The Z-transforms

Definition of the Z-Transform

Let $\{x(n)\}$ be a sequence defined for $n = 0, \pm 1, \pm 2, \ldots$

Then the two sided Z-transform of the sequence $x(n)$ is defined as

$$Z\{x(n)\} = X(z) = \sum_{n=-\infty}^{\infty} x(n) \cdot z^{-n}.$$

where z is a complex variable in general.

If $\{x(n)\}$ is a casual sequence, ie, $x(n) = 0$ for $n < 0$, then the Z-transform reduces to one-sided Z-transform. and its definition is

$$Z\{x(n)\} = X(z) = \sum_{n=0}^{\infty} x(n) \cdot z^{-n}.$$

Unit Sample Sequence:

The unit sample sequence $\delta(n)$ is defined as the sequence with values

$$\delta(n) = \begin{cases} 1 & \text{for } n = 0 \\ 0 & \text{for } n \neq 0 \end{cases}$$

The unit step sequence $u(n)$ has values

$$u(n) = \begin{cases} 1 & \text{for } n \geq 0 \\ 0 & \text{for } n < 0 \end{cases}$$

and

$$\delta(n-k) = \begin{cases} 1 & \text{if } k = n \\ 0 & \text{if } k \neq n \end{cases}$$

$$u(n-k) = \begin{cases} 1 & \text{for } (n-k) \geq 0 \\ 0 & \text{for } (n-k) < 0 \end{cases}$$

Definition:-

If $f(t)$ is a function defined for discrete values of t where $t = nT$, $n = 0, 1, 2, 3, \ldots,$ to ∞, T being the sampling period, then Z-transform of $f(t)$ is defined as

$$Z[f(t)] = F(z) = \sum_{n=0}^{\infty} f(nT) z^{-n}.$$

Properties and Theorems of Z-Transform.

Notations: $Z[f(t)] = F(z)$ and

$Z[x(n)] = X(z)$.

Theorem: 1

The Z-transform is linear. That is

$$Z[af(t) + bg(t)] = aZ[f(t)] + bZ[g(t)]$$

Proof:-

$$Z[af(t) + bg(t)] = \sum_{n=0}^{\infty} [af(nT) + bg(nT)] z^{-n}$$

$$= a\sum_{n=0}^{\infty} f(nT) z^{-n} + b\sum_{n=0}^{\infty} g(nT) z^{-n}$$ (by definition)

$$= aF(z) + bG(z)$$

$$= aZ[f(t)] + bZ[g(t)]$$

(or)

$$Z[a\{x(n)\} + b\{y(n)\}] = \sum_{n=0}^{\infty} [ax(n) + by(n)] z^{-n}$$

$$= a\sum_{n=0}^{\infty} x(n) z^{-n} + b\sum_{n=0}^{\infty} y(n) z^{-n}$$ (by defn).

$$= aX(z) + bY(z)$$

$$= aZ\{x(n)\} + bZ\{y(n)\}$$

Theorem : 2

$$Z[\delta(n)] = \sum_{n=\infty}^{\infty} \delta(n) z^{-n}$$

$$= 1 \quad \text{(by the definition of unit sample sequence)}$$

Theorem : 3

$$Z[u(n)] = \sum_{n=-\infty}^{\infty} u(n) z^{-n}$$

$$= \sum_{n=0}^{\infty} z^{-n} \quad \text{(by the definition of unit step sequence)}$$

$$= \sum_{n=0}^{\infty} 1/z^{n}$$

$$= 1 + 1/z + 1/z^2 + \cdots$$

$$= (1 - 1/z)^{-1} \quad \text{if } |1/z| < 1$$

$$= \left(\frac{z-1}{z}\right)^{-1} \quad \text{if } 1 < |z|$$

$$= \frac{z}{z-1} \quad \text{if } |z| > 1$$

[we know that $(1-x)^{-1} = 1 + x + x^2 + \cdots$].

