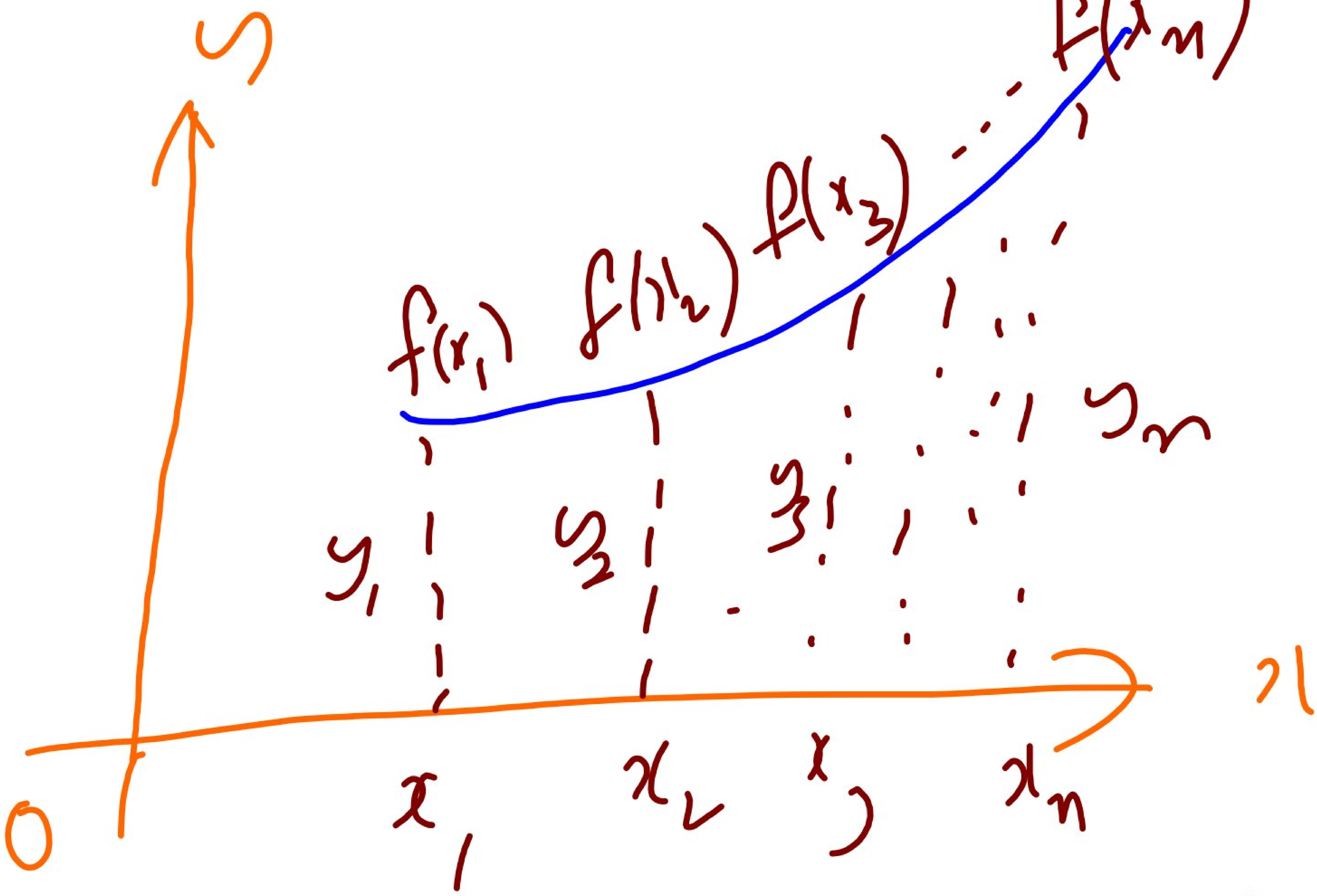
Multiple Integrals

Double Integration

Let $\int_a^b f(x)\,dx$ be a define integral, then the limit of the sum is

$$\int_a^b f(x)\,dx = \lim_{\substack{n \to \infty \\ \delta x \to 0}} \left[f(x_1)\,\delta x_1 + f(x_2)\,\delta x_2 + \cdots + f(x_n)\,\delta x_n \right]$$



Let us consider a function $f(x, y)$ of two variables x and y, defined in the finite region of xy-plane.

Divide the region 'A' into n elementary areas $\delta A_1, \delta A_2 \ldots$ ---- δA_n

Then

$$\iint_A f(x,y)\, dA = \lim_{\substack{n\to\infty \\ \delta A \to 0}} \Big[f(x_1,y_1)\,\delta A_1 + f(x_2,y_2)\,\delta A_2 + \cdots + f(x_n,y_n)\,\delta A_n \Big]$$

