SRM Institute of Science and Technology
Department of Mathematics
18MAB102T-Advanced Calculus and Complex Analysis
2020-2021 Even
Unit – I: Multiple Integrals
Tutorial Sheet - I

S.No.	Questions	Answers
Part – A [3 Marks]		
1	Evaluate $\int_0^3\int_0^2 xy(x+y)dxdy$	30
2	Evaluate $\int_2^4\int_1^2 \frac{dxdy}{xy}$	$(\log 2)^2$
3	Evaluate $\int_0^{\pi/2}\int_0^{2a\cos\theta} r dr d\theta$	$\frac{\pi}{8}$
4	Evaluate $\int_0^{\pi}\int_0^{a(1+\cos\theta)} r dr d\theta$	$\frac{3\pi a^2}{4}$
5	Change the order of integration $\int_0^3\int_1^{\sqrt{4-y}}(x+y)dxdy$	$\int_1^2\int_0^{4-x^2}(x+y)dydx$
Part – B [6 Marks]		
6	Evaluate $\int_0^1\int_0^{\sqrt{1+x^2}}\frac{dxdy}{1+x^2+y^2}$	$\frac{\pi}{4}\log(1+\sqrt{2})$
7	Evaluate $\int_0^{\pi/2}\int_{a(1-\cos\theta)}^{a} r^2 dr d\theta$	a^3
8	Change the order of integration $\int_0^a\int_y^a \frac{x}{x^2+y^2}dxdy$ and hence evaluate it	$\frac{\pi}{4}a$
9	Change the order of integration $\int_0^b\int_0^{\frac{a}{b}\sqrt{b^2-y^2}} xydxdy$ and hence evaluate it	$\frac{a^2b^2}{8}$
10	Change the order of integration and hence find the value of $\int_0^1\int_x^1 \frac{x}{x^2+y^2}dxdy$	$\frac{1}{2}\log 2$

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Branch: Common to ALL Branches of B.Tech. except B.Tech. (Business Systems)

UNIT II - VECTOR CALCULUS

Part – A

1.	If $\varphi(x,y,z) = xyz$, then $\nabla\varphi =$ (A) $yz\,\vec{i} + zx\,\vec{j} + xy\,\vec{k}$ (B) $xy\,\vec{i} + yz\,\vec{j} + xz\,\vec{k}$ (C) $xz\,\vec{i} + zy\,\vec{j} + xy\,\vec{k}$ (D) $x\,\vec{i} + y\,\vec{j} + z\,\vec{k}$	ANS **A**	(CLO-2, Apply)
2.	Curl ($grad\ \varphi$) = (A) $\vec{0}$ (B) 1 (C) 2 (D) – 1	ANS **A**	(CLO-2, Remember)
3.	The maximum directional derivative of $\varphi(x,y,z) = x^2 + y^2 + z^2$ at $(1, 1, 1)$ is (A) 0 (B) 3 (C) 2 (D) $2\sqrt{3}$	ANS **D**	(CLO-2, Apply)
4.	If $\vec{r}$ is the position vector of the point (x, y, z) with respect to the origin, then $\nabla \bullet \vec{r} =$ (A) 0 (B) 1 (C) 2 (D) 3	ANS **D**	(CLO-2, Apply)
5.	If $\vec{u}$ and $\vec{v}$ are irrotational, then $\vec{u} \times \vec{v}$ is (A) irrotational (B) solenoidal (C) zero vector (D) constant	ANS **B**	(CLO-2, Remember)
6.	The condition for $\vec{F}$ to be conservative is (A) $\nabla \bullet \vec{F} = 0$ (B) 0 (C) $\nabla \times \vec{F} = \vec{0}$ (D) 1	ANS **C**	(CLO-2, Remember)

7.	The relation between the surface integral and the volume integral is given by (A) Green's theorem (B) Stoke's theorem (C) Gauss Divergence theorem (D) Cauchy's theorem	ANS **C**	(CLO-2, Remember)
8.	By Stoke's theorem, $\int_C \vec{F} \bullet d\vec{r} =$ (A) $\iint_S \nabla \times \vec{F}\, dS$ (B) $\iint_S \nabla \bullet \vec{F}\, dS$ (C) $\iint_S (\nabla \bullet \vec{F})\, \hat{n}\, dS$ (D) $\iint_S (\nabla \times \vec{F}) \bullet \hat{n}\, dS$	ANS **D**	(CLO-2, Remember)
9.	The maximum value of the directional derivative is (A) $\lvert\nabla \varphi\rvert$ (B) curl φ (C) grad φ (D) $\lvert\nabla \times \varphi\rvert$	ANS **A**	(CLO-2, Remember)
10.	If $\vec{F}$ is irrotational, then $Curl\, \vec{F} =$ (A) 1 (B) 2 (C) 3 (D) $\vec{0}$	ANS **D**	(CLO-2, Apply)
11.	If the divergence of the vector is zero, then the vector is said to be (A) irrotational vector (B) constant vector (C) zero vector (D) solenoidal vector	ANS **D**	(CLO-2, Remember)
12.	The unit normal vector to the surface $x^2\, y + 2\, x\, z = 4$ at the point $(2, -2, 3)$ is (A) $-\frac{1}{3}\,\vec{\imath} + \frac{2}{3}\,\vec{\jmath} + \frac{2}{3}\,\vec{k}$ (B) $\frac{1}{3}\,\vec{\imath} + \frac{2}{3}\,\vec{\jmath} + \frac{2}{3}\,\vec{k}$ (C) $-\frac{1}{3}\,\vec{\imath} - \frac{2}{3}\,\vec{\jmath} + \frac{2}{3}\,\vec{k}$ (D) $\frac{1}{3}\,\vec{\imath} - \frac{2}{3}\,\vec{\jmath} - \frac{2}{3}\,\vec{k}$	ANS **A**	(CLO-2, Apply)
13.	If the vector $\vec{F} = (x + 3y)\,\vec{\imath} + (y - 2z)\,\vec{\jmath} + (x + a\,z)\,\vec{k}$ is solenoidal, then $a =$ (A) 2 (B) 0 (C) -2 (D) -1	ANS **C**	(CLO-2, Apply)
14.	The work done by the conservative force when it moves a particle around a closed curve is (A) $\nabla \bullet \vec{F} = 0$ (B) 0 (C) $\nabla \times \vec{F} = 0$ (D) $\nabla \bullet (\nabla \times \vec{F}) = 0$	ANS **C**	(CLO-2, Remember)

15.	The value of $\int_C \ x\,dy - y\,dx$ around the circle $x^2 + y^2 = 1$ is (A) π (B) $2\,\pi$ (C) $3\,\pi$ (D) 0	ANS **B**	(CLO-2, Apply)
16.	By Green's theorem, the area bounded by a simple closed curve is (A) $\int_C \ x\,dy - y\,dx$ (B) $\int_C \ x\,dy + y\,dx$ (C) $\int_C \ y\,dx - x\,dy$ (D) $\frac{1}{2}\left(\int_C \ x\,dy - y\,dx\right)$	ANS **D**	(CLO-2, Apply)
17.	To be conservative, $\vec{F}$ should be (A) solenoidal (B) irrotational (C) rotational (D) constant vector	ANS **B**	(CLO-2, Remember)
18.	The unit normal vector to the surface $x^2 + y^2 - z^2 = 1$ at the point $(1, 1, 1)$ is (A) $\frac{\vec{i}+\vec{j}+\vec{k}}{\sqrt{3}}$ (B) $\frac{\vec{i}+\vec{j}-\vec{k}}{\sqrt{3}}$ (C) $\frac{\vec{i}-\vec{j}-\vec{k}}{\sqrt{3}}$ (D) $\frac{\vec{i}+\vec{j}+\vec{k}}{\sqrt{2}}$	ANS **B**	(CLO-2, Apply)
19.	If $\vec{r}$ is the position vector of the point (x, y, z) with respect to the origin, then $div\,\vec{r} =$ (A) 0 (B) 1 (C) 2 (D) 3	ANS **D**	(CLO-2, Remember)
20.	If φ is a scalar function, then $\nabla \times \nabla\varphi =$ (A) $\vec{0}$ (B) solenoidal (C) irrotational (D) constant	ANS **A**	(CLO-2, Remember)
21.	The value of line integral $\int_C \ \vec{F} \bullet d\vec{r}$ where C is the line $y = x$ in XY plane from (1, 1) to (2, 2) is (A) 0 (B) 1 (C) 2 (D) 3	ANS **D**	(CLO-2, Apply)
22.	Angle between two level surfaces $\varphi_1 = C$ and $\varphi_2 = C$ is given by (A) $\sin\theta = \frac{\nabla\varphi_1 \bullet \nabla\varphi_2}{\lvert\nabla\varphi_1\rvert\,\lvert\nabla\varphi_2\rvert}$ (B) $\cos\theta = \frac{\nabla\varphi_1 \bullet \nabla\varphi_2}{\lvert\nabla\varphi_1\rvert\,\lvert\nabla\varphi_2\rvert}$ (C) $\tan\theta = \frac{\nabla\varphi_1 \bullet \nabla\varphi_2}{\lvert\nabla\varphi_1\rvert\,\lvert\nabla\varphi_2\rvert}$ (D) $\tan\theta = \frac{\nabla\varphi_1 \times \nabla\varphi_2}{\lvert\nabla\varphi_1\rvert\,\lvert\nabla\varphi_2\rvert}$	ANS **B**	(CLO-2, Apply)

23.	The condition for a vector $\vec{r}$ to be solenoidal is (A) $div\ \vec{r} = 0$ (B) $curl\ \vec{r} = 0$ (C) $div\ \vec{r} \neq 0$ (D) $curl\ \vec{r} \neq 0$	ANS **A**	(CLO-2, Remember)
24.	The unit normal vector to the surface $x^2 + 2\,y^2 + z^2 = 7$ at the point $(1, -1, 2)$ is (A) $\frac{\vec{\imath} - 2\,\vec{\jmath} - 2\,\vec{k}}{3}$ (B) $\frac{\vec{\imath} - 2\,\vec{\jmath} + 2\,\vec{k}}{3}$ (C) $\frac{\vec{\imath} + 2\,\vec{\jmath} + 2\,\vec{k}}{3}$ (D) $\frac{\vec{\imath} - 2\,\vec{\jmath} + 2\,\vec{k}}{3}$	ANS **D**	(CLO-2, Apply)
25.	If the integral $\int_A^B \vec{F} \bullet d\vec{r}$ depends only on the end points but not on the path C, then $\vec{F}$ is (A) neither solenoidal nor irrotational (B) solenoidal (C) irrotational (D) conservative	ANS **D**	(CLO-2, Remember)
26.	According to Gauss divergence theorem, $\int_C (P\,dx + Q\,dy) =$ (A) $\iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy$ (B) $\iint_R \left(\frac{\partial Q}{\partial y} - \frac{\partial P}{\partial x}\right) dx\,dy$ (C) $\iint_R \left(\frac{\partial P}{\partial x} - \frac{\partial Q}{\partial y}\right) dx\,dy$ (D) $\iint_R \left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right) dx\,dy$	ANS **A**	(CLO-2, Apply)
27.	By Green's theorem, $\frac{1}{2}\left(\int_C x\,dy - y\,dx\right) =$ (A) Area of a closed curve (B) 2 × Area of a closed curve (C) Volume of a closed curve (D) 3 × Volume of a closed curve	ANS **A**	(CLO-2, Apply)
28.	The value of $\iint_S \vec{r} \bullet \vec{n}\,dS$ where S is the surface of the sphere $x^2 + y^2 + z^2 = a^2$ is (A) $2\,\pi\,a^3$ (B) $3\,\pi\,a^3$ (C) $4\,\pi\,a^3$ (D) $5\,\pi\,a^3$	ANS **C**	(CLO-2, Apply)
29.	The maximum directional derivative of $\varphi\,(x, y, z) = xyz^2$ at (1, 0, 3) is (A) 9 (B) 1 (C) – 9 (D) 0	ANS **A**	(CLO-2, Apply)
30.	The relation between line integral and double integral is given by (A) Gauss divergence theorem (B) Cauchy's theorem (C) Green's theorem (D) Convolution theorem	ANS **C**	(CLO-2, Remember)

31.	If $\varphi\,(x,y,z) = x^2 + y^2 + z^2$, then $\nabla\varphi$ at $(1,1,1)$ = (A) $2\,\vec{i} + 2\,\vec{j} + 2\,\vec{k}$ (B) $2\,\vec{i} - 2\,\vec{j} + \vec{k}$ (C) $\vec{i} + \vec{j} + \vec{k}$ (D) $2\,\vec{i} - 2\,\vec{j} - 2\,\vec{k}$	ANS **A**	(CLO-2, Apply)
32.	If $\varphi\,(x,y,z) = xyz$, then $\nabla\varphi$ at $(1,1,1)$ is (A) $\vec{i} + \vec{j} + \vec{k}$ (B) $2\,\vec{i} + 2\,\vec{j} + 2\,\vec{k}$ (C) $2\,\vec{i} - 2\,\vec{j} + \vec{k}$ (D) $2\,\vec{i} - 2\,\vec{j} - 2\,\vec{k}$	ANS **A**	(CLO-2, Apply)
33.	The unit normal vector to the surface $\varphi = xy - yz - zx$ at the point (– 1, 1, 1) is (A) $-2\,\vec{j}$ (B) $-\vec{j}$ (C) $3\,\vec{i}$ (D) $4\,\vec{i}$	ANS **B**	(CLO-2, Apply)
34.	. $\nabla\, r^n =$ (A) $n\,\vec{r}$ (B) $n\,(n-1)\,\vec{r}$ (C) $n\,r^{n-2}\,\vec{r}$ (D) $n\,r^{n+2}\,\vec{r}$	ANS **C**	(CLO-2, Apply)
35.	The directional derivative of $\varphi = 2xy + z^2$ at (1, –1, 3) in the direction of $\vec{i} + 2\,\vec{j} + 2\,\vec{k}$ is (A) $\frac{14}{3}$ (B) $-\frac{14}{3}$ (C) $\frac{4}{3}$ (D) $\frac{3}{14}$	ANS **A**	(CLO-2, Apply)
36.	If $\vec{F} = (3x - 2y + z)\,\vec{i} + (4x + ay - 2)\,\vec{j} + (x - y + 2)\,\vec{k}$ is solenoidal, then $a =$ (A) 3 (B) 0 (C) -3 (D) -1	ANS **C**	(CLO-2, Apply)

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Unit 2 – Vector Calculus

Part – B (Each question carries 3 Marks)

1. Find $\nabla\phi$ if $\phi = \log\left(x^2+y^2+z^2\right)$.

Solution

$$\nabla\phi = \vec{i}\ \frac{\partial\phi}{\partial x} + \vec{j}\ \frac{\partial\phi}{\partial y} + \vec{k}\ \frac{\partial\phi}{\partial z}$$

$$= \vec{i}\frac{\partial}{\partial x}\left(\log(x^2+y^2+z^2)\right) + \vec{j}\frac{\partial}{\partial y}\log(x^2+y^2+z^2) + \vec{k}\frac{\partial}{\partial z}\log(x^2+y^2+z^2)$$

$$= \vec{i}\frac{2x}{(x^2+y^2+z^2)} + \vec{j}\frac{2y}{(x^2+y^2+z^2)} + \vec{k}\frac{2z}{(x^2+y^2+z^2)}$$

$$= \frac{2}{x^2+y^2+z^2}\left(x\vec{i}+y\vec{j}+z\vec{k}\right) = \frac{2\vec{r}}{r^2} \qquad \because (\vec{r} = x\vec{i}+y\vec{j}+z\vec{k}\ \&\ r^2 = x^2+y^2+z^2)$$

2. Find the unit normal vector to the surface $x^2+y^2=z$ at the point $(1,-2,5)$.

Solution

Given

$$\phi = x^2+y^2-z$$

$$\nabla\phi = \vec{i}\frac{\partial\phi}{\partial x} + \vec{j}\frac{\partial\phi}{\partial y} + \vec{k}\frac{\partial\phi}{\partial z} = 2x\vec{i} + 2y\vec{j} - \vec{k}$$

$$\nabla\phi \text{ at } (1,-2,5) = 2\vec{i} - 4\vec{j} - \vec{k}$$

$$|\nabla\phi| = \sqrt{4+4+1} = 3$$

Unit Normal vector is

$$\hat{n} = \frac{\nabla\phi}{|\nabla\phi|} = \frac{2\vec{i}+2\vec{j}-\vec{k}}{3}$$

3. Prove that $\boldsymbol{curl(grad\phi) = 0}$.

Solution

$$grad\phi = \nabla\phi = \vec{i}\frac{\partial\phi}{\partial x} + \vec{j}\frac{\partial\phi}{\partial y} + \vec{k}\frac{\partial\phi}{\partial z}$$

$$Curl\,(grad\,\varphi) = \nabla\times\nabla\varphi = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial\varphi}{\partial x} & \frac{\partial\varphi}{\partial y} & \frac{\partial\varphi}{\partial z} \end{vmatrix}$$

$$= \vec{i}\left(\frac{\partial^2\varphi}{\partial y\,\partial z} - \frac{\partial^2\varphi}{\partial z\,\partial y}\right) - \vec{j}\left(\frac{\partial^2\varphi}{\partial x\,\partial z} - \frac{\partial^2\varphi}{\partial z\,\partial x}\right) + \vec{k}\left(\frac{\partial^2\varphi}{\partial x\,\partial y} - \frac{\partial^2\varphi}{\partial y\,\partial x}\right)$$

$= 0\vec{i} + 0\vec{j} + 0\vec{k}$ **(Since mixed partial derivatives are equal.)**

4. Find $\boldsymbol{curl\vec{F}}$ if $\boldsymbol{\vec{F} = xy\vec{i} + yz\vec{j} + zx\vec{k}}$.

Solution

Given $\vec{F} = xy\vec{i} + yz\vec{j} + zx\vec{k}$

$$curl\vec{F} = \nabla\times\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xy & yz & zx \end{vmatrix} = \vec{i}\,(0-y) - \vec{j}(z-0) + \vec{k}\,(0-x)$$

$$= -y\,\vec{i} - z\,\vec{j} - x\,\vec{k}$$

5. In what direction from $(3,1,-2)$ is the directional derivative of $\phi = x^2y^2z^4$ maximum? Find also the magnitude of this maximum.

Solution

Given $\phi = x^2y^2z^4$

$$\nabla\phi = \vec{i}\frac{\partial\phi}{\partial x} + \vec{j}\frac{\partial\phi}{\partial y} + \vec{k}\frac{\partial\phi}{\partial z} = 2xy^2z^4\vec{i} + 2x^2yz^4\vec{j} + 4x^2y^2z^3\vec{k}$$

$$\nabla\phi \text{ at } (3,1,-2) = 92\vec{i} + 144\vec{j} - 92\vec{k}$$

$$|\nabla\phi| = \sqrt{92^2 + 144^2 + 92^2} = \sqrt{37664}$$

The directional derivative is maximum in the direction $\nabla\phi$ and the magnitude of this maximum is $|\nabla\phi| = \sqrt{37664}$.

6. Find the directional derivative of $\phi = x^2yz + 4xz^2$ at (1, – 2 , – 1) in the direction of $2\vec{i} - \vec{j} - 2\vec{k}$.

Solution

Given $\phi = x^2yz + 4xz^2$,

$$\vec{a} = 2\vec{i} - \vec{j} - 2\vec{k}, \ |\vec{a}| = \sqrt{4+1+4} = 3$$

$$\nabla\phi = (2xyz + 4z^2)\vec{i} + x^2z\vec{j} + (x^2y + 8xz)\vec{k}$$

$$(\nabla\phi)_{(1,-2,-1)} = 8\vec{i} - \vec{j} - 10\vec{k}$$

$$\text{D.D.} = \nabla\phi.\frac{\vec{a}}{|\vec{a}|} = (8\vec{i} - \vec{j} - 10\vec{k}).\frac{2\vec{i} - \vec{j} - 2\vec{k}}{3} = \frac{37}{3}$$

7. Find the directional derivative of $\phi = x^2 - y^2 + 2z^2$ at P (1, 2 , 3) in the direction of line PQ where Q is (5, 0, 4).

Solution

$$\nabla\varphi = grad\ \varphi = \vec{i}\frac{\partial\varphi}{\partial x} + \vec{j}\frac{\partial\varphi}{\partial y} + \vec{k}\frac{\partial\varphi}{\partial z}$$

$$\nabla\varphi = grad\ \varphi = \vec{i}\,2x + \vec{j}\,(-2y) + \vec{k}\,4z$$

$$\nabla\varphi \text{ at } (1,2,3) = 2\vec{i} - 4\vec{j} + 12\vec{k}$$

$$\vec{a} = OQ - OP = (5\vec{i} + 0\vec{j} + 4\vec{k}) - (\vec{i} + 2\vec{j} + 3\vec{k}) = 4\vec{i} - 2\vec{j} + \vec{k}$$

Directional derivative = $\nabla\varphi \bullet \dfrac{\vec{a}}{|\vec{a}|}$

$$= (2\vec{i} - 4\vec{j} + 12\vec{k}) \bullet \frac{4\vec{i} - 2\vec{j} + \vec{k}}{\sqrt{21}} = \frac{28}{\sqrt{21}}$$

8. Find the angle between the normals to the surfaces $x^2 = yz$ at the points (1, 1, 1) and (2, 4, 1).

Solution

Given $\varphi = x^2 - yz$

$\nabla\varphi = 2x\vec{i} - z\vec{j} - y\vec{k}$

$\nabla\varphi_1 /(1,1,1) = 2\vec{i} - \vec{j} - \vec{k}$ $\qquad$ $\nabla\varphi_2 /(2,4,1) = 4\vec{i} - \vec{j} - 4\vec{k}$

$$|\nabla\varphi_1| = \sqrt{4+1+1} = \sqrt{6} \qquad |\nabla\varphi_2| == \sqrt{16+1+16} = \sqrt{33}$$

$$\cos\theta = \frac{\nabla\varphi_1 \circ \nabla\varphi_2}{|\nabla\varphi_1||\nabla\varphi_2|} = \frac{(2\vec{i}-\vec{j}-\vec{k})\circ(4\vec{i}-\vec{j}-4\vec{k})}{\sqrt{6}\sqrt{33}} = \frac{13}{\sqrt{6}\sqrt{33}}.$$

9. Find a such that $\vec{F} = (3x-2y+z)\vec{i} + (4x+ay-z)\,\vec{j} + (x-y+2z)\vec{k}$ is solenoidal.

Solution

Given $\nabla\circ\vec{F} = 0 \Rightarrow \frac{\partial}{\partial x}(3x-2y+z) + \frac{\partial}{\partial y}(4x+ay-z) + \frac{\partial}{\partial z}(x-y+2z) = 0$

$$3+a+2=0 \Rightarrow a+5=0 \;\Rightarrow a=-5$$

10. Find the constant a, b, c so that $\vec{F} = (x+2y+az)\vec{i} + (bx-3y-z)\vec{j} + (4x+cy+2z)\vec{k}$ is irrotational.

Solution

Given $\vec{F}$ is irrotational **i.e.,** $\nabla\times\vec{F} = \vec{0}$

$$\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x+2y+az & bx-3y-z & 4x+cy+2z \end{vmatrix} = \vec{0}$$

$$\vec{i}\left(\frac{\partial}{\partial y}(4x+cy+2z) - \frac{\partial}{\partial z}(bx-3y-z)\right) - \vec{j}\left(\frac{\partial}{\partial x}(4x+cy+2z) - \frac{\partial}{\partial z}(x+2y+az)\right)$$
$$+\vec{k}\left(\frac{\partial}{\partial x}(bx-3y-z) - \frac{\partial}{\partial y}(x+2y+az)\right) = \vec{0}$$

$= i.e.,\quad \vec{i}\,(c+1) - \vec{j}(4-a) + \vec{k}(b-2) = 0$

$= \therefore c+1=0, 4-a=0, and\; b-2=0$

$\Rightarrow a=4, b=2, c=-1$

11. If $\vec{F} = x^3\vec{i} + y^3\vec{j} + z^3\vec{k}$, then find div curl $\vec{F}$.

Solution $div\ curl\,\vec{F} = \nabla\,.\,(\nabla\times\vec{F})$

$$\nabla\times\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ x^3 & y^3 & z^3 \end{vmatrix}$$

$$= \vec{i}(0-0) - \vec{j}(0-0) + \vec{k}(0-0) = \vec{0}$$

$$\nabla\times\vec{F} = \vec{0}$$

$$\therefore \nabla\,.(\nabla\times\vec{F}) = 0$$

12. Prove that $div\,\vec{r} = 3$.

Solution

$$\vec{r} = x\,\vec{i} + y\,\vec{j} + z\,\vec{k}$$

$$div\,\vec{r} = \nabla\bullet\vec{r} = \left(\vec{i}\,\frac{\partial}{\partial x} + \vec{j}\,\frac{\partial}{\partial y} + \vec{k}\,\frac{\partial}{\partial z}\right)\bullet\left(x\,\vec{i} + y\,\vec{j} + z\,\vec{k}\right)$$

$$= \frac{\partial}{\partial x}(x) + \frac{\partial}{\partial y}(y) + \frac{\partial}{\partial z}(z) = 1+1+1 = 3$$

13. Show that the vector $\vec{F} = \left(6xy + z^3\right)\vec{i} + \left(3x^2 - z\right)\vec{j} + \left(3xz^2 - y\right)\vec{k}$ is irrotational.

Solution

Given $\vec{F} = \left(6xy + z^3\right)\vec{i} + \left(3x^2 - z\right)\vec{j} + \left(3xz^2 - y\right)\vec{k}$

$curl\vec{F} = \nabla\times\vec{F} = \vec{0}$

$$\nabla\times\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ 6xy + z^3 & 3x^2 - z & 3xz^2 - y \end{vmatrix} = \vec{i}(-1+1) - \vec{j}(3z^2 - 3z^2) + \vec{k}(6x - 6x) = \vec{0}$$

$\therefore$ $\vec{F}$ is irrotational.

14. If $F=(3x^2+6y)\vec{i}-14yz\vec{j}+20xz^2\vec{k}$. **Evaluate** $\int_C \vec{F}\bullet d\vec{r}$ **from (0,0,0) to (1,1,1) along the curve**

x = t, y = t², z = t³.

Solution

The end points are (0,0,0) and (1,1,1).

These points correspond to t = 0 and t = 1.

$\therefore$ dx = dt, dy = 2t dt, dz = $3t^2$dt

$$\int_C \vec{F}\bullet d\vec{r}=\int_C (3x^2+6y)dx-14yzdy+20xz^2dz$$

$$=\int_0^1 (3t^2+6t^2)dt-14t^5(2tdt)+20t^7(3t^2)dt=\int_0^1 (9t^2-28t^6+60t^9)dt=5$$

15. If $F=ax\vec{i}+by\vec{j}+cz\vec{k}$, **a, b, c are constants, show that** $\iint_S \vec{F}\bullet\hat{n}\,ds=\frac{4\pi}{3}(a+b+c)$ **where S**

is the surface of a unit sphere.

Solution

W.K.T. Gauss's divergence theorem

$$\iint_S \vec{F}\bullet\hat{n}ds=\iiint_V \nabla\bullet\vec{F}dV=\iiint_V\left(\frac{\partial}{\partial x}(ax)+\frac{\partial}{\partial y}(by)+\frac{\partial}{\partial z}(cz)\right)dV$$

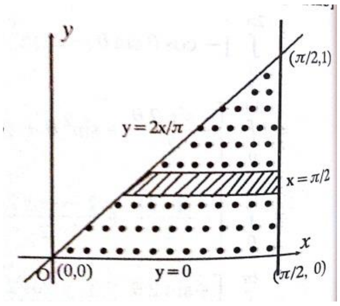
$$=\iiint_V (a+b+c)dV=(a+b+c)V=(a+b+c)\frac{4}{3}\pi(1)^3$$

$$\iint_S \vec{F}\bullet\hat{n}ds=\frac{4}{3}\pi(a+b+c)$$

16. Using Green's theorem, evaluate $\int_c (y-\sin x)\,dx + \cos x\,dy$ **where** c **is the triangle formed by** $y=0, x=\frac{\pi}{2}, y=\frac{2x}{\pi}$.

Solution

Using Green's theorem, we convert the line integral to double integral over the given region. $$ie., \int_C u\,dx + v\,dy = \iint_R \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) dxdy$$



$u = y - \sin x$ $\qquad$ $v = \cos x$

$\frac{\partial u}{\partial y} = 1$ $\qquad$ $\frac{\partial v}{\partial x} = -\sin x$

Hence, $$\int_C \{(y-\sin x)dx + \cos x dy\} = \iint_R (-\sin x - 1) dxdy$$

$$= \int_0^1 \int_{\pi y/2}^{\pi/2} (-\sin x - 1) dxdy = \int_0^1 [\cos x - x]_{\pi y/2}^{\pi/2}$$

$$= \int_0^1 \left(0 - \frac{\pi}{2} - \cos\frac{\pi y}{2} + \frac{\pi y}{2}\right) dy$$

$$= \left[\frac{-\pi y}{2} - \frac{\sin\frac{\pi y}{2}}{\frac{\pi}{2}} + \frac{\pi}{2}.\frac{y^2}{2}\right]_0^1 = \frac{-\pi}{2} - \frac{2}{\pi} + \frac{\pi}{4}$$

$$= \frac{-\pi^2 - 8}{4\pi} = -\left[\frac{\pi}{4} + \frac{2}{\pi}\right].$$

17. Using Green's theorem, evaluate $\int_c (3x^2 - 8y^2)\,dx + (4y - 6xy)\,dy$ **where** c **is the boundary of the triangle formed by the lines** $x=0, y=0, x+y=1$ **in the** xy **plane.**

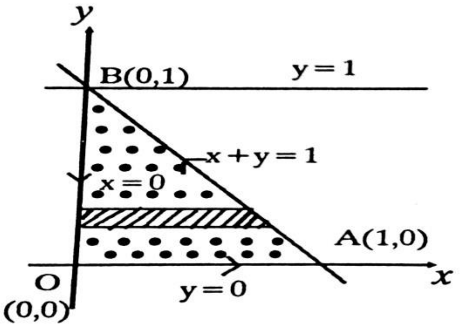
Solution

Using Green's theorem, we convert the line integral to double integral over the given region. $$ie., \int_C u\,dx + v\,dy = \iint_R \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) dxdy$$

$u = 3x - 8y^2$ $\quad\quad v = 4y - 6xy$

$\dfrac{\partial u}{\partial y} = -16y$ $\quad\quad \dfrac{\partial v}{\partial x} = -6y$

$$\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = -6y + 16y = 10y$$



Hence, $\displaystyle\iint_R \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) dxdy = \iint_R (10y) dxdy$

$$= 10\int_0^1 \int_0^{1-y} (y) dxdy = \int_0^1 y[x]_0^{1-y}\, dy$$

$$= 10\int_0^1 y(1-y)dy = 10\int_0^1 (y - y^2)dy$$

$$= 10\left(\frac{y^2}{2} - \frac{y^3}{3}\right)_0^1$$

$$= 10\left(\frac{1}{2} - \frac{1}{3}\right).$$

$$= 10\frac{3-2}{6} = \frac{10}{6} = \frac{5}{3}$$

18. Using Gauss divergence theorem evaluate $\displaystyle\iiint_V \nabla \circ \vec{F} dv$ **where** $\vec{F} = 4xz\vec{i} - y^2\,\vec{j} + yz\,\vec{k}$ **taken over the cube bounded by the planes** $x = 0,\ x = 1, y = 0,\ y = 1, z = 0, z = 1.$

Solution

$\vec{F} = 4xz\vec{i} - y^2\,\vec{j} + yz\,\vec{k}$

$$\nabla \circ \vec{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}$$

$$\nabla \circ \vec{F} = 4z - 2y + y = 4z - y$$

$$\iiint\limits_V \nabla \circ \vec{F} dv = \int_0^1\int_0^1\int_0^1 (4z - y)dxdydz = \int_0^1\int_0^1 [4zx - yx]_0^1 dydz = \int_0^1\int_0^1 [4z - y]dydz$$

$$= \int_0^1 \left[4zy - \frac{y^2}{2}\right]_0^1 dz = \int_0^1 \left[4z - \frac{1}{2}\right]dz = \left[4\frac{z^2}{2} - \frac{z}{2}\right]_0^1 = \frac{4}{2} - \frac{1}{2} = \frac{3}{2}$$

19. Using Gauss divergence theorem theorem evaluate $\iiint\limits_V \nabla \circ \vec{F} dv$ where

$\vec{F} = x^2\,\vec{i} + y^2\,\vec{j} + z^2\,\vec{k}$ taken over the cube bounded by the planes $x = 0,\ x = 1, y = 0,\ y = 1, z = 0, z = 1.$

Solution

$$\vec{F} = x^2\,\vec{i} + y^2\,\vec{j} + z^2\,\vec{k}$$

$$\nabla \circ \vec{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}$$

$$\nabla \circ \vec{F} = 2x + 2y + 2z = 2(x + y + z)$$

$$\iiint\limits_V \nabla \circ \vec{F} dv = 2\int_0^1\int_0^1\int_0^1 (x + y + z)dxdydz = 2\int_0^1\int_0^1 \left[\frac{x^2}{2} + xy + xz\right]_0^1 dydz = 2\int_0^1\int_0^1 \left[\frac{1}{2} + y + z\right]dydz$$

$$= 2\int_0^1 \left[\frac{y}{2} + \frac{y^2}{2} + yz\right]_0^1 dz = 2\int_0^1 \left[\frac{1}{2} + \frac{1}{2} + z\right]_0^1 dz = 2\int_0^1 [1 + z]_0^1 dz = 2\left[z + \frac{z^2}{2}\right]_0^1$$

$$= 2\left(1 + \frac{1}{2}\right) = 2\left(\frac{3}{2}\right) = 3$$

20. Using Stokes theorem find $\iint\limits_S curl\,\vec{F}\,ds$ where $\vec{F} = (x^2 - y^2)\,\vec{i} + 2xy\,\vec{j}$ in the rectangular region of $x = 0,\ y = 0,\ x = a$ and $y = a.$

Solution Stokes theorem $\int\limits_c \vec{F}.d\vec{r} = \iint\limits_s curl\,\vec{F}.d\vec{s}$

Given $\vec{F} = (x^2 - y^2)\vec{i} + 2xy\vec{j}$

$$curl\,\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ x^2 - y^2 & 2xy & 0 \end{vmatrix} = 4y\,\vec{k}$$

Here $\hat{n} = \vec{k}$

$$\iint_S curl\,\vec{F}.\hat{n}ds = \iint_S 4ydxdy = \int_0^b\int_0^a 4ydxdy = 2ab^2$$

21. Prove that the area bounded by a simple closed curve C is given by

$$\frac{1}{2}\int_C (xdy - ydx).$$

Solution

W.K.T. Green's theorem

$$\int_C (udx + vdy) = \iint_R \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)dxdy \qquad \dots 1$$

Here $\quad v = \frac{x}{2} \qquad u = -\frac{y}{2}$

$$\frac{\partial v}{\partial x} = \frac{1}{2} \qquad \frac{\partial u}{\partial y} = -\frac{1}{2}$$

$$(1) \Rightarrow \int_C \left(\frac{x}{2}dy - \frac{y}{2}dx\right) = \iint_R \left(\frac{1}{2} + \frac{1}{2}\right)dxdy$$

$$\frac{1}{2}\int_C (xdy - ydx) = \iint_R dxdy$$

22. Find the area of the ellipse x = a cos θ, y = b sin θ using Green's theorem.

Solution

Given $\quad x = a\cos\theta, \qquad y = b\sin\theta$

$dx = -a\sin\theta\, d\theta, \qquad dy = b\cos\theta\, d\theta$

θ varies from 0 to 2π.