Theorem: 4 If $Z\{f(t)] = F(z)$

then $Z[a^n f(t)] = F(z/a)$

Proof:

$$Z[a^n f(t)] = \sum_{n=0}^{\infty} a^n f(nT) \cdot z^{-n} \quad \text{(by defn.)}$$

$$= \sum_{n=0}^{\infty} f(nT) \cdot (z/a)^{-n}$$

$$= F(z/a)$$

(or)

If $Z\{x(n)\} = X(z)$ then $Z\{a^n x(n)\} = X(z/a)$

Proof:

$$Z[a^n x(n)] = \sum_{n=0}^{\infty} a^n x(n) z^{-n}$$

$$= \sum_{n=0}^{\infty} x(n) (z/a)^{-n}$$

$$= X(z/a).$$

Note: Theorem 4: we can say "Dampling rule."

Theorem: 5

$$Z[a^n u(n)] = \frac{z}{z-a} \quad \text{if } |z| > a.$$

Proof:

$$Z[a^n u(n)] = \sum_{n=0}^{\infty} a^n u(n) z^{-n}$$

$$= \sum_{n=0}^{\infty} a^n z^{-n} \quad \text{(by defn. of unit step seq.)}$$

$$= \sum_{n=0}^{\infty} (a/z)^n$$

$$= 1 + a/z + (a/z)^2 + \cdots$$

$$= (1 - a/z)^{-1} \quad \text{if } |a/z| < 1$$

$$= \left(\frac{z-a}{z}\right)^{-1} \quad \text{if } |a| < |z|$$

$$= \frac{z}{z-a} \quad \text{if } |z| > |a|.$$

Theorem : 6

$$Z[n f(t)] = -z \frac{dF(z)}{dz}$$

Proof :-

$$F(z) = Z[f(t)] = \sum_{n=0}^{\infty} f(nT) z^{-n} \quad (\text{defn.})$$

$$\frac{dF(z)}{dz} = \frac{d}{dz} \sum_{n=0}^{\infty} f(nT) z^{-n}$$

$$= \sum_{n=0}^{\infty} f(nT)(-n) z^{-n-1}$$

$$= -\frac{1}{z} \underbrace{\sum_{n=0}^{\infty} n f(nT) z^{-n}}$$

$$= \frac{-1}{z} Z[n f(t)]$$

$$\therefore -z \frac{dF(z)}{dz} = Z[n f(t)]$$

(ie). $Z[n f(t)] = -z \dfrac{dF(z)}{dz}$

$$= 1 + a/z + (a/z)^2 + \cdots$$

$$= (1 - a/z)^{-1} \quad \text{if } |a/z| < 1$$

$$= \left(\frac{z-a}{z}\right)^{-1} \quad \text{if } |a| < |z|$$

$$= \frac{z}{z-a} \quad \text{if } |z| > |a|.$$

Theorem : 6

$$Z[n f(t)] = -z \frac{dF(z)}{dz}$$

Proof :-

$$F(z) = Z[f(t)] = \sum_{n=0}^{\infty} f(nT) z^{-n} \quad \text{(defn.)}$$

$$\frac{dF(z)}{dz} = \frac{d}{dz} \sum_{n=0}^{\infty} f(nT) z^{-n}$$

$$= \sum_{n=0}^{\infty} f(nT)(-n) z^{-n-1}$$

$$= -\frac{1}{z} \underbrace{\sum_{n=0}^{\infty} n f(nT) z^{-n}}$$

$$= \frac{-1}{z} Z[n f(t)]$$

$$\therefore -z \frac{dF(z)}{dz} = Z[n f(t)]$$

(ie). $Z[n f(t)] = -z \dfrac{dF(z)}{dz}$

Theorem: 7

$$Z[f(t+kT)] = Z[f((n+k)T)] =$$

$$z^k \left[F(z) - f(0.T) - \frac{f(1.T)}{z} - \frac{f(2.T)}{z^2} - \dots - \frac{f((k-1)T)}{z^{k-1}} \right]$$

