Shifting Theorem: I

(i) If $Z[f(t)] = F(z)$, then $Z[e^{-at} f(t)] = F[ze^{aT}]$

(ii) If $Z[f(t)] = F(z)$, then $Z[e^{at} f(t)] = F[ze^{-aT}]$

Proof: (i) $F(z) = Z[f(t)] = \sum_{n=0}^{\infty} f(nT) z^{-n}$ (defn.)

$$Z[e^{-at} f(t)] = \sum_{n=0}^{\infty} e^{-anT} f(nT) z^{-n}$$

$$= \sum_{n=0}^{\infty} f(nT) (ze^{aT})^{-n}$$

$$= Z[f(t)]_{z \to ze^{aT}}$$

$$= F[ze^{aT}]$$

ie, $Z[e^{-at} f(t)] = Z[f(t)]_{z \to ze^{aT}}$

$$= F(z) \text{ where } z \to ze^{aT}.$$

Note:

Similarly, prove (ii)

1. Find $Z[e^{-at}]$

Soln:- we know that, $Z[e^{-at} f(t)] = [Z(f(t))]_{z \to ze^{aT}}$

$$Z[e^{-at} \cdot 1] = Z[1]_{z \to ze^{aT}}$$

$$= \left(\frac{z}{z-1}\right)_{z \to ze^{aT}} \qquad \left(\because Z(1) = \frac{z}{z-1}\right)$$

$$= \frac{ze^{aT}}{ze^{aT} - 1}$$

$$\therefore Z[e^{-at}] = \frac{ze^{aT}}{ze^{aT} - 1}$$

2. Find $[e^{-at} t]$

Soln:-

we know that, $Z[e^{-at} f(t)] = Z(f(t))_{z \to ze^{aT}}$

$$Z[e^{-at} t] = [Z(t)]_{z \to ze^{aT}}$$

$$= \left[\frac{Tz}{(z-1)^2}\right]_{z \to ze^{aT}} \qquad \left(\because Z(t) = \frac{Tz}{(z-1)^2}\right)$$

$$= \frac{Tze^{aT}}{(ze^{aT} - 1)^2}$$

Find

3. $Z[e^{-at}\sin bt]$

we know that $Z[e^{-at}f(t)] = Z[f(t)]_{z\to ze^{aT}}$

$$Z[e^{-at}\sin bt] = [Z[\sin bt]]_{z\to ze^{aT}}$$

$$= \left.\frac{z\sin bT}{z^2 - 2z\cos bT + 1}\right|_{z\to ze^{aT}}$$

$$= \frac{ze^{aT}\sin bT}{z^2e^{2aT} - 2ze^{aT}\cos bT + 1}$$

Find

4. $Z[e^{3t}\cos t]$.

$$Z[e^{3t}\cos t] = Z(\cos t)\Big|_{z\to ze^{-3T}}$$

$$= \left.\frac{z(z-\cos T)}{z^2 - 2z\cos T + 1}\right|_{z\to ze^{-3T}}$$

$$= \frac{ze^{-3T}(ze^{-3T} - \cos T)}{z^2e^{-6T} - 2ze^{-3T}\cos T + 1}$$

5. Find $Z[e^{-t}t^2]$

Soln: $Z[e^{-t}t^2] = Z(t^2)_{z \to ze^T}$

$$= \left. \frac{T^2 z(z+1)}{(z-1)^3} \right|_{z \to ze^T}$$

$$= \frac{T^2 \cdot ze^T(ze^T+1)}{(ze^T-1)^3}$$

If $Z\{f(t)\} = F(z)$ then $Z[a^n f(t)] = F(z/a)$

If $Z\{x(n)\} = X(z)$ then $Z[a^n x(n)] = X(z/a)$

1. Find $Z[a^n n]$

soln: we know that, $Z[a^n x(n)] = X(z/a)$

$$Z[a^n n] = [Z(n)]_{z \to z/a} \qquad \left[\because Z(n) = \frac{z}{(z-1)^2}\right]$$

$$= \left. \frac{z}{(z-1)^2} \right|_{z \to z/a}$$

$$= \frac{\frac{z}{a}}{\left(\frac{z}{a}-1\right)^2} = \frac{\frac{z}{a}}{\left(\frac{z-a}{a}\right)^2}$$

