Unit- I : Partial Differentiation

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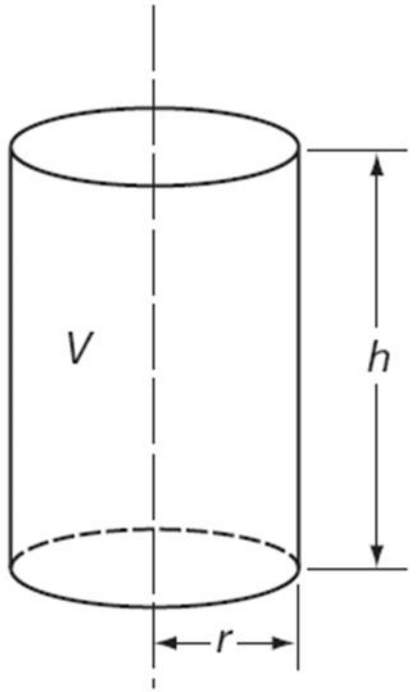
- Total derivative

Introduction

Consider the cylinder of radius r and height h.

The volume V of a cylinder is given by:

$$V = \pi r^2 h$$



If r is kept constant and h increases then V increases.

We can find the rate of change of V with respect to h

by differentiating V with respect to h, keeping r constant:

$$\left[\frac{dV}{dh}\right]_{r\text{ constant}} = \pi r^2$$

We write this as $\frac{\partial V}{\partial h} = \pi r^2$ and we read it as $dabba\ V$ by $dabba\ h$ or

$del\ V$ by $del\ h$.

This is called the first partial derivative of V with respect to h.

Similarly, if h is kept constant and r increases then V increases.

We can then find the rate of change of V by differentiating with respect to r keeping h constant:

$$\left[\frac{dV}{dr}\right]_{h\ \text{constant}} = 2\pi r h$$

We write this as $\frac{\partial V}{\partial r} = 2\pi r h$

We read it as $dabba\ V$ by $dabba\ r$ or $del\ V$ by $del\ r$

This is called the first partial derivative of V with respect to r.

How to find partial derivatives?

If $z = f(x, y)$ is a function of two real variables x and y, then

partial derivative of z w.r.to x is denoted by

$$\frac{\partial z}{\partial x} \text{ or } z_x \text{ or } \frac{\partial f}{\partial x} \text{ or } f_x$$

and **is the ordinary derivative of z w.r.to x by treating variable y as constant.**

Similarly partial derivative of z w.r.to y is denoted by

$$\frac{\partial z}{\partial y} \text{ or } z_y \text{ or } \frac{\partial f}{\partial y} \text{ or } f_y$$

and **is the ordinary derivative of z w.r.to y by treating variable x as constant.**

Note : (1) $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ are called as first order partial derivatives of z.

(2) $$\frac{\partial z}{\partial x} = \lim_{h \to 0} \frac{f(x+h,y) - f(x,y)}{h}$$

$$\frac{\partial z}{\partial y} = \lim_{k \to 0} \frac{f(x,y+k) - f(x,y)}{k}$$

(3) If $f(x,y,z)$ is a function of three variables x, y and z then partial derivative of f w.r.to any single variable is obtained by treating remaining two variables constant.

(4) Standard rules for derivatives of sum, difference, product and quotient are also applicable for partial derivatives.

1. Find the values of $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ at the point $(4,-5)$ if $f(x,y) = x^2 + 3xy + y - 1$

Solution: To find $\frac{\partial f}{\partial x}$, treat y as a constant and differentiate with respect to x.

$$\therefore \frac{\partial f}{\partial x} = 2x + 3y$$

To find $\frac{\partial f}{\partial y}$, treat x as a constant and differentiate with respect to y.

$$\therefore \frac{\partial f}{\partial y} = 3x + 1$$

$$\Rightarrow \left(\frac{\partial f}{\partial x}\right)_{(4,-5)} = 2(4) + 3(-5) = -7$$

$$\left(\frac{\partial f}{\partial y}\right)_{(4,-5)} = 3(4) + 1 = 13$$

2. Find the values of $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ where $f(x,y) = y\sin(xy)$.

Solution: $f(x,y) = y\sin(xy) \ldots\ldots\ldots (1)$

Differentiate (1) w.r.to x by treating y as a constant

$$\therefore \frac{\partial f}{\partial x} = y\cos(xy)\ \times y = y^2\cos(xy)$$

Differentiate (1) w.r.to y by treating x as a constant

$$\therefore\ \frac{\partial f}{\partial y} = \sin(xy) + y\cos(xy) \times x$$

$$=\ \sin(xy) + xy\cos\,(xy)$$

3. Find f_x and f_y if $f(x,y) = \dfrac{2y}{y+\cos x}$

Solution: $f_x = \dfrac{\partial f}{\partial x} = \dfrac{\partial}{\partial x}\left(\dfrac{2y}{y+\cos x}\right)$

$$= \frac{(y+\cos x)\frac{\partial}{\partial x}(2y) - (2y)\frac{\partial}{\partial x}(y+\cos x)}{(y+\cos x)^2}$$

$$= \frac{(y+\cos x)(0) - (2y)(0-\sin x)}{(y+\cos x)^2}$$

$$= \frac{0-(2y)(-\sin x)}{(y+\cos x)^2}$$

$$= \frac{2y\sin x}{(y+\cos x)^2}$$

$$f_y = \frac{\partial f}{\partial y} = \frac{\partial}{\partial y}\left(\frac{2y}{y+\cos x}\right)$$

$$= \frac{(y+\cos x)\frac{\partial}{\partial y}(2y) - (2y)\frac{\partial}{\partial y}(y+\cos x)}{(y+\cos x)^2}$$

$$= \frac{(y+\cos x)(2) - (2y)(1+0)}{(y+\cos x)^2}$$

$$= \frac{2(y+\cos x) - 2y}{(y+\cos x)^2}$$

$$= \frac{2y + 2\cos x - 2y}{(y+\cos x)^2}$$

$$= \frac{2\cos x}{(y+\cos x)^2}$$

Second-Order Partial Derivatives

By differentiating a function $z = f(x, y)$ twice,we get its second-order derivatives.
These derivatives are usually denoted by:

$$\frac{\partial^2 f}{\partial x^2} \text{ or } f_{xx} \text{ or } \frac{\partial^2 z}{\partial x^2} \text{ or } z_{xx}$$

$$\frac{\partial^2 f}{\partial y^2} \text{ or } f_{yy} \text{ or } \frac{\partial^2 z}{\partial y^2} \text{ or } z_{yy}$$

$$\frac{\partial^2 f}{\partial x \partial y} \text{ or } f_{xy} \text{ or } \frac{\partial^2 z}{\partial x \partial y} \text{ or } z_{xy}$$

$$\frac{\partial^2 f}{\partial y \partial x} \text{ or } f_{yx} \text{ or } \frac{\partial^2 z}{\partial y \partial x} \text{ or } z_{yx}$$

The defining equations are:

$$f_{xx} = \frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right)$$

$$f_{yy} = \frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right)$$

$$f_{xy} = \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)$$ → Differentiate first with respect to y, then with respect to x

$$f_{yx} = \frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)$$ → Differentiate first with respect to x, then with respect to y

Second Order Partial Derivatives

- Thus, **four second-order partial derivatives** can be obtained of a function of **two variables**:

$$
\begin{tikzcd}
 & & \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial x^2} \\
 & \frac{\partial f}{\partial x} \arrow[ur, "\frac{\partial}{\partial x}"] \arrow[r, "\frac{\partial}{\partial y}"'] & \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial y \partial x} \\
f \arrow[ur, "\frac{\partial}{\partial x}"] \arrow[dr, "\frac{\partial}{\partial y}"'] & & \\
 & \frac{\partial f}{\partial y} \arrow[r, "\frac{\partial}{\partial x}"] \arrow[dr, "\frac{\partial}{\partial y}"'] & \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial x \partial y} \\
 & & \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial y^2}
\end{tikzcd}
$$

$$\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial^2 f}{\partial x \partial y}$$

When both are continuous

1. If $f(x,y) = x\cos y + y\,e^x$, then find $\dfrac{\partial f}{\partial x}, \dfrac{\partial f}{\partial y}, \dfrac{\partial^2 f}{\partial x^2}, \dfrac{\partial^2 f}{\partial y^2}, \dfrac{\partial^2 f}{\partial x \partial y}, \dfrac{\partial^2 f}{\partial y \partial x}$

Solution:

$$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}(x\cos y + y\,e^x) = \cos y + y\,e^x$$

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}(x\cos y + y\,e^x) = -x\sin y + e^x$$

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial}{\partial x}(\cos y + y\,e^x) = 0 + y\,e^x = y\,e^x$$

$$\frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial}{\partial y}(-x\sin y + e^x) = -x\cos y + 0 = -x\cos y$$

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial}{\partial x}(-x \sin y + e^x)$$

$$= -\sin y + e^x$$

$$\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial}{\partial y}(\cos y + y\, e^x)$$

$$= -\sin y + e^x$$

2. If $u = \tan^{-1}\left(\frac{y}{x}\right)$ then find $\frac{\partial^2 u}{\partial x^2}, \frac{\partial^2 u}{\partial y^2}, \frac{\partial^2 u}{\partial x \partial y}$ and $\frac{\partial^2 u}{\partial y \partial x}$

Solution: $\frac{\partial u}{\partial x} = \frac{1}{1+\left(\frac{y}{x}\right)^2}\left(-\frac{y}{x^2}\right) = -\frac{y}{x^2+y^2}$

$$\frac{\partial u}{\partial y} = \frac{1}{1+\left(\frac{y}{x}\right)^2}\left(\frac{1}{x}\right) = \frac{x}{x^2+y^2}$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right) = \frac{\partial}{\partial x}\left(-\frac{y}{x^2+y^2}\right)$$

$$= -\frac{(x^2+y^2)\frac{\partial}{\partial x}(y) - y\frac{\partial}{\partial x}(x^2+y^2)}{(x^2+y^2)^2}$$

$$\frac{\partial^2 u}{\partial x^2} = -\frac{(x^2+y^2)(0) - y(2x)}{(x^2+y^2)^2} = \frac{2xy}{(x^2+y^2)^2}$$

$$\frac{\partial^2 u}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y}\right) = \frac{\partial}{\partial y}\left(\frac{x}{x^2+y^2}\right)$$

$$= \frac{(x^2+y^2)\frac{\partial}{\partial y}(x) - x\frac{\partial}{\partial y}(x^2+y^2)}{(x^2+y^2)^2}$$

$$= \frac{(x^2+y^2)(0) - x(2y)}{(x^2+y^2)^2} = -\frac{2xy}{(x^2+y^2)^2}$$

$$\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right) = \frac{\partial}{\partial x}\left(\frac{x}{x^2 + y^2}\right)$$

$$= \frac{(x^2+y^2)\frac{\partial}{\partial x}(x) - x\frac{\partial}{\partial x}(x^2 + y^2)}{(x^2 + y^2)^2}$$

$$= \frac{(x^2+y^2)(1) - x(2x)}{(x^2 + y^2)^2}$$

$$= \frac{x^2 + y^2 - 2x^2}{(x^2 + y^2)^2} = \frac{y^2 - x^2}{(x^2 + y^2)^2}$$

$$\frac{\partial^2 u}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}\right) = \frac{\partial}{\partial y}\left(\frac{-y}{x^2 + y^2}\right)$$

$$= -\frac{(x^2+y^2)\frac{\partial}{\partial y}(y) - y\frac{\partial}{\partial y}(x^2 + y^2)}{(x^2 + y^2)^2}$$

$$= -\frac{(x^2+y^2)(1) - y(2y)}{(x^2 + y^2)^2}$$

$$= -\frac{(x^2 + y^2 - 2y^2)}{(x^2 + y^2)^2} = \frac{y^2 - x^2}{(x^2 + y^2)^2}$$

3. Verify $\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 u}{\partial y \partial x}$ for i) $u = \log(x^2 + y^2)$ ii) $u = x^y + y^x$

Solution: i) $u = \log(x^2 + y^2)$

$$\frac{\partial u}{\partial x} = \frac{2x}{x^2 + y^2}$$

$$\frac{\partial u}{\partial y} = \frac{2y}{x^2 + y^2}$$

$$\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right) = \frac{\partial}{\partial x}\left(\frac{2y}{x^2 + y^2}\right) = 2y \frac{\partial}{\partial x}\left(\frac{1}{x^2 + y^2}\right)$$

$$= 2y\left[\frac{-1}{(x^2 + y^2)^2}\right](-2x) \qquad \left[\because \frac{d}{dx}\left(\frac{1}{f(x)}\right) = \frac{-1}{\left(f(x)\right)^2} f'(x)\right]$$

$$= \frac{-4xy}{(x^2 + y^2)^2}$$

$$\frac{\partial^2 u}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}\right) = \frac{\partial}{\partial y}\left(\frac{2x}{x^2+y^2}\right) = 2x\frac{\partial}{\partial y}\left(\frac{1}{x^2+y^2}\right)$$

$$= 2x\left[\frac{-1}{(x^2+y^2)^2}\right](-2y) \qquad \left[\because \frac{d}{dy}\left(\frac{1}{f(y)}\right) = \frac{-1}{\left(f(y)\right)^2} f'(y)\right]$$

$$= \frac{-4xy}{(x^2+y^2)^2}$$

$$\therefore \frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 u}{\partial y \partial x}$$

ii) $u = x^y + y^x$

$$\frac{\partial u}{\partial x} = yx^{y-1} + y^x \log y$$

$$\frac{\partial u}{\partial y} = x^y \log x + xy^{x-1}$$

$$\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right) = \frac{\partial}{\partial x}(x^y \log x + xy^{x-1})$$

$$= \frac{\partial}{\partial x}(x^y \log x) + \frac{\partial}{\partial x}(xy^{x-1})$$

$$= yx^{y-1} \log x + x^y . \frac{1}{x} + y^{x-1} + x\, y^{x-1} \log y$$

$$= x^{y-1} + y^{x-1} + yx^{y-1} \log x + x\, y^{x-1} \log y$$

$$\frac{\partial^2 u}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}\right) = \frac{\partial}{\partial y}(yx^{y-1} + y^x \log y)$$

$$= \frac{\partial}{\partial y}(yx^{y-1}) + \frac{\partial}{\partial y}(y^x \log y)$$

$$= x^{y-1} + y\, x^{y-1} \log x + xy^{x-1} \log y + y^x . \frac{1}{y}$$

$$= x^{y-1} + y^{x-1} + yx^{y-1} \log x + x\, y^{x-1} \log y$$

$$\therefore \frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 u}{\partial y \partial x}$$

