(1) If all the vertices of an undirected graph are each of odd degree k. Show that the number of edges of the graph is a multiple of k.

Soln: We know that the number of vertices of odd degree is even. Let the number of vertices of the given graph be $2n$. If e is the number of edges. Then by the hand-shaking theorem,

$$\sum_{i=1}^{2n} \deg(v_i) = 2e$$

$$\sum_{i=1}^{2n} k = 2e$$

$$\Rightarrow \quad 2nk = 2e$$

or $\quad e = nk$. a multiple of k.

(2) For each of the following degree sequences, find if there exists a graph. In each case, either draw a graph or explain why no graphs exists.

(i) 4, 4, 4, 3, 2 $\qquad$ (ii) 5, 5, 4, 3, 2, 1

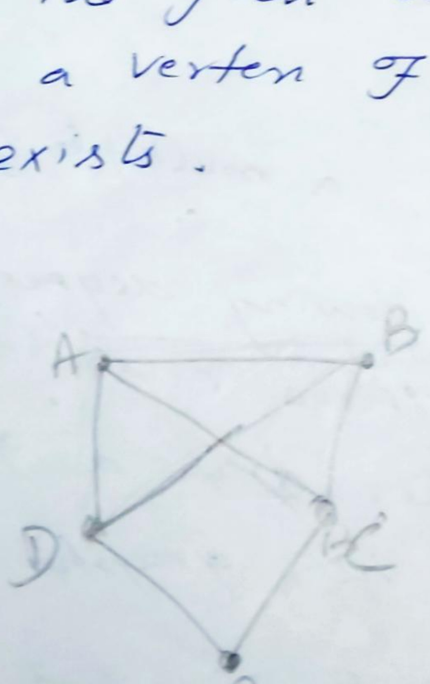
(iii) 3, 3, 3, 3, 2 $\qquad$ iv) 5, 1, 1, 1, 1

Soln sum of the degree of all the (i) vertices = 17 which is an odd number which is impossible by Hand shaking theorem.

No graph exists.

(ii) There are 6 vertices. Hence, a vertex of degree 5 must be adjacent to all other vertices. As there are two vertices of degree 5, all other vertices must be of degree at least 2. But in the given degree sequence there is a vertex of degree 1. ∴ No graph exists.

(iii)

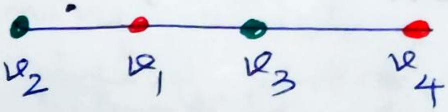


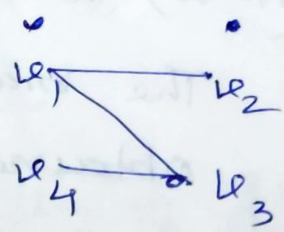
A simple graph with given degree sequence is possible.

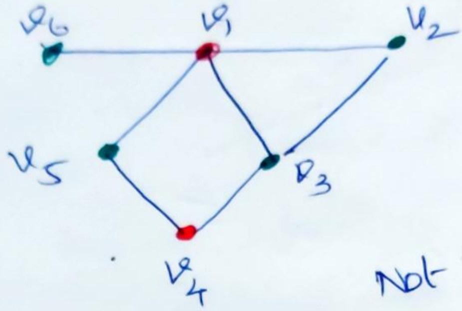
(iv) A simple graph is possible

Rules for Bipartite graph :-

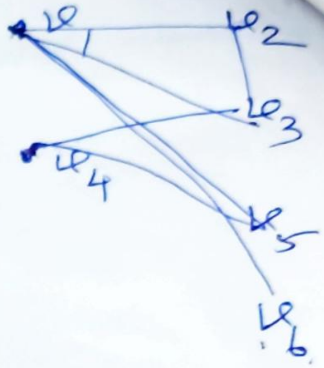
* use only 2 colours.
* Start from any vertex V_1 & use the first colour.
* Then colour the neighbours of V_1 with the second colour.
* continue the same for the next coloured vertex







Not Bipartite.



(3) Prove that the number of edges in a bipartite graph with n vertices is atmost $\frac{n^2}{4}$.

Proof :-

Let the vertex set be partitioned into the subsets V_1 & V_2. Let V_1 contain x vertices, Then V_2 contains $(n-x)$ vertices. This graph is having maximum number of edges when it is complete

The maximum number of edges will be obtained, when each of the x vertices [For a complete bipartite graph $K_{m,n-m}$. in V_1 is connected to each of the $n-x$ vertices of V_2

$f(x)$ No. of edges = product of the vertices

$$f(x) = x(n-x).$$

which is a function of x.

