Unit II : Applications of Partial Derivatives

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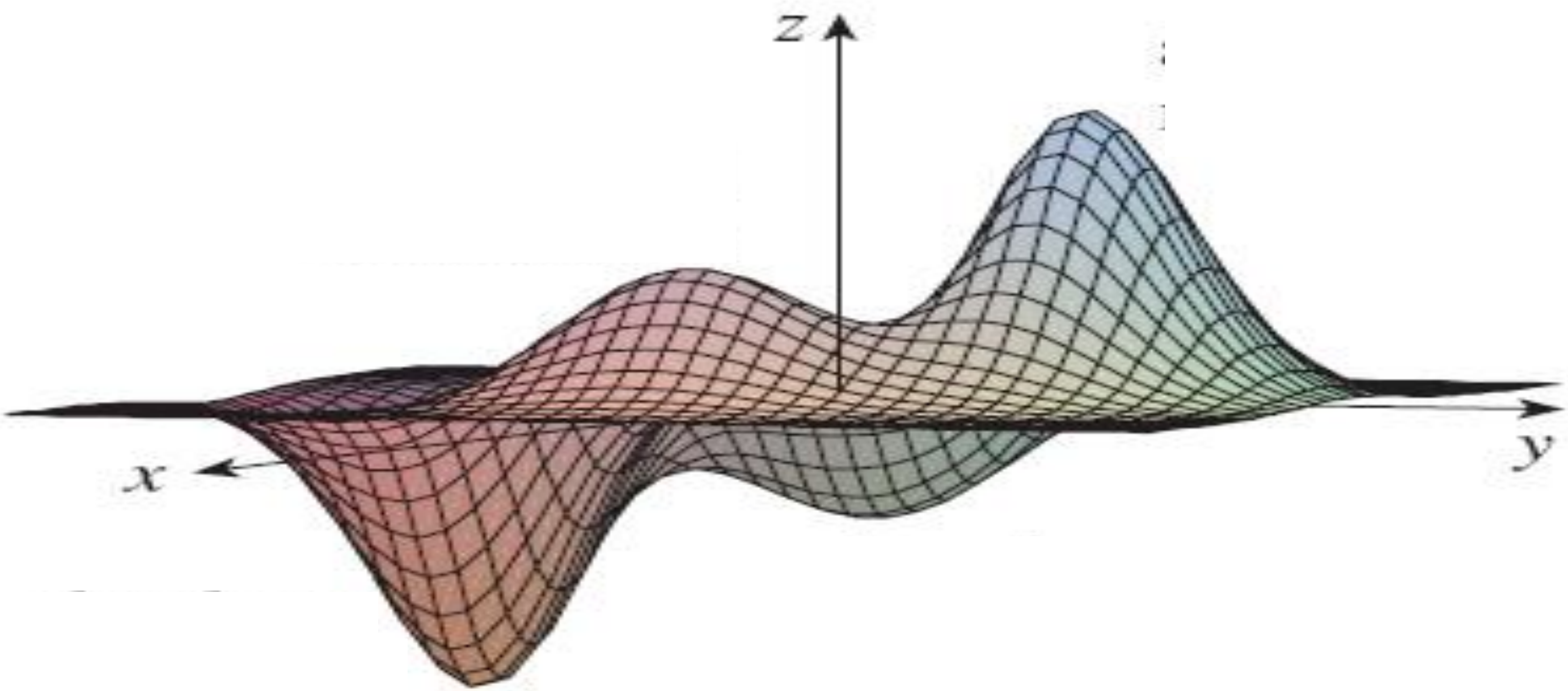
1. Maxima and Minima of function of two variables

Look at the hills and valleys in the graph of *f* shown here.

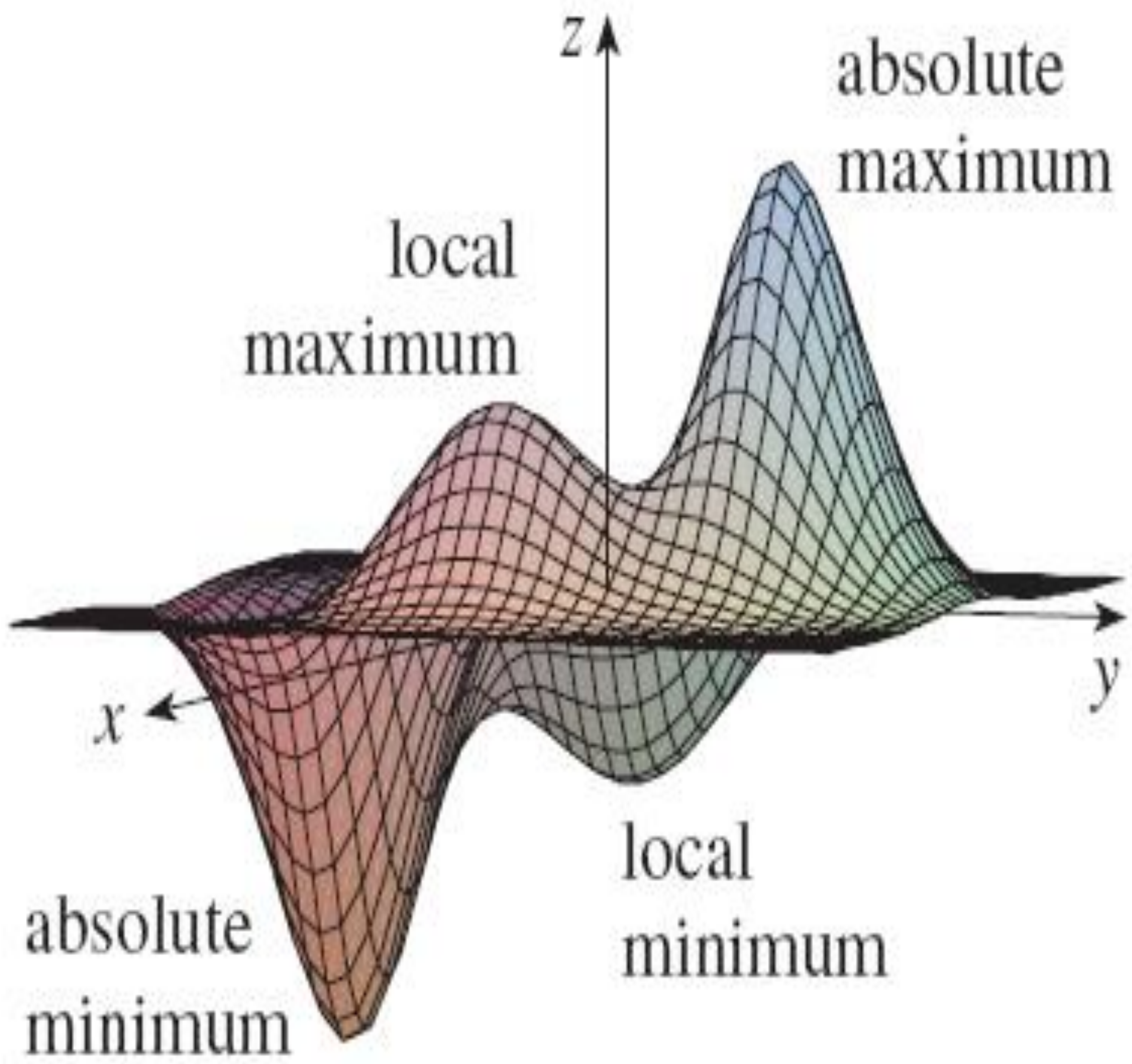


There are two points where f has a local maximum.

The larger of these two values is the **absolute maximum**.

Similarly there are two points where f has a local minimum.

The smaller of these two values is the **absolute minimum**.



Let f be a function defined on a region R containing the point (a, b).

Then, f has a local maximum at (a, b) if

$f(x, y) \leq f(a, b)$ **for all points** $(\boldsymbol{x}, \boldsymbol{y})$ **that are very near to** $(\boldsymbol{a}, \boldsymbol{b})$.

The number $f(a, b)$ is called a **local maximum value.**

Let f be a function defined on a region R containing the point (a, b).

If $f(x, y) \leq f(a, b)$ **for all points** $(\boldsymbol{x}, \boldsymbol{y})$ **in the domain** $\boldsymbol{R}$ of the function f

then number $f(a, b)$ is called a **absolute maximum value.**

Similarly, f has a local minimum at (a, b) if

$f(x, y) \geq f(a, b)$ for all points (x, y) that are very near to (a, b).

The number $f(a, b)$ is called a **local minimum value**.

If $f(x, y) \geq f(a, b)$ **for all points** $(\boldsymbol{x}, \boldsymbol{y})$ **in the domain** $\boldsymbol{R}$ of the function f

then number $f(a, b)$ is called a **absolute minimum value.**

Working rule for finding Extreme (Maximum/Minimum) values

Let $f(x, y)$ be a given function of x & y.

1. Find partial derivatives $p = \frac{\partial f}{\partial x}, q = \frac{\partial f}{\partial y}, r = \frac{\partial^2 f}{\partial x^2}, s = \frac{\partial^2 f}{\partial x \partial y}, t = \frac{\partial^2 f}{\partial y^2}$.

2. Find stationary points by solving equations $p = \frac{\partial f}{\partial x} = 0$ & $q = \frac{\partial f}{\partial y} = 0$

   simultaneously. Suppose (a, b) be the stationary point.

3. Find the values of $r = \frac{\partial^2 f}{\partial x^2}, s = \frac{\partial^2 f}{\partial x \partial y}, t = \frac{\partial^2 f}{\partial y^2}$ at point (a, b).

4. Find $rt - s^2$ at point (a, b). Then

a) $\boldsymbol{rt - s^2 > 0}$ and $\boldsymbol{r < 0}$ $\Rightarrow f(x, y)$ has **maximum** value at the point (a, b).

b) $\boldsymbol{rt - s^2 > 0}$ and $\boldsymbol{r > 0}$ $\Rightarrow f(x, y)$ has **minimum** value at the point (a, b).

c) $\boldsymbol{rt - s^2 < 0}$ $\Rightarrow f(x, y)$ has **neither a maximum nor a minimum** value at the point (a, b). Such a point is called as a **saddle point**.

d) $\boldsymbol{rt - s^2 = 0}$ then test gives no information. Further investigation is needed.

Note:

1. The point (a, b) at which $\frac{\partial f}{\partial x} = 0$ & $\frac{\partial f}{\partial y} = 0$ is called as **stationary point.**

2. Maximum/Minimum values occur only at stationary points.
However stationary points need not be maxima/minima.

Given function $f(x, y)$

Find $p = \dfrac{\partial f}{\partial x}$, $q = \dfrac{\partial f}{\partial y}$, $r = \dfrac{\partial^2 f}{\partial x^2}$, $s = \dfrac{\partial^2 f}{\partial x \partial y}$, $t = \dfrac{\partial^2 f}{\partial y^2}$

Find stationary points by solving equations $p = \dfrac{\partial f}{\partial x} = 0$ & $q = \dfrac{\partial f}{\partial y} = 0$ simultaneously

Calculate the value of $rt - s^2$ at stationary points

$rt - s^2 > 0$

$r > 0$

Minima

$r < 0$

Maxima

$rt - s^2 < 0$

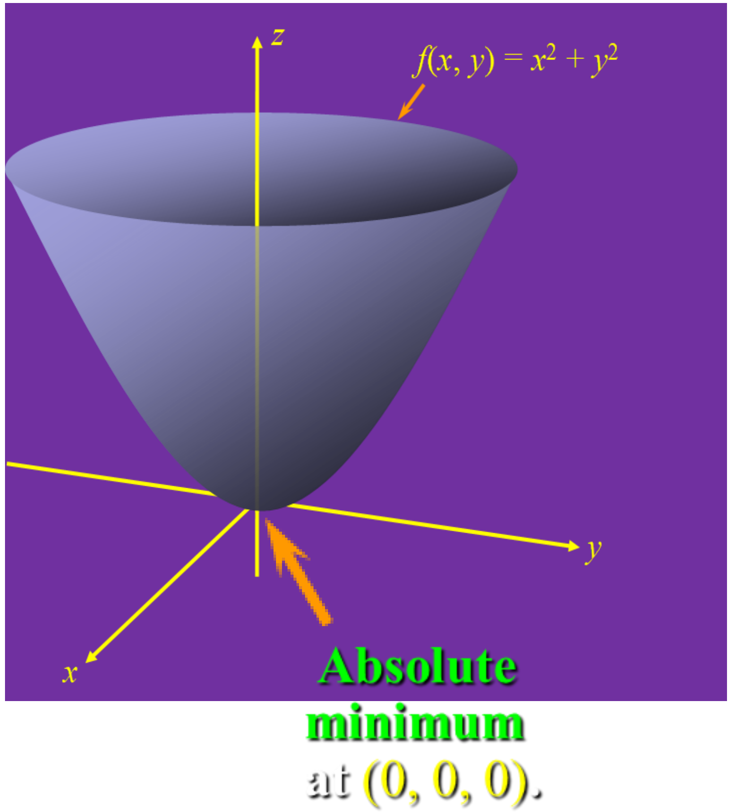
Neither maxima nor minima. Such a point is saddle point

$rt - s^2 = 0$

Test fails. Further investigation is required

Example 1. Find extreme values of the function $f(x, y) = x^2 + y^2$.

Solution : $f(x, y) = x^2 + y^2$



$p = \dfrac{\partial f}{\partial x} = 2x,$ $\qquad q = \dfrac{\partial f}{\partial y} = 2y$

$r = \dfrac{\partial^2 f}{\partial x^2} = 2,$ $\qquad s = \dfrac{\partial^2 f}{\partial x \partial y} = 0,$ $\qquad t = \dfrac{\partial^2 f}{\partial y^2} = 2$

$p = 0 \Longrightarrow 2x = 0 \Longrightarrow x = 0$

$q = 0 \Longrightarrow 2y = 0 \Longrightarrow y = 0$

Stationary point is $(0, 0)$.

At point $(0,0)$, $rt - s^2 = (2)(2) - (0)^2 = 4 > 0$ and $r = 2 > 0$.

$f(x, y)$ has minimum value at the point $(0,0)$.

$f_{min} = f(0,0) = 0$

Example 2. Find extreme values of the function $f(x,y) = 1 - x^2 - y^2$.

Solution : $f(x,y) = 1 - x^2 - y^2$

$$p = \frac{\partial f}{\partial x} = -2x, \qquad q = \frac{\partial f}{\partial y} = -2y$$

$$r = \frac{\partial^2 f}{\partial x^2} = -2, \qquad s = \frac{\partial^2 f}{\partial x \partial y} = 0, \qquad t = \frac{\partial^2 f}{\partial y^2} = -2$$

$$p = 0 \Rightarrow -2x = 0 \Rightarrow x = 0$$

$$q = 0 \Rightarrow -2y = 0 \Rightarrow y = 0$$

Stationary point is $(0,0)$.

At point (0,0), $rt - s^2 = (-2)(-2) - (0)^2 = 4 > 0$ and $r = -2 < 0$.

$f(x,y)$ has maximum value at the point (0,0).

$f_{max} = f(0,0) = 1$

Example 3. Find extreme values of the function $f(x,y) = xy$.

Solution : $f(x,y) = xy$

$p = \frac{\partial f}{\partial x} = y, \qquad q = \frac{\partial f}{\partial y} = x$

$r = \frac{\partial^2 f}{\partial x^2} = 0, \qquad s = \frac{\partial^2 f}{\partial x \partial y} = 1, \qquad t = \frac{\partial^2 f}{\partial y^2} = 0$

$p = 0 \Longrightarrow y = 0$ & $q = 0 \Longrightarrow x = 0$

Stationary point is $(0,0)$.

At point $(0,0)$, $rt - s^2 = (0)(0) - 1^2 = -1 < 0$ and $r = -2 < 0$.

$f(x,y)$ has neither maxima nor minima at point $(0,0)$.

So, $(0,0)$ is a saddle point of $f(x,y)$.

Example 4. Find extreme values of the function $f(x,y) = 1 - x^2y^2$.

Solution : $f(x,y) = 1 - x^2y^2$

$$p = \frac{\partial f}{\partial x} = -2xy^2, \qquad q = \frac{\partial f}{\partial y} = -2x^2y$$

$$r = \frac{\partial^2 f}{\partial x^2} = -2y^2, \qquad s = \frac{\partial^2 f}{\partial x \partial y} = -4xy, \qquad t = \frac{\partial^2 f}{\partial y^2} = -2x^2$$

$$p = 0 \Rightarrow -2xy^2 = 0 \Rightarrow x = 0, y = 0$$

$$q = 0 \Rightarrow -2x^2y = 0 \Rightarrow x = 0, y = 0$$

Stationary point is $(0,0)$.

$$rt - s^2 = (-2y^2)(-2x^2) - (-4xy)^2 = 4x^2y^2 - 4x^2y^2 = 0$$

At point (0,0), $\quad rt - s^2 = 0$

Test failed. Further investigation is needed.

Example 5. Find extreme values of $f(x,y) = xy - x^2 - y^2 - 2x - 2y + 4$

Solution : $f(x,y) = xy - x^2 - y^2 - 2x - 2y + 4$

$$p = \frac{\partial f}{\partial x} = y - 2x - 2\,, \qquad q = \frac{\partial f}{\partial y} = x - 2y - 2$$

$$r = \frac{\partial^2 f}{\partial x^2} = -2, \qquad s = \frac{\partial^2 f}{\partial x \partial y} = 1, \qquad t = \frac{\partial^2 f}{\partial y^2} = -2$$

$p = 0 \Longrightarrow y - 2x - 2 = 0 \ldots\ldots\ldots (1)$

$q = 0 \Longrightarrow x - 2y - 2 = 0 \ldots\ldots\ldots (2)$

Solving (1) and (2) simultaneously we get, $x = -2$ and $y = -2$

Therefore stationary point is $(-2, -2)$.

Thus $rt - s^2 = (-2)(-2) - (1)^2 = 4 - 1 = 3$

At point $(-2, -2)$, $rt - s^2 = 3 > 0$ and $r = -2 < 0$.

$\Longrightarrow f(x, y)$ has maximum value at the point $(-2, -2)$.

$$f_{max} = f(-2, -2)$$

$$= (-2)(-2) - (-2)^2 - (-2)^2 - 2(-2) - 2(-2) + 4$$

$$= 4 - 4 - 4 + 4 + 4 + 4$$

$$= 8$$

Example 6. Find extreme values of the function $f(x,y) = x^2 + y^2 - 4y + 9$

Solution : $f(x,y) = x^2 + y^2 - 4y + 9$

$p = \frac{\partial f}{\partial x} = 2x,$ $\quad q = \frac{\partial f}{\partial y} = 2y - 4$

$r = \frac{\partial^2 f}{\partial x^2} = 2,$ $\quad s = \frac{\partial^2 f}{\partial x \partial y} = 0,$ $\quad t = \frac{\partial^2 f}{\partial y^2} = 2$

$p = 0 \Longrightarrow 2x = 0 \Longrightarrow x = 0 \ldots\ldots\ldots(1)$

$q = 0 \Longrightarrow 2y - 4 = 0 \Longrightarrow y = 2 \ldots\ldots\ldots(2)$

From (1)and (2), stationary point is $(0,2)$.

Thus $rt - s^2 = (2)(2) - (0)^2 = 4$

At point $(0,2)$, $rt - s^2 = 4 > 0$ and $r = 2 > 0$.

$\Longrightarrow f(x,y)$ has minimum value at the point (0,2).