Evaluation of Double integral :-

Double integral over region A be evaluated by two successive integrations

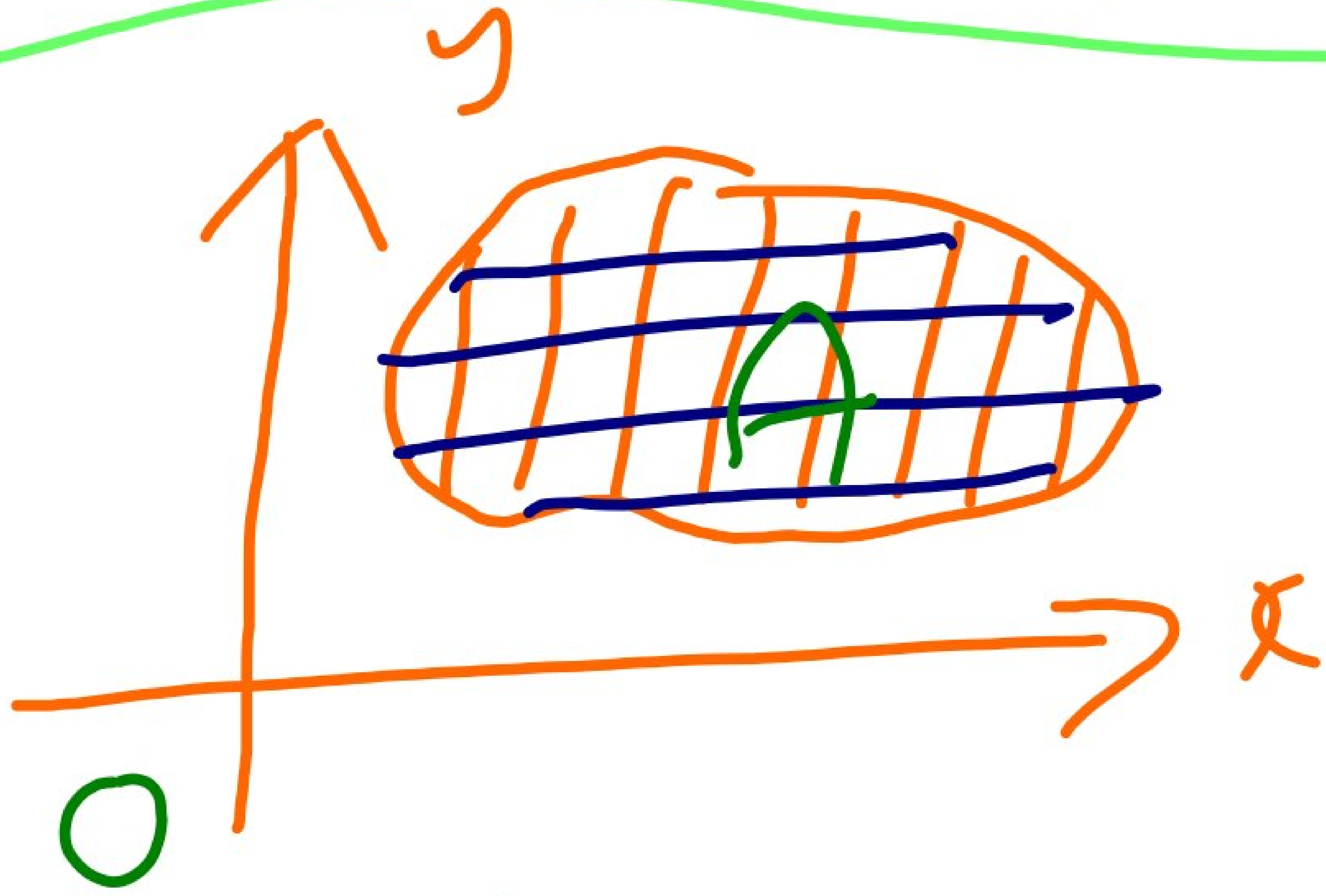
If A is described as

$$f(x_1) \le y \le f(x_2)$$

$$[\,y_1 \le y \le y_2\,],$$

and $a \le x \le b$

$$\boxed{\iint_A f(x,y)\,dA = \int_a^b \int_{y_1}^{y_2} f(x,y)\,dx\,dy}$$

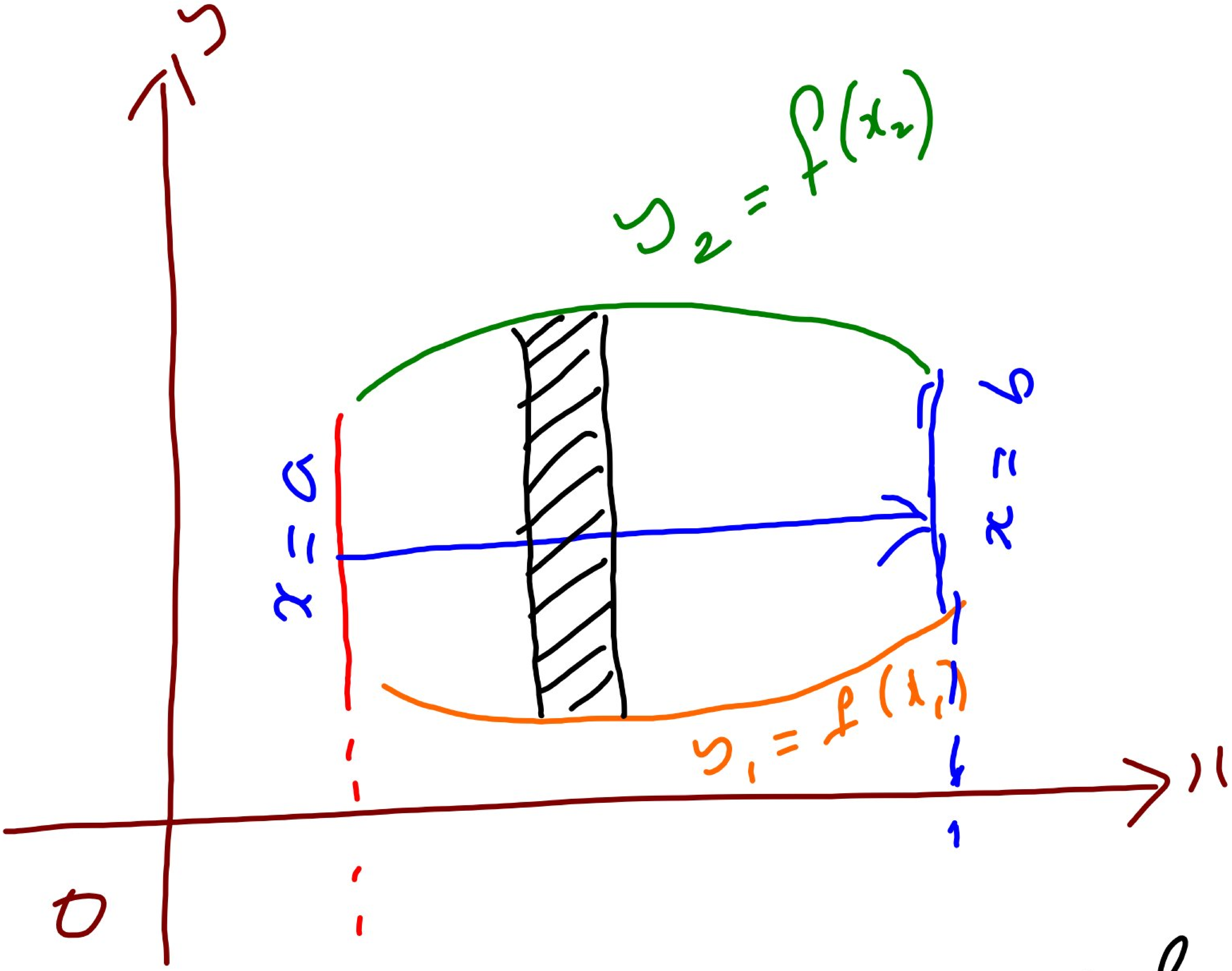


Method ① :-

$$\iint_A f(x,y)\,dx\,dy = \int_a^b \left\{ \int_{y_1}^{y_2} f(x,y)\,dy \right\} dx$$

ie $f(x,y)$ is first integrated with respect to 'y' treating x as constant b/w the limits y_1 and y_2.