To find the value of x for which it is maximum, we use calculus

$f'(x) = n - 2x$

$f''(x) = -2 < 0$

no. of edges = product of the vertices $= 2 \times 3 = 6$

$f'(x) = 0 \implies n - 2x = 0$

$n = 2x$

$\boxed{x = n/2}$ which is a

stationary point & $f''(n/2) < 0$

$\therefore f(x)$ is maximum at $x = n/2$

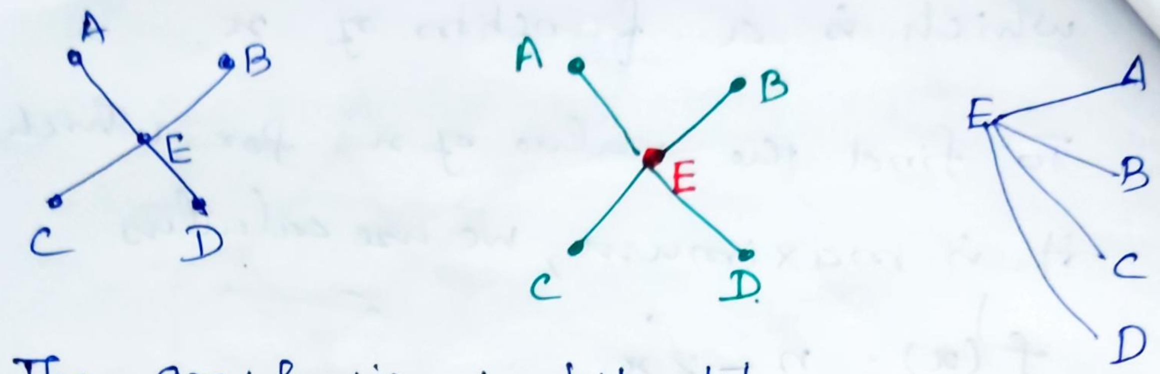
$f(x) = \frac{n}{2}\left(n - \frac{n}{2}\right)$

$= \frac{n}{2}\left(\frac{2n-n}{2}\right) = \frac{n}{2} \cdot \frac{n}{2}$

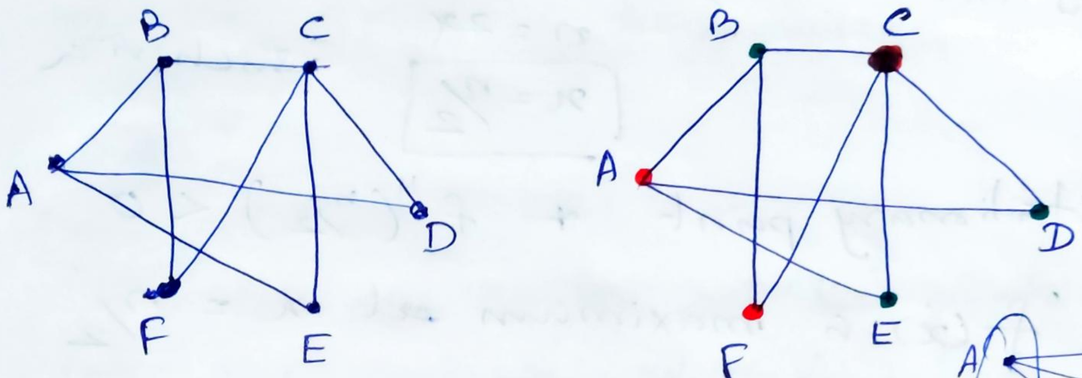
$= n^2/4$.

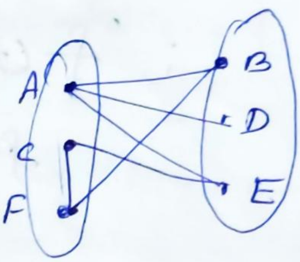
(4) Determine which of the following graphs are bipartite & which are not. If bipartite, state whether it is completely bipartite.

1



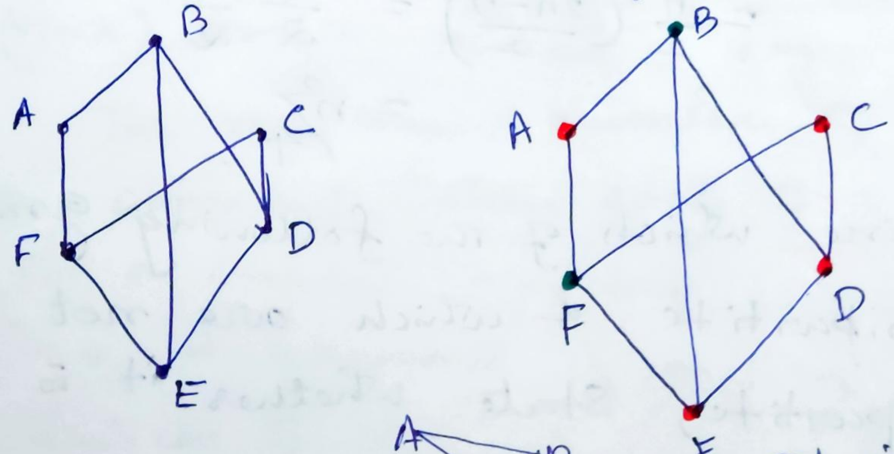
The graph (i) is bipartite, yes it is completely bipartite.

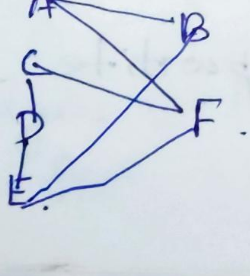




It is not a bipartite graph.