4. If $u = \log(\tan x + \tan y + \tan z)$ then show that

$$\sin 2x \frac{\partial u}{\partial x} + \sin 2y \frac{\partial u}{\partial y} + \sin 2z \frac{\partial u}{\partial z} = 2$$

Solution: $u = \log(\tan x + \tan y + \tan z)$

$$\frac{\partial u}{\partial x} = \frac{1}{\tan x + \tan y + \tan z} . \sec^2 x$$

$$\frac{\partial u}{\partial y} = \frac{1}{\tan x + \tan y + \tan z} . \sec^2 y$$

$$\frac{\partial u}{\partial z} = \frac{1}{\tan x + \tan y + \tan z} . \sec^2 z$$

$$\sin 2x \frac{\partial u}{\partial x} = \sin 2x \frac{\sec^2 x}{\tan x + \tan y + \tan z}$$

$$\sin 2x \frac{\partial u}{\partial x} = \frac{2\sin x \cos x \sec^2 x}{\tan x + \tan y + \tan z} = \frac{2\sin x \sec x}{\tan x + \tan y + \tan z}$$

$$= \frac{2\tan x}{\tan x + \tan y + \tan z}$$

$$\sin 2y \frac{\partial u}{\partial y} = \sin 2y \frac{\sec^2 y}{\tan x + \tan y + \tan z}$$

$$= \frac{2\sin y \cos y \sec^2 y}{\tan x + \tan y + \tan z} = \frac{2\sin y \sec y}{\tan x + \tan y + \tan z}$$

$$= \frac{2\tan y}{\tan x + \tan y + \tan z}$$

$$\sin 2z \frac{\partial u}{\partial z} = \sin 2z \frac{\sec^2 z}{\tan x + \tan y + \tan z} = \frac{2 \sin z \cos z \sec^2 z}{\tan x + \tan y + \tan z}$$

$$= \frac{2 \sin z \sec z}{\tan x + \tan y + \tan z} = \frac{2 \tan z}{\tan x + \tan y + \tan z}$$

$$\therefore \sin 2x \frac{\partial u}{\partial x} + \sin 2y \frac{\partial u}{\partial y} + \sin 2z \frac{\partial u}{\partial z}$$

$$= \frac{2 \tan x}{\tan x + \tan y + \tan z} + \frac{2 \tan y}{\tan x + \tan y + \tan z} + \frac{2 \tan z}{\tan x + \tan y + \tan z}$$

$$= \frac{2(\tan x + \tan y + \tan z)}{\tan x + \tan y + \tan z}$$

$$= 2$$

5. If $u = \dfrac{x^2+y^2}{x+y}$, show that $\left(\dfrac{\partial u}{\partial x} - \dfrac{\partial u}{\partial y}\right)^2 = 4\left(1 - \dfrac{\partial u}{\partial x} - \dfrac{\partial u}{\partial y}\right)$.

Solution: $u = \dfrac{x^2+y^2}{x+y}$

$$\frac{\partial u}{\partial x} = \frac{\partial}{\partial y}\left(\frac{x^2+y^2}{x+y}\right) = \frac{(x+y)\frac{\partial}{\partial x}(x^2+y^2) - (x^2+y^2)\frac{\partial}{\partial x}(x+y)}{(x+y)^2}$$

$$= \frac{(x+y)(2x) - (x^2+y^2)(1)}{(x+y)^2}$$

$$= \frac{2x^2+2xy-x^2-y^2}{(x+y)^2} = \frac{x^2+2xy-y^2}{(x+y)^2}$$

$$\frac{\partial u}{\partial y} = \frac{\partial}{\partial y}\left(\frac{x^2+y^2}{x+y}\right) = \frac{(x+y)\frac{\partial}{\partial y}(x^2+y^2) - (x^2+y^2)\frac{\partial}{\partial y}(x+y)}{(x+y)^2}$$

$$= \frac{(x+y)(2y) - (x^2+y^2)(1)}{(x+y)^2}$$

$$= \frac{2xy + 2y^2 - x^2 - y^2}{(x+y)^2}$$

$$= \frac{y^2 + 2xy - x^2}{(x+y)^2}$$

$$\text{LHS} = \left(\frac{\partial u}{\partial x} - \frac{\partial u}{\partial y}\right)^2 = \left(\frac{x^2 + 2xy - y^2}{(x+y)^2} - \frac{y^2 + 2xy - x^2}{(x+y)^2}\right)^2$$

$$= \left(\frac{2x^2 - 2y^2}{(x+y)^2}\right)^2$$

$$= 4\left(\frac{x^2 - y^2}{(x+y)^2}\right)^2$$

$$= 4\left(\frac{(x-y)(x+y)}{(x+y)^2}\right)^2$$

$$= 4\,\frac{(x-y)^2}{(x+y)^2}$$

$$\text{RHS} = 4\left(1 - \frac{\partial u}{\partial x} - \frac{\partial u}{\partial y}\right)$$

$$= 4\left(1 - \frac{(x^2 + 2xy - y^2)}{(x+y)^2} - \frac{(y^2 + 2xy - x^2)}{(x+y)^2}\right)$$

$$= 4\left(\frac{(x+y)^2 - (x^2 + 2xy - y^2) - (y^2 + 2xy - x^2)}{(x+y)^2}\right)$$

$$= 4\left(\frac{x^2 + 2xy + y^2 - x^2 - 2xy + y^2 - y^2 - 2xy + x^2}{(x+y)^2}\right)$$

$$= 4\left(\frac{x^2 - 2xy + y^2}{(x+y)^2}\right) = 4\frac{(x-y)^2}{(x+y)^2}$$

$$\text{LHS} = \text{RHS}$$

6. If $u = \log(x^3 + y^3 - x^2y - y^2x)$ then show that

$$\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\right)^2 u = -\frac{4}{(x+y)^2} \text{ or } \frac{\partial^2 u}{\partial x^2} + 2\frac{\partial^2 u}{\partial x \partial y} + \frac{\partial^2 u}{\partial y^2} = -\frac{4}{(x+y)^2}$$

Solution: $\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\right)^2 u = \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\right)\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\right)u$

$$= \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\right)\left(\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y}\right) \ldots\ldots\ldots(1)$$

$u = \log(x^3 + y^3 - x^2y - y^2x) = \log(x^3 - x^2y + y^3 - y^2x)$

$= \log[x^2(x-y) - y^2(x-y)]$

$= \log(x^2 - y^2)(x-y)$

$= \log(x+y)(x-y)(x-y) = \log(x+y)(x-y)^2$

$= \log(x+y) + \log(x-y)^2$

$= \log(x+y) + 2\log(x-y)$

Differentiating u w.r.to x we get,

$$\frac{\partial u}{\partial x} = \frac{1}{x+y} + \frac{2}{x-y}$$

Differentiating u w.r.to y we get,

$$\frac{\partial u}{\partial y} = \frac{1}{x+y} - \frac{2}{x-y}$$

$$\therefore \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} = \frac{1}{x+y} + \frac{2}{x-y} + \frac{1}{x+y} - \frac{2}{x-y} = \frac{2}{x+y}$$

$$\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\right)^2 u = \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\right)\left(\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y}\right) = \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\right)\left(\frac{2}{x+y}\right)$$

$$= \frac{\partial}{\partial x}\left(\frac{2}{x+y}\right) + \frac{\partial}{\partial y}\left(\frac{2}{x+y}\right)$$

$$= -\frac{2}{(x+y)^2} - \frac{2}{(x+y)^2} = -\frac{4}{(x+y)^2}$$

7. Find the value of n if $u = r^n(3\cos^2\theta - 1)$ satisfies the equation

$$\frac{\partial}{\partial r}\left(r^2\frac{\partial u}{\partial r}\right) + \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial u}{\partial\theta}\right) = 0.$$

Solution: $u = r^n(3\cos^2\theta - 1) \ldots\ldots\ldots (1)$

Diff.(1) w.r.to r, we get

$$\frac{\partial u}{\partial r} = nr^{n-1}(3\cos^2\theta - 1)$$

$$\therefore\ r^2\frac{\partial u}{\partial r} = r^2 nr^{n-1}(3\cos^2\theta - 1) = nr^{n+1}(3\cos^2\theta - 1)$$

$$\Rightarrow \frac{\partial}{\partial r}\left(r^2\frac{\partial u}{\partial r}\right) = \frac{\partial}{\partial r}\left(nr^{n+1}(3\cos^2\theta - 1)\right)$$

$$= n(n+1)r^n(3\cos^2\theta - 1) \ldots\ldots\ldots (2)$$

Diff.(1) w.r.to θ, we get

$$\frac{\partial u}{\partial \theta} = r^n(-6\cos\theta\sin\theta) = -6\, r^n \sin\theta\cos\theta$$

$$\therefore \sin\theta\frac{\partial u}{\partial \theta} = -6r^n\sin^2\theta\cos\theta$$

$$\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial u}{\partial \theta}\right) = \frac{\partial}{\partial \theta}(-6r^n\sin^2\theta\cos\theta)$$

$$= -6r^n\left[\cos\theta\frac{\partial}{\partial \theta}(\sin^2\theta) + \sin^2\theta\frac{\partial}{\partial \theta}(\cos\theta)\right]$$

$$= -6r^n[\cos\theta\,(2\sin\theta\,\cos\theta) + \sin^2\theta(-\sin\theta)]$$

$$= -6r^n[2\sin\theta\cos^2\theta - \sin^3\theta]$$

$$= -6r^n\sin\theta\,[2\cos^2\theta - \sin^2\theta]$$

$$\frac{\partial}{\partial\theta}\left(\sin\theta\,\frac{\partial u}{\partial\theta}\right) = -6r^n \sin\theta\,[2\cos^2\theta - \sin^2\theta]$$

$$= -6r^n \sin\theta\,[2\cos^2\theta - (1 - \cos^2\theta)]$$

$$= -6r^n \sin\theta\,[3\cos^2\theta - 1]$$

$$\therefore \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\,\frac{\partial u}{\partial\theta}\right) = \frac{1}{\sin\theta}\left[-6r^n \sin\theta\,[3\cos^2\theta - 1]\right]$$

$$= -6r^n[3\cos^2\theta - 1] \ldots\ldots\ldots (3)$$

Now $u = r^n(3\cos^2\theta - 1)$ satisfies the equation

$$\frac{\partial}{\partial r}\left(r^2\frac{\partial u}{\partial r}\right) + \frac{1}{sin\theta}\frac{\partial}{\partial \theta}\left(sin\theta\frac{\partial u}{\partial \theta}\right) = 0.$$

From eq.(2) and (3),

$$\therefore n(n+1)r^n(3\cos^2\theta - 1) - 6r^n[3\cos^2\theta - 1] = 0$$

$$n(n+1)r^n(3\cos^2\theta - 1) = 6r^n[3\cos^2\theta - 1]$$

$$\Longrightarrow n(n+1) = 6$$

$$\Longrightarrow n^2 + n - 6 = 0$$

$$\Longrightarrow (n+3)(n-2) = 0 \Longrightarrow n = -3, 2$$

8. Find the value of n if $\theta = t^n e^{\frac{-r^2}{4t}}$ satisfies the equation

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\theta}{\partial r}\right) = \frac{\partial\theta}{\partial t}.$$

Solution: $\theta = t^n e^{\frac{-r^2}{4t}}$

Taking logarithm on both sides,

$$\log\theta = \log\left(t^n e^{\frac{-r^2}{4t}}\right) = \log t^n + \log e^{\frac{-r^2}{4t}}$$

$$\log\theta = n\log t - \frac{r^2}{4t} \dots\dots\dots (1)$$

Diff.(1) w.r.to r, we get

$$\frac{1}{\theta}\frac{\partial\theta}{\partial r} = 0 - \frac{2r}{4t} = -\frac{r}{2t}$$

$$\therefore \frac{\partial \theta}{\partial r} = -\frac{r\theta}{2t}$$

$$\Longrightarrow r^2 \frac{\partial \theta}{\partial r} = -\frac{r^3\theta}{2t}$$

$$\frac{\partial}{\partial r}\left(r^2 \frac{\partial \theta}{\partial r}\right) = \frac{\partial}{\partial r}\left(-\frac{r^3\theta}{2t}\right) = -\frac{1}{2t}\left[r^3 \frac{\partial \theta}{\partial r} + \theta \frac{\partial}{\partial r}(r^3)\right]$$

$$= -\frac{1}{2t}\left[r^3 \left(-\frac{r\theta}{2t}\right) + 3r^2\theta\right]$$

$$\frac{\partial}{\partial r}\left(r^2 \frac{\partial \theta}{\partial r}\right) = \frac{r^4 \theta}{4t^2} - \frac{3r^2 \theta}{2t}$$

$$\frac{\partial}{\partial r}\left(r^2 \frac{\partial \theta}{\partial r}\right) = \frac{r^4 \theta}{4t^2} - \frac{3r^2 \theta}{2t}$$

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial \theta}{\partial r}\right) = \frac{1}{r^2}\left[\frac{r^4 \theta}{4t^2} - \frac{3r^2 \theta}{2t}\right]$$

$$= \frac{r^2 \theta}{4t^2} - \frac{3\theta}{2t} \ldots\ldots\ldots (2)$$

Now Diff.(1) w.r.to t,

$$\therefore \frac{1}{\theta}\frac{\partial\theta}{\partial t} = \frac{n}{t} + \frac{r^2}{4t^2}$$

$$\Longrightarrow \frac{\partial\theta}{\partial t} = \frac{n\theta}{t} + \frac{r^2\theta}{4t^2} \ldots\ldots\ldots(3)$$

$\theta = t^n e^{\frac{-r^2}{4t}}$ satisfies the equation $\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\theta}{\partial r}\right) = \frac{\partial\theta}{\partial t}$.