Area of the ellipse $= \frac{1}{2}\oint_C xdy - ydx$

$$= \frac{1}{2}\int_0^{2\pi} (a\cos\theta)(-b\cos\theta d\theta) - (b\sin\theta)(-a\sin\theta d\theta)$$

$$= \frac{1}{2}\int_0^{2\pi} [ab\cos\theta\cos\theta + ab\sin\theta\sin\theta] d\theta$$

$$= \frac{ab}{2}\int_0^{2\pi} (\cos^2\theta + \sin^2\theta) d\theta = \frac{ab}{2}\int_0^{2\pi} d\theta = \frac{ab}{2}[\theta]_{\theta=0}^{\theta=2\pi}$$

Area of the ellipse $= \frac{ab}{2}[2\pi] = \pi ab$

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18MAB102T
Advanced Calculus and Complex Analysis
Unit II -Vector Calculus

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Scalar and Vector Fields:

- A physical quantity expressible as a continuous function and which can assume one or more definite values at each point of a region of space, is called point function in the region and the region concerned is called a field.
- Point functions are classified as scalar point function and vector point function according as the nature of the quantity concerned is a scalar or a vector.
- At each point P of the field if the function denoted by f(P) is a scalar, it is known as scalar point function while if $\vec{f}(P)$ is a vector, then the function $\vec{f}(P)$ is called a vector point function. The concerned field is called a scalar field or a vector field respectively.

Example of Scalar Fields:

- The temperature distribution in a medium, the gravitational potential of a system of masses and the electrostatic potential of a system of charges.

Example of Vector Fields:

- The velocity of a moving particle, the electrostatic, the magneto static and gravitational fields.

Vector Differential Operator DEL(∇):

$$\nabla = \vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}$$

Gradient:

Let $\phi(x, y, z)$ defines a differentiable scalar field. (i.e) ϕ is differentiable at each point (x, y, z) is a certain region of space. Then the gradient of ϕ denoted by $\nabla\phi$ (or) grad ϕ is defined by

$$\nabla\phi = \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right)\phi = \vec{i}\frac{\partial\phi}{\partial x} + \vec{j}\frac{\partial\phi}{\partial y} + \vec{k}\frac{\partial\phi}{\partial z} = \sum \vec{i}\frac{\partial\phi}{\partial x}$$

Divergence :

If $\vec{F}(x, y, z)$ is defined and differentiable vector point function at each point (x, y, z) is a certain region of space, then the divergence of $\vec{F}$ denoted by $\nabla . \vec{F}$ (or) div$\vec{F}$ is defined by

$$\text{div}\vec{F} = \nabla . \vec{F} = \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right)\vec{F} = \sum \vec{i} . \frac{\partial \vec{F}}{\partial x}$$

If $\vec{F} = F_1\vec{i} + F_2\vec{j} + F_3\vec{k}$, then div$\vec{F} = \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right)\left(F_1\vec{i} + F_2\vec{j} + F_3\vec{k}\right)$

$$\text{div}\vec{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}$$

Solenoidal :

If $\vec{F}$ is a vector such that $\nabla.\vec{F} = 0$ for all points is a given region, then it is said to be a solenoidal vector in that region.

Curl :

If $\vec{F}(x, y, z)$ is a differentiable vector point function in a certain region of space, then the curl or rotation of $\vec{F}$ denoted by $\nabla \times \vec{F}$ (or) curl $\vec{F}$ (or) rot $\vec{F}$ is defined by

$$\nabla \times \vec{F} = curl\ \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix}$$

Irrotational :

If $\vec{F}$ is vector such that $\nabla \times \vec{F} = 0$ for all points in the region, then it is called an irrotational vector (or) Lamellar vector in that region.

Directional derivation : $\frac{\nabla\phi.\vec{a}}{|\vec{a}|}$

Unit normal vector : $\hat{n} = \frac{\nabla\phi}{|\nabla\phi|}$

Angle between the surfaces :

$$\cos\theta = \frac{\nabla\phi_1.\nabla\phi_2}{|\nabla\phi_1||\nabla\phi_2|}$$

Problem: 1

If $\phi = xyz$, find $\nabla\phi$ at $(1, 2, 3)$

Solution:

$$\nabla\phi = \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right)(xyz)$$

$$= \vec{i}\frac{\partial}{\partial x}(xyz) + \vec{j}\frac{\partial}{\partial y}(xyz) + \vec{k}\frac{\partial}{\partial z}(xyz)$$

$$= \vec{i}\, yz + \vec{j}\, xz + \vec{k}\, xy$$

$$\nabla\phi = yz\,\vec{i} + xz\,\vec{j} + xy\,\vec{k}\text{v}$$

$$\nabla\phi_{(1,2,3)} = 6\vec{i} + 3\vec{j} + 2\vec{k}.$$

Problem: 2

Prove that $\nabla(r^n) = nr^{n-2}\vec{r}$

Solution:

$$\nabla(r^n) = \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right)(r^n)$$

$$= \vec{i}\frac{\partial}{\partial x}(r^n) + \vec{j}\frac{\partial}{\partial y}(r^n) + \vec{k}\frac{\partial}{\partial z}(r^n)$$

$$= \vec{i}\, nr^{n-1}\frac{\partial r}{\partial x} + \vec{j}\, nr^{n-1}\frac{\partial r}{\partial y} + \vec{k}\, nr^{n-1}\frac{\partial r}{\partial z}$$

$$\nabla(r^n) = nr^{n-1}\left(\vec{i}\frac{\partial r}{\partial x} + \vec{j}\frac{\partial r}{\partial y} + \vec{k}\frac{\partial r}{\partial z}\right) \qquad (1)$$

$$\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$$

$$\vec{r}.\vec{r} = r^2 = x^2 + y^2 + z^2$$

$$2r\frac{\partial r}{\partial x} = 2x \qquad 2r\frac{\partial r}{\partial y} = 2y \qquad 2r\frac{\partial r}{\partial z} = 2z$$

$$\frac{\partial r}{\partial x} = \frac{x}{r} \qquad \frac{\partial r}{\partial y} = \frac{y}{r} \qquad \frac{\partial r}{\partial z} = \frac{z}{r} \qquad (2)$$

Sub (2) in (1),

$$\nabla(r^n) = nr^{n-1}\left(\vec{i}\frac{x}{r} + \vec{j}\frac{y}{r} + \vec{k}\frac{z}{r}\right)$$

$$\nabla(r^n) = nr^{n-2}\vec{r}.$$

Problem: 3

Find the directional derivative of $\phi = x^2yz + 4xz^2 + xyz$ at $(1,2,3)$ in the direction of $2\vec{i} + \vec{j} - \vec{k}$.

Solution:

Directional derivation $= \frac{\nabla\phi.\vec{a}}{|\vec{a}|}$

Given $\phi = x^2yz + 4xz^2 + xyz$

$$\nabla\phi = (2xyz + 4z^2 + yz)\vec{i} + (x^2z + xz)\vec{j} + (x^2 + 8xz + xy)\vec{k}$$

$$\nabla\phi_{(1,2,3)} = 54\vec{i} + 6\vec{j} + 28\vec{k}$$

Let $\vec{a} = 2\vec{i} + \vec{j} - \vec{k}$

$$|\vec{a}| = \sqrt{2^2 + 1^2 + (-1)^2}$$

$$|\vec{a}| = \sqrt{6}$$

Directional derivation $= \frac{\nabla\phi.\vec{a}}{|\vec{a}|}$

Directional derivation $= \frac{(54\vec{i}+6\vec{j}+28\vec{k}).(2\vec{i}+\vec{j}-\vec{k})}{\sqrt{6}}$

Directional derivation $= \frac{86}{\sqrt{6}}$

Problem: 4

Find a unit normal to the surface $x^2 + 2xz^2 = 8$ at the point $(1, 0, 2)$.

Solution:

Let $\phi = x^2 + 2xz^2 - 8$

$$\nabla\phi = (2xy + 2x^2)\vec{i} + x^2\vec{j} + 4xz\vec{k}$$

$$\nabla\phi_{(1,0,2)} = 8\vec{i} + \vec{j} + 8\vec{k}$$

$$|\nabla\phi| = \sqrt{8^2 + 1^2 + 8^2} = \sqrt{129}$$

Unit normal$= \hat{n} = \frac{\nabla\phi}{|\nabla\phi|} = \frac{8\vec{i}+\vec{j}+8\vec{k}}{\sqrt{129}}$.

Problem: 5

Find the angle between the surfaces $z = x^2 + y^2 - 3$ and $x^2 + y^2 + z^2 = 9$ at (2, -1, 2).

Solution:

Given $\phi_1 = x^2 + y^2 - 2 - 3$

$$\nabla\phi_1 = 2x\vec{i} + 2y\vec{j} - \vec{k}$$

$$\nabla\phi_1\,(2,-1,2) = 4\vec{i} - 2\vec{j} - \vec{k}$$

$$|\nabla\phi_1| = \sqrt{21}$$

$$\phi_2 = x^2 + y^2 + z^2 - 9$$

$$\nabla\phi_2 = 2x\vec{\imath} + 2y\vec{\jmath} + 2z\vec{k}$$

$$\nabla\phi_2\,(2,-1,2) = 4\vec{\imath} - 2\vec{\jmath} + 4\vec{k}$$

$$|\nabla\phi_2| = 6$$

$$\cos\theta = \frac{\nabla\phi_1 . \nabla\phi_2}{(\nabla\phi_1)(\nabla\phi_2)}$$

$$= \frac{\left(4\vec{\imath} - 2\vec{\jmath} - \vec{k}\right).(4\vec{\imath} - 2\vec{\jmath} + 4\vec{k})}{(\sqrt{21})\,(6)}$$

$$\cos\theta = \frac{8}{3\sqrt{21}}$$

$$\theta = \cos^{-1}\frac{8}{3\sqrt{21}}$$

Problem: 6

If $\nabla\phi = \left(yz\vec{i} + zx\vec{j} + xy\vec{k}\right)$, find ϕ.

Solution:

$$\nabla\phi = \left(yz\vec{i} + zx\vec{j} + xy\vec{k}\right)$$

$$\vec{i}\frac{\partial\phi}{\partial x} + \vec{j}\frac{\partial\phi}{\partial y} + \vec{k}\frac{\partial\phi}{\partial z} = \left(yz\vec{i} + zx\vec{j} + xy\vec{k}\right)$$

$$\frac{\partial\phi}{\partial x} = yz$$

function not involving x.

$$\frac{\partial \phi}{\partial y} = zx$$

$\phi = xyz + a$, function not involving y.

$$\frac{\partial \phi}{\partial z} = xy$$

$\phi = xyz + a$, function not involving z.

From the last three statements,

we conclude

$\phi = xyz + a$ is a constant.

Problem: 7

If $\vec{F} = x^2\vec{i} + y^2\vec{j} + z^2\vec{k}$, then find $\nabla \cdot \vec{F}$ and $\nabla \times \vec{F}$.

Solution:

$$\nabla \cdot \vec{F} = \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right) \cdot \left(x^2\vec{i} + y^2\vec{j} + z^2\vec{k}\right)$$

$$= \frac{\partial}{\partial x}(x^2) + \frac{\partial}{\partial y}(y^2) + \frac{\partial}{\partial z}(z^2)$$

$$\nabla \cdot \vec{F} = 2\mathrm{x} + 2\mathrm{y} + 2\mathrm{z}$$

$$\nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2 & y^2 & z^2 \end{vmatrix}$$

$$= \vec{i}\left[\frac{\partial}{\partial y}(z^2) - \frac{\partial}{\partial z}(y^2)\right] - \vec{j}\left[\frac{\partial}{\partial x}(z^2) - \frac{\partial}{\partial z}(x^2)\right] + \vec{k}\left[\frac{\partial}{\partial x}(y^2) - \frac{\partial}{\partial y}(x^2)\right]$$

$$= \vec{i}[0] - \vec{j}[0] + \vec{k}[0]$$

$$\nabla \times \vec{F} = 0.$$

Problem: 8

Prove that the vector $\vec{F} = z\vec{i} + x\vec{j} + y\vec{k}$ is solenoidal.

Solution:

$$\nabla \cdot \vec{F} = \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right) \cdot \left(z\vec{i} + x\vec{j} + y\vec{k}\right)$$

$$= \frac{\partial}{\partial x}(z) + \frac{\partial}{\partial y}(x) + \frac{\partial}{\partial z}(y)$$

$$\nabla \cdot \vec{F} = 0$$

$\therefore \vec{F}$ is solenoidal.

Problem: 9

If $\vec{F} = (x+3y)\vec{i} + (y-2z)\vec{j} + (x+\lambda z)\vec{k}$ is solenoidal, find the value of λ.

Solution:

$\nabla \,.\, \vec{F} = 0$

$$\frac{\partial}{\partial x}(x+3y) + \frac{\partial}{\partial y}(y-2z) + \frac{\partial}{\partial z}(x+\lambda z) = 0$$

$1 + 1 + \lambda = 0$

$\lambda = -2.$

Problem: 10

Show that $\vec{F} = \left(yz\vec{i} + zx\vec{j} + xy\vec{k}\right)$ is irrotational.

Solution:

$$\nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ yz & zx & xy \end{vmatrix}$$

$$= \vec{i}\left[\frac{\partial}{\partial y}(xy) - \frac{\partial}{\partial z}(xz)\right] - \vec{j}\left[\frac{\partial}{\partial x}(xy) - \frac{\partial}{\partial z}(yz)\right] + \vec{k}\left[\frac{\partial}{\partial x}(zx) - \frac{\partial}{\partial y}(yz)\right]$$

$$\nabla \times \vec{F} = 0$$

$\therefore \vec{F}$ is irrotational.

Laplace operator :

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

Problem: 11

Prove that $\nabla^2 \mathrm{r}^n = n(n+1)\mathrm{r}^{n-2}$ where $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$ and $r = |\vec{r}|$ and deduce $\nabla^2\left(\frac{1}{r}\right)$.

Solution:

$$r = |\vec{r}| = \sqrt{x^2 + y^2 + z^2}$$

$$\mathrm{r}^2 = x^2 + y^2 + z^2$$

$$\frac{\partial r}{\partial x} = \frac{x}{r}; \quad \frac{\partial r}{\partial y} = \frac{y}{r}; \quad \frac{\partial r}{\partial z} = \frac{z}{r}$$

$$\nabla^2 \mathrm{r}^n = \sum \frac{\partial^2}{\partial x^2}(\mathrm{r}^n) = \sum \frac{\partial}{\partial x}\left[\mathrm{nr}^{n-1}\frac{\partial r}{\partial x}\right]$$

$$= \sum \frac{\partial}{\partial x}\left[\mathrm{nr}^{n-1}\frac{x}{r}\right] = \sum \frac{\partial}{\partial x}\left[\mathrm{nr}^{n-2}x\right]$$

$$= \sum n\left[\left((\mathrm{n}-2)\mathrm{r}^{n-3}\frac{\partial r}{\partial x}\right)x + \mathrm{r}^{n-2}\right]$$

$$= \sum n\left[\left((\mathrm{n}-2)\mathrm{r}^{n-3}\frac{x}{r}\right)x + \mathrm{r}^{n-2}\right]$$

$$= \sum n[(x^2(n-2)r^{n-4}) + r^{n-2}]$$

$$= \sum[(n(n-2)r^{n-4}x^2) + nr^{n-2}]$$

$$= n(n-2)r^{n-4}(x^2 + y^2 + z^2) + 3nr^{n-2}$$

$$= n(n-2)r^{n-4}r^2 + 3nr^{n-2}$$

$$= n(n-2)r^{n-2} + 3nr^{n-2}$$

$$= nr^{n-2}[n-2+3]$$

$$\nabla^2(r^n) = n(n+1)r^{n-2}.$$

Line Integral

Problem: 12

Find the work done in moving a particle in the force field $\vec{F} = 3x^2\vec{\imath} + (2xz - y)\vec{\jmath} - z\vec{k}$ from $t = 0$ to $t = 1$ along the cone $x = 2t^2, y = t, z = 4t^3$.

Solution:

Work done $= \int_c \vec{F}.\overrightarrow{dr}$

$$\vec{F} = 3x^2\vec{\imath} + (2xz - y)\vec{\jmath} - z\vec{k}$$

$$\overrightarrow{dx} = dx\,\vec{\imath} + dy\,\vec{\jmath} + dz\,\vec{k}$$

$$\vec{F}.\overrightarrow{dr} = 3x^2dx + (2xz - y)dy - zdz$$

$x = 2t^2$	$y = t$	$z = 4t^3$
$dx = 4t\,dt$	$dy = dt$	$dz = 12t^2\,dt$

$$\vec{F}.\overrightarrow{dr} = 48\,t^5\,dt + (16\,t^5 - t)dt - 48\,t^5\,dt$$

$$\int_c \vec{F}.\overrightarrow{dr} = \int_0^1 (16\,t^5 - t)dt$$

$$= \left[16\,\frac{t^6}{6} - \frac{t^2}{2}\right]_0^1$$

$$= \frac{16}{6} - \frac{1}{2}$$

$$= \frac{13}{6}$$

Surface Integrals

$$\iint_S \vec{F}.\hat{n}\ ds = \iint_R \frac{\vec{F}.\hat{n}}{|\vec{n}.\hat{k}|}\ ds\ dy$$

Problem: 13

Evaluate $\iint_S \vec{F}.\hat{n}\ ds$ where $\vec{F} = z\,\vec{i} + x\,\vec{j} - y^2 z\,\vec{k}$ and S is the surface of the cylinder $x^2 + y^2 = 1$ included in the first octant between the planes $z = 0$ and $z = 2.$

Solution :

$$\vec{F} = z\,\vec{i} + x\,\vec{j} - y^2 z\,\vec{k}$$

$$\varphi = x^2 + y^2 - 1$$

$$|\nabla\varphi| = \sqrt{4x^2 + 4y^2} = 2$$

$$\hat{n} = \frac{\nabla\varphi}{|\nabla\varphi|}$$

$$= \frac{2x\,\vec{i} + 2y\,\vec{j}}{2}$$

$$\hat{n} = x\,\vec{i} + y\,\vec{j}$$

$$\vec{F}.\hat{n} = \left(z\,\vec{i} + x\,\vec{j} - y^2 z\,\vec{k}\right).(x\,\vec{i} + y\,\vec{j}) = xz + xy$$

$$\iint_S \vec{F}.\hat{n}\ ds = \iint_R \frac{\vec{F}.\hat{n}}{|\vec{n}.\hat{i}|}\ dy\,dz$$

Where R is the projection of S on yz plane.

$$= \iint_R (xz + xy) \; \frac{dy\,dz}{x}$$

$$= \iint_R (z + y) \; dy\,dz$$

$$= \int_0^2 \int_0^1 (z + y) \; dy\,dz$$

$$= \int_0^2 \left[zy + \frac{y^2}{2} \right]_0^1 \; dz$$

$$= \int_0^2 (z + \frac{1}{2}) \; dz$$

$$= \left[\frac{z^2}{2} + \frac{z}{2} \right]_0^2 = 3.$$

Volume Integrals

Problem: 14

If $\vec{F} = (2x^2 - 3x)\vec{i} - 2xy\vec{j} - 4x\vec{k}$. Evaluate $\iiint_v \nabla \times \vec{F}\, dv$ where v is the region bounded by $x = 0, y = 0, z = 0$ and $2x + 2y + z = 4$.

Solution:

$$\nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ 2x^2 - 3z & -2xy & -4x \end{vmatrix}$$

$$\nabla \times \vec{F} = \vec{j} - 2y\vec{k}$$

$$\iiint_v \nabla \times \vec{F}\, dv = \int_0^2 \int_0^{2-x} \int_0^{4-2x-2y} \left(\vec{j} - 2y\vec{k}\right) dzdydx$$

$$= \int_0^2 \int_0^{2-x} \left[z\vec{j} - 2yz\vec{k}\right]_0^{4-2x-2y} dydx$$

$$= \int_0^2 \int_0^{2-x} \left[(4 - 2x - 2y)\vec{j} - 2y(4 - 2x - 2y)\vec{k}\right] dydx$$

$$= \int_0^2 \left[\left(4y - 2xy - \frac{2y^2}{2}\right)\vec{j} - \left(4y^2 - 2xy^2 - \frac{4y^3}{3}\right)\vec{k}\right]_0^{2-x} dx$$

$$= \int_0^2 \left\{[4(2-x) - 2x(2-x) - (2-x)^2]\vec{j} - \left[4(2-x)^2 - 2x(2-x)^2 - \frac{4}{3}(2-x)^3\right]\vec{k}\right\} dx$$

$$= \int_0^2 \left[(4 - 4x + x^2)\vec{i} - \frac{\vec{k}}{3}(16 - 24x + 12x^2 - 2x^3)\right] dx$$

$$\iiint_v \nabla \times \vec{F}\, dv = \left[4x - 2x^2 + \frac{x^3}{3}\right]_0^2 \vec{i} - \frac{\vec{k}}{3}\left[16x - 12x^2 + 4x^3 - \frac{x^4}{2}\right]_0^2$$

$$= \left(8 - 8 + \frac{8}{3}\right)\vec{i} - \frac{\vec{k}}{3}(32 - 48 + 32 - 8)$$

$$= \frac{8}{3}\left(\vec{j} - \vec{k}\right).$$

Thank You

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18MAB102T- ADVANCED CALCULUS AND COMPLEX ANALYSIS; Unit II (Part-3) - Green's, Stoke's and Gauss Divergence theorem

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Outline

1. Green's theorem
2. Stoke's theorem
3. Gauss divergence theorem

Statement (Green's theorem):

Let C be a positively oriented, piecewise smooth, simple, closed curve and let R be the region enclosed by the curve C in the xy-plane. If $P(x, y)$ and $Q(x, y)$ have continuous first order partial derivatives on R, then

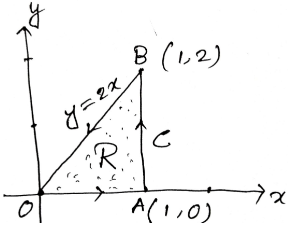
$$\oint_C Pdx + Qdy = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dxdy.$$

Applications of Green's theorem

Example 1:

Use Green's theorem to evaluate $\oint_C xydx + x^2y^3dy$, where C is the triangle with vertices $(0,0),(1,0),(1,2)$ with positive orientation.

Solution: Let $P = xy$, $Q = x^2y^3$ and the positive orientation curve C is as shown in the figure.



Applications of Green's theorem

Using Green's theorem,

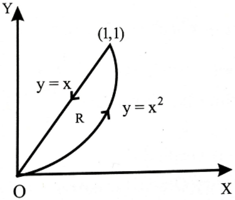
$$\oint_C xydx + x^2y^3dy = \oint_C Pdx + Qdy$$
$$= \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dxdy = \iint_R (2xy^3 - x)dxdy$$
$$= \int_0^1 \int_0^{2x} (2xy^3 - x)dydx = \int_0^1 \left[\frac{xy^4}{2} - xy\right]_0^{2x} dx$$
$$= \int_0^1 (8x^5 - 2x^2)dx = \left[\frac{4x^6}{3} - \frac{2x^3}{3}\right]_0^1 = \frac{2}{3}.$$

Applications of Green's theorem

Example 2:

Verify Green's theorem in the plane for $\oint_C [(xy + y^2)dx + x^2dy]$, where C is the closed curve of the region bounded by $y = x$ and $y = x^2$.

Solution: Let $P = xy + y^2$, $Q = x^2$ and the positive orientation curve C is as shown in the figure. The curves $y = x$ and $y = x^2$ intersect at $(0, 0)$ and $(1, 1)$.



Applications of Green's theorem

Using Green's theorem,

$$\oint_C [(xy + y^2)dx + x^2 dy] = \oint_C Pdx + Qdy$$

$$= \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dxdy = \iint_R (2x - x - 2y)dxdy$$

$$= \iint_R (x - 2y)dxdy = \int_0^1 \int_{y=x^2}^{x} (x - 2y)dydx$$

$$= \int_0^1 \left[xy - y^2 \right]_{y=x^2}^{x} dx = \int_0^1 (x^4 - x^3)dx$$

$$= \left[\frac{x^5}{5} - \frac{x^4}{4} \right]_0^1 = -\frac{1}{20}.$$

Applications of Green's theorem

Now let us evaluate the line integral along C. Along $y = x^2, dy = 2xdx$ and the line integral equals

$$\int_0^1 [(x(x^2) + x^4)dx + x^2(2x)dx] = \int_0^1 (3x^3 + x^4)dx$$
$$= \left[\frac{3x^4}{4} + \frac{x^5}{5}\right]_0^1 = \frac{19}{20}.$$

Along $y = x, dy = dx$ and the line integral equals

$$\int_1^0 [(x(x) + x^2)dx + x^2dx] = \int_1^0 (3x^2)dx = \left[\frac{3x^3}{3}\right]_1^0 = -1.$$

Therefore, the required line integral $= \frac{19}{20} - 1 = -\frac{1}{20}$. Hence the theorem is verified.

Statement (Stoke's theorem):

Let S be a smooth surface that is bounded by a simple closed, smooth boundary curve C with positive orientation and $\overrightarrow{F} = F_1 \overrightarrow{i} + F_2 \overrightarrow{j} + F_3 \overrightarrow{k}$ be any vector function having continuous first order partial derivatives, then

$$\oint_C \overrightarrow{F}.\overrightarrow{dr} = \iint_S curl \overrightarrow{F}.\hat{n} ds,$$

where $\hat{n}$ is the outward normal unit vector at any point of S.

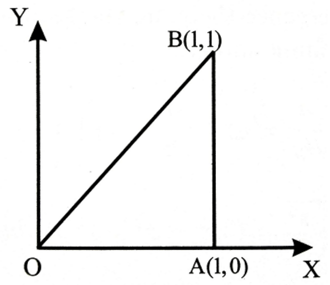
Applications of Stoke's theorem

Example 1:

Use Stoke's theorem to evaluate $\oint_C \overrightarrow{F}.\overrightarrow{dr}$, where $\overrightarrow{F} = y^2\overrightarrow{i} + x^2\overrightarrow{j} - (x+z)\overrightarrow{k}$ and C is the boundary of the triangle with vertices $(0,0,0),(1,0,0),(1,1,0)$ with positive orientation.

Solution: We note that z-coordinate of each vertex of the triangle is 0. Therefore, the triangle lies in the xy-plane. So $\widehat{n} = \overrightarrow{k}$ and the positive orientation curve C is as shown in the figure.

Applications of Stoke's theorem



Let $F_1 = y^2, F_2 = x^2, F_3 = -(x+z)$ and we have

$$curl\,\overrightarrow{F} = \begin{vmatrix} \overrightarrow{i} & \overrightarrow{j} & \overrightarrow{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y^2 & x^2 & -(x+z) \end{vmatrix} = 0\,\overrightarrow{i} + \overrightarrow{j} + 2(x-y)\,\overrightarrow{k}$$

Applications of Stoke's theorem

and $curl \overrightarrow{F}.\widehat{n} = [\overrightarrow{j} + 2(x-y)\overrightarrow{k}].\overrightarrow{k} = 2(x-y)$.
The equation of the line OB is $y = x$. Using Stoke's theorem,

$$\oint_C \overrightarrow{F}.\overrightarrow{dr} = \iint\limits_S curl \overrightarrow{F}.\widehat{n}ds = \int_0^1 \int_{y=0}^{x} 2(x-y)dxdy$$

$$= 2\int_0^1 \left[xy - \frac{y^2}{2}\right]_0^x dx = 2\int_0^1 \frac{x^2}{2}dx = \frac{1}{3}.$$

Applications of Stoke's theorem

Example 2:

Verify Stoke's theorem for $\overrightarrow{F} = (2x - y)\overrightarrow{i} - yz^2\overrightarrow{j} - y^2z\overrightarrow{k}$ over the upper half surface S of the sphere $x^2 + y^2 + z^2 = 1$ bounded by its projection on the xy-plane and C is its boundary.

Solution: The boundary C of S is a circle in the xy-plane of radius unity and centre at origin. Let $x = \cos t, y = \sin t$, $z = 0, 0 \leq t \leq 2\pi$ are parametric equations of C.

Applications of Stoke's theorem

Now

$$\oint_C \overrightarrow{F}.\overrightarrow{dr}$$
$$= \oint_C [(2x-y)\overrightarrow{i} - yz^2\overrightarrow{j} - y^2z\overrightarrow{k}].[dx\overrightarrow{i} + dy\overrightarrow{j} + dz\overrightarrow{k}]$$
$$= \oint_C (2x-y)dx - yz^2dy - y^2zdz = \oint_C (2x-y)dx$$
$$= -\int_0^{2\pi} (2\cos t - \sin t)\sin t dt = \pi. \quad (1)$$

Applications of Stoke's theorem

Also $\hat{n} = \overrightarrow{k}, ds = dxdy$,

$$curl\,\overrightarrow{F} = \begin{vmatrix} \overrightarrow{i} & \overrightarrow{j} & \overrightarrow{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2x-y & -yz^2 & -y^2z \end{vmatrix} = \overrightarrow{k}$$

and $curl\,\overrightarrow{F}.\hat{n} = \overrightarrow{k}.\overrightarrow{k} = 1$.
Using Stoke's theorem,

$$\iint_S curl\,\overrightarrow{F}.\hat{n}ds = \iint_S dxdy = \pi, \tag{2}$$

where $\pi(1)^2$ is the area of the circle C.
Hence from (1) and (2), the theorem is verified.