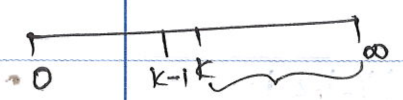
Proof: $$Z[f((k+n)T)] = \sum_{n=0}^{\infty} f((n+k)T) z^{-n} \quad \text{(by defn.)}$$

$$= \sum_{n=0}^{\infty} f((n+k)T) \cdot z^{-n} z^{k} \cdot z^{-k}$$

$$= z^k \sum_{n=0}^{\infty} f((n+k)T) z^{-(n+k)}$$

Put $n+k=m$.

$$= z^k \sum_{m=k}^{\infty} f(mT) z^{-m}$$



$$= z^k \left[\underbrace{\sum_{m=0}^{\infty} f(mT) z^{-m}} - \sum_{m=0}^{k-1} f(mT) z^{-m} \right]$$

$$= z^k \left[F(z) - f(0) - \frac{f(T)}{z} - \frac{f(2T)}{z^2} - \dots - \frac{f((k-1)T)}{z^{k-1}} \right]$$

Note:

If $f((n+k)T)$ is denoted by f_{n+k}, then

$$Z[f(t+kT)] = Z[f_{n+k}]$$

$$= z^k \left[F(z) - f_0 - \frac{f_1}{z} - \frac{f_2}{z^2} - \dots \frac{f_{k-1}}{z^{k-1}} \right]$$

Z- Transform of Standard Functions

Find the Z- transform of sequence $\{x(n)\}$ or $\{f_n\}$.

where $x(n)$ is given by

(1) $x(n) = k$	(6) $x(n) = \sin n\theta$	(11) $x(n) = n^2$
(2) $x(n) = (-1)^n$	(7) $x(n) = \cos n\theta$	
(3) $x(n) = a^n$	(8) $x(n) = r^n \sin n\theta$	
(4) $x(n) = n$	(9) $x(n) = r^n \cos n\theta$	
(5) $x(n) = na^n$	(10) $x(n) = n(n-1)$	

$$Z[x(n)] = \sum_{n=0}^{\infty} x(n) z^{-n}.$$

(1)
$$Z[k] = \sum_{n=0}^{\infty} k z^{-n} = k \sum_{n=0}^{\infty} z^{-n}$$

$$= k \sum_{n=0}^{\infty} 1/z^n$$

$$= k\left[1 + 1/z + 1/z^2 + \cdots\right]$$

$$= k\left[1 - 1/z\right]^{-1} \text{ if } |1/z| < 1$$

$$= k\left[\frac{z-1}{z}\right]^{-1} \text{ if } 1 < |z|$$

$$= \frac{kz}{z-1} \text{ if } |z| > 1.$$

Note:-

$$Z[1] = \frac{z}{z-1} \text{ if } |z| > 1$$

2) $Z\{(-1)^n\} = \sum_{n=0}^{\infty} (-1)^n z^{-n} = \sum_{n=0}^{\infty} (-1/z)^n$

$= 1 - 1/z + (1/z)^2 - (1/z)^3 + \cdots$

$= (1 + 1/z)^{-1}$ if $|1/z| < 1$

$= \left(\frac{z+1}{z}\right)^{-1}$ if $1 < |z|$.

$= \frac{z}{z+1}$ if $|z| > 1$

Note: $(1+x)^{-1} = 1 - x + x^2 - x^3 + \cdots$ if $|x| < 1$.

3) $Z[a^n] = \sum_{n=0}^{\infty} a^n z^{-n} = \sum_{n=0}^{\infty} (a/z)^n$

$= 1 + a/z + (a/z)^2 + \cdots$

$= (1 - a/z)^{-1}$ if $|a/z| < 1$

$= \left(\frac{z-a}{z}\right)^{-1}$ if $|a| < |z|$

$= \frac{z}{z-a}$ if $|z| > |a|$.