From eq.(2) and (3),

$$\frac{r^2\theta}{4t^2} - \frac{3\theta}{2t} = \frac{n\theta}{t} + \frac{r^2\theta}{4t^2} \Longrightarrow -\frac{3\theta}{2t} = \frac{n\theta}{t} \Longrightarrow n = -\frac{3}{2}$$

9. If $u = \log(x^3 + y^3 + z^3 - 3xyz)$ then show that

$$\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right)^2 u = -\frac{9}{(x+y+z)^2}$$

Solution: $\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right)^2 u = \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right)\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right) u$

$$= \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right)\left(\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z}\right) \ldots\ldots\ldots (1)$$

$$u = \log(x^3 + y^3 + z^3 - 3xyz)$$

$$\frac{\partial u}{\partial x} = \frac{3x^2 - 3yz}{x^3 + y^3 + z^3 - 3xyz}$$

$$\frac{\partial u}{\partial y} = \frac{3y^2 - 3xz}{x^3 + y^3 + z^3 - 3xyz}$$

$$\frac{\partial u}{\partial z} = \frac{3z^2 - 3xy}{x^3 + y^3 + z^3 - 3xyz}$$

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z}$$

$$= \frac{3x^2 - 3yz}{x^3 + y^3 + z^3 - 3xyz} + \frac{3y^2 - 3xz}{x^3 + y^3 + z^3 - 3xyz} + \frac{3z^2 - 3xy}{x^3 + y^3 + z^3 - 3xyz}$$

$$= \frac{3x^2 - 3yz + 3y^2 - 3xz + 3z^2 - 3xy}{x^3 + y^3 + z^3 - 3xyz}$$

$$\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}=\frac{3(x^2+y^2+z^2-xy-yz-xz)}{x^3+y^3+z^3-3xyz}$$

$$=\frac{3(x^2+y^2+z^2-xy-yz-xz)}{x^3+y^3+z^3-3xyz}\frac{(x+y+z)}{(x+y+z)}$$

$$(x^2+y^2+z^2-xy-yz-xz)(x+y+z)=x^3+y^3+z^3-3xyz$$

$$\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}=\frac{3(x^3+y^3+z^3-3xyz)}{x^3+y^3+z^3-3xyz}\frac{1}{(x+y+z)}$$

$$\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}=\frac{3}{x+y+z}$$

Putting value of $\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}$ in (1) we get,

$$\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)^2 u=\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)\left(\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}\right)$$

$$\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)^2 u=\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)\left(\frac{3}{x+y+z}\right)$$

$$=\frac{\partial}{\partial x}\left(\frac{3}{x+y+z}\right)+\frac{\partial}{\partial y}\left(\frac{3}{x+y+z}\right)+\frac{\partial}{\partial z}\left(\frac{3}{x+y+z}\right)$$

$$=-\frac{3}{(x+y+z)^2}-\frac{3}{(x+y+z)^2}-\frac{3}{(x+y+z)^2}$$

$$=-\frac{9}{(x+y+z)^2}$$

Test Your Knowledge

1. If $u = x^3y - xy^3$ then find the value of $\dfrac{1}{\frac{\partial u}{\partial x}} + \dfrac{1}{\frac{\partial u}{\partial y}}$ at $(1, 2)$. **Ans**: $-\dfrac{13}{22}$

2. If $u = \log(x^2 + y^2) + \tan^{-1}\left(\dfrac{y}{x}\right)$ then show that $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0.$

3. If $u = \log(e^x + e^y)$ then show that $\dfrac{\partial^2 u}{\partial x^2} . \dfrac{\partial^2 u}{\partial y^2} = \left(\dfrac{\partial^2 u}{\partial x \partial y}\right)^2.$

4. If $z^3 - zx - y = 0$ then find $\dfrac{\partial z}{\partial x}, \dfrac{\partial z}{\partial y}$ and show that $\dfrac{\partial^2 z}{\partial x \partial y} = -\dfrac{(3z^2+x)}{(3z^2-x)^3}.$

5. If $u = (x^2 + y^2 + z^2)^{-\frac{1}{2}}$ then show that $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0$.

6. If $z = \tan^{-1}\left(\frac{2xy}{x^2 - y^2}\right)$ then show that $\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = 0$.

7. If $z = \tan(y + ax) - (y - ax)^{\frac{3}{2}}$ then show that $\frac{\partial^2 z}{\partial x^2} = a^2 \frac{\partial^2 z}{\partial y^2}$.

8. Find the value of n if $u = Ae^{-gx}\sin(nt - gx)$ satisfies the equation $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$ where g and A are constants.

Variables to be treated as Constant

In some problems, it is difficult to identify which variable is to be treated as constant. In such cases, the variable to be treated as constant is written as the suffix of the bracket.

e. g. Consider the equations $x = r\cos\theta$ and $y = r\sin\theta$.

To find $\frac{\partial r}{\partial x}$ we need a relation between r and x.

$$x = r\cos\theta \Rightarrow r = x\sec\theta$$

$$\Rightarrow \frac{\partial r}{\partial x} = \sec\theta \ldots\ldots\ldots (1)$$

Now $x^2 + y^2 = r^2\cos^2\theta + r^2\sin^2\theta = r^2$

$$\therefore\ r^2 = x^2 + y^2$$

$$\therefore r = \sqrt{x^2 + y^2} \ldots\ldots\ldots (2)$$

Differentiating (2) w.r.to x keeping y constant we get,

$$\frac{\partial r}{\partial x} = \frac{1}{2\sqrt{x^2 + y^2}} 2x = \frac{x}{\sqrt{x^2 + y^2}}$$

$$\Longrightarrow \frac{\partial r}{\partial x} = \frac{x}{r} = \cos\theta \ldots\ldots\ldots (3)$$

From (1), $\frac{\partial r}{\partial x} = \sec\theta$ and from (3), $\frac{\partial r}{\partial x} = \cos\theta$

These two values of $\frac{\partial r}{\partial x}$ make confusion.

To avoid the confusion we use the following notations:

Notations:

1. $\left(\frac{\partial r}{\partial x}\right)_\theta$ means the partial derivative of r w. r. to x keeping θ constant in a relation expressing r as a function of x and θ.

From $r = x \sec\theta$, $\left(\frac{\partial r}{\partial x}\right)_\theta = \sec\theta$

2. $\left(\frac{\partial r}{\partial x}\right)_y$ means the partial derivative of r w. r. to x keeping y constant in a relation expressing r as a function of x and y.

From $r^2 = x^2 + y^2$, $\left(\frac{\partial r}{\partial x}\right)_y = \frac{x}{r} = \cos\theta$

Remember: To find $\left(\frac{\partial r}{\partial x}\right)_y$, first express r as a function of x and y and then take derivative of r w. r. to x keeping y constant

1. If $x = r\cos\theta$ & $y = r\sin\theta$ then find $\left(\frac{\partial r}{\partial x}\right)_y, \left(\frac{\partial r}{\partial y}\right)_x, \left(\frac{\partial \theta}{\partial x}\right)_y$ and $\left(\frac{\partial \theta}{\partial y}\right)_x$

Solution:

To find $\left(\frac{\partial r}{\partial x}\right)_y$ & $\left(\frac{\partial r}{\partial y}\right)_x$, express r in terms of x and y.

$$\therefore\ x^2 + y^2 = r^2\cos^2\theta + r^2\sin^2\theta = r^2 \Longrightarrow r^2 = x^2 + y^2$$

$$\Longrightarrow\ 2r\left(\frac{\partial r}{\partial x}\right)_y = 2x \Longrightarrow \left(\frac{\partial r}{\partial x}\right)_y = \frac{x}{r} = \cos\theta$$

Also $2r\left(\frac{\partial r}{\partial y}\right)_x = 2y \Longrightarrow \left(\frac{\partial r}{\partial y}\right)_x = \frac{y}{r} = \sin\theta$

To find $\left(\frac{\partial\theta}{\partial x}\right)_y$ and $\left(\frac{\partial\theta}{\partial y}\right)_x$, express θ in terms of x and y.

$x = r\cos\theta$ and $y = r\sin\theta$

$$\Longrightarrow \frac{y}{x} = \frac{r\sin\theta}{r\cos\theta} = \tan\theta$$

$$\Longrightarrow \theta = \tan^{-1}\left(\frac{y}{x}\right)$$

$$\left(\frac{\partial\theta}{\partial x}\right)_y = \frac{1}{1+\left(\frac{y}{x}\right)^2}\left(-\frac{y}{x^2}\right) = -\frac{y}{x^2+y^2}$$

$$\left(\frac{\partial\theta}{\partial y}\right)_x = \frac{1}{1+\left(\frac{y}{x}\right)^2}\left(\frac{1}{x}\right) = \frac{x}{x^2+y^2}$$

2. If $x^2 = au + bv, y^2 = au - bv$ then show that

$$\left(\frac{\partial u}{\partial x}\right)_y \left(\frac{\partial x}{\partial u}\right)_v = \frac{1}{2} = \left(\frac{\partial v}{\partial y}\right)_x \left(\frac{\partial y}{\partial v}\right)_u$$

Solution:

To find $\left(\frac{\partial u}{\partial x}\right)_y$, express u in terms of x and y.

$$x^2 = au + bv \;\; \& \;\; y^2 = au - bv$$

$$\Rightarrow x^2 + y^2 = au + bv + au - bv = 2au$$

$$\Rightarrow u = \frac{x^2 + y^2}{2a}$$

$$\therefore \left(\frac{\partial u}{\partial x}\right)_y = \frac{2x}{2a} = \frac{x}{a} \ldots\ldots\ldots (1)$$

To find $\left(\frac{\partial x}{\partial u}\right)_v$, express x in terms of u and v.

$$x^2 = au + bv \Longrightarrow x = \sqrt{au + bv}$$

$$\therefore \left(\frac{\partial x}{\partial u}\right)_v = \frac{a}{2\sqrt{au + bv}} = \frac{a}{2x} \ldots\ldots\ldots (2)$$

To find $\left(\frac{\partial v}{\partial y}\right)_x$, express v in terms of x and y.

$$x^2 = au + bv \ \& \ y^2 = au - bv$$

$$\Longrightarrow x^2 - y^2 = au + bv - (au - bv) = 2bv$$

$$\Longrightarrow v = \frac{x^2 - y^2}{2b}$$

$$\therefore \left(\frac{\partial v}{\partial y}\right)_x = \frac{-2y}{2b} = \frac{-y}{b} \ldots\ldots\ldots (3)$$

To find $\left(\frac{\partial y}{\partial v}\right)_u$, express y in terms of u and v.

$$y^2 = au - bv \Longrightarrow y = \sqrt{au - bv}$$

$$\therefore \left(\frac{\partial y}{\partial v}\right)_u = \frac{-b}{2\sqrt{au - bv}} = \frac{-b}{2y} \dots\dots\dots(4)$$

$$\left(\frac{\partial u}{\partial x}\right)_y \left(\frac{\partial x}{\partial u}\right)_v = \frac{x}{a} \times \frac{a}{2x} = \frac{1}{2}$$

$$\left(\frac{\partial v}{\partial y}\right)_x \left(\frac{\partial y}{\partial v}\right)_u = \frac{-y}{b} \times \frac{-b}{2y} = \frac{1}{2}$$

3. If $x = \frac{r}{2}\left(e^{\theta} + e^{-\theta}\right)$ and $y = \frac{r}{2}\left(e^{\theta} - e^{-\theta}\right)$ then show that $\left(\frac{\partial x}{\partial r}\right)_{\theta} = \left(\frac{\partial r}{\partial x}\right)_{y}$.