Statement (Gauss divergence theorem):

If V is the volume bounded by a closed surface S and $\overrightarrow{F}$ is a vector point function with continuous derivatives in V, then

$$\iint\limits_{S} curl\,\overrightarrow{F}.\hat{n}ds = \iiint\limits_{V} div\,\overrightarrow{F}\,dV,$$

where $\hat{n}$ is the outward normal unit vector at any point of S.

Applications of Gauss divergence theorem

Example 1:

Use Gauss divergence theorem to evaluate $\iint_S [(x^3 - yz)dydz - 2x^2ydzdx + zdxdy]$ over the surface S of a cube bounded by the coordinate planes and the plane $x = y = z = a$.

Solution: Let $F_1 = x^3 - yz, F_2 = -2x^2y, F_3 = z$. Using Gauss divergence theorem,

$$\iint_S curl\,\overrightarrow{F}.\hat{n}ds = \iiint_V div\,\overrightarrow{F}\,dV$$

$$= \iiint_V \left(\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}\right) dxdydz$$

Applications of Gauss divergence theorem

$$= \int_{x=0}^{a}\int_{y=0}^{a}\int_{z=0}^{a}(x^2+1)dxdydz = \int_{z=0}^{a}\int_{y=0}^{a}\left[\frac{x^3}{3}+x\right]_{x=0}^{a} dydz$$

$$= \left[\frac{a^3}{3}+a\right]\int_{z=0}^{a}\int_{y=0}^{a} dydz = a\left[\frac{a^3}{3}+a\right]\int_{z=0}^{a} dz = a^2\left[\frac{a^3}{3}+a\right].$$

Applications of Gauss divergence theorem

Example 2:

Use Gauss divergence theorem to evaluate $\iint\limits_S [(x+z)dydz + (y+z)dzdx + (x+y)dxdy]$ over the surface S of the sphere $x^2+y^2+z^2=4$.

Solution: Let $F_1 = x+z, F_2 = y+z, F_3 = x+y$. Using Gauss divergence theorem,

$$\iint\limits_S curl\,\overrightarrow{F}.\hat{n}ds = \iiint\limits_V div\,\overrightarrow{F}\,dV$$

$$= \iiint\limits_V \left(\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}\right) dV$$

Applications of Gauss divergence theorem

$$= \iiint_V 2dV = 2\iiint_V dV = 2V,$$

where V is the volume of the sphere $x^2+y^2+z^2=2^2$ ($\because$ the volume of a sphere of radius r is $\frac{4}{3}\pi r^3$).

$$= 2\left[\frac{4}{3}\pi(2)^3\right] = \frac{64}{3}\pi.$$

$= yz\vec{i} + xz\vec{j} + xy\vec{k}$

2) If $\phi = \log(x^2+y^2+z^2)$, find $\nabla\phi$.

$$\nabla\phi = \vec{i}\frac{\partial\phi}{\partial x} + \vec{j}\frac{\partial\phi}{\partial y} + \vec{k}\frac{\partial\phi}{\partial z}$$

$$= \vec{i}\left(\frac{1}{x^2+y^2+z^2}(2x)\right) + \vec{j}\left(\frac{2y}{x^2+y^2+z^2}\right) + \vec{k}\left(\frac{2z}{x^2+y^2+z^2}\right)$$

$$= \frac{2x}{x^2+y^2+z^2}\vec{i} + \frac{2y}{x^2+y^2+z^2}\vec{j} + \frac{2z}{x^2+y^2+z^2}\vec{k}$$

$$= 2\left[\frac{x\vec{i}+y\vec{j}+z\vec{k}}{x^2+y^2+z^2}\right]$$

$$= 2\left(\frac{\vec{r}}{r^2}\right)$$

3) If $\phi = x^2y + y^2x + z^2$, find $\nabla\phi$ at $(1,1,1)$.

$$\nabla\phi = \vec{i}(2xy+y^2) + \vec{j}(x^2+2yx) + \vec{k}(2z)$$

$$= \vec{i}(3) + \vec{j}(3) + (2)\vec{k}$$

$$= 3\vec{i} + 3\vec{j} + 2\vec{k}$$

4) Find ∇r.

$$\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$$

$$r^2 = |\vec{r}|^2 = x^2+y^2+z^2 \qquad 2r\frac{\partial r}{\partial x} = 2x$$

$$\cancel{\nabla\phi} \quad \nabla r = \vec{i}\frac{\partial r}{\partial x} + \vec{j}\frac{\partial r}{\partial y} + \vec{k}\frac{\partial r}{\partial z}$$

$$= \vec{i}\left(\frac{x}{r}\right) + \vec{j}\left(\frac{y}{r}\right) + \vec{k}\left(\frac{z}{r}\right)$$

$$= \frac{1}{r}(x\vec{i} + y\vec{j} + z\vec{k})$$

$$= \frac{\vec{r}}{r}$$

5) Find the unit normal vector to the surface $xy^3z^2 = 4$ at $(-1,-1,2)$.

Unit normal vector $(\hat{n}) = \dfrac{\nabla\phi}{|\nabla\phi|}$

$$\phi = xy^3z^2 - 4$$

$$\nabla\phi = \vec{i}(y^3z^2) + \vec{j}(3xy^2z^2) + \cancel{z}\,\vec{k}(2xy^3z)$$

SRM Institute of Science and Technology
Ramapuram Campus

Department of Mathematics

Year / Sem: I / II

Branch: Common to ALL Branches of B.Tech. except B.Tech. (Business Systems)

UNIT III - LAPLACE TRANSFORMS

Part – A

1.	$L[t] =$ (A) $\frac{1}{s}$ (B) $\frac{1}{s^2}$ (C) s (D) s^2	ANS **B**	(CLO-3, Apply)
2.	$L[\cos t] =$ (A) $\frac{1}{s^2-1}$ (B) $\frac{1}{s^2+1}$ (C) $\frac{s}{s^2-1}$ (D) $\frac{s}{s^2+1}$	ANS **D**	(CLO-3, Apply)
3.	$L[e^{3t}] =$ (A) $\frac{1}{s-3}$ (B) $\frac{s}{s^2+9}$ (C) $\frac{1}{s-\log 9}$ (D) $\frac{9}{s}$	ANS **A**	(CLO-3, Apply)
4.	If $L[f(t)] = F(s)$, then $L[e^{at} f(t)] =$ (A) $F(s+a)$ (B) $F(s-a)$ (C) $e^{as}F(s)$ (D) $e^{-as}F(s)$	ANS **B**	(CLO-3, Remember)
5.	$L[f(t) * g(t)] =$ (A) $F(s) - G(s)$ (B) $F(s) + G(s)$ (C) $F(s)\, G(s)$ (D) $F(s) \div G(s)$	ANS **C**	(CLO-3, Remember)
6.	$L[\sin t] =$ (A) $\frac{1}{s^2-1}$ (B) $\frac{1}{s^2+1}$ (C) $\frac{s}{s^2-1}$ (D) $\frac{s}{s^2+1}$	ANS **B**	(CLO-3, Apply)

7.	$L[e^{-3\,t}] =$ (A) $\frac{1}{s+3}$ (B) $\frac{s}{s^2+9}$ (C) $\frac{1}{s-\log 3}$ (D) $\frac{3}{s}$	ANS **A**	(CLO-3, Apply)
8.	$L^{-1}\left[\frac{1}{s}\right] =$ (A) t (B) s (C) 1 (D) $\delta(t)$	ANS **C**	(CLO-3, Apply)
9.	$L^{-1}\left[\frac{1}{s^2+9}\right] =$ (A) $\frac{\cos 3t}{3}$ (B) $\frac{\sin 3t}{3}$ (C) $\sin 3t$ (D) $\cos 3t$	ANS **B**	(CLO-3, Apply)
10.	$L^{-1}\left[\frac{s}{s^2+9}\right] =$ (A) $\frac{\cos 3t}{3}$ (B) $\frac{\sin 3t}{3}$ (C) $\sin 3t$ (D) $\cos 3t$	ANS **D**	(CLO-3, Apply)
11.	If $L[f(t)] = F(s)$, then $L[e^{-at} f(t)] =$ (A) $F(s+a)$ (B) $F(s-a)$ (C) $e^{as}F(s)$ (D) $e^{-as}F(s)$	ANS **A**	(CLO-3, Remember)
12.	$L[t^2] =$ (A) $\frac{1}{s}$ (B) $\frac{1}{s^2}$ (C) $\frac{2}{s^3}$ (D) $\frac{1}{s^3}$	ANS **C**	(CLO-3, Apply)
13.	$L[1] =$ (A) $\frac{1}{s}$ (B) $\frac{1}{s^2}$ (C) $\frac{2}{s^3}$ (D) $\frac{1}{s^3}$	ANS **A**	(CLO-3, Apply)
14.	$L[e^{-2\,t}] =$ (A) $\frac{1}{s\ +2}$ (B) $\frac{s}{s^2+4}$ (C) $\frac{1}{s-\log 4}$ (D) $\frac{4}{s}$	ANS **A**	(CLO-3, Apply)

15.	$L[\sin 3t] =$ (A) $\frac{1}{s^2-9}$ (B) $\frac{3}{s^2+9}$ (C) $\frac{s}{s^2-9}$ (D) $\frac{s}{s^2+9}$	ANS **B**	(CLO-3, Apply)
16.	$L[\sinh 2t] =$ (A) $\frac{2}{s^2-4}$ (B) $\frac{2}{s^2+4}$ (C) $\frac{1}{s^2-4}$ (D) $\frac{s}{s^2+4}$	ANS **A**	(CLO-3, Apply)
17.	$L[2^t] =$ (A) $\frac{1}{s-2}$ (B) $\frac{s}{s^2+4}$ (C) $\frac{1}{s-\log 2}$ (D) $\frac{2}{s}$	ANS **C**	(CLO-3, Apply)
18.	$L[t\,e^{2t}] =$ (A) $\frac{1}{s-2}$ (B) $\frac{1}{(s-2)^2}$ (C) $\frac{2}{(s-2)^3}$ (D) $\frac{1}{s^3}$	ANS **B**	(CLO-3, Apply)
19.	If $L[f(t)] = F(s)$, then $L[\,f(at)] =$ (A) $\frac{1}{a}\,F\left(\frac{s}{a}\right)$ (B) $F\left(\frac{s}{a}\right)$ (C) $F(s+a)$ (D) $F(s-a)$	ANS **A**	(CLO-3, Remember)
20.	$L^{-1}\left[\frac{s-2}{s^2-4s+13}\right] =$ (A) $\mathrm{e}^{-2t}\sin 3t$ (B) $\mathrm{e}^{-2t}\cos 3t$ (C) $\mathrm{e}^{2t}\sin 3t$ (D) $\mathrm{e}^{2t}\cos 3t$	ANS **D**	(CLO-3, Apply)
21.	If $L[f(t)] = F(s)$, then $L\left[\int_0^t f(u)du\right] =$ (A) $\frac{F(s)}{s}$ (B) $F\left(\frac{s}{a}\right)$ (C) $\frac{f(t))}{t}$ (D) $F(u)$	ANS **A**	(CLO-3, Remember)
22.	$L^{-1}[1] =$ (A) $\frac{1}{s}$ (B) s (C) 1 (D) $\delta(t)$	ANS **D**	(CLO-3, Apply)

23.	$L^{-1}\left[\frac{s-3}{s^2-6s+13}\right] =$ (A) $e^{-3t}\cos\ 3t$ (B) $e^{2t}\cos 3t$ (C) $e^{3t}\cos 2t$ (D) $e^{-2t}\cos 2t$	ANS **C**	(CLO-3, Apply)
24.	$L[4^{\,t}] =$ (A) $\frac{1}{s-4}$ (B) $\frac{s}{s^2+4}$ (C) $\frac{1}{s-\log 4}$ (D) $\frac{4}{s}$	ANS **C**	(CLO-3, Apply)
25.	$L[\cosh 3t] =$ (A) $\frac{s}{s^2+9}$ (B) $\frac{1}{s^2-9}$ (C) $\frac{s}{s^2-9}$ (D) $\frac{s}{s^2+9}$	ANS **C**	(CLO-3, Apply)
26.	$L[\mathrm{t}\cos at] =$ (A) $\frac{s^2+a^2}{(s^2-a^2)^2}$ (B) $\frac{s^2-a^2}{(s^2-a^2)^2}$ (C) $\frac{s^2-a^2}{(s^2+a^2)^2}$ (D) $\frac{s}{s^2+9}$	ANS **C**	(CLO-3, Apply)
27.	$L[t\sin 2t] =$ (A) $\frac{4\,s}{(s^2+4)^2}$ (B) $\frac{4\,s}{(s^2-4)^2}$ (C) $\frac{s}{(s^2+4)^2}$ (D) $\frac{4\,s}{(s^2-4)^2}$	ANS **A**	(CLO-3, Apply)
28.	$L[t\,e^t] =$ (A) $\frac{1}{s-1}$ (B) $\frac{1}{(s-2)^2}$ (C) $\frac{1}{(s-1)^2}$ (D) $\frac{1}{(s-1)^3}$	ANS **C**	(CLO-3, Apply)
29.	$L[2\,e^{-3\,t}] =$ (A) $\frac{2}{s+3}$ (B) $\frac{2}{(s-3)^2}$ (C) $\frac{1}{(s-3)^2}$ (D) $\frac{2}{(s-1)^3}$	ANS **A**	(CLO-3, Apply)
30.	$L[3] =$ (A) $\frac{1}{s-3}$ (B) $\frac{s}{s^2+9}$ (C) $\frac{1}{s+3}$ (D) $\frac{3}{s}$	ANS **D**	(CLO-3, Apply)

31.	$L[\sin 5t] =$ (A) $\frac{5}{s^2+29}$ (B) $\frac{5}{s^2+25}$ (C) $\frac{1}{s^2+29}$ (D) $\frac{s}{s^2+29}$	ANS **B**	(CLO-3, Apply)
32.	$L[\cos 2\,t] =$ (A) $\frac{1}{s^2-4}$ (B) $\frac{1}{s^2+4}$ (C) $\frac{s}{s^2-4}$ (D) $\frac{s}{s^2+4}$	ANS **D**	(CLO-3, Apply)
33.	$L[\cosh 2t] =$ (A) $\frac{s}{s^2+4}$ (B) $\frac{1}{s^2-4}$ (C) $\frac{s}{s^2-4}$ (D) $\frac{s}{s^2+4}$	ANS **C**	(CLO-3, Apply)
34.	$L^{-1}\left[\frac{1}{s-3}\right] =$ (A) e^{3t} (B) e^{-3t} (C) $\cos 3t$ (D) $\sin 3t$	ANS **A**	(CLO-3, Apply)
35.	$L^{-1}\left[\frac{s}{s^2-9}\right] =$ (A) $\cos 3t$ (B) $\sin 3t$ (C) $\cosh 3t$ (D) $\sinh 3t$	ANS **C**	(CLO-3, Apply)
36.	$L^{-1}\left[\frac{1}{(s-1)^2}\right] =$ (A) $t\,e^{t}$ (B) e^{t} (C) e^{-t} (D) $t\,e^{-t}$	ANS **A**	(CLO-3, Apply)

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SRM Institute of Science and Technology
Ramapuram Campus

Department of Mathematics

Year / Sem: I / II

Branch: Common to ALL Branches of B.Tech. except B.Tech. (Business Systems)

Unit 3 – Laplace Transforms

Part – B (Each question carries 3 Marks)

1. Find $L[2e^{-3t}]$.

Solution

$$L[e^{-at}] = \frac{1}{s+a}$$

$$L[2e^{-3t}] = 2L[e^{-3t}] = 2\left(\frac{1}{s+3}\right)$$

2. Find $L[e^{3t+5}]$.

Solution

$$L[e^{at}] = \frac{1}{s-a}$$

$$L\left[e^{3t}.e^{5}\right] = e^{5}L[e^{3t}] = e^{5}\left(\frac{1}{s-3}\right)$$

3. Find the Laplace transform of $f(t) = cos^2(3t)$.

Solution

$$L[\cos^2 3t] \quad = L\left[\frac{1+\cos 6t}{2}\right] = \frac{L(1)+L(\cos 6t)}{2} \qquad \because \cos^2 t = \frac{1+\cos 2t}{2}$$

$$= \frac{1}{2s} + \frac{s}{2(s^2+36)} \qquad \because L(1) = \frac{1}{s}, L(\cos at) = \frac{s}{s^2+a^2}$$

$\therefore$ $\quad$ **L[cos² 3t]** $= \dfrac{s^2+18}{s(s^2+36)}$

4. Find $L\left(t^2 - 4\sin 2t + 2\cos 3t\right)$.

Solution

$$L\left(t^2 - 4\sin 2t + 2\cos 3t\right) = \frac{2}{s^3} - 4\left(\frac{2}{s^2+4}\right) + 2\left(\frac{s}{s^2+9}\right)$$

5. Find the Laplace transform of e^{-t} **sin 2t.**

Solution

$$L[e^{-t}\sin 2t] = L[e^{-at} f(t)] = F(s+a) = F(s+1)$$

$$F(s) = L[f(t)] = L(\sin 2t) = \frac{2}{s^2+4}$$

$$F(s+1) = \frac{2}{(s+1)^2+4} = \frac{2}{s^2+2s+5}$$

6. Obtain the Laplace transform of $\boldsymbol{\sin 2t - 2t\cos 2t}$ **.**

Solution

$$L[\sin 2t - 2t\cos 2t] = L[\sin 2t] - 2L[t\cos 2t] = L[\sin 2t] - 2\left(-\frac{d}{ds}L[\cos 2t]\right)$$

$$= \frac{2}{s^2+4} + 2\frac{d}{ds}\left(\frac{s}{s^2+4}\right) = \frac{2}{s^2+4} + 2\left(\frac{\left(s^2+4\right)(1) - s(2s)}{\left(s^2+4\right)^2}\right)$$

$$= \frac{2\left(s^2+4\right) + 2\left(4-s^2\right)}{\left(s^2+4\right)^2}$$

$$\therefore L[\sin 2t - 2t\cos 2t] = \frac{16}{\left(s^2+4\right)^2}$$

7. Find $L\left(t e^t\right)$.

Solution

$$L\left(t\, f(t)\right) = -\frac{d}{ds} L(f(t))$$

$$L\left(t\,e^{t}\right)=-\frac{d}{ds}L(e^{t})$$

$$=-\frac{d}{ds}L\left(\frac{1}{s-1}\right)=\frac{1}{(s-1)^2}$$

8. Find $L(t\sin 2t)$**.**

Solution

$$L\left(t\ f(t)\right)=-\frac{d}{ds}L(f(t))$$

$$L\left(t\sin 2t\right)=-\frac{d}{ds}L(\sin 2t)$$

$$=-\frac{d}{ds}\left(\frac{2}{s^2+4}\right)=\frac{4s}{\left(s^2+4\right)^2}$$

9. Find the Laplace transform of $f(t)=t^2\cos t$.

Solution

$$L\left[t^2\cos t\right]=\left[\frac{d^2}{ds^2}L\left[\cos t\right]\right]=\frac{d^2}{ds^2}\left(\frac{s}{s^2+1}\right)$$

$$=\frac{d}{ds}\left(\frac{\left(s^2+1\right).1-1.2s.s}{\left(s^2+1\right)^2}\right)=\frac{d}{ds}\left(\frac{1-s^2}{\left(s^2+1\right)^2}\right)$$

$$=\frac{\left(s^2+1\right)^2(-2s)-\left(1-s^2\right)2\left(s^2+1\right)2s}{\left(s^2+1\right)^3}=\frac{-2s\left(3-s^2\right)}{\left(s^2+1\right)^3}$$

10. Find the Laplace transform of $\boldsymbol{f(t)=te^{-3t}cos2t}$

Solution

$$L[f(t)]=L[te^{-3t}cos2t]=-\frac{d}{ds}L[\cos 2t]_{s\to s+3}=-\frac{d}{ds}\left[\frac{s}{s^2+4}\right]_{s\to s+3}$$

$$=-\left[\frac{(s^2+4)(1)-s(2s)}{(s^2+4)^2}\right]_{s\to s+3}=\left[\frac{s^2-4}{(s^2+4)^2}\right]_{s\to s+3}$$

$$= \frac{(s+3)^2-4}{((s+3)^2+4)^2}$$

$$= \frac{s^2+6s+5}{(s^2+6s+13)^2}$$

11. Find the Laplace Transform of $f(t) = e^{-t}t\,cost$.

Solution

$$L[e^{-t}tcost] = -\frac{d}{ds}L[\cos t]_{s\to s+1} = -\frac{d}{ds}\left[\frac{s}{s^2+1}\right]_{s\to s+1}$$

$$= -\left[\frac{(s^2+1)(1)-s(2s)}{(s^2+1)^2}\right]_{s\to s+1}$$

$$= \left[\frac{s^2-1}{(s^2+1)^2}\right]_{s\to s+1}$$

$$= \frac{(s+1)^2-1}{((s+1)^2+1)^2} = \frac{s^2+2s}{(s^2+2s+2)^2}$$

$$= \frac{s(s+2)}{(s^2+2s+2)^2}$$

12. Find $L\left[\frac{sint}{t}\right]$.

Solution

$$L\left[\frac{sint}{t}\right]=L\left[\frac{f(t)}{t}\right]=\int_s^\infty F(s)ds$$

$$\text{F(s)} = \text{L}\,[sint]=\frac{1}{s^2+1^2}$$

$$\int_s^\infty F(s)ds=\int_s^\infty \frac{1}{s^2+1}ds=[tan^{-1}(s)]_s^\infty$$

$$=[tan^{-1}\infty - tan^{-1}s]=\left[\frac{\pi}{2} - tan^{-1}s\right] = cot^{-1}s$$

13. Find the Laplace transform of $f(t) = \dfrac{e^{-t}\sin t}{t}$.

Solution

$$L\left(\frac{e^{-t}\sin t}{t}\right) = \int_s^\infty L\left(e^{-t}\sin t\right)ds$$

$$= \int_s^\infty L(\sin t)_{s+1}\,ds = \int_s^\infty \left(\frac{1}{s^2+1}\right)_{s+1} ds = \int_s^\infty \frac{1}{(s+1)^2+1}ds$$

$$= \left[\tan^{-1}(s+1)\right]_s^\infty = \frac{\pi}{2} - \tan^{-1}(s+1) = \cot^{-1}(s+1)$$

14. Find the Laplace Transform of $\boldsymbol{f(t) = \frac{1-cost}{t}}$.

Solution

$$L[1-cost] = \frac{1}{s} - \frac{s}{s^2+1}$$

$$L\left[\frac{1-cost}{t}\right] = \int_s^\infty L[1-cost]ds = \int_s^\infty \left(\frac{1}{s} - \frac{s}{s^2+1}\right)ds$$

$$= \left[\log s - \frac{1}{2}\log(s^2+1)\right]_s^\infty$$

$$= -\frac{1}{2}[\log(s^2+1) - logs^2]_s^\infty$$

$$= -\frac{1}{2}\left[log\frac{s^2+1}{s^2}\right]_s^\infty = -\frac{1}{2}\left[log\left(1+\frac{1}{s^2}\right)\right]_s^\infty$$

$$= -\frac{1}{2}\log 1 + \frac{1}{2}log\left[1+\frac{1}{s^2}\right] = \frac{1}{2}log\left(\frac{s^2+1}{s^2}\right)$$

15. Find $\boldsymbol{L\left[\frac{cos\ at - cos\ bt}{t}\right]}$.

Solution

$$L\left[\frac{cosat - cosbt}{t}\right] = \int_s^\infty L[cosat - cosbt]ds$$

$$= \int_s^\infty \left(\frac{s}{s^2+a^2} - \frac{s}{s^2+b^2}\right)ds$$

$$= \left[\frac{1}{2}log(s^2+a^2) - \frac{1}{2}log(s^2+b^2)\right]_s^\infty$$

$$=\frac{1}{2}\left[log\,\frac{s^2+a^2}{s^2+b^2}\right]_s^\infty=\frac{1}{2}\left[log\,\frac{s^2\left(1+\frac{a^2}{s^2}\right)}{s^2\left(1+\frac{b^2}{s^2}\right)}\right]_s^\infty$$

$$=\frac{1}{2}\left[log1-log\left(\frac{1+\frac{a^2}{s^2}}{1+\frac{b^2}{s^2}}\right)\right]=\frac{1}{2}\ log\left(\frac{s^2+b^2}{s^2+a^2}\right)$$

16. Evaluate $\int_0^\infty t\,e^{-2t}\,sin\,t\,dt$ using Laplace transform.

Solution

$$\int_0^\infty te^{-2t}\sin t\,dt=\int_0^\infty e^{-st}f(t)dt=F(s)\text{ Here s = 2.}$$

$$F(s)\ =\ L\,[f\ (t)], F(s)\ =\ L\,[t\sin t]$$

$$=-\frac{d}{ds}\left[\frac{1}{s^2+1}\right]=\frac{2s}{(s^2+1)^2}$$

$$\int_0^\infty \boldsymbol{t\,e^{-2t}\,sint\,dt}\ =[\boldsymbol{F(s)}]_{s\,=\,2}=\frac{\mathbf{4}}{(\mathbf{4+1})^2}=\frac{\mathbf{4}}{\mathbf{25}}$$

17. Verify initial value theorem for the function *f* (*t*) = 2 – cos t.

Solution

Initial value theorem states that $\lim_{t\to 0} f(t)=\lim_{s\to\infty} sF(s)$

L. H. S. = $\lim_{t\to 0} f(t)=2-\cos 0=1$

R. H. S. = $\lim_{s\to\infty} s\,L(f(t))=\lim_{s\to\infty} s\,L(2-\cos t)$

$$=\lim_{s\to\infty} s\left(2-\frac{s^2}{s^2+1}\right)==\lim_{s\to\infty} s\left(2-\frac{1}{1+\frac{1}{s^2}}\right)=2-1=1$$

L.H.S=R.H.S

Initial value theorem verified.

18. Verify final value theorem for the function f(t) = 1 + e^{-t} (sin t + cos t).

Solution

$$L\left[f(t)\right]=F(s)$$

$$=\frac{1}{s}+L\left[\sin t+\cos t\right]_{s\to s+1}$$

$$=\frac{1}{s}+\frac{1}{(s+1)^2+1}+\frac{s+1}{(s+1)^2+1}=\frac{1}{s}+\frac{s+2}{(s+1)^2+1}$$

Final value theorem states that $\lim\limits_{t\to\infty} f(t)=\lim\limits_{s\to 0} sF(s)$

L.H.S. $=\lim\limits_{t\to\infty}\left[1+e^{-t}(\sin t+\cos t)\right]=1+0=1$

R. H. S. = $\lim\limits_{s\to 0} s\left[\frac{1}{s}+\frac{s+2}{(s+1)^2+1}\right]=\lim\limits_{s\to 0}\left[1+\frac{s^2+2s}{s^2+2s+2}\right]=1$

L.H.S.=R.H.S

Hence final value theorem verified

19. Find $L^{-1}\left(\frac{1}{s-3}+\frac{1}{s}+\frac{s}{s^2-9}\right)$**.**

Solution

$$L^{-1}\left(\frac{1}{s-3}+\frac{1}{s}+\frac{s}{s^2-9}\right)=e^{3t}+1+\cosh 3t$$

20. Find $L^{-1}\left(\frac{s}{(s+2)^2}\right)$**.**

Solution

$$L^{-1}\left(\frac{s}{(s+2)^2}\right)=L^{-1}\left(\frac{s+2-2}{(s+2)^2}\right)=L^{-1}\left(\frac{1}{(s+2)}\right)-2L^{-1}\left(\frac{1}{(s+2)^2}\right)=e^{-2t}-2te^{-2t}$$

21. Find $L^{-1}\left(\dfrac{1}{s^2+2s+5}\right)$**.**

Solution

$$L^{-1}\left(\frac{1}{s^2+2s+5}\right)=L^{-1}\left(\frac{1}{(s+1)^2+4}\right)=\frac{e^{-t}\ \sin 2t}{2}$$

22. Find $\boldsymbol{L^{-1}\left(\dfrac{s}{s^2+4s+5}\right)}$**.**

Solution

$$L^{-1}\left(\frac{s}{s^2+4s+5}\right)=L^{-1}\left(\frac{(s+2)-2}{(s+2)^2+1}\right)=e^{-2t}L^{-1}\left(\frac{s-2}{s^2+1}\right)$$

$$=e^{-2t}\left[L^{-1}\left(\frac{s}{s^2+1}\right)-2L^{-1}\left(\frac{1}{s^2+1}\right)\right]$$

$$=e^{-2t}\left[\cos t-2\sin t\right]$$

23. Find $L^{-1}\left(\dfrac{s-5}{s^2-3s+2}\right)$**.**

Solution:

$$L^{-1}\left(\frac{s-5}{s^2-3s+2}\right)=L^{-1}\left(\frac{A}{s-1}+\frac{B}{s-2}\right)=L^{-1}\left(\frac{4}{s-1}\right)+L^{-1}\left(\frac{-3}{s-2}\right)=4e^{t}-3e^{2t}$$

24. Find $\boldsymbol{L^{-1}\left[\dfrac{s+2}{s^2+2s+2}\right]}$**.**

Solution:

$$L^{-1}\left[\frac{s+2}{s^2+2s+2}\right]\ =\ L^{-1}\left[\frac{(s+1)+1}{(s+1)^2+1}\right]\ \because\ L^{-1}[F(s+a)]=e^{-at}L^{-1}[F(s)]$$

$$=\ L^{-1}\left[\frac{(s+1)}{(s+1)^2+1}\right]+L^{-1}\left[\frac{1}{(s+1)^2+1}\right]$$

$$=e^{-t}\left(L^{-1}\left[\frac{s}{s^2+1}\right]+L^{-1}\left[\frac{1}{s^2+1}\right]\right)=e^{-t}\ (\cos t+\sin t)$$

25. Find $L^{-1}\left[\frac{1}{s^2+6s+13}\right]$.

Solution

$$L^{-1}\left[\frac{1}{s^2+6s+13}\right] = L^{-1}\left[\frac{1}{(s+3)^2+4}\right] = L^{-1}\left[\frac{1}{(s+3)^2+2^2}\right]$$
$$= \frac{1}{2}L^{-1}\left[\frac{2}{(s+3)^2+2^2}\right] = \frac{1}{2}\,\mathrm{e}^{-3\mathrm{t}}\sin 2t.$$

26. Find $L^{-1}\left[\cot^{-1}(s+1)\right]$.

Solution:

Let $L^{-1}\left[\cot^{-1}(s+1)\right] = f(t)$

$$\therefore\ L\left[f(t)\right] = \cot^{-1}(s+1)$$

$$L\left[tf(t)\right] = -\frac{d}{ds}\left[\cot^{-1}(s+1)\right] = \frac{1}{(s+1)^2+1}$$

$$tf(t) = L^{-1}\left[\frac{1}{(s+1)^2+1}\right] = e^{-t}L^{-1}\left[\frac{1}{s^2+1}\right] = e^{-t}\sin t$$

$$\therefore\ f(t) = \frac{e^{-t}\sin t}{t}$$

27. Find the inverse Laplace transform of $\frac{s}{(s+2)^2}$.

Solution

$$L^{-1}\left(\frac{s}{(s+2)^2}\right) = L^{-1}\left(s.\frac{1}{(s+2)^2}\right)$$
$$= \frac{d}{dt}L^{-1}\left(\frac{1}{(s+2)^2}\right) = \frac{d}{dt}e^{-2t}L^{-1}\left(\frac{1}{s^2}\right)$$
$$= \frac{d}{dt}\left(e^{-2t}\,t\right) = e^{-2t} + t\left(-2e^{-2t}\right) = e^{-2t}(1-2t)$$

* * * * *

Module – 3 Laplace Transforms

Laplace Transforms of standard functions – Transforms properties – Transforms of Derivatives and Integrals – Initial value theorems (without proof) and verification for some problems – Final value theorems (without proof) and verification for some problems – Inverse Laplace transforms using partial fractions – Inverse Laplace transforms using second shifting theorem – LT using Convolution theorem – problems only – ILT using Convolution theorem – problems only – LT of periodic functions – problems only – Solve linear second order ordinary differential equations with constant coefficients only – Solution of Integral equation and integral equation involving convolution type – Application of Laplace Transform in Engineering.

Periodic function:

A function $f(t)$ is said to be periodic function if $f(t + p) = f(t)$ for all t. The least value of $p > 0$ is called the period of $f(t)$. For example, $\sin t$ and $\cos t$ are periodic functions with period 2π .