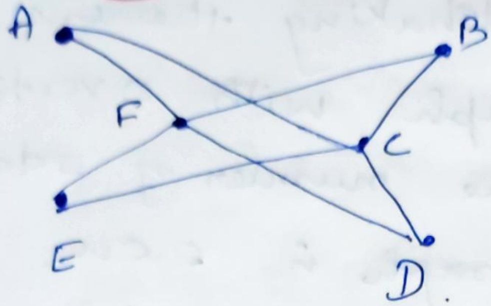
2

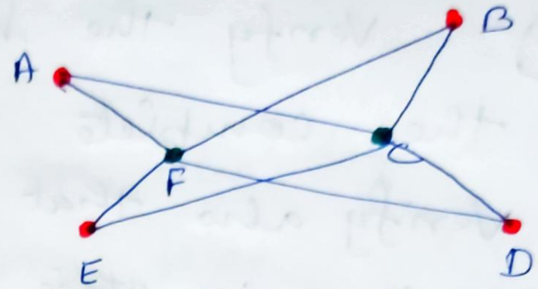


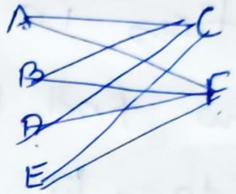


It is not bipartite graph.

3



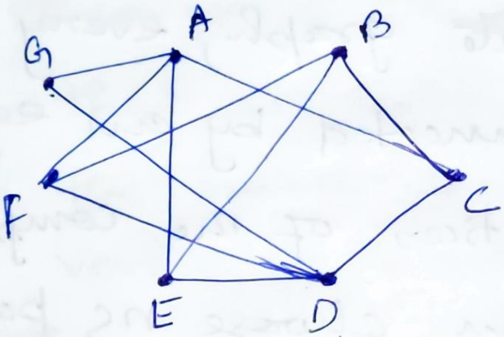


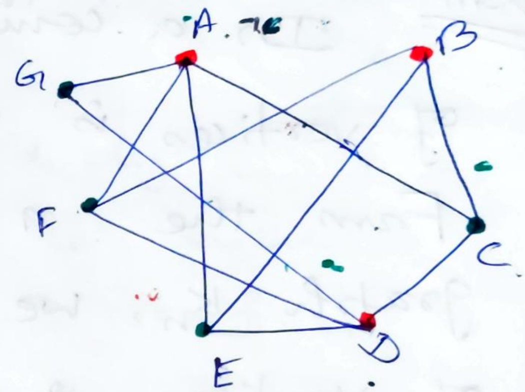


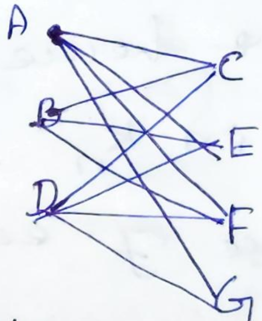
It is a bipartite graph. then it is a complete bipartite graph.

4

v)







It is a bipartite graph, but it is not complete bipartite graph

(5) Verify the handshaking theorem of the complete graph with n vertices. Verify also that the number of odd vertices in this graph is even. Also find the ratio of the number of edges to that of vertices (called the Beta index) for this graph.

Soln In a complete graph, every pair of vertices is connected by an edge.

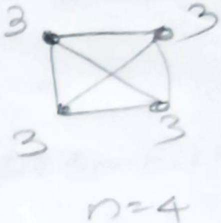
From the n vertices of the complete graph K_n, we can choose ${}^nC_2$ pairs of vertices & hence there are ${}^nC_2$ edges in K_n.

Also the degree of each of the n vertices $= n-1$

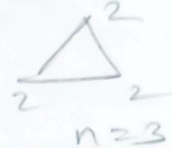
$$\therefore \sum_{i=1}^{n} \deg(v_i) = n(n-1)$$
$$= 2 \cdot {}^nC_2$$
$$= 2 \cdot \text{number of edges.}$$

Thus, the Handshaking theorem is verified.

Now, if n is even, the degree of each of these n vertices is $(n-1)$, i.e. odd. the number of odd degree vertices is even.



If n is odd, the degree of each of these n vertices is $(n-1)$, that is even.



∴ the number of odd degree vertices is zero i.e. even.

Thus the property is verified.

