# ELECTROSTATIC FIELDS

HISTORIES MAKE MEN WISE; POETS WITTY; THE MATHEMATICS SUBTLE; NATURAL PHILOSOPHY DEEP; MORAL GRAVE; LOGIC AND RHETORIC ABLE TO CONTEND.

—FRANCIS BACON

## 3.1 INTRODUCTION

Having mastered some essential mathematical tools needed for this course, we are now prepared to study the basic concepts of EM. We shall begin with those fundamental concepts that are applicable to static (or time-invariant) electric fields in free space (or vacuum). An electrostatic field is produced by a static charge distribution. A typical example of such a field is found in a cathode-ray tube.

Before we commence our study of electrostatics, it might be helpful to examine briefly the importance of such a study. Electrostatics is a fascinating subject that has grown up in diverse areas of application. Electric power transmission, X-ray machines, and lightning protection are associated with strong electric fields and will require a knowledge of electrostatics to understand and design suitable equipment. The devices used in solid-state electronics are based on electrostatics. These include resistors, capacitors, and active devices such as bipolar and field effect transistors, which are based on control of electron motion by electrostatic fields. Almost all computer peripheral devices, with the exception of magnetic memory, are based on electrostatic fields. Touch pads, capacitance keyboards, cathode-ray tubes, liquid crystal displays, and electrostatic printers are typical examples. In medical work, diagnosis is often carried out with the aid of electrostatics, as incorporated in electrocardiograms, electroencephalograms, and other recordings of the electrical activity of organs including eyes, ears, and stomachs. In industry, electrostatics is applied in a variety of forms such as paint spraying, electrodeposition, electrochemical machining, and separation of fine particles. Electrostatics is used in agriculture to sort seeds, for direct spraying of plants, to measure the moisture content of crops, to spin cotton, and for speed baking bread and smoking meat.[1,2]

We begin our study of electrostatics by investigating the two fundamental laws governing electrostatic fields: (1) Coulomb's law, and (2) Gauss's law. Both of these laws are

[1] For various applications of electrostatics, see J. M. Crowley, *Fundamentals of Applied Electrostatics.* New York: John Wiley & Sons, 1999; A. D. Moore, ed., *Electrostatics and Its Applications.* New York: John Wiley & Sons, 1973; and C. E. Jowett, *Electrostatics in the Electronics Environment.* New York: John Wiley & Sons, 1976.

[2] An interesting story on the magic of electrostatics is found in B. Bolton, *Electromagnetism and Its Applications: An Introduction.* London: Van Nostrand, 1980, p. 2.

based on experimental studies, and they are interdependent. Although Coulomb's law is applicable in finding the electric field due to any charge configuration, it is easier to use Gauss's law when charge distribution is symmetrical. Based on Coulomb's law, the concept of electric field intensity will be introduced and applied to cases involving point, line, surface, and volume charges. Special problems that can be solved with much effort using Coulomb's law will be solved with ease by applying Gauss's law. Throughout our discussion in this chapter, we will assume that the electric field is in a vacuum or free space. Electric field in material space will be covered in the next chapter.

## 3.2 COULOMB'S LAW AND FIELD INTENSITY

Coulomb's law is an experimental law formulated in 1785 by Charles Augustin de Coulomb, then a colonel in the French army. It deals with the force a point charge exerts on another point charge. By a *point charge* we mean a charge that is located on a body whose dimensions are much smaller than other relevant dimensions. For example, a collection of electric charges on a pinhead may be regarded as a point charge. The polarity of charges may be positive or negative, like charges repel, while unlike charges attract. Charges are generally measured in coulombs (C). One coulomb is approximately equivalent to $6 \times 10^{18}$ electrons; it is a very large unit of charge because one electron charge $e = -1.6019 \times 10^{-19}$ C.

**Coulomb's law** states that the force $F$ between two point charges $Q_1$ and $Q_2$ is:

1. Along the line joining them
2. Directly proportional to the product $Q_1Q_2$ of the charges
3. Inversely proportional to the square of the distance $R$ between them.[3]

Expressed mathematically,

$$F = \frac{k\,Q_1Q_2}{R^2} \tag{3.1}$$

where $k$ is the proportionality constant whose value depends on the choice of system of units. In SI units, charges $Q_1$ and $Q_2$ are in coulombs (C), the distance $R$ is in meters (m), and the force $F$ is in newtons (N) so that $k = 1/4\pi\varepsilon_o$. The constant $\varepsilon_o$ is known as the *permittivity of free space* (in farads per meter) and has the value

$$\boxed{\begin{aligned} \varepsilon_o &= 8.854 \times 10^{-12} \simeq \frac{10^{-9}}{36\pi} \text{ F/m} \\ \text{or} \quad k &= \frac{1}{4\pi\varepsilon_o} \simeq 9 \times 10^9 \text{ m/F} \end{aligned}} \tag{3.2}$$

[3] Further details of experimental verification of Coulomb's law can be found in W. F. Magie, *A Source Book in Physics*. Cambridge: Harvard Univ. Press, 1963, pp. 408–420.