$f_{min} = f(0,2) = 0 + (2)^2 - 4(2) + 9 = 5$

Example 7. Find extreme values of $f(x,y) = x^2 + 4y^3 - 12y^2 - 36y + 2$

Solution : $f(x,y) = x^2 + 4y^3 - 12y^2 - 36y + 2$

$$p = \frac{\partial f}{\partial x} = 2x \,, \qquad q = \frac{\partial f}{\partial y} = 12y^2 - 24y - 36$$

$$r = \frac{\partial^2 f}{\partial x^2} = 2, \qquad s = \frac{\partial^2 f}{\partial x \partial y} = 0, \qquad t = \frac{\partial^2 f}{\partial y^2} = 24y - 24 = 24(y-1)$$

$$\frac{\partial f}{\partial x} = 0 \Longrightarrow 2x = 0 \Longrightarrow \boldsymbol{x = 0} \ldots\ldots\ldots(1)$$

$$\frac{\partial f}{\partial y} = 0 \Longrightarrow 12y^2 - 24y - 36 = 0 \Longrightarrow \boldsymbol{y^2 - 2y - 3 = 0} \ldots\ldots\ldots(2)$$

From(2) we get, $y = -1$ or $y = 3$

Therefore stationary points are $(0,-1)$ and $(0,3)$

From(2) we get, $y = -1$ or $y = 3$

Therefore stationary points are $(0, -1)$ and $(0, 3)$

Now $rt - s^2 = 2\big(24(y-1)\big) - (0)^2 = 48(y-1)$

Point	$rt - s^2 = 48(y-1)$	$r = 2$	**Conclusion**
$(0, -1)$	$-96 < 0$	$-$	Neither maxima nor minima at $(0, -1)$. $(0, -1)$ is a saddle point.
$(0, 3)$	$96 > 0$	$2 > 0$	$f(x, y)$ has minimum value at $(0, 3)$. $f_{min} = f(0, 3)$ $= 0 + 4(3)^3 - 12(3)^2 - 36(3) + 2$ $= 108 - 108 - 108 + 2$ $= -106$

Example 8. Find extreme values of $f(x,y) = x^3 + y^3 - 3axy$ where $a > 0$

Solution : $f(x,y) = x^3 + y^3 - 3axy$

$$p = \frac{\partial f}{\partial x} = 3x^2 - 3ay, \qquad q = \frac{\partial f}{\partial y} = 3y^2 - 3ax$$

$$r = \frac{\partial^2 f}{\partial x^2} = 6x, \qquad s = \frac{\partial^2 f}{\partial x \partial y} = -3a, \qquad t = \frac{\partial^2 f}{\partial y^2} = 6y$$

$$p = 0 \Longrightarrow 3x^2 - 3ay = 0 \Longrightarrow \boldsymbol{x^2 = ay} \ldots\ldots\ldots(1)$$

$$q = 0 \Longrightarrow 3y^2 - 3ax = 0 \Longrightarrow \boldsymbol{y^2 = ax} \ldots\ldots\ldots(2)$$

$$x^2 = ay \Longrightarrow x^4 = a^2y^2 = a^2ax = a^3x$$

$$\Longrightarrow \quad x^4 - a^3x = 0 \Longrightarrow x(x^3 - a^3) = 0$$

$$\Longrightarrow x(x-a)(x^2 + ax + a^2) = 0$$

$$\Longrightarrow x = 0 \text{ or } x - a = 0 \text{ or } x^2 + ax + a^2 = 0$$

Now $x^2 + ax + a^2 = 0$ have complex (imaginary) roots.

$\Rightarrow x = 0$ or $x - a = 0$ $i.e.$ $x = a$

$x = 0$ & $x^2 = ay \Rightarrow y = 0$

$x = a$ & $x^2 = ay \Rightarrow y = a$

Therefore stationary points are $(0, 0)$ and (a, a)

Now $rt - s^2 = (6x)(6y) - (-3a)^2 = 36xy - 9a^2$

Point	$rt - s^2 = 36xy - 9a^2$	$r = 6x$	**Conclusion**
$(0, 0)$	$-9a^2 < 0$	$-$	Neither maxima nor minima at $(0, 0)$. $(0, 0)$ is a saddle point.
(a, a)	$27a^2 > 0$	$6a > 0$	$f(x, y)$ has minimum value at (a, a). $f_{min} = a^3 + a^3 - 3a^3 = -a^3$

Example 9. Find extreme values of $f(x,y) = x^4 + y^4 - 4xy + 1$

Solution : $f(x,y) = x^4 + y^4 - 4xy + 1$

$$p = \frac{\partial f}{\partial x} = 4x^3 - 4y, \qquad q = \frac{\partial f}{\partial y} = 4y^3 - 4x$$

$$r = \frac{\partial^2 f}{\partial x^2} = 12x^2, \qquad s = \frac{\partial^2 f}{\partial x \partial y} = -4, \qquad t = \frac{\partial^2 f}{\partial y^2} = 12y^2$$

$$p = 0 \Longrightarrow 4x^3 - 4y = 0 \Longrightarrow x^3 = y \ldots\ldots\ldots (1)$$

$$q = 0 \Longrightarrow 4y^3 - 4x = 0 \Longrightarrow y^3 = x \ldots\ldots\ldots (2)$$

$$x^3 = y \Longrightarrow (x^3)^3 = y^3 \Longrightarrow \quad \boldsymbol{x^9 = x}$$

$$\Longrightarrow \quad x^9 - x = 0 \Longrightarrow x(x^8 - 1) = 0$$

$$\Longrightarrow x(x^4 - 1)(x^4 + 1) = 0 \Longrightarrow x(x^2 - 1)(x^2 + 1)(x^4 + 1) = 0$$

$\Longrightarrow x = 0$ or $x^2 - 1 = 0$ or $x^2 + 1 = 0$ or $x^4 + 1 = 0$

Now $x^2 + 1 = 0$ and $x^4 + 1 = 0$ have complex (imaginary) roots.

$\therefore x = 0$ or $x^2 - 1 = 0 \Longrightarrow x = 0, \qquad x = \pm 1$

$x = 0$ & $x^3 = y \Longrightarrow y = 0$

$x = 1$ & $x^3 = y \Longrightarrow y = 1$

$x = -1$ & $x^3 = y \Longrightarrow y = -1$

Therefore stationary points are $(0, 0), (1,1)$ and $(-1, -1)$

Now $rt - s^2 = (12x^2)(12y^2) - (-4)^2 = 144x^2y^2 - 16$

Point	$\boldsymbol{rt - s^2 = 144x^2y^2 - 16}$	$\boldsymbol{r = 12x^2}$	**Conclusion**
$(0,0)$	$-16 < 0$	$-$	Neither maxima nor minima at $(0,0)$. $(0,0)$ is a saddle point.
$(1,1)$	$128 > 0$	$12 > 0$	f has minimum value at $(1,1)$. $f_{min} = 1^4 + 1^4 - 4(1)(1) + 1 = -1$
$(-1,-1)$	$128 > 0$	$12 > 0$	f has minimum value at $(-1,-1)$. $f_{min} = (-1)^4 + (-1)^4 - 4(-1)(-1) + 1 = -1$

Example 10. Find extreme values of $f(x,y) = xy(a-x-y)$ where $a > 0$

Solution : $p = \dfrac{\partial f}{\partial x} = ay - 2xy - y^2, \qquad q = \dfrac{\partial f}{\partial y} = ax - x^2 - 2xy$

$$r = \frac{\partial^2 f}{\partial x^2} = -2y, \qquad s = \frac{\partial^2 f}{\partial x \partial y} = a - 2x - 2y, \qquad t = \frac{\partial^2 f}{\partial y^2} = -2x$$

$$p = 0 \Longrightarrow ay - 2xy - y^2 = 0 \Longrightarrow y(a - 2x - y) = 0 \ldots\ldots\ldots (1)$$

$$q = 0 \Longrightarrow ax - x^2 - 2xy = 0 \Longrightarrow x(a - x - 2y) = 0 \ldots\ldots\ldots (2)$$

From(1), $y = 0$ or $a - 2x - y = 0$

From(2), $x = 0$ or $a - x - 2y = 0$

Considering following four pairs of equations of (1) and (2) , we get

$y = 0$ and $x = 0 \Longrightarrow (x, y) = (0, 0)$

$y = 0$ and $a - x - 2y = 0 \Longrightarrow (x, y) = (a, 0)$

$a - 2x - y = 0$ and $x = 0 \Longrightarrow (x, y) = (0, a)$

$a - 2x - y = 0$ and $a - x - 2y = 0 \Longrightarrow (x, y) = \left(\frac{a}{3}, \frac{a}{3}\right)$

Therefore stationary points are $(0, 0), (a, 0), (0, a)$and $\left(\frac{a}{3}, \frac{a}{3}\right)$.

Point	$rt - s^2 = 4xy - (a - 2x - 2y)^2$	$r = -2y$	Conclusion
$(0,0)$	$-a^2 < 0$	$-$	Neither maxima nor minima. $(0,0)$ is a saddle point.
$(a,0)$	$-a^2 < 0$	$-$	Neither maxima nor minima. $(a,0)$ is a saddle point.
$(0,a)$	$-a^2 < 0$	$-$	Neither maxima nor minima. $(0,a)$ is a saddle point.
$\left(\frac{a}{3}, \frac{a}{3}\right)$	$4\frac{a}{3}\frac{a}{3} - \left(a - \frac{2a}{3} - \frac{2a}{3}\right)^2 = \frac{a^2}{3} > 0$	$-\frac{2a}{3} < 0$	$f(x,y)$ has maximum value. $f_{max} = \frac{a}{3}\frac{a}{3}\left(a - \frac{a}{3} - \frac{a}{3}\right) = \frac{a^3}{27}$

Test Your Knowledge

Q.	Find extreme values of the following functions.	Stationary points	Extreme values
1.	$f(x,y) = x^2 + y^2 + 6x + 12$	$(-3,0)$	$f_{min} = 3$ at $(-3,0)$
2.	$f(x,y) = 3x^2 - y^2 + x^3$	$(0,0), (-2,0)$	$f_{max} = 4$ at $(-2,0)$
3.	$f(x,y) = xy + a^3\left(\frac{1}{x} + \frac{1}{y}\right)$	(a,a)	$f_{min} = 3a^2$ at (a,a)
4.	$f(x,y) = x^3 + 3xy^2 - 3x^2 - 3y^2 + 4$	$(0,0), (2,0),$ $(1,1), (1,-1)$	$f_{max} = 4$ at $(0,0)$ $f_{min} = 0$ at $(2,0)$
5.	$f(x,y) = x^3 + y^3 - 63(x+y) + 12xy$	$(-7,-7), (3,3),$ $(-1,5), (5,-1)$	$f_{max} = 784$ at $(-7,-7)$ $f_{min} = -216$ at $(3,3)$
6.	$f(x,y) = x^3 + y^3 - 3x - 12y + 20$	$(1,2), (1,-2),$ $(-1,2), (-1,-2)$	$f_{max} = 38$ at $(-1,2)$ $f_{min} = 2$ at $(1,2)$
7.	$f(x,y) = x^3 + 3xy^2 - 15x^2 - 15y^2 + 72x$	$(4,0), (6,0),$ $(5,1), (5,-1)$	$f_{max} = 112$ at $(4,0)$ $f_{min} = 108$ at $(6,0)$

Q. Find extreme values of the following functions.		Stationary points	Extreme values
8.	$f(x, y) = x^3y^2(1 - x - y)$	$(0,0), \left(\frac{1}{2}, \frac{1}{3}\right)$	$f_{max} = \frac{1}{432}$ at $\left(\frac{1}{2}, \frac{1}{3}\right)$
9.	$f(x, y) = x^4 + y^4 - 2x^2 + 4xy - 2y^2$	$(0,0), \left(\sqrt{2}, -\sqrt{2}\right),$ $\left(-\sqrt{2}, \sqrt{2}\right)$	$f_{min} = -8$ at $\left(\sqrt{2}, -\sqrt{2}\right)$ & $\left(-\sqrt{2}, \sqrt{2}\right)$
10.	$f(x, y) = 2(x^2 - y^2) - x^4 + y^4$	$(\pm 1,0), (0, \pm 1)$	$f_{max} = 1$ at $(\pm 1,0)$ $f_{min} = -1$ at $(0, \pm 1)$
11.	$f(x, y) = x^3 + 3x^2 + y^2 + 4xy.$		No Extreme values
12.	$f(x, y) = \sin x + \sin y + \sin(x + y)$	$\left(\frac{\pi}{3}, \frac{\pi}{3}\right), (-\pi, -\pi)$	$f_{max} = \frac{3\sqrt{3}}{2}$ at $\left(\frac{\pi}{3}, \frac{\pi}{3}\right)$
13.	$f(x, y) = \sin x \sin y \sin(x + y)$	$\left(\frac{\pi}{3}, \frac{\pi}{3}\right), (0,0)$	$f_{max} = \frac{3\sqrt{3}}{8}$ at $\left(\frac{\pi}{3}, \frac{\pi}{3}\right)$

2.Lagranges Method of Undermined Multiplier

Lagrange's Method of Undermined Multiplier is very convenient method for finding maximum or minimum values of function of several variables which are connected by given relations.

Working Rule:

Suppose we wish to find the values of x, y and z, for which $f(x, y, z)$ is maximum or minimum

1.Let $f(x, y, z)$ be a function of three variables x, y and z which are connected by the relation $\phi(x, y, z) = 0$.

2.Write an auxiliary function $F(x, y, z) = f(x, y, z) + \lambda\, \phi(x, y, z)$ where λ is an undermined multiplier to be determined.

3. Find $\frac{\partial F}{\partial x}, \frac{\partial F}{\partial y}$ and $\frac{\partial F}{\partial z}$

4. Eliminate λ by solving the equations $\frac{\partial F}{\partial x} = 0, \frac{\partial F}{\partial y} = 0$ and $\frac{\partial F}{\partial z} = 0$

   together with $\phi(x, y, z) = 0$.

The values of x, y, z so obtained will give the stationary (maximum or minimum) value of $f(x, y, z)$.

Note: This method is useful in finding stationary values of given function but it cannot determine nature of stationary points i.e. whether maximum or minimum.

Example 1: Find the maximum value of $f(x,y,z) = x^2y^3z^4$ subject to the condition $x + y + z = 5.$

Solution: Let $f(x,y,z) = xy^2z^3 \dots\dots\dots (1)$

$$x + y + z = 5 \quad \dots\dots\dots (2)$$

$\phi(x,y,z) = x + y + z - 5 = 0$

Consider an auxiliary function, $F(x,y,z) = f(x,y,z) + \lambda\, \phi(x,y,z)$

where λ is an undermined multiplier to be determined.

$$F(x,y,z) = x^2y^3z^4 + \lambda\,(x + y + z - 5)$$

$$\frac{\partial F}{\partial x} = 2xy^3z^4 + \lambda$$

$$\frac{\partial F}{\partial y} = 3x^2y^2z^4 + \lambda$$

$$\frac{\partial F}{\partial z} = 4x^2y^3z^3 + \lambda$$

$$\frac{\partial F}{\partial x} = 0 \Rightarrow 2xy^3z^4 + \lambda = 0 \Rightarrow \lambda = -2xy^3z^4 \dots\dots\dots (3)$$

$$\frac{\partial F}{\partial y} = 0 \Rightarrow 3x^2y^2z^4 + \lambda = 0 \Rightarrow \lambda = -3x^2y^2z^4 \dots\dots\dots (4)$$

$$\frac{\partial F}{\partial z} = 0 \Rightarrow 4x^2y^3z^3 + \lambda = 0 \Rightarrow \lambda = -4x^2y^3z^3 \dots\dots\dots (5)$$

From (3), (4) and (5)

$$-2xy^3z^4 = -3x^2y^2z^4 = -4x^2y^3z^3$$

$$\Rightarrow 2xy^3z^4 = 3x^2y^2z^4 = 4x^2y^3z^3$$

Dividing all by $x^2y^3z^4$ we get

$$\frac{2}{x} = \frac{3}{y} = \frac{4}{z} \Rightarrow y = \frac{3x}{2} \text{ and } z = 2x$$

$$\therefore x + y + z = 5 \Longrightarrow x + \frac{3x}{2} + 2x = 5$$

$$\Longrightarrow \frac{9x}{2} = 5 \quad \Longrightarrow x = \frac{10}{9}$$

$$\Longrightarrow y = \frac{3x}{2} = \frac{3}{2}\frac{10}{9} = \frac{15}{9}, \qquad z = 2x = \frac{20}{9}$$

Maximum value of $x^2y^3z^4 = \left(\frac{10}{9}\right)^2\left(\frac{15}{9}\right)^3\left(\frac{20}{9}\right)^4 = \frac{2^{10}\times 5^9}{3^{15}}$

Example 2: Divide 24 into three parts such that continued product of first, square of second and cube of third may be maximum.

Solution: Let x, y, z be three parts of 24 $\Longrightarrow$ $x + y + z = 24$

Continued product of first, square of second and cube of third $= xy^2z^3$

Let $f(x, y, z) = xy^2z^3$

We have to find maximum value of $f(x, y, z) = xy^2z^3$ subject to the condition $\phi(x, y, z) = x + y + z - 24 = 0$

Consider an auxiliary function $F(x, y, z) = f(x, y, z) + \lambda\, \phi(x, y, z)$

$$F(x, y, z) = xy^2z^3 + \lambda\,(x + y + z - 24)$$

$$\frac{\partial F}{\partial x} = y^2z^3 + \lambda$$

$$\frac{\partial F}{\partial y} = 2xyz^3 + \lambda$$

$$\frac{\partial F}{\partial z} = 3xy^2z^2 + \lambda$$

$$\frac{\partial F}{\partial x} = 0 \Longrightarrow y^2z^3 + \lambda = 0 \ \Longrightarrow \lambda = -y^2z^3 \ldots\ldots\ldots (1)$$

$$\frac{\partial F}{\partial y} = 0 \Longrightarrow 2xyz^3 + \lambda = 0 \ \Longrightarrow \lambda = -2xyz^3 \ldots\ldots\ldots (2)$$

$$\frac{\partial F}{\partial z} = 0 \Longrightarrow 3xy^2z^2 + \lambda = 0 \Longrightarrow \lambda = -3xy^2z^2 \ldots\ldots\ldots (3)$$

From (1), (2) and (3)

$$-y^2z^3 = -2xyz^3 = -3xy^2z^2 \Longrightarrow y^2\ z^3 = 2xyz^3 = 3xy^2z^2$$

Dividing all by xy^2z^3 we get,

$$\frac{1}{x} = \frac{2}{y} = \frac{3}{z}$$

$$\Longrightarrow y = 2x \ \ \& \ \ z = 3x$$

$\therefore x + y + z = 24 \Longrightarrow x + 2x + 3x = 24$

$\Longrightarrow 6x = 24 \;\; \Longrightarrow x = 4$

$\therefore x = 4 \;\; \Longrightarrow y = 2x = 8\,, \qquad z = 3x = 12$

So by dividing 24 into three parts namely 4, 8 and 12 continued product of first, square of second and cube of third will be maximum

Maximum value of continued product is $xy^2z^3 = 4(8)^2(12)^3 = 442368$

.

Example 3: Sum of three numbers is constant. Prove that their product is maximum when they are equal.

Solution : Let x, y, z be three numbers such that $x + y + z = a$

Let $f(x, y, z) = xyz$

We have to find maximum value of $f(x, y, z) = xyz$ subject to the condition $\phi(x, y, z) = x + y + z - a = 0$

Consider an auxiliary function $F(x, y, z) = f(x, y, z) + \lambda\, \phi(x, y, z)$

$$F(x, y, z) = xyz + \lambda\,(x + y + z - a)$$

$$\frac{\partial F}{\partial x} = yz + \lambda$$

$$\frac{\partial F}{\partial y} = xz + \lambda$$

$$\frac{\partial F}{\partial z} = xy + \lambda$$

$$\frac{\partial F}{\partial x} = 0 \Longrightarrow yz + \lambda = 0 \ \Longrightarrow \lambda = -yz \ldots\ldots\ldots (1)$$

$$\frac{\partial F}{\partial y} = 0 \Longrightarrow xz + \lambda = 0 \ \Longrightarrow \lambda = -xz \ldots\ldots\ldots (2)$$

$$\frac{\partial F}{\partial z} = 0 \Longrightarrow xy + \lambda = 0 \Longrightarrow \lambda = -xy \ldots\ldots\ldots (3)$$

From (1), (2) and (3)

$$-yz = -xz = -xy \Longrightarrow yz = xz = xy$$

Dividing all by xyz we get

$$\frac{1}{x} = \frac{1}{y} = \frac{1}{z}$$

$$\Longrightarrow x = y = z$$

$$\therefore x + y + z = a \Longrightarrow x + x + x = a \Longrightarrow 3x = a \Longrightarrow x = \frac{a}{3}$$

$$\Longrightarrow x = y = z = \frac{a}{3}$$

So Maximum value of product is obtained when $x = y = z$

Maximum value of product is $xyz = \frac{a}{3}\frac{a}{3}\frac{a}{3} = \frac{a^3}{27}$

Example 4. Find points on the surface $z^2 = xy + 1$ nearest to the origin.

Solution : Let (x, y, z) be any point on the surface $z^2 = xy + 1$.