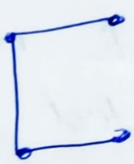
$$\text{Beta index } (\beta) = \frac{{}^{n}C_2}{n} = \frac{n(n-1)}{2n}$$

$$= \frac{n-1}{2}$$

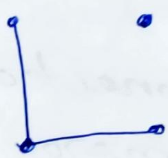
Connected graph :-

A Graph G is connected if for every pair of distinct Vertices $u, v \in V(G)$ there is a path from u to v in G
∃ at least one path

If there is no path from Vertex to another

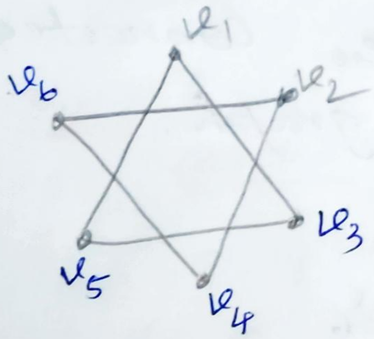


connected

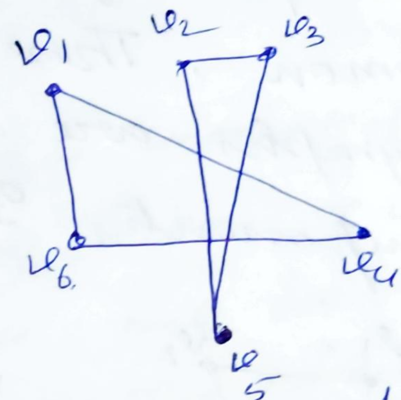


disconnected

disconnected.



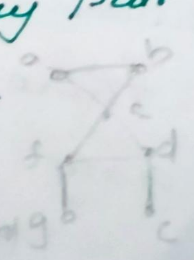
Disconnected

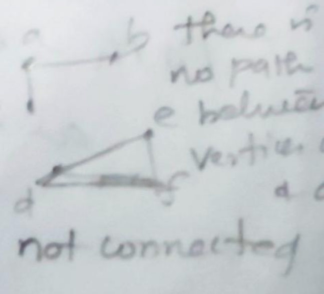


Disconnected.

A Graph that is not connected is called disconnected.

* An undirected Grap is called Connected if there is a path between every pair of distinct vertices of the graph.



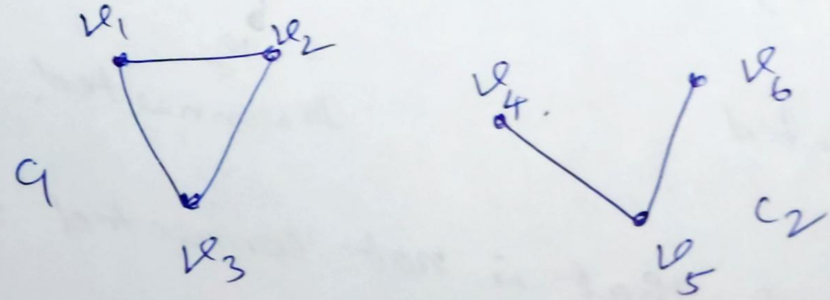


Note :-

1. Any graph with isolated vertic
is a disconnected graph.

2. A null graph is totally
disconnected.

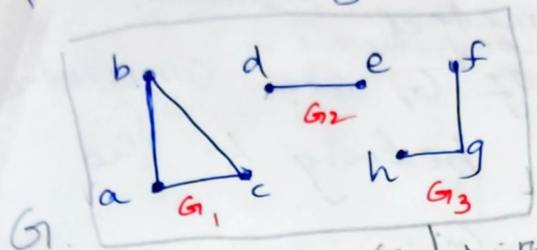
Clearly a disconnected graph is the union of two or more connected subgraphs each pair of which has no vertex in common. These disjoint connected subgraphs are called the connected components of the graph.



$G_3(V_3, E_3)$ is not connected as no path between v_3, v_4, v_2, v_4, v_1, v_4 & also there is no path between v_5, v_6 & any of v_1, v_2, v_3.

$G_3 = G_1 \cup G_2$

In G_3, G_1 & G_2 are the connected components.



A Graph G that is not connected has two or more connected components that are disjoint & have G as their union.

G ↳ is the union of 3 disjoint connected subgraphs. G_1, G_2, G_3 are the connected components of these subgraphs.

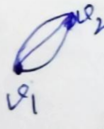
Theorem :-

If a Graph G (either connected G or not) has exactly two vertices of odd degree, there is a path joining these two vertices.

proof :-

case(i) :- Let G be connected

Let v_1, v_2 be the only vertices of G which are of odd degree. but we have already proved that the number of odd degree is even.



odd vertices of

clearly there is a path connecting v_1 & v_2. since G is connected.

case (ii) :-

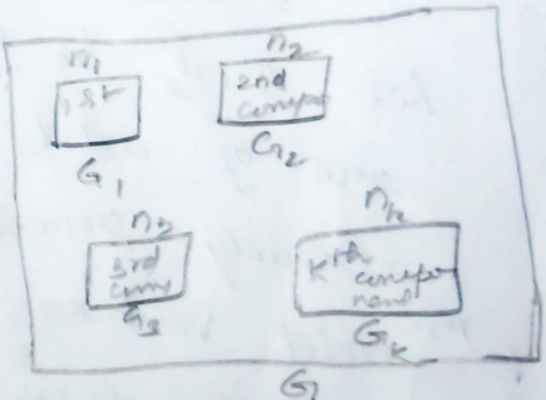
Let G be disconnected then G is a union of two or more components. Then the components of G are connected. Hence, v_1 & v_2 should belong to the same component of G.