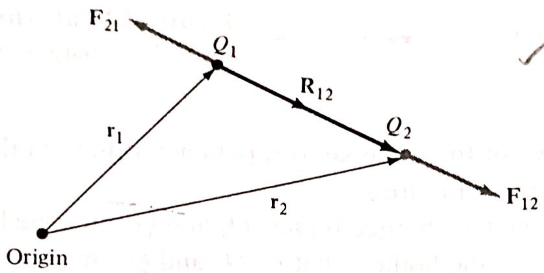



**Figure 3.1** Coulomb vector force on point changes $Q_1$ and $Q_2$.

Thus eq. (3.1) becomes

$$F = \frac{Q_1 Q_2}{4\pi\varepsilon_o R^2} \tag{3.3}$$

If point charges $Q_1$ and $Q_2$ are located at points having position vectors $\mathbf{r}_1$ and $\mathbf{r}_2$, then the force $\mathbf{F}_{12}$ on $Q_2$ due to $Q_1$, shown in Figure 3.1, is given by

$$\boxed{\mathbf{F}_{12} = \frac{Q_1 Q_2}{4\pi\varepsilon_o R^2}\mathbf{a}_{R_{12}}} \tag{3.4}$$

where

$$\mathbf{R}_{12} = \mathbf{r}_2 - \mathbf{r}_1 \tag{3.5a}$$

$$R = |\mathbf{R}_{12}| \tag{3.5b}$$

$$\mathbf{a}_{R_{12}} = \frac{\mathbf{R}_{12}}{R} \tag{3.5c}$$

By substituting eq. (3.5) into eq. (3.4), we may write eq. (3.4) as

$$\mathbf{F}_{12} = \frac{Q_1 Q_2}{4\pi\varepsilon_o R^3}\mathbf{R}_{12} \tag{3.6a}$$

or

$$\mathbf{F}_{12} = \frac{Q_1 Q_2 (\mathbf{r}_2 - \mathbf{r}_1)}{4\pi\varepsilon_o|\mathbf{r}_2 - \mathbf{r}_1|^3} \tag{3.6b}$$

It is worthwhile to note that

1. As shown in Figure 3.1, the force $\mathbf{F}_{21}$ on $Q_1$ due to $Q_2$ is given by

$$\mathbf{F}_{21} = |\mathbf{F}_{12}|\mathbf{a}_{R_{21}} = |\mathbf{F}_{12}|(-\mathbf{a}_{R_{12}})$$

or

$$\mathbf{F}_{21} = -\mathbf{F}_{12} \tag{3.7}$$

since

$$\mathbf{a}_{R_{21}} = -\mathbf{a}_{R_{12}}$$

**Figure 3.2** (a), (b) Like charges repel. (c) Unlike charges attract.

2. Like charges (charges of the same sign) repel each other, while unlike charges attract. This is illustrated in Figure 3.2.
3. The distance $R$ between the charged bodies $Q_1$ and $Q_2$ must be large compared with the linear dimensions of the bodies; that is, $Q_1$ and $Q_2$ must be point charges.
4. $Q_1$ and $Q_2$ must be static (at rest).
5. The signs of $Q_1$ and $Q_2$ must be taken into account in eq. (3.4) for like charges, $Q_1 Q_2 > 0$. For unlike charges, $Q_1 Q_2 < 0$.

If we have more than two point charges, we can use the *principle of superposition* to determine the force on a particular charge. The principle states that if there are $N$ charges $Q_1, Q_2, \ldots, Q_N$ located, respectively, at points with position vectors $\mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_N$, the resultant force $\mathbf{F}$ on a charge $Q$ located at point $\mathbf{r}$ is the vector sum of the forces exerted on $Q$ by each of the charges $Q_1, Q_2, \ldots, Q_N$. Hence:

$$\mathbf{F} = \frac{QQ_1(\mathbf{r} - \mathbf{r}_1)}{4\pi\varepsilon_o|\mathbf{r} - \mathbf{r}_1|^3} + \frac{QQ_2(\mathbf{r} - \mathbf{r}_2)}{4\pi\varepsilon_o|\mathbf{r} - \mathbf{r}_2|^3} + \cdots + \frac{QQ_N(\mathbf{r} - \mathbf{r}_n)}{4\pi\varepsilon_o|\mathbf{r} - \mathbf{r}_N|^3}$$

or

$$\boxed{\mathbf{F} = \frac{Q}{4\pi\varepsilon_o} \sum_{k=1}^{N} \frac{Q_k(\mathbf{r} - \mathbf{r}_k)}{|\mathbf{r} - \mathbf{r}_k|^3}} \tag{3.8}$$

We can now introduce the concept of *electric field intensity.*

> The **electric field intensity** (or **electric field strength**) **E** is the force per unit charge when placed in an electric field.

Thus

$$\mathbf{E} = \lim_{Q \to 0} \frac{\mathbf{F}}{Q} \tag{3.9}$$

or simply

$$\boxed{\mathbf{E} = \frac{\mathbf{F}}{Q}} \tag{3.10}$$

For $Q > 0$, the electric field intensity $\mathbf{E}$ is obviously in the direction of the force $\mathbf{F}$ and is measured in newtons per coulomb or volts per meter. The electric field intensity at point $\mathbf{r}$ due to a point charge located at $\mathbf{r}'$ is readily obtained from eqs. (3.6) and (3.10) as

$$\boxed{\mathbf{E} = \frac{Q}{4\pi\varepsilon_o R^2}\mathbf{a}_R = \frac{Q(\mathbf{r} - \mathbf{r}')}{4\pi\varepsilon_o|\mathbf{r} - \mathbf{r}'|^3}} \tag{3.11}$$

For $N$ point charges $Q_1, Q_2, \ldots, Q_N$ located at $\mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_N$, the electric field intensity at point $\mathbf{r}$ is obtained from eqs. (3.8) and (3.10) as

$$\mathbf{E} = \frac{Q_1(\mathbf{r} - \mathbf{r}_1)}{4\pi\varepsilon_o|\mathbf{r} - \mathbf{r}_1|^3} + \frac{Q_2(\mathbf{r} - \mathbf{r}_2)}{4\pi\varepsilon_o|\mathbf{r} - \mathbf{r}_2|^3} + \cdots + \frac{Q_N(\mathbf{r} - \mathbf{r}_N)}{4\pi\varepsilon_o|\mathbf{r} - \mathbf{r}_N|^3}$$

or

$$\mathbf{E} = \frac{1}{4\pi\varepsilon_o}\sum_{k=1}^{N}\frac{Q_k(\mathbf{r} - \mathbf{r}_k)}{|\mathbf{r} - \mathbf{r}_k|^3} \tag{3.12}$$

**EXAMPLE 3.1**

Point charges 1 mC and −2 mC are located at (3, 2, −1) and (−1, −1, 4), respectively. Calculate the electric force on a 10 nC charge located at (0, 3, 1) and the electric field intensity at that point.