Solution: To find $\left(\frac{\partial x}{\partial r}\right)_{\theta}$, express x in terms of r and θ.

$$x = \frac{r}{2}\left(e^{\theta} + e^{-\theta}\right) \Longrightarrow \left(\frac{\partial x}{\partial r}\right)_{\theta} = \frac{e^{\theta} + e^{-\theta}}{2} \dots\dots\dots (1)$$

To find $\left(\frac{\partial r}{\partial x}\right)_{y}$, express r in terms of x and y.

$$x = \frac{r}{2}\left(e^{\theta} + e^{-\theta}\right) \text{ and } y = \frac{r}{2}\left(e^{\theta} - e^{-\theta}\right)$$

$$\Longrightarrow x^2 - y^2 = \frac{r^2}{4}\left(e^{\theta} + e^{-\theta}\right)^2 - \frac{r^2}{4}\left(e^{\theta} - e^{-\theta}\right)^2$$

$$\Rightarrow x^2 - y^2 = \frac{r^2}{4}\left(e^{2\theta} + e^{-2\theta} + 2e^{\theta}e^{-\theta}\right) - \frac{r^2}{4}(e^{2\theta} + e^{-2\theta} - 2e^{\theta}e^{-\theta})$$

$$\Rightarrow x^2 - y^2 = \frac{r^2}{4}\left(e^{2\theta} + e^{-2\theta} + 2 - e^{2\theta} - e^{-2\theta} + 2\right)$$

$$\Rightarrow x^2 - y^2 = \frac{r^2}{4}(4) = r^2$$

$$\Rightarrow r = \sqrt{x^2 - y^2}$$

$$\therefore \left(\frac{\partial r}{\partial x}\right)_y = \frac{2x}{2\sqrt{x^2 - y^2}} = \frac{x}{\sqrt{x^2 - y^2}} = \frac{x}{r} = \frac{e^{\theta} + e^{-\theta}}{2} \ldots\ldots\ldots(2)$$

From (1) & (2), $\left(\frac{\partial x}{\partial r}\right)_\theta = \left(\frac{\partial r}{\partial x}\right)_y$

4. If $ux + vy = 0$ and $\frac{u}{x} + \frac{v}{y} = 1$ show that $\frac{u}{x}\left(\frac{\partial x}{\partial u}\right)_v + \frac{v}{y}\left(\frac{\partial y}{\partial v}\right)_u = 0$.

Solution:

To find $\left(\frac{\partial x}{\partial u}\right)_v$, express x in terms of u and v.

$$ux + vy = 0 \Longrightarrow y = -\frac{ux}{v} \ldots\ldots\ldots (1)$$

$$\frac{u}{x} + \frac{v}{y} = 1 \Longrightarrow \frac{v}{y} = 1 - \frac{u}{x} = \frac{x-u}{x}$$

$$\frac{y}{v} = \frac{x}{x-u} \Longrightarrow y = \frac{vx}{x-u} \ldots\ldots\ldots (2)$$

From (1) and (2), $$-\frac{ux}{v} = \frac{vx}{x-u}$$

$$-\frac{u}{v} = \frac{v}{x-u} \Longrightarrow -u(x-u) = v^2$$

$$\Longrightarrow -ux + u^2 = v^2 \Longrightarrow x = \frac{u^2 - v^2}{u}$$

$$y = -\frac{ux}{v} = -\frac{u}{v}\left(\frac{u^2 - v^2}{u}\right) = \frac{v^2 - u^2}{v}$$

$$x = \frac{u^2 - v^2}{u} \Longrightarrow \left(\frac{\partial x}{\partial u}\right)_v = \frac{u(2u) - (u^2 - v^2)(1)}{u^2}$$

$$\Longrightarrow \left(\frac{\partial x}{\partial u}\right)_v = \frac{2u^2 - u^2 + v^2}{u^2} = \frac{u^2 + v^2}{u^2}$$

$$y=\frac{v^2-u^2}{v}\Longrightarrow\left(\frac{\partial y}{\partial v}\right)_u=\frac{v(2v)-(v^2-u^2)(1)}{v^2}$$

$$\Longrightarrow\left(\frac{\partial y}{\partial v}\right)_u=\frac{2v^2-v^2+u^2}{v^2}=\frac{u^2+v^2}{v^2}$$

$$\frac{u}{x}\left(\frac{\partial x}{\partial u}\right)_v+\frac{v}{y}\left(\frac{\partial y}{\partial v}\right)_u=\frac{u}{x}\left(\frac{u^2+v^2}{u^2}\right)+\frac{v}{y}\left(\frac{u^2+v^2}{v^2}\right)$$

$$=\frac{u^2+v^2}{ux}+\frac{u^2+v^2}{vy}=(u^2+v^2)\left(\frac{1}{ux}+\frac{1}{vy}\right)$$

$$=(u^2+v^2)\left(\frac{vy+ux}{uxvy}\right)$$

$$=(u^2+v^2)\left(\frac{0}{uxvy}\right)=0$$

Test your Knowledge

1) If $u.x+v.y=0,\ \ \frac{u}{x}+\frac{v}{y}=1,$ prove that $\left(\frac{\partial u}{\partial x}\right)_y+\left(\frac{\partial v}{\partial y}\right)_x=\frac{x^2+y^2}{y^2-x^2}$

2) If $x=r\cos\theta, y=r\sin\theta,$ Prove that a) $\left(\frac{\partial r}{\partial x}\right)_y=\left(\frac{\partial x}{\partial y}\right)_\theta$, b) $\frac{1}{r}\left(\frac{\partial x}{\partial \theta}\right)_r=r\left(\frac{\partial \theta}{\partial x}\right)_y$, c) $\frac{\partial^2\theta}{\partial x^2}+\frac{\partial^2\theta}{\partial y^2}=0$

.

3) If $x=\frac{\cos\theta}{u},\ y=\frac{\sin\theta}{u}$, evaluate $\left(\frac{\partial x}{\partial u}\right)_\theta\left(\frac{\partial u}{\partial x}\right)_y+\left(\frac{\partial y}{\partial u}\right)_\theta\left(\frac{\partial u}{\partial y}\right)_x$.

4) If $x=u\tan v,\ y=u\sec v$, prove that $\left(\frac{\partial u}{\partial x}\right)_y\left(\frac{\partial v}{\partial x}\right)_y=\left(\frac{\partial u}{\partial y}\right)_x\left(\frac{\partial v}{\partial y}\right)_x$.

Euler's Theorem on Homogenous Functions

Homogenous function of degree *n* means?

A function $f(x,y)$ of two variables *x* and *y* is said to homogeneous function of degree *n* if

$$f(x,y) = x^n \phi\left(\frac{y}{x}\right) \quad \text{or} \quad f(x,y) = y^n \psi\left(\frac{x}{y}\right)$$

Alternately, function $f(x,y)$ of two variables *x* and *y* is said to homogeneous function of degree *n* if

$$f(tx,ty) = t^n f(x,y) \text{ where } t \text{ is a parameter.}$$

e.g.1) $f(x,y) = x^2 + y^2$ is a homogeneous function of degree 2.

$$f(tx,ty) = (tx)^2 + (ty)^2 = t^2(x^2 + y^2) = t^2 f(x,y)$$

2) $f(x,y) = \dfrac{x^2y^3}{x-y}$ is a homogeneous function of degree 4.

$$f(tx,ty) = \frac{(tx)^2(ty)^3}{tx-ty} = \frac{t^5x^2y^3}{t(x-y)} = t^4 f(x,y)$$

3) $f(x,y) = \log(x^2+y^2)$ is not a homogeneous function

$$f(tx,ty) = \log\left((tx)^2+(ty)^2\right) = \log t^2(x^2+y^2) \neq t^2\log(x^2+y^2)$$

4) $f(x,y) = \tan^{-1}\left(\dfrac{x}{y}\right)$ is a homogeneous function of degree 0.

$$f(tx,ty) = \tan^{-1}\left(\frac{tx}{ty}\right) = \tan^{-1}\left(\frac{x}{y}\right) = t^0 f(x,y)$$

Euler's Theorem

If u is a homogeneous function of x and y of degree n then

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = nu$$

Proof: u is a homogeneous function of x and y of degree n.

$$\Longrightarrow u = x^n f\left(\frac{y}{x}\right)$$

$$\Longrightarrow \frac{\partial u}{\partial x} = x^n f'\left(\frac{y}{x}\right)\left(-\frac{y}{x^2}\right) + nx^{n-1} f\left(\frac{y}{x}\right)$$

$$\therefore x\frac{\partial u}{\partial x} = -yx^{n-1} f'\left(\frac{y}{x}\right) + nx^n f\left(\frac{y}{x}\right) \ldots\ldots\ldots (1)$$

$$\frac{\partial u}{\partial y} = x^n f'\left(\frac{y}{x}\right)\left(\frac{1}{x}\right)$$

$$y\frac{\partial u}{\partial y} = yx^{n-1}f'\left(\frac{y}{x}\right) \dots \dots \dots \ (2)$$

Adding (1) and (2), we get

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = -yx^{n-1}f'\left(\frac{y}{x}\right) + nx^n f\left(\frac{y}{x}\right) + yx^{n-1}f'\left(\frac{y}{x}\right)$$

$$\Longrightarrow x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = nx^n f\left(\frac{y}{x}\right)$$

$$\therefore x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = nu \dots \dots \dots \text{Hence the proof}$$

Euler's Theorem for homogeneous function of three variables:

If u is a homogeneous function of three variables x, y and z of degree n then

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = nu$$

Deductions from Euler's Theorem:

Corollary 1:

If u is a homogeneous function of x and y of degree n then

$$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2} = n(n-1)u$$

1. Verify Euler's Theorem for $u = ax^2 + 2bxy + cy^2$.

Solution: $u = f(x, y) = ax^2 + 2bxy + cy^2$

Replacing x by tx and y by ty in $u = f(x, y)$,

$f(tx, ty) = a(tx)^2 + 2btxty + c(ty)^2 = t^2(ax^2 + 2bxy + cy^2)$

$f(tx, ty) = t^2 f(x, y)$

Thus, $u = ax^2 + 2bxy + cy^2$ is a homogeneous function of degree $n = 2$.

By Euler's Theorem,

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = nu$$

$$\Rightarrow x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 2u = 2(ax^2 + 2bxy + cy^2) \ldots\ldots\ldots (1)$$

Verification: $u = ax^2 + 2bxy + cy^2$

$$\frac{\partial u}{\partial x} = 2ax + 2by$$

$$\frac{\partial u}{\partial y} = 2bx + 2cy$$

$$\Longrightarrow x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = x(2ax + 2by) + y(2bx + 2cy)$$

$$\Longrightarrow x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 2ax^2 + 2bxy + 2bxy + 2cy^2$$

$$\Longrightarrow x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 2(ax^2 + 2bxy + cy^2) \ldots\ldots\ldots (2)$$

From (1) and (2), Euler's Theorem is verified

2. Verify Euler′s Theorem for $u = (\sqrt{x} + \sqrt{y})(x^n + y^n)$.

Solution: $u = f(x, y) = (\sqrt{x} + \sqrt{y})(x^n + y^n)$

Replacing x by tx and y by ty in $u = f(x, y)$,

$f(tx, ty) = \left(\sqrt{tx} + \sqrt{ty}\right)(t^n x^n + t^n y^n) = \sqrt{t}\,t^n(\sqrt{x} + \sqrt{y})(x^n + y^n)$

$f(tx, ty) = t^{n+\frac{1}{2}} f(x, y)$

Thus, $u = (\sqrt{x} + \sqrt{y})(x^n + y^n)$ is a homogeneous function of degree $n + \dfrac{1}{2}$.

By Euler's Theorem,

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \left(n + \frac{1}{2}\right)u$$

$$\Longrightarrow x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \left(n + \frac{1}{2}\right)(\sqrt{x} + \sqrt{y})(x^n + y^n) \ldots\ldots\ldots (1)$$

Verification: $u=(\sqrt{x}+\sqrt{y})(x^n+y^n)$

$$\frac{\partial u}{\partial x}=\frac{1}{2\sqrt{x}}(x^n+y^n)+nx^{n-1}(\sqrt{x}+\sqrt{y})$$

$$\frac{\partial u}{\partial y}=\frac{1}{2\sqrt{y}}(x^n+y^n)+ny^{n-1}(\sqrt{x}+\sqrt{y})$$

$$\Rightarrow x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}$$

$$=x\left[\frac{1}{2\sqrt{x}}(x^n+y^n)+nx^{n-1}(\sqrt{x}+\sqrt{y})\right]$$

$$+y\left[\frac{1}{2\sqrt{y}}(x^n+y^n)+ny^{n-1}(\sqrt{x}+\sqrt{y})\right]$$

$$= \frac{\sqrt{x}}{2}(x^n + y^n) + nx^n(\sqrt{x} + \sqrt{y}) + \frac{\sqrt{y}}{2}(x^n + y^n) + ny^n(\sqrt{x} + \sqrt{y})$$

$$= \frac{1}{2}(\sqrt{x} + \sqrt{y})(x^n + y^n) + n(x^n + y^n)(\sqrt{x} + \sqrt{y})$$

$$= \left(n + \frac{1}{2}\right)(\sqrt{x} + \sqrt{y})(x^n + y^n) \ldots \ldots \ldots (2)$$

From (1) and (2), Euler's Theorem is verified

2. Find $x\dfrac{\partial u}{\partial x}+y\dfrac{\partial u}{\partial y}$ if $u=8(x^2+y^2)(\log x-\log y)$.

Solution: $u=8(x^2+y^2)(\log x-\log y)=8(x^2+y^2)\log\left(\dfrac{x}{y}\right)$

Replacing x by tx and y by ty in $u=f(x,y)$,

$$f(tx,ty)=8[(tx)^2+(ty)^2]\log\left(\frac{tx}{ty}\right)=t^2\left[8(x^2+y^2)\log\left(\frac{x}{y}\right)\right]$$
$$=t^2 f(x,y)$$

Therfore, u is a homogeneous function of degree $n=2$.