Hence there is a path between v_1 & v_2.

Theorem 2 :-

The maximum number of edges in a simple disconnected graph G with n vertices & k components is

$$\frac{(n-k)(n-k+1)}{2}.$$



proof :-

Let $G_1, G_2, \ldots G_k$ be the k components of the disconnected graph with vertices $n_1, n_2, \ldots n_k$. Note that each G_i $(i=1$ to $k)$ itself is connected.

then, for 11, 22, 28, 43, 53, 55, 56, 62, 63, 79

$n_1 + n_2 + \cdots + n_k = n$ or

$$\sum_{i=1}^{k} n_i = n \quad \text{— (A)}$$

subtract 1 with each term in L.H.S (k times) so R.H.S also subtract k

i.e. $n_1 - 1 + n_2 - 1 + \cdots (n_k - 1) = n - k$

i.e. $\displaystyle\sum_{i=1}^{k} (n_i - 1) = n - k$.

$(a+b+c+\cdots)^2 = a^2+b^2+c^2+\cdots + 2(ab+bc+ac+\cdots)$

squaring on both sides

i.e. $\displaystyle\sum_{i=1}^{k} (n_i - 1)^2 + 2\sum_{i \neq j} (n_i - 1)(n_j - 1) = n^2 + k^2 - 2nk$.

i.e. $\displaystyle\sum_{i=1}^{k} (n_i - 1)^2 \leq n^2 - 2nk + k^2$

$(n_i - 1) \geq 0$
$(n_j - 1) \geq 0$
$n_i \geq 1$

as $\displaystyle\sum_{i \neq j} (n_i - 1)(n_j - 1) \geq 0$ → non negative terms

i.e. $\displaystyle\sum_{i=1}^{k} (n_i^2 - 2n_i + 1) \leq n^2 - 2nk + k^2$

i.e. $\displaystyle\sum_{i=1}^{k} n_i^2 - 2\sum_{i=1}^{k} n_i + \sum_{i=1}^{k} 1 \leq n^2 - 2nk + k^2$

i.e. $\displaystyle\sum_{i=1}^{k} n_i^2 - 2n + k \leq n^2 - 2nk + k^2$

ie $\sum_{i=1}^{k} n_i^2 \le n^2 - 2nk + k^2 + 2n - k$ — (1)

We know that Maximum no of edges in a complete graph with n vertices is $\frac{n(n-1)}{2}$.

Now the maximum number of edges in the ith component of $G = \frac{1}{2} n_i (n_i - 1)$

$\therefore$ Maximum number of edges in G_i are $= \frac{1}{2} \sum_{i=1}^{k} n_i (n_i - 1)$

Maximum no. of edges in G = Max no. of edges in $G_1 + G_2 + \cdots G_k$ $= \frac{n_1(n_1-1)}{2} + \frac{n_2(n_2-1)}{2} + \frac{n_k(n_k-1)}{2}$ $= \frac{n_1^2 + n_2^2 + \cdots}{2} - (n_1 + n_2 + \cdots)$

$$= \frac{1}{2} \sum_{i=1}^{k} n_i^2 - \frac{1}{2} \sum_{i=1}^{k} n_i \quad \text{— (2)}$$

$$= \frac{k}{2} \sum_{i=1}^{k} n_i^2 - \frac{1}{2} n \quad ; \text{by (A)}.$$

$$\le \frac{1}{2}(n^2 - 2nk + k^2 + 2n - k) - \frac{n}{2} \quad \text{From (1)}.$$

$$\le \frac{1}{2}\left[n^2 - 2nk + k^2 + 2n - k - n\right]$$

$$\le \frac{1}{2}\left[n^2 - 2nk + k^2 + n - k\right]$$

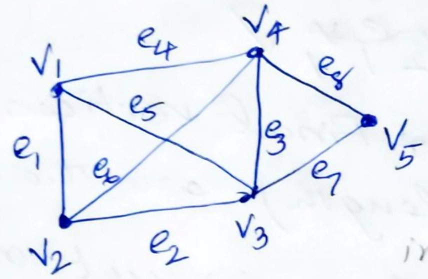
$$\le \frac{1}{2}\left((n-k)^2 + (n-k)\right)$$

$$\le \frac{1}{2}(n-k)(n-k+1)$$

Definition :-

path : A path in a graph is a finite alternating sequence of vertices & edges beginning & ending with vertices, such that each edge is incident on the vertices preceding & following it. (not necessarily distinct)

$P : v_1 e_1 v_2 e_2 v_3 e_5 v_1 e_1 v_2$



is a path joining v_1 to v_2, v_1 is called initial vertex & v_2 is called the final vertex of the path

Simple path :- If the edges in a path are distinct it is called a simple path.

$v_1 e_4 v_4 e_6 v_2 e_2 v_3 e_7 v_5$ is a simple path, as no edge appears more than once.

* The number of edges in a path (simple or general) is called the lengths of the path.

The length of both the paths given above is 4.