4) $Z[n] = \sum_{n=0}^{\infty} n z^{-n} = \sum_{n=0}^{\infty} n/z^n$

$= \frac{1}{z} + \frac{2}{z^2} + \frac{3}{z^3} + \frac{4}{z^4} + \cdots$

$= \frac{1}{z}\left[1 + \frac{2}{z} + \frac{3}{z^2} + \frac{4}{z^3} + \cdots\right]$

$$= \frac{1}{z}\left[1 + 2\left(\frac{1}{z}\right) + 3\left(\frac{1}{z}\right)^2 + 4\left(\frac{1}{z}\right)^3 + \cdots\right]$$

$$= \frac{1}{z}\left[1 - \frac{1}{z}\right)^{-2} \quad \text{if } \left|\frac{1}{z}\right| < 1$$

$$= \frac{1}{z}\left[\frac{z-1}{z}\right)^{-2} \quad \text{if } 1 < |z|.$$

$$= \frac{1}{z}\left[\frac{z}{z-1}\right]^2 \quad \text{if } |z| > 1$$

$$= \frac{1}{z}\left[\frac{z^2}{(z-1)^2}\right] \quad \text{if } |z| > 1$$

$$= \frac{z}{(z-1)^2} \quad \text{if } |z| > 1$$

Note :- $(1-x)^{-2} = 1 + 2x + 3x^2 + 4x^3 + \cdots$

if $|x| < 1$.

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(5) $Z[na^n] = \sum_{n=0}^{\infty} n a^n z^{-n}$

$$= \sum_{n=0}^{\infty} n \left(\frac{a}{z}\right)^n$$

$$\Rightarrow \frac{a}{z} + 2\left(\frac{a}{z}\right)^2 + 3\left(\frac{a}{z}\right)^3 + \cdots$$

$$= \frac{a}{z}\left[1 + 2\left(\frac{a}{z}\right) + 3\left(\frac{a}{z}\right)^2 + \cdots\right]$$

$$= \frac{a}{z}\left[\left(1 - \frac{a}{z}\right)^{-2}\right] \quad \text{if } \left|\frac{a}{z}\right| < 1$$

$$= \frac{a}{z}\left[\left(\frac{z-a}{z}\right)^{-2}\right] \quad \text{if } |a|<|z|$$

$$= \frac{a}{z}\left[\left(\frac{z}{z-a}\right)^{2}\right] \quad \text{if } |z|>|a|.$$

$$= \frac{a}{z}\left(\frac{z^2}{(z-a)^2}\right) \quad \text{if } |z|>|a|$$

$$= \frac{az}{(z-a)^2} \quad \text{if } |z|>|a|.$$

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(6) & (7) $\Rightarrow$.

we know that $Z[a^n] = \dfrac{z}{z-a}$ if $|z|>|a|$.

Taking $a = e^{i\theta}$

$$Z[(e^{i\theta})^n] = \frac{z}{z-e^{i\theta}}$$

$$Z[(e^{in\theta})] = \frac{z}{z-e^{i\theta}}.$$

$$Z[\cos n\theta + i\sin n\theta] = \frac{z}{z-(\cos\theta + i\sin\theta)}$$

$$= \frac{z}{(z-\cos\theta)-i\sin\theta} \times \frac{z-\cos\theta+i\sin\theta}{(z-\cos\theta)+i\sin\theta}$$

$$= \frac{z[z-\cos\theta+i\sin\theta]}{(z-\cos\theta)^2-(i\sin\theta)^2}$$

$$= \frac{z(z - \cos\theta + i\sin\theta)}{z^2 + \cos^2\theta - 2z\cos\theta + \sin^2\theta}$$

$$= \frac{z(z-\cos\theta) + iz\sin\theta}{z^2 - 2z\cos\theta + 1}$$

Equating real and imaginary parts, we get

$$Z[\cos n\theta] = \frac{z[z-\cos\theta]}{z^2 - 2z\cos\theta + 1} \quad \text{if } |z|>1$$

$$Z[\sin n\theta] = \frac{z\sin\theta}{z^2 - 2z\cos\theta + 1} \quad \text{if } |z|>1.$$