Then the result is integrated w.r.t 'x' b/w the limits a and b



here the path is parallel to y-axis

method ②:—

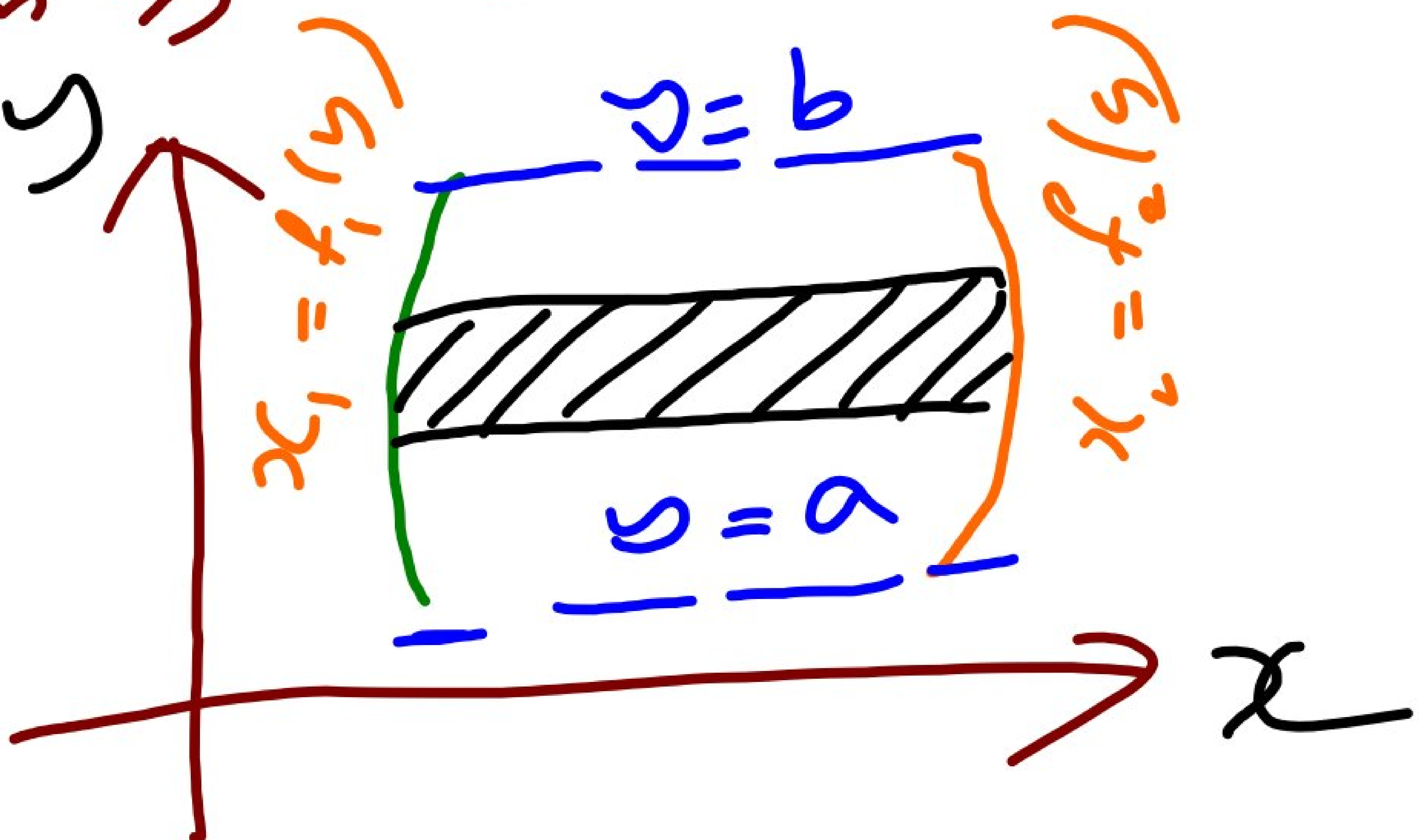
$$\iint_A f(x,y)\,dx\,dy = \int_a^b \left\{ \int_{x_1}^{x_2} f(x,y)\,dx \right\} dy$$

here $x_1 = f_1(y)$

$x_2 = f_2(y)$

$f(x,y)$ is integrated w.r.t x first, keeping y constant b/w the limits x_1 and x_2

Then the resulting expression is integrated w.r.t y b/w the limits a and b



here the path is parallel to x-axis.

problems:-

1) Evaluate $\int_0^1 \int_0^x e^{\frac{y}{x}}\, dx\, dy$

Sol:- Given $\int_{x=0}^{1} \int_{y=0}^{x} e^{\frac{y}{x}}\, dx\, dy$

$$\int_{x=0}^{1} \left\{ \int_{y=0}^{x} e^{\frac{y}{x}}\, dy \right\} dx$$

$$\int_{x=0}^{1} \left\{ \int_{y=0}^{x} e^{\left(\frac{1}{x}\right) y}\, dy \right\} dx$$

$$\int_{x=0}^{1} \left[\frac{e^{\left(\frac{1}{x}\right) y}}{\left(\frac{1}{x}\right)} \right]_{y=0}^{x} dx$$

$(\because$ taken $\frac{1}{x}$ as a constant$)$

$$\int_{x=0}^{1}\left[\frac{e^{\frac{1}{x}(x)}}{\frac{1}{x}}-\frac{e^{\frac{1}{x}(0)}}{\frac{1}{x}}\right]dx$$

$$\int_{x=0}^{1}\left(x e^{1}-x e^{0}\right)dx$$

$$\boxed{\frac{x}{\left(\frac{a}{b}\right)}=\frac{bx}{a}}$$

$\because e^{0}=1$

$$\int_{x=0}^{1} x(e-1)\,dx$$

$$(e-1)\int_{x=0}^{1} x\,dx$$

$$(e-1)\left(\frac{x^2}{2}\right)_0^1$$

$$(e-1)\left(\frac{1^2}{2}-\frac{0^2}{2}\right)$$

$$\frac{1}{2}(e-1)$$

2) Evaluate $\int_0^5 \int_0^{x^2} x(x^2+y^2)\,dx\,dy$

Sol:- Given $\int_{x=0}^{5} \int_{y=0}^{x^2} x(x^2+y^2)\,dx\,dy$

$$\int_{x=0}^{5} \left\{ \int_{y=0}^{x^2} x(x^2+y^2)\,dy \right\} dx$$

$$\int_{x=0}^{5} x \left(x^2 \int_{y=0}^{x^2} 1\,dy + \int_{y=0}^{x^2} y^2\,dy \right) dx$$

$$\int_{x=0}^{5} x \left[x^2 \left(y\right)_0^{x^2} + \left(\frac{y^3}{3}\right)_0^{x^2} \right] dx$$

$$\int_{x=0}^{5} x \left[x^2 \left[x^2 - 0\right] + \frac{(x^2)^3}{3} - \frac{0^3}{3} \right] dx$$

$$\int_{x=0}^{5} x \left(x^4 + \frac{x^6}{3} \right) dx$$

$$\int_{x=0}^{5} \left(x^5 + \frac{x^7}{3}\right) dx$$

$$\left(\frac{x^6}{6}\right)_0^5 + \frac{1}{3}\left(\frac{x^8}{8}\right)_0^5$$

$$\frac{5^6}{6} - \frac{0^6}{6} + \frac{1}{3}\left[\frac{5^8}{8} - \frac{0^8}{8}\right]$$

$$\frac{5^6}{6} + \frac{1}{3}\left(\frac{5^8}{8}\right)$$

$$5^6\left[\frac{1}{6} + \frac{5^2}{24}\right]$$

$$15625\left[\frac{4+25}{24}\right]$$

$$\frac{15625 \times 29}{24}$$

$$18880.2$$

3) $\int_1^2 \int_3^4 (xy + e^y)\, dy\, dx$

Sol:- $\int_1^2 \left(x \int_3^4 y\, dy + \int_3^4 e^y\, dy \right) dx$

$$\int_1^2 \left[x \left(\frac{y^2}{2} \right)_3^4 + \left(e^y \right)_3^4 \right] dx$$

$$\int_1^2 \left(x \left(\frac{4^2}{2} - \frac{3^2}{2} \right) + e^4 - e^3 \right) dx$$

$$\int_1^2 \left[x \left(8 - \frac{9}{2} \right) + \left(e^4 - e^3 \right) \right] dx$$

$$\frac{16-9}{2} \left(\frac{x^2}{2} \right)_1^2 + \left(e^4 - e^3 \right) \int_1^2 1\, dx$$

$$\frac{7}{2} \left(\frac{2^2}{2} - \frac{1^2}{2} \right) + \left(e^4 - e^3 \right) (x)_1^2$$