**Solution:**

$$\mathbf{F} = \sum_{k=1,2}\frac{QQ_k}{4\pi\varepsilon_o R^2}\mathbf{a}_R = \sum_{k=1,2}\frac{QQ_k(\mathbf{r} - \mathbf{r}_k)}{4\pi\varepsilon_o|\mathbf{r} - \mathbf{r}_k|^3}$$

$$= \frac{Q}{4\pi\varepsilon_o}\left\{\frac{10^{-3}[(0, 3, 1) - (3, 2, -1)]}{|(0, 3, 1) - (3, 2, -1)|^3} - \frac{2 \cdot 10^{-3}[(0, 3, 1) - (-1, -1, 4)]}{|(0, 3, 1) - (-1, -1, 4)|^3}\right\}$$

$$= \frac{10^{-3} \cdot 10 \cdot 10^{-9}}{4\pi \cdot \frac{10^{-9}}{36\pi}}\left[\frac{(-3, 1, 2)}{(9 + 1 + 4)^{3/2}} - \frac{2(1, 4, -3)}{(1 + 16 + 9)^{3/2}}\right]$$

$$= 9 \cdot 10^{-2}\left[\frac{(-3, 1, 2)}{14\sqrt{14}} + \frac{(-2, -8, 6)}{26\sqrt{26}}\right]$$

$$\mathbf{F} = -6.507\mathbf{a}_x - 3.817\mathbf{a}_y + 7.506\mathbf{a}_z \text{ mN}$$

At that point,

$$\mathbf{E} = \frac{\mathbf{F}}{Q}$$

$$= (-6.507, -3.817, 7.506) \cdot \frac{10^{-3}}{10 \cdot 10^{-9}}$$

$$\mathbf{E} = -650.7\mathbf{a}_x - 381.7\mathbf{a}_y + 750.6\mathbf{a}_z \text{ kV/m}$$

**PRACTICE EXERCISE 3.1**

Point charges 5 nC and −2 nC are located at (2, 0, 4) and (−3, 0, 5), respectively.

(a) Determine the force on a 1 nC point charge located at (1, −3, 7).

(b) Find the electric field $\mathbf{E}$ at (1, −3, 7).

**Answer:** (a) $-1.004\mathbf{a}_x - 1.284\mathbf{a}_y + 1.4\mathbf{a}_z$ nN.

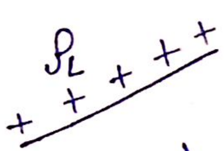
(b) $-1.004\mathbf{a}_x - 1.284\mathbf{a}_y + 1.4\mathbf{a}_z$ V/m.

## * ELECTRIC FIELDS DUE TO CONTINUOUS CHARGE DISTRIBUTION:-

The line charge density, surface charge density, and volume charge density are denoted by $\rho_L$ (in C/m), $\rho_S$ (in C/m²), and $\rho_V$ (in C/m³), respectively.

The charge element $dQ$ and the total charge $Q$ due to these charge distributions are obtained as

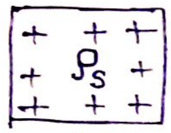
$$dQ = \rho_L \, dl \longrightarrow Q = \int_L \rho_L \, dl \quad \text{(Line charge)} \rightarrow ①$$



Line charge.

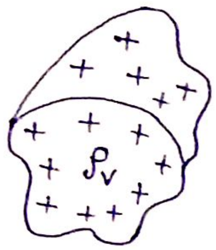
$$dQ = \rho_S \, ds \longrightarrow Q = \int_S \rho_S \, ds \quad \text{(Surface charge)} \rightarrow ②$$

$$dQ = \rho_V \, dv \longrightarrow Q = \int_V \rho_V \, dv \quad \text{(volume charge)} \rightarrow ③$$



Surface charge.

The electric field intensity due to each of the charge distributions $\rho_L$, $\rho_S$, and $\rho_V$ may be regarded as the summation of the field contributed by the numerous point charges making up the charge distribution.



Volume charge.

We know $$\vec{E} = \frac{Q}{4\pi\epsilon_0 R^2} \vec{a}_R \longrightarrow ④$$

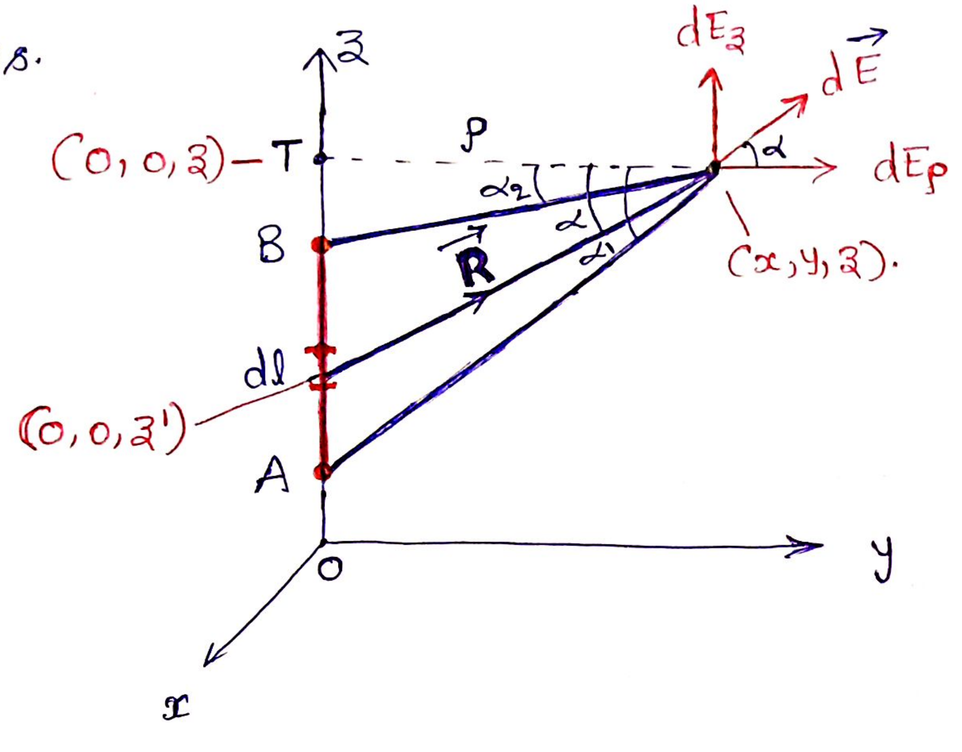
Replacing '$Q$' in eq. ④ with charge element $dQ = \rho_L \, dl$, $\rho_S \, ds$, or $\rho_V \, dv$ and integrating, we get

