# Unit III--Recurrence Relation and their solutions

Consider the numeric sequence $\{a_n\}$ given as $3, 5, 7, 9 \dots$. We can find a formula for the $n^{th}$ term of the sequence i.e. discrete numeric function by observing the pattern of the sequence

$$a_0 = 3 = 2 \times 1 + 1.$$

$$a_1 = 5 = 2 \times 2 + 1.$$

$$a_2 = 7 = 2 \times 3 + 1.$$

Thus, for the sequence $\{a_n\}$, $n^{\text{th}}$ term of the sequence is $a_n = 2n + 1$ for $n \geq 0$.This type of formula is called **explicit formula** or discrete numeric function for the sequence, because we can find any term of the sequence directly from the above derived formula. For example, $a_{100} = 2 \times 100 + 1 = 201$.

Let us take another example of a sequence defined as

$$1, 1, 2, 3, 5, 13, 21, \dots .$$

For this sequence, the explicit formula is not obvious. If we observe closely however, we find that pattern of the sequence is such that any term after the second term is the sum of the preceding two terms. That is,

3rd term $=$ 1st term $+$ 2nd term

4th term $=$ 2nd term $+$ 3rd term

5th term $=$ 3rd term $+$ 4th term

6th term $=$ 4th term $+$ 5th term

Here, the $n^{th}$ term of the sequence sequence can be expressed in the form of an equation

$$a_n = a_{n-1} + a_{n-2};\ n \geq 3.$$

Where,

$$a_1 = 1, a_2 = 1.$$

## Recurrence Relation

A recurrence relation for the sequence $\{a_n\}$ is an equation that express $a_n$ in terms one or more of the previous terms $a_0$,$a_{1,}$.....,$a_{n-1}$for all integers $n$ with $n \geq k$, where $k$ is a non negative integer. A sequence is called a solution of a recurrence relation if its terms satisfy the recurrence relation. The initial conditions for the recurrence relation are a set of values that explicitly define some of the members $a_0, a_1, .., a_{k-1}$. We say that we have solved the recurrence relation together with the initial conditions when we find an explicit formula, called a **closed formula**, for the terms of the sequence.

**Example:** The equation

$$a_n = a_{n-1} + a_{n-2}, n \geq 2 \text{ with } a_0 = 0, a_1 = 1$$

relates $a_n$ to $a_{n-1}$ and $a_{n-2}$. Here $k = 2$. So, this is a recurrence relation with initial conditions. The sequence defined by this recurrence relation is known as the **Fibonacci sequence**, after the Italian mathematician Fibonacci.

**Example:** Let $\{a_n\}$ be a sequence that satisfies the recurrence relation $a_n = a_{n-1} + 3$ for $n = 1, 2, 3, \ldots$, and suppose that $a_0 = 2$. What are $a_1, a_2$, and $a_3$ ?

*Solution:* We see from the recurrence relation that $a_1 = a_0 + 3 = 2 + 3 = 5$. It then follows that $a_2 = a_1 + 3 = 5 + 3 = 8$ and $a_3 = a_2 + 3 = 8 + 3 = 11$.

**Example:** Determine whether the sequence $\{a_n\}$, where $a_n = 3n$ for every nonnegative integer $n$, is a solution of the recurrence relation $a_n = 2a_{n-1} - a_{n-2}$ for $n = 2, 3, 4, \ldots$ . Answer the same question where $a_n = 2^n$ and where $a_n = 5$.

Solution: Suppose that $a_n = 3n$ for every nonnegative integer $n$. Then, for $n \geq 2$, we see that $2a_{n-1} - a_{n-2} = 2 \times 3 \times (n - 1) - 3 \times (n - 2) = 3n = a_n$. Therefore, $\{a_n\}$, where $a_n = 3n$, is a solution of the recurrence relation.

Suppose that $a_n = 2^n$ for every nonnegative integer $n$. Note that $a_0 = 1, a_1 = 2$, and $a_2 = 4$. Because $2a_1 - a_0 = 2 \times 2 - 1 = 3 \neq a_2$, we see that $\{a_n\}$, where $a_n = 2^n$, is not a solution of the recurrence relation.