$$\frac{7}{2}\left(2-\frac{1}{2}\right)+\left(e^4-e^3\right)(2-1)$$

$$\frac{7}{2}\left(\frac{4-1}{2}\right)+e^4-e^3$$

$$e^4-e^3+\frac{21}{4}$$

H.W

4) $\displaystyle\int_0^1\int_x^{\sqrt{x}}(x^2+y^2)\,dx\,dy$

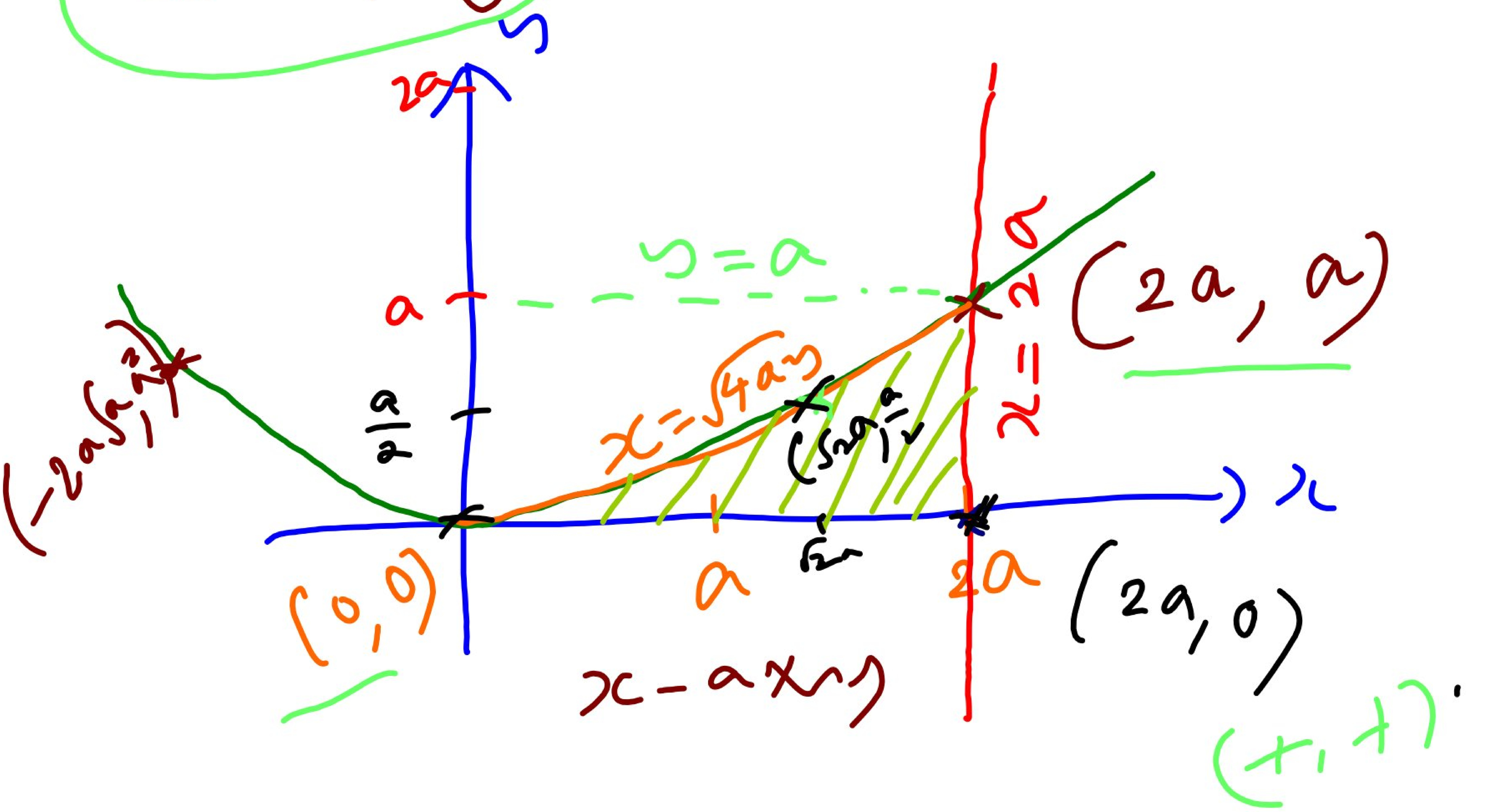
5) $\displaystyle\int_0^a\int_0^{\sqrt{ay}}xy\,dx\,dy$

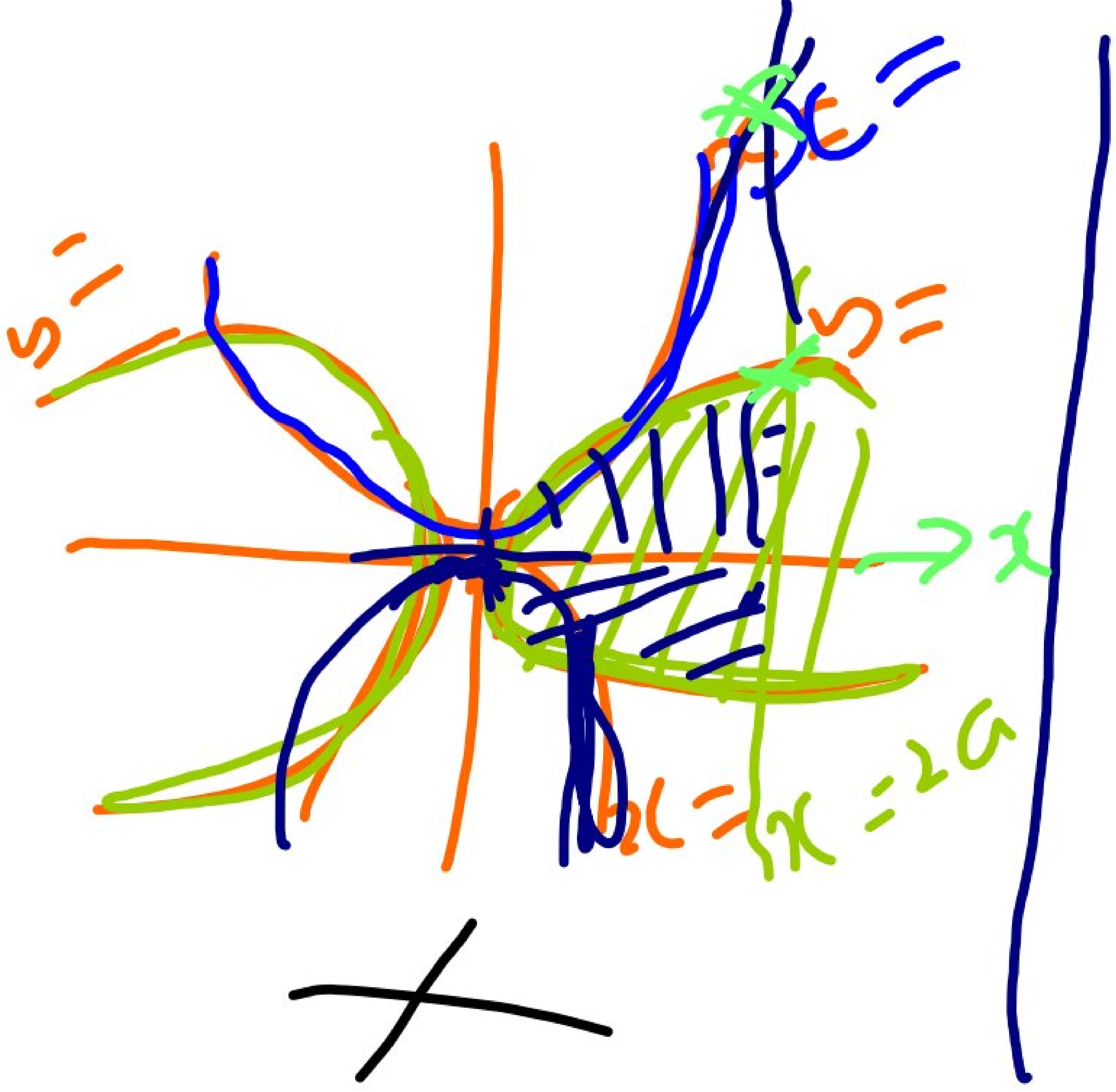
6) $\displaystyle\int_0^1\int_0^{\sqrt{1+x^2}}\frac{1}{1+x^2+y^2}\,dy\,dx$