$$\vec{E} = \int_L \frac{\rho_L \, dl}{4\pi\epsilon_0 R^2} \vec{a}_R \rightarrow (5) \text{ (line charge)}$$

$$\vec{E} = \int_S \frac{\rho_S \, ds}{4\pi\epsilon_0 R^2} \vec{a}_R \rightarrow (6) \text{ (surface charge)}.$$

$$\vec{E} = \int_V \frac{\rho_v \, dv}{4\pi\epsilon_0 R^2} \vec{a}_R \rightarrow (7) \text{ (volume charge)}.$$

**A LINE CHARGE :-** Consider a line charge with uniform charge density $\rho_L$ extending from A to B along the $z$- axis.




Note:- It is customary to denote the field point by $(x, y, z)$ and the source point by $(x', y', z')$.

The charge element $dQ$ associated with element $dl = dz'$ of the line is

$$dQ = \rho_L \, dl = \rho_L \, dz'.$$

$$\Rightarrow \quad Q = \int_{z_A}^{z_B} \rho_L \, dz'.$$

The electric field intensity E at an arbitrary point P (x, y, z) can be found by using eq. (5)

$$\vec{E} = \int_L \frac{\rho_L \, dl}{4\pi\epsilon_0 R^2} \vec{a}_R = \int_L \frac{\rho_L \, dz'}{4\pi\epsilon_0 R^2} \vec{a}_R \longrightarrow (8)$$

where $\vec{a}_R = \dfrac{\vec{\mathbf{R}}}{|\vec{\mathbf{R}}|} = \dfrac{\vec{\mathbf{R}}}{R}$.

$$\vec{\mathbf{R}} = (x, y, z) - (0, 0, z') = x\vec{a}_x + y\vec{a}_y + (z - z')\vec{a}_z$$

$$= \rho\vec{a}_\rho + (z - z')\vec{a}_z \rightarrow \text{CYLINDRICAL}$$

$$R = |\vec{\mathbf{R}}| = \sqrt{x^2 + y^2 + (z - z')^2}. = \sqrt{\rho^2 + (z - z')^2}.$$

$$\frac{\vec{a}_R}{R^2} = \frac{\vec{\mathbf{R}}}{R^2 \cdot |\vec{\mathbf{R}}|} = \frac{\vec{\mathbf{R}}}{R^3} = \frac{\rho\vec{a}_\rho + (z - z')\vec{a}_z}{[\rho^2 + (z - z')^2]^{3/2}}.$$

Substituting this in eq. (8) we get

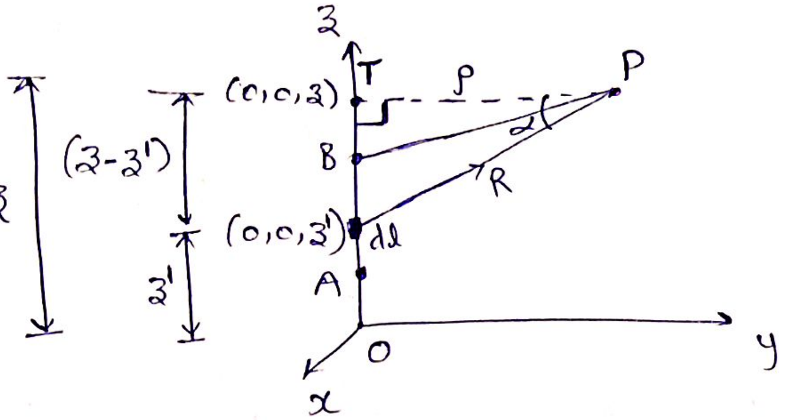
$$\vec{E} = \frac{\rho_L}{4\pi\epsilon_0} \int \frac{\rho\vec{a}_\rho + (z - z')\vec{a}_z}{[\rho^2 + (z - z')^2]^{3/2}} dz' \longrightarrow (9)$$

To evaluate this, it is convenient that we define $\alpha$, $\alpha_1$ and $\alpha_2$ in Fig.

