x,y)}{\partial(t,s)} = \frac{\partial(U,V)}{\partial(t,s)}$$

3. If U, V, W are functionally dependent function of three independent Variables x, y, z then

$$\frac{\partial(U,V,W)}{\partial(x,y,z)} = 0.$$

$f_{y/x}(a,b) = f_{y/x}|_{a,b}$

• Taylor's Series:

$h = x - a, \quad k = y - b$

$$f(x,y) = f(a,b) + \frac{1}{1!}\left[h f_x(a,b) + k f_y(a,b)\right] + \frac{1}{2!}\left[h^2 f_{xx}|_{(a,b)} + 2hk\, f_{xy}|_{a,b} + k^2 f_{yy}|_{a,b}\right] + \frac{1}{3!}\left[h^3 f_{xxx}|_{a,b} + 3h^2k\, f_{xxy}|_{a,b} + 3k^2h\, f_{yyx}|_{a,b} + k^3 f_{yyy}|_{a,b}\right] + \cdots$$

• Maxima and Minima for Functions of two Variables:

If $f_x = 0$ or DNE

$f_y = 0$ or DNE

and $f_{xx} = A$

$f_{xy} = B$

$f_{yy} = C$ then,

(i) ~~If~~ $f(a,b)$ is Maximum value if $AC - B^2 > 0$ and $A < 0$ (or $B < 0$)

(ii) $f(a,b)$ is minimum value if $AC - B^2 > 0$ and $A > 0$ (or $B > 0$)

(a,b) is known as Stationary point

(iii) $f(a,b)$ is not an extremum (Saddle) if $AC - B^2 < 0$

(a,b) is known as a saddle point

(iv) $AC - B^2 = 0$, then test is inclusive or further investigation is required.

• Steps to Find extermum Values:

1. Find f_x and f_y
2. $f_x = 0$ and $f_y = 0$
3. Find Stationary points that satisfy the equation.
4. Find $f_{xx} = r$, $f_{yy} = t$, $f_{xy} = s$ and find $rt - s^2$
5. Make plot →

Stationary pt.	$rt - s^2$	r	Conclusion
(x_1, y_1)			
(x_2, y_2)			

- Method of Lagrange Multipliers

To find the Maximum and Minimum value of the function $f(x, y, z)$ subject to Constraint $g(x, y, z) = 0$

Consider $F = f + \lambda g$

Form the eqs $\frac{\partial F}{\partial x} = 0$, $\frac{\partial F}{\partial y} = 0$, $\frac{\partial F}{\partial z} = 0$ and $\frac{\partial F}{\partial \lambda} = 0$.

On solving these eqs we get the Maximum and Minimum points.

FUNCTIONS OF SEVERAL VARIABLES

Total differentiation:

Suppose f is an fn of x and y and x and y are fn of r and t.

$$\begin{array}{c} f \\ x \quad y \\ r \; t \quad r \; t \end{array}$$

then,

$$\frac{df}{dt} = \frac{df}{dx}\cdot\frac{dx}{dt} + \frac{df}{dy}\cdot\frac{dy}{dt}$$

$$\frac{df}{dr} = \frac{df}{dx}\cdot\frac{dx}{dr} + \frac{df}{dy}\cdot\frac{dy}{dr}$$

Partial Derivatives

$z = f(x, y)$

$\frac{\partial z}{\partial x} = p$ $\quad \frac{\partial z}{\partial y} = q$

$\frac{\partial^2 z}{\partial x^2} = r$ $\quad \frac{\partial^2 z}{\partial x \partial y} = s$

$\frac{\partial^2 z}{\partial y^2} = t$ It basically means that, one variable is kept constant and the other one is differentiated

Function of 2 independent Variables

$z = f(x, y)$

z → Dependent; x, y → Independent

Limits:

$\lim_{\substack{x \to a \\ y \to b}} f(x, y) = L$ is true when L is path independent

$\lim_{\substack{x \to a \\ y \to b}} [f(x, y) \pm g(x, y)] = L \pm M$

" $f/g = \frac{L}{M}$

" $f \cdot g = L \cdot M$

" $f^{n/m} = L^{n/m}$

" $kf = kL$

Eulers Theorem for Homogenous functions:

If u is a homogenous fn of degree n in x and y then $u = f(x, y)$

$$x\frac{du}{dx} + y\frac{du}{dy} = n \cdot u$$

(n ↳ degree)

Jacobian:

To convert from fn of one system to other.

$dx\,dy\,dz = |J|\,du\,dv\,dz$

$$\frac{\partial(x, y, z)}{\partial(u, v, z)} = |J| = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial z} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial z} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial z} \end{vmatrix}$$

Properties:

$$\frac{\partial(u, v)}{\partial(x, y)} \cdot \frac{\partial(x, y)}{\partial(u, v)} = 1$$

$$\frac{\partial(u, v)}{\partial(x, y)} \cdot \frac{\partial(x, y)}{\partial(r, s)} = \frac{\partial(u, v)}{\partial(r, s)}$$

$\frac{\partial(u, v)}{\partial(x, y)} = 0$ (Functionally Dependent)

Taylor Series:

Used to find the approximate value of a non-polynomial function in terms of a polynomial function

For a $f(x, y)$ value expanded to a and b from x and y. Its value is.

$$f(x, y) = f(a, b) + \frac{1}{1!}\left[h f_x(a, b) + k f_y|_{(a,b)}\right] + \frac{1}{2!}\left[h^2 f_{xx}|_{(a,b)} + hk f_{xy}|_{(a,b)} + k^2 f_{yy}|_{(a,b)}\right] + \frac{1}{3!}\left[h^3 f_{xxx}|_{(a,b)} + 3h^2k f_{xxy}|_{(a,b)} + 3k^2h f_{yyx}|_{(a,b)} + k^3 f_{yyy}|_{(a,b)}\right] + \dots\dots$$

where $h = (x - a)$, $k = (y - b)$

Total derivatives:

$$\frac{dy}{dx} = -\frac{\frac{\partial f}{\partial x}}{\frac{\partial f}{\partial y}} \text{ (or) } -\frac{f_x}{f_y}$$

(Implicity Differention)

⚠ Note: $f_x = \frac{\partial f}{\partial x}$; $f_y = \frac{\partial f}{\partial y}$

FUNCTION OF SEVERAL VARIABLES.

Maxima and Minima for fn of two variables.

If f_x or $f_y = 0$ or DNE
and $f_{xx} = A$
$f_{xy} = B$
$f_{yy} = C$ then,

(a, b) are stationary points

i) $f(a,b) \to$ Max if $AC - B^2 > 0$
$A < 0$ or $(B < 0)$

ii) $f(a,b) \to$ Min if $AC - B^2 > 0$
$A > 0$ or $B > 0$

iii) $f(a,b)$ in not an extremum
if $AC - B^2 < 0$ and the points
(a,b) are known as saddle points.

iv) $f(a,b) = 0$; further investigation required.

Steps to Find extremum Values

1. Find f_x and f_y
2. $f_x = 0$ and $f_y = 0$
3. Find Stationary points to satisfy $f_x = 0$ and $f_y = 0$.
4. Find $f_{xx} = r$
   $f_{yy} = t$ and find $rt - s^2$
   $f_{xy} = s$
5. Make plot and find values

Stationary pt	$rt - s^2$	r	Conclusion

Method of Lagrange Multipliers.

To find extremum values of any fn $f(x, y, z)$ subject to constraint $g(x, y, z) = 0$.
Consider,

$$F = f + \lambda g$$

and find and put.

$$\frac{\partial F}{\partial x} = 0, \quad \frac{\partial F}{\partial y} = 0, \quad \frac{\partial F}{\partial z} = 0; \quad \frac{\partial F}{\partial \lambda} = 0$$

and solve the equations to find the points.

UNIT-III

ORDINARY DIFFERENTIAL EQUATIONS

- A differential equation is a mathematical equation involving an unknown function and its derivatives

Eg: $\frac{d^2y}{dx^2} + \frac{3dy}{dx} + 2y = e^{-2x}$

$$\left(\frac{d^2y}{dx^2}\right)^2 + 2\frac{dy}{dx} + y = 5x$$

- Order and Degree of a differential Equation:

The Order of a differential equation is the order of the highest derivative of the unknown function involved in the equation.

The degree of a differential equation is the degree of the highest derivative of the unknown function involved in the equation.

Eg. $\frac{d^3y}{dx^3} + \left(\frac{d^2y}{dx^2}\right)^5 + \left(\frac{dy}{dx}\right)^2 + y = 3x$

In the above eqs Order = 3 while degree = 1.

or, $e^{\frac{dy}{dx}} + y = 0$ or $\sin\left(\frac{dy}{dx}\right) = -y$

Has order = 1, but no degree.

- Linear Differential Equations with Constant Coefficients:

An equation of the form,

$$\boxed{a_0\frac{d^ny}{dx^n} + a_1\frac{d^{n-1}y}{dx^{n-1}} + a_2\frac{d^{n-2}y}{dx^{n-2}} + \cdots + a_n y = F(x)}$$

where $a_0, a_1 \cdots a_n$ are Constants, is Called a Linear differential equation of order n with Constant Coefficients.