By Euler's Theorem,

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=nu=2u$$
$$=16(x^2+y^2)(\log x-\log y).$$

3. If $u = e^{\frac{x}{y}} \sin\left(\frac{x}{y}\right) + e^{\frac{y}{x}} \cos\left(\frac{y}{x}\right)$, prove that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 0$.

Solution: $u = e^{\frac{x}{y}} \sin\left(\frac{x}{y}\right) + e^{\frac{y}{x}} \cos\left(\frac{y}{x}\right)$

Replacing x by tx and y by ty in $u = f(x, y)$,

$$f(tx, ty) = e^{\frac{tx}{ty}} \sin\left(\frac{tx}{ty}\right) + e^{\frac{ty}{tx}} \cos\left(\frac{ty}{tx}\right) = e^{\frac{x}{y}} \sin\left(\frac{x}{y}\right) + e^{\frac{y}{x}} \cos\left(\frac{y}{x}\right)$$

$$= t^0 f(x, y)$$

Therfore, u is a homogeneous function of degree $n = 0$.

By Euler's Theorem,

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = nu = 0u = 0$$

4. If $u = \frac{1}{x^2} + \frac{1}{y^2} + \frac{\log x - \log y}{x^2 + y^2}$ then show that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + 2u = 0.$

Solution: $u = \frac{1}{x^2} + \frac{1}{y^2} + \frac{\log x - \log y}{x^2 + y^2} = \frac{1}{x^2} + \frac{1}{y^2} + \frac{\log\left(\frac{x}{y}\right)}{x^2 + y^2}$

Replacing x by tx and y by ty in $u = f(x, y)$,

$$f(tx, ty) = \frac{1}{(tx)^2} + \frac{1}{(ty)^2} + \frac{\log\left(\frac{tx}{ty}\right)}{(tx)^2 + (ty)^2}$$

$$= \frac{1}{t^2x^2} + \frac{1}{t^2y^2} + \frac{\log\left(\frac{x}{y}\right)}{t^2x^2 + t^2y^2}$$

$$= \frac{1}{t^2}\left[\frac{1}{x^2} + \frac{1}{y^2} + \frac{\log\left(\frac{x}{y}\right)}{x^2 + y^2}\right]$$

$$f(tx,ty)=\frac{1}{t^2}\left[\frac{1}{x^2}+\frac{1}{y^2}+\frac{\log\left(\frac{x}{y}\right)}{x^2+y^2}\right]$$

$$=\frac{1}{t^2}f(x,y)$$

$$f(tx,ty)=t^{-2}f(x,y)$$

Therfore, $u=f(x,y)=\frac{1}{x^2}+\frac{1}{y^2}+\frac{\log x-\log y}{x^2+y^2}$ is a homogeneous function of degree $n=-2$.

By Euler's Theorem,

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=nu$$

$$\Longrightarrow x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=-2u\Longrightarrow x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}+2u=0$$

5. If $u = \dfrac{x^3y^3}{x^3+y^3}$ then show that $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} = 3u$

and $x^2\dfrac{\partial^2 u}{\partial x^2} + 2xy\dfrac{\partial^2 u}{\partial x \partial y} + y^2\dfrac{\partial^2 u}{\partial y^2} = 6u.$

Solution: $u = f(x,y) = \dfrac{x^3y^3}{x^3+y^3}$

Replacing x by tx and y by ty in $u = f(x,y)$,

$$f(tx,ty) = \frac{(tx)^3(ty)^3}{(tx)^3+(ty)^3} = \frac{t^6x^3y^3}{t^3(x^3+y^3)} = t^3\frac{x^3y^3}{(x^3+y^3)}$$

$$f(tx,ty) = t^3 f(x,y)$$

Therfore, $u = \dfrac{x^3y^3}{x^3+y^3}$ is a homogeneous function of degree $n = 3$.

By Euler's Theorem,

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = nu$$

$$\Longrightarrow x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 3u$$

$$\text{Also } x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2} = n(n-1)u$$

$$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2} = 3(3-1)u = 6u$$

6. If $u = x^2 e^{\frac{y}{x}} + y^2 \tan^{-1}\left(\frac{x}{y}\right)$ then show that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 2u$ and $x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2} = 2u.$

Solution: $u = f(x,y) = x^2 e^{\frac{y}{x}} + y^2 \tan^{-1}\left(\frac{x}{y}\right)$

Replacing x by tx and y by ty in $u = f(x,y)$,

$$f(tx,ty) = (tx)^2 e^{\frac{ty}{tx}} + (ty)^2 \tan^{-1}\left(\frac{tx}{ty}\right) = t^2x^2 e^{\frac{y}{x}} + t^2y^2 \tan^{-1}\left(\frac{x}{y}\right)$$

$$f(tx,ty) = t^2\left[x^2 e^{\frac{y}{x}} + y^2 \tan^{-1}\left(\frac{x}{y}\right)\right] = t^2 f(x,y)$$

Thus, $u = x^2 e^{\frac{y}{x}} + y^2 \tan^{-1}\left(\frac{x}{y}\right)$ is a homogeneous function of degree $n = 2$.

By Euler's Theorem,

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=nu$$

$$\Longrightarrow x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=2u$$

$$\text{Also } x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2}=n(n-1)u$$

$$\therefore x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2}=2(2-1)u=2u$$

7. If $T = \sin\left(\frac{xy}{x^2+y^2}\right) + \sqrt{x^2+y^2} + \frac{x^2y}{x+y}$, find the value of $x\frac{\partial T}{\partial x} + y\frac{\partial T}{\partial y}$

Solution: Let $u = \sin\left(\frac{xy}{x^2+y^2}\right), v = \sqrt{x^2+y^2}, w = \frac{x^2y}{x+y}$

$\therefore T = u + v + w$

Replacing x by tx and y by ty in u, v and w,

$$u = \sin\left(\frac{tx\,ty}{(tx)^2+(ty)^2}\right) = \sin\left(\frac{xy}{x^2+y^2}\right) = t^0\sin\left(\frac{xy}{x^2+y^2}\right)$$

$$v = \sqrt{(tx)^2+(ty)^2} = \sqrt{t^2(x^2+y^2)} = t\sqrt{x^2+y^2}$$

$$w = \frac{(tx)^2\,ty}{tx+ty} = t^2\frac{x^2y}{x+y}$$

Thus, u, v and w are homogeneous functions of degree 0,1 & 2 respectively

By Euler's Theorem,

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=0u=0$$

$$x\frac{\partial v}{\partial x}+y\frac{\partial v}{\partial y}=1v=v$$

$$x\frac{\partial w}{\partial x}+y\frac{\partial w}{\partial y}=2w$$

$$T=u+v+w\Rightarrow\frac{\partial T}{\partial x}=\frac{\partial u}{\partial x}+\frac{\partial v}{\partial x}+\frac{\partial w}{\partial x}\ \text{ and}$$

$$\frac{\partial T}{\partial y}=\frac{\partial u}{\partial y}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial y}$$

$$\therefore x\frac{\partial T}{\partial x} + y\frac{\partial T}{\partial y} = x\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial x} + \frac{\partial w}{\partial x}\right) + y\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial y}\right)$$

$$= \left(x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}\right) + \left(x\frac{\partial v}{\partial x} + y\frac{\partial v}{\partial y}\right) + \left(x\frac{\partial w}{\partial x} + y\frac{\partial w}{\partial y}\right)$$

$$= 0 + v + 2w$$

$$= \sqrt{x^2 + y^2} + \frac{2x^2 y}{x + y}$$

Test Your Knowledge

1. Verify Euler′s Theorem for (i) $u = \dfrac{x^2 + y^2}{x + y}$ (ii) $u = x^4 y^2 \sin^{-1}\left(\dfrac{y}{x}\right)$

   (iii) $u = 3x^2yz + 5xy^2z + 4z^4$ (iv) $u = \sqrt{x} + \sqrt{y} + \sqrt{z}$

2. If $u = \sin^{-1}\left[\dfrac{x}{y}\right] + \tan^{-1}\left[\dfrac{y}{x}\right]$ then show that $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} = 0.$

3. If $u = f\left(\dfrac{y}{x}\right) + \sqrt{x^2 + y^2}$ then show that $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} = \sqrt{x^2 + y^2}.$

4. If $u = \dfrac{x}{y+z} + \dfrac{y}{z+x} + \dfrac{z}{x+y}$ then show that $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} + z\dfrac{\partial u}{\partial z} = 0.$

5. If $u = \dfrac{x^3y^3z^3}{x^3 + y^3 + z^3}$ then show that $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} + z\dfrac{\partial u}{\partial z} = 6u.$

6. If $u = \log\left(\frac{\sqrt{x^2+y^2}}{x+y}\right)$ find the value of $x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2}$. (**Ans**: 0)

7. If $u = x^3 e^{-\frac{x}{y}}$ find the value of $x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2}$. (**Ans**: 6)

8. If $u = \frac{x^3+y^3}{y\sqrt{x}}$ find the value of $x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2} + x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}$ at (1,2).

9. If $u = x^3 y^2 \sin^{-1}\left(\frac{y}{x}\right)$ find the value of $x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2} + x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}$.
(**Ans**: $25u$)

10. If $u = (x^2+y^2)^{\frac{2}{3}}$ find the value of $x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2}$. (**Ans**: $\frac{4u}{9}$)

Deductions from Euler's Theorem

Corollary 2: If u is not a homogeneous function of x and y but $z = f(u)$ is a homogeneous function of x and y of degree n then

1) $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} = \dfrac{n\,f(u)}{f'(u)}$

2) $x^2\dfrac{\partial^2 u}{\partial x^2} + 2xy\dfrac{\partial^2 u}{\partial x \partial y} + y^2\dfrac{\partial^2 u}{\partial y^2} = g(u)[g'(u) - 1]$ where $g(u) = \dfrac{n\,f(u)}{f'(u)}$

Note: If u is not a homogeneous function of x, y and z but $w = f(u)$ is a homogeneous function of degree n in x, y and z then

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = \frac{n\,f(u)}{f'(u)}$$

1. If $u = e^{x^2+y^2}$ then show that $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} = 2u\log u$.

Solution: Replacing x by tx and y by ty in u,

$e^{(tx)^2+(ty)^2} = e^{t^2(x^2+y^2)} \neq t^2 e^{(x^2+y^2)}$

$\therefore\ u = e^{x^2+y^2}$ is not a homogeneous function of x and y.

Now $u = e^{x^2+y^2} \Longrightarrow \log u = x^2 + y^2$

Let $f(u) = \log u$

Therfore, $f(u) = \log u$ is a homogeneous function of degree $n = 2$.

By Deduction of Euler's Theorem,

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \frac{nf(u)}{f'(u)}$$

$$\Longrightarrow x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \frac{2\log u}{1/u} = 2u\log u$$

2. If $u = \log\left[\frac{x^3+y^3}{x^2+y^2}\right]$, prove that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 1$ and

$$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2} = -1.$$

Solution: Replacing x by tx and y by ty in u,

$$\log\left[\frac{(tx)^3+(ty)^3}{(tx)^2+(ty)^2}\right] = \log\left[\frac{t^3}{t^2}\left(\frac{x^3+y^3}{x^2+y^2}\right)\right] = \log\left[t\left(\frac{x^3+y^3}{x^2+y^2}\right)\right]$$

$$\neq t\log\left[\left(\frac{x^3+y^3}{x^2+y^2}\right)\right]$$

$$\therefore u = \log\left[\frac{x^3+y^3}{x^2+y^2}\right] \text{ is not a homogeneous function of } x \text{ and } y.$$

Now $u = \log\left[\frac{x^3+y^3}{x^2+y^2}\right] \Longrightarrow e^u = \frac{x^3+y^3}{x^2+y^2}$

Let $f(u) = e^u$

Therfore, $f(u) = e^u$ is a homogeneous function of degree $n = 1$.

By Deduction of Euler's Theorem,

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \frac{nf(u)}{f'(u)}$$

$$\Longrightarrow x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \frac{1e^u}{e^u} = 1$$

Also $x^2 \dfrac{\partial^2 u}{\partial x^2} + 2xy \dfrac{\partial^2 u}{\partial x \partial y} + y^2 \dfrac{\partial^2 u}{\partial y^2} = g(u)[g'(u) - 1]$

where $g(u) = \dfrac{n\, f(u)}{f'(u)} = \dfrac{1e^u}{e^u} = 1$

$\Longrightarrow g'\ (u) = 0$

$\therefore\ x^2 \dfrac{\partial^2 u}{\partial x^2} + 2xy \dfrac{\partial^2 u}{\partial x \partial y} + y^2 \dfrac{\partial^2 u}{\partial y^2} = 1[0 - 1] = -1.$

3. If $u = \sin^{-1}\left(\sqrt{x^2+y^2}\right)$, prove that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \tan u$ and

$$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2} = \tan^3 u.$$

Solution: Replacing x by tx and y by ty in u,

$$\sin^{-1}\left(\sqrt{(tx)^2+(ty)^2}\right) = \sin^{-1}\left(\sqrt{t^2(x^2+y^2)}\right) = \sin^{-1}\left(t\sqrt{x^2+y^2}\right)$$

$$\neq t\sin^{-1}\left(\sqrt{x^2+y^2}\right)$$

$\therefore u = \sin^{-1}\left(\sqrt{x^2+y^2}\right)$ is not a homogeneous function of x and y.