Circuit or cycle :-

If the initial & final vertices of a simple path of non zero length are the same, the simple path is called a simple circuit or a simple cycle.

$\cancel{v_1 e_1 v_2 e_2 v_3 e_3 v_4 e_6 v_2 e_1 v_1}$ ✓

If the initial & final vertices of a path (of non-zero length) are the same the path is called a circuit or cycle (or) closed path.

$v_1 e_5 v_3 e_7 v_5 e_8 v_4 e_4 v_1$ is a simple circuit of length 4.

$v_1 e_1 v_2 e_2 v_3 e_3 v_4 e_6 v_2 e_1 v_1$ is a circuit of length 5.

Eulerian & Hamiltonian graphs :-

(Vertices may be repeated)

A path of graph G is called an **Eulerian path**, if it includes each edge of G exactly once.

A circuit of a graph G is called an Eulerian circuit, if it includes each edge of G exactly once.

A graph containing an Eulerian circuit is called an **Eulerian graph**.

Eg :-

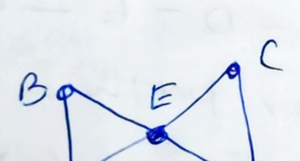
G_1

G_1 contains an Eulerian path between B & D

path between B – A – D

B – D – C – B – A – D

since it includes each of the edges exactly once.



G_2

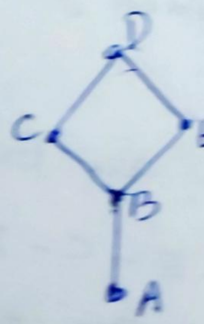
Graph G_2 contains an Eulerian circuit namely A – E – C – D – E – B – A since it includes each of the edges exactly once.

G_2 is an Euler graph, as it contains an Eulerian circuit.

Note : Necessary & Sufficient condition for Euler circuits & paths :-

* An Euler path visits every edge once.
* An Euler circuit or cycle visits every edge once & begins & ends at the same vertex.
* An Euler circuit is possible if every vertex has Even degree.
* An Euler path is possible, but an Euler circuit is not possible, if exactly two vertices have odd degree.
* No Euler path is possible if more than two vertices have an odd degree.

(1)



Euler path : A – B – C – D – E – B

B – C – D – E – B – A.

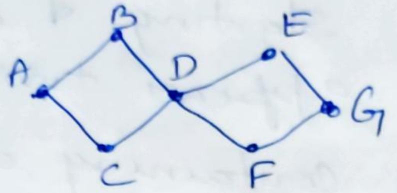
G_1 have Euler path.

Do not have Euler circuit.

∴ deg A = 1, deg B = 3.

Exactly two vertices have odd degree.

∴ Not Eulerian graph.

(2) 

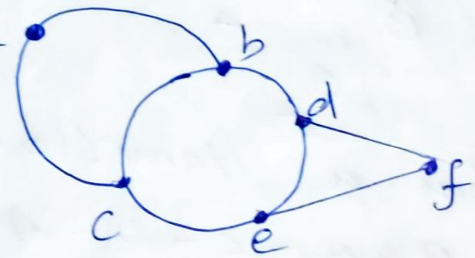
G_2

Euler path
A – B – D – E – G – F – D – C – A.

(i) path
(ii) ~~E~~ Euler circuit.

∴ Eulerian graph.

All vertices are of even degree, ∴ Euler circuit is possible.

(iii) 

G_3

Not Euler path
" Euler circuit.
∴ G_3 is not Eulerian graph.

Since all vertices are not even degree
More than two vertices have odd degree.

(Edges may be repeated)

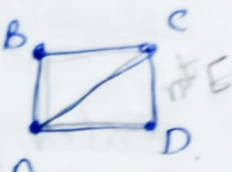
Hamiltonian Graph :-

A path of a graph G is called a Hamiltonian path, if it includes each vertex of G exactly once.
except the terminal vertices.

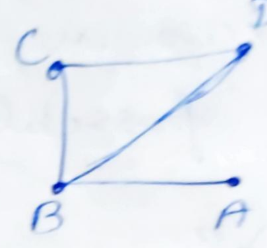
A circuit of a graph G is called a Hamiltonian circuit if it includes each vertex of G exactly once except the starting & end vertices which appears twice.

A Graph containing a Hamiltonian circuit is called a Hamiltonian graph.

For ex :-



G_1



G_2

The graph G_1 has a Hamiltonian circuit namely A - B - C - D - A.

In this circuit all the vertices appear only one. but not all edges.

The graph G_2 has a Hamiltonian path namely A - B - C - D, but not a hamiltonian circuit.

Note :-

* From the above graph, it is clear that the path obtained by deleting any one edge from a Hamiltonian circuit is a Hamiltonian path.

* Also, a Hamiltonian circuit contains a Hamiltonian path, but a graph containing a Hamiltonian path need not have a Hamiltonian circuit.