Let $\frac{d}{dx} = D$, $\frac{d^2}{dx^2} = D^2$ where D is known as the 'differential operator' be replaced in the above equation; We get

$$[a_0 D^n + a_1 D^{n-1} + a_2 D^{n-2} + \cdots + a_n] y = F(x)$$

(or) $$[\phi(D)] y = F(x)$$

→ The general or Complete Solution of the above equation Consists of two parts namely,

(i) Complementary Function (CF) &

(ii) Particular Integral (PI).

That is, $y = CF + PI$.

• To find the Complementary function:

1. Form the 'Auxiliary eqn' by putting $D = m$
2. Put $\phi(m) = 0$ and find the roots of the polynomial equation.
3. We get roots, $m_1, m_2, m_3 \ldots m_n$.

→ **Case I:**

If the roots are real and unequal, ie if $m_1 \neq m_2 \neq m_3 \ldots \neq m_n$, then

$$CF = C_1 e^{m_1 x} + C_2 e^{m_2 x} + C_3 e^{m_3 x} + \cdots + C_n e^{m_n x}$$

→ **Case II:**

If $m_1 = m_2 = m$ and the remaining be real and unequal, then,

$$CF = (C_1 + C_2 x) e^{mx} + C_3 e^{m_3 x} + C_4 e^{m_4 x} + \cdots$$

→ **Case III:**

If $m_1 = m_2 = m_3 = m$ and the remaining be real and unequal, then,

$$CF = (C_1 + C_2 x + C_3 x^2) e^{mx} + C_4 e^{m_4 x} + \cdots + C_n e^{m_n x}$$

→ Case IV:

If roots are imaginary, if $m = \alpha \pm i\beta$, Then
$CF = e^{\alpha x}(C_1 \cos\beta x + C_2 \sin\beta x)$.

• To find the Particular Integral:

Let the given differential eqs be $\phi(D) y = F(x)$.
If RHS $= 0$; $F(x) = 0$, then There is no particular integral and the solution of the ODE is simply its CF ($y = CF$)

On the other hand if $F(x) \neq 0$, Then we have a PI ;
The PI is given by,
$$PI = \frac{1}{\phi(D)} F(x)$$

→ Type-I: If $\boxed{F(x) = e^{ax}}$: $\boxed{PI = \frac{1}{\phi(D)} F(x)}$.
$$PI = \frac{1}{\phi(D)} e^{ax}$$

1. in $\phi(D)$ replace D by a ;
where a is the coeff of x in e^{ax}
$$PI = \frac{1}{\phi(a)} \times e^{ax} \quad (\phi(a) \neq 0)$$

2. If $\phi(a) = 0$, then;
$$PI = x \frac{1}{\phi'(D)} e^{ax}$$

3. in $\phi'(D)$ replace D by a again,
then,
$$PI = x \frac{1}{\phi'(a)} e^{ax} \quad (\phi'(a) \neq 0)$$

If $\phi'(a) = 0$; then repeat step 2 again.
i.e; $x^2 \frac{1}{\phi'(D)} e^{ax}$ and so on.

Hyperbolic functions:
$\sinh ax = (e^{ax} - e^{-ax})/2$
$\cosh ax = (e^{ax} + e^{-ax})/2$

→ Type 2: If $\boxed{F(x) = \sin ax \text{ (or) } \cos ax}$

$PI = \frac{1}{\phi(D)} \sin ax$ (or) $\cos ax$

Since $y = \sin ax$
$Dy = \cos ax \cdot a$
$D^2 y = -\sin ax \cdot a^2$
$\therefore D^2 = -(a^2)$

1. in $\phi(D)$ replace D^2 by $-(a^2)$.

$PI = \frac{1}{\phi(-a^2)} \sin ax$ or $\cos ax$; $(\phi(-a^2) \neq 0)$

2. If $\phi(-a^2) = 0$, then

$PI = x \cdot \frac{1}{\phi'(D)} \sin ax$ or $\cos ax$

Aliter Method:
$\sin ax = \text{Im}\, e^{iax}$

3. in $\phi'(D)$, replace D^2 by $-(a^2)$

$PI = x \frac{1}{\phi'(-a^2)} \sin ax$ or $\cos ax$.

If $\phi'(-a^2) = 0$, then repeat Step 2.

→ Type 3: $\boxed{\text{If } F(x) = x^n}$

$PI = \frac{1}{\phi(D)} x^n = \frac{1}{[1 \pm f(D)]} x^n = [1 \pm f(D)]^{-1} x^n$.

1. Express $\phi(D)$ as $1 \pm f(D)$;
2. Bring it to the numerator as $[1 \pm f(D)]^{-1}$
3. Expand $[1 \pm f(D)]^{-1}$ as a Binomial Series.
4. Multiply x^n in each term and evaluate.

Binomial Expansions

$(1+x)^{-1} = 1 - x + x^2 - x^3 + x^4 + \cdots$

$(1-x)^{-1} = 1 + x + x^2 + x^3 + x^4$

$(1+x)^{-2} = 1 - 2x + 3x^2 - 4x^3 + 5x^4 + \cdots$

$(1-x)^{-2} = 1 + 2x + 3x^2 + 4x^3 + 5x^4 + \cdots$

→ **Type 4:** If $\boxed{F(x) = e^{ax} f(x)}$; $f(x) = e^{ax}$ (or) $\sin ax$ (or) $\cos ax$

$$PI = \frac{1}{\phi(D)} e^{ax} f(x)$$

1. in $\phi(D)$ replace D by $D+a$, where a is the coeff of x in e^{ax} and bring out e^{ax}

$$PI = e^{ax} \frac{1}{\phi(D+a)} f(x)$$

2. Equate $\frac{1}{\phi(D+a)} f(x)$ as their respective functions.

→ **Type 5:** If $\boxed{F(x) = x^n \sin ax \text{ (or) } x^n \cos ax}$, then

$$PI = \frac{1}{\phi(D)} x^n \sin ax \text{ or } \cos ax$$

wkt,

$$e^{iax} = \cos ax + i \sin ax$$

where, $\times x^n$

$$\cdot\ x^n e^{iax} = x^n \cos ax + i x^n \sin ax$$

$$\times \frac{1}{\phi(D)}$$

$$\frac{1}{\phi(D)} x^n e^{iax} = \frac{1}{\phi(D)} x^n \cos ax + \frac{1}{\phi(D)} i x^n \sin ax$$

$\therefore$

$$\left.\begin{aligned} \frac{1}{\phi(D)} x^n \cos x &= \text{Real Part} \left[\frac{1}{\phi(D)} x^n e^{iax} \right] \\ \frac{1}{\phi(D)} x^n \sin x &= \text{Imaginary Part} \left[\frac{1}{\phi(D)} x^n e^{iax} \right] \end{aligned}\right\} \text{Equate like Type-4.}$$

(or Alternatively)

When, $\boxed{F(x) = x^n \sin ax \text{ (or) } x^n \cos ax}$

$$PI = \frac{1}{\phi(D)} x^n \sin ax \text{ or } x^n \cos ax$$

$$PI = x \frac{1}{\phi(D)} \sin ax - \frac{\phi'(D)}{[\phi(D)]^2} \sin ax$$

- Bernoulli's Formula:

$$\int u \cdot dv = uv - u'v_1 + u''v_2 - u'''v_3 \ldots$$

$u', u'', u''' \rightarrow$ diff of u w.r.t x

$v, v_1, v_2 \rightarrow$ Integral of dv w.r.t x.

Order of Priority:

Inverse Logarithic Arithmetic Trigonometric Exponential → ILATE

- Integration by Parts:

$$\int u \cdot dv = u \int dv - \int \left(u' \cdot \int dv\right)$$

$$\int u \cdot dv = uv - \int (u'v)$$

- Section - II

Linear ODE with Variable Coefficients.

Type - I Euler Type (Couchey)	Type - II Legendre Type
$x^n \frac{d^n y}{dx^n} + a_1 x^{n-1} \frac{d^{n-1} y}{dx^{n-1}} + a_2 x^{n-2} \frac{d^{n-2} y}{dx^{n-2}} + \cdots + a^n y = F(x)$	$(ax+b)^n \frac{d^n y}{dx^n} + (ax+b)^{n-1} \frac{d^{n-1} y}{dx^{n-1}} + py = F(x)$

$\Rightarrow$ Euler Type:

Eg: Second order ODE with Variable Coefficients.

$$x^2 \frac{d^2y}{dx^2} + a_1 x \frac{dy}{dx} + a_2 y = F(x).$$

To Solve:

Let $x = e^z$ $\qquad Z = \log_e x \Rightarrow \frac{dz}{dx} = \frac{1}{x}$

$\rightarrow \frac{dy}{dx} = \frac{dy}{dz} \cdot \frac{dz}{dx}$ $\qquad \frac{dy}{dx} = Dy \quad D \equiv \frac{d}{dx}$

$= \frac{dy}{dz} \frac{1}{x}$ $\qquad D' = \frac{d}{dz}$

$xDz = D'y$

$\boxed{xD = D'}$

$\rightarrow \frac{d^2y}{dx^2} = \frac{d}{dx}\left[\frac{dy}{dx}\right]$

$= \frac{d}{dx}\left[\frac{dy}{dz} \cdot \frac{1}{x}\right]$

$= \frac{dy}{dz}\left[\frac{-1}{x^2}\right] + \frac{1}{x} \frac{d^2y}{dx\,dz}$

$= \frac{1}{x} \frac{d}{dz}\left[\frac{dy}{dz}\right] \frac{dz}{dx} - \frac{dy}{dz} \frac{1}{x^2}$