Laplace Transform:

Let f(t) be a given function which is defined for all positive values of t, if

$L\{f(t)\}= F(s) = \int_0^\infty e^{-st} f(t)\, dt$ exists, then F(s) is called *Laplace transform* of f(t).

Sufficient condition for the existence of Laplace transform:

The Laplace transform of $f(t)$ exists if

- **i.** $f(t)$ is piecewise continuous in $[a, b]$ where $a > 0$.
- **ii.** $f(t)$ is of exponential order.

Laplace transform for some basic functions

S.No	f(t)	L{f(t)}
1	e^{at}	$\frac{1}{s-a},\ s-a>0$
2	e^{-at}	$\frac{1}{s+a},\ s+a>0$
3	$\sin at$	$\frac{a}{s^2+a^2},\ s>0$
4	$\cos at$	$\frac{s}{s^2+a^2},\ s>0$

5	$\sinh at$	$\frac{a}{s^2-a^2}$, $s>\lvert a\rvert$
6	$\cosh at$	$\frac{s}{s^2-a^2}$, $s>\lvert a\rvert$
7	1	$\frac{1}{s}$
8	t	$\frac{1}{s^2}$
9	t^n	$\frac{n!}{s^{n+1}}$
10	Periodic function with period 'p'	$\frac{1}{1-e^{-ps}}\int_0^p e^{-at} f(t)\,dt$

Properties of Laplace transform:

Sl. No.	Property	Laplace Transform
1	Linear Property	$L(af(t)\pm bg(t))=aL(f(t))\pm bL(g(t))$
2	First shifting theorem	$L(e^{-at}f(t))=F(s+a)$ $L(e^{at}f(t))=F(s-a)$
3	Change of scale property	$L(f(at))=\frac{1}{a}F\left(\frac{s}{a}\right),\ a>0$
4	Multiplication by t	$L(t^n f(t))=(-1)^n\frac{d^n}{ds^n}F(s)$
5	Division by t	$L\left(\frac{f(t)}{t}\right)=\int_s^\infty F(s)\,ds$, provided $\lim_{t\to 0}\frac{f(t)}{t}$ exists
6	Transforms of integrals	$L\left(\int_0^t f(t)dt\right)=\frac{L[f(t)]}{s}$

Inverse Laplace transform for some basic functions:

S.No	F(s)	$f(t)=L^{-1}(F(s))$
1	$\frac{1}{s-a},\ s-a>0$	e^{at}
2	$\frac{1}{s+a},\ s+a>0$	e^{-at}
3	$\frac{a}{s^2+a^2},\ s>0$	$\sin at$
4	$\frac{s}{s^2+a^2},\ s>0$	$\cos at$
5	$\frac{a}{s^2-a^2},\ s>\lvert a\rvert$	$\sinh at$
6	$\frac{s}{s^2-a^2},\ s>\lvert a\rvert$	$\cosh at$
7	$\frac{1}{s}$	1
8	$\frac{1}{s^2}$	t
9	$\frac{n!}{s^{n+1}}$	t^n

Initial Value theorem:

If $L(f(t))=F(s)$ then $\lim\limits_{t\to 0} f(t)=\lim\limits_{s\to\infty} sF(s)$

Final value theorem:

If $L(f(t))=F(s)$ then $\lim\limits_{t\to\infty} f(t)=\lim\limits_{s\to 0} sF(s)$

Convolution:

The convolution of two functions f(t) and g(t) is defined as $\int_0^t f(u)g(t-u)\,du = f(t)*g(t)$

Convolution theorem:

The Laplace transform of convolution of two functions is equal to the product of their Laplace transforms.

(i.e) $L\left[f(t)*g(t)\right]=L\left[f(t)\right]L\left[g(t)\right]$.

1. **Obtain the Laplace transform of** $\boldsymbol{\sin 2t-2t\cos 2t}$.

Solution: $L[\sin 2t-2t\cos 2t] = L[\sin 2t]-2L[t\cos 2t]=L[\sin 2t]-2\left(-\dfrac{d}{ds}L[\cos 2t]\right)$

$$=\frac{2}{s^2+4}+2\frac{d}{ds}\left(\frac{s}{s^2+4}\right) = \frac{2}{s^2+4}+2\left(\frac{(s^2+4)(1)-s(2s)}{(s^2+4)^2}\right)$$

$$=\frac{2(s^2+4)+2(4-s^2)}{(s^2+4)^2}$$

$\therefore L[\sin 2t-2t\cos 2t] = \dfrac{16}{(s^2+4)^2}$

2. **Find the Laplace transform** $\boldsymbol{\sin^3(2t)}$

Solution: $L\left[\sin^3(2t)\right]=\dfrac{1}{4}L[3\sin 2t-\sin 6t]=\dfrac{3}{4}L[\sin 2t]-\dfrac{1}{4}L[\sin 6t]$

$\left(\because \sin^3 t=\dfrac{1}{4}[3\sin t-\sin 3t]\right)$

$$=\frac{3}{4}\left(\frac{2}{s^2+4}\right)-\frac{1}{4}\left(\frac{6}{s^2+36}\right) = \frac{6}{4}\left(\frac{1}{s^2+4}-\frac{1}{s^2+36}\right).$$

Find the Laplace transform of $\boldsymbol{f(t)=\cos^2(3t)}$.

3.

Solution: $L[\cos^2 3t] = L\left[\dfrac{1+\cos 6t}{2}\right]=\dfrac{L(1)+L(\cos 6t)}{2}\ \because \cos^2 t=\dfrac{1+\cos 2t}{2}$

$$=\frac{1}{2s}+\frac{s}{2(s^2+36)}\ \because L(1)=\frac{1}{s}, L(\cos at)=\frac{s}{s^2+a^2}$$

$\therefore \quad L[\cos^2 3t] = \dfrac{s^2+18}{s(s^2+36)}$

4. **Find the Laplace transform of unit step function**

Solution: The Unit step function is $u_a(t) = \begin{cases} 0, & t < a \\ 1, & t > a, \; a \geq 0 \end{cases}$

The Laplace transform $L\left[f(t)\right] = \int_0^\infty e^{-st} f(t)\,dt = \int_a^\infty e^{-st}(1)\,dt = \left[\frac{e^{-st}}{-s}\right]_a^\infty = -\frac{1}{s}\left[e^{-\infty} - e^{-as}\right] = \frac{e^{-as}}{s}.$

5. **Find the Laplace transform of the following functions (i) $\boldsymbol{\frac{e^{-t}\sin t}{t}}$** (ii) $\boldsymbol{t^2 \cos t}$

Solution:

(i) To find $\frac{e^{-t}\sin t}{t}$

$$L\left(\frac{e^{-t}\sin t}{t}\right) = \int_s^\infty L\left(e^{-t}\sin t\right)ds$$

$$= \int_s^\infty L(\sin t)_{s+1}\,ds = \int_s^\infty \left(\frac{1}{s^2+1}\right)_{s+1} ds = \int_s^\infty \frac{1}{(s+1)^2+1}\,ds$$

$$= \left[\tan^{-1}(s+1)\right]_s^\infty = \frac{\pi}{2} - \tan^{-1}(s+1) = \cot^{-1}(s+1)$$

(ii) $t^2 \cos t$

$$L\left[t^2\cos t\right] = \left[\frac{d^2}{ds^2}L[\cos t]\right] = \frac{d^2}{ds^2}\left(\frac{s}{s^2+1}\right)$$

$$= \frac{d}{ds}\left(\frac{(s^2+1).1 - 1.2s.s}{(s^2+1)^2}\right) = \frac{d}{ds}\left(\frac{1-s^2}{(s^2+1)^2}\right)$$

$$= \frac{(s^2+1)^2(-2s) - (1-s^2)2(s^2+1)2s}{(s^2+1)^3} = \frac{-2s(3-s^2)}{(s^2+1)^3}$$

6. **Find the Laplace transform of $\boldsymbol{e^{-2t}t^{1/2}}$.**

Solution: $L\left[e^{-2t}t^{1/2}\right] = L\left[t^{1/2}\right]_{s \to s+2}$

$$\because \textit{If } L\left[f(t)\right] = F(s), \text{then } L\left[e^{-at}f(t)\right] = F(s)\big|_{s \to s+a}$$

$$= \left[\frac{\Gamma\left(\frac{1}{2}+1\right)}{s^{3/2}}\right]_{s\to s+2} = \left[\frac{\frac{1}{2}\Gamma\left(\frac{1}{2}\right)}{s^{3/2}}\right]_{s\to s+2}$$

$$= \frac{\frac{1}{2}\sqrt{\pi}}{(s+2)^{3/2}} \quad \left(\because \Gamma\left(\frac{1}{2}\right)=\sqrt{\pi}, \Gamma n+1 = n\Gamma n\right)$$

7. **Find $L\left[t^2 e^{-t}\cos t\right]$**

Solution:

$$L\left[t^2 e^{-t}\cos t\right] = L\left[t^2\cos t\right]_{s\to s+1}$$

$$= \left[(-1)^2 \frac{d^2}{ds^2} L[\cos t]\right]_{s\to s+1} = \left[\frac{d^2}{ds^2}\left[\frac{s}{s^2+1}\right]\right]_{s\to s+1}$$

$$= \left[\frac{d}{ds}\frac{\left(s^2+1\right)1 - s.2s}{\left(s^2+1\right)^2}\right]_{s\to s+1}$$

$$= \left[\frac{d}{ds}\frac{1-s^2}{\left(s^2+1\right)^2}\right]_{s\to s+1}$$

$$= \left[\frac{2s^3-6s}{\left(s^2+1\right)^3}\right]_{s\to s+1}$$

$$= \frac{2(s+1)^3 - 6(s+1)}{\left((s+1)^2+1\right)^3}$$

8. **Find $L\left[t^2 e^{t}\sin t\right]$**

Solution:

$$L\left[t^2 e^{t}\sin t\right] = (-1)^2 \frac{d^2}{ds^2} L\left[e^t \sin t\right] \quad \dots (1)$$

Now $L\left[e^{t}\sin t\right]=\left[L[\sin t]\right]_{s\to(s-1)}=\dfrac{1}{(s-1)^{2}+1}$ ... (2)

Substituting (2) in (1) we get

$$L\left[t^{2}e^{t}\sin t\right]=\frac{d}{ds}\left[\frac{0-2(s-1)}{\left((s-1)^{2}+1\right)^{2}}\right]=\frac{d}{ds}\left[\frac{-2(s-1)}{\left(s^{2}-2s+2\right)^{2}}\right]$$

$$=\frac{\left(s^{2}-2s+2\right)^{2}(-2)+2(s-1)2\left(s^{2}-2s+2\right)(2s-2)}{\left(s^{2}-2s+2\right)^{4}}$$

$$=\frac{2\left(s^{2}-2s+2\right)\left[-\left(s^{2}-2s+2\right)+4(s-1)^{2}\right]}{\left(s^{2}-2s+2\right)^{4}}$$

$$=\frac{2\left(s^{2}-2s+2\right)\left[-s^{2}+2s-2+4s^{2}+4-8s\right]}{\left(s^{2}-2s+2\right)^{4}}$$

$$\therefore F(s)=\frac{2\left(s^{2}-2s+2\right)\left[3s^{2}-6s+2\right]}{\left(s^{2}-2s+2\right)^{4}}=\frac{2\left(3s^{2}-6s+2\right)}{\left(s^{2}-2s+2\right)^{3}}$$

9. **Find** $\boldsymbol{L\left[\dfrac{\sin^{2}t}{t}\right]}$

Solution:

$$L\left[\frac{\sin^{2}t}{t}\right]=L\left[\frac{1-\cos 2t}{2t}\right]=\frac{1}{2}L\left[\frac{1-\cos 2t}{t}\right]=\frac{1}{2}\int_{s}^{\infty}L[1-\cos 2t]\;ds$$

$$=\frac{1}{2}\int_{s}^{\infty}\left\{L[1]-L[\cos 2t]\right\}ds=\frac{1}{2}\int_{s}^{\infty}\left[\frac{1}{s}-\frac{s}{s^{2}+4}\right]ds$$

$$=\frac{1}{2}\left[\log s-\frac{1}{2}\log\left(s^{2}+4\right)\right]_{s}^{\infty}=\frac{1}{2}\left[\log\frac{s}{\sqrt{s^{2}+4}}\right]_{s}^{\infty}$$

$$= \frac{1}{2}\left[\log \frac{1}{\sqrt{1+\frac{4}{s^2}}}\right]_s^\infty = \frac{1}{2}\left[\log 1 - \log \frac{1}{\sqrt{1+\frac{4}{s^2}}}\right] = \frac{1}{2}\left[0 - \log \frac{s}{\sqrt{s^2+4}}\right]$$

$$F(s) = \frac{1}{2}\log\left(\frac{s}{\sqrt{s^2+4}}\right)^{-1} = \frac{1}{2}\log\left(\frac{\sqrt{s^2+4}}{s}\right)$$

10. **Using Laplace transform, Evaluate** $\int_0^\infty t\, e^{-2t} \sin t\, dt$

Solution: $\int_0^\infty e^{-2t} f(t)dt = \left[\int_0^\infty e^{-st} f(t)dt\right]_{s=2} = [L[t\sin t]]_{s=2} = \left[-\frac{d}{ds} L[\sin t]\right]_{s=2}$

$$= -\frac{d}{ds}\left(\frac{1}{s^2+1}\right) = -\left(\frac{-2s}{(s^2+1)^2}\right) = \frac{4}{25}$$

11. **Evaluate** $\int_0^t \sin u \cos(t-u)\, du$ **using Laplace Transform.**

Solution: Let $L\left[\int_0^t \sin u \cos(t-u)du\right] = L[\sin t * \cos t]$

= L[sin t] L[cos t] (by Convolution theorem)

$$= \frac{1}{(s^2+1)}\frac{s}{(s^2+1)} = \frac{s}{(s^2+1)^2}$$

$$\int_0^t \sin u \cos(t-u)du = L^{-1}\left[\frac{s}{(s^2+1)^2}\right] = \frac{1}{2}L^{-1}\left[\frac{2s}{(s^2+1)^2}\right] = \frac{t}{2}\sin t \quad \left(\because L^{-1}\left[\frac{2s}{(s^2+a^2)^2}\right] = t\sin at\right)$$

12. **Find the Laplace transform of** $\int_0^t t\, e^{-t} \sin t\, dt$

Solution:

$$L[\sin t] = \frac{1}{s^2+1}$$

$$L[t\sin t] = -\frac{d}{ds}\left(\frac{1}{s^2+1}\right) = -\left(\frac{\left(s^2+1\right)0-2s}{\left(s^2+1\right)^2}\right) = \frac{2s}{\left(s^2+1\right)^2}$$

$$\therefore\ L\left[te^{-t}\ \sin t\right] = \left.\frac{2s}{\left(s^2+1\right)^2}\right|_{s\to s+1} = \frac{2(s+1)}{\left((s+1)^2+1\right)^2} = \frac{2(s+1)}{\left(s^2+2s+2\right)^2}$$

$$L\left[\int_0^t t\,e^{-t}\sin t\,dt\right] = \frac{1}{s}L\left[te^{-t}\ \sin t\right]$$

$$\therefore\quad = \frac{1}{s}\frac{2(s+1)}{\left(s^2+2s+2\right)^2}$$

13. **Find the Laplace transform of** $e^{-t}\int_0^t t\cos t\,dt$

$$L\left[e^{-t}\int_0^t t\cos t\,dt\right] = \left[L\left(\int_0^t t\cos t\,dt\right)\right]_{s\to s+1} = \left[\frac{1}{s}L(t\cos t)\right]_{s\to(s+1)}$$

$$= \left[\frac{1}{s}\left(-\frac{d}{ds}L(\cos t)\right)\right]_{s\to(s+1)} = \left[-\frac{1}{s}\frac{d}{ds}\left(\frac{s}{s^2+1}\right)\right]_{s\to(s+1)}$$

$$= \left[-\frac{1}{s}\left(\frac{s^2+1-2s^2}{\left(s^2+1\right)^2}\right)\right]_{s\to(s+1)} = \left[-\frac{1}{s}\left(\frac{1-s^2}{\left(s^2+1\right)^2}\right)\right]_{s\to(s+1)}$$

$$\therefore F(s) = \left[\frac{s^2-1}{s\left(s^2+1\right)^2}\right]_{s\to(s+1)} = \left[\frac{(s+1)^2-1}{(s+1)\left[(s+1)^2+1\right]^2}\right] = \frac{s^2+2s}{(s+1)\left(s^2+2s+2\right)^2}$$

14. **Find the Laplace transform of** $e^{-4t}\int_0^t t\ \sin 3t dt$

Solution:

$$L[\sin 3t] = \frac{3}{s^2+9}$$

$$L\left[t\sin 3t\right] = -\frac{d}{ds}\left(\frac{3}{s^2+9}\right) = -\left(\frac{\left(s^2+9\right)0-3(2s)}{\left(s^2+9\right)^2}\right) = \frac{6s}{\left(s^2+9\right)^2}$$

$$L\left(\int_0^t t\sin 3t dt\right) = \frac{L(t\sin 3t)}{s} = \frac{6}{(s^2+9)^2}$$

$$L\left(e^{-4t}\int_0^t t\sin 3t dt\right) = L\left(\int_0^t t\sin 3t dt\right)\Bigg|_{s\to s+4} = \frac{6}{\left(\left(s+4\right)^2+9\right)^2} = \frac{6}{\left(s^2+8s+16+9\right)^2}$$

$$\therefore\ L\left(e^{-4t}\int_0^t t\sin 3t dt\right) = \frac{6}{\left(s^2+8s+25\right)^2}$$

15. **Verify initial and final value theorems for the function f(t) = 1 + e^{-t} (sin t + cos t)**

Solution:

Initial value theorem states that $\lim\limits_{t\to 0} f(t) = \lim\limits_{s\to\infty} sF(s)$

$$L\left[f(t)\right] = F(s)$$

$$= \frac{1}{s} + L\left[\sin t + \cos t\right]_{s\to s+1}$$

$$= \frac{1}{s} + \frac{1}{\left(s+1\right)^2+1} + \frac{s+1}{\left(s+1\right)^2+1} = \frac{1}{s} + \frac{s+2}{\left(s+1\right)^2+1}$$

L.H.S. $= \lim\limits_{t\to 0} f(t) = 1+1 = 2$

R.H.S $= \lim\limits_{s\to\infty} s\left[\frac{1}{s} + \frac{s+2}{\left(s+1\right)^2+1}\right] = \lim\limits_{s\to\infty}\left[1 + \frac{s\left(s+2\right)}{\left(s+1\right)^2+1}\right]$

$$= \lim_{s\to\infty}\left[1 + \frac{s^2\left(1+\frac{2}{s}\right)}{s^2\left[1+\frac{2}{s}+\frac{2}{s^2}\right]}\right] = \lim_{s\to\infty}\left[1 + \frac{1+\frac{2}{s}}{1+\frac{2}{s}+\frac{2}{s^2}}\right] = 1+1 = 2$$

L.H.S=R.H.S

Initial value theorem verified.

Final value theorem states that $\lim_{t\to\infty} f(t) = \lim_{s\to 0} sF(s)$

L.H.S. $= \lim_{t\to\infty}\left[1+e^{-t}(\sin t+\cos t)\right] = 1+0 = 1$

R.H.S $= \lim_{s\to 0}\left[1+\dfrac{s(s+2)}{(s+1)^2+1}\right] = 1+0 = 1$

L.H.S.=R.H.S

Hence final value theorem verified

16. **Find the Laplace transform of the square wave function defined by** $\boldsymbol{f(t) = \begin{cases} E, & 0 < t < \frac{a}{2} \\ -E, & \frac{a}{2} < t < a \end{cases} \;\&\; f(t+a) = f(t)}$

Solution:

$$L[f(t)] = \frac{1}{1-e^{-as}}\int_0^a e^{-st} f(t)\,dt$$

$$= \frac{1}{1-e^{-as}}\left[\int_0^{a/2} e^{-st} f(t)\,dt + \int_{a/2}^{a} e^{-st} f(t)\,dt\right]$$

$$= \frac{1}{1-e^{-as}}\left[\int_0^{a/2} Ee^{-st}\,dt + \int_{a/2}^{a} e^{-st}(-E)\,dt\right] = \frac{E}{1-e^{-as}}\left[\left(\frac{e^{-st}}{-s}\right)_0^{a/2} - \left(\frac{e^{-st}}{-s}\right)_{a/2}^{a}\right]$$

$$= \frac{E}{s\left(1-e^{-as}\right)}\left[-\left(e^{-\frac{as}{2}}-1\right)+\left(e^{-as}-e^{-\frac{as}{2}}\right)\right]$$

$$= \frac{E}{s\left(1-e^{-as}\right)}\left[-e^{-\frac{as}{2}}+1+e^{-as}-e^{-\frac{as}{2}}\right]$$

$$= \frac{E}{s\left(1-e^{-\frac{as}{2}}\right)\left(1+e^{-\frac{as}{2}}\right)}\left(1-e^{-\frac{as}{2}}\right)^2 = \frac{E}{s}\left(\frac{1-e^{-\frac{as}{2}}}{1+e^{-\frac{as}{2}}}\right)$$

$$\therefore F(s) = \frac{E}{s}\left[\frac{e^{sa/4} - e^{-sa/4}}{e^{sa/4} + e^{-sa/4}}\right] = \frac{E}{s}\tanh\left(\frac{sa}{4}\right)$$

17. **Find the Laplace transform of the rectangular wave given by** $f(t) = \begin{cases} 1, & 0 < t < b \\ -1, & b < t < 2b \end{cases}$

Given $f(t) = \begin{cases} 1, & 0 < t < b \\ -1, & b < t < 2b \end{cases}$

This function is periodic in the interval $(0, 2b)$ with period $2b$.

$$L[f(t)] = \frac{1}{1 - e^{-2bs}} \int_0^{2b} e^{-st} f(t)\,dt$$

$$= \frac{1}{1 - e^{-2bs}}\left[\int_0^{b} e^{-st} f(t)\,dt + \int_b^{2b} e^{-st} f(t)\,dt\right]$$

$$= \frac{1}{1 - e^{-2bs}}\left[\int_0^{b} e^{-st}\,dt + \int_b^{2b} e^{-st}(-1)\,dt\right] = \frac{1}{1 - e^{-2bs}}\left[\left(\frac{e^{-st}}{-s}\right)_0^b - \left(\frac{e^{-st}}{-s}\right)_b^{2b}\right]$$

$$= \frac{1}{s\left(1 - e^{-2bs}\right)}\left[-\left(e^{-bs} - 1\right) + \left(e^{-2bs} - e^{-bs}\right)\right]$$

$$= \frac{1}{s\left(1 - e^{-2bs}\right)}\left[-e^{-bs} + 1 + \left(e^{-bs}\right)^2 - e^{-bs}\right]$$

$$= \frac{1}{s\left(1 - e^{-bs}\right)\left(1 + e^{-bs}\right)}\left(1 - e^{-bs}\right)^2 = \frac{1}{s}\left(\frac{1 - e^{-bs}}{1 + e^{-bs}}\right)$$

$$\therefore F(s) = \frac{1}{s}\left[\frac{e^{sb/2} - e^{-sb/2}}{e^{sb/2} + e^{-sb/2}}\right] = \frac{1}{s}\tanh\left(\frac{sb}{2}\right)$$

18. **Find the Laplace transform of f(t) =** $\begin{cases} t, & 0 \le t \le a \\ 2a - t, & a \le t \le 2a \end{cases}$ **and** $f(t+2a) = f(t)$ **for all** t

Solution:

$$L[f(t)] = \frac{1}{1 - e^{-2as}} \int_0^{2a} e^{-st} f(t)\,dt$$

$$= \frac{1}{1-e^{-2as}}\left[\int_0^a e^{-st}\, f(t)\,dt + \int_a^{2a} e^{-st}\, f(t)\,dt\right]$$

$$= \frac{1}{1-e^{-2as}}\left[\int_0^a e^{-st}\, t\,dt + \int_a^{2a} e^{-st}\,(2a-t)\,dt\right]$$

$$= \frac{1}{1-e^{-2as}}\left[\left[t\left(\frac{e^{-st}}{-s}\right)-(1)\left(\frac{e^{-st}}{s^2}\right)\right]_0^a + \left[(2a-t)\left(\frac{e^{-st}}{-s}\right)-(-1)\left(\frac{e^{-st}}{s^2}\right)\right]_a^{2a}\right]$$

$$= \frac{1}{1-e^{-2as}}\left[\left[-t\left(\frac{e^{-st}}{s}\right)-\left(\frac{e^{-st}}{s^2}\right)\right]_0^a + \left[-(2a-t)\left(\frac{e^{-st}}{s}\right)+\left(\frac{e^{-st}}{s^2}\right)\right]_a^{2a}\right]$$

$$= \frac{1}{1-e^{-2as}}\left[\left[\left(-a\,\frac{e^{-as}}{s}-\frac{e^{-as}}{s^2}\right)-\left(-\frac{1}{s^2}\right)\right]+\left[\frac{e^{-2as}}{s^2}-\left(-\frac{ae^{-as}}{s}+\frac{e^{-as}}{s^2}\right)\right]\right]$$

$$= \frac{1}{1-e^{-2as}}\left[\frac{-ae^{-as}}{s}-\frac{e^{-as}}{s^2}+\frac{1}{s^2}+\frac{e^{-2as}}{s^2}+\frac{ae^{-as}}{s}-\frac{e^{-as}}{s^2}\right]$$

$$= \frac{1}{1-e^{-2as}}\left[\frac{1+e^{-2as}-2e^{-as}}{s^2}\right] = \frac{\left(1-e^{-sa}\right)^2}{s^2\left(1-e^{-as}\right)\left(1+e^{-as}\right)}$$

$$\therefore\ F(s) = \frac{1-e^{-sa}}{s^2\left(1+e^{-as}\right)} = \frac{1}{s^2}\tanh\left(\frac{as}{2}\right)$$

19. **Find the Laplace transform of the rectangular wave given by** $f(t)=\begin{cases}\sin\omega t, & 0<t<\dfrac{\pi}{\omega}\\ 0, & \dfrac{\pi}{\omega}<t<\dfrac{2\pi}{\omega}\end{cases}$

Solution:

This function is periodic function with period $\frac{2\pi}{\omega}$ in the interval $\left(0,\frac{2\pi}{\omega}\right)$

$$L\left[f(t)\right] = \frac{1}{1-e^{\frac{-2\pi s}{\omega}}}\int_0^{\frac{2\pi}{\omega}} e^{-st}\, f(t)\,dt$$

$$= \frac{1}{1-e^{\frac{-2\pi s}{\omega}}}\left[\int_0^{\frac{\pi}{\omega}} e^{-st}\sin\omega t\,dt + 0\right]$$

$$= \frac{1}{1-e^{\frac{-2\pi s}{\omega}}}\left[\frac{e^{-st}}{s^2+\omega^2}\left[-s\sin\omega t - \omega\cos\omega t\right]\right]_0^{\frac{\pi}{\omega}}$$

$$= \frac{1}{1-e^{\frac{-2\pi s}{\omega}}}\left[\frac{e^{\frac{-s\pi}{\omega}}\omega + \omega}{s^2+\omega^2}\right]$$

$$= \frac{\omega\left(e^{\frac{-s\pi}{\omega}}+1\right)}{\left(1-e^{\frac{-\pi s}{\omega}}\right)\left(1+e^{\frac{-\pi s}{\omega}}\right)\left(s^2+\omega^2\right)} = \frac{\omega}{\left(1-e^{\frac{-\pi s}{\omega}}\right)\left(s^2+\omega^2\right)}$$

20. **Find $L^{-1}\left(\frac{s}{s^2+4s+5}\right)$**

Solution:

$$L^{-1}\left(\frac{s}{s^2+4s+5}\right) = L^{-1}\left(\frac{(s+2)-2}{(s+2)^2+1}\right) = e^{-2t}L^{-1}\left(\frac{s-2}{s^2+1}\right)$$

$$= e^{-2t}\left[L^{-1}\left(\frac{s}{s^2+1}\right) - 2L^{-1}\left(\frac{1}{s^2+1}\right)\right]$$

$$= e^{-2t}\left[\cos t - 2\sin t\right]$$

21. **Find $L^{-1}\left[\frac{s+2}{s^2+2s+2}\right]$**

Solution: $L^{-1}\left[\frac{s+2}{s^2+2s+2}\right] = L^{-1}\left[\frac{(s+1)+1}{(s+1)^2+1}\right] \because L^{-1}\left[F(s+a)\right] = e^{-at}L^{-1}\left[F(s)\right]$

$$= L^{-1}\left[\frac{(s+1)}{(s+1)^2+1}\right] + L^{-1}\left[\frac{1}{(s+1)^2+1}\right]$$

$$= e^{-t}\left(L^{-1}\left[\frac{s}{s^2+1}\right] + L^{-1}\left[\frac{1}{s^2+1}\right]\right)$$

$$\therefore\ L^{-1}\left[\frac{s+2}{s^2+2s+2}\right] = e^{-t}\,(\cos t + \sin t)$$

22. **Find** $L^{-1}\left(\frac{s}{(s+2)^3}\right)$

Solution: $L^{-1}\left(\frac{s}{(s+2)^3}\right) = L^{-1}\left(\frac{s+2-2}{(s+2)^3}\right)$

$$= L^{-1}\left(\frac{1}{(s+2)^2}\right) - 2\ L^{-1}\left(\frac{1}{(s+2)^3}\right)$$

$$= e^{-2t}L^{-1}\left(\frac{1}{s^2}\right) - e^{-2t}L^{-1}\left(\frac{2}{s^3}\right)$$

$$= e^{-2t}\left(t - t^2\right).$$

23. **Find** $L^{-1}\left[\tan^{-1}\left(\frac{1}{s}\right)\right]$

Solution: Let F(s) = $\tan^{-1}\left(\frac{1}{s}\right)$

$$F'(s) = \frac{1}{1+(1/s)^2}\left(\frac{-1}{s^2}\right) = \frac{-1}{s^2+1}$$

By property $L^{-1}\left[F'(s)\right] = -L^{-1}\left[\frac{1}{s^2+1}\right] = -\sin t$

$$\therefore\ L^{-1}\left(F'(s)\right) = -\sin t\,;\ L^{-1}\left(F(s)\right) = \frac{-1}{t}L^{-1}\left[F'(s)\right]$$

$$\therefore L^{-1}\left[\tan^{-1}\left(\frac{1}{s}\right)\right] = \frac{\sin t}{t}$$

24. **Find the inverse Laplace transform of** $\frac{s}{(s+2)^2}$

Solution:

$$L^{-1}\left(\frac{s}{(s+2)^2}\right)=L^{-1}\left(s.\frac{1}{(s+2)^2}\right)$$

$$=\frac{d}{dt}L^{-1}\left(\frac{1}{(s+2)^2}\right)=\frac{d}{dt}e^{-2t}L^{-1}\left(\frac{1}{s^2}\right)$$

$$=\frac{d}{dt}\left(e^{-2t}\,t\right)=e^{-2t}+t\left(-2e^{-2t}\right)=e^{-2t}\left(1-2t\right)$$

25. **Find $L^{-1}\left[\cot^{-1}(s+1)\right]$**

Let $L^{-1}\left[\cot^{-1}(s+1)\right]=f(t)$

$\therefore\ L\left[f(t)\right]=\cot^{-1}(s+1)$

$$L\left[tf(t)\right]=-\frac{d}{ds}\left[\cot^{-1}(s+1)\right]=\frac{1}{(s+1)^2+1}$$

$$tf(t)=L^{-1}\left[\frac{1}{(s+1)^2+1}\right]=e^{-t}L^{-1}\left[\frac{1}{s^2+1}\right]=e^{-t}\sin t$$

$$\therefore\ f(t)=\frac{e^{-t}\sin t}{t}$$

26. **Find the inverse Laplace transform of $\log\left(\frac{1+s}{s^2}\right)$**

Solution:

Let $L^{-1}\left[\log\left(\frac{1+s}{s^2}\right)\right]=f(t)$

$\therefore\ L\left[f(t)\right]=\log\left(\frac{1+s}{s^2}\right)$

$$L\left[t\,f(t)\right]=\frac{-d}{ds}\left[\log\left(\frac{1+s}{s^2}\right)\right]=\frac{-d}{ds}\left[\log(1+s)-\log\left(s^2\right)\right]=-\frac{1}{1+s}+\frac{1}{s^2}2s$$

$$L\left[t\,f(t)\right]=\frac{2}{s}-\frac{1}{s+1}$$

$$t\,f(t)=L^{-1}\left[\frac{2}{s}-\frac{1}{s+1}\right]=2L^{-1}\left[\frac{1}{s}\right]-L^{-1}\left[\frac{1}{s+1}\right]=2(1)-e^{-t}$$

$$\therefore\ f(t)=\frac{2-e^{-t}}{t}$$

$$\therefore\ L^{-1}\left[\log\left(\frac{1+s}{s^2}\right)\right]=\frac{2-e^{-t}}{t}$$