Distance of the point (x, y, z) from origin is

$d = \sqrt{(x-0)^2 + (y-0)^2 + (z-0)^2} = \sqrt{x^2 + y^2 + z^2}$

$d^2 = x^2 + y^2 + z^2$

Let $f(x, y, z) = x^2 + y^2 + z^2$

We have to find minimum value of $f(x, y, z) = x^2 + y^2 + z^2$ subject to the condition $\phi(x, y, z) = z^2 - xy - 1 = 0$

Let $F(x, y, z) = f(x, y, z) + \lambda\, \phi(x, y, z)$

$$F(x, y, z) = x^2 + y^2 + z^2 + \lambda\, (z^2 - xy - 1)$$

$$\frac{\partial F}{\partial x} = 2x - \lambda y, \qquad \frac{\partial F}{\partial y} = 2y - \lambda x \quad \text{and} \quad \frac{\partial F}{\partial z} = 2z + 2\lambda z$$

$$\frac{\partial F}{\partial x} = 0 \Longrightarrow 2x - \lambda y = 0 \ \Longrightarrow \lambda = \frac{2x}{y} \dots \dots \dots (1)$$

$$\frac{\partial F}{\partial y} = 0 \Longrightarrow 2y - \lambda x = 0 \ \Longrightarrow \lambda = \frac{2y}{x} \dots \dots \dots (2)$$

$$\frac{\partial F}{\partial z} = 0 \Longrightarrow 2z + 2\lambda z = 0 \Longrightarrow z(1 + \lambda) = 0 \Longrightarrow 1 + \lambda = 0 \Longrightarrow \lambda = -1$$

Putting value of λ in (1) and (2),

$$\frac{2x}{y} = -1 \Longrightarrow 2x + y = 0 \qquad \text{and} \qquad \frac{2y}{x} = -1 \Longrightarrow x + 2y = 0$$

$$2x + y = 0 \ \text{ and } x + 2y = 0 \Longrightarrow x = 0, y = 0$$

$$\Longrightarrow z^2 = xy + 1 = 0 + 1 = 1$$

$$\Longrightarrow z = \pm 1$$

Points of the surface $z^2 = xy + 1$ nearest to origin are $(0,0,1)$ and $(0,0,-1)$

Example 5. Find minimum distance of the plane $3x + 2y + z = 12$ from origin.

Solution: Let (x, y, z) be any point on the plane $3x + 2y + z = 12$.

Distance of the point (x, y, z) from origin is

$d = \sqrt{(x-0)^2 + (y-0)^2 + (z-0)^2} = \sqrt{x^2 + y^2 + z^2}$

$d^2 = x^2 + y^2 + z^2$

Let $f(x, y, z) = x^2 + y^2 + z^2$

We have to find minimum value of $f(x, y, z) = x^2 + y^2 + z^2$ subject to the condition $\phi(x, y, z) = 3x + 2y + z - 12 = 0$

Let $F(x, y, z) = f(x, y, z) + \lambda\, \phi(x, y, z)$

$$F(x, y, z) = x^2 + y^2 + z^2 + \lambda\,(3x + 2y + z - 12)$$

$$\frac{\partial F}{\partial x} = 2x + 3\lambda, \qquad \frac{\partial F}{\partial y} = 2y + 2\lambda \quad \text{and} \quad \frac{\partial F}{\partial z} = 2z + \lambda$$

$$\frac{\partial F}{\partial x} = 0 \Longrightarrow 2x + 3\lambda = 0 \Longrightarrow \lambda = -\frac{2x}{3} \dots \dots \dots (1)$$

$$\frac{\partial F}{\partial y} = 0 \Longrightarrow 2y + 2\lambda = 0 \Longrightarrow \lambda = -y \dots \dots \dots (2)$$

$$\frac{\partial F}{\partial z} = 0 \Longrightarrow 2z + \lambda = 0 \Longrightarrow \lambda = -2z \dots \dots \dots (3)$$

From (1), (2) and (3),

$$-\frac{2x}{3} = -y = -2z \Longrightarrow \frac{2x}{3} = y = 2z$$

$$\Longrightarrow y = \frac{2x}{3}, \qquad z = \frac{x}{3}$$

$$3x + 2y + z = 12 \Longrightarrow 3x + 2\left(\frac{2x}{3}\right) + \frac{x}{3} = 12$$

$$\Longrightarrow \frac{14x}{3} = 12$$

$$\Longrightarrow 14x = 36 \Longrightarrow x = \frac{36}{14} = \frac{18}{7}$$

$$\Longrightarrow y = \frac{2x}{3} = \frac{2}{3} \times \frac{18}{7} = \frac{12}{7}, \qquad z = \frac{x}{3} = \frac{1}{3} \times \frac{18}{7} = \frac{6}{7}$$

Minimum distance of the plane $3x + 2y + z = 12$ from origin is

$$d = \sqrt{x^2 + y^2 + z^2} = \sqrt{\left(\frac{18}{7}\right)^2 + \left(\frac{12}{7}\right)^2 + \left(\frac{6}{7}\right)^2} = \sqrt{\frac{324 + 144 + 36}{49}}$$

$$d = \sqrt{\frac{504}{49}} = \sqrt{\frac{36 \times 14}{49}}$$

$$d = \frac{6\sqrt{14}}{7}$$

Example 6. In triangle ΔABC, show that value of $\cos A \cos B \cos C$ is maximum when triangle is equilateral.

Solution: Note that $A + B + C = \pi$

Let $f(A, B, C) = \cos A \cos B \cos C$

We have to find maximum value of $f(A, B, C) = \cos A \cos B \cos C$ subject to the condition $\phi(A, B, C) = A + B + C - \pi = 0$

Let $F(A, B, C) = f(A, B, C) + \lambda\, \phi(A, B, C)$

$F(A, B, C) = \cos A \cos B \cos C + \lambda\,(A + B + C - \pi)$

$$\frac{\partial F}{\partial A} = -\sin A \cos B \cos C + \lambda,$$

$$\frac{\partial F}{\partial B} = -\cos A \sin B \cos C + \lambda$$

$$\frac{\partial F}{\partial C} = -\cos A \cos B \sin C + \lambda$$

$$\frac{\partial F}{\partial A} = 0 \Rightarrow -\sin A \cos B \cos C + \lambda = 0 \Rightarrow \lambda = \sin A \cos B \cos C \ldots \ldots \ldots (1)$$
$$\frac{\partial F}{\partial B} = 0 \Rightarrow -\cos A \sin B \cos C + \lambda = 0 \Rightarrow \lambda = \cos A \sin B \cos C \ldots \ldots \ldots (2)$$
$$\frac{\partial F}{\partial C} = 0 \Rightarrow -\cos A \cos B \sin C + \lambda = 0 \Rightarrow \lambda = \cos A \cos B \sin C \ldots \ldots \ldots (3)$$

From (1), (2) and (3),

$\sin A \cos B \cos C = \cos A \sin B \cos C = \cos A \cos B \sin C$

Dividing all by $\cos A \cos B \cos C$ we get,

$\tan A = \tan B = \tan C$

$\Rightarrow A = B = C = \frac{\pi}{3} \Rightarrow$ Triangle is equilateral.

Maximum value of $\cos A \cos B \cos C = \cos\frac{\pi}{3} \cos\frac{\pi}{3} \cos\frac{\pi}{3} = \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} = \frac{1}{8}$

Example 7. Show that if the perimeter of triangle is constant then triangle has maximum area when it is equilateral.

Solution: Let x, y and z be the sides of the triangle.

Perimeter of the triangle is, $s = \dfrac{x+y+z}{2}$

Area of the triangle $A = \sqrt{s(s-x)(s-y)(s-z)}$

$$A^2 = s(s-x)(s-y)(s-z)$$

Let $f(x,y,z) = A^2 = s(s-x)(s-y)(s-z)$

Let $\phi(x,y,z) = x + y + z - 2s = 0.$

We have to find maximum value of $f(x,y,z) = s(s-x)(s-y)(s-z)$

subject to the condition $\phi(x,y,z) = x + y + z - 2s = 0$

Consider an auxiliary function, $F(x,y,z) = f(x,y,z) + \lambda\, \phi(x,y,z)$

$$\therefore F(x, y, z) = s(s-x)(s-y)(s-z) + \lambda\,(x+y+z-2s)$$

$$\frac{\partial F}{\partial x} = -s(s-y)(s-z) + \lambda$$

$$\frac{\partial F}{\partial y} = -s(s-x)(s-z) + \lambda$$

$$\frac{\partial F}{\partial z} = -s(s-x)(s-y) + \lambda$$

$$\frac{\partial F}{\partial x} = 0 \Longrightarrow -s(s-y)(s-z) + \lambda = 0 \;\Longrightarrow \lambda = s(s-y)(s-z) \ldots\ldots\ldots (1)$$

$$\frac{\partial F}{\partial y} = 0 \Longrightarrow -s(s-x)(s-z) + \lambda = 0 \;\Longrightarrow \lambda = s(s-x)(s-z) \ldots\ldots\ldots (2)$$

$$\frac{\partial F}{\partial z} = 0 \Longrightarrow -s(s-x)(s-y) + \lambda = 0 \Longrightarrow \lambda = s(s-x)(s-y) \ldots\ldots\ldots (3)$$

From (1), (2) and (3),

$$s(s-y)(s-z) = s(s-x)(s-z) = s(s-x)(s-y)$$

$$s(s-y)(s-z) = s(s-x)(s-z) \quad \text{and} \quad s(s-x)(s-z) = s(s-x)(s-y)$$

$$\Longrightarrow s-y = s-x \quad \text{and} \quad s-z = s-y$$

$$\Longrightarrow y = x \quad \text{and} \quad z = y$$

$$\Longrightarrow x = y = z$$

Hence, the triangle is equilateral.

Example 8. Show that the rectangular solid of maximum volume that can be inscribed in a sphere is a cube.

Solution: Let x, y and z be the length, breadth and height of the rectangular solid.

Volume of solid is, $V = xyz.$

Equation of sphere is $x^2 + y^2 + z^2 = a^2$

Let $f(x, y, z) = V = xyz$

Let $\phi(x, y, z) = x^2 + y^2 + z^2 - a^2 = 0.$

Consider an auxiliary function, $F(x, y, z) = f(x, y, z) + \lambda\,\phi(x, y, z)$

$\therefore F(x, y, z) = xyz + \lambda\,(x^2 + y^2 + z^2 - a^2)$

$$\frac{\partial F}{\partial x} = yz + 2x\lambda$$

$$\frac{\partial F}{\partial y} = xz + 2y\lambda \qquad and \qquad \frac{\partial F}{\partial z} = xy + 2z\lambda$$

$$\frac{\partial F}{\partial x} = 0 \Longrightarrow yz + 2x\lambda = 0 \ \Longrightarrow \lambda = -\frac{yz}{2x} \ldots\ldots\ldots(1)$$

$$\frac{\partial F}{\partial y} = 0 \Longrightarrow xz + 2y\lambda = 0 \ \Longrightarrow \lambda = -\frac{xz}{2y} \ldots\ldots\ldots(2)$$

$$\frac{\partial F}{\partial z} = 0 \Longrightarrow xy + 2z\lambda = 0 \ \Longrightarrow \lambda = -\frac{xy}{2z} \ldots\ldots\ldots(3)$$

From (1), (2) and (3),

$$-\frac{yz}{2x} = -\frac{xz}{2y} = -\frac{xy}{2z} \Longrightarrow \frac{x}{yz} = \frac{y}{xz} = \frac{z}{xy}$$

Multiplying all by xyz we get,

$$\Longrightarrow x^2 = y^2 = z^2 \Longrightarrow x = y = z$$

Hence, the rectangular solid is a cube.

Test Your Knowledge

1. Find the maximum value of $x^2y^3z^4$ subject to the condition $x+y+z=5$.
2. Find the maximum value of x^2yz^3 subject to the condition $2x+y+3z=a$.
3. Prove that the stationary value of $x^my^nz^p$ under the condition

$x+y+z=a$ is $m^mn^np^p\left(\dfrac{a}{m+n+p}\right)^{m+n+p}$

4. If $u=\dfrac{x^2}{a^3}+\dfrac{y^2}{b^3}+\dfrac{z^2}{c^3}$ where $x+y+z=1$ then prove that the stationary value

of u is given by $x=\dfrac{a^3}{a^3+b^3+c^3}, y=\dfrac{b^3}{a^3+b^3+c^3}, z=\dfrac{c^3}{a^3+b^3+c^3}$

5. Show that the stationary value of $a^3x^2+b^3y^2+c^3z^2$, where $\dfrac{1}{x}+\dfrac{1}{y}+\dfrac{1}{z}=1$

is given by $x=\dfrac{a+b+c}{a}, y=\dfrac{a+b+c}{b}, z=\dfrac{a+b+c}{c}$

6.Divide 120 into three parts such that sum of their products taken two at a time shall be maximum.

7.Find minimum distance from the point $\left(0,0,\frac{25}{9}\right)$ to the surface $z = xy$. **Ans:** $\frac{\sqrt{41}}{3}$

8.The temperature $u(x, y, z)$ at any point in space is $u = 400\,xyz^2$. Find the highest temperature on surface of the sphere $x^2 + y^2 + z^2 = 1$. **Ans:** 50

9.A space probe in the shape of the ellipsoid $4x^2 + y^2 + 4z^2 = 16$ enters the earth's atmosphere and its surface begins to heat. After 1 hour, the temperature at point (x, y, z) on the surface of the probe $T(x, y, z) = 4x^2 + 4yz - 16z + 600$. Find the hottest points on the probe's surface. **Ans:** $\left(\pm\frac{4}{3}, -\frac{4}{3}, -\frac{4}{3}\right)$

10. Find a point in the plane $x + 2y + 3z = 13$ nearest to the point $(1,1,1)$.

3. Jacobian

Let $u = f(x, y)$ and $v = g(x, y)$ be two functions of x and y.

Then Jacobian of u, v with respect to x, y is given by $J = \frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix}$

Similarly if $u = f(x, y, z)$, $v = g(x, y, z)$ and $w = h(x, y, z)$ are three functions of x , y and z. Then Jacobian of u, v, w with respect to x, y, z is given by

$$J = \frac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix}$$

Types of Problems on Jacobians

Jacobian of Explicit functions

$u = f(x,y)$, $v = g(x,y)$

Then find

$$J = \frac{\partial(u,v)}{\partial(x,y)}$$

by applying definition of Jacobian

To verify $JJ' = 1$

Find

$$J = \frac{\partial(u,v)}{\partial(x,y)}$$

Write down x & y in terms of u & v.

Then find

$$J' = \frac{\partial(x,y)}{\partial(u,v)}$$

Verify $JJ' = 1$

Jacobian of Composite functions

$u = f(x,y)$, $v = g(x,y)$ & $x = \phi(r,s)$, $y = \psi(r,s)$

$$\frac{\partial(u,v)}{\partial(r,s)} = \frac{\partial(u,v)}{\partial(x,y)} \cdot \frac{\partial(x,y)}{\partial(r,s)}$$

Functional Dependence

Find

$$J = \frac{\partial(u,v)}{\partial(x,y)}$$

$J = 0$ then F.D

Find relation between u & v

Jacobian of Implicit functions

$f_1(u,v,x,y) = 0$

$f_2(u,v,x,y) = 0$

$$\frac{\partial(u,v)}{\partial(x,y)} = (-1)^2 \frac{\dfrac{\partial(f_1,f_2)}{\partial(x,y)}}{\dfrac{\partial(f_1,f_2)}{\partial(u,v)}}.$$

Apply definition of Jacobian for numerator & denominator

Partial derivatives using Jacobian

$$\frac{\partial u}{\partial x} = -\frac{\dfrac{\partial(f_1,f_2)}{\partial(x,v)}}{\dfrac{\partial(f_1,f_2)}{\partial(u,v)}}.$$

Apply definition of Jacobian for numerator & denominator

Jacobian of Explicit functions

e.g. 1) $u = x^2 - y^2$ and $v = 2xy$ then

$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} = \begin{vmatrix} 2x & -2y \\ 2y & 2x \end{vmatrix} = 4x^2 + 4y^2$$

e.g. 2) $x = r\cos\theta$ and $y = r\sin\theta$ then

$$\frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r$$

e.g. 3) $u = x\sin y$ and $v = y\sin x$ then

$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} = \begin{vmatrix} \sin y & x\cos y \\ y\cos x & \sin x \end{vmatrix} = \sin x \sin y - xy\cos x\cos y$$

e.g. 4) $u = \dfrac{x+y}{1-xy}$ and $v = \tan^{-1}x + \tan^{-1}y$ then

$$\frac{\partial u}{\partial x} = \frac{(1-xy)(1)-(x+y)(-y)}{(1-xy)^2} = \frac{1-xy+xy+y^2}{(1-xy)^2} = \frac{1+y^2}{(1-xy)^2}$$

$$\frac{\partial u}{\partial y} = \frac{(1-xy)(1)-(x+y)(-x)}{(1-xy)^2} = \frac{1-xy+xy+x^2}{(1-xy)^2} = \frac{1+x^2}{(1-xy)^2}$$

$$\frac{\partial v}{\partial x} = \frac{1}{1+x^2} \quad \text{and} \quad \frac{\partial v}{\partial y} = \frac{1}{1+y^2}$$

$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} = \begin{vmatrix} \frac{1+y^2}{(1-xy)^2} & \frac{1+x^2}{(1-xy)^2} \\ \frac{1}{1+x^2} & \frac{1}{1+y^2} \end{vmatrix}$$

$$= \frac{1+y^2}{(1-xy)^2}\frac{1}{1+y^2} - \frac{1+x^2}{(1-xy)^2}\frac{1}{1+x^2}$$

$$= \frac{1}{(1-xy)^2} - \frac{1}{(1-xy)^2}$$

$$= 0$$

e.g.5) $u = xyz, \;\; v = x^2 + y^2 + z^2, \;\; w = x + y + z$ then

$$J = \frac{\partial(u, v, w)}{\partial(x, y, z)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix}$$

By $C_1 - \; C_3$ and $C_2 - \; C_3$

$$J = 2 \begin{vmatrix} yz - xy & xz - xy & xy \\ x - z & y - z & 2z \\ 0 & 0 & 1 \end{vmatrix}$$

e.g.5) $u = xyz, \;\; v = x^2 + y^2 + z^2, \;\; w = x + y + z$ then

$$J = \frac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix} = \begin{vmatrix} yz & xz & xy \\ 2x & 2y & 2z \\ 1 & 1 & 1 \end{vmatrix}$$

By $C_1 - C_3$ and $C_2 - C_3$

$$J = 2\begin{vmatrix} yz - xy & xz - xy & xy \\ x - z & y - z & 2z \\ 0 & 0 & 1 \end{vmatrix}$$

$$J = 2\begin{vmatrix} y(z-x) & x(z-y) & xy \\ x-z & y-z & 2z \\ 0 & 0 & 1 \end{vmatrix}$$

$$J = -2\begin{vmatrix} 0 & 0 & 1 \\ x-z & y-z & 2z \\ y(z-x) & x(z-y) & xy \end{vmatrix}$$

$$= -2[x(x-z)(z-y) - y(z-x)(y-z)]$$

$$= -2(x-z)(z-y)[x - y(-1)(-1)]$$
$$= -2(x-y)(y-z)(z-x)$$

e.g.6) $u = \dfrac{yz}{x}, v = \dfrac{zx}{y}$ and $w = \dfrac{xy}{z}$ then

$$J = \frac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} \dfrac{\partial u}{\partial x} & \dfrac{\partial u}{\partial y} & \dfrac{\partial u}{\partial z} \\ \dfrac{\partial v}{\partial x} & \dfrac{\partial v}{\partial y} & \dfrac{\partial v}{\partial z} \\ \dfrac{\partial w}{\partial x} & \dfrac{\partial w}{\partial y} & \dfrac{\partial w}{\partial z} \end{vmatrix} = \begin{vmatrix} -\dfrac{yz}{x^2} & \dfrac{z}{x} & \dfrac{y}{x} \\ \dfrac{z}{y} & -\dfrac{zx}{y^2} & \dfrac{x}{y} \\ \dfrac{y}{z} & \dfrac{x}{z} & -\dfrac{xy}{z^2} \end{vmatrix}$$

$$= \frac{1}{xyz} \begin{vmatrix} -\dfrac{yz}{x} & z & y \\ z & -\dfrac{zx}{y} & x \\ y & x & -\dfrac{xy}{z} \end{vmatrix}$$