7) Evaluate $\iint_A xy \, dx \, dy$, where A is the domain bounded by x-axis, ordinate $x = 2a$ and the curve $x^2 = 4ay$

Sol:- Given the equation of a line $x = 2a$ curve in the form of the parabola

$x^2 = 4ay$





$\because x^2 = 4ay$

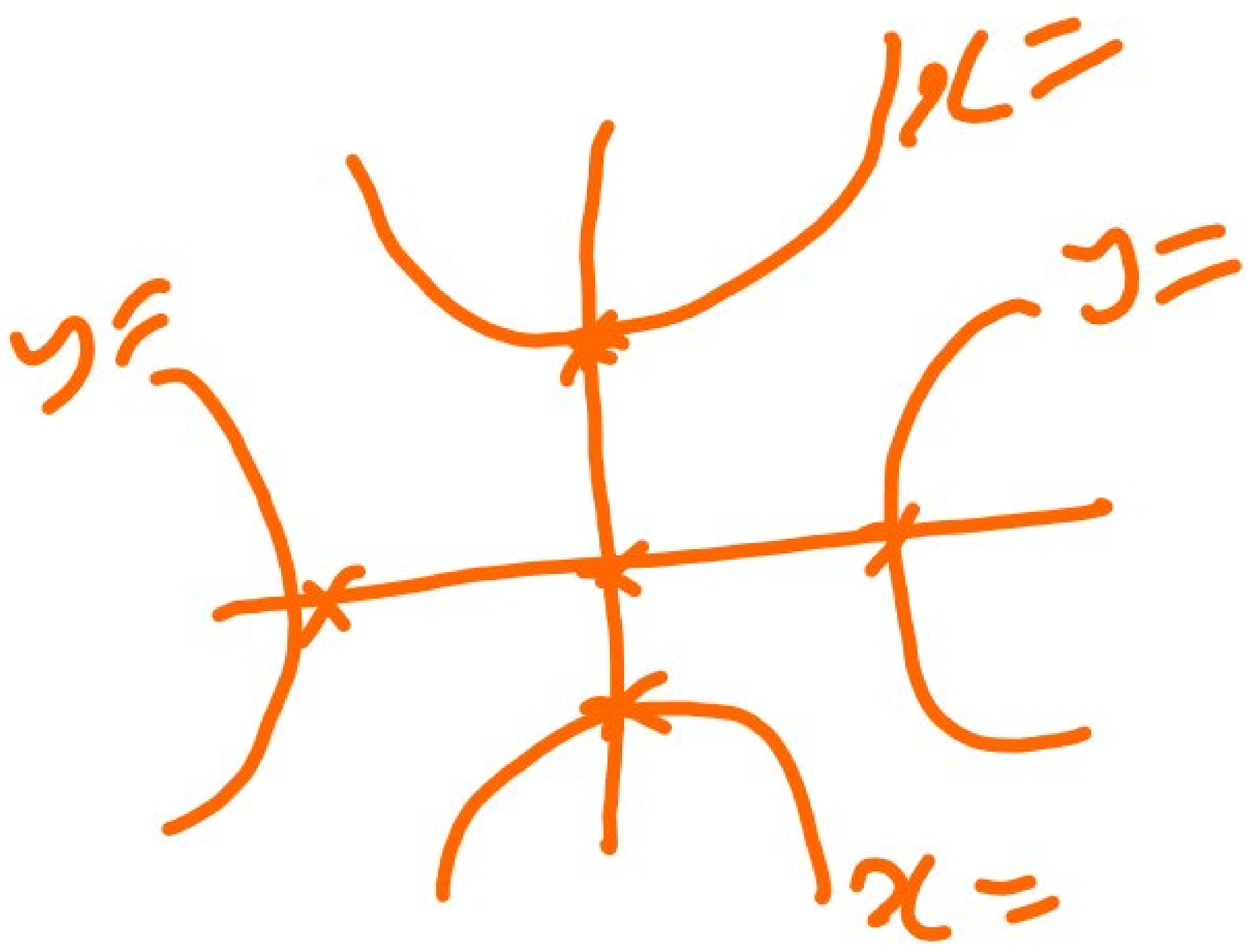
$(2a)^2 = 4ay \quad (\because x = 2a)$

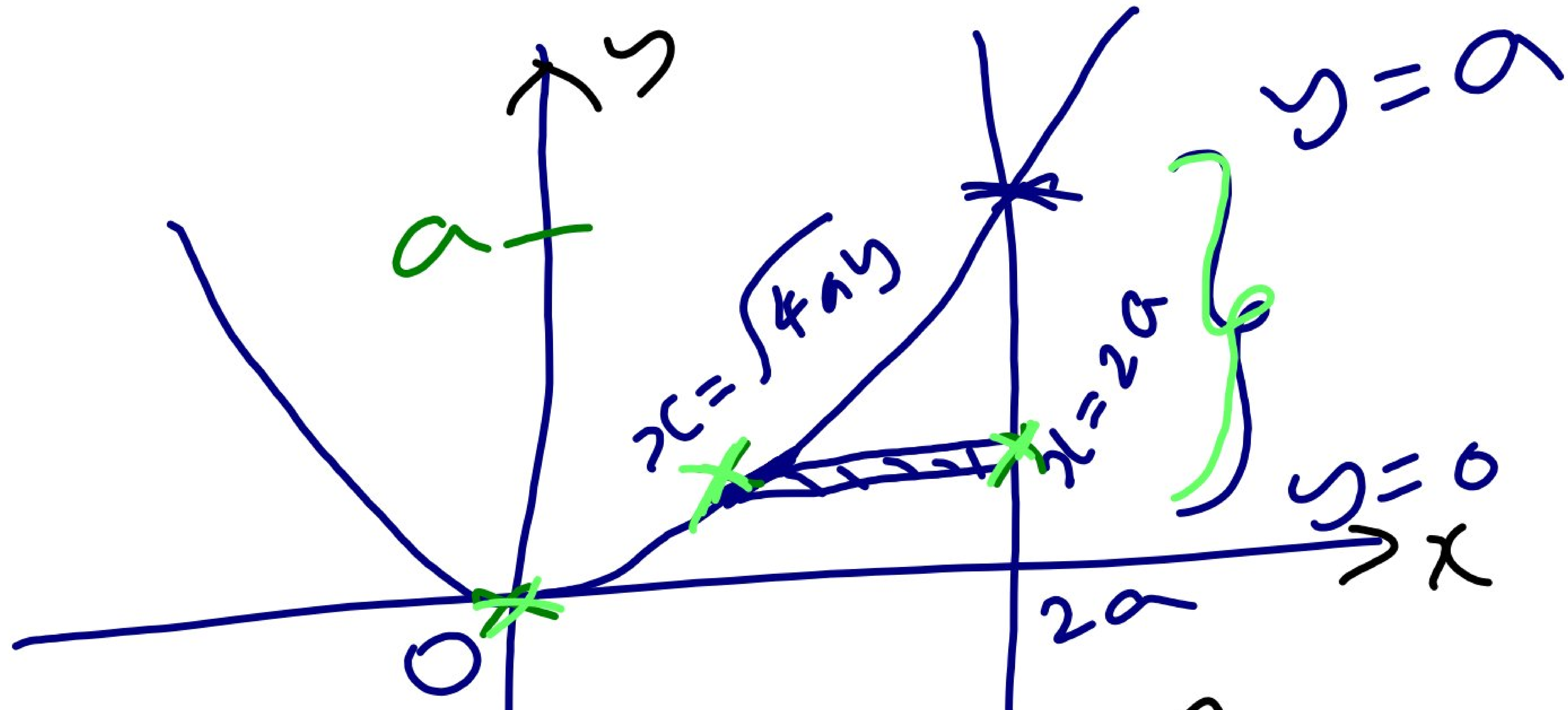
$4a^2 = 4ay$

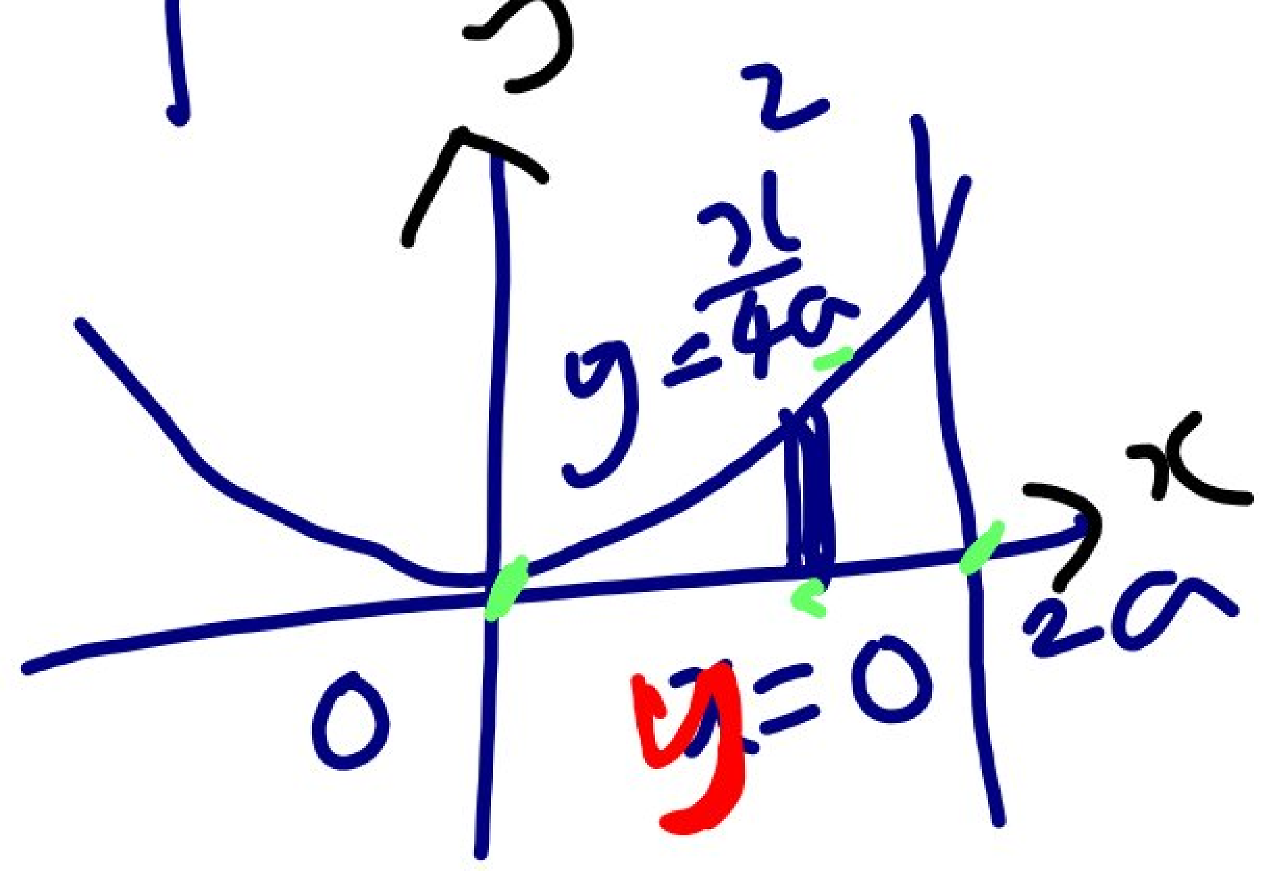
$y = a \Rightarrow (2a, a)$

$x^2 = 4ay \Rightarrow \boxed{y = \frac{x^2}{4a}}$ (or) $x = \sqrt{4ay}$

y	$(-a)^2$	0	a	$\frac{a}{2}$
x	$-2a\sqrt{a}$	0	$2a$	$\sqrt{2}a$







The path is parallel to x-axis

x is varies from

$\sqrt{4ay}$ to $2a$

y is varies from 0 to a

$$\iint_A xy\,dx\,dy = \int_{x=\sqrt{4ay}}^{2a}\int_{y=0}^{a} xy\,dx\,dy$$

$$= \int_{y=0}^{a} y\left\{\int_{x=\sqrt{4ay}}^{2a} x\,dx\right\}dy$$