$$= \frac{az}{(z-a)^2}.$$

(2) Find $Z\left[\frac{a^n}{n!}\right]$

Note: In page NO. 20, problem NO. 13. The same problem can be solved also by formula

soln:- we know that, $Z[a^n x(n)] = X(z/a)$

$$Z\left[a^n \frac{1}{n!}\right] = \left[Z\left[1/n!\right]\right]_{z \to z/a}$$

$$= e^{1/z}\Big|_{z \to z/a} \quad \because Z\left[1/n!\right] = e^{1/z}$$

$$= e^{1/z/a} = e^{a/z}.$$

(3) Find $Z\left[\frac{a^n}{n}\right]$

soln:- we know that, $Z[a^n x(n)] = X(z/a)$

$$Z\left[a^n \cdot \frac{1}{n}\right] = \left[Z\left[1/n\right]\right]_{z \to z/a}$$

$$= \log \frac{z}{z-1}\Big|_{z \to z/a} \quad \because Z\left[\frac{1}{n}\right] = \log\left(\frac{z}{z-1}\right)$$

$$= \log\left(\frac{z/a}{z/a - 1}\right)$$

$$= \log\left(\frac{z/a}{\frac{z-a}{a}}\right) = \log\left[\frac{z}{z-a}\right]$$

(4) Find $Z[a^n t]$

Soln:- we know that, $Z[a^n f(t)] = F(z/a)$

$$Z(a^n t) = (Z(t))_{z \to z/a}$$

$$= \left.\frac{Tz}{(z-1)^2}\right|_{z \to z/a}$$

$$= \frac{\frac{Tz}{a}}{\left(\frac{z}{a}-1\right)^2} = \frac{\frac{Tz}{a}}{\left(\frac{z-a}{a}\right)^2}$$

$$= \frac{Tz}{a} \times \frac{a^2}{(z-a)^2}$$

$$= \frac{Taz}{(z-a)^2}$$

(5) Find $Z[a^n e^{at}]$

Soln:- we know that, $Z[a^n f(t)] = F(z/a)$

$$Z[a^n e^{at}] = [Z(e^{at})]_{z \to z/a}$$

$$= \left.\frac{z}{z-e^{aT}}\right|_{z \to z/a}$$

$$= \frac{\frac{z}{a}}{\frac{z}{a}-e^{aT}} = \frac{\frac{z}{a}}{\frac{z-ae^{aT}}{a}} = \frac{z}{z-ae^{aT}}$$

Shifting Theorem II.

If $Z[f(t)] = F(z)$, then $Z[f(t+T)] = z[F(z) - f(0)]$.

Proof :-

$$Z[f(t+T)] = \sum_{n=0}^{\infty} f(nT+T) z^{-n}$$

$$= \sum_{n=0}^{\infty} f((n+1)T) z^{-n}$$

$$= \sum_{n=0}^{\infty} f((n+1)T) . z^{-n} . z^{1} . z^{-1}$$

$$= z \sum_{n=0}^{\infty} f((n+1)T) . z^{-(n+1)}$$

Put $n+1 = k$.

$$= z \sum_{k=1}^{\infty} f(kT) z^{-k}$$

$$= z \left[\underbrace{\sum_{k=1}^{\infty} f(kT) z^{-k} + f(0)} - f(0) \right]$$

$$= z \left[\sum_{k=0}^{\infty} f(kT) z^{-k} - f(0) \right]$$

$$= z \left[F(z) - f(0) \right]$$

(1) Find $Z[e^{3(t+T)}]$

Soln: we know that, $Z[f(t+T)] = z[F(z) - f(0)]$

$$= z[Z[f(t)] - f(0)]$$

here $f(t) = e^{3t}$, $f(0) = e^0 = 1$

$$Z[f(t)] = Z(e^{3t}) = \frac{z}{z - e^{3T}}$$

$$\therefore Z[e^{3(t+T)}] = z[F(z) - f(0)]$$

$$= z\left[\frac{z}{z - e^{3T}} - 1\right]$$

$$= z\left[\frac{z - z + e^{3T}}{z - e^{3T}}\right]$$

$$= \frac{z e^{3T}}{z - e^{3T}}$$

(or)

$$Z[e^{3(t+T)}] = Z[e^{3t+3T}]$$

$$= Z[e^{3T} e^{3t}]$$

$$= e^{3T} Z[e^{3t}]$$

$$= e^{3T}\left[\frac{z}{z - e^{3T}}\right] = \frac{z e^{3T}}{z - e^{3T}}.$$

(2) Find $Z[\sin(t+T)]$

we know that, $Z[f(t+T)] = z(F(z) - f(0))$

$= z[Z[f(t)] - f(0)]$

here $f(t) = \sin t$, $f(0) = 1$.