By taking $\dfrac{1}{x}, \dfrac{1}{y}, \dfrac{1}{z}$ common from R_1, R_2, R_3

$$J=\frac{1}{xyz}\begin{vmatrix} -\frac{yz}{x} & z & y \\ z & -\frac{zx}{y} & x \\ y & x & -\frac{xy}{z} \end{vmatrix}$$

$$=\frac{1}{xyz}\left[-\frac{yz}{x}(x^2-x^2)-z(-xy-xy)+y(zx+zx)\right]$$

$$=\frac{1}{xyz}[0+2xyz+2xyz]$$

$$=\frac{1}{xyz}(4xyz)$$

$$=4$$

e.g.7) $x = r\sin\theta\cos\phi$, $y = r\sin\theta\sin\phi, z = r\cos\theta$ then

$$\frac{\partial(x,y,z)}{\partial(r,\theta,\phi)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \phi} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \phi} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial \phi} \end{vmatrix} = \begin{vmatrix} \sin\theta\cos\phi & r\cos\theta\cos\phi & -r\sin\theta\sin\phi \\ \sin\theta\sin\phi & r\cos\theta\sin\phi & r\sin\theta\cos\phi \\ \cos\theta & -r\sin\theta & 0 \end{vmatrix}$$

$$= \sin\theta\cos\phi\,[0 + r^2\sin^2\theta\cos\phi] - r\cos\theta\cos\phi\,[0 - r\sin\theta\cos\theta\cos\phi]$$

$$-r\sin\theta\sin\phi[-r\sin^2\theta\sin\phi - r\cos^2\theta\sin\phi]$$

$$= r^2\sin^3\theta\cos^2\phi + r^2\sin\theta\cos^2\theta\cos^2\phi + r^2\sin^3\theta\sin^2\phi + r^2\sin\theta\cos^2\theta\sin^2\phi$$

$$= r^2\sin^3\theta + r^2\sin\theta\cos^2\theta$$

$$= r^2\sin\theta$$

Test Your Knowledge

Q. Find the Jacobian $\dfrac{\partial(u,v)}{\partial(x,y)}$ for each of the following functions:

1) $u = x + y$ and $v = x - y$ $\quad$ $\boldsymbol{Ans}: -1/2$

2) $u = x^2$ and $v = y^2$ $\quad$ $\boldsymbol{Ans}: 4xy$

3) $u = 3x + 5y$ and $v = 4x - 3y$ $\quad$ $\boldsymbol{Ans}: -29$

4) $u = x + \dfrac{y^2}{x}$ and $v = \dfrac{y^2}{x}$ $\quad$ $\boldsymbol{Ans}: \dfrac{2y}{x}$

5) $u = \dfrac{y^2}{x}$ and $v = \dfrac{x^2}{y}$ $\quad$ $\boldsymbol{Ans}: -3$

6) $u = \dfrac{y^2}{2x}$ and $v = \dfrac{x^2 + y^2}{2x}$ $\quad$ $\boldsymbol{Ans}: \dfrac{-y}{2x}$

7) $u = \dfrac{x - y}{1 + xy}$ and $v = \tan^{-1}x - \tan^{-1}y$ $\quad$ $\boldsymbol{Ans}: 0$

8) $u = x(1+y)$ and $v = y(1+x)$ **Ans**: $1+x+y$

9) $u = xy$ and $v = \dfrac{x+y}{x-y}$ **Ans**: $\dfrac{(x-y)^2}{4xy}$

10) $u = \dfrac{2yz}{x}, v = \dfrac{3zx}{y}$ and $w = \dfrac{4xy}{z}$ **Ans**: $\dfrac{1}{96}$

11) $u = \dfrac{x}{y-z}, v = \dfrac{y}{z-x}$ and $w = \dfrac{z}{x-y}$ **Ans**: 0

12) $u = x(1-y), v = xy(1-z)$ and $w = xyz$ **Ans**: x^2y

13) $u = xyz, v = xy+yz+zx$ and $w = x+y+z$ **Ans**: $(x-y)(y-z)(z-x)$

14) $u = x^2, v = \sin y$ and $w = e^{-3z}$ **Ans**: $-6xe^{-3z}\cos y$

15) $u = \dfrac{x^2-y^2}{2}, v = xy$ and $w = z$ **Ans**: $\dfrac{1}{x^2+y^2}$

Properties of Jacobian

Property No.1: $JJ' = 1$

1. If $u=f(x,y)$ and $v=g(x,y)$ be two functions of x and y and if $\frac{\partial(u,v)}{\partial(x,y)} \neq 0$

then $\frac{\partial(u,v)}{\partial(x,y)} \cdot \frac{\partial(x,y)}{\partial(u,v)} = 1$

$i.e.$ if $J = \frac{\partial(u,v)}{\partial(x,y)}$ and $J' = \frac{\partial(x,y)}{\partial(u,v)}$ then $JJ' = 1$

2. If $u = f(x,y,z)$, $v = g(x,y,z)$ and $w = h(x,y,z)$ be three functions of x, y and z then

$$\frac{\partial(u,v,w)}{\partial(x,y,z)} \cdot \frac{\partial(x,y,z)}{\partial(u,v,w)} = 1$$

Example : Verify $JJ' = 1$ for the following functions.

1) $x = r\cos\theta\,, y = r\sin\theta$

2) $u = e^x$ secy , $v = e^x$ tany

3) $u = x(1-y), \qquad v = xy$

4) $u = x^2, v = \sin y, \qquad w = e^z$

Solution:

1) $x = r\cos\theta\,, y = r\sin\theta$

$$J = \frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \dfrac{\partial x}{\partial r} & \dfrac{\partial x}{\partial \theta} \\ \dfrac{\partial y}{\partial r} & \dfrac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r$$

Now to find $J' = \dfrac{\partial(r,\theta)}{\partial(x,y)}$, we need to express r and θ in terms of x and y.

$x^2 + y^2 = r^2\cos^2\theta + r^2\sin^2\theta = r^2$

$$\Longrightarrow \boldsymbol{r = \sqrt{x^2 + y^2}}$$

$$\frac{\partial r}{\partial x} = \frac{2x}{2\sqrt{x^2 + y^2}} = \frac{x}{\sqrt{x^2 + y^2}}$$

$$\frac{\partial r}{\partial y} = \frac{2y}{2\sqrt{x^2 + y^2}} = \frac{y}{\sqrt{x^2 + y^2}}$$

Also $\frac{y}{x} = \frac{r\sin\theta}{r\cos\theta} = \tan\theta \Longrightarrow \boldsymbol{\theta = \tan^{-1}\left(\frac{y}{x}\right)}$

$$\frac{\partial \theta}{\partial x} = \frac{1}{1 + \left(\frac{y}{x}\right)^2}\left(-\frac{y}{x^2}\right) = \frac{-y}{x^2 + y^2}$$

$$\frac{\partial \theta}{\partial y} = \frac{1}{1 + \left(\frac{y}{x}\right)^2}\left(\frac{1}{x}\right) = \frac{x}{x^2 + y^2}$$

$$J' = \frac{\partial(r,\theta)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial r}{\partial x} & \frac{\partial r}{\partial y} \\ \frac{\partial \theta}{\partial x} & \frac{\partial \theta}{\partial y} \end{vmatrix} = \begin{vmatrix} \frac{x}{\sqrt{x^2+y^2}} & \frac{y}{\sqrt{x^2+y^2}} \\ \frac{-y}{x^2+y^2} & \frac{x}{x^2+y^2} \end{vmatrix}$$

$$= \frac{x^2}{(x^2+y^2)\sqrt{x^2+y^2}} + \frac{y^2}{(x^2+y^2)\sqrt{x^2+y^2}}$$

$$= \frac{x^2+y^2}{(x^2+y^2)\sqrt{x^2+y^2}}$$

$$= \frac{1}{\sqrt{x^2+y^2}}$$

$$= \frac{1}{\sqrt{r^2}} = \frac{1}{r}$$

$$\therefore \; \boldsymbol{JJ'} = r.\frac{1}{r} = \mathbf{1}$$

2) $u = e^x \sec y$, $v = e^x \tan y$

$$J = \frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} = \begin{vmatrix} e^x \sec y & e^x \sec y \tan y \\ e^x \tan y & e^x \sec^2 y \end{vmatrix}$$

$$= e^{2x} \sec^3 y - e^{2x} \sec y \tan^2 y$$

$$= e^{2x} \sec y \left(\sec^2 y - \tan^2 y\right)$$

$$= e^{2x} \sec y$$

Now to find $J' = \frac{\partial(x,y)}{\partial(u,v)}$, we need to express x and y in terms of u and v.

$$u^2 - v^2 = e^{2x} \sec^2 y - e^{2x} \tan^2 y = e^{2x}\left(\sec^2 y - \tan^2 y\right) = e^{2x}$$

$$u^2 - v^2 = e^{2x} \Rightarrow \boldsymbol{x = \frac{1}{2} \log(u^2 - v^2)}$$

$$\boldsymbol{x = \frac{1}{2}\log(u^2 - v^2)} \Longrightarrow \frac{\partial x}{\partial u} = \frac{1}{2}\frac{2u}{u^2 - v^2} = \frac{u}{u^2 - v^2}$$

$$\frac{\partial x}{\partial v} = \frac{1}{2}\frac{-2v}{u^2 - v^2} = \frac{-v}{u^2 - v^2}$$

$$\frac{v}{u} = \frac{e^x \tan y}{e^x \sec y} = \cos y \tan y = \sin y$$

$$\Longrightarrow \boldsymbol{y = \sin^{-1}\left(\frac{v}{u}\right)}$$

$$\Longrightarrow \frac{\partial y}{\partial u} = \frac{1}{\sqrt{1 - \frac{v^2}{u^2}}}\left(-\frac{v}{u^2}\right) = \frac{-v}{u}\frac{1}{\sqrt{u^2 - v^2}}$$

$$\frac{\partial y}{\partial v} = \frac{1}{\sqrt{1 - \frac{v^2}{u^2}}}\left(\frac{1}{u}\right) = \frac{1}{\sqrt{u^2 - v^2}}$$

$$J' = \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} \frac{u}{u^2 - v^2} & \frac{-v}{u^2 - v^2} \\ \frac{-v}{u} \frac{1}{\sqrt{u^2 - v^2}} & \frac{1}{\sqrt{u^2 - v^2}} \end{vmatrix}$$

$$J' = \frac{u}{(u^2 - v^2)\sqrt{u^2 - v^2}} - \frac{v^2}{u(u^2 - v^2)\sqrt{u^2 - v^2}}$$

$$= \frac{1}{(u^2 - v^2)\sqrt{u^2 - v^2}} \left[u - \frac{v^2}{u} \right]$$

$$= \frac{1}{(u^2 - v^2)\sqrt{u^2 - v^2}} \left[\frac{u^2 - v^2}{u} \right]$$

$$J' = \frac{1}{(u^2 - v^2)\sqrt{u^2 - v^2}}\left[\frac{u^2 - v^2}{u}\right]$$

$$= \frac{1}{u\sqrt{u^2 - v^2}}$$

$$= \frac{1}{e^x \sec y \sqrt{e^{2x}}}$$

$$= \frac{1}{e^{2x} \sec y}$$

$$\therefore \boldsymbol{JJ'} = e^{2x} \sec y \,.\, \frac{1}{e^{2x} \sec y} = \mathbf{1}$$

3) $u = x(1-y), \qquad v = xy$

$$J = \frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} = \begin{vmatrix} 1-y & -x \\ y & x \end{vmatrix} = x(1-y) + xy = x$$

Now to find $J' = \frac{\partial(x,y)}{\partial(u,v)}$, we need to express x and y in terms of u & v.

$u = x(1-y) = x - xy = x - v$

$\Longrightarrow \boldsymbol{x = u + v}$

$v = xy = (u+v)y \Longrightarrow \boldsymbol{y = \frac{v}{u+v} = 1 - \frac{u}{u+v}}$

$$J' = \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} \frac{1}{\frac{-v}{(u+v)^2}} & \frac{1}{\frac{u}{(u+v)^2}} \end{vmatrix}$$

$$J' = \frac{u}{(u+v)^2} + \frac{v}{(u+v)^2} = \frac{u+v}{(u+v)^2}$$

$$J' = \frac{1}{u+v} = \frac{1}{x}$$

$$\therefore \boldsymbol{JJ'} = x.\frac{1}{x} = \mathbf{1}$$

4) $u = x^2, v = \sin y, \qquad w = e^z$

$$J = \frac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix} = \begin{vmatrix} 2x & 0 & 0 \\ 0 & \cos y & 0 \\ 0 & 0 & e^z \end{vmatrix} = 2xe^z \cos y$$

Now to find $J' = \dfrac{\partial(x,y,z)}{\partial(u,v,w)}$, we need to express x, y, z in terms of u, v, w.

$u = x^2 \Longrightarrow x = \sqrt{u}$

$v = \sin y \Longrightarrow y = \sin^{-1} v$

$w = e^z \Longrightarrow z = \log w$

$$J' = \frac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix} = \begin{vmatrix} \frac{1}{2\sqrt{u}} & 0 & 0 \\ 0 & \frac{1}{\sqrt{1-v^2}} & 0 \\ 0 & 0 & \frac{1}{w} \end{vmatrix}$$

$$J' = \frac{1}{2\sqrt{u}} \frac{1}{\sqrt{1-v^2}} \frac{1}{w}$$

$$J' = \frac{1}{2x} \frac{1}{\sqrt{1-\sin^2 y}} \frac{1}{e^z} = \frac{1}{2xe^z \cos y}$$

$$\therefore \ \boldsymbol{JJ'} = 2xe^z \cos y \,.\, \frac{1}{2xe^z \cos y} = \mathbf{1}$$

Test Your Knowledge

Q. Verify $JJ' = 1$ for the following functions.

1. $x = e^u \cos v \,, y = e^u \sin v$
2. $x + y = 2e^u \cos v \,, x - y = 2ie^u \sin v$
3. $x = uv, y = \dfrac{u}{v}$
4. $x = u + v, y = uv$
5. $x = \dfrac{u^2 + v^2}{x}, y = \dfrac{v^2}{u}$
6. $x = uv, y = \dfrac{u + v}{u - v}$
7. $x = u \cos v \,, y = u \sin v$
8. $x = \sin u \cos v \,, y = \sin u \sin v$
9. $x = e^y\ \sec u\ \,, y = e^y \tan u$
10. $x = u \,, y = u \tan v,\ z = w$
11. $x = v^2 + w^2, y = w^2 + u^2, z = u^2 + v^2$

Property No.2: Functional Dependence

1. If $u=f(x,y)$ and $v=g(x,y)$ be two functions of x & y then u & v are **functionally dependent** if $\frac{\partial(u,v)}{\partial(x,y)} = 0.$

In such a case there always exists a relation between u & v.

2. If $u = f(x,y,z), v = g(x,y,z)$ & $w = h(x,y,z)$ be three functions of x, y & z then u, v & w are **functionally dependent** if $\frac{\partial(u,v,w)}{\partial(x,y,z)} = 0.$

In such a case there always exists a relation between u, v & w.

Example: Find Jacobian of the following composite functions

1. $u = \dfrac{x+y}{1-xy}, \qquad v = \tan^{-1} x + \tan^{-1} y$

2. $u = \sin^{-1} x + \sin^{-1} y, \qquad v = x\sqrt{1-y^2} + y\sqrt{1-x^2}$

3. $u = \dfrac{x-y}{x+y}, \qquad v = \dfrac{x+y}{x}$

4. $u = \dfrac{x-y}{x+y}, \qquad v = \dfrac{xy}{(x-y)^2}$

5. $u = e^x \sin y, \qquad v = x + \log \sin y$

6. $u = x^2 + y^2 + 2xy + 2x + 2y, \qquad v = e^x e^y$

7. $u = x + y + z, \qquad v = x - y + z, \qquad w = x^2 + y^2 + z^2 + 2xz$

8. $u = y + z, \qquad v = x + 2z^2, \qquad w = x - 4yz - 2y^2$

9. $u = x + y + z, \qquad v = x^2 + y^2 + z^2 - xy - 2yz - 2zx,$
$w = x^3 + y^3 + z^3 - 3xyz$

Solution:

1) $u = \dfrac{x+y}{1-xy}, \qquad v = \tan^{-1} x + \tan^{-1} y$

$$\frac{\partial u}{\partial x} = \frac{(1-xy)(1)-(x+y)(-y)}{(1-xy)^2} = \frac{1-xy+xy+y^2}{(1-xy)^2} = \frac{1+y^2}{(1-xy)^2}$$

$$\frac{\partial u}{\partial y} = \frac{(1-xy)(1)-(x+y)(-x)}{(1-xy)^2} = \frac{1-xy+xy+x^2}{(1-xy)^2} = \frac{1+x^2}{(1-xy)^2}$$

$$\frac{\partial v}{\partial x} = \frac{1}{1+x^2} \quad \text{and} \quad \frac{\partial v}{\partial y} = \frac{1}{1+y^2}$$

$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \dfrac{\partial u}{\partial x} & \dfrac{\partial u}{\partial y} \\ \dfrac{\partial v}{\partial x} & \dfrac{\partial v}{\partial y} \end{vmatrix} = \begin{vmatrix} \dfrac{1+y^2}{(1-xy)^2} & \dfrac{1+x^2}{(1-xy)^2} \\ \dfrac{1}{1+x^2} & \dfrac{1}{1+y^2} \end{vmatrix}$$

$$\frac{\partial(u,v)}{\partial(x,y)} = \frac{1+y^2}{(1-xy)^2}\frac{1}{1+y^2} - \frac{1+x^2}{(1-xy)^2}\frac{1}{1+x^2}$$

$$= \frac{1}{(1-xy)^2} - \frac{1}{(1-xy)^2}$$

$$= 0$$

Therefore u & v are functionally dependent.

Relation between $\boldsymbol{u}$ & $\boldsymbol{v}$:

$$v = \tan^{-1} x + \tan^{-1} y = \tan^{-1}\left(\frac{x+y}{1-xy}\right) = \tan^{-1} u$$

$$\Rightarrow \boldsymbol{u = \tan v}$$

2) $u = \sin^{-1} x + \sin^{-1} y\,,\qquad v = x\sqrt{1-y^2} + y\sqrt{1-x^2}$

$$\frac{\partial u}{\partial x} = \frac{1}{\sqrt{1-x^2}}, \qquad \frac{\partial u}{\partial y} = \frac{1}{\sqrt{1-y^2}}$$

$$\frac{\partial v}{\partial x} = \sqrt{1-y^2} + y\frac{(-2x)}{2\sqrt{1-x^2}} = \sqrt{1-y^2} - \frac{xy}{\sqrt{1-x^2}}$$

$$\frac{\partial v}{\partial y} = x\frac{(-2y)}{2\sqrt{1-y^2}} + \sqrt{1-x^2} = \sqrt{1-x^2} - \frac{xy}{\sqrt{1-y^2}}$$

$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} = \begin{vmatrix} \frac{1}{\sqrt{1-x^2}} & \frac{1}{\sqrt{1-y^2}} \\ \sqrt{1-y^2} - \frac{xy}{\sqrt{1-x^2}} & \sqrt{1-x^2} - \frac{xy}{\sqrt{1-y^2}} \end{vmatrix}$$

$$\frac{\partial(u,v)}{\partial(x,y)} = \frac{1}{\sqrt{1-x^2}}\left[\sqrt{1-x^2} - \frac{xy}{\sqrt{1-y^2}}\right] - \frac{1}{\sqrt{1-y^2}}\left[\sqrt{1-y^2} - \frac{xy}{\sqrt{1-x^2}}\right]$$

$$= 1 - \frac{xy}{\sqrt{1-x^2}\sqrt{1-y^2}} - 1 + \frac{xy}{\sqrt{1-y^2}\sqrt{1-x^2}}$$

$$= 0$$

Therefore u & v are functionally dependent.