Now $u = \sin^{-1}\left(\sqrt{x^2 + y^2}\right) \Longrightarrow \sin u = \sqrt{x^2 + y^2}$

Let $f(u) = \sin u$

Therfore, $f(u) = \sin u$ is a homogeneous function of degree $n = 1$.

By Deduction of Euler's Theorem,

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \frac{nf(u)}{f'(u)}$$

$$\Longrightarrow x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \frac{1 \sin u}{\cos u} = \tan u$$

Also $x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2}=g(u)[g'(u)-1]$

where $g(u)=\frac{n\,f(u)}{f'(u)}=\frac{1\sin u}{\cos u}=\tan u$

$\Longrightarrow g'(u)=\sec^2 u$

$$\therefore x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2}=\tan u\,[\sec^2 u\ -1]=\tan u\,[\tan^2 u\ ]$$

$$=\tan^3 u$$

5. If $x = e^u \tan v$ and $y = e^u \sec v$ then find the value of $\left(x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}\right)\left(x\frac{\partial v}{\partial x} + y\frac{\partial v}{\partial y}\right)$.

Solution: $x = e^u \tan v$ and $y = e^u \sec v$

$\therefore\ y^2 - x^2 = e^{2u}\sec^2 v - e^{2u}\tan^2 v = e^{2u}(\sec^2 v - \tan^2 v) = e^{2u}$

$e^{2u} = y^2 - x^2$

Let $f(u) = e^{2u}$

Therfore, $f(u)$ is a homogeneous function of degree $n = 2$.

By Deduction of Euler's Theorem,

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \frac{nf(u)}{f'(u)}$$

$$\Rightarrow x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \frac{2e^{2u}}{2e^{2u}} = 1$$

Now $x = e^u \tan v$ and $y = e^u \sec v$

$$\therefore \frac{x}{y} = \frac{e^u \tan v}{e^u \sec v} = \cos v \tan v = \sin v$$

$$\Longrightarrow v = \sin^{-1}\left(\frac{x}{y}\right)$$

$\Longrightarrow v$ is a homogeneous function of of x and y with degree $n = 0$

By Euler's Theorem,

$$x\frac{\partial v}{\partial x} + y\frac{\partial v}{\partial y} = nv = 0.\, v = 0$$

$$\therefore \left(x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}\right)\left(x\frac{\partial v}{\partial x} + y\frac{\partial v}{\partial y}\right) = 1 \times 0 = 0$$

6. If $u = \sin^{-1}\left[\frac{x+y}{\sqrt{x}+\sqrt{y}}\right]$ then prove that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \frac{1}{2}\tan u$ and

$$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2} = \frac{1}{4}\tan^3 u - \frac{1}{4}\tan u$$

Solution: Replacing x by tx and y by ty in u,

$$\sin^{-1}\left[\frac{tx+ty}{\sqrt{tx}+\sqrt{ty}}\right] = \sin^{-1}\left[\frac{t}{\sqrt{t}}\left(\frac{x+y}{\sqrt{x}+\sqrt{y}}\right)\right] = \sin^{-1}\left[t^{1/2}\left(\frac{x+y}{\sqrt{x}+\sqrt{y}}\right)\right]$$

$$\neq t^{1/2}\sin^{-1}\left[\left(\frac{x+y}{\sqrt{x}+\sqrt{y}}\right)\right]$$

$\therefore u = \sin^{-1}\left[\frac{x+y}{\sqrt{x}+\sqrt{y}}\right]$ is not a homogeneous function of x and y.

Now $u = \sin^{-1}\left[\dfrac{x+y}{\sqrt{x}+\sqrt{y}}\right] \Longrightarrow \sin u = \dfrac{x+y}{\sqrt{x}+\sqrt{y}}$

Let $f(u) = \sin u$

Therfore, $f(u)$ is a homogeneous function of degree $n = \dfrac{1}{2}$.

By Deduction of Euler's Theorem,

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \frac{nf(u)}{f'(u)}$$

$$\Longrightarrow x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \frac{1}{2}\frac{\sin u}{\cos u} = \frac{1}{2}\tan u$$

Also $x^2 \dfrac{\partial^2 u}{\partial x^2} + 2xy \dfrac{\partial^2 u}{\partial x \partial y} + y^2 \dfrac{\partial^2 u}{\partial y^2} = g(u)[g'(u) - 1]$

where $g(u) = \dfrac{n\, f(u)}{f'(u)} = \dfrac{1}{2} \dfrac{\sin u}{\cos u} = \dfrac{1}{2} \tan u$

$\Longrightarrow g'\ (u) = \dfrac{1}{2} \sec^2 u$

$\therefore\ x^2 \dfrac{\partial^2 u}{\partial x^2} + 2xy \dfrac{\partial^2 u}{\partial x \partial y} + y^2 \dfrac{\partial^2 u}{\partial y^2} = \dfrac{1}{2} \tan u \left[\dfrac{1}{2} \sec^2 u - 1\right]$

$= \dfrac{1}{2} \tan u \left[\dfrac{1}{2} (1 + \tan^2 u) - 1\right]$

$= \dfrac{1}{2} \tan u \left[\dfrac{1}{2} \tan^2 u - \dfrac{1}{2}\right]$

$= \dfrac{1}{4} \tan^3 u - \dfrac{1}{4} \tan u$

7. If $u = \operatorname{cosec}^{-1}\left[\sqrt{\dfrac{x^{1/2}+y^{1/2}}{x^{1/3}+y^{1/3}}}\right]$, prove that

$$x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2}=\frac{\tan u}{144}(13+\tan^2 u).$$

Solution: Replacing x by tx and y by ty in u,

$$\operatorname{cosec}^{-1}\left[\sqrt{\frac{(tx)^{1/2}+(ty)^{1/2}}{(tx)^{1/3}+(ty)^{1/3}}}\right]=\operatorname{cosec}^{-1}\left[\sqrt{\frac{t^{1/2}}{t^{1/3}}\left(\frac{x^{1/2}+y^{1/2}}{x^{1/3}+y^{1/3}}\right)}\right]=\operatorname{cosec}^{-1}\left[\sqrt{t^{\frac{1}{6}}\left(\frac{x^{1/2}+y^{1/2}}{x^{1/3}+y^{1/3}}\right)}\right]$$

$$=\operatorname{cosec}^{-1}\left[t^{\frac{1}{12}}\sqrt{\left(\frac{x^{1/2}+y^{1/2}}{x^{1/3}+y^{1/3}}\right)}\right]\neq t^{\frac{1}{12}}\operatorname{cosec}^{-1}\left[\sqrt{\left(\frac{x^{1/2}+y^{1/2}}{x^{1/3}+y^{1/3}}\right)}\right]$$

$\therefore\ u = \operatorname{cosec}^{-1}\left[\sqrt{\dfrac{x^{1/2}+y^{1/2}}{x^{1/3}+y^{1/3}}}\right]$ is not a homogeneous function of x and y.

Now $u = \operatorname{cosec}^{-1}\left[\sqrt{\dfrac{x^{1/2}+y^{1/2}}{x^{1/3}+y^{1/3}}}\right] \Longrightarrow \operatorname{co}sec\, u = \sqrt{\dfrac{x^{1/2}+y^{1/2}}{x^{1/3}+y^{1/3}}}$

Let $f(u) = \operatorname{cosec} u$

Therfore, $f(u)$ is a homogeneous function of degree $n = \dfrac{1}{12}$.

By Deduction of Euler's Theorem,

$$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2} = g(u)[g'(u) - 1] \quad \text{where } g(u) = \frac{n\, f(u)}{f'(u)}$$

$$g(u) = \frac{n\,f(u)}{f'(u)} = \frac{1}{12}\frac{\operatorname{cosec} u}{(-\operatorname{cosec} u \cot u)} = -\frac{1}{12}\tan u$$

$$\Longrightarrow g'\,(u) = -\frac{1}{12}\sec^2 u$$

$$\Longrightarrow x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2} = -\frac{1}{12}\tan u\left[-\frac{1}{12}\sec^2 u - 1\right]$$

$$= \frac{1}{12}\tan u\left[\frac{1}{12}(1 + \tan^2 u) + 1\right]$$

$$= \frac{1}{12}\tan u\left[\frac{1}{12}\tan^2 u + \frac{13}{12}\right]$$

$$= \frac{\tan u}{144}[\tan^2 u + 13]$$

8. If $u = \sec^{-1}\left(\frac{x^3+y^3}{x+y}\right)$ then show that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 2\cot u$.

Also find the value of $x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2}$.

Solution: Replacing x by tx and y by ty in u,

$$\sec^{-1}\left(\frac{(tx)^3+(ty)^3}{tx+ty}\right)$$

$$= \sec^{-1}\left(\frac{t^3}{t}\,\frac{x^3+y^3}{x+y}\right) = \sec^{-1}\left(t^2\,\frac{x^3+y^3}{x+y}\right) \neq t^2\sec^{-1}\left(\frac{x^3+y^3}{x+y}\right)$$

$\therefore u = \sec^{-1}\left(\frac{x^3+y^3}{x+y}\right)$ is not a homogeneous function of x and y.

$$u = \sec^{-1}\left(\frac{x^3 + y^3}{x + y}\right) \Longrightarrow \sec u = \frac{x^3 + y^3}{x + y}$$

Let $f(u) = \sec u$

Therfore, $f(u) = \sec u$ is a homogeneous function of degree $n = 2.$

By Deduction of Euler's Theorem,

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \frac{n\,f(u)}{f'(u)}$$

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \frac{2\sec u}{\sec u\tan u} = \frac{2}{\tan u}$$

$$= 2\cot u$$

Also $x^2 \frac{\partial^2 u}{\partial x^2} + 2xy \frac{\partial^2 u}{\partial x \partial y} + y^2 \frac{\partial^2 u}{\partial y^2} = g(u)[g'(u) - 1]$

where $g(u) = \frac{n\,f(u)}{f'(u)} = \frac{2 \sec u}{\sec u \tan u} = \frac{2}{\tan u} = 2 \cot u$

$\Longrightarrow g'\ (u) = -2\,\text{cosec}^2\, u$

$\therefore\ x^2 \frac{\partial^2 u}{\partial x^2} + 2xy \frac{\partial^2 u}{\partial x \partial y} + y^2 \frac{\partial^2 u}{\partial y^2} = 2 \cot u\,[-2\,\text{cosec}^2\, u - 1]$

$$= 2 \cot u\,[-2\,(1 + \cot^2 u) - 1]$$

$$= 2 \cot u\,[-2 - 2\cot^2 u - 1]$$

$$= 2 \cot u\,[-3 - 2\cot^2 u]$$

$$= -6 \cot u - 4 \cot^3 u$$

Test Your Knowledge

1. If $u = \cos^{-1}\left[\dfrac{x+y}{\sqrt{x}+\sqrt{y}}\right]$ then show that $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} = -\dfrac{1}{2}\cot u$.

2. If $u = \sin^{-1}\left[\dfrac{x+2y+3z}{\sqrt{x^8+y^8+z^8}}\right]$ then show that $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} + z\dfrac{\partial u}{\partial z} = -3\tan u$

3. If $u = \sin^{-1}\left[\dfrac{x+y}{\sqrt{x}+\sqrt{y}}\right]$ then show that

$$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2} = -\frac{\sin u \cos 2u}{\cos^3 u}$$

4. If $u = \tan^{-1}\left(\dfrac{x^3+y^3}{x-y}\right)$ then show that

$$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2} = (1-4\sin^2 u)\sin 2u$$

5. If $u = \sin^{-1}\left[\sqrt{\dfrac{x^{1/3}+y^{1/3}}{x^{1/2}+y^{1/2}}}\right]$ then prove that

$$x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2}=\frac{\tan u}{144}(13+\tan^2 u).$$

6. If $u = \sin^{-1}(x^3+y^3)^{2/5}$ then show that

$$x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2}=\frac{6}{5}\tan u\left(\frac{6}{5}\sec^2 u-1\right)$$

7. If $u = \tan^{-1}\left(\dfrac{y^2}{x}\right)$ then show that $x^2\dfrac{\partial^2 u}{\partial x^2}+2xy\dfrac{\partial^2 u}{\partial x\partial y}+y^2\dfrac{\partial^2 u}{\partial y^2}=-\sin^2 u\sin 2u$

8. If $u = \sin^{-1}(xyz)$ then show that $x\dfrac{\partial u}{\partial x}+y\dfrac{\partial u}{\partial y}+z\dfrac{\partial u}{\partial z}=-3\tan u$

Partial Derivative of Composite Function

If $z = f(u)$ is a function of u and u is again a function of two variables x and y, i.e., $u = \phi(x\,,y)$ then z is called the composite function of x and y.

$$\begin{tikzcd} z \arrow[r] \arrow[rr, bend left] \arrow[r, bend right] & u \arrow[r] \arrow[r, bend right] & x,\, y \end{tikzcd}$$

$$\frac{\partial z}{\partial x} = \frac{dz}{du} \cdot \frac{\partial u}{\partial x} = \frac{df}{du} \cdot \frac{\partial u}{\partial x} = f'(u)\frac{\partial u}{\partial x}$$

$$\frac{\partial z}{\partial y} = \frac{dz}{du} \cdot \frac{\partial u}{\partial y} = \frac{df}{du} \cdot \frac{\partial u}{\partial y} = f'(u)\frac{\partial u}{\partial y}$$

If $u = f(x,y)$ and $x = \phi(t),\ y = \psi(t)$ then u is called the composite function of a single variable t.

$$\begin{tikzcd}
u \arrow[r] \arrow[r, bend right] \arrow[rr, bend left] & x,y \arrow[r] \arrow[r, bend right] & t
\end{tikzcd}$$

$$\frac{du}{dt} = \frac{\partial u}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial u}{\partial y}\cdot\frac{dy}{dt}$$

$\frac{du}{dt}$ is called as total derivative of u.