$\frac{d^2y}{dx^2} = \frac{1}{x} \frac{d^2y}{dz^2} \times \frac{1}{x} - \frac{1}{x^2} \frac{dy}{dz}$

$x^2 \frac{d^2y}{dx^2} = \frac{d^2y}{dz^2} - \frac{dy}{dz}$

$x^2 D^2 y = (D'^2 - D')y$

$\boxed{x^2 D^2 = D'^2 - D'}$ or $\boxed{x^2 D^2 = D'(D' - 1)}$

$\boxed{x^3 D^3 = D'(D'-1)(D'-2)}$ $\qquad D' = \frac{d}{dz}$ //

⇒ Legendre Type:

$$(ax+b)^n \frac{d^n y}{dx^n} + (ax+b)^{n-1}\frac{d^{n-1}y}{dx^{n-1}} + (ax+b)\frac{dy}{dx} + y = F(x)$$

Eg: Second order ODE with Varying Coefficients.

$$(ax+b)^2 \frac{d^2y}{dx^2} + (ax+b)\frac{dy}{dx} + y = F(x)$$

Let $ax+b = e^z$

$z = \log_e(ax+b)$

Let $D = \frac{d}{dx}$ and $D' = \frac{d}{dz}$.

$(ax+b)D = aD'$
$(ax+b)^2 D^2 = a^2 D'(D'-1)$
$(ax+b)^3 D^3 = a^3 D'(D'-1)(D'-2)$

- Linear ODE with Const Coeff:
  Method of Variation of Parameters:

⇒ Working Rule.

$$[a_0 D^2 + a_1 D + a_2]y = F(x)$$

$$C.F = \begin{cases} Ae^{m_1x} + Be^{m_2x} \\ (Ax+B)e^{mx} \\ e^{\alpha x}[A\cos\beta x + B\sin\beta x] \end{cases}$$

(Assume)

$$CF: \begin{cases} Ay_1 + By_2 \Rightarrow y_1 = e^{m_1x}, \quad y_2 = e^{m_2x} \\ Ay_1 + By_2 \Rightarrow y_1 = xe^{mx} \quad y_2 = e^{mx} \\ Ay_1 + By_2 \Rightarrow y_1 = e^{\alpha x}\cos\beta x \quad y_2 = e^{\alpha x}\sin\beta x \end{cases}$$

(i) Decide y_1 and y_2

(ii) Find Wronskian

$$W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$$

(iii) Let $PI = Py_1 + Qy_2$

$$P = -\int \frac{y_2 F(x)}{W(y_1,y_2)}\,dx$$

$$Q = \int \frac{y_1 F(x)}{W(y_1,y_2)}$$

Solving Simultaneous Differential Equations

Eg Solve $\frac{dx}{dt} = y+1$; $\frac{dy}{dt} = x+1$

Let $\frac{d}{dt} = D$

then, $Dx = y+1$ — ① and $Dy = x+1$ — ②

① × D + ② : $D^2x = Dy$

$Dy = x+1$

$D^2x - x = 1$

$(D^2-1)x = 1$

C.F : $m^2 - 1$

$m = \pm 1$

$= Ae^{t} + Be^{-t}$

P.I $= \frac{1}{D^2-1} e^{0t} = \frac{1}{-1} e^{0t} = -1$

Solution $x = Ae^{t} + Be^{-t} - 1$

From eq ① sub x,

$\frac{d}{dt}(Ae^{t} + Be^{-t} - 1) = y+1$

$Ae^{t} - Be^{-t} = y+1$

$y = Ae^{t} - Be^{-t} - 1$

▽ Try to cancel out one of the variable and then find C.F and P.I for that one variable alone. Then substitute it in other equations.

ORDINARY DIFFERENTIAL EQUATIONS

$\frac{d^2y}{dx^2} + \left[\frac{dy}{dx}\right]^3 + 10y = 3x$ ORDER : 2 DEGREE : 1

(or)

$D^2y + [2Dy]^3 + 10y = 3x$

DEGREE : power of the highest order derivative

ORDER - the highest order derivative

Solution of a Differential Equation

$y = CF + PI$

CF - Complimentary Function

PI - Particular Integral

Types of Ordinary Differential Equations

1. Linear ODE with Constant Coeff.
$[a_0D^n + a_1D^{n-1} + a_2D^{n-2} + a_n]y = F(x)$

2. Linear ODE with Variable Coeff.

a. Euler Type
$[x^nD^n + a_1x^{n-1}D^{n-1} + a_2x^{n-2}D^{n-2} + \dots + a_n]y = F(x)$

b. Legendre Type
$[(ax+b)^nD^n + (ax+b)^{n-1}D^{n-1} + (ax+b)^{n-2}D^{n-2} + \dots + a_n]y = F(x)$

Linear ODE with Const. Coeff. Finding CE:

$[a_0D^n + a_1D^{n-1} + a_2D^{n-2} + \dots a_n]y = F(x)$

1. Convert into auxiliary eqs by replacing differential operator with m ie $D = m$.
2. Find the roots for the equation $a_0m^n + a_1m^{n-1} + a_2m^{n-2} + \dots + a_n = 0$
3. a) If roots are real and unequal, $m = m_1, m_2$
$\boxed{C.F = Ae^{m_1x} + Be^{m_2x}}$
b) If roots are real and equal $m = m, m$
$\boxed{CF = (Ax + B)e^{mx}}$
c) If roots are imaginary $m = \alpha \pm i\beta$
$\boxed{CF = e^{\alpha x}[A\cos\beta x + B\sin\beta x]}$

Linear ODE with Const. Coeff Finding PI:

$a_0D^n + a_1D^{n-1} + \dots = \phi(D)$

$\phi(D)\, y = F(x)$

$\boxed{y = \frac{1}{\phi(D)}F(x) = PI}$

Type - I $F(x) = e^{ax}$

1. $PI = \frac{1}{\phi(D)}e^{ax}$
2. $PI = \frac{1}{\phi(a)}e^{ax}$ if $\phi(a) \neq 0$
3. $PI = x\frac{1}{\phi'(a)}e^{ax}$

Type - II: $\sin ax$ or $\cos ax$

1. $PI = \frac{1}{\phi(D)}\sin ax$
$= \frac{1}{\phi(-a^2)}\sin ax$

∇ only $D^2 = -(a^2)$.

if $\frac{1}{D}$ integrate or D differentiate

Type - III x^m

Use binomial theorem to solve.

Formulas:

$(1-x)^{-1} = 1 + x + x^2 + x^3 + x^4 + \dots$

$(1+x)^{-1} = 1 - x + x^2 - x^3 + x^4 - x^5 - \dots$

$(1-x)^{-2} = 1 + 2x + 3x^2 + 4x^3 + \dots$

$(1+x)^{-2} = 1 - 2x + 3x^2 - 4x^3 + 5x^4 \dots$

1. Make $\phi(D)$ as $[1 + F(D)]^{-1 \text{ or } -2}$
2. Fullwork by $\times x^m$ inside.

Type - IV: $e^{ax}F(x)$

$F(x) = x^m$ or $\sin ax$ or $\cos ax$.

1. $PI = \frac{1}{\phi(D)}e^{ax}F(x)$
$= e^{ax}\frac{1}{\phi(D+a)}F(x)$

Solve like above types.

$= e^{ax}\frac{1}{\phi(D+a)}F(x)$

Type - V: $x^m \sin ax$ or $x^m \cos ax$

$PI = x\frac{1}{\phi(D)}\sin ax - \frac{\phi'(D)}{[\phi(D)]^2}\sin ax$

(or)

Solve using Aliter Method:

$e^{iax} = \cos ax + i\sin ax$

- $\frac{1}{\phi(D)}x^n\cos ax = \text{Real part} \times \left[\frac{1}{\phi(D)}x^ne^{iax}\right]$
- $\frac{1}{\phi(D)}x^n\sin ax = \text{Im part} \times \left[\frac{1}{\phi(D)}x^ne^{iax}\right]$

Solving Euler Type:

$[x^nD^n + a_1x^{n-1}D^{n-1} + a_n]y = F(x)$

$D = \frac{d}{dx}$ $D' = \frac{d}{dz}$

1. Replace
$xD = D'$
$x^2D^2 = D'(D'-1)$
$x^3D^3 = D'(D'-1)(D'-2)$
2. Solve like Linear ODE with Const Variable.

Solving Legendre Type:

$[(ax+b)^nD^n + (ax+b)^{n-1}D^{n-1}]y = F(x)$

$D = \frac{d}{dx}$ $D' = \frac{d}{dz}$

$(ax+b)D = aD'$

$(ax+b)^2D^2 = a^2D'(D'-1)$

$(ax+b)^3D^3 = a^3D'(D'-1)(D'-2)$

Method of Variation of Parameters

(For PI neither from T-I-V)

1. $CF = Ae^{m_1x} + Be^{m_2x}$
$= (Ax+B)e^{mx}$
$= e^{\alpha x}[A\cos\beta x + B\sin\beta x]$

Convert to:
$Ay_1 + By_1$
$y_1 = e^{m_1x}, xe^{mx}$ etc
$y_2 = e^{m_2x}, e^{mx}$ etc.