$$Z[f(t)] = Z(\sin t) = \frac{z\sin T}{z^2 - 2z\cos T + 1}$$

$$\therefore Z[\sin(t+T)] = z\left[\frac{z\sin T}{z^2 - 2z\cos T + 1} - 0\right]$$

$$= \frac{z^2\sin T}{z^2 - 2z\cos T + 1}$$

(or)

$$Z[\sin(t+T)] = Z[\sin t\cos T + \cos t\sin T]$$

$$= \cos T\, Z[\sin t] + \sin T\, Z(\cos t)$$

$$= \cos T\left[\frac{z\sin T}{z^2 - 2z\cos T + 1}\right] + \sin T\left[\frac{z(z-\cos T)}{z^2 - 2z\cos T + 1}\right]$$

$$= \frac{z\sin T\cos T + z^2\sin T - z\sin T\cos T}{z^2 - 2z\cos T + 1}$$

$$= \frac{z^2\sin T}{z^2 - 2z\cos T + 1}$$

Initial value Theorem :-

If $Z[f(t)] = F(z)$, then $f(0) = \lim\limits_{z \to \infty} F(z)$.

Proof :-

$$F(z) = Z[f(t)] = \sum_{n=0}^{\infty} f(nT) z^{-n}$$

$$= f(0.T) + \frac{f(1.T)}{z} + \frac{f(2.T)}{z^2} + \cdots \text{ to } \infty$$

$$= f(0) + \frac{1}{z} f(T) + \frac{1}{z^2} f(2T) + \cdots \text{ to } \infty$$

Taking limit as $z \to \infty$

$$\lim_{z \to \infty} F(z) = \lim_{z \to \infty} \left[f(0) + \frac{1}{z} f(T) + \frac{1}{z^2} f(2T) + \cdots + \text{to } \infty \right]$$

$$= f(0) \qquad \because \lim_{n \to \infty} 1/n \to 0.$$

$$\therefore \lim_{z \to \infty} F(z) = f(0).$$

Final value theorem:

If $Z[f(t)] = F(z)$ then $\lim_{t \to \infty} f(t) = \lim_{z \to 1} (z-1) F(z)$

Proof:

$$Z[f(t+T) - f(t)] = \sum_{n=0}^{\infty} [f(nT+T) - f(nT)] z^{-n}$$

$$Z[f(t+T)] - Z[f(t)] = \sum_{n=0}^{\infty} [f((n+1)T) - f(nT)] z^{-n}$$

$$zF(z) - zf(0) - F(z) = \sum_{n=0}^{\infty} [f((n+1)T) - f(nT)] z^{-n}$$

$$(z-1)F(z) - zf(0) = \sum_{n=0}^{\infty} [f((n+1)T) - f(nT)] z^{-n}$$

Taking limit as $z \to 1$,

$$\lim_{z \to 1} [(z-1)F(z) - zf(0)] = \lim_{z \to 1} \sum_{n=0}^{\infty} [f((n+1)T) - f(nT)] z^{-n}$$

$$\lim_{z \to 1} (z-1)F(z) - f(0) = \sum_{n=0}^{\infty} f((n+1)T) - f(nT)$$

$$= (f(\cancel{T}) - f(0)) + [f(\cancel{2T}) - f(\cancel{T})] + f(3T) - f(\cancel{2T}) + \dots + [f((n+1)T) - f(nT)] + \dots + f(\infty)$$

$$= f(\infty) - f(0)$$

$$= \lim_{t \to \infty} f(t) - f(0)$$

$$\lim_{z \to 1} (z-1) F(z) - \cancel{f(0)} = \lim_{t \to \infty} f(t) - \cancel{f(0)}$$