Note:

$|z| > |e^{i\theta}|$

$|z| > |\cos\theta + i\sin\theta|$

$|z| > \sqrt{\cos^2\theta + \sin^2\theta}$

$|z| > 1.$

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(8) & (9) $\Rightarrow$:-

$$\underbrace{x(n) = r^n \sin n\theta \quad \& \quad x(n) = r^n \cos n\theta.}_{\text{Exercise:}}$$

Note:

$$Z[a^n] = \frac{z}{z-a} \quad \text{if } |z|>|a|$$

Put $a = re^{i\theta}$

$$Z[r^n \cos n\theta] = \frac{z(z - r\cos\theta)}{z^2 - 2zr\cos\theta + r^2} \quad \text{if } |z| > |r|$$

$$Z[r^n \sin n\theta] = \frac{zr\sin\theta}{z^2 - 2zr\cos\theta + r^2} \quad \text{if } |z| > |r|$$

$\Big\}$ Answer :-

(10) $Z[n^2] = Z[n \cdot n] =$

WKT $Z[n f(t)] = -z \frac{d}{dz} F(z)$

$$= -z \frac{d}{dz}\left[Z[f(t)]\right] = -z \frac{d}{dz}\left[Z(n)\right]$$

$$= -z \frac{d}{dz}\left(\frac{z}{(z-1)^2}\right) \quad u/v$$

$$= -z\left[\frac{(z-1)^2(1) - z(2(z-1)(1))}{(z-1)^4}\right] \quad (2z-2)$$

$$= -z\left[\frac{z^2+1-\cancel{2z}-2z^2+\cancel{2z}}{(z-1)^4}\right]$$

$$= -z\left[\frac{-z^2+1}{(z-1)^4}\right]$$

$$= +z\left[\frac{z^2-1}{(z-1)^4}\right]$$

$$= z\left(\frac{(z+1)\cancel{(z-1)}}{(z-1)^{\cancel{4}}}\right) = \frac{z(z+1)}{(z-1)^3}.$$

(11) $$Z[n(n-1)] = Z(n^2-n)$$
$$= Z(n^2) - Z(n)$$
$$= \frac{z(z+1)}{(z-1)^3} - \frac{z}{(z-1)^2}$$
$$= \frac{z(z+1) - z(z-1)}{(z-1)^3}$$
$$= \frac{\cancel{z^2}+z-\cancel{z^2}+z}{(z-1)^3} = \frac{2z}{(z-1)^3}.$$

Find the Z-transform of :

① $\{1/n\}$

② $\left\{\cos\frac{n\pi}{2}\right\}$

③ $\left\{\frac{1}{n(n+1)}\right\}, n\geq 1$

④ $u(n-1)$

⑤ $3^n\,\delta(n-1)$

⑥ $\cos\frac{n\pi}{2}\,u(n)$

⑦ $\delta(n-k)$

⑧ $ab^n\ (a\neq 0, b\neq 0)$

⑨ $x(n)=\begin{cases} n, & n\geq 0 \\ 0 & n<0\end{cases}$

⑩ $\frac{(n+1)(n+2)}{2}$

⑪ $x(n)=\begin{cases} 1 & \text{for } n=k \\ 0 & \text{otherwise}\end{cases}$

⑫ $x(n)=\begin{cases} 0 & \text{if } n>0 \\ 1 & \text{if } n\leq 0\end{cases}$

⑬ $x(n)=\begin{cases} \frac{a^n}{n!} & \text{for } n\geq 0 \\ 0 & \text{otherwise}\end{cases}$