2. Find Wronskian
$W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$
3. $PI = Py_1 + Qy_2$

where,

$P = -\int \frac{y_2 F(x)}{W}\,dx$

$Q = +\int \frac{y_1 F(x)}{W}\,dx$

Bernoulli's Formula:

$\int u\,dv = uv_1 - u'v_2 + u''v_3 + \dots$

$v = \int dv \dots$

$u' = d/dx\, u$

Intergration by Parts ILATE

$\int u\,dv = u\int dv - \int(u' \times \int dv)$

UNIT - IV

APPLICATION OF DIFFERENTIAL EQUATIONS
(or)
GEOMETRICAL APPLICATION OF DIFFERENTIAL CALCULUS

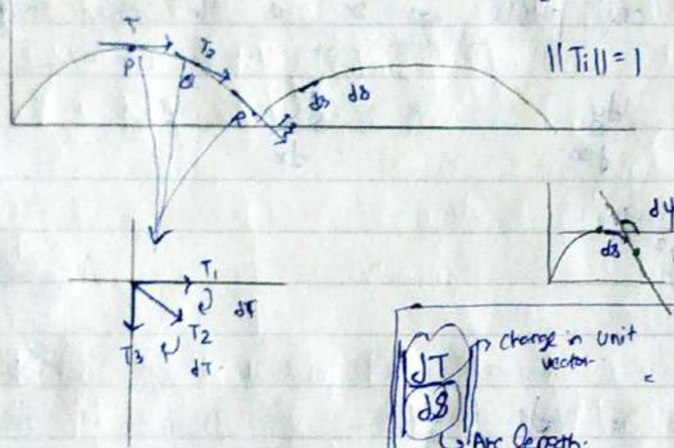
Curvature:

The amount by which a curve deviates from that of a straight line (or) The amount by which a surface deviates from that of a plane.

$K = \frac{1}{R}$ (Kappa) (Radius of curvature)

Mathematically,

$$\vec{s} = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} t - \sin t \\ 1 - \cos t \end{bmatrix} \quad \text{cycloid}$$



$\|T_i\| = 1$ how quickly that unit tangent vector changes direction

$$\boxed{\left|\frac{dT}{ds}\right| = K} \quad \text{or} \quad \boxed{\left|\frac{d\psi}{ds}\right| = K}$$

change in unit vector; Arc length.

Tangent vector fn $= \frac{d\vec{s}}{dt}$; unit tangent vector $= \frac{s'(t)}{|s'(t)|}$

$$\left|\frac{dT}{ds}\right| = \frac{|dT/dt|}{|ds/dt|}$$

Eg:  $\vec{s} = \begin{bmatrix} \cos t\, R \\ \sin t\, R \end{bmatrix}$

$$\frac{d\vec{s}}{dt} = \begin{bmatrix} -\sin t\, R \\ \cos t\, R \end{bmatrix} \text{ - tangent}$$

$$T(t) = \frac{\frac{d\vec{s}}{dt} \text{ or } \vec{s}\,'}{|s'|} = \frac{s'}{R} = \begin{bmatrix} -\sin t \\ \cos t \end{bmatrix}$$

$$\frac{dT}{dt} = \begin{bmatrix} -\cos t \\ \sin t \end{bmatrix}$$

or

$$\left|\frac{dT}{ds}\right| = \frac{|dT/dt|}{|ds/dt|} = \frac{1}{R} = K$$

- <u>Radius of Curvature</u>

The reciprocal of radius of curvature is the curvature and vice versa.

$$\underset{(rho)}{\rho} = \frac{1}{K_{(kappa)}}$$

$$\rho = \left|\frac{ds}{d\psi}\right| \text{ or } \left|\frac{ds}{dT}\right|$$

Slope at P = $\tan\psi = \frac{dy}{dx}$ $\qquad \frac{dy}{dx}\Big|_{(a,b)} = y_1 \quad \psi = \tan^{-1}(y_1)$

⇒ <u>Case - I</u>: $y = f(x)$ is the given curve.

Diff w.r.t x only possible as x is the only independent variable

$$\frac{d\psi}{dx} = \frac{1}{1+y_1^2}\,\frac{dy_1}{dx} \qquad \frac{dy_1}{dx} = y_2$$

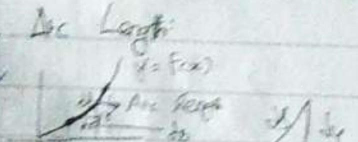
$$= \frac{1}{1+y_1^2}\, y_2$$

$$\rho = \frac{ds}{d\psi}$$

$$= \frac{ds}{d\psi}\cdot\frac{dx}{d\psi} = \frac{ds}{dx} \Big/ \frac{d\psi}{dx}$$

$$= \frac{\frac{ds}{dx}}{\frac{y_2}{1+y_1^2}}$$

Arc Length



Arc length = $\int ds$

$$\frac{ds}{dx} = \sqrt{1+y_1^2}$$

Length of an arc.

$$\frac{ds}{dx} = \sqrt{1+\left(\frac{dy}{dx}\right)^2} = \sqrt{1+y_1^2}$$

$$\rho = \frac{(1+y_1^2)^{1/2}}{\frac{y_2}{1+y_1^2}} = \boxed{\frac{(1+y_1^2)^{3/2}}{y_2} = \rho}$$

⟹ Case - II : The curve is in its parametric form:

$$x = x(t)$$
$$y = y(t)$$

1 Find dx/dt and dy/dt

2 $y_1 = \dfrac{dy/dt}{dx/dt}$ (or) $\dfrac{y'}{x'}$.

3 $y_2 = \dfrac{d}{dt}(y_1)\dfrac{dt}{dx}$

(or)

Substitute x' and y' in the Formula

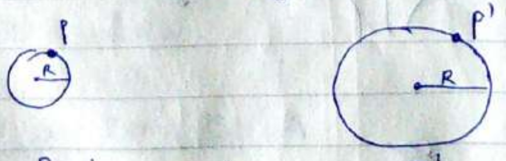
$$\boxed{\rho = \frac{(x'^2 + y'^2)^{3/2}}{x'y'' - y'x''}}$$

to Find the radius of Curvature

- Notes:

1 For a Straight line, the curvature is 0. $K = 0$
2. For a Circle, the curvature will be reciprocal of its radius $K = 1/R$.
3. Radius of Curvature of a Circle is R.
4. The curve which has more bend at a point will have more Curvature and will have a smaller radius of Curvature
5. If $\dfrac{dy}{dx}$ at the point is ∞,

then $$\rho = \frac{\left[1 + \left[\frac{dx}{dy}\right]^2\right]^{3/2}}{\frac{d^2x}{dy^2}}$$ i.e $x = f(y)$



P - Large Curvature
Small radius.

P' - Small Curvature
Large Radius.

• Parametric form of Various Curves:

1. Circle:

Eqs: $(x-x_1)^2 + (y-y_1)^2 = r^2$ or $x^2+y^2=r^2$

PF: $x = r\cos\theta$

$y = r\sin\theta$

2. Ellipse:

Eqs: $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

$a > b$

$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

$a < b$

PF: $x = a\cos\theta$

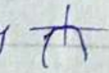
$y = b\sin\theta$

3. Parabola

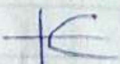
std eqs $(y-k)^2 = 4a(x-h)$

Eqs: $x^2 = 4ay$

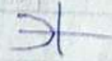
$x^2 = -4ay$



$y^2 = 4ax$



PM = $x = at^2$ for $y^2 = 4ax$ $\quad$ $x = 2at$ for $x^2 = 4ay$. $\quad$ $y^2 = -4ax$

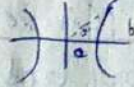


$y = 2at$ $\quad$ $y = at^2$

$\left(\overset{(-)}{x} = at^2 \text{ for } y^2 = -4ax,\ y = +2at\right)$ $\quad$ $(x = 2at$ for $x^2 = -4ay$. $y = -at^2)$

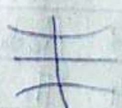
4. Hyperbola:

Eqs: $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$



transverse on x-axis

$\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$



transverse on y-axis.

PF: $x = a\sec\theta$

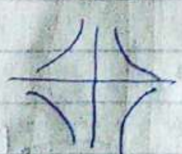
$y = b\tan\theta$

5. Astroid:

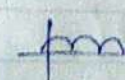
$x = a\cos^3\theta$

$y = a\sin^3\theta$

$x^{2/3} + y^{2/3} = a^{2/3}$



6. Cycloid



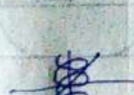
$x = a(\theta - \sin\theta)$

$y = a(1-\cos\theta)$

7. Helix

$x = \cos\theta \quad z = \theta/5$

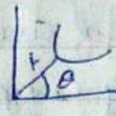
$y = \sin\theta$



- **Radius of Curvature (Polar Form)**

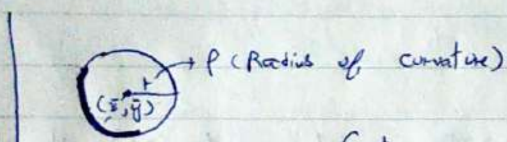
Let the curve be in the form.