$$= \int_{y=0}^{a} y\left(\frac{x^2}{2}\right)_{\sqrt{4ay}}^{2a} dy$$

The path is parallel to y-axis

x is varies from 0 to $2a$

y is varies from 0 to $\frac{x^2}{4a}$

$$\iint_A xy\,dx\,dy$$

$$= \int_{x=0}^{2a}\int_{y=0}^{\frac{x^2}{4a}} xy\,dx\,dy$$

$$= \int_{x=0}^{2a} x\left\{\int_{y=0}^{\frac{x^2}{4a}} y\,dy\right\}dx$$

$$= \int_{y=0}^{a} y\left(\frac{(2a)^2}{2} - \frac{(\sqrt{4ay})^2}{2}\right)dy$$

$$= \int_{y=0}^{a} y\left(\frac{4a^2}{2} - \frac{4ay}{2}\right)dy$$

$$= \frac{4a}{2}\int_{y=0}^{a} y(a-y)\,dy$$

$$= 2a\int_{y=0}^{a}(ay - y^2)\,dy$$

$$= 2a\left[a\,\frac{y^2}{2} - \frac{y^3}{3}\right]_{y=0}^{a}$$

$$= 2a\left[a\,\frac{a^2}{2} - \frac{a^3}{3} - 0\right]$$

$$= 2a\left[\frac{a^3}{2} - \frac{a^3}{3}\right]$$

$$= \cancel{2}a\left[\frac{3a^3 - 2a^3}{\cancel{6}_3}\right] = \frac{a(a^3)}{3} = \frac{a^4}{3}$$