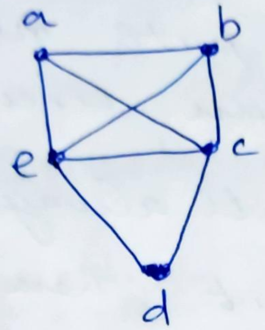
* A complete Graph K_n will always have a Hamiltonian circuit, when $n \geq 3$, due to the fact that an edge exists between any two vertices & a circuit can be formed by beginning at any vertex & by visting the remaining vertices in any order.

* A given graph may contain more than one Hamiltonian circuit.

* If any vertices having degree one, it is not Hamiltonian graph.

Example :-

(1)



Hamilton path

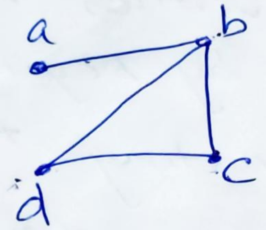
a – b – c – d – e

Hamilton circuit

a – b – c – d – e – a

Hamiltonian graph.

(2).



Hamilton path.

a – b – c – d

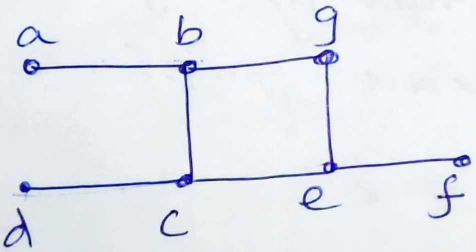
a – b – d – c

Hamilton circuit

a – b – c – d. does not exists.

∴ It is not Hamiltonian graph.

(3).



Hamilton path.

a – b – g – e – c – d.

d – c – e – f

Graph donot contain any Hamilton path.

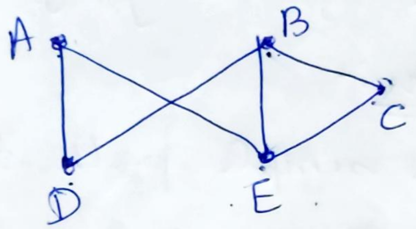
do not contain any Hamilton circuit

" " " Hamiltonian graph.

Neither a Hamiltonian circuit nor a Hamilton Path.

problems :-

1. Find which of the following vertex sequences are simple paths, paths, closed path (circuits) & simple circuit.



a) A – D – E – B – C
b) A – D – B – C – E
c) A – E – C – B – E – A
d) C – B – D – A – E – C
e) A – B – B – E – C – B

a) A – D – E – B – C is not a path, since DE is not an edge of the given graph

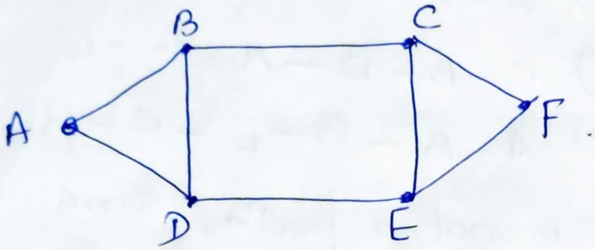
b) A – D – B – C – E — is a simple path between the vertices A & E. Since the vertices & edges involved are distinct.

c) A – E – C – B – E – A — is a closed path (circuit) since the initial & final vertices are the same & the vertex E appears twice.

d) C – B – D – A – E – C — is a simple circuit since the initial & final vertices are the same & the vertices & edges are distinct.

e) A – D – B – E – C – B – is a path (but not a simple path) as the vertex B appears twice.

(2). Find all the simple paths from A to F & all the circuits in the following graph.



Ans :-

The simple path from A to F are the

(i) A – B – C – F

(ii) A – D – E – F.

(iii) A – B – D – E – F

(iv) A – D – B – C – F

(v) A – B – C – E – F

(vi) A – D – E – C – F

(vii) A – B – D – E – C – F

8) A – D – B – C – E – F.

The circuit in the graph

1. A – B – D – A

2. C – F – E – C

3. B – C – E – D – B

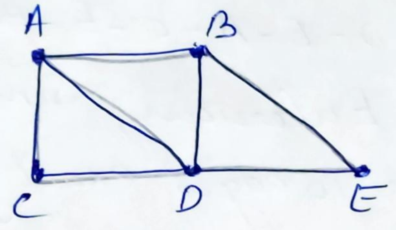
4. A – B – C – E – D – A

5. B – C – F – E – D – B

6. A – B – C – F – E – D – A.

(3) Find an Euler path or an Euler circuit, if it exists in each of the three graphs. If it does not exist, Explain why?

Soln



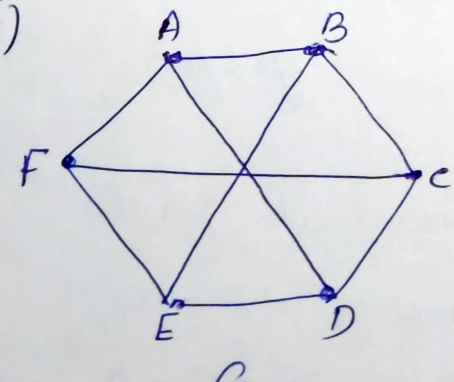
G_1

In G_1, there are only two vertices, namely A & B of degree 3 & other vertices are of even degree.