$$R^2 = \rho^2 + (z - z')^2$$

and $\cos\alpha = \dfrac{\rho}{R}$

$$\Rightarrow \quad R = \frac{\rho}{\cos\alpha} = \rho \sec\alpha$$




$$\therefore \quad R = \left[\rho^2 + (z - z')^2\right]^{1/2} = \rho \sec\alpha \longrightarrow (10)$$

$$\tan\alpha = \frac{(z - z')}{\rho} \quad \Rightarrow \quad z - z' = \rho \tan\alpha \longrightarrow (11)$$

$$\Rightarrow \quad z' = z - \rho \tan\alpha$$

$$\Rightarrow \quad z' = OT - \rho \tan\alpha$$

$$\therefore \quad dz' = -\rho \sec^2\alpha \, d\alpha \longrightarrow (12)$$

From eq. (9) $\quad \vec{E} = \dfrac{\rho_L}{4\pi\varepsilon_0} \displaystyle\int \dfrac{\rho \vec{a}_\rho + (z - z')\vec{a}_z}{\left[\left(\rho^2 + (z - z')^2\right)^{1/2}\right]^3} dz'$

substituting (10), (11) and (12) ~~we get~~ in above equation we get

$$\Rightarrow \quad \vec{E} = \frac{\rho_L}{4\pi\varepsilon_0} \int_{\alpha_1}^{\alpha_2} \frac{\left[\rho \vec{a}_\rho + \rho \tan\alpha \, \vec{a}_z\right]}{\left[\rho \sec\alpha\right]^3} \left[-\rho \sec^2\alpha \, d\alpha\right]$$

$$\Rightarrow \quad \vec{E} = \frac{-\rho_L}{4\pi\varepsilon_0} \int_{\alpha_1}^{\alpha_2} \frac{\rho^2 \sec^2\alpha \left[\cos\alpha \, \vec{a}_\rho + \sin\alpha \, \vec{a}_z\right] d\alpha}{\rho^3 \sec^3\alpha \cos\alpha}$$

$$\Rightarrow \quad \vec{E} = \frac{-\rho_L}{4\pi\varepsilon_0 \rho} \int_{\alpha_1}^{\alpha_2} \left[\cos\alpha \, \vec{a}_\rho + \sin\alpha \, \vec{a}_z\right] d\alpha$$

From figure $\uparrow a_z$

Thus for a finite line charge

$$\Rightarrow \vec{E} = \frac{-\rho_L}{4\pi\varepsilon_0\rho}\left[\sin\alpha\,\vec{a_\rho} - \cos\alpha\,\vec{a_z}\right]_{\alpha_1}^{\alpha_2}$$

$$\Rightarrow \vec{E} = \frac{-\rho_L}{4\pi\varepsilon_0\rho}\left[(\sin\alpha_2 - \sin\alpha_1)\vec{a_\rho} - (\cos\alpha_2 - \cos\alpha_1)\vec{a_z}\right]$$

$$\Rightarrow \vec{E} = \frac{\rho_L}{4\pi\varepsilon_0\rho}\left[-(\sin\alpha_2 - \sin\alpha_1)\vec{a_\rho} + (\cos\alpha_2 - \cos\alpha_1)\vec{a_z}\right]$$

$$\therefore \vec{E} = \frac{\rho_L}{4\pi\varepsilon_0\rho}\left[-(\sin\alpha_2 - \sin\alpha_1)\vec{a_\rho} + (\cos\alpha_2 - \cos\alpha_1)\vec{a_z}\right] \rightarrow (13)$$

$$\vec{E_\rho} = \frac{-\rho_L}{4\pi\varepsilon_0\rho}(\sin\alpha_2 - \sin\alpha_1) \longrightarrow (14)$$

and
$$\vec{E_z} = \frac{\rho_L}{4\pi\varepsilon_0\rho}(\cos\alpha_2 - \cos\alpha_1) \longrightarrow (15)$$

Note:- As a special case, for an infinite line charge, point B is at $(0, 0, \infty)$ and A at $(0, 0, -\infty)$ so that $\alpha_1 = \frac{\pi}{2}$, $\alpha_2 = \frac{-\pi}{2}$; the z- component vanishes and eq. (13) becomes

$$\boxed{\vec{E} = \frac{\rho_L}{2\pi\varepsilon_0\rho}\vec{a_\rho}}$$

(INFINITE LINE CHARGE along z-axis).

**Note:-** If the line is not along the $z$-axis, '$\rho$' is the perpendicular distance from the line to the point of interest, and $\vec{a_\rho}$ is a unit vector along that distance directed from the line charge to the field point.

## A SURFACE CHARGE:—

Consider an infinite sheet of charge in the $xy$-plane with uniform charge density $\rho_s$. The charge associated with an ~~external~~ elemental area $ds$ is

$$dQ = \rho_s \, ds \longrightarrow (16)$$

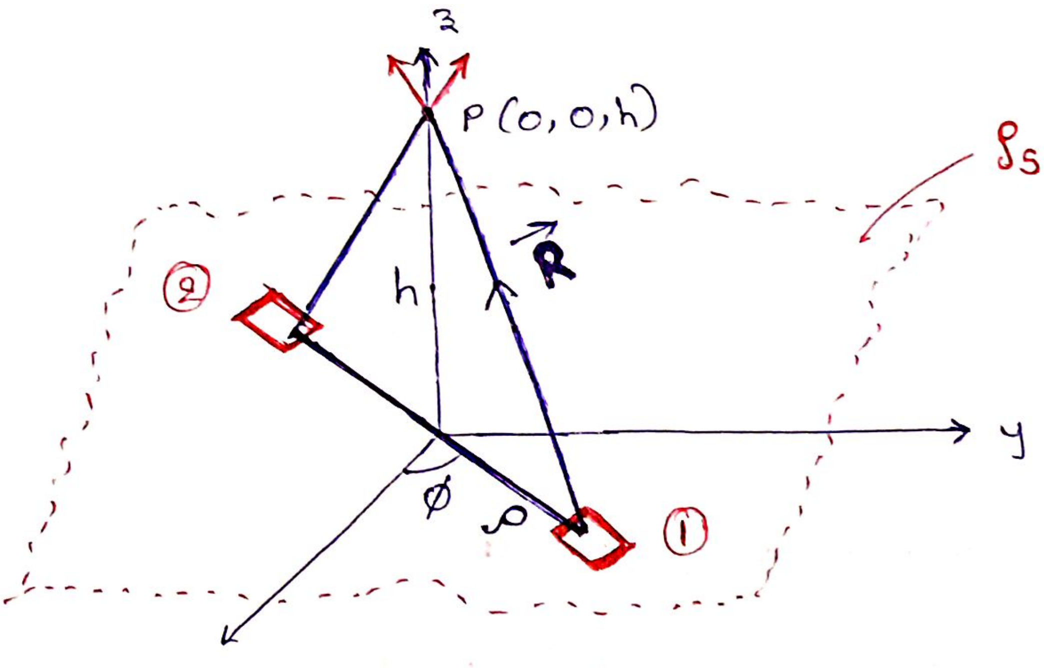



Fig:- Evaluation of the E-field due to infinite sheet of charge.