$$= \int_{x=0}^{2a} x\left(\frac{y^2}{2}\right)_0^{\frac{x^2}{4a}} dx$$

$$= \int_{x=0}^{2a} x\left[\frac{\left(\frac{x^2}{4a}\right)^2}{2} - 0\right]dx$$

$$= \int_{x=0}^{2a} x\,\frac{\left(\frac{x^4}{16a^2}\right)}{\left(\frac{2}{1}\right)}\,dx$$

$$= \int_{x=0}^{2a} \frac{x\,(x^4)}{2(16)a^2}\,dx$$

$$= \frac{1}{32a^2}\int_{x=0}^{2a} x^5\,dx$$

$$= \frac{1}{32a^2}\left(\frac{x^6}{6}\right)_0^{2a}$$

$$= \frac{1}{32a^2}\left(\frac{(2a)^6}{6} - \frac{(0)^6}{6}\right)$$

$$= \frac{1}{\cancel{32}_{1}\,a^2}\left[\frac{\cancel{64}^{\,2}\,a^6}{\cancel{6}_{3}}\right]$$

$$= \frac{a^4}{3}$$

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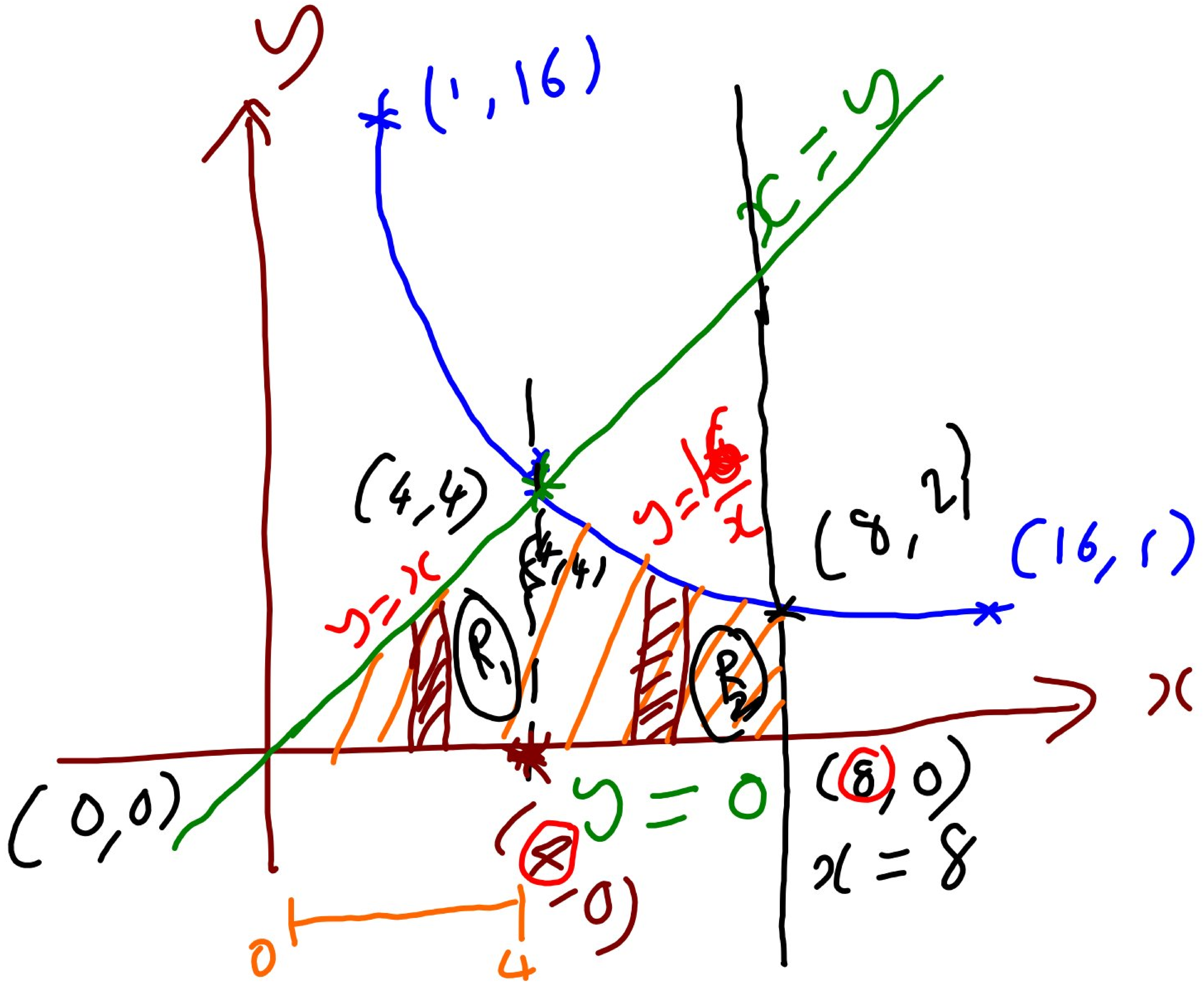
8) Evaluate $\iint_R x^2\,dx\,dy$ where R is the region in the first quadrant bounded by the lines $x=y$, $y=0$, $x=8$ and the curve $xy=16$