$r = f(\theta)$



$$\boxed{\rho = \frac{(r^2 + r_1^2)^{3/2}}{r^2 + 2r_1^2 - r r_2}}$$; $r_1 = \frac{dr}{d\theta}$ $r_2 = \frac{d^2 r}{d\theta^2}$, at the point (r, θ).

- **Centre of Curvature and Circle of Curvature:**



Centre of curvature = $(\bar{x}, \bar{y})$.

The Circle of Curvature is given by the equation,

$$(x - \bar{x})^2 + (y - \bar{y}^2) = \rho^2$$

where,

$$\bar{x} = x - \left[\frac{y_1(1 + y_1^2)}{y_2}\right]$$

$$\bar{y} = y + \left[\frac{(1 + y_1^2)}{y_2}\right]$$

At the given point.

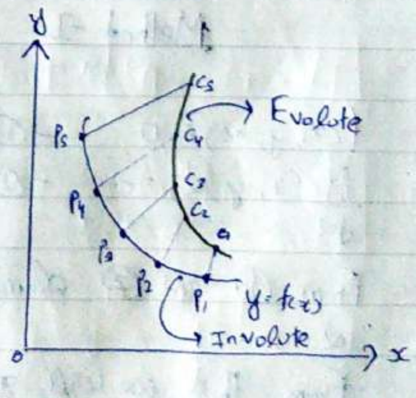
- **Involute & Evolute:**

⇒ Involute:

The curve $y = f(x)$ is known as the involute

⇒ Evolute:

The locus of all Centre of Curvatures for the Curve $y = f(x)$ is called evolutes.



⇒ Steps to Find Evolute:

Step 1: Get the Curve $y = f(x)$ to its parametric form. $x = x(t)$
$y = y(t)$

Step 2: Find the Centre of curvature $(\bar{x}, \bar{y})$ for all points using
$$\bar{x} = x - \left[\frac{y_1 (1 + y_1^2)}{y_2} \right] ; \quad \bar{y} = y + \left[\frac{(1 + y_1^2)}{y_2} \right]$$

Step 3: Eliminate the parameter t from $\bar{x}$ and $\bar{y}$ by equating them together.

Step 4: Replace $\bar{x}$ with x and $\bar{y}$ with y in the final equation to get the required evolute.

- Envelopes:

If an equation containing a parameter (α), then for different Values of α, a family of curves will be obtained.
The envelope of a family of curves is the curve which touches each member of the family.
Mathematically,

$$\left.\begin{array}{ll} f(x, y, \alpha) = 0 & -① \\ f(x, y, (\alpha + d\alpha)) = 0 & -② \end{array}\right] \text{- different members of the Same family.}$$

Then locus of their points of the intersection is called envelope of that family.

Rules to find the envelope of family of curves.

Method - I	Method - II
$f(x, y, \alpha) = 0 \quad -①$ $\frac{df}{d\alpha}(x, y, \alpha) = 0 \quad -②$ From ① and ② eliminate the parameter α. From this we will get the envelope.	If it is possible to make the given family such like $A\alpha^2 + B\alpha + C = 0$; where $A = A(x, y)$, $B = B(x, y)$, and $C = C(x, y)$. Then the envelope is the discriminant of the quadratic eqn. $\boxed{B^2 - 4AC = 0}$

• Beta and Gamma Functions

→ Gamma Function:

$$\boxed{\Gamma(n) = \int_0^\infty e^{-x} x^{n-1} \cdot dx}$$ where n is any positive integer.

Properties of Gamma function:

1. $\Gamma(1) = \int_0^\infty e^{-x} x^{1-1} \cdot dx = \int_0^\infty e^{-x} \cdot 1 \cdot dx = -[e^{-x}]_0^\infty = -[0-1] = 1$

$\Gamma(1) = 1$

2. $\Gamma(n+1) = \int_0^\infty e^{-x} x^n \cdot dx$ (now integrating by parts and applying ILATE)

$= x^n \int_0^\infty e^{-x} - \int_0^\infty n x^{n-1} \cdot \int e^{-x}$ $\quad \int u \cdot dv = uv - \int v \cdot du$

$= [x^n e^{-x}]_0^\infty + n \int_0^\infty e^{-x} x^{n-1}$

$= \cancel{-x}[0 - 1\times 0] + n\,\Gamma(n) \rightarrow (-\infty^n e^{-\infty}) - (-0^n e^0) + n\,\Gamma(n)$

$\boxed{\Gamma(n+1) = n\,\Gamma(n)}$ → Recurrence Relation / Formula.

3. $\Gamma(2) = \cancel{2\,\Gamma(1) = 2\times 1 = 2}$ (Wrong)
$\Gamma(3) = \cancel{3\,\Gamma(2) = 3\times 2\times \Gamma(1) = 3\times 2\times 1} = 6$

$\boxed{\Gamma(n+1) = n!}$ → Factorial relation / Formula.

4. $$\boxed{\begin{array}{l} \Gamma(n) = \Gamma(n-1) \times (n-1) \\ \Gamma(n) = (n-1)! \end{array}}$$

5. For $n = \frac{p}{q}$; $q \neq 0$.

Here Factorial relation is not possible but,

6. $\boxed{\Gamma(1/2) = \sqrt{\pi}}$ (Sqrt of pi)

(using Recurrence relation)

7. $\Gamma(\frac{3}{2}) = (\frac{3}{2} - 1)\,\Gamma(\frac{3}{2} - 1)$

$= \frac{1}{2}\,\Gamma(1/2) = \frac{\sqrt{\pi}}{2}$

8. $$\boxed{\Gamma(n) \cdot \Gamma(1-n) = \frac{\pi}{\sin n\pi}}$$

$\Gamma(\frac{1}{4})\,\Gamma(3/4) = \sqrt{2}(\pi)$

- Beta Function: ($\beta(m,n)$)

$$\boxed{\beta(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1} \cdot dx}$$

⇒ Other forms of beta function:

i) $\beta(m,n) = \int_0^\infty \frac{y^{m-1}}{(1+y)^{m+n}} \cdot dy$ by replacing $x = \frac{y}{1+y}$

ii) $\beta(m,n) = 2\int_0^{\pi/2} \sin^{2m-1}\theta \cdot \cos^{2n-1}\theta \cdot d\theta$

by replacing $x = \sin^2\theta$

- Relation Between β and Γ functions:

$$\boxed{\beta(m,n) = \frac{\Gamma(m) * \Gamma(n)}{\Gamma(m+n)}}$$

∴ Result: $\beta(m,n) = \beta(n,m)$

Proof:

$\beta(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1} \cdot dx$

Let $y = 1-x$

$dy = -dx$

$= \int_1^0 (1-y)^{m-1} y^{n-1}(-dy)$

$x \to 0 \quad y \to 1$

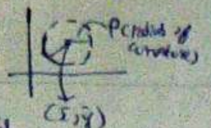
$x \to 1 \quad y \to 0$

$= \int_0^1 y^{n-1}(1-y)^{m-1} \cdot dy$

$= \beta(n,m)$ //

APPLICATION OF DIFFERENTIAL EQUATIONS

CENTRE OF CURVATURE AND CIRCLE OF CURVATURE:



The Circle of Curvature is

$$\boxed{(x-\bar{x})^2 + (y-\bar{y})^2 = \rho^2}$$

where,

$$\bar{x} = x - \left[\frac{y_1(1+y_1^2)}{y_2}\right]$$

$$\bar{y} = y + \left[\frac{(1+y_1^2)}{y_2}\right]$$

at the points (x,y)

CURVATURE:

The degree to which a curve/surface deviates from a straight line/plane

$$\underset{(\text{Kappa})}{K} = \left|\frac{d\psi}{ds}\right|$$

$d\psi$ → change in angle

ds → change in arc length

Conversely,

$$K = \left|\frac{1}{R}\right|$$ → Radius of Curvature

RADIUS OF CURVATURE:

$$\underset{(\text{rho})}{\rho} = \left|\frac{ds}{d\psi}\right|$$

$$\boxed{\rho = \frac{(1+y_1^2)^{3/2}}{y_2}}$$

$$\rho = \frac{(x'^2 + y'^2)^{3/2}}{x'y'' - y'x''}$$

(equation for parametric form of a curve)

$y_1 = \left.\frac{dy}{dx}\right|_{(x,y)}$

$y_2 = \left.\frac{d^2y}{dx^2}\right|_{(x,y)}$

$x' = \left.\frac{dx}{dt}\right|_{(x,y)}$

$y' = \left.\frac{dy}{dt}\right|_{(x,y)}$

Notes:

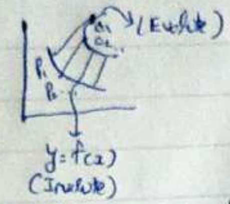
1. Straight line, $K=0$
2. Circle, $K = \frac{1}{R}$
3. Circle, Radius = R
4. More bend at a curve means more curvature and vice-versa
5. If $\frac{dy}{dx} = \infty$, then

$$\boxed{\rho = \frac{\left[1+\left(\frac{dx}{dy}\right)^2\right]^{3/2}}{\frac{d^2x}{dy^2}}}$$ ie $x = f(y)$

INVOLUTE AND EVOLUTE:

Involute: The curve $y=f(x)$ is known as the involute.

Evolute: The locus of all centre of curvature for the curve $y=f(x)$ is called evolute.