Relation between $\boldsymbol{u}$ & $\boldsymbol{v}$:

Let $\alpha = \sin^{-1} x$ & $\beta = \sin^{-1} y \implies x = \sin\alpha$ & $y = \sin\beta$

$$v = x\sqrt{1-y^2} + y\sqrt{1-x^2} = \sin\alpha\sqrt{1-\sin^2\beta} + \sin\beta\sqrt{1-\sin^2\alpha}$$

$$v = \sin\alpha\cos\beta + \sin\beta\cos\alpha = \sin(\alpha+\beta) = \sin u$$

$$\implies \boldsymbol{v = \sin u}$$

3) $u = \dfrac{x-y}{x+y}, \qquad v = \dfrac{x+y}{x} = 1 + \dfrac{y}{x}$

$$\frac{\partial u}{\partial x} = \frac{(x+y)-(x-y)}{(x+y)^2} = \frac{2y}{(x+y)^2}$$

$$\frac{\partial u}{\partial y} = \frac{(x+y)(-1)-(x-y)}{(x+y)^2} = \frac{-x-y-x+y}{(x+y)^2} = \frac{-2x}{(x+y)^2}$$

$$\frac{\partial v}{\partial x} = \frac{-y}{x^2} \qquad \text{and} \qquad \frac{\partial v}{\partial y} = \frac{1}{x}$$

$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \dfrac{\partial u}{\partial x} & \dfrac{\partial u}{\partial y} \\ \dfrac{\partial v}{\partial x} & \dfrac{\partial v}{\partial y} \end{vmatrix} = \begin{vmatrix} \dfrac{2y}{(x+y)^2} & \dfrac{-2x}{(x+y)^2} \\ \dfrac{-y}{x^2} & \dfrac{1}{x} \end{vmatrix}$$

$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \frac{2y}{(x+y)^2} & \frac{-2x}{(x+y)^2} \\ \frac{-y}{x^2} & \frac{1}{x} \end{vmatrix} = \frac{2y}{(x+y)^2}\frac{1}{x} - \frac{2y}{(x+y)^2}\frac{1}{x} = 0$$

Therefore u & v are functionally dependent.

Relation between $\boldsymbol{u}$ & $\boldsymbol{v}$:

$$uv = \frac{x-y}{x+y}\,\frac{x+y}{x} = \frac{x-y}{x} = 1 - \frac{y}{x} \ldots\ldots\ldots\ldots (i)$$

$$v = 1 + \frac{y}{x} \Longrightarrow \frac{y}{x} = v - 1 \ldots\ldots\ldots\ldots (ii)$$

From (i) & (ii), $uv = 1 - (v-1) = 2 - v$

$$\Longrightarrow \boldsymbol{uv + v = 2}$$

4) $u = \dfrac{x-y}{x+y}, \qquad v = \dfrac{xy}{(x-y)^2}$

$$\frac{\partial u}{\partial x} = \frac{(x+y)-(x-y)}{(x+y)^2} = \frac{2y}{(x+y)^2}$$

$$\frac{\partial u}{\partial y} = \frac{(x+y)(-1)-(x-y)}{(x+y)^2} = \frac{-x-y-x+y}{(x+y)^2} = \frac{-2x}{(x+y)^2}$$

$$\frac{\partial v}{\partial x} = \frac{y(x-y)^2-2xy(x-y)}{(x-y)^4} = \frac{y(x-y)-2xy}{(x-y)^3} = \frac{xy-y^2-2xy}{(x-y)^3} = \frac{-y^2-xy}{(x-y)^3}$$

$$\frac{\partial v}{\partial y} = \frac{x(x-y)^2-2xy(x-y)(-1)}{(x-y)^4} = \frac{x(x-y)+2xy}{(x-y)^3} = \frac{x^2-xy+2xy}{(x-y)^3} = \frac{x^2+xy}{(x-y)^3}$$

$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} = \begin{vmatrix} \frac{2y}{(x+y)^2} & \frac{-2x}{(x+y)^2} \\ \frac{-y^2-xy}{(x-y)^3} & \frac{x^2+xy}{(x-y)^3} \end{vmatrix}$$

$$= \frac{2}{(x+y)^2(x-y)^3} \begin{vmatrix} y & -x \\ -y^2-xy & x^2+xy \end{vmatrix}$$

$$= \frac{2}{(x+y)^2(x-y)^3} [y(x^2+xy) + x(-y^2-xy)]$$

$$= \frac{2}{(x+y)^2(x-y)^3} [x^2y + xy^2 - xy^2 - x^2y]$$

$$= 0$$

Therefore u & v are functionally dependent.

Relation between u & v:

$$u = \frac{x-y}{x+y} \Longrightarrow u^2 = \frac{(x-y)^2}{(x+y)^2} \Longrightarrow \frac{1}{u^2} = \frac{(x+y)^2}{(x-y)^2}$$

$$\frac{1}{u^2} - 4v = \frac{(x+y)^2}{(x-y)^2} - \frac{4xy}{(x-y)^2} = \frac{(x+y)^2}{(x-y)^2} - \frac{4xy}{(x-y)^2} = \frac{(x+y)^2 - 4xy}{(x-y)^2}$$

$$= \frac{x^2 + y^2 + 2xy - 4xy}{(x-y)^2} = \frac{x^2 + y^2 - 2xy}{(x-y)^2} = \frac{(x-y)^2}{(x-y)^2} = 1$$

$$\Longrightarrow \frac{1}{u^2} - 4v = 1$$

$$\Longrightarrow \boldsymbol{\frac{1}{u^2} = 1 + 4v}$$

5) $u = e^x \sin y, \qquad v = x + \log \sin y$

$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \dfrac{\partial u}{\partial x} & \dfrac{\partial u}{\partial y} \\ \dfrac{\partial v}{\partial x} & \dfrac{\partial v}{\partial y} \end{vmatrix} = \begin{vmatrix} e^x \sin y & e^x \cos y \\ 1 & \dfrac{\cos y}{\sin y} \end{vmatrix} = e^x \cos y - e^x \cos y = 0$$

Therefore u & v are functionally dependent.

Relation between $\boldsymbol{u}$ & $\boldsymbol{v}$:

$$e^v = e^{x+\log \sin y} = e^x e^{\log \sin y} = e^x \sin y = u$$

$$\Longrightarrow \boldsymbol{u = e^v}$$

6) $u = x^2 + y^2 + 2xy + 2x + 2y, \qquad v = e^x e^y$

$$\frac{\partial u}{\partial x} = 2x + 2y + 2, \qquad \frac{\partial u}{\partial y} = 2y + 2x + 2$$

$$\frac{\partial v}{\partial x} = e^x e^y, \qquad \frac{\partial v}{\partial y} = e^x e^y$$

$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} = \begin{vmatrix} 2x + 2y + 2 & 2y + 2x + 2 \\ e^x e^y & e^x e^y \end{vmatrix} = 0$$

Therefore u & v are functionally dependent.

Relation between $\boldsymbol{u}$ & $\boldsymbol{v}$:

$$u = x^2 + y^2 + 2xy + 2x + 2y = (x+y)^2 + 2(x+y)$$

$$v = e^x e^y = e^{x+y} \Longrightarrow x + y = \log v$$

$$\boldsymbol{\Longrightarrow u = (\log v)^2 + 2\log v}$$

7) $u = x + y + z,\ v = x - y + z,\ w = x^2 + y^2 + z^2 + 2xz$

$$J = \frac{\partial(u, v, w)}{\partial(x, y, z)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ 2x + 2z & 2y & 2z + 2x \end{vmatrix} = 0 \quad (\because C_1 = C_2)$$

Therefore u, v & w are functionally dependent.

Relation between u, v & w:

$u^2 + v^2 = (x + y + z)^2 + (x - y + z)^2$

$= x^2 + y^2 + z^2 + 2(xy + yz + zx) + x^2 + y^2 + z^2 + 2(-xy - yz + zx)$

$= 2x^2 + 2y^2 + 2z^2 + 4xz = 2w$

$\Longrightarrow \boldsymbol{u^2 + v^2 = 2w}$

8) $u = y + z,\ v = x + 2z^2,\ w = x - 4yz - 2y^2$

$$J = \frac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix} = \begin{vmatrix} 0 & 1 & 1 \\ 1 & 0 & 4z \\ 1 & -4z - 4y & -4y \end{vmatrix}$$

$$= -4y - 4z - (-4y - 4z) = 0$$

Therefore u, v & w are functionally dependent.

Relation between $\boldsymbol{u, v}$ & $\boldsymbol{w}$:

$$v - 2u^2 = x + 2z^2 - 2(y+z)^2 = x + 2z^2 - 2(y^2 + z^2 + 2yz)$$

$$= x - 2y^2 - 4yz = w$$

$$\boldsymbol{\Longrightarrow v - 2u^2 = w}$$

$$J = 2\begin{vmatrix} y(z-x) & x(z-y) & xy \\ x-z & y-z & 2z \\ 0 & 0 & 1 \end{vmatrix}$$

$$J = -2\begin{vmatrix} 0 & 0 & 1 \\ x-z & y-z & 2z \\ y(z-x) & x(z-y) & xy \end{vmatrix}$$

$$= -2[x(x-z)(z-y) - y(z-x)(y-z)]$$

$$= -2(x-z)(z-y)[x - y(-1)(-1)]$$
$$= -2(x-y)(y-z)(z-x)$$

$$\frac{\partial(u,v)}{\partial(x,y)} = \frac{1}{\sqrt{1-x^2}}\left[\sqrt{1-x^2} - \frac{xy}{\sqrt{1-y^2}}\right] - \frac{1}{\sqrt{1-y^2}}\left[\sqrt{1-y^2} - \frac{xy}{\sqrt{1-x^2}}\right]$$

$$= 1 - \frac{xy}{\sqrt{1-x^2}\sqrt{1-y^2}} - 1 + \frac{xy}{\sqrt{1-y^2}\sqrt{1-x^2}}$$

$$= 0$$

Therefore u & v are functionally dependent.

Relation between $\boldsymbol{u}$ & $\boldsymbol{v}$:

Let $\alpha = \sin^{-1} x \quad \& \;\; \beta = \sin^{-1} y \;\; \Longrightarrow x = \sin\alpha \;\; \& \;\; y = \sin\beta$

$$v = x\sqrt{1-y^2} + y\sqrt{1-x^2} = \sin\alpha\sqrt{1-\sin^2\beta} + \sin\beta\,\sqrt{1-\sin^2\alpha}$$

$$v = \sin\alpha\cos\beta + \sin\beta\cos\alpha = \sin(\,\alpha+\beta) = \sin u$$

$$\Longrightarrow \boldsymbol{v = \sin u}$$

Test Your Knowledge

Q. Verify whether the following functions are functionally dependent and if so, find the relation between (among) them.

1. $u = e^x \sin y, \qquad v = e^x \cos y$
2. $x = uv, y = \dfrac{u}{v}$
3. $u = \dfrac{y-x}{1+xy}, \qquad v = \tan^{-1} y - \tan^{-1} x$
4. $u = \dfrac{x+y}{x-y}, \qquad v = \dfrac{xy}{(x-y)^2}$
5. $u = \dfrac{x-y}{x+y}, \qquad v = \dfrac{2xy}{(x+y)^2}$
6. $u = \dfrac{x^2-y^2}{x^2+y^2}, \qquad v = \dfrac{2xy}{x^2+y^2}$
7. $u = \sin(x+y), \qquad v = \sin x + \sin y$

Test Your Knowledge

8. $u = x + y + z, \qquad v = x^2 + y^2 + z^2, \qquad w = xy + yz + zx$

9. $u = x + y - z, \qquad v = x - y + z, \qquad w = x^2 + y^2 + z^2 - 2yz$

10. $u = x + y + z, \qquad v = x^2 + y^2 + z^2, \qquad w = x^3 + y^3 + z^3 - 3xyz$

11. $u = x^2 e^{-y} \cos z\,, \qquad v = x^2 e^{-y} \sin z\,, \qquad w = 3x^4 e^{-2y}$

12. $u = \dfrac{3x^2}{2(y+z)}, \qquad v = \dfrac{2(y+z)}{3(x-y)^2}, \qquad w = \dfrac{x-y}{x}$

Answers:

Property No.3: Jacobian of Composite Functions

If $u = f(x,y)$ & $v = g(x,y)$ be two functions of x & y and if $x = \phi(r,s)$ & $y = \psi(r,s)$ then u & v are called composite functions of r & s.

$$\begin{tikzcd}
u, v \arrow[r] \arrow[r, bend right] \arrow[rr, bend left] & x, y \arrow[r] \arrow[r, bend right] & r, s
\end{tikzcd}$$

$$\therefore \frac{\partial(u,v)}{\partial(r,s)} = \frac{\partial(u,v)}{\partial(x,y)} \cdot \frac{\partial(x,y)}{\partial(r,s)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial s} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial s} \end{vmatrix}$$

If $u = f(x, y, z), v = g(x, y, z)$ & $w = h(x, y, z)$ be three functions of x, y & z and if $x = \phi(r, s, t), y = \psi(r, s, t)$ & $z = \chi(r, s, t)$ then u, v & w are called composite functions of r, s & t.

$$\therefore \frac{\partial(u, v, w)}{\partial(r, s, t)} = \frac{\partial(u, v, w)}{\partial(x, y, z)} \cdot \frac{\partial(x, y, z)}{\partial(r, s, t)}$$

$$= \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial w}{\partial z} \\ \frac{\partial u}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial w}{\partial z} \\ \frac{\partial u}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix} \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial s} & \frac{\partial x}{\partial t} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial s} & \frac{\partial y}{\partial t} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial s} & \frac{\partial z}{\partial t} \end{vmatrix}$$

Example: Find Jacobian of the following composite functions.

1. $u = x^2 + y^2, v = 2xy$ and $x = r\cos\theta, y = r\sin\theta$
2. $x = a(u+v), y = b(u-v)$ and $u = r^2\cos 2\theta, v = r^2\sin 2\theta$
3. $u = \sqrt{yz}, v = \sqrt{xz}, w = \sqrt{xy}$ and $x = r\sin\theta\cos\phi,$
   $y = r\sin\theta\sin\phi,\ z = r\cos\theta$

Solution:

1) $u = x^2 + y^2, v = 2xy$ and $x = r\cos\theta, y = r\sin\theta$

Here u & v are composite functions of r & θ.

$$\therefore \frac{\partial(u,v)}{\partial(r,\theta)} = \frac{\partial(u,v)}{\partial(x,y)} \cdot \frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix}$$

$$\frac{\partial(u,v)}{\partial(r,\theta)} = \begin{vmatrix} 2x & 2y \\ 2y & 2x \end{vmatrix} \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix}$$

$$= (4x^2 - 4y^2)(r\cos^2\theta + r\sin^2\theta)$$

$$= 4(r^2\cos^2\theta - r^2\sin^2\theta)(r)$$

$$= 4r^3(\cos^2\theta - \sin^2\theta)$$

$$= 4r^3\cos 2\theta$$

2) $x = a(u+v), y = b(u-v)$ and $u = r^2\cos 2\theta\,, v = r^2\sin 2\theta$

Here x & y are composite functions of r & θ.

$$\frac{\partial(x,y)}{\partial(r,\theta)} = \frac{\partial(x,y)}{\partial(u,v)} \cdot \frac{\partial(u,v)}{\partial(r,\theta)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} \begin{vmatrix} \frac{\partial u}{\partial r} & \frac{\partial u}{\partial \theta} \\ \frac{\partial v}{\partial r} & \frac{\partial v}{\partial \theta} \end{vmatrix}$$

$$= \begin{vmatrix} a & a \\ b & -b \end{vmatrix} \begin{vmatrix} 2r\cos 2\theta & -2r^2\sin 2\theta \\ 2r\sin 2\theta & 2r^2\cos 2\theta \end{vmatrix}$$

$$\frac{\partial(x,y)}{\partial(u,v)} = (-ab - ab)(4r^3\cos^2\theta + 4r^3\sin^2\theta)$$

$$= (-2ab)(4r^3)$$

$$= -8abr^3$$

3) $u = \sqrt{yz}, v = \sqrt{xz}, w = \sqrt{xy}$ and

$x = r\sin\theta\cos\phi$, $y = r\sin\theta\sin\phi$, $z = r\cos\theta$

Here u, v & w are composite functions of r, θ & ϕ.

$$\therefore \frac{\partial(u,v,w)}{\partial(r,\theta,\phi)} = \frac{\partial(u,v,w)}{\partial(x,y,z)} \cdot \frac{\partial(x,y,z)}{\partial(r,\theta,\phi)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial w}{\partial z} \\ \frac{\partial u}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial w}{\partial z} \\ \frac{\partial u}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix} \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \phi} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \phi} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial \phi} \end{vmatrix}$$

$$\begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial w}{\partial z} \\ \frac{\partial u}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial w}{\partial z} \\ \frac{\partial u}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix} = \begin{vmatrix} 0 & \frac{1}{2}\sqrt{\frac{z}{y}} & \frac{1}{2}\sqrt{\frac{y}{z}} \\ \frac{1}{2}\sqrt{\frac{z}{x}} & 0 & \frac{1}{2}\sqrt{\frac{x}{z}} \\ \frac{1}{2}\sqrt{\frac{y}{x}} & \frac{1}{2}\sqrt{\frac{x}{y}} & 0 \end{vmatrix} = \frac{1}{8}\frac{1}{\sqrt{xyz}}\begin{vmatrix} 0 & \sqrt{z} & \sqrt{y} \\ \sqrt{z} & 0 & \sqrt{x} \\ \sqrt{y} & \sqrt{x} & 0 \end{vmatrix}$$

$$= \frac{1}{8}\frac{1}{\sqrt{xyz}}\left[0 - \sqrt{z}\,(0 - \sqrt{xy}\,) + \sqrt{y}\,(\sqrt{xz} - 0)\right]$$

$$= \frac{1}{8}\frac{1}{\sqrt{xyz}}\left[\sqrt{xyz}\, + \sqrt{xyz}\right]$$

$$= \frac{1}{4}$$

$$\frac{\partial(x,y,z)}{\partial(r,\theta,\phi)}=\begin{vmatrix}\frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \phi}\\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \phi}\\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial \phi}\end{vmatrix}=\begin{vmatrix}\sin\theta\cos\phi & r\cos\theta\cos\phi & -r\sin\theta\sin\phi\\ \sin\theta\sin\phi & r\cos\theta\sin\phi & r\sin\theta\cos\phi\\ \cos\theta & -r\sin\theta & 0\end{vmatrix}$$

$$= \sin\theta\cos\phi\,[0+r^2\sin^2\theta\cos\phi]-r\cos\theta\cos\phi\,[0-r\sin\theta\cos\theta\cos\phi]$$

$$-r\sin\theta\sin\phi[-r\sin^2\theta\sin\phi-r\cos^2\theta\sin\phi]$$

$$=r^2\sin^3\theta\cos^2\phi+r^2\sin\theta\cos^2\theta\cos^2\phi+r^2\sin^3\theta\sin^2\phi+r^2\sin\theta\cos^2\theta\sin^2\phi$$

$$=r^2\sin^3\theta+r^2\sin\theta\cos^2\theta$$

$$=r^2\sin\theta$$

$$\therefore\ \frac{\partial(u,v,w)}{\partial(r,\theta,\phi)}=\frac{\partial(u,v,w)}{\partial(x,y,z)}.\frac{\partial(x,y,z)}{\partial(r,\theta,\phi)}=\frac{1}{4}r^2\sin\theta$$

Test Your Knowledge

Q. Find Jacobian of the following composite functions.

1. $u = 2xy, \qquad v = x^2 - y^2$ and $x = r\cos\theta, \qquad y = r\sin\theta$

   Ans: $\dfrac{\partial(u,v)}{\partial(r,\theta)} = -4r^3$

2. $u = x^2 y, \qquad v = xy^2$ and $x = r^2 - s^2, \qquad y = rs$

   Ans: $\dfrac{\partial(u,v)}{\partial(r,s)} = 6r^2 s^2 (r^2 - s^2)^3$

3. $u = e^x \cos y, \qquad v = e^x \sin y$ and $x = r + s, \qquad y = r - s$

   Ans: $\dfrac{\partial(u,v)}{\partial(r,s)} = -2e^{2x}$

4. $u = x^2 - 2y^2, \qquad v = 2x^2 - y^2$ and $x = r\cos\theta, \qquad y = r\sin\theta$

   Ans: $\dfrac{\partial(u,v)}{\partial(r,\theta)} = 6r^3 \sin 2\theta$

Property No.4: Jacobian of Implicit Functions

If u & v are implicit functions of x & y connected by f_1 and f_2 such that $f_1(u, v, x, y) = 0$ & $f_2(u, v, x, y) = 0$ then

$$\frac{\partial(\boldsymbol{u}, \boldsymbol{v})}{\partial(\boldsymbol{x}, \boldsymbol{y})} = (-1)^2 \frac{\frac{\partial(f_1, f_2)}{\partial(x, y)}}{\frac{\partial(f_1, f_2)}{\partial(u, v)}}$$