Sol:- Given curve $xy=16$

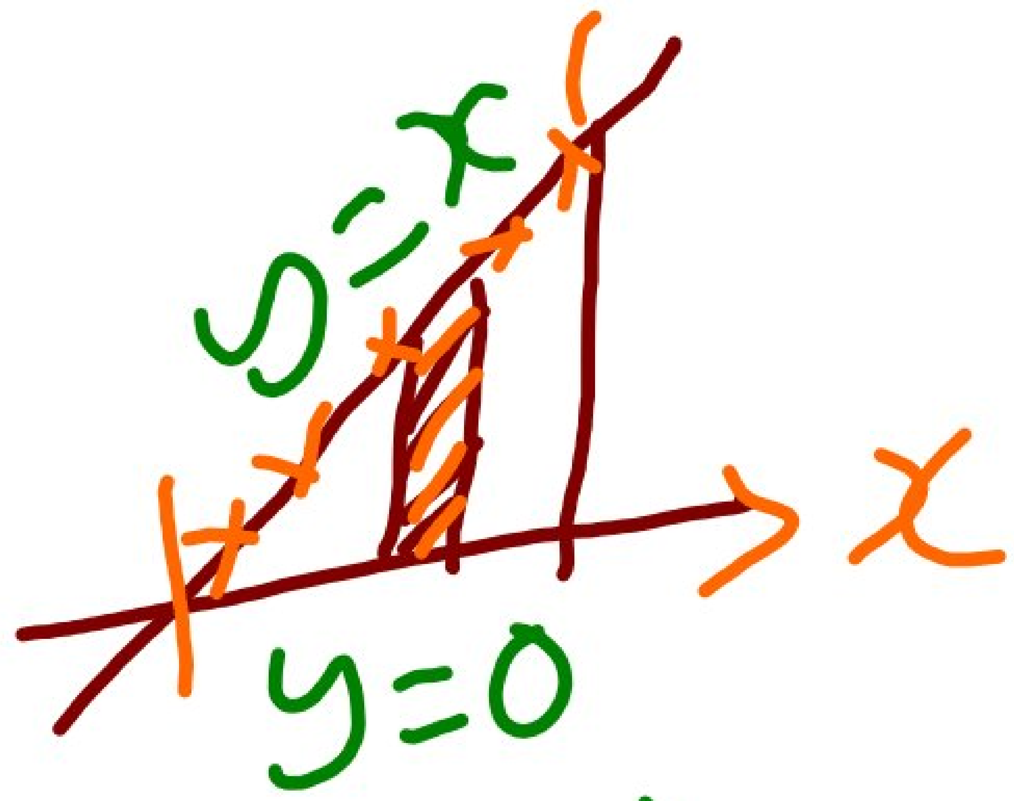
$$y = \frac{16}{x}$$

x	1	2	4	8	16
y	16	8	4	2	1

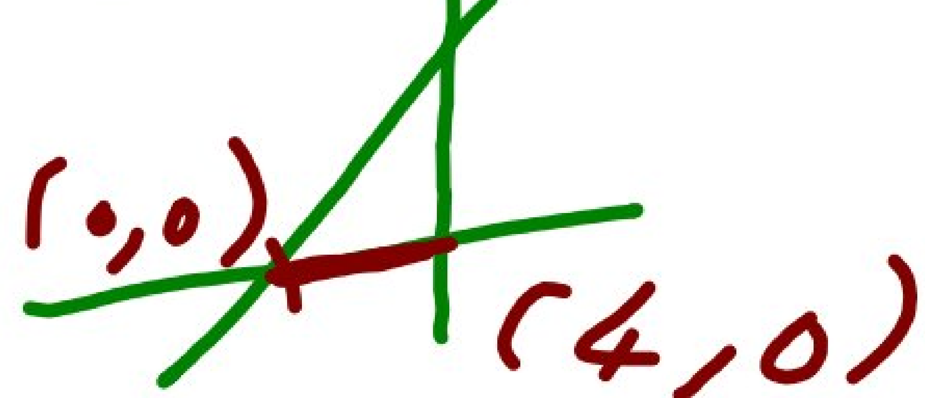


$R_1 \Rightarrow \iint_{R_1} x^2 \, dx \, dy$

The path is parallel to y-axis



y is varying from 0 to x



x is varying from 0 to 4

$$= \int_{x=0}^{4} \int_{y=0}^{x} x^2 \, dx \, dy$$

$$= \int_{x=0}^{4} x^2 \left\{ \int_{y=0}^{x} (1) \, dy \right\} dx$$

$$= \int_{x=0}^{4} x^2 \left(x \right)_0^x \, dx$$

$$= \int_{x=0}^{4} x^2 (x - 0) \, dx$$

$$= \int_{x=0}^{4} x^3 \, dx$$

$$= \left(\frac{x^4}{4}\right)_0^4$$

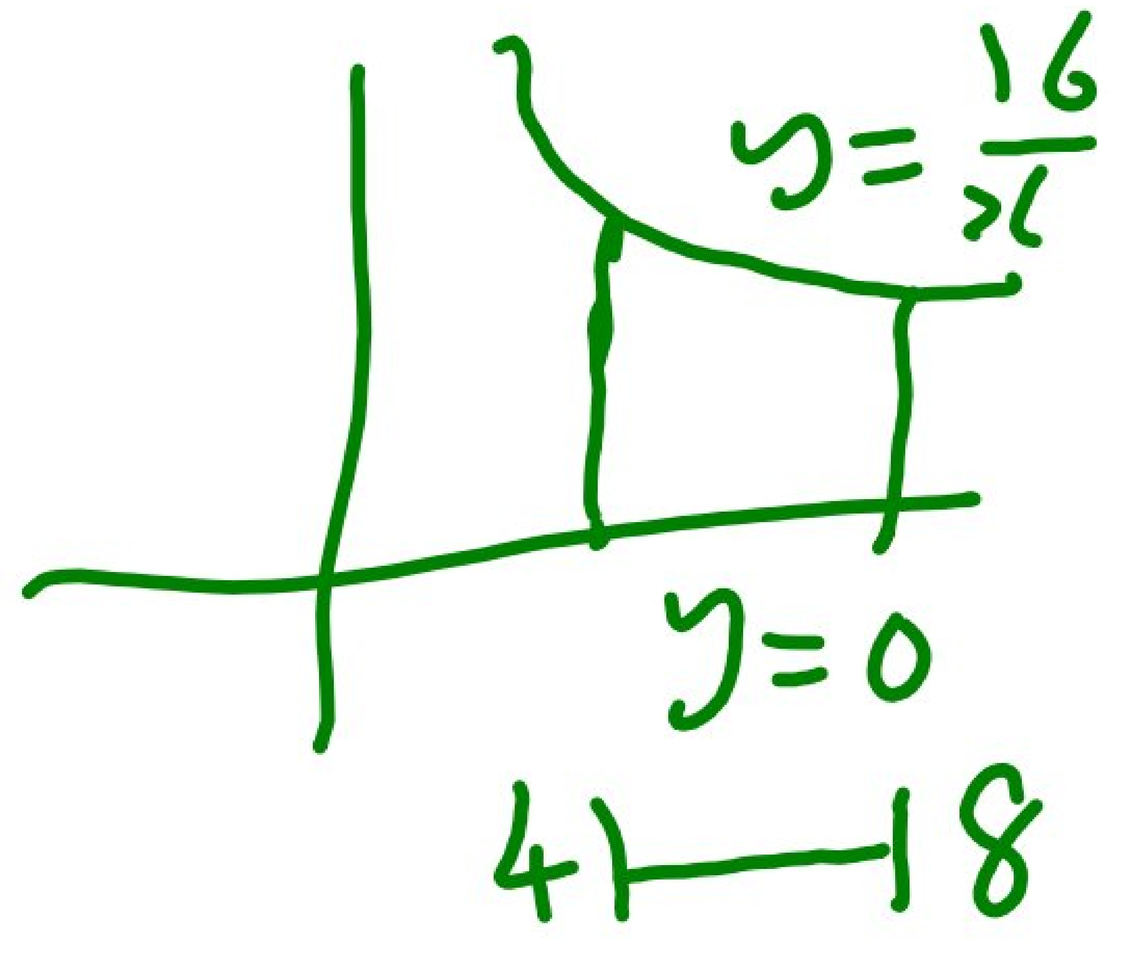
$$= \frac{4^4}{4} - 0$$

$$= 4^3$$

$$R_2 \Rightarrow \iint_{R_2} x^2 \, dx \, dy$$

The path is parallel to y-axis

y varies from 0 to $\frac{16}{x}$



x varies from 4 to 8

$$\int_{x=4}^{8}\int_{y=0}^{\frac{16}{x}} x^2 \, dx \, dy$$

$$\int_{x=4}^{8} x^2 \left\{ \int_{y=0}^{\frac{16}{x}} 1 \, dy \right\} dx$$

$$\int_{x=4}^{8} x^2 \left| x \right)_{0}^{\frac{16}{x}} dx$$

$$\int_{x=4}^{8} x^2 \left(\frac{16}{x} - 0 \right) dx$$

$$\int_{x=4}^{8} 16x \, dx$$

$$16 \left(\frac{x^2}{2} \right)_4^8$$

$$16\left(\frac{8^2}{2} - \frac{4^2}{2}\right)$$

$$16\,(32 - 8)$$

$$16 \times (24)$$

$$\iint_R x^2\,dx\,dy = \left(\iint_{R_1} + \iint_{R_2}\right) x^2\,dx\,dy$$

$$= 4^3 + (16 \times 24)$$

$$= 64 + 384$$

$$= 448$$