If $u = f(x,y,z)$ and $x = \phi(t), y = \psi(t), z = \xi(t)$ then total derivative of u is

$$\frac{du}{dt} = \frac{\partial u}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial u}{\partial y}\cdot\frac{dy}{dt} + \frac{\partial u}{\partial z}\cdot\frac{dz}{dt}$$

1. If $u = y^2 - 4ax, x = at^2, y = 2at$ then find $\frac{du}{dt}$.

Solution: $u = y^2 - 4ax, x = at^2, y = 2at$

Here u is a function of x and y. Also x and y are functions of t.

Therefore u is the composite function of a single variable t.

$$\therefore \frac{du}{dt} = \frac{\partial u}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial u}{\partial y} \cdot \frac{dy}{dt}$$

$$= (-4a)(2at) + (2y)(2a)$$

$$= -8a^2t + 2(2at)(2a)$$

$$= -8a^2t + 8a^2t$$

$$= 0$$

2. If $u = \sin\left(\frac{x}{y}\right), x = e^t, y = t^2$ then find $\frac{du}{dt}$.

Solution: Here u is a function of x and y. Also x and y are functions of t.

Therefore u is the composite function of a single variable t.

$$\therefore \frac{du}{dt} = \frac{\partial u}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial u}{\partial y} \cdot \frac{dy}{dt}$$

$$= \frac{1}{y} \cos\left(\frac{x}{y}\right) . (e^t) + \left(-\frac{x}{y^2}\right) \cos\left(\frac{x}{y}\right) . (2t)$$

$$= \frac{1}{t^2} \cos\left(\frac{e^t}{t^2}\right) . (e^t) + \left(-\frac{e^t}{(t^2)^2}\right) \cos\left(\frac{e^t}{t^2}\right) . (2t)$$

$$\frac{du}{dt} = \frac{e^t}{t^2}\cos\left(\frac{e^t}{t^2}\right) + \left(-\frac{2e^t}{t^3}\right)\cos\left(\frac{e^t}{t^2}\right)$$

$$= \frac{e^t}{t^2}\cos\left(\frac{e^t}{t^2}\right).\left(1 - \frac{2}{t}\right)$$

Composite Function of Two Variables

If $z = f(u,v)$ and $u = \phi(x,y)$, $v = \psi(x,y)$ then z is called the composite function of two variables x and y.

$$\begin{tikzcd} z \arrow[r] \arrow[rr, bend left] \arrow[r, bend right] & u,v \arrow[r] \arrow[r, bend right] & x,y \end{tikzcd}$$

$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u} \cdot \frac{\partial u}{\partial x} + \frac{\partial z}{\partial v} \cdot \frac{\partial v}{\partial x}$$

$$\frac{\partial z}{\partial y} = \frac{\partial z}{\partial u} \cdot \frac{\partial u}{\partial y} + \frac{\partial z}{\partial v} \cdot \frac{\partial v}{\partial y}$$

1. If $z = f(u, v), u = \log(x^2 + y^2), v = \dfrac{y}{x}$ then show that

$$x\frac{\partial z}{\partial y} - y\frac{\partial z}{\partial x} = (1 + v^2)\frac{\partial z}{\partial v}.$$

Solution: $z = f(u, v), u = \log(x^2 + y^2), v = \dfrac{y}{x}$

Here z is a function of u and v. Also u and v both are functions of x and y.

Therefore z is a composite function of two variables x and y .

$$\therefore \frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}.\frac{\partial u}{\partial x} + \frac{\partial z}{\partial v}.\frac{\partial v}{\partial x}$$ and

$$\frac{\partial z}{\partial y} = \frac{\partial z}{\partial u}.\frac{\partial u}{\partial y} + \frac{\partial z}{\partial v}.\frac{\partial v}{\partial y}$$

$$\frac{\partial z}{\partial x}=\frac{\partial z}{\partial u}.\frac{\partial u}{\partial x}+\frac{\partial z}{\partial v}.\frac{\partial v}{\partial x}$$

$$\frac{\partial z}{\partial x}=\frac{\partial z}{\partial u}.\left(\frac{1}{x^2+y^2}.2x\right)+\frac{\partial z}{\partial v}.\left(-\frac{y}{x^2}\right)=\frac{2x}{x^2+y^2}\frac{\partial z}{\partial u}-\frac{y}{x^2}\frac{\partial z}{\partial v}$$

$$\therefore y\frac{\partial z}{\partial x}=\frac{2xy}{x^2+y^2}\frac{\partial z}{\partial u}-\frac{y^2}{x^2}\frac{\partial z}{\partial v}\dots\dots\dots\dots\dots(1)$$

Now $\frac{\partial z}{\partial y}=\frac{\partial z}{\partial u}.\frac{\partial u}{\partial y}+\frac{\partial z}{\partial v}.\frac{\partial v}{\partial y}$

$$=\frac{\partial z}{\partial u}.\left(\frac{1}{x^2+y^2}.2y\right)+\frac{\partial z}{\partial v}.\left(\frac{1}{x}\right)=\frac{2y}{x^2+y^2}\frac{\partial z}{\partial u}+\frac{1}{x}\frac{\partial z}{\partial v}$$

$$\frac{\partial z}{\partial y} = \frac{2y}{x^2+y^2}\frac{\partial z}{\partial u} + \frac{1}{x}\frac{\partial z}{\partial v}$$

$$\therefore x\frac{\partial z}{\partial y} = \frac{2xy}{x^2+y^2}\frac{\partial z}{\partial u} + \frac{\partial z}{\partial v} \dots\dots\dots\dots\dots (2)$$

Subtracting Eq.(1) from (2),

$$LHS = x\frac{\partial z}{\partial y} - y\frac{\partial z}{\partial x} = \left[\frac{2xy}{x^2+y^2}\frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}\right] - \left[\frac{2xy}{x^2+y^2}\frac{\partial z}{\partial u} - \frac{y^2}{x^2}\frac{\partial z}{\partial v}\right]$$

$$= \left(1 + \frac{y^2}{x^2}\right)\frac{\partial z}{\partial v}$$

$$= (1+v^2)\frac{\partial z}{\partial v} = RHS$$

2. If $z = f(u, v), u = x^2 + y^2, v = 2xy$ then show that

$$x\frac{\partial z}{\partial x} - y\frac{\partial z}{\partial y} = 2\sqrt{u^2 - v^2}\ \frac{\partial z}{\partial u}.$$

Solution: $z = f(u, v), u = x^2 + y^2, v = 2xy$

Here z is a function of u and v. Also u and v both are functions of x and y.

Therefore z is a composite function of two variables x and y .

$$\therefore \frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}.\frac{\partial u}{\partial x} + \frac{\partial z}{\partial v}.\frac{\partial v}{\partial x}$$ and

$$\frac{\partial z}{\partial y} = \frac{\partial z}{\partial u}.\frac{\partial u}{\partial y} + \frac{\partial z}{\partial v}.\frac{\partial v}{\partial y}$$

$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}.\frac{\partial u}{\partial x} + \frac{\partial z}{\partial v}.\frac{\partial v}{\partial x}$$

$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}.(2x) \;+ \frac{\partial z}{\partial v}.(2y)$$

$$\therefore x\frac{\partial z}{\partial x} = 2x^2\frac{\partial z}{\partial u} + 2xy\frac{\partial z}{\partial v} \ldots\ldots\ldots\ldots\ldots(1)$$

Now $\frac{\partial z}{\partial y} = \frac{\partial z}{\partial u}.\frac{\partial u}{\partial y} + \frac{\partial z}{\partial v}.\frac{\partial v}{\partial y}$

$$= \frac{\partial z}{\partial u}.(2y) \;+ \frac{\partial z}{\partial v}.(2x)$$

$$\therefore y\frac{\partial z}{\partial y} = 2y^2\frac{\partial z}{\partial u} + 2xy\frac{\partial z}{\partial v} \ldots\ldots\ldots\ldots\ldots(2)$$

Subtracting Eq.(2) from (1),

$$x\frac{\partial z}{\partial x} - y\frac{\partial z}{\partial y} = \left[2x^2\frac{\partial z}{\partial u} + 2xy\frac{\partial z}{\partial v}\right] - \left[2y^2\frac{\partial z}{\partial u} + 2xy\frac{\partial z}{\partial v}\right]$$

$$= 2(x^2 - y^2)\frac{\partial z}{\partial u}$$

$$= 2\sqrt{(x^2 - y^2)^2}\,\frac{\partial z}{\partial u}$$

$$= 2\sqrt{(x^2 + y^2)^2 - 4x^2y^2}\;\frac{\partial z}{\partial u}$$

$$= 2\sqrt{u^2 - v^2}\;\frac{\partial z}{\partial u}$$

3. If $z = f(u, v), u = x\cos\theta - y\sin\theta, v = x\sin\theta + y\cos\theta$ then show that

$$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = u\frac{\partial z}{\partial u} + v\frac{\partial z}{\partial v}.$$

Solution: $z = f(u, v), u = x\cos\theta - y\sin\theta, v = x\sin\theta + y\cos\theta$.

Here z is a function of u and v. Also u and v both are functions of x and y.

Therefore z is a composite function of two variables x and y.

$$\therefore \frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}.\frac{\partial u}{\partial x} + \frac{\partial z}{\partial v}.\frac{\partial v}{\partial x}$$ and

$$\frac{\partial z}{\partial y} = \frac{\partial z}{\partial u}.\frac{\partial u}{\partial y} + \frac{\partial z}{\partial v}.\frac{\partial v}{\partial y}$$

$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}.\frac{\partial u}{\partial x} + \frac{\partial z}{\partial v}.\frac{\partial v}{\partial x}$$

$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}(\cos\theta) \ + \frac{\partial z}{\partial v}(\sin\theta)$$

$$\therefore x\frac{\partial z}{\partial x} = x\cos\theta\,\frac{\partial z}{\partial u} + x\sin\theta\,\frac{\partial z}{\partial v} \dots\dots\dots\dots\dots(1)$$

$$\text{Now } \ \frac{\partial z}{\partial y} = \frac{\partial z}{\partial u}.\frac{\partial u}{\partial y} + \frac{\partial z}{\partial v}.\frac{\partial v}{\partial y}$$

$$= \frac{\partial z}{\partial u}(-\sin\theta) \ + \frac{\partial z}{\partial v}(\cos\theta)$$

$$\therefore y\frac{\partial z}{\partial y} = -y\sin\theta\,\frac{\partial z}{\partial u} + y\cos\theta\,\frac{\partial z}{\partial v} \dots\dots\dots\dots\dots(2)$$

Adding Eqs.(1) and (2),

$$x\frac{\partial z}{\partial x}+y\frac{\partial z}{\partial y}=\left[x\cos\theta\frac{\partial z}{\partial u}+x\sin\theta\frac{\partial z}{\partial v}\right]-\left[-y\sin\theta\frac{\partial z}{\partial u}+y\cos\theta\frac{\partial z}{\partial v}\right]$$

$$=(x\cos\theta+y\sin\theta)\frac{\partial z}{\partial u}+(x\sin\theta-y\cos\theta)\frac{\partial z}{\partial v}$$

$$=u\frac{\partial z}{\partial u}+v\frac{\partial z}{\partial v}$$

4. If $z = f(x, y), x = e^u + e^{-v}, y = e^{-u} - e^v$ then show that

$$\frac{\partial z}{\partial u} - \frac{\partial z}{\partial v} = x\frac{\partial z}{\partial x} - y\frac{\partial z}{\partial y}.$$

Solution: $z = f(x, y), x = e^u + e^{-v}, y = e^{-u} - e^v$

Here z is a function of x and y. Also x and y both are functions of u and v.