STEPS TO FIND EVOLUTE:

1. Convert $y=f(x)$ to parametric form, i.e. $x = x(t)$ and $y = y(t)$
2. Find the Centre of Curvature $(\bar{x},\bar{y})$ for all points using $\bar{x}$ and $\bar{y}$ formula
3. Equate $\bar{x}$ and $\bar{y}$ to eliminate parameter t.
4. Replace $\bar{x} \to x$ and $\bar{y} \to y$ and the final eqn is the req. evolute.

ENVELOPES:

- The envelope of a family of curves is the curve which touches each member of the family.
- Mathematically
  $f(x,y,\alpha)=0$
  $f(x,y,\alpha+d\alpha)=0$] diff mem of same family
  where α is the parameter
- The locus of their points of intersection is called envelope of the family

→ Rules to find the envelope to a family of curves

1. $f(x,y,\alpha)=0$ — ①
   $\frac{\partial f}{\partial \alpha}(x,y,\alpha)=0$ — ②
   From ① and ② eliminating the parameter α and the envelope will be obtained
2. Make the family of curves as,
   $A\alpha^2 + B\alpha + C = 0$
   where $A = A(x,y)$; $B = B(x,y)$ and $C = C(x,y)$
   The envelope is given by the discriminant as
   $$\boxed{B^2 - 4AC = 0}$$

Gamma Function $[\Gamma(n)]$

$$\boxed{\Gamma(n) = \int_0^\infty e^{-x}x^{n-1}\,dx}$$

1. $\Gamma(1) = 1$
2. $\Gamma(n+1) = n\Gamma(n)$ — Recurrence Relation
3. $\Gamma(n+1) = n!$
4. $\Gamma(n) = (n-1)\,\Gamma(n-1)$
   $\Gamma(n) = (n-1)!$
5. $\Gamma(1/2) = \sqrt{\pi}$ (cont. Γ ...)
6. $\Gamma\left(\frac{p}{q}\right) = \Gamma\left(\frac{p}{q}-1\right)\left(\frac{p}{q}-1\right)$
7. $\Gamma(n)\,\Gamma(1-n) = \frac{\pi}{\sin n\pi}$

Beta Function $[\beta(m,n)] = \beta(n,m)$

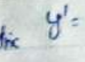
$$\beta(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}\,dx$$

$$\beta(m,n) = \int_0^\infty \frac{y^{m-1}}{(1+y)^{m+n}}\,dy \quad \left(x = \frac{y}{1+y}\right)$$

$$\beta(m,n) = 2\int_0^{\pi/2} \sin^{2m-1}\theta\,\cos^{2n-1}\theta\,d\theta \quad (x = \sin^2\theta)$$

RADIUS OF CURVATURE:

(~~Parametric~~ Polar form)



$$r = f(\theta)$$

$$\boxed{\rho = \frac{(r^2 + r_1^2)^{3/2}}{r^2 + 2r_1^2 - r r_2}}$$

$r_1 = \left.\frac{dr}{d\theta}\right|_{(r,\theta)}$

$r_2 = \left.\frac{d^2r}{d\theta^2}\right|_{(r,\theta)}$

Relation btwn $\beta_{(m,n)}$ and $\Gamma_{(n)}$

$$\beta(m,n) = \frac{\Gamma(m) * \Gamma(n)}{\Gamma(m+n)}$$

Parametric form of Various Curves:

Circle

$(x-x_1)^2 + (y-y_1)^2 = r^2$ or $x^2+y^2=r^2$

PF $x = r\cos\theta$
$y = r\sin\theta$

Helix:

$x = \cos\theta$
$y = \sin\theta$ $\quad z = \theta/s$

Hyperbola:

$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

$x = a\sec\theta$
$y = b\tan\theta$

Parabola

$x^2 = \pm 4ay$ $\qquad$ $y^2 = \pm 4ax$

PF: $x = 2at$ $\qquad$ $x = at^2$
$y = at^2$ $\qquad$ $y = 2at$

Ellipse:

$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

PF $a\cos\theta = x$
$b\sin\theta = y$

Cycloid:

$x = a(\theta - \sin\theta)$
$y = a(1-\cos\theta)$

Astroid:

$x = a\cos^3\theta$
$y = a\sin^3\theta$
$x^{2/3} + y^{2/3} = a^{2/3}$

UNIT - V

SEQUENCE AND SERIES

- **Sequence:**

  It is a collection of objects, where repitition of objects are allowed and the order of the collection matters. The number of terms / elements is called as its length and can range from $0 \to \infty$.

  Eg: $1, 2, 3, 4, 5, + \ldots$ or $\{a_n\}_{n=1}^{\infty}$ where $a_n = n$.

- **Series:**

  An operation of adding infinitely many quantities, one after the other.

  $$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \cdots$$

- **SEQUENCE OF REAL NUMBERS:**

  Let $\{a_n\}_{n=1}^{\infty}$ be a sequence of real numbers.

Eg: $a_n = \frac{1}{n}$; $a_1 = \frac{1}{1}$; $a_2 = \frac{1}{2}$; $a_3 = \frac{1}{3}$ ..... $\{\frac{1}{n}\}_{n=1}^{\infty}$

Eg: [Constant Sequence]

$a_n = k$; $a_1 = k$; $a_2 = k$; $a_3 = k$ ..... $\{k\}_{n=1}^{\infty}$

⇒ **Limit of Sequence $\{a_n\}_{n=1}^{\infty}$ (or) $\{a_n\}_{n=1}^{\infty}$ Converges to l**

Let $\{a_n\}_{n=1}^{\infty}$ be a sequence of real numbers and if given ε (epsilon) > 0 (small or near to 0), then there exists an integer $N > 0$, such that

$$|a_n - l| < \varepsilon \quad \forall \text{ (for all) } n \geq N \quad \text{(or)} \quad \lim_{n\to\infty} a_n = l$$

Eg. $\lim_{n\to\infty} \frac{1}{n} = 0$, $l = 0$

$\lim_{n\to\infty} k = k$, $l = k$

⇒ **Bounded Below and Bounded Above:**

1. Bounded Below	2. Bounded Above
Let $\{a_n\}_{n=1}^{\infty}$ be a sequence of real numbers. If we say $\{a_n\}_{n=1}^{\infty}$ is said to have bounded below, then $a_n \geq m$, m - any real no. $\forall n$	Let $\{a_n\}_{n=1}^{\infty}$ be a sequence of real numbers. If we say $\{a_n\}_{n=1}^{\infty}$ is said to have a bounded above, then $a_n \leq m$, m - any real no $\forall n$.
$\{\frac{1}{n}\}_{n=1}^{\infty}$ has the bounded below 0.	$\{\frac{1}{n}\}_{n=1}^{\infty}$ has the bounded above as 1

⇒ Bounded Sequence:

If $\{a_n\}_{n=1}^{\infty}$ has both bounded above and banded below

ie $m_1 \le a_n \le m_2$ $\forall n$; $m_1, m_2 \to$ real no.

then, $\{a_n\}_{n=1}^{\infty}$ is said to be a bounded sequence.

Eg. $\left\{\frac{1}{n}\right\}_{n=1}^{\infty}$ is bounded as $0 \le \frac{1}{n} \le 1$ | $\{n^3\}_{n=1}^{\infty}$ is not bounded as $1 \le n^3 \le \infty$

⇒ Divergent Sequence:

(i) $\{a_n\}_{n=1}^{\infty}$ diverges to $+\infty$, if given positive real number M, then there exists positive integer N; such that $a_n > M$ for all $n \ge N$ (or) $\lim_{n \to \infty} a_n = +\infty$.

(ii) $\{a_n\}_{n=1}^{\infty}$ diverges to $-\infty$; ... $\lim_{n \to \infty} a_n = -\infty$

Eg. $\{n_2\}_{n=1}^{\infty} \to$ diverges to $+\infty$; $\{(-n)\}_{n=1}^{\infty}$ diverges to $-\infty$

⇒ Monotonically increasing and Decreasing Sequences.

Monotonically Increasing	Monotonically Decreasing
A sequence $\{a_n\}_{n=1}^{\infty}$ is said to be monotonically increasing if $a_n \le a_{n+1}$ $\forall n$ ie $a_1 \le a_2 \le a_3 \cdots$	A sequence $\{a_n\}_{n=1}^{\infty}$ is said to be monotonically decreasing if $a_n \ge a_{n+1}$ $\forall n$ i.e $a_1 \ge a_2 \ge a_3 \cdots$

Some Results:

(i) A monotonically increasing sequence with bounded above will be a Convergent sequence.

(ii) A monotonically decreasing sequence with bounded below will be a Convergent sequence.

⇒ Oscillatory Sequence:

A sequence $\{a_n\}_{n=1}^{\infty}$ that neither converges nor diverges to $\pm\infty$.