27. **Find** $\boldsymbol{L^{-1}\left[\dfrac{5s^2-15s-11}{(s+1)(s-2)^3}\right]}$

Solution:

$$\frac{5s^2-15s-11}{(s+1)(s-2)^3}=\frac{A}{s+1}+\frac{B}{s-2}+\frac{C}{(s-2)^2}+\frac{D}{(s-2)^3}$$

$$5s^2-15s-11=A(s-2)^3+B(s+1)(s-2)^2+C(s+1)(s-2)+D(s+1)$$

Put $s=-1\Rightarrow \boxed{A=-\frac{1}{3}}$

Equating the coefficients of $s^3\Rightarrow \boxed{B=\frac{1}{3}}$

Put $s=2\Rightarrow \boxed{D=-7}$

Put $s=0\Rightarrow \boxed{C=4}$

$$\therefore\ \frac{5s^2-15s-11}{(s+1)(s-2)^3}=\frac{-1/3}{s+1}+\frac{1/3}{s-2}+\frac{4}{(s-2)^2}-\frac{7}{(s-2)^3}$$

$$L^{-1}\left[\frac{5s^2-15s-11}{(s+1)(s-2)^3}\right]=-\frac{1}{3}L^{-1}\left[\frac{1}{s+1}\right]+\frac{1}{3}L^{-1}\left[\frac{1}{s-2}\right]+4L^{-1}\left[\frac{1}{(s-2)^2}\right]-7L^{-1}\left[\frac{1}{(s-2)^3}\right]$$

$$=-\frac{1}{3}e^{-t}+\frac{1}{3}e^{2t}+4e^{2t}L^{-1}\left[\frac{1}{s^2}\right]-7e^{2t}L^{-1}\left[\frac{1}{s^3}\right]$$

$$=-\frac{1}{3}e^{-t}+\frac{1}{3}e^{2t}+4e^{2t}\,t-\frac{7}{2}e^{2t}L^{-1}\left[\frac{2}{s^3}\right]$$

$$\therefore\ f(t)=-\frac{1}{3}e^{-t}+\frac{1}{3}e^{2t}+4e^{2t}\,t-\frac{7}{2}e^{2t}t^2$$

28. **Using Convolution theorem find** $L^{-1}\left[\dfrac{s}{\left(s^2+a^2\right)^2}\right]$

Solution:

$$L^{-1}\left[F(s)G(s)\right]=L^{-1}\left[F(s)\right]*L^{-1}\left[G(s)\right]$$

$$L^{-1}\left[\frac{s}{\left(s^2+a^2\right)^2}\right]=L^{-1}\left[\frac{s}{s^2+a^2}\right]*L^{-1}\left[\frac{1}{s^2+a^2}\right]=L^{-1}\left[\frac{s}{s^2+a^2}\right]*\frac{1}{a}L^{-1}\left[\frac{a}{s^2+a^2}\right]$$

$$=\cos at*\frac{1}{a}\sin at=\frac{1}{a}\left[\cos at*\sin at\right]$$

$$=\frac{1}{a}\int_0^t\cos au\ \sin a(t-u)\ du=\frac{1}{a}\int_0^t\sin(at-au)\ \cos au\ \ du$$

$$=\frac{1}{a}\int_0^t\frac{\sin(at-au+au)+\sin(at-au-au)}{2}\ du$$

$$=\frac{1}{2a}\int_0^t\left[\sin at+\sin a(t-2u)\right]\ du$$

$$=\frac{1}{2a}\left[\sin at\ u+\left(\frac{-\cos a(t-2u)}{-2a}\right)\right]_0^t$$

$$=\frac{1}{2a}\left[u\ \sin at+\left(\frac{\cos a(t-2u)}{2a}\right)\right]_0^t$$

$$=\frac{1}{2a}\left[t\ \sin at+\left(\frac{\cos at}{2a}\right)-\left(0+\frac{\cos at}{2a}\right)\right]$$

$$f(t)=\frac{1}{2a}\left[t\sin at+\frac{\cos at}{2a}-\frac{\cos at}{2a}\right]=\frac{1}{2a}t\sin at$$

29. **Find the inverse Laplace transform of** $\dfrac{s}{(s^2+a^2)(s^2+b^2)}$ **using convolution theorem.**

Solution:

$$L^{-1}\left[F(s)G(s)\right] = L^{-1}\left[F(s)\right] * L^{-1}\left[G(s)\right]$$

$$\therefore\ L^{-1}\left[\frac{s}{\left(s^2+a^2\right)\left(s^2+b^2\right)}\right] = L^{-1}\left[\frac{s}{s^2+a^2}\right] * L^{-1}\left[\frac{1}{s^2+b^2}\right]$$

$$= \frac{1}{b}\cos at * \sin bt$$

$$= \frac{1}{b}\int_0^t \cos au\ \ \sin b(t-u)\ du$$

$$= \frac{1}{2b}\int_0^t \left[\sin(au+bt-bu) - \sin(au-bt+bu)\right]\ du$$

$$= \frac{1}{2b}\int_0^t \left[\sin((a-b)u+bt) - \sin((a+b)u-bt)\right]\ du$$

$$= \frac{1}{2b}\left[\frac{-\cos(bt+(a-b)u)}{a-b} + \frac{\cos((a+b)u-bt)}{a+b}\right]_0^t$$

$$= \frac{1}{2b}\left[\left(\frac{-\cos(bt+at-bt)}{a-b} + \frac{\cos(at+bt-bt)}{a+b}\right) - \left(\frac{-\cos bt}{a-b} + \frac{\cos bt}{a+b}\right)\right]$$

$$= \frac{1}{2b}\left[\left(\frac{-\cos(at)}{a-b} + \frac{\cos(at)}{a+b}\right) - \left(\frac{-\cos bt}{a-b} + \frac{\cos bt}{a+b}\right)\right]$$

$$= \frac{1}{2b}\left(\frac{-2b\cos at}{a^2-b^2} + \frac{2b\cos bt}{a^2-\mathrm{b}^2}\right)$$

$$f(t) = \frac{\cos bt - \cos at}{a^2-b^2}$$

30. **Find the inverse Laplace transform of** $\boldsymbol{\frac{s^2}{(s^2+a^2)(s^2+b^2)}}$ **using convolution theorem.**

Solution:

$$L^{-1}\left[F(s)G(s)\right] = L^{-1}\left[F(s)\right] * L^{-1}\left[G(s)\right]$$

$$L^{-1}\left[\frac{s^2}{\left(s^2+a^2\right)\left(s^2+b^2\right)}\right]=L^{-1}\left[\frac{s}{s^2+a^2}\right]*L^{-1}\left[\frac{s}{s^2+b^2}\right]=\cos at*\cos bt$$

$$=\int_0^t \cos au\ \ \cos b\left(t-u\right)\,du$$

$$=\frac{1}{2}\int_0^t\left[\cos\left(au+bt-bu\right)+\cos\left(au-bt+bu\right)\right]\,du$$

$$=\frac{1}{2}\int_0^t\left[\cos\left(\left(a-b\right)u+bt\right)+\cos\left(\left(a+b\right)u-bt\right)\right]\,du$$

$$=\frac{1}{2}\left[\frac{\sin\left(bt+\left(a-b\right)u\right)}{a-b}+\frac{\sin\left(\left(a+b\right)u-bt\right)}{a+b}\right]_0^t$$

$$=\frac{1}{2}\left[\left(\frac{\sin\left(bt+at-bt\right)}{a-b}+\frac{\sin\left(at+bt-bt\right)}{a+b}\right)-\left(\frac{\sin bt}{a-b}-\frac{\sin bt}{a+b}\right)\right]$$

$$=\frac{1}{2}\left[\left(\frac{\sin\left(at\right)}{a-b}+\frac{\sin\left(at\right)}{a+b}\right)-\left(\frac{\sin bt}{a-b}-\frac{\sin bt}{a+b}\right)\right]$$

$$=\frac{1}{2}\left(\frac{2a\sin\left(at\right)}{a^2-b^2}-\frac{2b\sin\left(bt\right)}{a^2-\mathrm{b}^2}\right)$$

$$f\left(t\right)=\frac{a\sin\left(at\right)-b\sin\left(bt\right)}{a^2-b^2}$$

31. **Find the inverse Laplace transform of** $\dfrac{s}{\left(s^2+1\right)\left(s^2+4\right)}$

Solution:

$$L^{-1}\left[\frac{s}{\left(s^2+1\right)\left(s^2+4\right)}\right]=L^{-1}\left[\frac{s}{s^2+1}\ \frac{1}{s^2+4}\right]=L^{-1}\left[\frac{s}{s^2+1}\right]*\frac{1}{2}L^{-1}\left[\frac{2}{s^2+4}\right]$$

$$=\frac{1}{2}\cos t*\sin 2t$$

$$=\frac{1}{2}\int_0^t \cos u\ \sin 2(t-u)\,du$$

$$=\frac{1}{4}\int_0^t \left[\sin(u+2t-2u)-\sin(u-2t+2u)\right]du \qquad \left(\because 2\cos A\sin B=\sin(A+B)-\sin(A-B)\right)$$

$$=\frac{1}{4}\int_0^t \left[\sin(2t-u)-\sin(u-2t)\right]du$$

$$=\frac{1}{4}\left[\frac{-\cos(2t-u)}{-1}+\frac{\cos(u-2t)}{1}\right]_0^t$$

$$=\frac{1}{4}\left[\cos t-\cos 2t+\cos t-\cos 2t\right]$$

$$=\frac{1}{4}\left[2\cos t-2\cos 2t\right]$$

$$\therefore\ f(t)=\frac{1}{2}\left[\cos t-\cos 2t\right]$$

32. **Using Convolution theorem find the inverse Laplace transform of** $\mathbf{\dfrac{2}{(s+1)(s^2+4)}}$

Solution:

$$L^{-1}\left[\frac{2}{(s+1)(s^2+4)}\right]=L^{-1}\left[\frac{1}{s+1}\ \frac{2}{s^2+4}\right]=L^{-1}\left[\frac{1}{s+1}\right]*L^{-1}\left[\frac{2}{s^2+4}\right]$$

$$=e^{-t}*\sin 2t$$

$$=\int_0^t e^{-u}\ \sin 2(t-u)\,du$$

$$=\int_0^t e^{-u}\ \sin(2t-2u)\,du$$

$$=\int_0^t e^{-u}\ \left[\sin 2t\ \cos 2u-\cos 2t\ \sin 2u\right]du$$

$$=\int_0^t e^{-u}\ \sin 2t\ \cos 2u\ du-\int_0^t e^{-u}\ \cos 2t\ \sin 2u\ du$$

$$= \sin 2t \int_0^t e^{-u} \cos 2u \; du - \cos 2t \int_0^t e^{-u} \sin 2u \; du$$

$$= \sin 2t \left[\frac{e^{-u}}{1+4} \left(-\cos 2u + 2\sin 2u \right) \right]_0^t - \cos 2t \left[\frac{e^{-u}}{1+4} \left(-\sin 2u - 2\cos 2u \right) \right]_0^t$$

$$= \sin 2t \left[\left(\frac{e^{-t}}{5} \left(-\cos 2t + 2\sin 2t \right) \right) - \left(\frac{1}{5}(-1) \right) \right] - \cos 2t \left[\frac{e^{-t}}{5} \left(-\sin 2t - 2\cos 2t \right) - \left(\frac{1}{5}(-2) \right) \right]$$

$$= \sin 2t \left[\frac{e^{-t}}{5} \left(-\cos 2t + 2\sin 2t \right) + \frac{1}{5} \right] - \cos 2t \left[\frac{e^{-t}}{5} \left(-\sin 2t - 2\cos 2t + \frac{2}{5} \right) \right]$$

$$= \frac{e^{-t}}{5} \left[-\sin 2t \cos 2t + 2\sin^2 2t + \sin 2t \cos 2t + 2\cos^2 2t \right] + \frac{1}{5}\sin 2t - \frac{2}{5}\cos 2t$$

$$= \frac{e^{-t}}{5} \left[2(1) \right] + \frac{1}{5}\sin 2t - \frac{2}{5}\cos 2t$$

$$f(t) = \frac{1}{5}\left[2e^{-t} + \sin 2t - 2\cos 2t \right]$$

33. **Find the inverse Laplace transform of** $\dfrac{s^2}{\left(s^2+1\right)\left(s^2+4\right)}$

Solution:

$$L^{-1}\left[F(s) G(s) \right] = L^{-1}\left[F(s) \right] * L^{-1}\left[G(s) \right]$$

$$\therefore \; L^{-1}\left[\frac{s^2}{\left(s^2+1^2\right)\left(s^2+2^2\right)} \right] = L^{-1}\left[\frac{s}{s^2+1^2} \right] * L^{-1}\left[\frac{s}{s^2+2^2} \right]$$

$$= \cos t * \cos 2t$$

$$= \int_0^t \cos u \; \cos 2(t-u) \; du$$

$$= \frac{1}{2}\int_0^t \left[\cos(u+2t-2u) + \cos(u-2t+2u) \right] \; du$$

$$= \frac{1}{2}\int_0^t \left[\cos(-u+2t) + \cos(3u-2t) \right] \; du$$

$$=\frac{1}{2}\left[\frac{\sin(2t-u)}{-1}+\frac{\sin(3u-2t)}{3}\right]_0^t$$

$$=\frac{1}{2}\left[\left(\frac{\sin t}{-1}+\frac{\sin t}{3}\right)-\left(\frac{\sin 2t}{-1}-\frac{\sin 2t}{3}\right)\right]$$

$$=\frac{1}{2}\left(\frac{2\sin t}{-3}-\frac{4\sin 2t}{-3}\right)$$

$$f(t)=\frac{\sin t-2\sin 2t}{-3}$$

34. Find $L^{-1}\left(\dfrac{e^{-2s}}{\left(s^2+s+1\right)^2}\right)$

Solution:

$$L^{-1}\left(\frac{e^{-2s}}{\left(s^2+s+1\right)^2}\right)=L^{-1}\left(\frac{e^{-s}}{s^2+s+1}\,\frac{e^{-s}}{s^2+s+1}\right)$$

$$=L^{-1}\left(\frac{1}{s^2+s+1}\right)_{t\to t-1}*L^{-1}\left(\frac{1}{s^2+s+1}\right)_{t\to t-1}$$

$$=L^{-1}\left(\frac{1}{\left(s+\frac{1}{2}\right)^2+\frac{3}{4}}\right)_{t\to t-1}*L^{-1}\left(\frac{1}{\left(s+\frac{1}{2}\right)^2+\frac{3}{4}}\right)_{t\to t-1}$$

$$=e^{-t/2}L^{-1}\left(\frac{1}{s^2+\left(\frac{\sqrt{3}}{2}\right)^2}\right)_{t\to t-1}*e^{-t/2}L^{-1}\left(\frac{1}{s^2+\left(\frac{\sqrt{3}}{2}\right)^2}\right)_{t\to t-1}$$

$$=\left[e^{-t/2}\,\frac{\sin\left(\frac{\sqrt{3}}{2}t\right)}{\frac{\sqrt{3}}{2}}*e^{-t/2}\,\frac{\sin\left(\frac{\sqrt{3}}{2}t\right)}{\frac{\sqrt{3}}{2}}\right]_{t\to t-1}$$

$$=\frac{2}{\sqrt{3}}e^{-(t-1)/2}\sin\left(\frac{\sqrt{3}}{2}(t-1)\right)*\frac{2}{\sqrt{3}}e^{-(t-1)/2}\sin\left(\frac{\sqrt{3}}{2}(t-1)\right)$$

$$=\frac{4}{3}\left[e^{-(t-1)/2}\sin\left(\frac{\sqrt{3}}{2}(t-1)\right)*e^{-(t-1)/2}\sin\left(\frac{\sqrt{3}}{2}(t-1)\right)\right]$$

$$=\frac{4}{3}\int_0^t e^{-\frac{u-1}{2}}e^{-\frac{t-u-1}{2}}\sin\left(\frac{\sqrt{3}}{2}u-\frac{\sqrt{3}}{2}\right)\ \sin\left(\frac{\sqrt{3}}{2}t-\frac{\sqrt{3}}{2}u-\frac{\sqrt{3}}{2}\right)du$$

$$=\frac{4}{3}\int_0^t e^{-\left(\frac{t}{2}-1\right)}\frac{1}{2}\ \cos\left(\frac{\sqrt{3}}{2}u-\frac{\sqrt{3}}{2}t\right)-\cos\left(\frac{\sqrt{3}}{2}t-\sqrt{3}\right)du$$

$$=\frac{2}{3}e^{-\left(\frac{t-2}{2}\right)}\left[\frac{\sin\left(\frac{\sqrt{3}}{2}u-\frac{\sqrt{3}}{2}t\right)}{\frac{\sqrt{3}}{2}}-\cos\left(\frac{\sqrt{3}}{2}t-\sqrt{3}\right)u\right]_0^t$$

$$=e^{-\left(\frac{t-2}{2}\right)}\left[\frac{4}{3\sqrt{3}}\sin\frac{\sqrt{3}}{2}t-\frac{2}{3}t\cos\left(\frac{\sqrt{3}}{2}t-\sqrt{3}\right)\right]$$

35. **Solve using Laplace transform $\frac{dy}{dt}+y=e^{-t}$ given that y(0) = 0.**

Solution: Taking L.T. on both sides, we get $L\left[y'(t)\right]+L\left[y(t)\right]=L\left[e^{-t}\right]$

$$sL\left[y(t)\right]-y(0)+L\left[y(t)\right]=L\left[e^{-t}\right]$$

$$sL\left[y(t)\right]-0+L\left[y(t)\right]=\frac{1}{s+1}$$

$$(s+1)\ L[y(t)]=\frac{1}{s+1}$$

$$L[y(t)]=\frac{1}{(s+1)^2}$$

$$\therefore y(t)=L^{-1}\left(\frac{1}{(s+1)^2}\right)=e^{-t}L\left(\frac{1}{s^2}\right)=e^{-t}\,t \qquad \left(\because\ L\left[e^{-at}f(t)\right]=F(s+a)\right)$$

36. **Using Laplace transform to solve the differential equation**
$\boldsymbol{y'' + y' = t^2 + 2t,}$ **given** $\boldsymbol{y = 4,\ y' = -2}$ **when** $\boldsymbol{t = 0}$

Solution:

Given $y'' + y' = t^2 + 2t$

$$L\left[y'' + y'\right] = L\left[t^2 + 2t\right]$$

$$\left[s^2 L\left[y(t)\right] - sy(0) - y'(0)\right] + \left[sL\left[y(t)\right] - y(0)\right] = \frac{2}{s^3} + \frac{2}{s^2}$$

$$L\left[y(t)\right]\left(s^2 + s\right) = \frac{2}{s^3} + \frac{2}{s^2} + 4s - 2 + 4$$

$$L\left[y(t)\right]s(s+1) = \frac{2}{s^3} + \frac{2}{s^2} + 4s + 2$$

$$L\left[y(t)\right] = \frac{2 + 2s + 4s^4 + 2s^3}{s^4(s+1)}$$

$$L\left[y(t)\right] = \frac{2}{s} + \frac{2}{s^4} + \frac{2}{s+1}$$

$$y(t) = L^{-1}\left[\frac{2}{s} + \frac{2}{s^4} + \frac{2}{s+1}\right]$$

$$= 2 + 2\frac{t^3}{6} + 2e^{-t}$$

$$y(t) = 2 + \frac{t^3}{3} + 2e^{-t}$$

37. **Solve** $\boldsymbol{\left(D^2 + 3D + 2\right)y = e^{-3t}}$**, given** $\boldsymbol{y(0) = 1,}$ **and** $\boldsymbol{y'(0) = -1}$ **using Laplace Transforms**

Solution:

Given $y'' + 3y' + 2y = e^{-3t}$

Taking Laplace transforms on both side

$$L\left(y'' + 3y' + 2y\right) = L\left(e^{-3t}\right)$$

$$L\left[y''(t)\right] + 3L\left[y'(t)\right] + 2L\left[y(t)\right] = \frac{1}{s+3}$$

$$\left[s^2L\left[y(t)\right]-sy(0)-y^{'}(0)\right]+3\left[sL\left[y(t)\right]-y(0)\right]+2L\left[y(t)\right]=\frac{1}{s+3}$$

$$\left[s^2L\left[y(t)\right]-s(1)-(-1)\right]+3\left[sL\left[y(t)\right]-1\right]+2L\left[y(t)\right]=\frac{1}{s+3}$$

$$L\left[y(t)\right]\left[s^2+3s+2\right]=\frac{1}{s+3}+s+2$$

$$L\left[y(t)\right]=\frac{s^2+5s+7}{(s+3)\left(s^2+3s+2\right)}\ ,\ y(t)=L^{-1}\left[\frac{s^2+5s+7}{(s+1)(s+2)(s+3)}\right]$$

$$y(t)=L^{-1}\left[\frac{3/2}{s+1}-\frac{1}{s+2}+\frac{1/2}{s+3}\right]$$

$$y(t)=\frac{3}{2}L^{-1}\left[\frac{1}{s+1}\right]-L^{-1}\left[\frac{1}{s+2}\right]+\frac{1}{2}L^{-1}\left[\frac{1}{s+3}\right]$$

$$y(t)=\frac{3}{2}e^{-t}-e^{-2t}+\frac{1}{2}e^{-3t}$$

38. **Solve $y^{''}+2y^{'}-3y=\sin t$, given $y(0)=0,\ y^{'}(0)=0$**

Solution:

Given $y^{''}+2y^{'}-3y=\sin t$

$$L\left[y^{''}(t)+2y^{'}(t)-3y(t)\right]=L[\sin t]$$

$$L\left[y^{''}(t)\right]+2L\left[y^{'}(t)\right]-3L\left[y(t)\right]=L[\sin t]$$

$$\left[s^2L\left[y(t)\right]-sy(0)-y^{'}(0)\right]+2\left[sL\left[y(t)\right]-y(0)\right]-3L\left[y(t)\right]=\frac{1}{s^2+1}$$

$$\left[s^2L\left[y(t)\right]-s(0)-0\right]+2\left[sL\left[y(t)\right]-(0)\right]-3L\left[y(t)\right]=\frac{1}{s^2+1}$$

$$s^2L\left[y(t)\right]+2sL\left[y(t)\right]-3L\left[y(t)\right]=\frac{1}{s^2+1}$$

$$L\left[y(t)\right]\left(s^2+2s-3\right)=\frac{1}{s^2+1}$$

$$L\left[y(t)\right]=\frac{1}{\left(s^2+1\right)\left(s^2+2s-3\right)}$$

$$y(t)=L^{-1}\left[\frac{1}{\left(s^2+1\right)\left(s^2+2s-3\right)}\right]==L^{-1}\left[\frac{1}{(s-1)(s+3)\left(s^2+1\right)}\right]$$

Now

$$\frac{1}{(s-1)(s+3)\left(s^2+1\right)}=\frac{A}{s-1}+\frac{B}{s+3}+\frac{Cs+D}{\left(s^2+1\right)}$$

$$1=A(s+3)\left(s^2+1\right)+B(s-1)\left(s^2+1\right)+(Cs+D)(s-1)(s+3)$$

Put $s=1\Rightarrow \boxed{A=\frac{1}{8}}$

Put $s=-3\Rightarrow \boxed{B=\frac{-1}{40}}$

Equating coeff. of $s^3\Rightarrow \boxed{C=\frac{-1}{10}}$

Equating the constant terms $\Rightarrow \boxed{D=\frac{-1}{5}}$

$$\therefore\ \frac{1}{(s-1)(s+3)\left(s^2+1\right)}=\frac{1/8}{s-1}+\frac{-1/40}{s+3}+\frac{(-1/10)s-1/5}{\left(s^2+1\right)}$$

$$L^{-1}\left[\frac{1}{(s-1)(s+3)\left(s^2+1\right)}\right]=L^{-1}\left[\frac{1/8}{s-1}+\frac{-1/40}{s+3}+\frac{(-1/10)s-1/5}{\left(s^2+1\right)}\right]$$

$$=\frac{1}{8}L^{-1}\left[\frac{1}{s-1}\right]-\frac{1}{40}L^{-1}\left[\frac{1}{s+3}\right]-\frac{1}{10}L^{-1}\left[\frac{s+2}{s^2+1}\right]$$

$$=\frac{1}{8}e^{t}-\frac{1}{40}e^{-3t}-\frac{1}{10}\left[L^{-1}\left[\frac{s}{s^2+1}\right]+L^{-1}\left[\frac{2}{s^2+1}\right]\right]$$

$$=\frac{1}{8}e^{t}-\frac{1}{40}e^{-3t}-\frac{1}{10}[\cos t+2\sin t]$$

39. **Solve the equation y″+ 9y = cos2t with y(0) =1, y $\left(\frac{\pi}{2}\right)$ = –1**

Solution:

Given $\left(D^2+9\right)y=\cos 2t$

Taking Laplace transforms on both sides

$$L\left[y''(t)\right]+9L\left[y(t)\right]=L[\cos 2t]$$

$$s^2L\left[y(t)\right]-sy(0)-y'(0)+9L\left[y(t)\right]=\frac{s}{s^2+4}$$

Using the initial conditions

$y(0)=1,$ and taking $y'(0)=k$

We have

$$s^2L\left[y(t)\right]-(s)(1)-k+9L\left[y(t)\right]=\frac{s}{s^2+4}$$

$$\Rightarrow L\left[y(t)\right]=\frac{s}{\left(s^2+4\right)\left(s^2+9\right)}+\frac{s+k}{s^2+9}$$

$$=\frac{s}{5\left(s^2+4\right)}-\frac{s}{5\left(s^2+9\right)}+\frac{s}{s^2+9}+\frac{k}{s^2+9}$$

$$\therefore\ y(t)=\frac{1}{5}L^{-1}\left[\frac{s}{s^2+4}\right]-\frac{1}{5}L^{-1}\left[\frac{s}{s^2+9}\right]+L^{-1}\left[\frac{s}{s^2+9}\right]+k\,L^{-1}\left[\frac{s}{s^2+9}\right]$$

$$=\frac{1}{5}\cos 2t-\frac{1}{5}\cos 3t+\cos 3t+\frac{k}{3}\sin 3t$$

Put $t=\frac{\pi}{2}$ we get $y\left(\frac{\pi}{2}\right)=\frac{1}{5}(-1)-\frac{1}{5}(0)+0+\frac{k}{3}(-1)=-\frac{1}{5}-\frac{k}{3}$

But given $y\left(\frac{\pi}{2}\right)=-1$

$$\therefore\ -1=-\frac{1}{5}-\frac{k}{3}$$

$$\Rightarrow k=\frac{12}{5}$$

$$\therefore\ y(t)=\frac{1}{5}\cos 2t-\frac{1}{5}\cos 3t+\cos 3t+\frac{4}{5}\sin 3t$$

$$y(t)=\frac{4}{5}[\cos 3t+\sin 3t]+\frac{1}{5}\cos 2t$$

40. **Solve $x'' + 2x' + 5x = e^{-t}\sin t$, where $x(0)=0,\ x'(0)=1$ using Laplace Transforms**

Solution:

Given $x'' + 2x' + 5x = e^{-t}\sin t$

Taking Laplace transforms on both side

$$L\left[x'' + 2x' + 5x\right] = L\left[e^{-t}\sin t\right]$$

$$L\left[x''(t)\right] + 2L\left[x'(t)\right] + 5L\left[x(t)\right] = \frac{1}{s^2+2s+2}$$

$$\left[s^2L\left[x(t)\right] - sx(0) - x'(0)\right] + 2\left[sL\left[x(t)\right] - x(0)\right] + 5L\left[x(t)\right] = \frac{1}{s^2+2s+2}$$

$$\left[s^2L\left[x(t)\right] - s(0) - 1\right] + 2\left[sL\left[x(t)\right] - (0)\right] + 5L\left[x(t)\right] = \frac{1}{s^2+2s+2}$$

$$L\left[x(t)\right]\left[s^2+2s+5\right] = \frac{1}{s^2+2s+2} + 1$$

$$L\left[x(t)\right]\left[s^2+2s+5\right] = \frac{s^2+2s+3}{s^2+2s+2}$$

$$L\left[x(t)\right] = \frac{s^2+2s+3}{\left(s^2+2s+2\right)\left(s^2+2s+5\right)} = \frac{(s+1)^2+2}{\left((s+1)^2+1\right)\left((s+1)^2+4\right)}$$

$$x(t) = L^{-1}\left[\frac{(s+1)^2+2}{\left((s+1)^2+1\right)\left((s+1)^2+4\right)}\right]$$

$$x(t) = e^{-t}L^{-1}\left[\frac{s^2+2}{\left(s^2+1\right)\left(s^2+4\right)}\right]$$

$$x(t) = e^{-t}L^{-1}\left[\frac{1/3}{s^2+1} + \frac{2/3}{s^2+4}\right]$$

$$= e^{-t}\left[\frac{1}{3}\sin t + \frac{1}{3}\sin 2t\right]$$

$$= \frac{e^{-t}}{3}[\sin t + \sin 2t]$$

41. **Using Laplace transform to solve the differential equation**

$y'' - 3y' + 2y = 4t + e^{3t}$, where $y(0) = 1,\ y'(0) = -1$

Solution:

Given $y'' - 3y' + 2y = 4t + 3e^{t}$

$$L\left[y'' - 3y' + 2y\right] = L\left[4t + 3e^{t}\right]$$

$$L\left[y''(t)\right] - 3L\left[y'(t)\right] + 2L\left[y(t)\right] = 4L[t] + 3L\left[e^{3t}\right]$$

$$\left[s^2L\left[y(t)\right] - sy(0) - y'(0)\right] - 3\left[sL\left[y(t)\right] - y(0)\right] + 2L\left[y(t)\right] = \frac{4}{s^2} + \frac{3}{s-3}$$

$$\left[s^2L\left[y(t)\right] - s(1) - (-1)\right] - 3\left[sL\left[y(t)\right] - 1\right] + 2L\left[y(t)\right] = \frac{4}{s^2} + \frac{3}{s-3}$$

$$\left[s^2L\left[y(t)\right] - s + 1\right] - 3\left[sL\left[y(t)\right] - 1\right] + 2L\left[y(t)\right] = \frac{4}{s^2} + \frac{3}{s-3}$$

$$L\left[y(t)\right]\left(s^2 - 3s + 2\right) = s - 4 + \frac{4}{s^2} + \frac{3}{s-3}$$

$$L\left[y(t)\right]\left(s^2 - 3s + 2\right) = \frac{(s-4)s^2(s-3) + 4(s-4) + 3s^2}{s^2(s-3)}$$

$$L\left[y(t)\right] = \frac{s^4 - 7s^3 + 13s^2 + 4s - 12}{\left(s^2 - 3s + 2\right)s^2(s-3)}$$

$$L\left[y(t)\right] = \frac{s^4 - 7s^3 + 13s^2 + 4s - 12}{(s-2)(s-1)s^2(s-3)}$$

$$y(t) = L^{-1}\left[\frac{s^4 - 7s^3 + 13s^2 + 4s - 12}{(s-2)(s-1)s^2(s-3)}\right]$$

$$= L^{-1}\left[\frac{A}{s}+\frac{B}{s^2}+\frac{C}{s-1}+\frac{D}{s-2}+\frac{E}{s-3}\right]$$

$$= L^{-1}\left[\frac{3}{s}+\frac{2}{s^2}+\frac{-1/2}{s-1}+\frac{-2}{s-2}+\frac{1/2}{s-3}\right]$$

$$y(t) = 3+2t-\frac{1}{2}e^{t}-2e^{2t}+\frac{1}{2}e^{3t}$$

42. **Solve $y'' - 3y' + 2y = e^{2t}$, $y(0) = -3$, $y'(0) = 5$**

Solution:

Given $y'' - 3y' + 2y = e^{2t}$

$$L\left[y''-3y'+2y\right] = L\left[e^{2t}\right]$$

$$L\left[y''\right]-3L\left[y'\right]+2L[y] = L\left[e^{2t}\right]$$

$$\left[s^2L\left[y(t)\right]-sy(0)-y'(0)\right]-3\left[sL\left[y(t)\right]-y(0)\right]+2L\left[y(t)\right] = \frac{1}{s-2}$$

$$\left[s^2L\left[y(t)\right]-s(-3)-5\right]-3\left[sL\left[y(t)\right]-(-3)\right]+2L\left[y(t)\right] = \frac{1}{s-2}$$

$$s^2L\left[y(t)\right]+3s-5-3sL\left[y(t)\right]-9+2L\left[y(t)\right] = \frac{1}{s-2}$$

$$L\left[y(t)\right]\left[s^2-3s+2\right]+3s-14 = \frac{1}{s-2}$$

$$\therefore \quad L\left[y(t)\right]\left[s^2-3s+2\right] = \frac{1}{s-2}-3s+14$$

$$\therefore \quad L\left[y(t)\right] = \frac{-3s^2+20s-27}{(s-2)\left(s^2-3s+2\right)}$$

$$y(t) = L^{-1}\left[\frac{-3s^2+20s-27}{(s-2)\left(s^2-3s+2\right)}\right]$$

$$y(t) = L^{-1}\left[\frac{-3s^2+20s-27}{(s-1)(s-2)^2}\right]$$

$$\frac{-3s^2+20s-27}{(s-1)(s-2)^2} = \frac{A}{s-1}+\frac{B}{s-2}+\frac{C}{(s-2)^2}$$

$$-3s^2+20s-27=A(s-2)^2+B(s-1)(s-2)+C(s-1)$$

Put $s=1 \Rightarrow \boxed{A=-10}$

Put $s=2 \Rightarrow \boxed{C=1}$

Equating the coeff. of $s^2 \Rightarrow \boxed{B=7}$

$$\therefore \frac{-3s^2+20s-27}{(s-1)(s-2)^2}=\frac{-10}{s-1}+\frac{7}{s-2}+\frac{1}{(s-2)^2}$$

$$L^{-1}\left[\frac{-3s^2+20s-27}{(s-1)(s-2)^2}\right]=L^{-1}\left[\frac{-10}{s-1}\right]+L^{-1}\left[\frac{7}{s-2}\right]+L^{-1}\left[\frac{1}{(s-2)^2}\right]$$

$$=-10e^t+7e^{2t}+e^{2t}L^{-1}\left[\frac{1}{s^2}\right]$$

$$=-10e^t+7e^{2t}+te^{2t}$$

43. **Solve $\frac{dx}{dt}-2x+3y=0;\ \frac{dy}{dt}-y+2x=0$ with $x(0)=8,\ y(0)=3$**

The given differential equation canbe written as

$$x'(t)-2x+3y=0 \quad y'(t)-y+2x=0$$

Taking Laplace transforms weget,

$$L\left[x'(t)-2x+3y\right]=L[0]$$

$$sL[x(t)]-x(0)-2L[x(t)]+3L[y(t)]=0$$

$$sL[x(t)]-8-2L[x(t)]+3L[y(t)]=0$$

$$L[x(t)](s-2)+3L[y(t)]=8 \qquad (1)$$

And $L\left[y'(t)-y+2x\right]=L[0]$

$$sL[y(t)]-y(0)-L[y(t)]+2L[x(t)]=0$$

$$sL[y(t)]-3-L[y(t)]+2L[x(t)]=0$$

$$2L[x(t)]+(s-1)L[y(t)]=3 \qquad (2)$$

Solving (1) and (2) we get,

$$L\left[x(t)\right]=\frac{8s-17}{(s+1)(s-4)}=\frac{5}{s+1}+\frac{3}{s-4},$$

$$\therefore\ x(t)=L^{-1}\left[\frac{5}{s+1}+\frac{3}{s-4}\right],$$

$$\boxed{x(t)=5e^{-t}+3e^{4t}}$$

and $$L\left[y(t)\right]=\frac{3s-22}{(s+1)(s-4)}=\frac{5}{s+1}-\frac{2}{s-4}$$

$$y(t)=L^{-1}\left[\frac{5}{s+1}-\frac{2}{s-4}\right]=5e^{-t}-2e^{4t}$$

44. **Determine y which satisfies the equation** $\boldsymbol{\frac{dy}{dt}+2y+\int_0^t y\,dt=2\cos t,\ \ y(0)=1}$

Solution:

Given $y'(t)+2y(t)+\int_0^t y(t)\,dt=2\cos t,\ \ y(0)=1$

$$L\left[y'(t)\right]+2L\left[y(t)\right]+L\left[\int_0^t y(t)\,dt\right]=L\left[2\cos t\right]$$

$$sL\left[y(t)\right]-y(0)+2L\left[y(t)\right]+\frac{1}{s}L\left[y(t)\right]=\frac{2s}{s^2+1}$$

$$sL\left[y(t)\right]-1+2L\left[y(t)\right]+\frac{1}{s}L\left[y(t)\right]=\frac{2s}{s^2+1}$$

$$L\left[y(t)\right]=\frac{s}{s^2+1}$$

$$y(t)=L^{-1}\left[\frac{s}{s^2+1}\right]=\cos t$$

SRM Institute of Science and Technology
Kattankulathur

DEPARTMENT OF MEATHEMATICS

18MAB102T ADVANCED CALCULUS & COMPLEX ANALYSIS

UNIT –IV ANALYTIC FUNCTIONS

	Tutorial Sheet -2	**Answers**
	Part – A	
1	Find the image of the circle |z|=3 under the transformation w=2z	6
2	Find a function w such that w=u+iv is analytic, if $u = e^x \sin y$	$f(z) = -ie^z + c$
3	Determine the analytic function u+iv whose real part $u = x^3 - 3xy^2 + 3x^2 - 3y^2 + 1$	$f(z) = z^3 + 3z^2 + c$
	Part – B	
4	Find the analytic function $f(z) = u + iv$ if $u - v = e^x(cosy - siny)$	$f(z) = e^z + c$
5	Find the analytic function $f(z) = u + iv$ if $u - v = \dfrac{sin2x}{cosh2y - cos2x}$	$f(z) = \dfrac{\cot z}{1+i} + c$
6	Determine the region D' of the w-plane into which the triangular region D enclosed by the lines x=0, y=0, x+y=1 is transformed under the transformation w=2z	
7	Find an analytic function $f(z) = u + iv$, given that $2u + 3v = \dfrac{sin\ 2x}{cos\ h2y - \cos x}$	$f(z) = \dfrac{(2+3i)\cot z}{13} + c$

Module – 5 Complex Integration

Cauchy's integral formulae – Problems – Taylor's expansions with simple problems – Laurent's expansions with simple problems – Singularities – Types of Poles and Residues – Cauchy's residue theorem (without proof) – Contour integration: Unit circle, semicircular contour – Application of Contour integration in Engineering.

Cauchy's Integral Theorem

If $f(z)$ is analytic at every point of the region R bounded by a simple closed curve C and if $f'(z)$ is continuous at all points inside and on C, then $\int_C f(z)\,dz = 0$

Cauchy's integral formula for n^{th} derivative

If $f(z)$ is analytic inside and on a simple closed curve C and $z = a$ is any interior point of the region R enclosed by C, then $f^{(n)}(a) = \frac{n!}{2\pi i}\int_C \frac{f(z)}{(z-a)^{n+1}}dz$

$$(i.e.)\quad \boxed{\int_C \frac{f(z)}{(z-a)^{n+1}}dz = \frac{2\pi i}{n!}f^{(n)}(a)}$$

Taylor's series

If $f(z)$ is analytic inside a circle C with centre at a then Taylor's series about $z = a$ is

$$f(z) = f(a) + \frac{f'(a)}{1!}(z-a) + \frac{f''(a)}{2!}(z-a)^2 + \ldots$$

Laurent's series

If C_1, C_2 are two concentric circles with centre at z = a and radii r_1 and r_2 ($r_1 < r_2$) and if f(z) is analytic inside and on the circles and within the annular region between C_1 and C_2, then for any z in the annular region, we have

$$f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n + \sum_{n=1}^{\infty} b_n (z-a)^{-n},$$

where $a_n = \frac{1}{2\pi i}\int_{C_1}\frac{f(z)}{(z-a)^{n+1}}dz$ and $b_n = \frac{1}{2\pi i}\int_{C_2}\frac{f(z)}{(z-a)^{-n+1}}dz$

Cauchy's Residue theorem

If f(z) is analytic inside a closed curve C except at a finite number of isolated singular points $a_1, a_2, \ldots a_n$ inside C, then

$\int_C f(z)dz = 2\pi i \times$(sum of the residues of f(z) at these singular points).

Contour Integration

Type I:

$$\boxed{\int_0^{2\pi} f\left(\cos\theta, \sin\theta\right) d\theta}$$

$$\text{Let } z = e^{i\theta}, \quad dz = ie^{i\theta} d\theta = iz d\theta \Rightarrow d\theta = \frac{dz}{iz}$$

Then we have

$$\cos\theta = \frac{1}{2}\left(z + \frac{1}{z}\right); \qquad \sin\theta = \frac{1}{2i}\left(z - \frac{1}{z}\right)$$

$$\cos 2\theta = \text{Real part of } z^2 \ ; \qquad \cos n\theta = \text{Real part of } z^n$$

$$\sin 2\theta = \text{Im part of } z^2; \qquad \sin n\theta = \text{Im part of } z^n$$

$$\cos^2\theta = \frac{1+\cos 2\theta}{2} = \text{Real part of} \left[\frac{1+z^2}{2}\right];$$

$$\sin^2\theta = \frac{1-\cos 2\theta}{2} = \text{Real part of} \left[\frac{1-z^2}{2}\right]$$

$\therefore$

$$\int_0^{2\pi} f\left(\cos\theta, \sin\theta\right) d\theta = \int_C f\left(z\right) dz, \text{ where } C \text{ is } |z| = 1 \text{ and solve by known method.}$$

Type II:

$$\boxed{\int_{-\infty}^{\infty} f\left(x\right) dx}$$

Using Cauchy's integral formula, find $\int_C \frac{z+4}{z^2+2z+5} dz$**, where C is** $|z+1-i| = 2$

Solution:

$| \ z + 1 - i \ | = 2$

$|x + iy + 1 - i| = 2$

$\left|\left(x+1\right) + i\left(y-1\right)\right| = 2, \qquad \sqrt{\left(x+1\right)^2 + \left(y-1\right)^2} = 2$

Squaring on both sides,

$$(x+1)^2+(y-1)^2=4$$

This is equation of circle with centre $(-1,1)$ and radius 2.

$$z^2+2z+5=0$$

$$z=\frac{-2\pm\sqrt{4-4(1)(5)}}{2(1)}=\frac{-2\pm 4i}{2}=-1\pm 2i$$

$$\int_C \frac{z+4}{z^2+2z+5}\,dz=\int_C \frac{z+4}{\left[z-(-1+2i)\right]\left[z-(-1-2i)\right]}\,dz$$

Here –1+2*i* lies inside the circle c and –1–2*i* lies outside the circle c.

Let $a=-1+2i$

By Cauchy's integral formula, $f(a)=\dfrac{1}{2\pi i}\displaystyle\int_C \frac{f(z)}{z-a}\,dz$

Substituting for *a*, $f(-1+2i)=\dfrac{1}{2\pi i}\displaystyle\int_C \frac{f(z)}{z-(-1+2i)}\,dz$ ......(1)

Comparing equation (1) with given problem,

$$f(z)=\frac{z+4}{z-(-1-2i)}$$

$$f(-1+2i)=\frac{-1+2i+4}{-1+2i-(-1-2i)}=\frac{2i+3}{-1+2i+1+2i}=\frac{2i+3}{4i}$$

Substituting for $f(-1+2i)$ in (1)

$$\frac{2i+3}{4i}=\frac{1}{2\pi i}\int_C \frac{z+4}{z^2+2z+5}\,dz$$

Cross multiplying

$$\int_C \frac{z+4}{z^2+2z+5}\,dz=\frac{(2i+3)(2\pi i)}{4i}=\frac{\pi}{2}(3+2i)$$

Using Cauchy's integral formula, evaluate $\displaystyle\int_C \frac{\sin \pi z^2+\cos \pi z^2}{(z-2)\ (z-1)}\,dz$**, where C is** $|z|=3$

Solution:

We know that, Cauchy's integral formula is $f\left(a\right)=\frac{1}{2\pi i}\int_C \frac{f\left(z\right)}{z-a}\,dz$

$(i.e)\ 2\pi i\ f\left(a\right)=\int_C \frac{f\left(z\right)}{z-a}\,dz$

Given: $\int_C \frac{\sin \pi z^2+\cos \pi z^2}{(z-1)(z-2)}dz$ Here, $f(z)=\sin \pi z^2+\cos \pi z^2$

The points $a_1=1, a_2=2$ lies inside $|z|=3$

Now, $\frac{1}{(z-1)(z-2)}=\frac{-1}{(z-1)}+\frac{1}{(z-2)}$ (by Partial fraction method)

$$\therefore \int_C \frac{\sin \pi z^2+\cos \pi z^2}{(z-1)(z-2)}dz=-\int_C \frac{\sin \pi z^2+\cos \pi z^2}{(z-1)}dz+\int_C \frac{\sin \pi z^2+\cos \pi z^2}{(z-2)}dz$$
$$=-2\pi i\ f(1)+2\pi i\ f(2)$$

$$\text{f}(z)=\sin \pi z^2+\cos \pi z^2$$
$$\text{f}(1)=\sin \pi+\cos \pi=-1 \text{ and } \text{f}(2)=\sin 4\pi+\cos 4\pi=1$$

$$\therefore \int_C \frac{\sin \pi z^2+\cos \pi z^2}{(z-1)(z-2)}dz=-2\pi i(-1)+2\pi i(1)=4\pi i$$

Using Cauchy's integral formula, evaluate $\int_C \frac{1}{(z-2)\ (z+1)^2}dz$, where C is $|z|=\frac{3}{2}$

Solution:

Here $z=-1$ is a pole lies inside the circle
$z=2$ is a pole lies out side the circle

$$\therefore \int_C \frac{dz}{\left(z+1\right)^2\left(z-2\right)}=\int \frac{\frac{1}{z-2}}{\left(z+1\right)^2}\,dz$$

Here $\text{f}\left(z\right)=\frac{1}{z-2}$, $f'\left(z\right)=-\frac{1}{(z-2)^2}$

Hence by Cauchy's integral formula

$$\int_C \frac{f\left(z\right)}{\left(z-a\right)^{n+1}}\,dz=\frac{2\pi i}{n!}f^n\left(a\right)$$

$$\int_C \frac{dz}{(z+1)^2(z-2)} = \int_C \frac{\frac{1}{z-2}}{[z-(-1)]^2}\,dz = \frac{2\pi i}{1!} f'(-1)$$

$$= 2\pi i\left[\frac{-1}{(-1-2)^2}\right] \left(\because f'(z) = \frac{-1}{(z-2)^2}\right) = 2\pi i\left[\frac{-1}{9}\right]$$

$$\int_C \frac{1}{(z-2)\ (z+1)^2}dz = \frac{-2}{9}\pi i.$$

Using Cauchy's integral formula, evaluate $\int_c \frac{z}{z^2+1}dz$ where C is $|z+i|=1$.

Solution:

Consider the curve

$|z+i|=1 \Rightarrow |x+iy+i|=1$

$|x+i(y+1)|=1 \Rightarrow x^2+(y+1)^2=1$

Which is a circle with centre (0,–1) and radius 1

The poles are obtained by $z^2+1=0$

$\Rightarrow z=i$ is a simple pole which lies outside C.

$z=-i$ is a simple pole which lies inside C.

$$\int_c \frac{z}{z^2+1}dz = \int_c \frac{z}{(z+i)(z-i)}dz = \int_c \frac{\frac{z}{(z-i)}}{(z+i)} = 2\pi i f(-i) \ldots (1)$$

$$f(z) = \frac{z}{(z-i)},\ f(-i) = \frac{-i}{(-i-i)} = \frac{-i}{-2i} = \frac{1}{2}$$

$$(1) \Rightarrow \int_c \frac{z}{z^2+1}dz = 2\pi i f(-i) = 2\pi i\left(\frac{1}{2}\right) = \pi i$$

Expand $f(z) = \log(1+z)$ in Taylor's series about $z=0$

Solution: Let $f(z) = \log(1+z)$ $f(0) = \log 1 = 0$

$$f'(z) = \frac{1}{1+z} \quad f'(0) = \frac{1}{1+0} = 1$$

$$f''(z) = \frac{-1}{(1+z)^2} \qquad f''(0) = -1$$

$$f'''(z) = \frac{2}{(1+z)^3} \quad f'''(0) = 2$$

$$f^{iv}(z) = \frac{-6}{(1+z)^4} \qquad f^{iv}(0) = -6$$

$$\log(1+z) = f(0) + \frac{f'(0)}{1!}z + \frac{f''(0)}{2!}z^2 + ... = z - \frac{z^2}{2} + \frac{z^3}{3} - \frac{z^4}{4} + ...$$

Find the Taylor's series expansion of f(z) = $\frac{z}{(z+1)(z-3)}$, in the region $|z| < 1$

Solution:

Splitting f(z) into partial fractions, we have

$$f(z) = \frac{z}{(z+1)(z-3)} = \frac{A}{(z+1)} + \frac{B}{(z-3)}$$

$$\Rightarrow z = A(z-3) + B(z+1)$$

$$\text{put } z = -1, \text{ we get } A = \frac{1}{4}$$

$$\text{put } z = 3, \text{ we get } B = \frac{3}{4}$$

$$\therefore \quad f(z) = \frac{1}{4}\left(\frac{1}{z+1}\right) + \frac{3}{4}\left(\frac{1}{z-3}\right) = \frac{1}{4}\left(\frac{1}{1+z}\right) + \frac{3}{4}\left(\frac{1}{-3}\right)\left(\frac{1}{1-\frac{z}{3}}\right)$$

$$= \frac{1}{4}\left[(1+z)^{-1} - \left(1-\frac{z}{3}\right)^{-1}\right]$$

$$= \frac{1}{4}\left[(1 - z + z^2 - ....) - \left(1 + \frac{z}{3} + \frac{z^2}{9} + ...\right)\right]$$

$$= \frac{1}{4}\left[\left((-1) - \frac{1}{3}\right)z + \left((-1)^2 - \left(\frac{1}{3}\right)^2\right)z^2 + ...\right]$$

$$\therefore f(z) = \frac{1}{4}\sum_{n=1}^{\infty}\left((-1)^n - \left(\frac{1}{3}\right)^n\right)z^n$$

Obtain Taylor's Series to represent the function $f(z) = \frac{z^2-1}{(z+2)(z+3)}$ in the region | z | < 2

Solution:

$$f(z)=\frac{z^2-1}{(z+2)(z+3)}=\frac{z^2-1}{z^2+5z+6}$$

Since the degree of the numerator and denominator are same we have to divide and apply partial fractions.

$$\frac{z^2-1}{z^2+5z+6}=1+\frac{-5z-7}{z^2+5z+6}=1+\frac{-5z-7}{(z+3)(z+2)}$$

$$|z|<2 \quad \Rightarrow \frac{|z|}{2}<1 \text{ and } \quad \therefore \frac{|z|}{3}<1$$

Consider

$$\frac{-5z-7}{(z+3)(z+2)}=\frac{3}{z+2}-\frac{8}{z+3}=\frac{3}{2\left(1+\frac{z}{2}\right)}-\frac{8}{3\left(1+\frac{z}{3}\right)}=\frac{3}{2}\left(1+\frac{z}{2}\right)^{-1}-\frac{8}{3}\left(1+\frac{z}{3}\right)^{-1}$$

$$=\frac{3}{2}\left(1-\frac{z}{2}+\frac{z^2}{2}-....\right)-\frac{8}{3}\left(1-\frac{z}{3}+\frac{z^2}{9}-...\right)$$

$$\therefore \frac{z^2-1}{z^2+5z+6}=1+\frac{-5z-7}{z^2+5z+6}=1+\frac{3}{2}\left(1-\frac{z}{2}+\frac{z^2}{2}-....\right)-\frac{8}{3}\left(1-\frac{z}{3}+\frac{z^2}{9}-...\right)$$

Find the Laurent's series expansion of $\frac{1}{(z-2)(z-1)}$ **valid in the regions** $|z|>2$ **and** $0<|z-1|<1$

Solution:

$$f(z)=\frac{1}{(z-2)(z-1)}=\frac{A}{(z-1)}+\frac{B}{(z-2)}=\frac{A(z-2)+B(z-1)}{(z-2)(z-1)}$$

$$\Rightarrow 1=A(z-2)+B(z-1)$$

Put $z=1,\ A=-1$

$z=2,\ B=1$

$$\therefore f(z)=\frac{-1}{(z-1)}+\frac{1}{(z-2)}$$

Region 1:

$$|z| > 2 \Rightarrow 2 < |z|$$

$$\Rightarrow \left|\frac{2}{z}\right| < 1$$

$$f(z) = \frac{-1}{z\left(1-\frac{1}{z}\right)} + \frac{1}{z\left(1-\frac{2}{z}\right)}$$

$$= -\frac{1}{z}\left(1-\frac{1}{z}\right)^{-1} + \frac{1}{z}\left(1-\frac{2}{z}\right)^{-1}$$

$$= -\frac{1}{z}\left(1+\frac{1}{z}+\left(\frac{1}{z}\right)^2+\ldots\right) + \frac{1}{z}\left(1+\frac{2}{z}+\left(\frac{2}{z}\right)^2+\ldots\right)$$

$$= -\frac{1}{z}\sum_{n=0}^{\infty}\left(\frac{1}{z}\right)^n + \frac{1}{z}\sum_{n=0}^{\infty}\left(\frac{2}{z}\right)^n$$

$$= -\sum_{n=0}^{\infty}\frac{1}{z^{n+1}} + \sum_{n=0}^{\infty}\frac{2^n}{z^{n+1}}$$

Region 2:

Put $z-1=t \Rightarrow z=1+t$

$$0<|z-1|<1 \Rightarrow 0<|t|<1$$

$$\Rightarrow |t|<1$$

$$f(z) = \frac{-1}{(z-1)} + \frac{1}{(z-2)}$$

$$= \frac{-1}{t} + \frac{1}{t-1}$$

$$= \frac{-1}{t} + \frac{1}{-(1-t)}$$

$$= \frac{-1}{t} - (1-t)^{-1}$$

$$= \frac{-1}{t} - \left(1+t+t^2+\ldots\right)$$

$$=\frac{-1}{(z-1)}-\left(1+(z-1)+(z-1)^2+\ldots\right)$$

$$=\frac{-1}{(z-1)}-\sum_{n=0}^{\infty}(z-1)^n$$

Expand f(z) = $\dfrac{z^2-1}{z^2+5z+6}$ in a Laurent's series expansion for $|z|>3$ and $2<|z|<3$

Solution:

$$\frac{z^2-1}{z^2+5z+6}=1+\frac{-5z-7}{z^2+5z+6}=1+\frac{-5z-7}{(z+3)(z+2)}$$

Consider $\dfrac{-5z-7}{(z+3)(z+2)}$

$$\frac{-5z-7}{(z+3)(z+2)}=\frac{A}{z+2}+\frac{B}{z+3}=\frac{A(z+3)+B(z+2)}{(z+3)(z+2)}$$

$$-5z-7=A(z+3)+B(z+2)$$

Put $z=-2$ then $A=3$

Put $z=-3$ then $B=-8$

Substituting we get, $\dfrac{-5z-7}{(z+3)(z+2)}=\dfrac{3}{z+2}-\dfrac{8}{z+3}$

$$\frac{z^2-1}{z^2+5z+6}=1+\frac{3}{z+2}-\frac{8}{z+3}$$

(i) **Given** $|z|>3 \quad \Rightarrow \quad \dfrac{3}{|z|}<1$

$$\frac{z^2-1}{z^2+5z+6}=1+\frac{3}{z+2}-\frac{8}{z+3}=1+\frac{3}{z\left(1+\frac{2}{z}\right)}-\frac{8}{z\left(1+\frac{3}{z}\right)}$$

$$=1+\frac{3}{z}\left(1+\frac{2}{z}\right)^{-1}-\frac{8}{z}\left(1+\frac{3}{z}\right)^{-1}$$

$$=1+\frac{3}{z}\left(1-\frac{2}{z}+\frac{4}{z^2}-\ldots\right)-\frac{8}{z}\left(1-\frac{3}{z}+\frac{9}{z^2}-\ldots\right)$$

(ii) **Given** $2<|z|<3\Rightarrow \quad \dfrac{2}{|z|}<1$ and $\dfrac{|z|}{3}<1$

$$\frac{z^2-1}{z^2+5z+6}=1+\frac{3}{z+2}-\frac{8}{z+3}=1+\frac{3}{z\left(1+\frac{2}{z}\right)}-\frac{8}{3\left(1+\frac{z}{3}\right)}$$

$$=1+\frac{3}{z}\left(1+\frac{2}{z}\right)^{-1}-\frac{8}{3}\left(1+\frac{z}{3}\right)^{-1}$$

$$=1+\frac{3}{z}\left(1-\frac{2}{z}+\frac{4}{z^2}-...\right)-\frac{8}{3}\left(1-\frac{z}{3}+\frac{z^2}{9}-...\right)$$

Obtain the Laurent's series expansion for the function f(z) = $\frac{4z}{\left(z^2-1\right)(z-4)}$ **in**

$|z-1|>4$ **and** $2<|z-1|<3$

Solution:

Put $z-1=u \Rightarrow z=u+1$

Now, $f(z)=\frac{4z}{\left(z^2-1\right)(z-4)}=\frac{4z}{(z-1)(z+1)(z-4)}$

Hence $f(u)=\frac{4(u+1)}{u(u+2)(u-3)}$

$$\frac{4(u+1)}{u(u+2)(u-3)}=\frac{A}{u}+\frac{B}{u+2}+\frac{C}{u-3}=\frac{A(u+2)(u-3)+Bu(u-3)+Cu(u+2)}{u(u+2)(u-3)}$$

$$4(u+1)=A(u+2)(u-3)+Bu(u-3)+Cu(u+2)$$

Put $u=0$ then $A=\frac{-2}{3}$

Put $u=-2$ then $B=\frac{-2}{5}$

Put $u=3$ then $C=\frac{16}{15}$

$$f(u)=\frac{4(u+1)}{u(u+2)(u-3)}=\frac{-2/3}{u}+\frac{-2/5}{u+2}+\frac{16/15}{u-3}$$

(i) $|u|>4 \quad \Rightarrow \quad \frac{4}{|u|}<1$

$$f(u)=\frac{-2/3}{u}-\frac{2/5}{u+2}+\frac{16/15}{u-3}$$

$$f(u) = -\frac{2}{3}\left(\frac{1}{u}\right) - \frac{2}{5}\left(\frac{1}{u\left(1+\frac{2}{u}\right)}\right) + \frac{16}{15}\left(\frac{1}{u\left(1-\frac{3}{u}\right)}\right)$$

$$= -\frac{2}{3}\left(\frac{1}{u}\right) - \frac{2}{5}\left(\frac{1}{u}\right)\left(1+\frac{2}{u}\right)^{-1} + \frac{16}{15}\left(\frac{1}{u}\right)\left(1-\frac{3}{u}\right)^{-1}$$

$$= \frac{1}{u}\left[-\frac{2}{3} - \frac{2}{5}\left(1 - \frac{2}{u} + \frac{4}{u^2} - ....\right) + \frac{16}{15}\left(1 + \frac{3}{u} + \frac{9}{u^2} + ...\right)\right]$$

$$\therefore f(z) = \frac{1}{(z-1)}\left[-\frac{2}{3} - \frac{2}{5}\left(1 - \frac{2}{(z-1)} + \frac{4}{(z-1)^2} - ....\right) + \frac{16}{15}\left(1 + \frac{3}{(z-1)} + \frac{9}{(z-1)^2} + ...\right)\right]$$

(ii) $2 < |u| < 3 \Rightarrow \frac{2}{|u|} < 1 \text{ and } \frac{|u|}{3} < 1$

$$f(u) = -\frac{2}{3}\left(\frac{1}{u}\right) - \frac{2}{5}\left(\frac{1}{u\left(1+\frac{2}{u}\right)}\right) + \frac{16}{15}\left(\frac{1}{-3\left(1-\frac{u}{3}\right)}\right)$$

$$= -\frac{2}{3}\left(\frac{1}{u}\right) - \frac{2}{5}\left(\frac{1}{u}\right)\left(1+\frac{2}{u}\right)^{-1} - \frac{16}{45}\left(1-\frac{u}{3}\right)^{-1}$$

$$= \frac{1}{u}\left[-\frac{2}{3} - \frac{2}{5}\left(1 - \frac{2}{u} + \frac{4}{u^2} - ....\right) - \frac{16}{45}\left(1 + \frac{u}{3} + \frac{u^2}{9} + ...\right)\right]$$

$$\therefore f(z) = \frac{1}{(z-1)}\left[-\frac{2}{3} - \frac{2}{5}\left(1 - \frac{2}{(z-1)} + \frac{4}{(z-1)^2} - ....\right) - \frac{16}{45}\left(1 + \frac{(z-1)}{3} + \frac{(z-1)^2}{9} + ...\right)\right]$$

Find the Laurent's series expansion of $f(z) = \frac{7z-2}{z(z-2)(z+1)}$ in 1 < | z + 1| < 3

Solution:

The singular points are z = 0, z = 2, z = –1

$$\frac{7z-2}{z(z-2)(z+1)} = \frac{A}{z} + \frac{B}{z-2} + \frac{C}{z+1}$$

$\Rightarrow$ 7z – 2 = A(z – 2) (z + 1) + B z (z + 1) + C z (z – 2)

Put z = 0, –2 = A(–2) $\Rightarrow$ A = 1

z = 2, 14 – 2 = B 2(2 + 1) $\Rightarrow$ B = 2

$z = -1,\quad -7-2 = C(-1)(-1-2) \Rightarrow C = -3$

$$\frac{7z-2}{z(z-2)(z+1)} = \frac{1}{z}+\frac{2}{z-2}-\frac{3}{z+1}$$

Put t = z + 1 $\Rightarrow$ z = t – 1

$\therefore 1 < |t| < 3$

$1 < |t| \Rightarrow \left|\frac{1}{t}\right| < 1$ and $\left|\frac{t}{3}\right| < 1$

$$f(z) = \frac{1}{z}+\frac{2}{z-2}-\frac{3}{z+1}$$

$$= \frac{1}{t-1}+\frac{2}{t-3}-\frac{3}{t}$$

$$= \frac{1}{t\left(1-\frac{1}{t}\right)}+\frac{2}{(-3)\left(1-\frac{t}{3}\right)}-\frac{3}{t}$$

$$= \frac{1}{t}\left(1-\frac{1}{t}\right)^{-1}-\frac{2}{3}\left(1-\frac{t}{3}\right)^{-1}-\frac{3}{t}$$

$$= \frac{1}{t}\left[1+\frac{1}{t}+\frac{1}{t^2}+\frac{1}{t^3}+\ldots\right]-\frac{2}{3}\left[1+\frac{t}{3}+\left(\frac{t}{3}\right)^2+\left(\frac{t}{3}\right)^3+\ldots\right]-\frac{3}{t}$$

$$= -\frac{2}{t}+\frac{1}{t^2}+\frac{1}{t^3}+\ldots-\frac{2}{3}\left[1+\frac{t}{3}+\left(\frac{t}{3}\right)^2+\left(\frac{t}{3}\right)^3+\ldots\right]$$

$$= -2(z+1)^{-1}+(z+1)^{-2}+(z+1)^{-3}+\ldots-\frac{2}{3}\left[1+\frac{z+1}{3}+\left(\frac{z+1}{3}\right)^2+\left(\frac{z+1}{3}\right)^3+\ldots\right]$$

Evaluate $\int_C \frac{z\ dz}{(z-1)(z-2)^2}$, where C is the circle $|z-2| = \frac{1}{2}$ by Cauchy Residue theorem.

Solution:

The poles are obtained by $(z-1)(z-2)^2 = 0$

$\Rightarrow$ z = 1 is a simple pole and z = 2 is a pole of order 2.

C is the circle $|z-2| = \frac{1}{2}$

Here z = 1 lies outside C and z = 2 lies inside C.