Similarly if u, v & w are implicit functions of x, y & z connected by f_1, f_2 and f_3 such that $f_1(u, v, w, x, y, z) = 0$,$f_2(u, v, w, x, y, z) = 0$ & $f_3(u, v, w, x, y, z) = 0$ then

$$\frac{\partial(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w})}{\partial(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z})} = (-1)^3 \frac{\frac{\partial(f_1, f_2, f_3)}{\partial(x, y, z)}}{\frac{\partial(f_1, f_2, f_3)}{\partial(u, v, w)}}$$

Example 1: If $x^2+y^2+u^2-v^2=0$ & $uv+xy=0$ then prove that

$$\frac{\partial(u,v)}{\partial(x,y)}=\frac{x^2-y^2}{u^2+v^2}$$

Solution: Let $f_1 \equiv x^2+y^2+u^2-v^2$

$$f_2 \equiv uv+xy$$

$$\frac{\partial(u,v)}{\partial(x,y)}=(-1)^2\frac{\frac{\partial(f_1,f_2)}{\partial(x,y)}}{\frac{\partial(f_1,f_2)}{\partial(u,v)}}=\frac{\begin{vmatrix}\frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y}\\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y}\end{vmatrix}}{\begin{vmatrix}\frac{\partial f_1}{\partial u} & \frac{\partial f_1}{\partial v}\\ \frac{\partial f_2}{\partial u} & \frac{\partial f_2}{\partial v}\end{vmatrix}}=\frac{\begin{vmatrix}2x & 2y\\ y & x\end{vmatrix}}{\begin{vmatrix}2u & -2v\\ v & u\end{vmatrix}}=\frac{2(x^2-y^2)}{2(u^2+v^2)}=\frac{x^2-y^2}{u^2+v^2}$$

Example 2: If $u^3+v^3=x+y$ & $u^2+v^2=x^3+y^3$ then prove that

$$\frac{\partial(u,v)}{\partial(x,y)}=\frac{y^2-x^2}{2uv(u-v)}$$

Solution: Let $f_1\equiv u^3+v^3-x-y$ and $f_2\equiv u^2+v^2-x^3-y^3$

$$\frac{\partial(u,v)}{\partial(x,y)}=(-1)^2\frac{\frac{\partial(f_1,f_2)}{\partial(x,y)}}{\frac{\partial(f_1,f_2)}{\partial(u,v)}}=\frac{\begin{vmatrix}\frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y}\\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y}\end{vmatrix}}{\begin{vmatrix}\frac{\partial f_1}{\partial u} & \frac{\partial f_1}{\partial v}\\ \frac{\partial f_2}{\partial u} & \frac{\partial f_2}{\partial v}\end{vmatrix}}=\frac{\begin{vmatrix}-1 & -1\\ -3x^2 & -3y^2\end{vmatrix}}{\begin{vmatrix}3u^2 & 3v^2\\ 2u & 2v\end{vmatrix}}=\frac{3(y^2-x^2)}{6(u^2v-uv^2)}$$

$$=\frac{y^2-x^2}{2uv(u-v)}$$

Example 3: If $x + y + z = u, y + z = uv, z = uvw$ prove that $\dfrac{\partial(x,y,z)}{\partial(u,v,w)} = u^2 v$

Solution: Let $f_1 \equiv x + y + z - u,\quad f_2 \equiv y + z - uv$ and $f_3 \equiv z - uvw$

$$\frac{\partial(x,y,z)}{\partial(u,v,w)} = (-1)^3 \frac{\dfrac{\partial(f_1,f_2,f_3)}{\partial(u,v,w)}}{\dfrac{\partial(f_1,f_2,f_3)}{\partial(x,y,z)}} = -\frac{\begin{vmatrix} \dfrac{\partial f_1}{\partial u} & \dfrac{\partial f_1}{\partial v} & \dfrac{\partial f_1}{\partial w} \\ \dfrac{\partial f_2}{\partial u} & \dfrac{\partial f_2}{\partial v} & \dfrac{\partial f_2}{\partial w} \\ \dfrac{\partial f_3}{\partial u} & \dfrac{\partial f_3}{\partial v} & \dfrac{\partial f_3}{\partial w} \end{vmatrix}}{\begin{vmatrix} \dfrac{\partial f_1}{\partial x} & \dfrac{\partial f_1}{\partial y} & \dfrac{\partial f_1}{\partial z} \\ \dfrac{\partial f_2}{\partial x} & \dfrac{\partial f_2}{\partial y} & \dfrac{\partial f_2}{\partial z} \\ \dfrac{\partial f_3}{\partial x} & \dfrac{\partial f_3}{\partial y} & \dfrac{\partial f_3}{\partial z} \end{vmatrix}} = -\frac{N}{D}$$

$$N = \begin{vmatrix} \frac{\partial f_1}{\partial u} & \frac{\partial f_1}{\partial v} & \frac{\partial f_1}{\partial w} \\ \frac{\partial f_2}{\partial u} & \frac{\partial f_2}{\partial v} & \frac{\partial f_2}{\partial w} \\ \frac{\partial f_3}{\partial u} & \frac{\partial f_3}{\partial v} & \frac{\partial f_3}{\partial w} \end{vmatrix} = \begin{vmatrix} -1 & 0 & 0 \\ -v & -u & 0 \\ -vw & -uw & -uv \end{vmatrix} = (-1)(-u)(-uv) = -u^2 v$$

$$D = \begin{vmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} & \frac{\partial f_1}{\partial z} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} & \frac{\partial f_2}{\partial z} \\ \frac{\partial f_3}{\partial x} & \frac{\partial f_3}{\partial y} & \frac{\partial f_3}{\partial z} \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{vmatrix} = 1$$

$$\therefore \frac{\partial(x, y, z)}{\partial(u, v, w)} = -\frac{N}{D} = -\frac{(-u^2 v)}{1} = u^2 v$$

Example 4: If $u^3+v^3+w^3=x+y+z$, $u^2+v^2+w^2=x^3+y^3+z^3$, $u+v+w=x^2+y^2+z^2$ prove that $\dfrac{\partial(u,v,w)}{\partial(x,y,z)}=\dfrac{(x-y)(y-z)(z-x)}{(u-v)(v-w)(w-u)}$

Solution: Let $f_1\equiv u^3+v^3+w^3-x-y-z$, $\quad f_2\equiv u^2+v^2+w^2-x^3-y^3-z^3$ and $f_3\equiv u+v+w-x^2-y^2-z^2$

$$\frac{\partial(u,v,w)}{\partial(x,y,z)}=(-1)^3\frac{\dfrac{\partial(f_1,f_2,f_3)}{\partial(x,y,z)}}{\dfrac{\partial(f_1,f_2,f_3)}{\partial(u,v,w)}}=-\frac{\begin{vmatrix}\dfrac{\partial f_1}{\partial x} & \dfrac{\partial f_1}{\partial y} & \dfrac{\partial f_1}{\partial z}\\ \dfrac{\partial f_2}{\partial x} & \dfrac{\partial f_2}{\partial y} & \dfrac{\partial f_2}{\partial z}\\ \dfrac{\partial f_3}{\partial x} & \dfrac{\partial f_3}{\partial y} & \dfrac{\partial f_3}{\partial z}\end{vmatrix}}{\begin{vmatrix}\dfrac{\partial f_1}{\partial u} & \dfrac{\partial f_1}{\partial v} & \dfrac{\partial f_1}{\partial w}\\ \dfrac{\partial f_2}{\partial u} & \dfrac{\partial f_2}{\partial v} & \dfrac{\partial f_2}{\partial w}\\ \dfrac{\partial f_3}{\partial u} & \dfrac{\partial f_3}{\partial v} & \dfrac{\partial f_3}{\partial w}\end{vmatrix}}=-\frac{N}{D}$$

$$N = \begin{vmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} & \frac{\partial f_1}{\partial z} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} & \frac{\partial f_2}{\partial z} \\ \frac{\partial f_3}{\partial x} & \frac{\partial f_3}{\partial y} & \frac{\partial f_3}{\partial z} \end{vmatrix} = \begin{vmatrix} -1 & -1 & -1 \\ -3x^2 & -3y^2 & -3z^2 \\ -2x & -2y & -2z \end{vmatrix}$$

$$= -6 \begin{vmatrix} 1 & 1 & 1 \\ x^2 & y^2 & z^2 \\ x & y & z \end{vmatrix} \qquad \text{(By taking } -1, -3 \text{ \& } -2 \text{ common from } R_1,\ R_2 \text{ \& } R_3\text{)}$$

$$= 6 \begin{vmatrix} 1 & 1 & 1 \\ x & y & z \\ x^2 & y^2 & z^2 \end{vmatrix} \qquad \text{(By } R_2 \leftrightarrow R_3\text{)}$$

$$= 6(x-y)(y-z)(z-x)$$

$$D = \begin{vmatrix} \frac{\partial f_1}{\partial u} & \frac{\partial f_1}{\partial v} & \frac{\partial f_1}{\partial w} \\ \frac{\partial f_2}{\partial u} & \frac{\partial f_2}{\partial v} & \frac{\partial f_2}{\partial w} \\ \frac{\partial f_3}{\partial u} & \frac{\partial f_3}{\partial v} & \frac{\partial f_3}{\partial w} \end{vmatrix} = \begin{vmatrix} 3u^2 & 3v^2 & 3w^2 \\ 2u & 2v & 2w \\ 1 & 1 & 1 \end{vmatrix} = 6 \begin{vmatrix} u^2 & v^2 & w^2 \\ u & v & w \\ 1 & 1 & 1 \end{vmatrix}$$

$$= -6 \begin{vmatrix} 1 & 1 & 1 \\ u & v & w \\ u^2 & v^2 & w^2 \end{vmatrix} \qquad (\text{ By } R_1 \leftrightarrow R_3)$$

$$= -6(u-v)(v-w)(w-u)$$

$$\frac{\partial(x,y,z)}{\partial(u,v,w)} = -\frac{N}{D} = -\frac{6(x-y)(y-z)(z-x)}{-6(u-v)(v-w)(w-u)} = \frac{(x-y)(y-z)(z-x)}{(u-v)(v-w)(w-u)}$$

Example 5: If $u = \frac{x}{k}, v = \frac{y}{k}, w = \frac{z}{k}$ where $k = \sqrt{1-(x^2+y^2+z^2)}$ find $\frac{\partial(x,y,z)}{\partial(u,v,w)}$ in terms of k.

Solution: $k = \sqrt{1-(x^2+y^2+z^2)} \Longrightarrow k^2 + x^2 + y^2 + z^2 = 1$

$u = \frac{x}{k} \Longrightarrow uk = x \Longrightarrow u\sqrt{1-(x^2+y^2+z^2)} = x$

$\Longrightarrow u^2(1-x^2-y^2-z^2) = x^2 \Longrightarrow u^2(1-x^2-y^2-z^2) - x^2 = 0$

Similarly $v^2(1-x^2-y^2-z^2) - y^2 = 0$

$w^2(1-x^2-y^2-z^2) - z^2 = 0$

Let $f_1 \equiv u^2(1-x^2-y^2-z^2) - x^2 = u^2k^2 - x^2$

$f_2 \equiv v^2(1-x^2-y^2-z^2) - y^2 = v^2k^2 - y^2$

$f_3 \equiv w^2(1-x^2-y^2-z^2) - z^2 = w^2k^2 - z^2$

$$\frac{\partial(x,y,z)}{\partial(u,v,w)} = (-1)^3 \frac{\frac{\partial(f_1,f_2,f_3)}{\partial(u,v,w)}}{\frac{\partial(f_1,f_2,f_3)}{\partial(x,y,z)}} = -\frac{N}{D}$$

$$\text{where } N = \begin{vmatrix} \frac{\partial f_1}{\partial u} & \frac{\partial f_1}{\partial v} & \frac{\partial f_1}{\partial w} \\ \frac{\partial f_2}{\partial u} & \frac{\partial f_2}{\partial v} & \frac{\partial f_2}{\partial w} \\ \frac{\partial f_3}{\partial u} & \frac{\partial f_3}{\partial v} & \frac{\partial f_3}{\partial w} \end{vmatrix} = \begin{vmatrix} 2uk^2 & 0 & 0 \\ 0 & 2vk^2 & 0 \\ 0 & 0 & 2wk^2 \end{vmatrix}$$

$$= 8uvwk^6$$

$$D = \begin{vmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} & \frac{\partial f_1}{\partial z} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} & \frac{\partial f_2}{\partial z} \\ \frac{\partial f_3}{\partial x} & \frac{\partial f_3}{\partial y} & \frac{\partial f_3}{\partial z} \end{vmatrix} = \begin{vmatrix} -2u^2x - 2x & -2u^2y & -2u^2z \\ -2v^2x & -2v^2y - 2y & -2v^2z \\ -2w^2x & -2w^2y & -2w^2z - 2z \end{vmatrix}$$

$$= (-2x)(-2y)(-2z) \begin{vmatrix} u^2 + 1 & u^2 & u^2 \\ v^2 & v^2 + 1 & v^2 \\ w^2 & w^2 & w^2 + 1 \end{vmatrix}$$

(By taking $-2x, -2y, -2z$ $common\ from\ C_1, C_2, C_3$)

$$= -8xyz \begin{vmatrix} 1 & u^2 & 0 \\ -1 & v^2 + 1 & -1 \\ 0 & w^2 & 1 \end{vmatrix} \qquad \text{(By } C_1 - C_2,\ C_3 - C_2 \text{)}$$

$$D = -8xyz[(1 + w^2 + v^2) + u^2(1)]$$

$$= -8xyz[u^2 + v^2 + w^2 + 1]$$

$$= -8xyz\left[\frac{x^2}{k^2} + \frac{y^2}{k^2} + \frac{z^2}{k^2} + 1\right]$$

$$= -8xyz\left[\frac{x^2 + y^2 + z^2 + k^2}{k^2}\right]$$

$$= -8xyz\left[\frac{1}{k^2}\right]$$

$$\frac{\partial(x,y,z)}{\partial(u,v,w)} = -\frac{N}{D} = -\frac{8uvwk^6}{-8xyz\left[\frac{1}{k^2}\right]} = k^8\frac{u}{x}\frac{v}{y}\frac{w}{z} = k^8\frac{1}{k}\frac{1}{k}\frac{1}{k} = k^5$$

Example 6: If $u+v+w=x+y+z, uv+vw+wu=x^2+y^2+z^2$,

$uvw=\frac{1}{3}(x^3+y^3+z^3)$ find $\frac{\partial(u,v,w)}{\partial(x,y,z)}$

Solution: Let $f_1 \equiv u+v+w-x-y-z,\quad f_2 \equiv uv+vw+wu-x^2-y^2-z^2$

and $f_3 \equiv uvw-\frac{1}{3}x^3-\frac{1}{3}y^3-\frac{1}{3}z^3$

$$\frac{\partial(u,v,w)}{\partial(x,y,z)}=(-1)^3\frac{\frac{\partial(f_1,f_2,f_3)}{\partial(x,y,z)}}{\frac{\partial(f_1,f_2,f_3)}{\partial(u,v,w)}}=-\frac{\begin{vmatrix}\frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} & \frac{\partial f_1}{\partial z}\\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} & \frac{\partial f_2}{\partial z}\\ \frac{\partial f_3}{\partial x} & \frac{\partial f_3}{\partial y} & \frac{\partial f_3}{\partial z}\end{vmatrix}}{\begin{vmatrix}\frac{\partial f_1}{\partial u} & \frac{\partial f_1}{\partial v} & \frac{\partial f_1}{\partial w}\\ \frac{\partial f_2}{\partial u} & \frac{\partial f_2}{\partial v} & \frac{\partial f_2}{\partial w}\\ \frac{\partial f_3}{\partial u} & \frac{\partial f_3}{\partial v} & \frac{\partial f_3}{\partial w}\end{vmatrix}}=-\frac{N}{D}$$

$$N = \begin{vmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} & \frac{\partial f_1}{\partial z} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} & \frac{\partial f_2}{\partial z} \\ \frac{\partial f_3}{\partial x} & \frac{\partial f_3}{\partial y} & \frac{\partial f_3}{\partial z} \end{vmatrix} = \begin{vmatrix} -1 & -1 & -1 \\ -2x & -2y & -2z \\ -x^2 & -y^2 & -z^2 \end{vmatrix}$$

$$= -2 \begin{vmatrix} 1 & 1 & 1 \\ x & y & z \\ x^2 & y^2 & z^2 \end{vmatrix}$$

$$= -2(x-y)(y-z)(z-x)$$

$$D = \begin{vmatrix} \dfrac{\partial f_1}{\partial u} & \dfrac{\partial f_1}{\partial v} & \dfrac{\partial f_1}{\partial w} \\ \dfrac{\partial f_2}{\partial u} & \dfrac{\partial f_2}{\partial v} & \dfrac{\partial f_2}{\partial w} \\ \dfrac{\partial f_3}{\partial u} & \dfrac{\partial f_3}{\partial v} & \dfrac{\partial f_3}{\partial w} \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 \\ v+w & u+w & u+v \\ vw & uw & uv \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 0 & 0 \\ v+w & u-v & u-w \\ vw & (u-v)w & (u-w)v \end{vmatrix} \quad (\text{By } C_2 - C_1, C_3 - C_1)$$

$$= (u-v)(u-w) \begin{vmatrix} 1 & 0 & 0 \\ v+w & 1 & 1 \\ vw & w & v \end{vmatrix} = (u-v)(u-w)(v-w)$$

$$\frac{\partial(x,y,z)}{\partial(u,v,w)} = -\frac{N}{D} = -\frac{-2(x-y)(y-z)(z-x)}{(u-v)(u-w)(v-w)} = \frac{2(x-y)(y-z)(z-x)}{(u-v)(v-w)(w-u)}$$

Test Your Knowledge

1. If $u = x + y + z, \qquad u^2 v = y + z, \qquad u^3 w = z,$ show that $\dfrac{\partial(u,v,w)}{\partial(x,y,z)} = \dfrac{1}{u^5}$

2. If $u^3 + v + w = x + y^2 + z^2, \qquad u + v^3 + w = x^2 + y + z^2,$
   $u + v + w^3 = x^2 + y^2 + z,$ show that
   $$\frac{\partial(u,v,w)}{\partial(x,y,z)} = \frac{1 - 4(xy + yz + zx) + 16xyz}{27u^2v^2w^2 - 3(u^2 + v^2 + w^2) + 2}$$

3. If u, v, w are roots of the equation $\dfrac{x}{a+\lambda} + \dfrac{y}{b+\lambda} + \dfrac{z}{c+\lambda} = 1$ in λ
   then show that $\dfrac{\partial(x,y,z)}{\partial(u,v,w)} = -\dfrac{(u-v)(v-w)(w-u)}{(a-b)(b-c)(c-a)}$

4. If u, v, w are roots of the equation $(\lambda - x)^3 + (\lambda - y)^3 + (\lambda - z)^3 = 1$ in λ
   then show that $\dfrac{\partial(u,v,w)}{\partial(x,y,z)} = \dfrac{2(x-y)(y-z)(z-x)}{(u-v)(v-w)(w-u)}$

Partial Derivatives of Implicit Functions using Jacobian

If u & v are implicit functions of x & y connected by f_1 and f_2 such that $f_1(u,v,x,y) = 0$ & $f_2(u,v,x,y) = 0$ then

$$\frac{\partial u}{\partial x} = -\frac{\frac{\partial(f_1,f_2)}{\partial(x,v)}}{\frac{\partial(f_1,f_2)}{\partial(u,v)}}, \quad \frac{\partial u}{\partial y} = -\frac{\frac{\partial(f_1,f_2)}{\partial(y,v)}}{\frac{\partial(f_1,f_2)}{\partial(u,v)}}, \quad \frac{\partial v}{\partial x} = -\frac{\frac{\partial(f_1,f_2)}{\partial(u,x)}}{\frac{\partial(f_1,f_2)}{\partial(u,v)}}, \quad \frac{\partial v}{\partial y} = -\frac{\frac{\partial(f_1,f_2)}{\partial(u,y)}}{\frac{\partial(f_1,f_2)}{\partial(u,v)}}$$

$$\frac{\partial x}{\partial u} = -\frac{\frac{\partial(f_1,f_2)}{\partial(u,y)}}{\frac{\partial(f_1,f_2)}{\partial(x,y)}}, \quad \frac{\partial x}{\partial v} = -\frac{\frac{\partial(f_1,f_2)}{\partial(v,y)}}{\frac{\partial(f_1,f_2)}{\partial(x,y)}}, \quad \frac{\partial y}{\partial u} = -\frac{\frac{\partial(f_1,f_2)}{\partial(x,u)}}{\frac{\partial(f_1,f_2)}{\partial(x,y)}}, \quad \frac{\partial y}{\partial v} = -\frac{\frac{\partial(f_1,f_2)}{\partial(x,v)}}{\frac{\partial(f_1,f_2)}{\partial(x,y)}}$$