Hence there exists an Euler path between A & B. The actual path is A – B – E – D – A – C – D – B

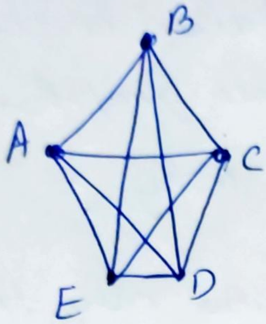
This is an Eulerian path, as it includes each of the 7 edges exactly once.

(ii)



G_2

In G_2, there are 6 vertices of odd degree. Hence, G_2 contains neither an Euler path nor an Euler circuit.



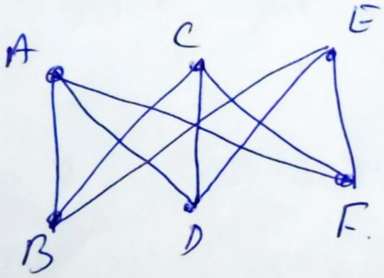
G_3

In G_3, all the vertices are of even degree. Hence there exists an Euler circuit in G_3.

It is A-B-C-D-E-A-C-E-B-D-A

This circuit is Eulerian, Since it includes each of the 10 edges exactly once.

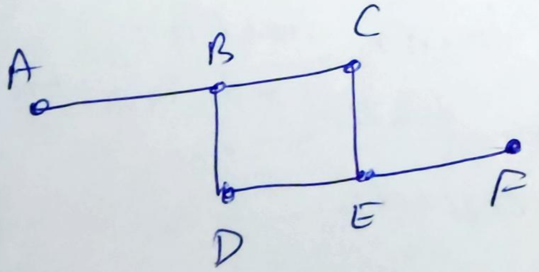
(4) Find a Hamiltonian path or a Hamiltonian circuit if it exists in each of the graph. If it does not exist, explain why?



G_1 contains Hamiltonian circuit

A-B-C-D-E-F-A.

Hamiltonian graph.

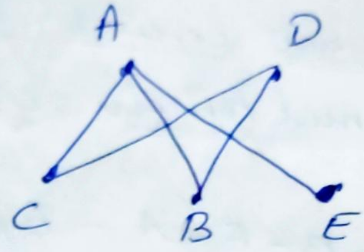


Hamiltonian path

A-B-D-E-F-C

A-B-C-E-F-D.

G_2 contains neither a Hamiltonian path nor a Hamiltonian circuit.



Hamiltonian path

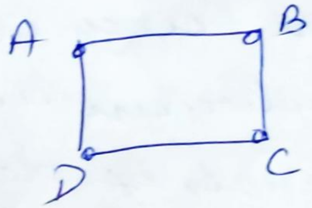
E – A – C – D – B.

E – A – B – D – C

but no Hamiltonian circuit.

⑤ Give an Example of a graph which contains:

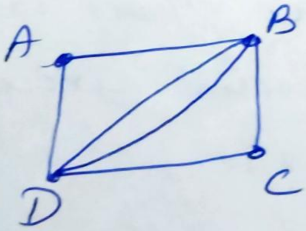
(i) an Eulerian circuit that is also a Hamiltonian circuit :-



A – B – C – D – A – Euler circuit

A – B – C – D – A – Hamiltonian circuit.

(2) an Eulerian circuit & Hamiltonian circuit that are distinct :-

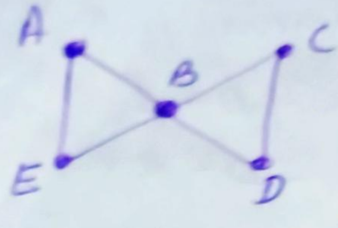


A – B – D – B – C – D – A

Eulerian circuit

A – B – C – D – A

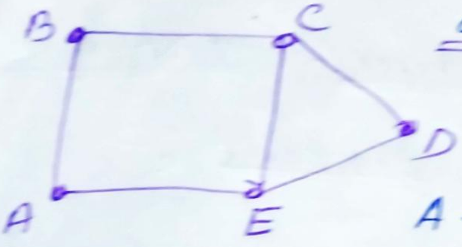
– Hamiltonian circuit.

(3). 

an Eulerian circuit, but not a Hamiltonian circuit

A – B – C – D – B – E – A – Eulerian circuit

But this circuit is not Hamiltonian as vertex B is repeated twice.

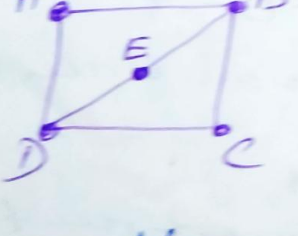
(4) 

a Hamiltonian circuit but not a Eulerian circuit

A – B – C – D – E – A is a Hamiltonian circuit.

But it does not contain Eulerian circuit as there are 4 vertices each of degree 3.

(5) Neither an Eulerian circuit nor a Hamiltonian circuit :-



deg B = 3
deg D = 3

Hence there is no Euler circuit in it.

Also no circuit passes through each of the vertices exactly once.