From eq. (6), the contribution to the E field at point P (0, 0, h) by the charge $dQ$ on the elemental surface (1) shown in Fig. is

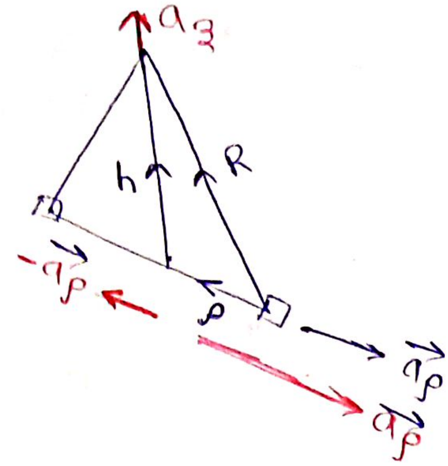
$$d\vec{E} = \frac{dQ}{4\pi\epsilon_0 R^2}\vec{a}_R \longrightarrow (17)$$

From figure

$$\vec{R} = \rho(-\vec{a}_\rho) + h\vec{a}_z \longrightarrow (18)$$

$$R = |\vec{R}| = \sqrt{\rho^2 + h^2} \longrightarrow (19)$$

$$\vec{a}_R = \frac{\vec{R}}{R} \longrightarrow (20)$$




$$dQ = \rho_s\, dS = \rho_s\, d\rho\, \rho\, d\phi = \rho_s\, \rho\, d\phi\, d\rho \longrightarrow (21)$$

Substituting (18) - (21) in (17) gives

$$d\vec{E} = \frac{\rho_s \rho\, d\phi\, d\rho\, [-\rho\vec{a}_\rho + h\vec{a}_z]}{4\pi\varepsilon_0 [\rho^2 + h^2]^{3/2}} \longrightarrow (22)$$

$$\Rightarrow d\vec{E} = \frac{-\rho_s \rho^2\, d\phi\, d\rho}{4\pi\varepsilon_0 [\rho^2 + h^2]^{3/2}}\vec{a}_\rho + \frac{\rho_s \rho h\, d\phi\, d\rho}{4\pi\varepsilon_0 [\rho^2 + h^2]^{3/2}}\vec{a}_z.$$

Owing to the symmetry of the charge distribution, for every element 1, there is a corresponding element 2 whose contribution along $\vec{a}_\rho$ cancels that of of element 1. Thus the contributions to $E_\rho$ add up to zero so that E has only z-component.

$$\therefore \vec{E} = \int_S dE_z = \int_S \frac{\rho_s \rho h\, d\phi\, d\rho}{4\pi\varepsilon_0 [\rho^2 + h^2]^{3/2}}\vec{a}_z$$

$$\Rightarrow \vec{E} = \frac{\rho_s}{4\pi\varepsilon_0} \int_{\phi=0}^{2\pi} \int_{\rho=0}^{\infty} \frac{h\,\rho\, d\rho\, d\phi}{[\rho^2+h^2]^{3/2}}\, \vec{a}_z$$

$$\Rightarrow E = \frac{\rho_s h}{4\pi\varepsilon_0} \times 2\pi \int_0^{\infty} \frac{\rho\, d\rho}{[\rho^2+h^2]^{3/2}}\, \vec{a}_z$$

Note:- $d(\rho^2) = 2\rho\, d\rho$

$\Rightarrow \rho\, d\rho = \frac{1}{2} d(\rho^2).$

$$\Rightarrow \vec{E} = \frac{\rho_s h}{2\varepsilon_0} \int_0^{\infty} [\rho^2+h^2]^{-3/2} \frac{1}{2}\, d(\rho^2)\, \vec{a}_z$$

$$\Rightarrow \vec{E} = \frac{\rho_s h}{4\varepsilon_0} \left[ \frac{(\rho^2+h^2)^{-\frac{3}{2}+1}}{-\frac{3}{2}+1} \right]_0^{\infty} \vec{a}_z$$

$$\Rightarrow \vec{E} = \frac{-\rho_s h}{2\varepsilon_0} \left[ \frac{1}{\sqrt{\rho^2+h^2}} \right]_0^{\infty} \vec{a}_z$$

$$\Rightarrow \vec{E} = \frac{-\rho_s h}{2\varepsilon_0} \left[ 0 - \frac{1}{h} \right] \vec{a}_z$$

$$\Rightarrow \vec{E} = \frac{\rho_s}{2\varepsilon_0} \vec{a}_z. \longrightarrow (23)$$

That is, $\vec{E}$ has only $z$- component if the charge is in the $xy$- plane. Eq. (23) is valid for $h>0$; for $h<0$, we would need to replace $\vec{a}_z$ with $-\vec{a}_z$.

In general, for an infinite sheet of charge

$$\boxed{\vec{E} = \frac{\rho_s}{2\varepsilon_0} \vec{a}_n} \longrightarrow (24)$$

where '$\vec{a}_n$' is a unit vector normal to the sheet.

From eq. (23) and (24), we notice that the electric field is normal to the sheet and it is surprisingly independent of the distance between the sheet and the point of observation P.