Eg. $a_n = \{(-1)^n\}_{n=1}^{\infty}$ or $-1, +1, -2, +2, -3, +3, \ldots$

Some Results:

(i) If $|x| < 1$, then $\lim_{n\to\infty} x^n = 0$ // $x<1$ as take $\left(\frac{1}{x}\right)^n = \frac{1}{x^n} = \frac{1}{\infty} = 0$

i.e Convergent

(ii) If $x > 1$, $\lim_{n\to\infty} x^n = \infty$

i.e Divergent

(iii) If $\lim_{n\to\infty} a_n = a$; $\lim_{n\to\infty} b_n = b$.

then,

$$\{a_n \pm b_n\}_{n=1}^{\infty} = a \pm b$$

$$\{a_n \cdot b_n\}_{n=1}^{\infty} = ab$$

$$\{a_n / b_n\}_{n=1}^{\infty} = a/b \quad \text{provided } b \neq 0$$

⇒ Cauchy's General Principle of Convergence:

Let $\{a_n\}_{n=1}^{\infty}$ be a sequence of real numbers; then given $\varepsilon > 0$ there exists a positive integer N, such that

$$|a_n - a_m| < \varepsilon \quad \forall\ n, m \geq N \text{ also / and } n > m$$

Then, $\{a_n\}_{n=1}^{\infty}$ be a Convergent Sequence.

• INFINITE SERIES:

If $\{a_n\}_{n=1}^{\infty}$ be a sequence of real no; Then an infinite Series is,

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \cdots a_{n-1} + a_n \ldots$$

(Summation)

⇒ How to know if a Series Converges or diverges?

Let $\sum_{n=1}^{\infty} a_n$ be a Series and $\{S_n\}_{n=1}^{\infty}$ be a sequence of partial Sum of the Series such that,

$S_1 = a_1$

$S_2 = a_1 + a_2$

$S_3 = a_1 + a_2 + a_3$ and so on

Now,

(i) $\sum_{n=1}^{\infty} a_n$ Converges if and only if $\{S_n\}_{n=1}^{\infty}$ Converges.

(ii) $\sum_{n=1}^{\infty} a_n$ diverges if and only if $\{S_n\}_{n=1}^{\infty}$ diverges.

(iii) $\sum_{n=1}^{\infty} a_n$ Oscillates if and only if $\{S_n\}_{n=1}^{\infty}$ Oscillates.

- **Series with Positive Terms:**

In a Series $\sum_{n=1}^{\infty} a_n$, here every $a_n \geq 0 \; \forall \, n$.

⇒ **Theorem:**

If $\sum_{n=1}^{\infty} a_n$ is a Converging Series, then $\lim_{n\to\infty} a_n = 0$

But the Corvase is not true.

↳ if $\lim_{n\to\infty} a_n = 0$ that does not necessairily mean $\sum_{n=1}^{\infty} a_n$ is a Converging Series.

Eg. $\lim_{n\to\infty} \frac{1}{n} = 0$, but $\sum_{n=1}^{\infty} \frac{1}{n}$ does not Converge.

- **Tests for Positive Termed Series:**

1. Integral Test $\quad \int_1^{\infty} f(x) = \text{finite} \rightarrow \text{cgs} \quad \int_1^{\infty} f(x) = \text{Infinite} \rightarrow \text{dgs}$
2. * Comparision Test $\quad \lim_{n\to\infty} \left(U_n \leq K V_n \text{ or } \frac{U_n}{V_n} \leq K\right)$, U_n cgs if V_n dgs and vice versa
3. * Ratio / D'Alembertz Ratio Test. $\quad \lim_{n\to\infty} \frac{U_{n+1}}{U_n} = L$, $\quad L < 1$ cgs, $L > 1$ dgs, $\quad L = 1$ Test fails
4. * Rabbe's Test $\quad \lim_{n\to\infty} n\left[\frac{U_n}{U_{n+1}} - 1\right] = L \quad L <$ dgs, $L >$ cgs. $\quad L = 1$ no effect
5. Root / Cauchy's Root Test
6. Logarithmic Test

⇒ **1. Integral Test:**

If $\sum f(n) = f(1) + f(2) + \cdots$ also $f(n)$ decreses as n increases,
then $\sum f(n)$ Converges if $\int_1^{\infty} f(n) \cdot dx$ is finite.
$\sum f(n)$ diverges if $\int_1^{\infty} f(n) \cdot dx$ is Infinite.

Results:

(i) $\sum_{n=1}^{\infty} 1/n$ diverges

(ii) $\sum_{n=1}^{\infty} \frac{1}{n^2}$ Converges

(iii) $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ diverges

(iv) $\sum_{n=1}^{\infty} \frac{1}{n^p} \begin{cases} p \leq 1 & \text{diverges} \\ p > 1 & \text{Converges} \end{cases}$

(v) $\frac{1}{n^{p+q-1}}$ — Converges $p+q-1 > 1$; diverges $p+q-1 \leq 1$

2. Comparision Test:

If $\sum_{n=1}^{\infty} U_n$ and $\sum_{n=1}^{\infty} V_n$ are two positive termed series and if $U_n \leq K V_n$ for all n, Then

(i) $\sum_{n=1}^{\infty} U_n$ will Converge if $\sum_{n=1}^{\infty} V_n$ Converges.

(ii) $\sum_{n=1}^{\infty} U_n$ will diverge if $\sum_{n=1}^{\infty} V_n$ diverges.

Other forms of Comparision Test:

If $\sum_{n=1}^{\infty} U_n$ and $\sum_{n=1}^{\infty} V_n$ are two positive termed series and $\frac{U_n}{V_n} \leq K$ which is a finite value and $\neq 0$ as $n \to \infty$.

$\lim_{n\to\infty} \frac{U_n}{V_n}$ to find K.

(i) $\sum_{n=1}^{\infty} U_n$ → Converges if $\sum V_n$ Converges.
→ Diverges if $\sum V_n$ diverges.

Note: $V_n = \frac{\text{the degree of } n \text{ in numerator}}{\text{the degree of } n \text{ in denominator}}$

Eg: $U_n = \frac{n^2}{(1+n^3)(n+1)}$

then $V_n = \frac{n^2}{n^4} = \frac{1}{n^2}$

3. D'Alemberts Ratio Test or Ratio Test:

(Very useful for problems with unknown)

Rule: If i) $U_n \geq 0$,

ii) $\lim_{n\to\infty} \frac{U_{n+1}}{U_n} = L$

Suppose, $U_n = \frac{1}{2} + \frac{2}{3} + \frac{3}{4} + \dots$

$U_n = \frac{1 \cdot 2 \cdot 3 \cdots n}{2 \cdot 3 \cdot 4 \cdots (n+1)}$

$U_{n+1} = \frac{1 \cdot 2 \cdot 3 \cdots n\,(n+1)}{2 \cdot 3 \cdot 4 \cdots (n+1)(n+2)}$

The previous terms are also included

$n+1$ simply means next terms. They are also included.

Then, the series $\sum_{n=1}^{\infty} U_n$,

i) U_n Converges if $L < 1$

ii) U_n diverges if $L > 1$

iii) Test fails if $L = 1$

4. Rabbe's Test:

If $\sum_{n=1}^{\infty} U_n$ be a positive termed series, then

$$\lim_{n\to\infty} \left\{ n\left[\frac{U_n}{U_{n+1}} - 1\right]\right\} = L \to \begin{cases} L < 1 & \text{diverges} \\ L > 1 & \text{Converges} \end{cases}$$

⇒ 5. Root Test (or) Cauchy's Root Test:

If i) $U_n \geq 0$ (positive termed Series)

ii) $L = \lim_{n\to\infty} U_n^{1/n}$

Then, i) If $L < 1$; Series Converges

ii) If $L > 1$; Series diverges.

$(n+1)! = (n+1)\,n!$

as $n! = n \times (n-1) \times (n-2) \cdots$

so $(n+1)! = (n+1)(n+1-1)(n+1-2) \cdots$

Note:

1.1 $\lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n = e$ also $\lim_{n\to\infty} \left(1 + \frac{k}{n}\right)^n = e^k$

- **Series of Various non-polynomial functions:**

1. $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \cdots$
2. $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \frac{x^{10}}{10!} + \cdots$
3. $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$
4. $\tan x = x + \frac{x^3}{3} + \frac{2x^5}{15} + \frac{17x^7}{315} + \cdots$
5. $\log(x+1) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \cdots$

⇒ 6. Logarithmic Test:

Let $\sum_{n=1}^{\infty} U_n$ be a positive termed Series ($U_n \geq 0$)

Then, $L = \lim_{n\to\infty} n \log \frac{U_n}{U_{n+1}}$

If $L > 1 \rightarrow \sum_{n=1}^{\infty} U_n$ Converges

$L < 1 \rightarrow \sum_{n=1}^{\infty} U_n$ Diverges.

- ALTERNATING SERIES:

A series in which the terms are alternatively positive and negative, is called an alternating series.

Eg: $U_1 - U_2 + U_3 - U_4 + U_5 - \cdots$

may be represented as; $\sum_{n=1}^{\infty} (-1)^{n-1} U_n$.

⇒ Absolute Series:

In the alternating series, (mod of an alternating series)

$\sum_{n=1}^{\infty} |(-1)^{n-1} U_n|$ → which means that the series will contain all terms as positive termed series.

⇒ Absolute Convergence of the alternating Series:

Rule:

$\sum_{n=1} U_n$ Converges and $\sum_{n=1}^{\infty} |U_n|$ should also converge,

Then we can say that the given alternating series is absolutely converging.

⇒ Conditionally Converges (or) Conditional Convergence of the alternating $\sum U_n$

$\sum_{n=1}^{\infty} U_n$ Converges, whereas, $\sum_{n=1}^{\infty} |U_n|$ diverges,

Then the alternating series Conditionally Converges.