$$\therefore \lim_{z \to 1} (z-1) F(z) = \lim_{t \to \infty} f(t).$$

Problems:-

① If $F(z) = \dfrac{5z}{(z-2)(z-3)}$ find $f(0)$ and $\lim_{t \to \infty} f(t)$

soln:- By initial value thm,

$$f(0) = \lim_{z \to \infty} F(z)$$

$$= \lim_{z \to \infty} \frac{5z}{(z-2)(z-3)}$$

$$= \lim_{z \to \infty} \frac{5z}{z^2 - 5z + 6} = \frac{\infty}{\infty}.$$

$$= \lim_{z \to \infty} \frac{5}{2z - 5} \quad \text{(by L' hospital's rule)}$$

$$= 0$$

By Final value theorem,

$$\lim_{t\to\infty} f(t) = \lim_{z\to 1} (z-1)\, F(z)$$

$$= \lim_{z\to 1} (z-1) \frac{5z}{(z-2)(z-3)} = 0.$$

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(2) If $F(z) = \dfrac{2z}{z-e^{-T}}$ find $\lim\limits_{t\to\infty} f(t)$ and $f(0)$.

Soln: By initial value thm,

$$f(0) = \lim_{z\to\infty} F(z)$$

$$= \lim_{z\to\infty} \frac{2z}{z-e^{-T}}$$

$$= \lim_{z\to\infty} \frac{\cancel{z}}{\cancel{z}} \left[\frac{2}{1-\frac{e^{-T}}{z}}\right] \quad \left[\because \lim_{z\to\infty} \frac{e^{-T}}{z} \to 0\right]$$

$$= 2$$

By Final value theorem,

$$\lim_{t\to\infty} f(t) = \lim_{z\to 1} (z-1)\, F(z)$$

$$= \lim_{z\to 1} (z-1) \frac{2z}{z-e^{-T}} = 0.$$

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(3) If $F(z) = \dfrac{z(z-\cos aT)}{z^2 - 2z\cos aT + 1}$ find $f(0)$ also find $\lim\limits_{t\to\infty} f(t)$.

soln:-

By initial value thm,

$$f(0) = \lim_{z\to\infty} F(z)$$

$$= \lim_{z\to\infty} \frac{z(z-\cos aT)}{z^2 - 2z\cos aT + 1} = \frac{\infty}{\infty}$$

$z^2 - z\cos aT$

$$\Rightarrow \lim_{z\to\infty} \frac{2z - \cos aT}{2z - 2\cos aT} \quad \text{[by L-Hospital's rule)}$$

$$= \lim_{z\to\infty} \frac{\cancel{z}\left[2 - \frac{\cos aT}{z}\right]}{\cancel{z}\left[2 - \frac{\cos aT}{z}\right]} \qquad \lim_{z\to\infty} \frac{\cos aT}{z} \to 0$$

$$= \lim_{z\to\infty} \frac{2}{2} = 1.$$

By Final value thm,

$$\lim_{t\to\infty} f(t) = \lim_{z\to 1} (z-1) F(z)$$

$$= \lim_{z\to 1} (z-1) \frac{z(z-\cos aT)}{z^2 - 2z\cos aT + 1}$$

$$= 0.$$

Shift property.

$Z\{x(n-m)\} = z^{-m} X(z)$ where $x(n)$ is a causal sequence and m is a positive integer.

Proof:

$$Z\{x(n-m)\} = \sum_{n=0}^{\infty} x(n-m) z^{-n} \quad \text{(by defn)}$$

Put $n-m = k$

$n = m+k$.

$$\Rightarrow \sum_{k=-m}^{\infty} x(k) z^{-(m+k)}$$

$$= \sum_{k=0}^{\infty} x(k) z^{-m} \cdot z^{-k} \quad [\because \{x(n)\} \text{ is a causal seq.)}$$

ie, $x(n) = 0$ for $n < 0$.

$$= z^{-m} \sum_{k=0}^{\infty} x(k) z^{-n}$$

$$= z^{-m} X(z).$$

$$\therefore Z\{x(n-m)\} = z^{-m} X(z).$$

Note:

$$Z^{-}[z^{-m} X(z)] = \{x(n-m)\} = [z^{-1} X(z)]_{n \to n-m}.$$