If u, v & w are implicit functions of x, y & z connected by f_1, f_2 and f_3 such that $f_1(u,v,w,x,y,z) = 0$,$f_2(u,v,w,x,y,z) = 0$ & $f_3(u,v,w,x,y,z) = 0$ then

$$\frac{\partial u}{\partial x} = -\frac{\frac{\partial(f_1,f_2,f_3)}{\partial(x,v,w)}}{\frac{\partial(f_1,f_2,f_3)}{\partial(u,v,w)}}, \quad \frac{\partial u}{\partial y} = -\frac{\frac{\partial(f_1,f_2,f_3)}{\partial(y,v,w)}}{\frac{\partial(f_1,f_2,f_3)}{\partial(u,v,w)}}, \quad \frac{\partial u}{\partial z} = -\frac{\frac{\partial(f_1,f_2,f_3)}{\partial(z,v,w)}}{\frac{\partial(f_1,f_2,f_3)}{\partial(u,v,w)}}$$

$$\frac{\partial v}{\partial x} = -\frac{\frac{\partial(f_1,f_2,f_3)}{\partial(u,x,w)}}{\frac{\partial(f_1,f_2,f_3)}{\partial(u,v,w)}}, \quad \frac{\partial v}{\partial y} = -\frac{\frac{\partial(f_1,f_2,f_3)}{\partial(u,y,w)}}{\frac{\partial(f_1,f_2,f_3)}{\partial(u,v,w)}}, \quad \frac{\partial v}{\partial z} = -\frac{\frac{\partial(f_1,f_2,f_3)}{\partial(u,z,w)}}{\frac{\partial(f_1,f_2,f_3)}{\partial(u,v,w)}}$$

$$\frac{\partial w}{\partial x} = -\frac{\frac{\partial(f_1,f_2,f_3)}{\partial(u,v,x)}}{\frac{\partial(f_1,f_2,f_3)}{\partial(u,v,w)}}, \quad \frac{\partial w}{\partial y} = -\frac{\frac{\partial(f_1,f_2,f_3)}{\partial(u,v,y)}}{\frac{\partial(f_1,f_2,f_3)}{\partial(u,v,w)}}, \quad \frac{\partial w}{\partial z} = -\frac{\frac{\partial(f_1,f_2,f_3)}{\partial(u,v,z)}}{\frac{\partial(f_1,f_2,f_3)}{\partial(u,v,w)}}$$

$$\frac{\partial x}{\partial u} = -\frac{\frac{\partial(f_1,f_2,f_3)}{\partial(u,y,z)}}{\frac{\partial(f_1,f_2,f_3)}{\partial(x,y,z)}}, \quad \frac{\partial x}{\partial v} = -\frac{\frac{\partial(f_1,f_2,f_3)}{\partial(v,y,z)}}{\frac{\partial(f_1,f_2,f_3)}{\partial(x,y,z)}}, \quad \frac{\partial x}{\partial w} = -\frac{\frac{\partial(f_1,f_2,f_3)}{\partial(w,y,z)}}{\frac{\partial(f_1,f_2,f_3)}{\partial(x,y,z)}}$$

$$\frac{\partial y}{\partial u} = -\frac{\frac{\partial(f_1,f_2,f_3)}{\partial(x,u,z)}}{\frac{\partial(f_1,f_2,f_3)}{\partial(x,y,z)}}, \quad \frac{\partial y}{\partial v} = -\frac{\frac{\partial(f_1,f_2,f_3)}{\partial(x,v,z)}}{\frac{\partial(f_1,f_2,f_3)}{\partial(x,y,z)}}, \quad \frac{\partial y}{\partial w} = -\frac{\frac{\partial(f_1,f_2,f_3)}{\partial(x,w,z)}}{\frac{\partial(f_1,f_2,f_3)}{\partial(x,y,z)}}$$

$$\frac{\partial z}{\partial u} = -\frac{\frac{\partial(f_1,f_2,f_3)}{\partial(x,y,u)}}{\frac{\partial(f_1,f_2,f_3)}{\partial(x,y,z)}}, \quad \frac{\partial z}{\partial v} = -\frac{\frac{\partial(f_1,f_2,f_3)}{\partial(x,y,v)}}{\frac{\partial(f_1,f_2,f_3)}{\partial(x,y,z)}}, \quad \frac{\partial z}{\partial w} = -\frac{\frac{\partial(f_1,f_2,f_3)}{\partial(x,y,w)}}{\frac{\partial(f_1,f_2,f_3)}{\partial(x,y,z)}}$$

Example 1: If $x^2 + y^2 + u^2 - v^2 = 0$ & $uv + xy = 0$ find $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}$ & $\frac{\partial v}{\partial y}$

Solution: Let $f_1 \equiv x^2 + y^2 + u^2 - v^2$

$f_2 \equiv uv + xy$

$$\frac{\partial u}{\partial x} = -\frac{\frac{\partial(f_1, f_2)}{\partial(x, v)}}{\frac{\partial(f_1, f_2)}{\partial(u, v)}} = \frac{\begin{vmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial v} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial v} \end{vmatrix}}{\begin{vmatrix} \frac{\partial f_1}{\partial u} & \frac{\partial f_1}{\partial v} \\ \frac{\partial f_2}{\partial u} & \frac{\partial f_2}{\partial v} \end{vmatrix}} = \frac{\begin{vmatrix} 2x & -2v \\ y & u \end{vmatrix}}{\begin{vmatrix} 2u & -2v \\ v & u \end{vmatrix}} = \frac{2(ux + vy)}{2(u^2 + v^2)} = \frac{ux + vy}{u^2 + v^2}$$

$$\frac{\partial u}{\partial y} = -\frac{\frac{\partial(f_1,f_2)}{\partial(y,v)}}{\frac{\partial(f_1,f_2)}{\partial(u,v)}} = \frac{\begin{vmatrix} \frac{\partial f_1}{\partial y} & \frac{\partial f_1}{\partial v} \\ \frac{\partial f_2}{\partial y} & \frac{\partial f_2}{\partial v} \end{vmatrix}}{\begin{vmatrix} \frac{\partial f_1}{\partial u} & \frac{\partial f_1}{\partial v} \\ \frac{\partial f_2}{\partial u} & \frac{\partial f_2}{\partial v} \end{vmatrix}} = \frac{\begin{vmatrix} 2y & -2v \\ x & u \end{vmatrix}}{\begin{vmatrix} 2u & -2v \\ v & u \end{vmatrix}} = \frac{2(uy+vx)}{2(u^2+v^2)} = \frac{uy+vx}{u^2+v^2}$$

$$\frac{\partial v}{\partial x} = -\frac{\frac{\partial(f_1,f_2)}{\partial(u,x)}}{\frac{\partial(f_1,f_2)}{\partial(u,v)}} = \frac{\begin{vmatrix} \frac{\partial f_1}{\partial u} & \frac{\partial f_1}{\partial x} \\ \frac{\partial f_2}{\partial u} & \frac{\partial f_2}{\partial x} \end{vmatrix}}{\begin{vmatrix} \frac{\partial f_1}{\partial u} & \frac{\partial f_1}{\partial v} \\ \frac{\partial f_2}{\partial u} & \frac{\partial f_2}{\partial v} \end{vmatrix}} = \frac{\begin{vmatrix} 2u & 2x \\ v & y \end{vmatrix}}{\begin{vmatrix} 2u & -2v \\ v & u \end{vmatrix}} = \frac{2(uy-vx)}{2(u^2+v^2)} = \frac{uy-vx}{u^2+v^2}$$

$$\frac{\partial v}{\partial y} = -\frac{\frac{\partial(f_1, f_2)}{\partial(u, y)}}{\frac{\partial(f_1, f_2)}{\partial(u, v)}} = \frac{\begin{vmatrix} \frac{\partial f_1}{\partial u} & \frac{\partial f_1}{\partial y} \\ \frac{\partial f_2}{\partial u} & \frac{\partial f_2}{\partial y} \end{vmatrix}}{\begin{vmatrix} \frac{\partial f_1}{\partial u} & \frac{\partial f_1}{\partial v} \\ \frac{\partial f_2}{\partial u} & \frac{\partial f_2}{\partial v} \end{vmatrix}} = \frac{\begin{vmatrix} 2u & 2y \\ v & x \end{vmatrix}}{\begin{vmatrix} 2u & -2v \\ v & u \end{vmatrix}} = \frac{2(ux - vy)}{2(u^2 + v^2)} = \frac{ux - vy}{u^2 + v^2}$$

Example 2: If $u = xyz, v = x^2 + y^2 + z^2, w = x + y + z$ find $\frac{\partial u}{\partial x}$ & $\frac{\partial x}{\partial u}$

Solution: Let $f_1 \equiv u - xyz,\ f_2 \equiv v - x^2 - y^2 - z^2$ & $f_3 \equiv w - x - y - z$

$$\frac{\partial u}{\partial x} = -\frac{\frac{\partial(f_1,f_2,f_3)}{\partial(x,v,w)}}{\frac{\partial(f_1,f_2,f_3)}{\partial(u,v,w)}} = -\frac{\begin{vmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial v} & \frac{\partial f_1}{\partial w} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial v} & \frac{\partial f_2}{\partial w} \\ \frac{\partial f_3}{\partial x} & \frac{\partial f_3}{\partial v} & \frac{\partial f_3}{\partial w} \end{vmatrix}}{\begin{vmatrix} \frac{\partial f_1}{\partial u} & \frac{\partial f_1}{\partial v} & \frac{\partial f_1}{\partial w} \\ \frac{\partial f_2}{\partial u} & \frac{\partial f_2}{\partial v} & \frac{\partial f_2}{\partial w} \\ \frac{\partial f_3}{\partial u} & \frac{\partial f_3}{\partial v} & \frac{\partial f_3}{\partial w} \end{vmatrix}} = -\frac{\begin{vmatrix} -yz & 0 & 0 \\ -2x & 1 & 0 \\ -1 & 0 & 1 \end{vmatrix}}{\begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix}} = -\frac{-yz}{1} = yz$$

$$\frac{\partial x}{\partial u} = -\frac{\frac{\partial(f_1, f_2, f_3)}{\partial(u, y, z)}}{\frac{\partial(f_1, f_2, f_3)}{\partial(x, y, z)}} = -\frac{\begin{vmatrix} \frac{\partial f_1}{\partial u} & \frac{\partial f_1}{\partial y} & \frac{\partial f_1}{\partial z} \\ \frac{\partial f_2}{\partial u} & \frac{\partial f_2}{\partial y} & \frac{\partial f_2}{\partial z} \\ \frac{\partial f_3}{\partial u} & \frac{\partial f_3}{\partial y} & \frac{\partial f_3}{\partial z} \end{vmatrix}}{\begin{vmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} & \frac{\partial f_1}{\partial z} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} & \frac{\partial f_2}{\partial z} \\ \frac{\partial f_3}{\partial x} & \frac{\partial f_3}{\partial y} & \frac{\partial f_3}{\partial z} \end{vmatrix}} = -\frac{\begin{vmatrix} 1 & -xz & -xy \\ 0 & -2y & -2z \\ 0 & -1 & -1 \end{vmatrix}}{\begin{vmatrix} -yz & -xz & -xy \\ -2x & -2y & -2z \\ -1 & -1 & -1 \end{vmatrix}}$$

$$= -\frac{(2y - 2z)}{2\begin{vmatrix} 1 & 1 & 1 \\ x & y & z \\ yz & xz & zx \end{vmatrix}} = -\frac{(y - z)}{(x - y)(y - z)(z - x)} = \frac{1}{(x - y)(x - z)}$$

Example 3: If $u = x + y^2, v = y + z^2$ & $w = z + x^2$ then find $\dfrac{\partial x}{\partial u}$

Solution: Let $f_1 \equiv u - x - y^2,\ f_2 \equiv v - y - z^2$ & $f_3 \equiv w - z - x^2$

$$\frac{\partial x}{\partial u} = -\frac{\dfrac{\partial(f_1, f_2, f_3)}{\partial(u, y, z)}}{\dfrac{\partial(f_1, f_2, f_3)}{\partial(x, y, z)}} = -\frac{\begin{vmatrix} \dfrac{\partial f_1}{\partial u} & \dfrac{\partial f_1}{\partial y} & \dfrac{\partial f_1}{\partial z} \\ \dfrac{\partial f_2}{\partial u} & \dfrac{\partial f_2}{\partial y} & \dfrac{\partial f_2}{\partial z} \\ \dfrac{\partial f_3}{\partial u} & \dfrac{\partial f_3}{\partial y} & \dfrac{\partial f_3}{\partial z} \end{vmatrix}}{\begin{vmatrix} \dfrac{\partial f_1}{\partial x} & \dfrac{\partial f_1}{\partial y} & \dfrac{\partial f_1}{\partial z} \\ \dfrac{\partial f_2}{\partial x} & \dfrac{\partial f_2}{\partial y} & \dfrac{\partial f_2}{\partial z} \\ \dfrac{\partial f_3}{\partial x} & \dfrac{\partial f_3}{\partial y} & \dfrac{\partial f_3}{\partial z} \end{vmatrix}} = -\frac{\begin{vmatrix} 1 & -2y & 0 \\ 0 & -1 & -2z \\ 0 & 0 & -1 \end{vmatrix}}{\begin{vmatrix} -1 & -2y & 0 \\ 0 & -1 & -2z \\ -2x & 0 & -1 \end{vmatrix}} = \frac{-1}{1 + 8xyz}$$

Example 4: If $u = x + y + z, v = x^2 + y^2 + z^2, w = x^3 + y^3 + z^3$

then show that $\dfrac{\partial x}{\partial u} = -\dfrac{yz}{(x-y)(z-x)}$

Solution: Let $f_1 \equiv u - x - y - z,\ f_2 \equiv v - x^2 - y^2 - z^2$ & $f_3 \equiv w - x^3 - y^3 - z^3$

$$\frac{\partial x}{\partial u} = -\frac{\dfrac{\partial(f_1, f_2, f_3)}{\partial(u, y, z)}}{\dfrac{\partial(f_1, f_2, f_3)}{\partial(x, y, z)}} = -\frac{\begin{vmatrix} \dfrac{\partial f_1}{\partial u} & \dfrac{\partial f_1}{\partial y} & \dfrac{\partial f_1}{\partial z} \\ \dfrac{\partial f_2}{\partial u} & \dfrac{\partial f_2}{\partial y} & \dfrac{\partial f_2}{\partial z} \\ \dfrac{\partial f_3}{\partial u} & \dfrac{\partial f_3}{\partial y} & \dfrac{\partial f_3}{\partial z} \end{vmatrix}}{\begin{vmatrix} \dfrac{\partial f_1}{\partial x} & \dfrac{\partial f_1}{\partial y} & \dfrac{\partial f_1}{\partial z} \\ \dfrac{\partial f_2}{\partial x} & \dfrac{\partial f_2}{\partial y} & \dfrac{\partial f_2}{\partial z} \\ \dfrac{\partial f_3}{\partial x} & \dfrac{\partial f_3}{\partial y} & \dfrac{\partial f_3}{\partial z} \end{vmatrix}} = -\frac{\begin{vmatrix} 1 & -1 & -1 \\ 0 & -2y & -2z \\ 0 & -3y^2 & -3z^2 \end{vmatrix}}{\begin{vmatrix} -1 & -1 & -1 \\ -2x & -2y & -2z \\ -3x^2 & -3y^2 & -3z^2 \end{vmatrix}}$$

$$\frac{\partial x}{\partial u} = -\frac{6yz^2 - 6y^2z}{(-6)\begin{vmatrix} 1 & 1 & 1 \\ x & y & z \\ x^2 & y^2 & z^2 \end{vmatrix}}$$

$$= \frac{yz(z-y)}{(x-y)(y-z)(z-x)}$$

$$= -\frac{yz}{(x-y)(z-x)}$$

Example 5: If $x = u + v, y = v^2 + w^2, z = w^3 + u^3$, show that $\dfrac{\partial u}{\partial x} = \dfrac{vw}{vw + u^2}$

Solution: Let $f_1 \equiv x - u - v,\ f_2 \equiv y - v^2 - w^2$ & $f_3 \equiv z - w^3 - u^3$

$$\frac{\partial u}{\partial x} = -\frac{\dfrac{\partial(f_1, f_2, f_3)}{\partial(x, v, w)}}{\dfrac{\partial(f_1, f_2, f_3)}{\partial(u, v, w)}} = -\frac{\begin{vmatrix} \dfrac{\partial f_1}{\partial x} & \dfrac{\partial f_1}{\partial v} & \dfrac{\partial f_1}{\partial w} \\ \dfrac{\partial f_2}{\partial x} & \dfrac{\partial f_2}{\partial v} & \dfrac{\partial f_2}{\partial w} \\ \dfrac{\partial f_3}{\partial x} & \dfrac{\partial f_3}{\partial v} & \dfrac{\partial f_3}{\partial w} \end{vmatrix}}{\begin{vmatrix} \dfrac{\partial f_1}{\partial u} & \dfrac{\partial f_1}{\partial v} & \dfrac{\partial f_1}{\partial w} \\ \dfrac{\partial f_2}{\partial u} & \dfrac{\partial f_2}{\partial v} & \dfrac{\partial f_2}{\partial w} \\ \dfrac{\partial f_3}{\partial u} & \dfrac{\partial f_3}{\partial v} & \dfrac{\partial f_3}{\partial w} \end{vmatrix}} = -\frac{\begin{vmatrix} 1 & -1 & 0 \\ 0 & -2v & -2w \\ 0 & 0 & -3w^2 \end{vmatrix}}{\begin{vmatrix} -1 & -1 & 0 \\ 0 & -2v & -2w \\ -3u^2 & 0 & -3w^2 \end{vmatrix}}$$

$$= -\frac{6vw^2}{(-1)(6vw^2) + (1)(-6u^2w)} = \frac{vw}{vw + u^2}$$

Test Your Knowledge

1.If $u+v^2=x, v+w^2=y$ & $w+u^2=z$ then show that $\dfrac{\partial u}{\partial x}=-\dfrac{1}{1+8uvw}$

2.If $x=u+v+w, y=u^2+v^2+w^2, z=u^3+v^3+w^3$

then show that $\dfrac{\partial u}{\partial x}=\dfrac{vw}{(u-v)(u-w)}$

3. If $x=u^2-v^2, y=uv$ then find $\dfrac{\partial u}{\partial x},\dfrac{\partial u}{\partial y},\dfrac{\partial v}{\partial x}$ & $\dfrac{\partial v}{\partial y}$.

Ans: $\dfrac{u}{2(u^2+v^2)},\dfrac{v}{2(u^2+v^2)},\dfrac{-v}{2(u^2+v^2)}$ & $\dfrac{u}{2(u^2+v^2)}$

4. If $u^2+xv^2=x+y, v^2+yu^2=x-y$ then find $\dfrac{\partial u}{\partial x}$ & $\dfrac{\partial v}{\partial y}$.

Ans: $\dfrac{\partial u}{\partial x}=\dfrac{1-x-v^2}{2u(1-xy)}$ & $\dfrac{\partial v}{\partial y}=\dfrac{1+y+u^2}{-2v(1-xy)}$

4. Errors and Approximations

Errors and approximations is an interesting topic for all engineers because all measurements taken from instruments or studies in real world are not exact which causes deviations (errors) on the results that we obtain from the solutions.

e.g The actual length of a field is 500 feet. A measuring instrument shows the length to be 508 feet.

$$\text{Error} = \text{Measured value} - \text{Actual value}$$

$$\text{Error} = 8 \text{ feet.}$$

Suppose $f(x, y)$ is a continuous function in x and y. If δx and δy are increments (small changes) in x and y then increment δf in f is given by

$$\delta f = f(x + \delta x, y + \delta y) - f(x, y)$$

1. δx is called **absolute error** in x
2. $\dfrac{\delta x}{x}$ is called **relative error** in x
3. $\dfrac{\delta x}{x} \times 100$ is called **percentage error** in x

Note: If $u = f(x, y)$ then $\delta u = \dfrac{\partial u}{\partial x}\delta x + \dfrac{\partial u}{\partial y}\delta y$

Note: If $u = f(x, y, z)$ then $\delta u = \dfrac{\partial u}{\partial x}\delta x + \dfrac{\partial u}{\partial y}\delta y + \dfrac{\partial u}{\partial z}\delta z$

Example 1: Find the percentage error in calculating the area of a rectangle when an error of 3% is made in measuring each of its sides.