Note:

i) An absolutely converging series will always converge if, $\sum_{n=1}^{\infty} |U_n|$ Converges, i.e if $\sum_{n=1}^{\infty} |U_n|$ Converges then $\sum_{n=1} U_n$ also Cgs.

The Converse may or may not be true.

$\Rightarrow$ Leibnitz Test: (for checking the convergence of alternating series):

The alternating series, $\sum_{n=1}^{\infty} (-1)^{n-1} U_n$ converges if,

(i) $\{U_n\}_{n=1}^{\infty}$ is a monotonically decreasing ($U_n \geq U_{n+1}$) sequence. Each term is numerically less than the preceding term.

(ii) $\lim_{n\to\infty} U_n = 0$. Then $U_1 - U_2 + U_3 - U_4 \dots$ converges by Leibnitz Test.

If $\lim_{n\to\infty} U_n \neq 0$, $U_1 - U_2 + U_3 - U_4 + \dots$ <u>Oscillates</u>.

SEQUENCES AND SERIES

SERIES

Operation of adding infinitely many quantities one after the other

$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \cdots a_{n-1} + a_n$

→ Infinite Series

How to know if a Series Cgs or Dgs!

$\sum_{n=1}^{\infty} a_n$ be a series

$\{S_n\}_{n=1}^{\infty}$ be a Sequence of partial Sum of Series

such that $S_1 = a_1$, $S_2 = a_1 + a_2$, $S_3 = a_1 + a_2 + a_3$ and so on

If (i) $\{S_n\}_{n=1}^{\infty}$ Cgs, then $\sum_{n=1}^{\infty} a_n$ Cgs

(ii) $\{S_n\}_{n=1}^{\infty}$ dgs, then $\sum_{n=1}^{\infty} a_n$ dgs

(iii) $\{S_n\}_{n=1}^{\infty}$ oscillates, then $\sum_{n=1}^{\infty} a_n$ oscillates

Series with positive Terms:

$\sum_{n=1}^{\infty} a_n$, for every $a_n \geq 0 \ \forall n$.

Theorem: If $\sum_n a_n$ is a Converging Series, then $\lim_{n\to\infty} a_n = 0$. But the Converse is not true.

Tests for Positive Termed Series

Integral Test: $\sum f(n) = f(1) + f(2) + \cdots$, $f(n)$ decreases as n increases

then, $\sum f(n)$ cgs if $\int_1^{\infty} f(n)\,dx$ is finite

$\sum f(n)$ dgs if $\int_1^{\infty} f(n)\,dx$ is Infinite

Results:

$\sum_{n=1}^{\infty} \frac{1}{n^p}$ — cgs $p>1$ → $\frac{1}{n^2}$ Cgs; dgs $p \leq 1$ → $\frac{1}{n}, \frac{1}{n^{1/2}}$ dgs.

$\sum_{n=1}^{\infty} \frac{1}{n^{p+q-1}}$ — dgs $p+q-1 \leq 1$; cgs $p+q-1>1$

Comparision Test: Two Series $\sum U_n$ & $\sum V_n$.

if $U_n \leq k V_n \ \forall n$,

(i) $\sum_{n=1}^{\infty} U_n$ will Cgs if $\sum_{n=1}^{\infty} V_n$ Cgs.

(ii) $\sum_{n=1}^{\infty} U_n$ will dgs if $\sum_{n=1}^{\infty} V_n$ dgs

also, $\frac{U_n}{V_n} \leq k$ (Some Finite value $\neq 0$)

$\sum_{n=1}^{\infty} U_n$ — cgs if $\sum_{n=1}^{\infty} V_n$ Cgs; dgs if $\sum_{n=1}^{\infty} V_n$ dgs

∇ V_n = the degree of n in numerator / the degree of n in denominator.

D'Alemberts Ratio Test:

if i) $U_n \geq 0$

ii) $\boxed{\lim_{n\to\infty} \frac{U_{n+1}}{U_n} = L}$

Then

$\sum_{n=1}^{\infty} U_n$ — Cgs if $L<1$; dgs if $L>1$; Test fails if $L=1$

Raabes Test:

i) $U_n \geq 0$

ii) $\boxed{L = \lim_{n\to\infty}\left\{n\left[\frac{U_n}{U_{n+1}} - 1\right]\right\} = L}$

$\sum_{n=1}^{\infty} U_n$ — cgs if $L>1$; dgs if $L<1$

Cauchys Root Test:

If i) $U_n \geq 0$

ii) $\boxed{\lim_{n\to\infty} U_n^{1/n} = L}$

$\sum_{n=1}^{\infty} U_n$ — Cgs if $L<1$; dgs if $L>1$

Note:

i) $\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e$.

Logarithmic Test:

If i) $U_n \geq 0$

ii) $\boxed{\lim_{n\to\infty} n \log \frac{U_n}{U_{n+1}} = L}$

$\sum_{n=1}^{\infty} U_n$ — Cgs $L>1$; dgs $L<1$

Alternating Series:

$\sum_{n=1}^{\infty} a_n$ with alternating +ve and -ve terms

Eg: $U_1 - U_2 + U_3 - U_4 + U_5 - U_6 \cdots$

$\Rightarrow \sum_{n=1}^{\infty} (-1)^{n-1} U_n$

$\sum_{n=1}^{\infty} (-1)^{n-1} a_n$, $a_n \to U_n$

Absolute Series	Conditional Series
$\sum_{n=1}^{\infty} \lvert U_n \rvert$ → all +ve terms	$\sum U_n$ → Cgs
Rules: $\sum U_n$ Cgs and $\sum \lvert U_n \rvert$ should also Cgs. alternating Then given Series is Absolutely Cgs.	$\sum \lvert U_n \rvert$ → dgs. Then alternating Series is Conditionally Cgs.

Note:

If $\sum_{n=1}^{\infty} \lvert U_n \rvert$ Cgs which implies $\sum_{n=1}^{\infty} U_n$ also Cgs. The Converse may not be true.

$\sum \lvert U_n \rvert \not\Leftarrow \sum U_n$

Leibnitz Test: (for checking the Cgs of alternating Series.)

$\sum_{n=1}^{\infty} (-1)^{n-1} U_n$ Cgs if,

i) $\{U_n\}_{n=1}^{\infty}$ monotonically decreasing sequence, $U_n \geq U_{n+1}$

ii) $\lim_{n\to\infty} U_n = 0$.

If $\lim_{n\to\infty} U_n \neq 0$ then, alternating Series alternates.

SEQUENCE

Collection of objects in an order

$\{a_n\}_{n=1}^{\infty}$ is a Sequence.

→ $a_1, a_2, a_3, a_4, \ldots$

Types of Sequences

Constant Sequence

For $\{a_n\}_{n=1}^{\infty}$ $a_n = k$.

Converging Sequence:

For $\{a_n\}_{n=1}^{\infty}$

$\varepsilon > 0$

$N > 0$

$|a_n - l| < \varepsilon \ \forall n \geq N$

(or)

$\lim_{n\to\infty} a_n = l$.

Diverging Sequence:

For $\{a_n\}_{n=1}^{\infty}$, given value m, $a_n > m$, $n \geq N$

$\lim_{n\to\infty} a_n = \pm\infty$

Bounded Below	Bounded above
For $\{a_n\}_{n=1}^{\infty}$	
$a_n \geq m$ $\forall n$ (the lowest possible value)	$a_n \leq m$ $\forall n$ (the highest possible value)

Bounded Sequence:

For $\{a_n\}_{n=1}^{\infty}$

$m_1 \leq a_n \leq m_2 \ \forall n$

(lowest possible and highest possible value)

Monotonically Increasing	Monotonically Decreasing
For $\{a_n\}_{n=1}^{\infty}$	
$a_n \leq a_{n+1}$, $a_1 \leq a_2 \leq a_3$	$a_n \geq a_{n+1}$, $a_1 \geq a_2 \geq a_3$

Results:

(i) A monotonically increasing Sequence with bounded above will be a Convergent Sequence

(ii) A monotonically decreasing sequence with bounded below will be a Convergent Sequence.

Oscillatory Sequence:

$\{a_n\}_{n=1}^{\infty}$ neither Converges nor diverges

Eg: $a_n \{(-1)^n\}_{n=1}^{\infty}$ → $-1, 1, -1, 1, -1, 1, \ldots$

Results

(i) $|x|<1$, $\lim_{n\to\infty} x^n = 0$ ie Convergent

(ii) $x>1$, $\lim_{n\to\infty} x^n = \infty$ ie Divergent

(iii) $\lim_{n\to\infty} a_n = a$, $\lim_{n\to\infty} b_n = b$

$[a_n \pm b_n]_{n=1}^{\infty} = a \pm b$

$[a_n \cdot b_n]_{n=1}^{\infty} = ab$

$[a_n / b_n]_{n=1}^{\infty} = a/b$, $b \neq 0$.

Cauchys General Principle of Convergence:

For $\{a_n\}_{n=1}^{\infty}$, $\varepsilon > 0$, some N,

$|a_n - a_m| < \varepsilon \ \forall n, m \geq N$, $n > m$

then $\{a_n\}_{n=1}^{\infty}$ be a convergent Sequence.