Residue at *z*=2: (Pole of order 2)

$$\text{Res } f(z) = \underset{z\to 2}{lt}\frac{d}{dz}(z-2)^2\frac{z}{(z-1)(z-2)^2} = \underset{z\to 2}{lt}\frac{z-1-z}{(z-1)^2} = -1$$

By Cauchy Residue theorem,

$$\int_C \frac{z\,dz}{(z-1)(z-2)^2} = 2\pi i(-1) = -2\pi i$$

Using Cauchy's residue theorem evaluate $\displaystyle\int_C \frac{3z^2+z-1}{(z^2-1)(z-3)}dz$ **, where C is** $|z| = 2$

Solution:

$|z| = 2$ is the equation of the circle with centre at origin and radius 2.

$$(z^2-1)(z-3) = 0$$

$$\begin{aligned} (z^2-1) &= 0, & (z-3) &= 0 \\ z^2 &= 1, & z &= 3 \\ z &= \pm 1, & z &= 3 \end{aligned}$$

$z = 1, -1$ lies inside the circle and $z = 3$ lies outside the circle

Residue at $z = 1$ is

$$= Lt_{z\to 1}\left((z-1)\frac{3z^2+z-1}{(z+1)(z-1)(z-3)}\right)$$

$$= Lt_{z\to 1}\left(\frac{3z^2+z-1}{(z+1)(z-3)}\right) = -\frac{3}{4}$$

Residue at $z = -1$ is

$$= Lt_{z\to -1}\left((z+1)\frac{3z^2+z-1}{(z+1)(z-1)(z-3)}\right)$$

$$= Lt_{z\to -1}\left(\frac{3z^2+z-1}{(z-1)(z-3)}\right) = \frac{1}{8}$$

By Cauchy's Residue theorem,

$$\int_C f(z)\,dz = 2\pi i\left(\text{Sum of the Residues of } f(z) \text{ at each of its poles which lies inside } C\right)$$

$$\therefore \int_C \frac{3z^2+z-1}{\left(z^2-1\right)(z-3)}dz = 2\pi i\left(\frac{1}{8}-\frac{3}{4}\right) = -\frac{5\pi i}{4}$$

Evaluate $\int_C \frac{z-1}{(z+1)^2(z-2)}dz$, where C is $|z-i|=2$ using Cauchy's residue theorem

Solution:

Let $f(z)=\dfrac{z-1}{(z+1)^2(z-2)}$

poles of $f(z)$ are z = - 1 (pole of order 2) and z = 2 (simple pole)

Given: $|z-i|=2$

$$|x+iy-i|=2 \Rightarrow |x+i(y-1)|=2$$

Squaring on both sides $\sqrt{x^2+(y-1)^2}=2 \Rightarrow x^2+(y-1)^2=4$

This is equation of circle with centre $(0,1)$ and radius 2

Hence, The pole $z=2$ lies outside C and $z=-1$ lies inside C

Residue of f(z) at $z=-1$

$$= Lt_{z\to -1}\frac{1}{1!}\frac{d}{dz}\left((z+1)^2\frac{(z-1)}{(z+1)^2(z-2)}\right)$$

$$= Lt_{z\to -1}\frac{1}{1!}\frac{d}{dz}\left(\frac{(z-1)}{(z-2)}\right) = Lt_{z\to -1}\left(\frac{(z-2)(1)-(z-1)(1)}{(z-2)^2}\right)$$

$$= Lt_{z\to -1}\left(\frac{-1}{(z-2)^2}\right) = -\frac{1}{9}$$

By Cauchy's Residue theorem,

$$\int_C f(z)\,dz = 2\pi i\left(\text{Sum of the Residues of } f(z) \text{ at each of its poles which lies inside } C\right)$$

$$\therefore \int_C \frac{(z-1)}{(z+1)^2(z-2)}\,dz = 2\pi i\left(0-\frac{1}{9}\right) = -\frac{2\pi i}{9}$$

Using Cauchy's residue theorem, find $\int_C \frac{z+1}{(z-3)(z-1)}\,dz$ **, where C is** $|z| = 2$

Solution:

The singular points are given by (z – 1) (z – 3) = 0 $\Rightarrow$ z = 1 , 3

Given C is | z | = 2

If z = 1 then | z | = | 1 | = 1 < 2

If z = 3 then | z | = | 3 | = 3 > 2

$$\int_C f(z)\,dz = 2\pi i\left(\text{Sum of the Residues of } f(z) \text{ at each of its poles which lies inside } C\right)$$

Residue at z=1:

$$\text{Res}\Big|_{z=1} = \lim_{z\to 1}(z-1)\,f(z) = \lim_{z\to 1}(z-1)\frac{z+1}{(z-3)(z-1)} = -1$$

$$\therefore \int_C \frac{z+1}{(z-3)(z-1)}\,dz = 2\pi i(-1) = -2\pi i$$

Evaluate $\int_0^{2\pi} \frac{d\theta}{13+5\sin\theta}$ **by using Contour integration.**

Solution:

Consider the unit circle | z | = 1 as contour C.

Put $z = e^{i\theta}$, then $\frac{1}{z} = e^{-i\theta}$

$$\therefore d\theta = \frac{dz}{iz},\ \sin\theta = \frac{z-\frac{1}{z}}{2i} = \frac{z^2-1}{2iz}$$

$$\therefore I = \int_C \frac{\frac{dz}{iz}}{13+5\frac{(z^2-1)}{2iz}} = \int_C \frac{\frac{dz}{iz}}{\frac{26iz+5z^2-5}{2iz}} = 2\int_C \frac{dz}{5z^2+26iz-5}$$

Let $f(z) = \frac{1}{5z^2+26iz-5}$ $\quad\quad \therefore I = 2\int_C f(z)dz$

The poles of f(z) are given by $5z^2 + 26iz - 5 = 0$

$$z = \frac{-26i \pm \sqrt{(26i)^2 - 4\ .\ 5(-5)}}{10} = \frac{-26i \pm \sqrt{-676+100}}{10} = \frac{-26i \pm \sqrt{-576}}{10} = \frac{-26i \pm 24i}{10}$$

$$z = -\frac{i}{5}\ ,\ -5i$$

which are simple poles.

Now $5z^2 + 26iz - 5 = 5\left(z+\frac{i}{5}\right)(z+5i)$

Since $\left|\frac{-i}{5}\right| = \frac{1}{5} < 1,$ the pole $z = \frac{-i}{5}$ lies inside C

and $|-5i| = 5 > 1,$ $\therefore$ the pole $z = -5i$ lies outside C.

Now $R\left(-\frac{i}{5}\right) = \lim_{z\to\frac{-i}{5}}\left(z+\frac{i}{5}\right)f(z) = \lim_{z\to\frac{-i}{5}}\left(z+\frac{i}{5}\right)\frac{1}{5\left(z+\frac{i}{5}\right)(z+5i)} = \lim_{z\to\frac{-i}{5}}\frac{1}{5(z+5i)}$

$$= \lim_{z\to\frac{-i}{5}}\frac{1}{5\left(-\frac{i}{5}+5i\right)} = \frac{1}{24i}$$

By Cauchy's residue theorem,

$$\int_C f(z)dz = 2\pi i\left(\frac{1}{24i}\right) = \frac{\pi}{12}$$

$$\therefore I = 2.\frac{\pi}{12} = \frac{\pi}{6}$$

Evaluate $\int_0^{2\pi} \frac{d\theta}{13+12\cos\theta}$ by using Contour integration.

Solution:

Consider the unit circle | z | = 1 as contour C.

Put $z = e^{i\theta}$, then $\frac{1}{z} = e^{-i\theta}$

$$\therefore d\theta = \frac{dz}{iz}, \quad \cos\theta = \frac{z^2+1}{2z}$$

$$\therefore I = \int_C \frac{\frac{dz}{iz}}{13+12\frac{(z^2+1)}{2z}} = \int_C \frac{dz}{\frac{iz(13z+6z^2+6)}{z}} = \int_C \frac{dz}{i(6z^2+13z+6)} = \frac{1}{i6}\int_C \frac{dz}{(z^2+\frac{13}{6}z+1)}$$

Let $f(z) = \int_C \frac{dz}{(z^2+\frac{13}{6}z+1)}$ $\qquad \therefore I = \frac{1}{6i}\int_C f(z)dz$

The poles of f(z) are given by $z^2 + \frac{13}{6}z + 1 = 0$

By solving we get $z = -\frac{2}{3}, \quad -\frac{3}{2}$

which are simple poles.

Now $z^2 + \frac{13}{6}z + 1 = \left(z+\frac{2}{3}\right)\left(z+\frac{3}{2}\right)$

Since $\left|\frac{-2}{3}\right| = \frac{2}{3} < 1$, the pole $z = \frac{-2}{3}$ lies inside C

and $\left|\frac{-3}{2}\right| = 1.5 > 1$, $\therefore$ the pole $z = \frac{-3}{2}$ lies outside C.

$$Now \quad R\left(-\frac{2}{3}\right) = \lim_{z\to\frac{-2}{3}}\left(z+\frac{2}{3}\right)f(z) = \lim_{z\to\frac{-2}{3}}\left(z+\frac{2}{3}\right)\frac{1}{\left(z+\frac{2}{3}\right)\left(z+\frac{3}{2}\right)} = \lim_{z\to\frac{-2}{3}}\frac{1}{\left(z+\frac{3}{2}\right)}$$

$$= \lim_{z\to\frac{-2}{3}}\frac{1}{\left(z+\frac{3}{2}\right)} = \frac{6}{5}$$

By Cauchy's residue theorem,

$$\int_C f(z)dz = 2\pi i\left(\frac{6}{5}\right) = \frac{12\pi i}{5}, \qquad \therefore I = \frac{1}{6i}\times\left(\frac{12\pi i}{5}\right) = \frac{2\pi}{5}.$$

Evaluate $\int_0^{2\pi}\frac{\cos 3\theta\, d\theta}{5-4\cos\theta}$ by using Contour integration

Solution:

Consider the unit circle | z | = 1 as contour C.

$$Put\ z = e^{i\theta},\ then\ \frac{1}{z} = e^{-i\theta}$$

$$\therefore d\theta = \frac{dz}{iz}, \quad \cos\theta = \frac{z^2+1}{2z}$$

$\cos 3\theta$=R.P. of $e^{i3\theta}$=R.P. of $(e^{i\theta})^3$=R.P. of z^3

$$\therefore \qquad I = \int_C \frac{R.P. of\ z^3 \frac{dz}{iz}}{5 - 4\frac{(z^2+1)}{2z}} = R.P.\ of\ \int_C \frac{z^3 dz}{\frac{iz(5z-2z^2-2)}{z}}$$

$$= R.P.\ of \int_C \frac{z^3 dz}{i(-2z^2+5z-2)}$$

$$= R.P.\ of \int_C \frac{z^3 dz}{-i(2z^2-5z+2)}$$

$$= R.P. of\ \frac{-1}{2i}\int_C \frac{z^3 dz}{(2z-1)(z-2)}$$

Let $\int_C f(z)dz = \int_C \frac{z^3 dz}{(2z-1)(z-2)}$ $\qquad \therefore I = R.P.$ of $\frac{-1}{2i}\int_C f(z)dz$

The poles of f(z) are given by

$(2z-1)(z-2) = 0$

$z = \frac{1}{2},\ z = 2$

$z = \frac{1}{2},\ z = 2$ (simple poles)

$z = \frac{1}{2}$ is a pole lies inside c.

$z = 2$ is a pole lies outside c.

$$\text{Now} \quad \operatorname{Res}\left(z=\frac{1}{2}\right) = \lim_{z\to\frac{1}{2}}\left(z-\frac{1}{2}\right) f(z) = \lim_{z\to\frac{1}{2}}\left(z-\frac{1}{2}\right)\frac{z^3}{\left(z-\frac{1}{2}\right)\ (z-2)} = \frac{-1}{12}$$

By Cauchy's residue theorem,

$$\int_C f(z)dz = 2\pi i \left(\frac{-1}{12}\right) = \frac{-\pi i}{6}$$

$$\therefore I = R.P.of\ \frac{-1}{2i}\ .\frac{-\pi i}{6} = R.P.of\ \frac{\pi}{12} = \frac{\pi}{12}$$

Evaluate $\int_0^{2\pi} \frac{d\theta}{1-2p\sin\theta + p^2}$ **, | p | < 1**

Solution: Let $z = e^{i\theta}$, $dz = i e^{i\theta} d\theta \Rightarrow d\theta = \frac{dz}{iz}$, $\sin\theta = \frac{z^2-1}{2iz}$

$$\int_0^{2\pi} \frac{d\theta}{1-2p\sin\theta+p^2} = \int_C \frac{\left(dz/iz\right)}{1-2p\left(\frac{z^2-1}{2iz}\right)+p^2} \quad ,C\ \text{is}\ |z|=1$$

$$= \int_C \frac{dz}{iz - p(z^2-1)+izp^2} = -\int_C \frac{dz}{pz^2 - iz(p^2+1)-p} = -\frac{1}{p}\int_C \frac{dz}{z^2 - iz\left(p+\frac{1}{p}\right)-1}$$

$$\int_0^{2\pi} \frac{d\theta}{1-2p\sin\theta+p^2} = -\frac{1}{p}\int_C \frac{dz}{(z-ip)\left(z-\frac{i}{p}\right)} \quad ....(1)$$

The poles are given by $z = ip\ \&\ z = \frac{i}{p}$

| z | = | i p | = p < 1. $\therefore$ z= ip lies inside C and $z = \frac{i}{p}$ lies outside C.

$$\therefore \left[\text{Res of } f(z)\right]_{z=ip} = \underset{z\to ip}{\text{Lt}}\ (z-ip)\left[\frac{1}{(z-ip)\left(z-\frac{i}{p}\right)}\right] = \underset{z\to ip}{\text{Lt}}\left(\frac{1}{z-\frac{i}{p}}\right) = \frac{1}{i\left(p-\frac{1}{p}\right)} = \frac{ip}{1-p^2}$$

By Cauchy Residue Theorem $\int_C \frac{dz}{(z-ip)\left(z-\frac{i}{p}\right)} = 2\pi i\left(\frac{ip}{1-p^2}\right) = \frac{-2\pi p}{1-p^2}$

From (1) $\int_0^{2\pi} \frac{d\theta}{1-2p\sin\theta+p^2} = -\frac{1}{p}\left(-\frac{2\pi p}{1-p^2}\right) = \frac{2\pi}{1-p^2}$

Evaluate $\int_0^{\infty} \frac{dx}{(x^2+a^2)^2}, (a>0)$ **using contour integration**

Solution:

Let $f(z)=\dfrac{1}{(z^2+a^2)^2}$. Consider $\int\limits_c f(z)dz$

where C is the contour consists of the upper half circle c_1 of $|z|=R$ & the real axix from –R to R.

$$\therefore \int\limits_c f(z)dz=\int\limits_{c_1} f(z)dz+\int\limits_{-R}^{R} f(z)dz\text{.......... ....} \quad (1)$$

The poles of $f(z)$ are given by $(z^2+a^2)^2=0 \Rightarrow z=\pm ai$ (twice)

z = ai is a pole of order 2 & lies inside C

z = -ai is a pole of order 2 & lies outside C

$$\mathrm{Res}\left[f(z),ai\right]=\lim_{z\to ai}\frac{d}{dz}\left[(z-ai)^2\frac{1}{(z+ai)^2(z-ai)^2}\right]=\lim_{z\to ai}\frac{d}{dz}\left[\frac{1}{(z+ai)^2}\right]=\frac{-2}{(2ai)^3}=\frac{1}{4a^3 i}$$

By Cauchy's Residue Theorem $\int f(z)dz=2\pi i(\dfrac{1}{4a^3 i})=\dfrac{\pi}{2a^3}$

In (1) $R\to\infty$, then $\int\limits_{c_1} f(z)dz=0$

$$\therefore (1)\Rightarrow \int\limits_c f(z)dz=\int\limits_{-\infty}^{\infty} f(x)dx$$

$$=\int\limits_{-\infty}^{\infty}\frac{dx}{(x^2+a^2)^2}=\frac{\pi}{2a^3}$$

$$=2\int\limits_{0}^{\infty}\frac{dx}{(x^2+a^2)^2}=\frac{\pi}{2a^3}$$

$$\int\limits_{0}^{\infty}\frac{dx}{(x^2+a^2)^2}=\frac{\pi}{4a^3}$$

Evaluate $\displaystyle\int\limits_0^\infty \frac{\cos ax\,dx}{x^2+1}$, $a>0$, using contour integration.

Solution:

$$\int\limits_0^\infty \frac{\cos ax\ dx}{1+x^2}=\frac{1}{2}\int\limits_{-\infty}^{\infty}\frac{\cos ax\ dx}{1+x^2}$$

Now $\displaystyle\int\limits_{-\infty}^{\infty}\frac{\cos ax\ dx}{1+x^2}=\int\limits_{-\infty}^{\infty}\frac{\text{RP of } e^{iax}}{1+x^2}dx \qquad \left\{\because e^{i\theta}=\cos\theta+i\sin\theta\right\}$

Consider $\int_c f(z)\,dz = \text{R.P}\int_c \frac{e^{iaz}}{1+z^2}\,dz$

Where c is the upper half of the semi-circle Γ with the bounding diameter [–R, R]. By Cauchy's residue theorem, we have

$$\int_c f(z)\,dz = \int_{-R}^{R} f(x)\,dx + \int_\Gamma f(z)\,dz$$

The poles of f(z) are at $1+z^2=0$

$z^2=-1 \Rightarrow z=\pm i$

The point $z=i$ lies inside the semi-circle and the point $z=-i$ lies outside the semi-circle

Residue at $z=i$ is given by

$$Lt_{z\to i}(z-i)\,f(z) = Lt_{z\to i}(z-i)\frac{e^{iaz}}{(z-i)(z+i)}$$

$$= Lt_{z\to i}\frac{e^{iaz}}{(z+i)} = \frac{e^{ia(i)}}{i+i} = \frac{e^{ai^2}}{2i} = \frac{e^{-a}}{2i}$$

By Cauchy Residue theorem,

$$R.P\int_c \frac{e^{iaz}\,dz}{1+z^2} = \text{R.P of } 2\pi i\left(\frac{e^{-a}}{2i}\right) = \text{R.P of } \pi e^{-a} = \pi e^{-a}$$

$$\therefore \int_{-R}^{R} f(x)\,dx + \int_\Gamma f(z)\,dz = \pi e^{-a}$$

If $R\to\infty$, then $\int_\Gamma f(z)\,dz \to 0$

Hence $\int_{-\infty}^{\infty} f(x)\,dx = \pi e^{-a}$

$$\int_0^\infty \frac{\cos ax\,dx}{1+x^2} = \frac{1}{2}\int_{-\infty}^{\infty}\frac{\cos ax\,dx}{1+x^2} = \frac{\pi e^{-a}}{2}$$

Evaluate $\int_{-\infty}^{\infty}\frac{x^2-x+2}{x^4+10x^2+9}\,dx$**, using contour integration.**

Solution:

Let $f\left(z\right)=\dfrac{z^2-z+2}{z^4+10z^2+9}$

Consider $\int_c f\left(z\right)dz=\int_c \dfrac{z^2-z+2}{z^4+10z^2+9}dz$

Where c is the upper half of the semi-circle Γ with the bounding diameter [-R, R]. By Cauchy's residue theorem, we have

$$\int_c f\left(z\right)dz=\int_{-R}^{R} f(x)\,dx+\int_{\Gamma} f(z)\,dz$$

The poles f(z) are at $z^4+10z^2+9=0$

$$\left(z^2+1\right)\left(z^2+9\right)=0$$

$$z^2=-1; \qquad z^2=-9$$

$$z=\pm i; \qquad z=\pm 3i$$

The poles are at $3i,\ -3i,\ i,\ -i$

Here the poles $3i$ and i lie inside the semi-circle.

Residue at $z=3i$ is given by

$$=Lt_{z\to 3i}\left(z-3i\right)f\left(z\right)$$

$$=Lt_{z\to 3i}\left(z-3i\right)\frac{z^2-z+2}{\left(z^2+9\right)\left(z^2+1\right)}$$

$$=Lt_{z\to 3i}\left(z-3i\right)\frac{z^2-z+2}{\left(z-3i\right)\left(z+3i\right)\left(z^2+1\right)}$$

$$=Lt_{z\to 3i}\frac{z^2-z+2}{\left(z+3i\right)\left(z^2+1\right)}=\frac{7+3i}{48i}$$

Residue at $z=i$ is given by

$$=Lt_{z\to i}\left(z-i\right)f\left(z\right)$$

$$=Lt_{z\to i}\left(z-i\right)\frac{z^2-z+2}{\left(z^2+9\right)\left(z^2+1\right)}$$

$$=Lt_{z\to i}\left(z-i\right)\frac{z^2-z+2}{\left(z-i\right)\left(z+i\right)\left(z^2+9\right)}$$

$$= Lt_{z\to i} \frac{z^2 - z + 2}{(z+i)(z^2+9)} = \frac{1-i}{16i}$$

By Cauchy Residue theorem,

$$\int_c \frac{z^2 - z + 2}{z^4 + 10z^2 + 9} dz = 2\pi i\left[\frac{7+3i}{48i} + \frac{1-i}{16i}\right] = 2\pi i\left[\frac{7+3i+3-3i}{48i}\right] = 2\pi i\left[\frac{10}{48i}\right] = \frac{5\pi}{12}$$

$$\therefore \int_{-R}^{R} f(x)\,dx + \int_{\Gamma} f(z)\,dz = \frac{5\pi}{12}$$

If $R\to\infty$, then $\int_{\Gamma} f(z)\,dz \to 0$

Hence $\int_{-\infty}^{\infty} f(x)\,dx = \frac{5\pi}{12}$

$$\therefore \int_{-\infty}^{\infty} \frac{x^2 - x + 2}{x^4 + 10x^2 + 9} dx = \frac{5\pi}{12}$$

Evaluate $\int_0^{\infty} \frac{x \sin mx \;\; dx}{(x^2 + a^2)}$, where a > 0 , m > 0

Solution:

Let $f(z) = \int_0^{\infty} \frac{x \sin mx dx}{(x^2+a^2)} = \frac{1}{2}\int_{-\infty}^{\infty} \frac{x \sin mx dx}{(x^2+a^2)}$

$$= \frac{1}{2}\int_{-\infty}^{\infty} \frac{x \sin mx dx}{(x^2+a^2)} = \frac{1}{2}\,\mathrm{IP}\int_{-\infty}^{\infty} \frac{xe^{imx} dx}{(x^2+a^2)} = \frac{1}{2}\,\mathrm{IP}(\mathrm{I}_1)$$

$$\mathrm{I}_1 = \int_{-\infty}^{\infty} \frac{xe^{imx}}{x^2+a^2} dx = \int_{-\infty}^{\infty} F(x)dx$$

Here $F(x) = \frac{xe^{imx}}{x^2+a^2}$ let $F(z) = \frac{ze^{imx}}{z^2+a^2}$

The poles of F(z) are given by

$\Rightarrow$ z = ±ia are poles of order 1

$\Rightarrow$ z = ia lies inside C

Consider $\int_C f(z)\,dz$ where C is the contour consists of the upper half circle C, of | z| = R. and the real axis from –R to R.

$$\therefore \int_C f(z)\,dz = \int_{C_1} f(z)\,dz + \int_{-R}^{R} f(x)\,dx \quad ----(1)$$

$$\therefore \left[\text{Res of } f(z)\right]_{z=ai} = \underset{z\to ia}{\text{Lt}}\,(z-ia)\,\frac{ze^{imz}}{(z+ib)(z-ib)}$$

$$= \frac{e^{-ma}(ia)}{2ia} = \frac{e^{-ma}}{2}$$

$$I_1 = 2\pi i\left(\frac{e^{-ma}}{2}\right) + \pi i(0) = i\pi e^{-ma}$$

$$I = \frac{1}{2}IP(I_1) = \frac{1}{2}IP(i\pi e^{-ma}) = \frac{\pi e^{-ma}}{2}$$

By Cauchy's Residue Theorem

$$\therefore (1) \Rightarrow \int_C f(z)\,dz = \int_{-\infty}^{\infty} f(x)\,dx \qquad \because \int_C f(z)\,dz \to 0 \text{ as } R \to \infty$$

$$\Rightarrow \int_0^{\infty} f(x)\,dx = \frac{\pi e^{-ma}}{2}$$

Evaluate $\int_0^{\infty} \frac{\cos x\,dx}{(x^2+a^2)(x^2+b^2)}$**, a > 0 , b > 0**

Solution:

Let $f(z) = \text{Real Part of } \frac{e^{iz}}{(z^2+a^2)\;(z^2+b^2)}$

Consider $\int_C f(z)\,dz$ where C is the contour consists of the upper half circle C, of | z| = R. and the real axis from –R to R.

$$\therefore \int_C f(z)\,dz = \int_{C_1} f(z)\,dz + \int_{-R}^{R} f(x)\,dx \quad ----(1)$$

The poles of $f(z)$ are given by $(z^2+a^2)(z^2+b^2) = 0$

$\Rightarrow z = \pm ia, \pm ib$

$\Rightarrow z = ia, ib$ lies inside C and $z = -ia, -ib$ lies in lower half plane

$$\therefore \left[\operatorname{Res} \text{ of } f(z)\right]_{z=ai} = \underset{z\to ia}{\operatorname{Lt}} (z-ia)\,\frac{e^{iz}}{(z+ia)(z-ia)(z^2+b^2)}$$

$$= \frac{e^{-a}}{2ia(b^2-a^2)}$$

$$\left[\operatorname{Res} \text{ of } f(z)\right]_{z=bi} = \underset{z\to ia}{\operatorname{Lt}} (z-ib)\,\frac{e^{iz}}{(z+ib)(z-ib)(z^2+a^2)}$$

$$= \frac{e^{-a}}{2ib(a^2-b^2)}$$

By Cauchy's Residue Theorem

$$\int_C \frac{e^{iz}}{(z^2+a^2)(z^2+b^2)}\,dz = 2\pi i\left[\frac{e^{-a}}{2ia(b^2-a^2)} + \frac{e^{-b}}{2ib(a^2-b^2)}\right]$$

$$= \frac{\pi}{(a^2-b^2)}\left[\frac{e^{-b}}{b} - \frac{e^{-a}}{a}\right]$$

In (1) if $R \to \infty$, $\int_{C_1} f(z)\,dz \to 0$

$$\therefore (1) \Rightarrow \int_C f(z)\,dz = \int_{-\infty}^{\infty} f(x)\,dx$$

$$\int_{-\infty}^{\infty} \frac{e^{ix}}{(x^2+a^2)(x^2+b^2)}\,dx = \frac{\pi}{a^2-b^2}\left(\frac{e^{-b}}{b} - \frac{e^{-a}}{a}\right)$$

$$\int_{-\infty}^{\infty} \frac{\cos x}{(x^2+a^2)(x^2+b^2)}\,dx = \text{Real Part of } \frac{\pi}{a^2-b^2}\left(\frac{e^{-b}}{b} - \frac{e^{-a}}{a}\right)$$

$$\therefore \int_{-\infty}^{\infty} \frac{\cos x}{(x^2+a^2)(x^2+b^2)}\,dx = \frac{\pi}{a^2-b^2}\left(\frac{e^{-b}}{b} - \frac{e^{-a}}{a}\right)$$

SRM OF INSTITUTE OF SCIENCE AND TECHNOLOGY
FACULTY OF ENGINEERING AND TECHNOLOGY
18MAB102T- ADVANCED CALCULUS AND COMPLEX ANALYSIS
PART - A : MULTIPLE CHOICE QUESTIONS

UNIT – I: MULTIPLE INTEGRALS

1. Evaluation of $\int_0^1\int_0^1 dxdy$ is

   (a) 1 (b) 2 (c) 0 (d) 4

2. The curve $y^2 = 4x$ is a

   (a) parabola (b) hyperbola (c) straight line (d) ellipse

3. Evaluation of $\int_0^\pi\int_0^\pi d\theta d\phi$ is

   $a)\ 1$ $b)\ 0$ $c)\ \pi/2$ $d)\ \pi^2$

4. The area of an ellipse is

   $a)\ \pi r^2$ $b)\ \pi a^2 b$ $c)\ \pi ab^2$ $d)\ \pi ab$

5. $\int_1^b\int_1^a \frac{dxdy}{xy}$ is equal to

   a) $\log a + \log b$ b) $\log a$ c) $\log b$ d) $\log a \log b$

6. $\int_0^1\int_0^x dxdy$ is equal to

   $a)\ 1$ $b)\ 1/2$ $c)\ 2$ $d)\ 3$

7. $\int_0^1\int_0^2 dxdy$ is equal to

   $a)\ \int_0^2\int_0^1 dydx$ $b)\ -\int_0^1\int_0^2 dxdy$ $c)\ \int_2^0\int_0^1 dydx$ $d)\ \int_1^0\int_0^2 dydx$

8. If R is the region bounded $x = 0,\ y = 0,\ x + y = 1$ then $\iint_R dxdy$ is equal to

   $a)\ 1$ $b)\ 1/2$ $c)\ 1/3$ $d)\ 2/3$

9. Area of the double integral in cartesian co-ordinate is equal to

   $a)\ \iint_R dydx$ $b)\ \iint_R rdrd\theta$ $c)\ \iint_R xdxdy$ $d)\ \iint_R x^2dxdy$

10. Change the order of integration in $\int_0^a \int_0^x dxdy$ is

$a)\ \int_0^a \int_0^x dxdy$ $b)\ \int_0^a \int_0^x xdydx$ $c)\ \int_0^a \int_y^a dxdy$ $d)\ \int_0^a \int_0^y dxdy$

11. Area of the double integral in polar co-ordinate is equal to

$a)\ \iint_R drd\theta$ $b)\ \iint_R r^2 drd\theta$ $c)\ \iint_R (r+1)drd\theta$ $d)\ \iint_R rdrd\theta$

12. $\int_0^1 \int_0^2 \int_0^3 dxdydz$ is equal to

$a)\ 3$ $b)\ 4$ $c)\ 2$ $d)\ 6$

13. The name of the curve $r = a(1+\cos\theta)$ is

a) lemniscate b) cycloid c) cardioid d) hemicircle

14. The volume integral in cartesian coordinates is equal to

$a)\ \iiint_V dxdydz$ $b)\ \iiint_V drd\theta d\phi$ $c)\ \iint_R drd\theta$ $d)\ \iint_R rdrd\theta$

15. $\int_0^1 \int_0^2 x^2 ydxdy$ is equal to

$a)\ \frac{2}{3}$ $b)\ \frac{1}{3}$ $c)\ \frac{4}{3}$ $d)\ \frac{8}{3}$

16. $\int_0^1 \int_0^1 (x+y)dxdy$ is equal to

$a)\ 1$ $b)\ 2$ $c)\ 3$ $d)\ 4$

17. After changing the double integral $\int_0^\infty \int_0^\infty e^{-(x^2+y^2)}dxdy$ into polar coordinates, we have

$a)\ \int_0^{\pi/2} \int_0^\infty e^{-r^2} drd\theta$ $b)\ \int_0^{\pi/4} \int_0^\infty e^{-r} drd\theta$ $c)\ \int_0^{\pi/2} \int_0^\infty e^{-r^2} rdrd\theta$ $d)\ \int_0^{\pi/2} \int_0^\infty e^{-r} drd\theta$

18. $\int_0^\infty \int_0^y \frac{e^{-y}}{y} dxdy$ is equal to

$a)\ 1$ $b)\ 0$ $c)\ -1$ $d)\ 2$

19. The value of the integral $\int_0^2 \int_0^1 xydxdy$ is

(a) 1 (b) 2 (c) 3 (d) 4

20. The value of the integral $\int_0^{\pi/2}\int_0^{\pi/2} \sin(\theta+\phi)\,d\theta d\phi$

(a) 1 (b) 2 (c) 3 (d) 4

21. The region of integration of the integral $\int_{-b}^{b}\int_{-a}^{a} f(x,y)dxdy$ is

(a) square (b) circle (c) rectangle (d) triangle

22. The region of integration of the integral $\int_0^1\int_0^x f(x,y)dxdy$ is

(a) square (b) rectangle (c) triangle (d) circle

23. The limits of integration is the double integral $\iint_R f(x,y)dxdy$, where R is in the first quadrant and bounded by $x=0,\ y=0,\ x+y=1$ are

$(a)\ \int_{x=0}^{1}\int_{y=0}^{1-x} f(x,y)dydx$ $(b)\ \int_{y=1}^{2}\int_{x=0}^{1-y} f(x,y)dxdy$

$(c)\ \int_{y=0}^{1}\int_{x=1}^{y} f(x,y)dxdy$ $(d)\ \int_{y=0}^{2}\int_{x=0}^{1-y} f(x,y)dxdy$

ANSWERS:

1	a	6	b	11	d	16	a	21	c
2	a	7	a	12	d	17	c	22	c
3	d	8	b	13	c	18	a	23	a
4	d	9	a	14	a	19	a		
5	d	10	c	15	c	20	b		