Solution : Let a and b be the sides of the rectangle and A is its area.

$$A = ab$$

$$\log A = \log a + \log b$$

Differentiating we get , $\frac{1}{A}\delta A = \frac{1}{a}\delta a + \frac{1}{b}\delta b$

$$\Rightarrow \frac{\delta A}{A} \times 100 = \left(\frac{\delta a}{a} \times 100\right) + \left(\frac{\delta b}{b} \times 100\right)$$

Given: $\frac{\delta a}{a} \times 100 = 3\%$ and $\frac{\delta b}{b} \times 100 = 3\%$

$$\frac{\delta A}{A} \times 100 = 3\% + 3\% = 6\%$$

$\therefore$ Percentage error in calculating the area is 6%

Example 2: Find the percentage error in the area of an ellipse when errors of 2% and 3% are made in measuring its major and minor axes respectively.

Solution : Let $2a$ and $2b$ be the major & minor axes of the ellipse and A is its area.

$$A = \pi ab$$

$$\log A = \log \pi + \log a + \log b$$

Differentiating we get , $\frac{1}{A}\delta A = 0 + \frac{1}{a}\delta a + \frac{1}{b}\delta b$

$$\Rightarrow \frac{\delta A}{A} \times 100 = \left(\frac{\delta a}{a} \times 100\right) + \left(\frac{\delta b}{b} \times 100\right)$$

Given: $\frac{\delta a}{a} \times 100 = 2\%$ and $\frac{\delta b}{b} \times 100 = 3\%$

$$\frac{\delta A}{A} \times 100 = 2\% + 3\% = 5\%$$

$\therefore$ Percentage error in calculating the area is 5%

Example 3: Find the percentage error in computing parallel resistance r of two resistances r_1 & r_2 from the formula $\frac{1}{r} = \frac{1}{r_1} + \frac{1}{r_2}$ where r_1 & r_2 both are in error by 2% each.

Solution : We have $\frac{1}{r} = \frac{1}{r_1} + \frac{1}{r_2}$

On differentiating we get, $$-\frac{1}{r^2}\delta r = -\frac{1}{{r_1}^2}\delta r_1 - \frac{1}{{r_2}^2}\delta r_2$$

$$\frac{1}{r}\left(\frac{1}{r}\delta r\right) = \frac{1}{r_1}\left(\frac{1}{r_1}\delta r_1\right) + \frac{1}{r_2}\left(\frac{1}{r_2}\delta r_2\right)$$

$$\Longrightarrow \frac{1}{r}\left(\frac{\delta r}{r} \times 100\right) = \frac{1}{r_1}\left(\frac{\delta r_1}{r_1} \times 100\right) + \frac{1}{r_2}\left(\frac{\delta r_2}{r_2} \times 100\right)$$

Given: $\frac{\delta r_1}{r_1} \times 100 = 2\%$ and $\frac{\delta r_2}{r_2} \times 100 = 2\%$

$$\therefore \ \frac{1}{r}\left(\frac{\delta r}{r} \times 100\right) = \frac{1}{r_1}(2) + \frac{1}{r_2}(2)$$

$$\Longrightarrow \frac{1}{r}\left(\frac{\delta r}{r} \times 100\right) = 2\left(\frac{1}{r_1} + \frac{1}{r_2}\right) = \frac{2}{r}$$

$$\Longrightarrow \frac{\delta r}{r} \times 100 = 2$$

Percentage error in resistance r is also 2%

Example 4: The focal length f of a mirror is found from the formula $\frac{2}{f} = \frac{1}{v} - \frac{1}{u}$. Find percentage error in f if u and v both of error by 2% each.

Solution : We have $\frac{2}{f} = \frac{1}{v} - \frac{1}{u}$

On differentiating we get, $$-\frac{2}{f^2}\delta f = -\frac{1}{v^2}\delta v + \frac{1}{u^2}\delta u$$

$$-\frac{2}{f}\left(\frac{1}{f}\delta f\right) = -\frac{1}{v}\left(\frac{1}{v}\delta v\right) + \frac{1}{u}\left(\frac{1}{u}\delta u\right)$$

$$-\frac{2}{f}\left(\frac{\delta f}{f}\times 100\right) = -\frac{1}{v}\left(\frac{\delta v}{v}\times 100\right) + \frac{1}{u}\left(\frac{\delta u}{u}\times 100\right)$$

Given: $\frac{\delta v}{v} \times 100 = 2\%$ and $\frac{\delta u}{u} \times 100 = 2\%$

$$\therefore -\frac{2}{f}\left(\frac{\delta f}{f} \times 100\right) = -\frac{1}{v}(2) + \frac{1}{u}(2)$$

$$\Longrightarrow \frac{1}{f}\left(\frac{\delta f}{f} \times 100\right) = \frac{1}{v} - \frac{1}{u}$$

$$\Longrightarrow \frac{1}{f}\left(\frac{\delta f}{f} \times 100\right) = \frac{2}{f}$$

$$\Longrightarrow \frac{\delta f}{f} \times 100 = 2$$

Percentage error in focal length f is also 2%

Example 5: The resonant frequency in a series electrical circuit is given by $f = \dfrac{1}{2\pi\sqrt{LC}}$. If the measurement of L and C are in error by 2% and -1% respectively, find the percentage error in f.

Solution: $f = \dfrac{1}{2\pi\sqrt{LC}} \Longrightarrow \log f = \log\dfrac{1}{2\pi} + \log L^{-1/2} + \log C^{-1/2}$

$$= -\log 2\pi - \frac{1}{2}\log L - \frac{1}{2}\log C$$

Differentiating we get, $\dfrac{1}{f}\delta f = 0 - \dfrac{1}{2}\dfrac{1}{L}\delta L - \dfrac{1}{2}\dfrac{1}{C}\delta C$

$$\frac{\delta f}{f} \times 100 = -\frac{1}{2}\left[\left(\frac{\delta L}{L} \times 100\right) + \left(\frac{\delta C}{C} \times 100\right)\right]$$

$$\frac{\delta f}{f} \times 100 = -\frac{1}{2}[2 - 1] = -0.5\%$$

Hence, percentage error in $f = -0.5$ %.

Example 6: Area of a triangle is $\Delta = \frac{1}{2} bc \sin A$. Find percentage error in area if percentage errors in b, c and A are 1% , 2% and 3% . (Given $A = \frac{\pi}{4}$)

Solution : $\Delta = \frac{1}{2} bc \sin A$

Taking logarithm on both sides, we get,

$$\log \Delta = \log\left(\frac{1}{2}\right) + \log b + \log c + \log \sin A$$

Differentiating we get, $\frac{1}{\Delta}\delta\Delta = 0 + \frac{1}{b}\delta b + \frac{1}{c}\delta c + \frac{\cos A}{\sin A}\delta A$

$$\frac{\delta\Delta}{\Delta} \times 100 = \left(\frac{\delta b}{b} \times 100\right) + \left(\frac{\delta c}{c} \times 100\right) + \cot A\,(\delta A \times 100)$$

Given: $\dfrac{\delta b}{b} \times 100 = 1\%,$ $\dfrac{\delta c}{c} \times 100 = 2\%$ and $\dfrac{\delta A}{A} \times 100 = 3\%$

$$\frac{\delta \Delta}{\Delta} \times 100 = \left(\frac{\delta b}{b} \times 100\right) + \left(\frac{\delta c}{c} \times 100\right) + \cot A\,(\delta A \times 100)$$

$$\Longrightarrow \frac{\delta \Delta}{\Delta} \times 100 = 1 + 2 + \cot A\ \ (3 \times A)$$

$$\Longrightarrow \frac{\delta \Delta}{\Delta} \times 100 = 3 + \frac{3\pi}{4}\cot\frac{\pi}{4}$$

Percentage error in area Δ is $\left(3 + \dfrac{3\pi}{4}\right)\%$

Example 7: In calculating the volume of a right circular cone, errors of 2% and 1% are made in height and radius of base respectively. Find the percentage error in calculating the volume.

Solution: Let r be the radius of base, h height & V volume of a right circular cone.

$$V = \frac{1}{3}\pi r^2 h$$

$$\Longrightarrow \log V = \log\frac{\pi}{3} + 2\log r + \log h$$

Differentiating we get, $$\frac{1}{V}\delta V = 0 + \frac{2}{r}\delta r + \frac{1}{h}\delta h$$

$$\frac{\delta V}{V}\times 100 = 2\left(\frac{\delta r}{r}\times 100\right) + \left(\frac{\delta h}{h}\times 100\right)$$

$$\frac{\delta V}{V}\times 100 = 2(1\%) + 2\% = 4\%$$

Hence, percentage error in volume $= 4\%$.

Example 8: The diameter and the altitude of a can in the shape of a right circular cylinder are measured as 4 cm and 6 cm respectively. The possible error in each measurement is 0.1 cm. Find approximately the possible error in the values computed for volume and lateral surface area.

Solution: Let d and h are diameter and height of the right circular cylinder respectively and V be its volume.

$$V = \pi r^2 h = \pi \left(\frac{d}{2}\right)^2 h = \frac{\pi}{4} d^2 h$$

$$\Longrightarrow \log V = \log \frac{\pi}{4} + 2 \log d + \log h$$

Differentiating we get, $$\frac{1}{V}\delta V = 0 + \frac{2}{d}\delta d + \frac{1}{h}\delta h = \frac{2}{d}\delta d + \frac{1}{h}\delta h$$

$$\therefore \quad \delta V = V\left(\frac{2}{d}\delta d + \frac{1}{h}\delta h\right)$$

Given : $d = 4$ cm, $h = 6$ cm , $\delta d = 0.1$ cm & $\delta h = 0.1$ cm

$$\therefore V = \frac{\pi}{4} d^2 h = \frac{\pi}{4} \times 4^2 \times 6 = 75.36 \text{ cm}^3$$

$$\therefore \quad \delta V = (75.36)\left(\frac{2}{4}(0.1) + \frac{1}{6}(0.1)\right)$$

$$= (75.36)(0.067)$$

$$= 5.05 \text{ cm}^3$$

Hence, error in volume $= 5.05 \text{ cm}^3$.

Lateral surface area is, $\quad S = 2\pi r h = \pi d h$

$$\Longrightarrow \log S = \log 2\pi + \log d + \log h$$

$$\Longrightarrow \frac{1}{S}\delta S = 0 + \frac{1}{d}\delta d + \frac{1}{h}\delta h = \frac{1}{d}\delta d + \frac{1}{h}\delta h$$

$$\Longrightarrow \delta S = S\left(\frac{1}{d}\delta d + \frac{1}{h}\delta h\right)$$

Given : $d = 4 \text{ cm}, h = 6 \text{ cm}, \delta d = 0.1 \text{ cm} \;\&\; \delta h = 0.1 \text{ cm}$

$$\therefore S = \pi d h = \pi \times 4 \times 6 = 75.36 \text{ cm}^2$$

$$\delta S = S\left(\frac{1}{d}\delta d + \frac{1}{h}\delta h\right)$$

$$\therefore \quad \delta S = (75.36)\left(\frac{1}{4}(0.1) + \frac{1}{6}(0.1)\right)$$

$$= (75.36)(0.0416)$$

$$= 3.14 \text{ cm}^2$$

Hence, error in lateral surface area $= 3.14 \text{ cm}^2$.

Example 9: If $D = \dfrac{a^2}{b} + \dfrac{c^2}{2}$, Find the percentage error in D if error in measuring a is 0.5% and in measuring b and c are 1% each.

Solution: $D = \dfrac{a^2}{b} + \dfrac{c^2}{2}$

Differentiating we get, $\delta D = \dfrac{2a}{b}\delta a - \dfrac{a^2}{b^2}\delta b + \dfrac{2c}{2}\delta c$

$$\frac{\delta D}{D} \times 100 = \frac{100}{D}\left[\frac{2a}{b}\delta a - \frac{a^2}{b^2}\delta b + c\,\delta c\right]$$

$$\frac{\delta D}{D} \times 100 = \frac{1}{D}\left[\frac{2a^2}{b}\left(\frac{\delta a}{a} \times 100\right) - \frac{a^2}{b}\left(\frac{\delta b}{b} \times 100\right) + c^2\left(\frac{\delta c}{c} \times 100\right)\right]$$

Given : $\frac{\delta a}{a} \times 100 = 0.5, \qquad \frac{\delta b}{b} \times 100 = 1, \qquad \frac{\delta c}{c} \times 100 = 1$

$$\frac{\delta D}{D} \times 100 = \frac{1}{D}\left[\frac{2a^2}{b}(0.5) - \frac{a^2}{b}(1) + c^2(1)\right]$$

$$\frac{\delta D}{D} \times 100 = \frac{c^2}{D}$$

$$= \frac{c^2}{\frac{a^2}{b} + \frac{c^2}{2}}$$

$$= \frac{2bc^2}{2a^2 + 2c^2}$$

Hence, percentage error in D $= \frac{2bc^2}{2a^2 + 2c^2}\%$

Example 10: If the kinetic energy is calculated by the formula $T = \frac{mv^2}{2}$ and m changes from 49 to 49.5 and v changes from 1600 to 1590, find

a) approximately the change in T

b) percentage error in kinetic energy T

Solution: $T = \frac{mv^2}{2} \Longrightarrow \log T = \log m + 2\log v - \log 2$

Differentiating we get, $$\frac{\delta T}{T} = \frac{\delta m}{m} + 2\frac{\delta v}{v} - 0$$

$$\delta T = T\left[\frac{\delta m}{m} + 2\frac{\delta v}{v}\right]$$

Given : $m = 49, v = 1600$

$\delta m = 49.5 - 49 = 0.5$

$\delta v = 1590 - 1600 = -10$

$$T = \frac{mv^2}{2} = \frac{49 \times 1600^2}{2} = 6{,}27{,}20{,}000$$

$$\delta T = T\left[\frac{\delta m}{m} + 2\frac{\delta v}{v}\right] = (6{,}27{,}20{,}000)\left[\frac{0.5}{49} + 2\frac{(-10)}{1600}\right]$$

$$= -144000$$

Hence, approximate change in $T = -144000.$

Percentage error in T is given by,

$$\frac{\delta T}{T} \times 100 = \frac{-144000}{6{,}27{,}20{,}000} \times 100 = -0.2295$$

Hence, percentage error in $T = -0.2295\ \%$

Example 11: The period of a simple pendulum with small oscillations is $T = 2\pi\sqrt{\frac{l}{g}}$ if T is computed using $l = 8\,m, g = 32m/s^2$ find the % error in T if the true values of l and g are 8.05 m and 32.01 m/s^2 respectively.

Solution: $T = 2\pi\sqrt{\frac{l}{g}} \Longrightarrow \log T = \log 2\pi + \frac{1}{2}\log l - \frac{1}{2}\log g$

Differentiating we get $\frac{\delta T}{T} = 0 + \frac{1}{2}\frac{\delta l}{l} - \frac{1}{2}\frac{\delta g}{g}$

$$\delta T = \frac{T}{2}\left[\frac{\delta l}{l} - \frac{\delta g}{g}\right]$$

Given : $l = 8m, g = 32\frac{m}{s^2}, \delta l = 8.05 - 8 = 0.05m,$

$$\delta g = 32.01 - 32 = 0.01m/s^2$$

$$T = 2\pi\sqrt{\frac{8}{32}} = 3.14s$$

$$\delta T = \frac{T}{2}\left[\frac{\delta l}{l} - \frac{\delta g}{g}\right]$$

$$= \frac{(3.14)}{2}\left[\frac{0.05}{8} - \frac{0.01}{32}\right]$$

$$\delta T = 0.00932$$

Hence, error in $T = 0.00932\ s$

Example 12: Find approximate value of $[(3.82)^2 + 2(2.1)^3]^{1/5}$

Solution: Let $u = (x^2 + 2y^3)^{\frac{1}{5}}$

$$\delta u = \frac{\partial u}{\partial x}\delta x + \frac{\partial u}{\partial y}\delta y$$

$$\delta u = \frac{2x}{5}(x^2 + 2y^3)^{-\frac{4}{5}}\delta x + \frac{6y}{5}(x^2 + 2y^3)^{-\frac{4}{5}}\delta y$$

Let $x = 4,\qquad y = 2,$

$$\delta x = 3.82 - 4 = -0.18,$$

$$\delta y = 2.1 - 2 = 0.1$$

$$x^2 + 2y^3 = 4^2 + 2 \times 2^3 = 32$$

$$u = (x^2 + 2y^3)^{\frac{1}{5}} = (32)^{\frac{1}{5}} = 2$$

$$\delta u = \frac{2x}{5}(x^2 + 2\mathrm{y}^3)^{-\frac{4}{5}}\,\delta x + \frac{6y}{5}(x^2 + 2\mathrm{y}^3)^{-\frac{4}{5}}\,\delta y$$

$$\delta u = \frac{2 \times 4}{5}(32)^{-\frac{4}{5}}(-0.18) + \frac{6 \times 2}{5}(32)^{-\frac{4}{5}}(0.1)$$

$$\delta u = 0.012$$

$$\text{Approximate value} = u + \delta u$$

$$= 2 + 0.012$$

$$= 2.012$$

Example 13: Find approximate value of $[(0.98)^2 + (2.01)^2 + (1.94)^2]^{1/2}$

Solution: Let $$u = \sqrt{x^2 + y^2 + z^2}$$

$$\delta u = \frac{\partial u}{\partial x}\delta x + \frac{\partial u}{\partial y}\delta y + \frac{\partial u}{\partial z}\delta \mathrm{z}$$

$$\delta u = \frac{x}{\sqrt{x^2 + y^2 + z^2}}\delta x + \frac{x}{\sqrt{x^2 + y^2 + z^2}}\delta y + \frac{x}{\sqrt{x^2 + y^2 + z^2}}\delta \mathrm{z}$$

Let $x = 1,\quad y = 2,\quad z = 2,$

$$\delta x = 0.98 - 1 = -0.02,$$

$$\delta y = 2.01 - 2 = 0.01,$$

$$\delta z = 1.94 - 2 = -0.06$$

$$u = \sqrt{1^2 + 2^2 + 2^2} = 3$$

$$\delta u = \frac{x}{\sqrt{x^2+y^2+z^2}}\delta x + \frac{x}{\sqrt{x^2+y^2+z^2}}\delta y + \frac{x}{\sqrt{x^2+y^2+z^2}}\delta \mathrm{z}$$

$$\delta u = \frac{1}{3}(-0.02) + \frac{1}{3}(0.01) + \frac{1}{3}(-.006)$$

$$\delta u = -0.04$$

$$\text{Approximate value} = u + \delta u$$

$$= 3 - 0.04$$

$$= 2.96$$

Test Your Knowledge

1. In calculating the volume of a right circular cylinder, errors of 2% and 1% are found in measuring height and base radius respectively. Find the percentage error in calculated volume of the cylinder.
2. Find the possible percentage error in computing the parallel resistance R of three resistances R_1, R_2, R_3 if R_1, R_2, R_3 are each in error by 1.2%.
3. The diameter of a right circular cylinder are measured to be 5 inches and 8 inches respectively. If each of these dimensions were in error by 0.1 inch, find the % error in the volume of the cylinder.
4. The torsional rigidity N of a length l of a wire is obtained from the formula $N = \dfrac{8\pi l}{r^4 t^2}$. Find the % error in N due to -2 % error in t, 2 % error in r and 1.5 % error in l.
5. Find approximate value of $[(2.92)^3 + (5.87)^3]^{1/5}$
6. Find approximate value of $(1.99)^2\ (3.01)^3 (0.98)^{1/10}$

7. The deflection at a centre of a rod of length l and diameter d, supported at its ends and loaded at the centre with a weight w, varies as wl^3d^{-4}. What is the percentage increase in the deflection corresponding to the percentage increase in w, l and d of 3, 2 and 1 respectively.

8. The quantity Q of water flowing over a triangular notch is given by the formula $Q = CH^{5/2}$ where H is head of water and C is constant . Find the % error in Q if the error in H is that it is measured 0.198 instead of 0.2.

9. If the Horse power required to propel a steamer varies as the cube of the velocity and the square of length. If there is 3 % increase in velocity and 4 % increase in length, find the % increase in HP.

10. The time of swing t of a pendulum of length l under certain condition is given by $t = 2\pi\sqrt{\frac{l}{g'}}$ where $g' = g\left(\frac{r}{r+h}\right)^2$.Find % error in t, due to error p % in h and q % in l where r is constant.