<u>Note</u>:- In a parallel plate capacitor, the electric field existing between the two plates having equal and opposite charges is given by

$$\vec{E} = \frac{\rho_s}{2\varepsilon_0} \vec{a}_n + \frac{-\rho_s}{2\varepsilon_0}(-\vec{a}_n) = \frac{\rho_s}{\varepsilon_0} \vec{a}_n.$$

<u>A VOLUME CHARGE</u>:- Consider a sphere of radius 'a' centered at the origin. Let the volume of the sphere be filled uniformly with a volume charge density $\rho_v$ (in $C/m^3$).

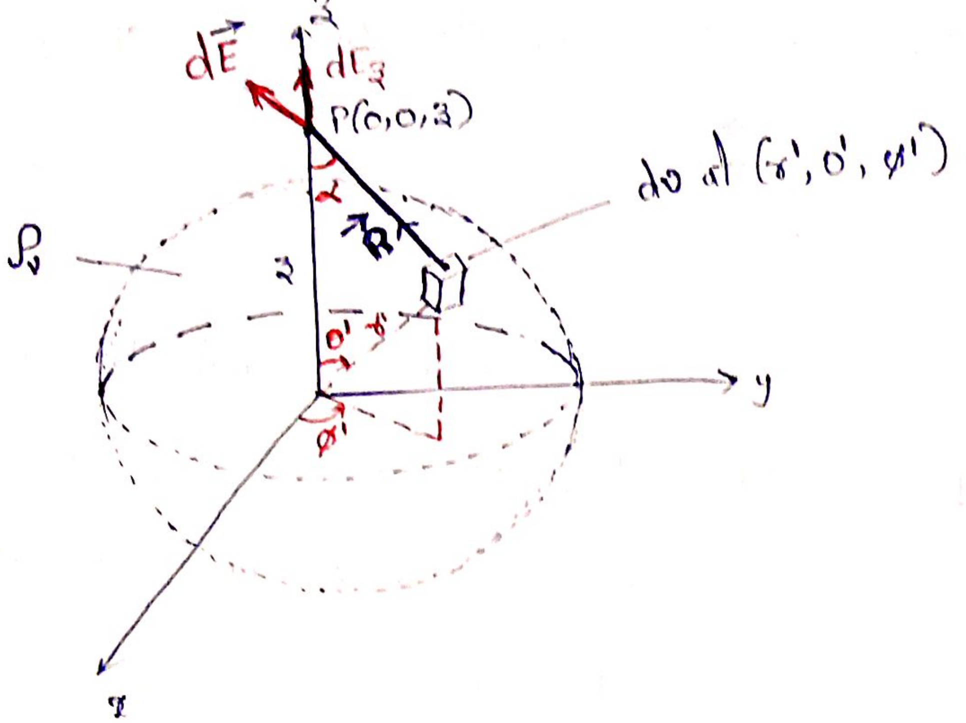



Fig:- Evaluation of the E field due to a volume charge distribution.

The charge dQ associated with the elemental volume dv is

$$dQ = \rho_v \, dv$$

and hence the total charge in a sphere of radius 'a' is

$$Q = \int_v \rho_v \, dv = \rho_v \int_v dv$$

$$\boxed{\therefore \; Q = \rho_v \frac{4\pi a^3}{3}} \longrightarrow (1)$$

The electric field $d\vec{E}$ outside the sphere at P(0,0,z) due to the elementary volume charge is

$$\boxed{d\vec{E} = \frac{\rho_v \, dv}{4\pi\varepsilon_0 R^2} \vec{a}_R} \longrightarrow (2)$$

where $\vec{a}_R = \cos\alpha\, \vec{a}_z + \sin\alpha\, \vec{a}_\rho$.

Owing to the symmetry of the charge distribution, the contributions to $E_x$ & $E_y$ add up to zero. We are left only with $E_z$, given by

$$d\vec{E} = \frac{\rho_v\, dv}{4\pi\epsilon_0 R^2}\left(\cos\alpha\, \vec{a}_z + \underbrace{\sin\alpha\, \vec{a}_\rho}_{0}\right)$$

$$\Rightarrow d\vec{E}_z = \frac{\rho_v\, dv}{4\pi\epsilon_0 R^2}\cos\alpha\, \vec{a}_z$$

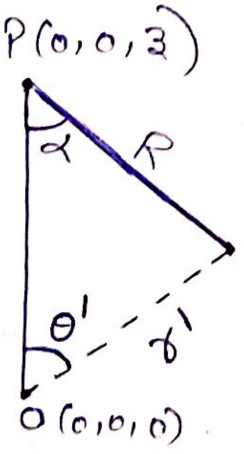
$$\Rightarrow dE_z = d\vec{E}\cdot a_z = \frac{\rho_v\, dv}{4\pi\epsilon_0 R^2}\cos\alpha$$

$$\Rightarrow E_z = \int \frac{\rho_v\, dv}{4\pi\epsilon_0 R^2}\cos\alpha = \frac{\rho_v}{4\pi\epsilon_0}\int \frac{dv\cos\alpha}{R^2}. \longrightarrow (3)$$

Again, we need to derive expressions for $dv$, $R^2$ and $\cos\alpha$.

$$dv = r'^2 \sin\theta'\, dr'\, d\theta'\, d\phi' \longrightarrow (4)$$

Applying the cosine rule to Fig, we have




$$R^2 = z^2 + r'^2 - 2zr'\cos\theta' \longrightarrow (5)$$

$$r'^2 = z^2 + R^2 - 2zR\cos\alpha \longrightarrow (6)$$

It is convenient to evaluate the integral in (3) in terms of $R$ and $r'$. Hence we express $\cos\theta'$, $\cos\alpha$, and $\sin\theta'\, d\theta'$

interms of $R$ and $r'$, that is,

$$\cos\alpha = \frac{z^2+R^2-r'^2}{2zR} \longrightarrow (7)$$

$$\cos\theta' = \frac{z^2+r'^2-R^2}{2zr'} \longrightarrow (8)$$

Differentiating eq (8) with respect to $\theta'$ and keeping $z$ and $r'$ fixed, we obtain

$$\sin\theta' d\theta' = \frac{R\,dR}{zr'} \longrightarrow (9)$$

As $\theta'$ varies from $0$ to $\pi$, $R$ varies from $(z-r')$ to $(z+r')$ if $P$ is outside the sphere.