UNIT – II: VECTOR CALCULUS

1. The directional derivative of $\phi = xy + yz + zx$ at the point (1,2,3) along x - axis is
   (a) 4 (b) 5 (c) 6 (d) 0

2. In what direction from (3, 1, -2) is the directional derivative of $\phi = x^2 y^2 z^4$ maximum?
   a) $\frac{1}{\sqrt{19}}(\vec{i} + 3\vec{j} - \vec{k})$ (b) $19(\vec{i} + 3\vec{j} - 3\vec{k})$
   (c) $96(\vec{i} + 3\vec{j} - 3\vec{k})$ d) $\frac{1}{\sqrt{19}}(3\vec{i} + 3\vec{j} - \vec{k})$

3. If $\vec{r}$ is the position vector of the point (x, y, z) w. r. to the origin, then $\nabla \cdot \vec{r}$ is
   (a) 2 (b) 3 (c) 0 (d) 1

4. If $\vec{r}$ is the position vector of the point (x, y, z) w. r. to the origin, then $\nabla \times \vec{r}$ is
   a) $\nabla \times \vec{r} = 0$ b) $x\vec{i} + y\vec{j} + z\vec{k} = 0$ c) $\nabla \times \vec{r} \neq 0$ d) $\vec{i} + \vec{j} + \vec{k} = 0$

5. The unit vector normal to the surface $x^2 + y^2 - z^2 = 1$ at (1, 1, 1) is
   a) $\frac{\vec{i} + \vec{j} - \vec{k}}{\sqrt{3}}$ b) $\frac{2\vec{i} + 2\vec{j} - 2\vec{k}}{\sqrt{2}}$ c) $\frac{3\vec{i} + 3\vec{j} - 3\vec{k}}{2\sqrt{3}}$ d) $\frac{\vec{i} + \vec{j} - \vec{k}}{3\sqrt{2}}$

6. If $\phi = xyz,$ then $\nabla\phi$ is
   a) $yz\vec{i} + zx\vec{j} + xy\vec{k}$ b) $xy\vec{i} + yz\vec{j} + zx\vec{k}$ c) $zx\vec{i} + xy\vec{j} + yz\vec{k}$ d) 0

7. If $\vec{F} = (x+3y)\vec{i} + (y-3z)\vec{j} + (x-2z)\vec{k}$ then $\vec{F}$ is
   a) solenoidal b) irrotational c) constant vector
   d) both solenoidal and irrotational

8. If $\vec{F} = \left(axy - z^3\right)\vec{i} + (a-2)x^2\vec{j} + (1-a)xz^2\vec{k}$ is irrotational then the value of a is
   a) 0 b) 4 c) -1 d) 2

9. If $\vec{u}$ and $\vec{v}$ are irrotational then $\vec{u} \times \vec{v}$ is
   a) solenoidal b) irrotational c) constant vector d) zero vector

10. If ϕ and ψ are scalar functions then $\nabla\phi \times \nabla\psi$ is

a) solenoidal b) irrotational c) constant vector
d) both solenoidal and irrotational

11. If $\vec{F} = \left(y^2 - z^2 + 3yz - 2x\right)\vec{i} + \left(3xz + 2xy\right)\vec{j} + \left(3xy - 2xz + 2z\right)\vec{k}$ then $\vec{F}$ is

a) solenoidal b) irrotational c) both solenoidal and irrotational
d) neither solenoidal nor irrotational

12. If $\vec{a}$ is a constant vector and $\vec{r}$ is the position vector of the point (x, y, z) w. r. to the origin then $grad(\vec{a} \cdot \vec{r})$ is

a) 0 b) 1 c) $\vec{a}$ d) $\vec{r}$

13. If $\vec{a}$ is a constant vector and $\vec{r}$ is the position vector of the point (x, y, z) w. r. to the origin then $div(\vec{a} \times \vec{r})$ is

a) 0 b) 1 c) $\vec{a}$ d) $\vec{r}$

14. If $\vec{a}$ is a constant vector and $\vec{r}$ is the position vector of the point (x, y, z) w. r. to the origin then $curl(\vec{a} \times \vec{r})$ is

a) 0 b) 1 c) $2\vec{a}$ d) $2\vec{r}$

15. If ϕ scalar functions then $curl(grad\phi)$ is

a) solenoidal b) irrotational c) constant vector d) 0

16. If the value of $\int_A^B \vec{F} \cdot d\vec{r}$ does not depend on the curve C, but only on the terminal points A and B then $\vec{F}$ is called

a) solenoidal vector b) irrotational vector c) conservative vector
d) neither conservative nor irrotational

17. The condition for $\vec{F}$ to be Conservative is, $\vec{F}$ should be

a) solenoidal vector b) irrotational vector c) rotational
d) neither solenoidal nor irrotational

18. The value of $\int_C \vec{r} \cdot d\vec{r}$ where C is the line $y = x$ in the xy-plane from (1,1) to (2,2) is

a) 0 b) 1 c) 2 d) 3

19. The work done by the conservative force when it moves a particle around a closed curve is

a) $\nabla \cdot \vec{F} = 0$ b) $\nabla \times \vec{F} = 0$ c) 0 d) $\nabla \cdot (\nabla \times \vec{F}) = 0$

20. The connection between a line integral and a double integral is known as
a) Green's theorem b) Stoke's theorem c) Gauss Divergence theorem
d) convolution theorem

21. The connection between a line integral and a surface integral is known as
a) Green's theorem b) Stoke's theorem c) Gauss Divergence theorem
d) Residue theorem

22. The connection between a surface integral and a volume integral is known as
a) Green's theorem b) Stoke's theorem c) Gauss Divergence theorem
d) Cauchy's theorem

23. Using Gauss divergence theorem, find the value of $\iint_S \vec{r}\, \vec{ds}$ where $\vec{r}$ is the position vector and V is the volume

a) $4V$ b) 0 c) $3V$ d) volume of the given surface

24. If S is any closed surface enclosing the volume V and if $\vec{F} = ax\,\vec{i} + by\,\vec{j} + cz\,\vec{k}$ then the value of $\iint_S \vec{F} \cdot \vec{n}\, dS$ is

a) $abcV$ b) $(a+b+c)V$ c) 0 d) $abc(a+b+c)V$

ANSWERS:

1	b	6	a	11	c	16	c	21	b
2	c	7	a	12	c	17	b	22	c
3	b	8	b	13	a	18	d	23	c
4	a	9	a	14	a	19	c	24	b
5	a	10	a	15	d	20	a		

UNIT-III LAPLACE TRANSFORMS

1. L(1) =

   (a) $\frac{1}{s}$ (b) $\frac{1}{s^2}$ (c) 1 (d) s

2. $L(e^{3t}) =$

   (a) $\frac{1}{s+3}$ (b) $\frac{1}{s-3}$ (c) $\frac{3}{s+3}$ (d) $\frac{s}{s-3}$

3. $L(e^{-at}) =$

   (a) $\frac{1}{s+1}$ (b) $\frac{1}{s-1}$ (c) $\frac{1}{s+a}$ (d) $\frac{1}{s-a}$

4. $L(\cos 2t) =$

   (a) $\frac{s}{s^2+4}$ (b) $\frac{s}{s^2+2}$ (c) $\frac{2}{s^2+2}$ (d) $\frac{4}{s^2+4}$

5. $L(t^4) =$

   (a) $\frac{4!}{s^5}$ (b) $\frac{3!}{s^4}$ (c) $\frac{4!}{s^4}$ (d) $\frac{5!}{s^4}$

6. $L(a^t) =$

   (a) $\frac{1}{s-\log a}$ (b) $\frac{1}{s+\log a}$ (c) $\frac{1}{s-a}$ (d) $\frac{1}{s+a}$

7. $L(\sinh \omega t) =$

   (a) $\frac{s}{s^2+\omega^2}$ (b) $\frac{\omega}{s^2+\omega^2}$ (c) $\frac{s}{s^2-\omega^2}$ (d) $\frac{\omega}{s^2-\omega^2}$

8. An example of a function for which the Laplace transforms does not exists is

   (a) $f(t)=t^2$ (b) $f(t)=\tan t$ (c) $f(t)=\sin t$ (d) $f(t)=e^{-at}$

9. If $L(f(t)) = F(s)$, then $L(e^{-at} f(t)) =$

   (a) $F(s+a)$ (b) $F(s-a)$ (c) $F(s)$ (d) $\frac{1}{a}F\left(\frac{s}{a}\right)$

10. $L(e^{-at}\cos bt) =$

    (a) $\frac{s+b}{(s+b)^2+a^2}$ (b) $\frac{s+a}{(s+a)^2+b^2}$ (c) $\frac{a}{s^2+a^2}$ (d) $\frac{s}{s^2+b^2}$

11. $L(te^{t}) =$

$(a)\ \dfrac{1}{(s+1)^2}$ $(b)\ \dfrac{1}{s+1}$ $(c)\ \dfrac{1}{s-1}$ $(d)\ \dfrac{1}{(s-1)^2}$

12. $L(t\sin at) =$

$(a)\ \dfrac{2as}{(s^2+a^2)^2}$ $(b)\ \dfrac{2s}{(s^2+a^2)^2}$ $(c)\ \dfrac{s^2-a^2}{(s^2+a^2)^2}$ $(d)\ \dfrac{1}{s^2+a^2}$

13. $L(\sin 3t) =$

$(a)\ \dfrac{3}{s^2+3}$ $(b)\ \dfrac{3}{s^2+9}$ $(c)\ \dfrac{s}{s^2+3}$ $(d)\ \dfrac{s}{s^2+9}$

14. $L(\cosh t) =$

$(a)\ \dfrac{s}{s^2+1}$ $(b)\ \dfrac{s}{s^2-1}$ $(c)\ \dfrac{1}{s^2+1}$ $(d)\ \dfrac{1}{s^2-1}$

15. $L(t^{1/2}) =$

$(a)\ \dfrac{\Gamma(3/2)}{s^{1/2}}$ $(b)\ \dfrac{\Gamma(1/2)}{s^{3/2}}$ $(c)\ \dfrac{\Gamma(1/2)}{s^{1/2}}$ $(d)\ \dfrac{\Gamma(3/2)}{s^{3/2}}$

16. $L(t^{-1/2}) =$

$(a)\ \sqrt{\dfrac{\pi}{s}}$ $(b)\ \sqrt{\dfrac{\pi}{2s}}$ $(c)\ \sqrt{\dfrac{1}{s}}$ $(d)\ \dfrac{1}{s}$

17. $L[te^{2t}] =$

$(a)\ \dfrac{1}{(s-2)^2}$ $(b)\ -\dfrac{1}{(s-2)^2}$ $(c)\ \dfrac{1}{(s-1)^2}$ $(d)\ \dfrac{1}{(s+1)^2}$

18. If $L[f(t)] = F(s)$ then $L\left\{f\left(\dfrac{t}{a}\right)\right\}$ is

(a) $aF(as)$ (b) $\dfrac{1}{a}F\left(\dfrac{s}{a}\right)$ (c) $F(s+a)$ (d) $\dfrac{1}{a}F(as)$

19. $L\left(\int_0^t \sin t\,dt\right)$ is

(a) $\dfrac{1}{s^2+1}$ (b) $\dfrac{s}{s^2+1}$ (c) $\dfrac{1}{(s^2+1)^2}$ (d) $\dfrac{1}{s(s^2+1)}$

20. $L(\sin t\cos t)$ is

(a) $L(\sin t)..L(\cos t)$ (b) $L(\sin t)+L(\cos t)$ (c) $L(\sin t)-L(\cos t)$ (d) $\dfrac{L(\sin 2t)}{2}$

21. If $L[f(t)]=F[s]$ then $L[tf(t)]=$

$(a)\ \dfrac{d}{ds}F(s)$ $(b)\ -\dfrac{d}{ds}F(s)$ $(c)\ (-1)^n\dfrac{d}{ds}F(s)$ $(d)\ -\dfrac{d^2}{ds^2}F(s)$

22. If $L[f(t)]=F[s]$ then $L\left[\dfrac{f(t)}{t}\right]=$

$(a)\ \int_0^\infty F(s)\,ds$ $(b)\ \int_s^\infty F(s)\,ds$ $(c)\ \int_{-\infty}^\infty F(s)\,ds$ $(d)\ \int_a^\infty F(s)\,ds$

23. $L\left[\dfrac{\cos t}{t}\right]=$

$(a)\ \dfrac{s}{s^2+a^2}$ $(b)\ \dfrac{1}{s^2+a^2}$ (c) *does not exist* $(d)\ \dfrac{s^2-a^2}{(s^2+a^2)^2}$

24. If $L[f(t)]=F[s]$ then $L[t^n f(t)]=$

$(a)\ (-1)^n\dfrac{d^n}{ds^n}F(s)$ $(b)\ \dfrac{d^n}{ds^n}F(s)$ $(c)\ -\dfrac{d^n}{ds^n}F(s)$ $(d)\ (-1)^{n-1}\dfrac{d^n}{ds^n}F(s)$

25. $L\left[\dfrac{1-e^{-t}}{t}\right]=$

$(a)\ \log\left(\dfrac{s}{s-1}\right)$ $(b)\ \log\left(\dfrac{s}{s+1}\right)$ $(c)\ \log\left(\dfrac{s+1}{s}\right)$ $(d)\ \log\left(\dfrac{s-1}{s}\right)$

26. $L(u_a(t))$ is

(a) $\dfrac{e^{as}}{s}$ (b) $\dfrac{e^{-as}}{s}$ (c) $-\dfrac{e^{-as}}{s}$ (d) $-\dfrac{e^{as}}{s}$

27. If $L[f(t)]=F[s]$ then $L[f'(t)]=$

(a) $sL[f(t)]-f(0)$ (b) $sL[f(t)]-sf(0)$ (c) $L[f(t)]-f(0)$ (d) $sL[f(t)]-f'(0)$

28. Using the initial value theorem, find the value of the function $f(t)=ae^{-bt}$

$(a)\ a$ $(b)\ a^2$ $(c)\ ab$ $(d)\ 0$

29. Using the initial value theorem, find the value of $f(t)=e^{-2t}\sin t$

$(a)\ 0$ $(b)\ \infty$ $(c)\ 1$ $(d)\ 2$

30. Using the initial value theorem, find the value of the function $f(t) = \sin^2 t$

(a) 0 (b) ∞ (c) 1 (d) 2

31. Using the initial value theorem, find the value of the function $f(t) = 1 + e^{-t} + t^2$

(a) 2 (b) 1 (c) 0 (d) ∞

32. Using the initial value theorem, find the value of the function $f(t) = 3 - 2\cos t$

(a) 3 (b) 2 (c) 1 (d) 0

33. Using the final value theorem, find the value of the function $f(t) = 1 + e^{-t}(\sin t + \cos t)$

(a) 1 (b) 0 (c) ∞ (d) -2

34. Using the final value theorem, find the value of the function $f(t) = t^2 e^{-3t}$

(a) 0 (b) ∞ (c) 1 (d) -1

35. Using the final value theorem, find the value of the function $f(t) = 1 - e^{-at}$

(a) 0 (b) 1 (c) 2 (d) ∞

36. The period of $\tan t$ is

(a) π (b) $\frac{\pi}{2}$ (c) 2π (d) $\frac{\pi}{4}$

37. The period of $|\sin \omega t|$ is

(a) $\frac{\pi}{\omega}$ (b) $\frac{2\pi}{\omega}$ (c) 2π (d) $2\pi\omega$

38. Inverse Laplace transform of $\frac{1}{(s-1)^2}$ is

(a) te^{-t} (b) te^{t} (c) $t^2 e^{t}$ (d) t

39. Inverse Laplace transform of $\frac{2}{s-b}$ is

(a) $2e^{-bt}$ (b) $2e^{bt}$ (c) $2te^{bt}$ (d) $2bt$

40. If $L^{-1}[F(s)] = f(t)$ then $L^{-1}\left(\frac{F(s)}{s}\right)$ is

(a) $\int_0^\infty f(t)dt$ (b) $\int_0^t f(t)dt$ (c) $\int_{-\infty}^\infty f(t)dt$ (d) $\int_{-a}^a f(t)dt$

41. If $L^{-1}[F(s)] = f(t)$ then $L^{-1}\left(\frac{1}{s^2+4}\right)$ is

(a) $\frac{\sin 2t}{2}$ (b) $\frac{\sin\sqrt{2}t}{\sqrt{2}}$ (c) $\sin 2t$ (d) $\sin\sqrt{2}t$

42. Inverse Laplace transform of $\frac{1}{s^2-a^2}$ is

(a) $\frac{\sin at}{a}$ (b) $\frac{\sinh at}{a}$ (c) $\sin at$ (d) $\sinh at$

43. If $L^{-1}[F(s)] = f(t)$ then $L^{-1}\left(\frac{1}{s^2}\right)$ is

(a) t (b) $2t$ (c) $3t$ (d) t^2

44. Inverse Laplace transform of $\frac{s}{s^2-9}$ is

(a) $\cos 9t$ (b) $\cos 3t$ (c) $\cosh 9t$ (d) $\cosh 3t$

45. If $L^{-1}[F(s)] = f(t)$ then $L^{-1}(F(as))$ is

(a) $\frac{f(t)}{a}$ (b) $\frac{1}{a} f\left(\frac{t}{a}\right)$ (c) $f\left(\frac{t}{a}\right)$ (d) $f(at)$

46. Inverse Laplace transform of $\frac{1}{s^3}$ is

(a) $\frac{t}{2}$ (b) t (c) $\frac{t^2}{2}$ (d) t^2

47. Inverse Laplace transform of $\frac{s+3}{(s+3)^2+9}$ is

(a) $e^{3t}\cos 3t$ (b) $e^{-3t}\cos 3t$ (c) $e^{-3t}\cosh 3t$ (d) $e^{-3t}\cos 9t$

48. Inverse Laplace transform of $\frac{b}{s+a}$ is

(a) ae^{-bt} (b) be^{-bt} (c) ae^{bt} (d) be^{at}

49. The value of $e^{-t} * \sin t =$

$(a)\left(\frac{\sin t - \cos t}{2}\right)$ $(b)\left(\frac{\cos t - \sin t}{2}\right)$ $(c)\left(\frac{e^{-t}}{2}\right)+\left(\frac{\sin t - \cos t}{2}\right)$ $(d)\left(\frac{e^{-t}}{2}\right)$

50. The value of $1 * e^t$ is

(a) $\mathrm{e}^{\mathrm{t}} - 1$ (b) $\mathrm{e}^{\mathrm{t}} + 1$ (c) e^{t} (d) e

ANSWERS:

1	a	11	d	21	b	31	a	41	a
2	b	12	a	22	b	32	c	42	b
3	c	13	b	23	c	33	a	43	a
4	a	14	b	24	a	34	a	44	d
5	a	15	d	25	c	35	b	45	b
6	a	16	a	26	b	36	a	46	c
7	d	17	a	27	a	37	a	47	b
8	b	18	a	28	a	38	b	48	b
9	a	19	d	29	a	39	b	49	c
10	b	20	d	30	a	40	b	50	a

UNIT– IV: ANALYTIC FUNCTIONS

1. Cauchy – Riemann equation in polar co-ordinates are
   $(a)\ ru_r = v_\theta, u_\theta = -rv_r$ $(b)\ -ru_r = v_\theta, u_\theta = rv_r$
   $(c)\ -ru_r = v_\theta, u_\theta = rv_r$ $(d)\ u_r = rv_\theta, ru_\theta = v_r$

2. If $w = f(z)$ is analytic function of z, then
   $(a)\ \dfrac{\partial w}{\partial z} = i\dfrac{\partial w}{\partial x}$ $(b)\ \dfrac{\partial w}{\partial z} = i\dfrac{\partial w}{\partial y}$ $(c)\ \dfrac{\partial^2 w}{\partial z \partial \overline{z}} = 0$ $(d)\ \dfrac{\partial w}{\partial \overline{z}} = 0$

3. The function $f(z) = u + iv$ is analytic if
   $(a)\ u_x = v_y, u_y = -v_x$ $(b)\ u_x = -v_y, u_y = v_x$
   $(c)\ u_x + v_y = 0, u_y - v_x = 0$ $(d)\ u_y = v_y, u_x = v_x$

4. The function $w = \sin x \cosh y + i \cos x \sinh y$ is
   (a) need not be analytic (b) analytic (c) discontinuous
   (d) differentiable only at origin

5. If u and v are harmonic, then $u + iv$ is
   (a) harmonic (b) need not be analytic (c) analytic (d) continuous

6. If a function $u(x, y)$ satisfies $u_{xx} + u_{yy} = 0,$ then u is
   (a) analytic (b) harmonic (c) differentiable (d) continuous

7. If $u + iv$ is analytic, then the curves $u = c_1$ and $v = c_2$ are
   (a) cut orthogonally (b) intersect each other (c) are parallel
   (d) coincides

8. The invariant point of the transformation $w = \dfrac{1}{z - 2i}$ is
   (a) $z = i$ $(b)\ z = -i$ $(c)\ z = 1$ $(d)\ z = -1$

9. The transformation $w = cz$ where c is real constant represents
   (a) rotation (b) reflection (c) magnification (d) magnification and rotation

10. The complex function $w = az$ where a is complex constant represents
    (a) rotation (b) magnification and rotation (c) translation (d) reflection

11. The values of $C_1 \& C_2$ such that the function $f(z) = C_1 xy + i[C_2 x^2 + y^2]$ is analytic are
    $(a)\ C_1 = 0, C_2 = 1$ $(b)\ C_1 = 2, C_2 = -1$
    $(c)\ C_1 = -2, C_2 = 1$ $(d)\ C_1 = -2, C_2 = 0$

12. The real part of $f(z)=e^{2z}$ is
(a) $e^x \cos y$ (b) $e^x \sin y$ (c) $e^{2x}\cos 2y$ (d) $e^{2x}\sin 2y$

13. If *f(z)* is analytic where $f(z)=r^2\cos 2\theta + ir^2 \sin p\theta$, the value of p is
(a) $p=1$ (b) $p=-2$ (c) $p=-1$ (d) $p=2$

14. The points at which the function $f(z)=\dfrac{1}{z^2+1}$ fails to be analytic an
(a) $z=\pm 1$ (b) $z=\pm i$ (c) $z=0$ (d) $z=\pm 2$

15. The critical point of transformation $w=z^2$ is
(a) $z=2$ (b) $z=0$ (c) $z=1$ (d) $z=-2$

16. An analytic function with constant modulus is
(a) zero (b) analytic (c) constant (d) harmonic

17. The image of the rectangular region in the z-plane bounded by the lines $x=0$, $y=0$, $x=2$ and $y=1$ under the transformation $w=2z$.
(a) parabola (b) circle (c) straight line (d) rectangle is magnified twice

18. If $f(z)$ and $\overline{f(z)}$ are analytic function of z, then $f(z)$ is
(a) analytic (b) zero (c) constant (d) discontinuous

19. The invariant points of the transformation $w=-\left(\dfrac{2z+4i}{iz+1}\right)$ are
(a) $z=4i,-i$ (b) $z=-4i,i$ (c) $z=2i,i$ (d) $z=-2i,i$

20. The function $|z|^2$ is
(a) differentiable at the origin (b) analytic (c) constant (d) differentiable everywhere

21. If f(z) is regular function of z then,
(a) $\left(\dfrac{\partial^2}{\partial x^2}+\dfrac{\partial^2}{\partial y^2}\right)|f(z)|^2=|f'(z)|^2$ (b) $\left(\dfrac{\partial^2}{\partial x^2}+\dfrac{\partial^2}{\partial y^2}\right)|f(z)|^2=4|f'(z)|^2$
(c) $\left(\dfrac{\partial}{\partial x}+\dfrac{\partial}{\partial y}\right)|f(z)|^2=4|f'(z)|^2$ (d) $\left(\dfrac{\partial^2}{\partial x^2}+\dfrac{\partial^2}{\partial y^2}\right)|f(z)|^2=4|f'(z)|$

22. The transformation $w=z+c$ where c is a complex constant represents
(a) rotation (b) magnification (c) translation (d) magnification & rotation

23. The mapping $w = \frac{1}{z}$ is

(a) conformal (b) not conformal at $z = 0$ (c) conformal every where
(d) orthogonal

24. The function $u + iv = \frac{x - iy}{x - iy + a}$ $(a \neq 0)$ is not analytic function of z where as $u - iv$ is

(a) need not be analytic (b) analytic at all points (c) analytic except at $z = a$
(d) continuous everywhere

25. If z_1, z_2, z_3, z_4 are four points in the z-plane then the cross-ratio of these point is

(a) $\frac{(z_1 - z_2)(z_4 - z_3)}{(z_1 - z_4)(z_2 - z_3)}$ (b) $\frac{(z_1 - z_2)(z_3 - z_4)}{(z_1 - z_4)(z_3 - z_2)}$
(c) $\frac{(z_1 - z_2)(z_4 - z_3)}{(z_1 - z_4)(z - z_3)}$ (d) $\frac{(z_1 - z_2)(z_3 - z_4)}{(z_4 - z_1)(z_3 - z_2)}$

26. The invariant points of the transformation $w = \frac{1 - iz}{z - i}$

(a) 0 (b) $\pm i$ (c) ± 2 (d) ± 1

ANSWERS:

1	a	6	b	11	b	16	c	21	b	26	d
2	d	7	a	12	c	17	d	22	c		
3	a	8	a	13	d	18	c	23	b		
4	b	9	c	14	b	19	a	24	c		
5	b	10	b	15	b	20	a	25	b		

UNIT – V: COMPLEX INTEGRATION

1. A curve which does not cross itself is called a
   (a) curve (b) closed curve (c) simple closed curve (d) multiple curve

2. The value of $\int_c \frac{zdz}{z-2}$ where c is the circle $|z|=1$ is
   (a) 0 (b) $\frac{\pi}{2}i$ (c) $\frac{\pi}{2}$ (d) 2

3. The value of $\int_c \frac{z}{(z-1)^2}dz$ where c is the circle $|z|=2$ is
   (a) πi (b) $2\pi i$ (c) $4\pi i$ (d) 0

4. The value of $\int_c (z-2)^n dz;\ (n \neq 1)$ where c is the circle $|z-2|=4$ is
   a. 2^n (b) n^2 (c) 0 (c) n

5. The value of $\int_c \frac{1}{2z+1}dz$ where c is the circle $|z|=1$ is
   (a) 0 (b) πi (c) $\frac{\pi}{2}i$ (d) 2

6. The value of $\int_c \frac{1}{3z+1}dz$ where c is the circle $|z|=1$ is
   (a) 0 (b) πi (c) $\frac{2\pi}{3}i$ (d) 2

7. If $f(z)$ is analytic inside and on c, the value of $\int_c \frac{f(z)}{z-a}dz$, where c is the simple closed curve and a is any point within c, is
   (a) $f(a)$ (b) $2\pi i f(a)$ (c) $\pi i f(a)$ (d) 0

8. If $f(z)$ is analytic inside and on c, the value of $\int_c f(z)dz$, where c is the simple closed curve, is
   (a) $f(a)$ (b) $2\pi i f(a)$ (c) $\pi i f(a)$ (d) 0

9. If $f(z)$ is analytic inside and on c, the value of $\int_c \frac{f(z)}{(z-a)^2}dz$, where c is the simple closed curve and a is any point within c, is
   (a) $f'(a)$ (b) $2\pi i f'(a)$ (c) $\pi i f'(a)$ (d) 0

10. If $f(z)$ is analytic inside and on c, the value of $\int_c \frac{f(z)}{(z-a)^3} dz$, where c is the simple closed curve and a is any point within c, is

(a) $f''(a)$ (b) $2\pi i f''(a)$ (c) $\pi i f''(a)$ (d) 0

11. Let $C: |z-a| = r$ be a circle, the f(z) can be expanded as a Taylor's series if

(a) $f(z)$ is a defined function within c
(b) $f(z)$ is a analytic function within c
(c) $f(z)$ is not a analytic function within c
(d) $f(z)$ is a analytic function outside c

12. Let $C_1: |z-a| = R_1$ and $C_2: |z-a| = R_2$ be two concentric circles ($R_2 < R_1$), the $f(z)$ can be expanded as a Laurent's series if

(a) $f(z)$ is analytic within C_2
(b) $f(z)$ is not analytic within C_2
(c) $f(z)$ is analytic in the annular region
(d) $f(z)$ is not analytic in the annular region

13. Let $C_1 : |z-a| = R_1$ and $C_2 : |z-a| = R_2$ be two concentric circles ($R_2 < R_1$), the annular region is defined as

(a) within C_1 (b) within C_2
(c) within C_2 and outside C_1 (d) within C_1 and outside C_2

14. The part $\sum_{n=0}^{\infty} a_n (z-a)^n$ consisting of positive integral powers of $(z-a)$ is called as

(a) The analytic part of the Laurent's series
(b) The principal part of the Laurent's series
(c) The real part of the Laurent's series
(d) The imaginary part of the Laurent's series

15. The part $\sum_{n=1}^{\infty} b_n (z-a)^{-n}$ consisting of negative integral powers of $(z-a)$ is called as

(a) The analytic part of the Laurent's series
(b) The principal part of the Laurent's series
(c) The real part of the Laurent's series
(d) The imaginary part of the Laurent's series

16. The annular region for the function $f(z) = \frac{1}{z(z-1)}$ is

(a) $0 < |z| < 1$ (b) $1 < |z| < 2$ (c) $1 < |z| < 0$ (d) $|z| < 1$

17. The annular region for the function $f(z)=\dfrac{1}{(z-1)(z-2)}$ is

(a) $0<|z|<1$ (b) $1<|z|<2$ (c) $1<|z|<0$ (d) $|z|<1$

18. The annular region for the function $f(z)=\dfrac{1}{z^2-z-6}$ is

(a) $0<|z|<1$ (b) $1<|z|<2$ (c) $2<|z|<3$ (d) $|z|<3$

19. If $f(z)$ is not analytic at $z=z_0$ and there exists a neighborhood of $z=z_0$ containing no other singularity, then

(a) The point $z=z_0$ is isolated singularity of $f(z)$

(b) The point $z=z_0$ is a zero point of $f(z)$

(c) The point $z=z_0$ is nonzero of $f(z)$

(d) The point $z=z_0$ is non isolated singularity of $f(z)$

20. If $f(z)=\dfrac{\sin z}{z}$, then

(a) $z=0$ is a simple pole (b) $z=0$ is a pole of order 2

(c) $z=0$ is a removable singularity (d) $z=0$ is a zero of $f(z)$

21. If $f(z)=\dfrac{\sin z-z}{z^3}$, then

(a) $z=0$ is a simple pole (b) $z=0$ is a pole of order 2

(c) $z=0$ is a removable singularity (d) $z=0$ is a zero of $f(z)$

22. If $\lim_{z\to a}(z-a)^n f(z)\neq 0$ then

(a) $z=a$ is a simple pole (b) $z=a$ is a pole of order n

(c) $z=a$ is a removable singularity (d) $z=a$ is a zero of $f(z)$

23. If $f(z)=\dfrac{1}{(z-4)^2(z-3)^3(z-1)}$, then

(a) 4 is a simple pole, 3 is a pole of order 3 and 1 is a pole of order 2

(b) 3 is a simple pole, 1 is a pole of order 3 and 4 is a pole of order 2

(c) 1 is a simple pole, 3 is a pole of order 3 and 4 is a pole of order 2

(d) 3 is a simple pole, 4 is a pole of order 1 and 4 is a pole of order 2

24. If $f(z)=e^{\frac{1}{z-4}}$ then

(a) $z=4$ is removable singularity (b) $z=4$ is pole of order 2

(c) $z=4$ is an essential singularity (d) $z=4$ is zero of $f(z)$

25. Let $z = a$ is a simple pole for $f(z)$ and $b = \lim_{z \to a}(z-a)f(z)$, then

(a) b is a simple pole
(b) b is a residue at a
(c) b is removable singularity
(d) b is a residue at a of order n

26. The residue of $f(z) = \dfrac{1-e^{2z}}{z^3}$ is

(a) 0 (b) 2 (c) -2 (d) 1

27. The residue of $f(z) = \dfrac{e^{2z}}{(z+1)^2}$ is

(a) e^{-2} (b) -2 e^{-2} (c) -1 (d) 2 e^{-2}

28. The residue of $f(z) = \cot z$ is

(a) π (b) 1 (c) -1 (d) 0

ANSWERS:

1	c	6	c	11	b	16	a	21	c	26	c
2	a	7	b	12	c	17	b	22	b	27	d
3	b	8	d	13	d	18	c	23	c	28	b
4	c	9	b	14	a	19	a	24	c		
5	b	10	b	15	b	20	c	25	b		