Therefore z is a composite function of two variables u and v .

$$\therefore \frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}.\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}.\frac{\partial y}{\partial u}$$

and

$$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}.\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}.\frac{\partial y}{\partial v}$$

$$\frac{\partial z}{\partial u}=\frac{\partial z}{\partial x}.\frac{\partial x}{\partial u}+\frac{\partial z}{\partial y}.\frac{\partial y}{\partial u}$$

$$\frac{\partial z}{\partial u}=\frac{\partial z}{\partial x}(e^{u})\ +\frac{\partial z}{\partial y}(-e^{-u})\ \ldots\ldots\ldots\ldots\ldots(1)$$

Now $\frac{\partial z}{\partial v}=\frac{\partial z}{\partial x}.\frac{\partial x}{\partial v}+\frac{\partial z}{\partial y}.\frac{\partial y}{\partial v}$

$$\frac{\partial z}{\partial v}=\frac{\partial z}{\partial x}(-e^{-v})+\frac{\partial z}{\partial y}(-e^{v})\ \ldots\ldots\ldots\ldots\ldots(2)$$

Subtracting Eq.(2) from (1),

$$\frac{\partial z}{\partial u}-\frac{\partial z}{\partial v}=\left[\frac{\partial z}{\partial x}(e^{u})\ +\frac{\partial z}{\partial y}(-e^{-u})\right]-\left[\frac{\partial z}{\partial x}(-e^{-v})+\frac{\partial z}{\partial y}(-e^{v})\right]$$

$$\frac{\partial z}{\partial u}-\frac{\partial z}{\partial v}=\left[\frac{\partial z}{\partial x}(e^{u}) \; +\frac{\partial z}{\partial y}(-e^{-u})\right]-\left[\frac{\partial z}{\partial x}(-e^{-v})+\frac{\partial z}{\partial y}(-e^{v})\right]$$

$$=(e^{u}+e^{-v})\frac{\partial z}{\partial x}-(e^{-u}-e^{v})\frac{\partial z}{\partial x}$$

$$=x\frac{\partial z}{\partial x}-y\frac{\partial z}{\partial x}$$

5. If $z = f(x,y), x^2 = au + bv, y^2 = au - bv$ then show that

$$u\frac{\partial z}{\partial u} + v\frac{\partial z}{\partial v} = \frac{1}{2}\left(x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y}\right).$$

Solution: $z = f(x,y), x^2 = au + bv \Longrightarrow x = \sqrt{au + bv},$

$y^2 = au - bv \Longrightarrow y = \sqrt{au - bv}$

Here z is a function of x and y. Also x and y both are functions of u and v.

Therefore z is a composite function of two variables u and v .

$$\therefore \frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}.\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}.\frac{\partial y}{\partial u}$$ and

$$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}.\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}.\frac{\partial y}{\partial v}$$

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial u} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial u}$$

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\left(\frac{a}{2\sqrt{au + bv}}\right) + \frac{\partial z}{\partial y}\left(\frac{a}{2\sqrt{au - bv}}\right)$$

$$\frac{\partial z}{\partial u} = \frac{a}{2x}\frac{\partial z}{\partial x} + \frac{a}{2y}\frac{\partial z}{\partial y}$$

$$\therefore u\frac{\partial z}{\partial u} = \frac{au}{2x}\frac{\partial z}{\partial x} + \frac{au}{2y}\frac{\partial z}{\partial y} \ldots\ldots\ldots\ldots\ldots (1)$$

Now $\frac{\partial z}{\partial v}=\frac{\partial z}{\partial x}.\frac{\partial x}{\partial v}+\frac{\partial z}{\partial y}.\frac{\partial y}{\partial v}$

$$\frac{\partial z}{\partial v}=\frac{\partial z}{\partial x}\left(\frac{b}{2\sqrt{au+bv}}\right)+\frac{\partial z}{\partial y}\left(\frac{-b}{2\sqrt{au-bv}}\right)$$

$$\frac{\partial z}{\partial v}=\frac{b}{2x}\frac{\partial z}{\partial x}-\frac{b}{2y}\frac{\partial z}{\partial y}$$

$$\therefore v\frac{\partial z}{\partial v}=\frac{bv}{2x}\frac{\partial z}{\partial x}-\frac{bv}{2y}\frac{\partial z}{\partial y}\dots\dots\dots\dots\dots(2)$$

Adding Eqs.(1) and (2),

$$u\frac{\partial z}{\partial u}+v\frac{\partial z}{\partial v}=\left[\frac{au}{2x}\frac{\partial z}{\partial x}+\frac{au}{2y}\frac{\partial z}{\partial y}\right]+\left[\frac{bv}{2x}\frac{\partial z}{\partial x}-\frac{bv}{2y}\frac{\partial z}{\partial y}\right]$$

$$=\left(\frac{au+bv}{2x}\right)\frac{\partial z}{\partial x}+\left(\frac{au-bv}{2y}\right)\frac{\partial z}{\partial y}$$

$$=\left(\frac{x^2}{2x}\right)\frac{\partial z}{\partial x}+\left(\frac{y^2}{2y}\right)\frac{\partial z}{\partial y}$$

$$=\frac{1}{2}\left(x\frac{\partial z}{\partial x}+y\frac{\partial z}{\partial y}\right)$$

6. If $u = f(x-y, y-z, z-x)$, then show that $\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = 0$.

Solution: Let $x - y = l,\ \ y - z = m,\ \ z - x = n$

$u = f(x-y, y-z, z-x) = f(l, m, n)$

Here u is a function of l, m and n. Also l, m and n are functions of x, y and z.

Therefore u is a composite function of three variables x, y and z.

$$\therefore \frac{\partial u}{\partial x} = \frac{\partial u}{\partial l}\cdot\frac{\partial l}{\partial x} + \frac{\partial u}{\partial m}\cdot\frac{\partial m}{\partial x} + \frac{\partial u}{\partial n}\cdot\frac{\partial n}{\partial x}$$

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial l}\cdot\frac{\partial l}{\partial y} + \frac{\partial u}{\partial m}\cdot\frac{\partial m}{\partial y} + \frac{\partial u}{\partial n}\cdot\frac{\partial n}{\partial y}$$

$$\frac{\partial u}{\partial z} = \frac{\partial u}{\partial l}\cdot\frac{\partial l}{\partial z} + \frac{\partial u}{\partial m}\cdot\frac{\partial m}{\partial z} + \frac{\partial u}{\partial n}\cdot\frac{\partial n}{\partial z}$$

$$\frac{\partial u}{\partial x}=\frac{\partial u}{\partial l}.\frac{\partial l}{\partial x}+\frac{\partial u}{\partial m}.\frac{\partial m}{\partial x}+\frac{\partial u}{\partial n}.\frac{\partial n}{\partial x}$$

$$\frac{\partial u}{\partial x}=\frac{\partial u}{\partial l}(1)+\frac{\partial u}{\partial m}(0)+\frac{\partial u}{\partial n}(-1)$$

$$\frac{\partial u}{\partial x}=\frac{\partial u}{\partial l}-\frac{\partial u}{\partial n}\ \ldots\ldots\ldots\ldots\ldots(1)$$

$$\frac{\partial u}{\partial y}=\frac{\partial u}{\partial l}.\frac{\partial l}{\partial y}+\frac{\partial u}{\partial m}.\frac{\partial m}{\partial y}+\frac{\partial u}{\partial n}.\frac{\partial n}{\partial y}$$

$$\frac{\partial u}{\partial y}=\frac{\partial u}{\partial l}(-1)+\frac{\partial u}{\partial m}(1)+\frac{\partial u}{\partial n}(0)$$

$$\frac{\partial u}{\partial y}=-\frac{\partial u}{\partial l}+\frac{\partial u}{\partial m}\ \ldots\ldots\ldots\ldots\ldots(2)$$

$$\frac{\partial u}{\partial z} = \frac{\partial u}{\partial l} \cdot \frac{\partial l}{\partial z} + \frac{\partial u}{\partial m} \cdot \frac{\partial m}{\partial z} + \frac{\partial u}{\partial n} \cdot \frac{\partial n}{\partial z}$$

$$\frac{\partial u}{\partial z} = \frac{\partial u}{\partial l}(0) + \frac{\partial u}{\partial m}(-1) + \frac{\partial u}{\partial n}(1)$$

$$\frac{\partial u}{\partial z} = -\frac{\partial u}{\partial m} + \frac{\partial u}{\partial n} \quad \dots \dots \dots \dots \dots (3)$$

Adding Eqs.(1), (2) and (3),

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = \left(\frac{\partial u}{\partial l} - \frac{\partial u}{\partial n}\right) + \left(-\frac{\partial u}{\partial l} + \frac{\partial u}{\partial m}\right) + \left(-\frac{\partial u}{\partial m} + \frac{\partial u}{\partial n}\right)$$

$$\Rightarrow \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = 0$$

7. If $u = f\left(\frac{x}{y}, \frac{y}{z}, \frac{z}{x}\right)$, then show that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = 0$.

Solution: Let $\frac{x}{y} = l, \qquad \frac{y}{z} = m, \qquad \frac{z}{x} = n$

$$u = f\left(\frac{x}{y}, \frac{y}{z}, \frac{z}{x}\right) = f(l, m, n)$$

Here u is a function of l, m and n. Also l, m and n are functions of x, y and z.

Therefore u is a composite function of three variables x, y and z.

$$\therefore \frac{\partial u}{\partial x} = \frac{\partial u}{\partial l} \cdot \frac{\partial l}{\partial x} + \frac{\partial u}{\partial m} \cdot \frac{\partial m}{\partial x} + \frac{\partial u}{\partial n} \cdot \frac{\partial n}{\partial x}$$

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial l} \cdot \frac{\partial l}{\partial y} + \frac{\partial u}{\partial m} \cdot \frac{\partial m}{\partial y} + \frac{\partial u}{\partial n} \cdot \frac{\partial n}{\partial y}$$

$$\frac{\partial u}{\partial z} = \frac{\partial u}{\partial l} \cdot \frac{\partial l}{\partial z} + \frac{\partial u}{\partial m} \cdot \frac{\partial m}{\partial z} + \frac{\partial u}{\partial n} \cdot \frac{\partial n}{\partial z}$$

$$Now\ \frac{\partial u}{\partial x} = \frac{\partial u}{\partial l}\left(\frac{1}{y}\right) + \frac{\partial u}{\partial m}(0) + \frac{\partial u}{\partial n}\left(-\frac{z}{x^2}\right)$$

$$\therefore x\frac{\partial u}{\partial x} = \frac{x}{y}\frac{\partial u}{\partial l} - \frac{z}{x}\frac{\partial u}{\partial n} \dots\dots\dots\dots\dots (1)$$

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial l}\left(-\frac{x}{y^2}\right) + \frac{\partial u}{\partial m}\left(\frac{1}{z}\right) + \frac{\partial u}{\partial n}(0)$$

$$\therefore\ y\frac{\partial u}{\partial y} = -\frac{x}{y}\frac{\partial u}{\partial l} + \frac{y}{z}\frac{\partial u}{\partial m} \ldots\ldots\ldots\ldots\ldots (2)$$

$$\frac{\partial u}{\partial z} = \frac{\partial u}{\partial l}(0) + \frac{\partial u}{\partial m}\left(-\frac{y}{z^2}\right) + \frac{\partial u}{\partial n}\left(\frac{1}{x}\right)$$

$$\therefore\ z\frac{\partial u}{\partial z} = -\frac{y}{z}\frac{\partial u}{\partial m} + \frac{z}{x}\frac{\partial u}{\partial n} \ldots\ldots\ldots\ldots\ldots (2)$$

Adding Eqs.(1), (2) and (3),

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}+z\frac{\partial u}{\partial z}$$

$$=\left(\frac{x}{y}\frac{\partial u}{\partial l}-\frac{z}{x}\frac{\partial u}{\partial n}\right)+\left(-\frac{x}{y}\frac{\partial u}{\partial l}+\frac{y}{z}\frac{\partial u}{\partial m}\right)+\left(-\frac{y}{z}\frac{\partial u}{\partial m}+\frac{z}{x}\frac{\partial u}{\partial n}\right)$$

$$\Longrightarrow x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}+z\frac{\partial u}{\partial z}=0$$

Implicit Functions

A function $f(x, y) = c$ is called an implicit function if y is a function of x but y can not be expressed in terms of x.

$e.g.\ x^3 + y^3 = 3axy,\ \ y \sin x = x \cos y,\ \ x^y + y^x = c$ are implicit functions.

If $f(x, y) = c$ is an implicit function then

$$\frac{dy}{dx} = -\frac{\frac{\partial f}{\partial x}}{\frac{\partial f}{\partial y}} \qquad \text{and}$$

$$\frac{d^2y}{dx^2} = \frac{1}{q^3}\begin{vmatrix} r & s & p \\ s & t & q \\ p & q & 0 \end{vmatrix} = -\frac{1}{q^3}[q^2r - 2pqs + p^2t]$$

where $p = \dfrac{\partial f}{\partial x}, \qquad q = \dfrac{\partial f}{\partial y}, \qquad r = \dfrac{\partial^2 f}{\partial x^2}, \qquad s = \dfrac{\partial^2 f}{\partial x \partial y}$ and $t = \dfrac{\partial^2 f}{\partial y^2}$

Q.Find $\frac{dy}{dx}$ $if\ (i)x^3 + y^3 = 3axy, (ii) y\sin x = x\cos y\,, (iii)x^y + y^x = c$

Solution: (i) Let $f(x, y) = x^3 + y^3 - 3axy$

$$\frac{dy}{dx} = -\frac{\frac{\partial f}{\partial x}}{\frac{\partial f}{\partial y}} = -\frac{(3x^2 - 3ay)}{(3y^2 - 3ax)} = \frac{ay - x^2}{y^2 - ax}$$

(ii) Let $f(x, y) = y\sin x - x\cos y$

$$\frac{dy}{dx} = -\frac{\frac{\partial f}{\partial x}}{\frac{\partial f}{\partial y}} = -\frac{(y\cos x - \cos y)}{(\sin x + x\sin y)} = \frac{\cos y - y\cos x}{\sin x + x\sin y}$$

(ii) Let $f(x, y) = x^y + y^x - c$

$$\frac{dy}{dx} = -\frac{\frac{\partial f}{\partial x}}{\frac{\partial f}{\partial y}} = -\frac{(yx^{y-1} + y^x\log y)}{(x^y\log x + x^{y-1})}$$