Substituting eq. (4) - (9) in (3) yields

$$E_z = \frac{\rho_v}{4\pi\epsilon_0}\int \frac{dv\cos\alpha}{R^2}$$

$$\Rightarrow E_z = \frac{\rho_v}{4\pi\epsilon_0}\int \frac{r'^2\sin\theta'\,dr'\,d\theta'\,d\phi'\cos\alpha}{R^2}$$

$$\Rightarrow E_z = \frac{\rho_v}{4\pi\epsilon_0}\int_{\phi'=0}^{2\pi}\int_{r'=0}^{a}\int_{R=z-r'}^{z+r'} r'^2\frac{R\,dR}{zr'}\,dr'\,\frac{z^2+R^2-r'^2}{2zR}\,\frac{1}{R^2}$$

$$\Rightarrow E_z = \frac{\rho_v}{4\pi\epsilon_0}\int_{\phi'=0}^{2\pi}d\phi'\int_{r'=0}^{a}\int_{R=z-r'}^{z+r'} r'\left(\frac{z^2+R^2-r'^2}{2z^2R^2}\right)dR\,dr'$$

$$\Rightarrow E_z = \frac{\rho_v}{4\pi\varepsilon_0 z^2} \times \frac{2\pi}{2} \int_{r'=0}^{a} \int_{R=z-r'}^{z+r'} r' \left[1 + \frac{z^2 - r'^2}{R^2}\right] dR\, dr'.$$

$$\Rightarrow E_z = \frac{\rho_v \pi}{4\pi\varepsilon_0 z^2} \int_{r'=0}^{a} r' \left[R - \frac{(z^2 - r'^2)}{R}\right]_{z-r'}^{z+r'} dr'$$

$$\Rightarrow E_z = \frac{\rho_v \pi}{4\pi\varepsilon_0 z^2} \int_0^a r' \left[z + r' - \frac{(z^2 - r'^2)}{z + r'} - z + r' + \frac{(z^2 - r'^2)}{z - r'}\right] dr'$$

$$\Rightarrow E_z = \frac{\rho_v \pi}{4\pi\varepsilon_0 z^2} \int_0^a r' \left[2r' + (z^2 - r'^2)\left(\frac{1}{z - r'} - \frac{1}{z + r'}\right)\right] dr'$$

$$\Rightarrow E_z = \frac{\rho_v \pi}{4\pi\varepsilon_0 z^2} \int_0^a r' \left[2r' + (z^2 - r'^2)\left(\frac{z + r' - z + r'}{z^2 - r'^2}\right)\right] dr'$$

$$\Rightarrow E_z = \frac{\rho_v \pi}{4\pi\varepsilon_0 z^2} \int_0^a 4 r'^2\, dr'$$

$$\Rightarrow E_z = \frac{\rho_v \pi}{4\pi\varepsilon_0 z^2} \times 4 \frac{r'^3}{3} \Big|_0^a$$

$$\Rightarrow E_z = \frac{1}{4\pi\varepsilon_0} \cdot \frac{1}{z^2} \left(\rho_v \frac{4\pi a^3}{3}\right).$$

$$\Rightarrow E_z = \frac{Q}{4\pi\varepsilon_0 z^2}$$

या $\boxed{\vec{E} = E_z \vec{a}_z = \frac{Q}{4\pi\varepsilon_0 z^2} \vec{a}_z}$ → (10)

This result is obtained for $\vec{E}$ at $P(0,0,z)$. Owing to the symmetry of the charge distribution, the electric field at $P(r, \theta, \phi)$ is readily obtained from eq. (10) as

$$\boxed{\vec{E} = \frac{Q}{4\pi\epsilon_0 r^2} \vec{a}_r}$$

which is identical to the electric field at the same point due to a point charge Q located at the origin or center of the spherical charge distribution.

*. ELECTRIC FLUX DENSITY :- [ELECTRIC DISPLACEMENT]

- The flux due to the electric field can be calculated by using the general definition of flux.
- For practical reasons, however, this quantity is not usually considered to be the most useful flux in electrostatics.
- The electric field intensity is dependent on the medium in which the charge is placed.

Suppose a new vector field $\vec{D}$ is defined by

$$\boxed{\vec{D} = \epsilon_0 \vec{E}} \longrightarrow (1)$$

The electric flux $\psi$ can be defined interms of $\vec{D}$, as

$$\boxed{\psi = \int_S \vec{D} \cdot d\vec{S}} \longrightarrow (2)$$

In SI units, one line of electric flux emanates from +1C and terminates on -1C. Therefore electric flux is measured in Coulombs. Hence the vector field '$\vec{D}$' is called the electric flux density and is measured in Coulombs per square meter.

<u>Definition</u>:- Electric flux density is electric flux passing through a unit area perpendicular to the direction of the flux.

(OR).

It is a measure of the strength of an electric field generated by a free electric charge, corresponding to the number of electric lines of force passing through a given area.

For an infinite sheet of charge $\vec{D} = \frac{\rho_s}{2} \vec{a_n}$

and for volume charge distribution $\vec{D} = \int_v \frac{\rho_v dv}{4\pi R^2} \vec{a_R}$.

'$\vec{D}$' is a function of charge and position only; it is independent of the medium.