Suppose that $a_n = 5$ for every nonnegative integer $n$. Then for $n \geq 2$, we see that $a_n = 2a_{n-1} - a_{n-2} = 2 \times 5 - 5 = 5 = a_n$. Therefore, $\{a_n\}$, where $a_n = 5$, is a solution of the recurrence relation.

**Example:** Show that

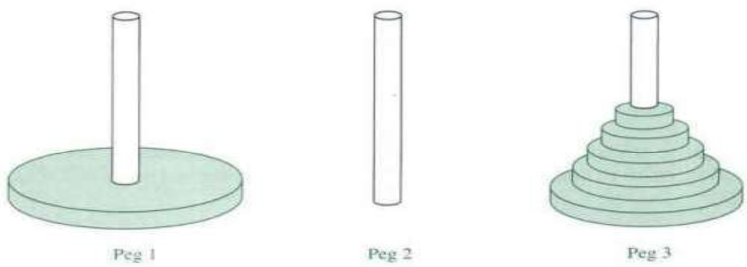
(i) For the Recurrence Relation $a_n = 2a_{n-1}, n \geq 1, a_n = 2^n$ is the solution.

(ii) For the Recurrence Relation $a_n - 7a_{n-1} + 10a_{n-2} = 0, n \geq 2, a_n = c_1 2^n + c_2 5^n$ is a solution, where $c_1$ and $c_2$ arbitrary constants.

## Application of Recurrence Relation

Recurrence relations can be used to study and to solve counting problems. To start with, let us demonstrate the **Tower of Hanoi** puzzle.

Tower of Hanoi: In the nineteenth century, a game called the Tower of Hanoi became popular in Europe. This game represents work that is under way in the temple of Brahma. There are three pegs, with one peg containing 64 golden disks. Each golden disk is slightly smaller than the disk below it. The task is to move all 64 disks from the first peg to the third peg.




The rules for moving the disks are as follows:

1. Only one disk can be moved at a time.
2. The removed disk must be placed on one of the pegs.
3. A larger disk cannot be placed on top of a smaller disk.

The objective is to determine the minimum number of moves required to transfer the disks from the first peg to the third peg. Let peg 1 contains $n \geq 1$ disks. The move of the disks from peg 1 to peg 3 can be described as follows.

1. Move the top $n - 1$ disks from peg 1 to peg 2 using peg 3 as an intermediate peg.
2. Move the largest disk ($n^{th}$ number) from peg 1 to peg 3.
3. Move the $n - 1$ disks from peg 2 to peg 3 using peg 1 as the intermediate peg.

The problem can be formulated as a Recurrence Relation for the sequence $\{H_n\}_{n=1}^{\infty}$ as follows. Let $H_n$ be the number of moves required to move $n$ disks, $n \geq 1$, from one peg to another peg. Step 1 requires us to move $n-1$ disks form peg 1 to peg 2, which requires $H_{n-1}$ moves. Step 2 requires to move the $n^{th}$ disk from peg 1 to peg 3, which requires one move. Step 3 requires us to move $n-1$ disks form peg 2 to peg 3, which requires $H_{n-1}$ moves. Thus, it follows that

$$H_n = H_{n-1} + 1 + H_{n-1} = 2H_{n-1} + 1, \text{ if } n \geq 2$$

The initial condition is $H_1 = 1$, because one disk can be transferred from peg 1 to peg 3, according to the rules of the puzzle, in one move.

Many methods have been developed for solving recurrence relations. Here, we will introduce a straightforward method known as **iteration** or **substitution** as follows:

$$\begin{aligned} H_n &= 2H_{n-1} + 1 \\ &= 2(2H_{n-2} + 1) + 1 \\ &= 2^2 H_{n-2} + 2 + 1 \\ &= 2^2(2H_{n-3} + 1) + 2 + 1 \\ &= 2^3 H_{n-3} + 2^2 + 2 + 1 \\ &\vdots \\ &= 2^{n-1}H_1 + 2^{n-2} + 2^{n-3} + \cdots + 2 + 1 \\ &= 2^{n-1} + 2^{n-2} + \cdots + 2 + 1 \\ &= 2^n - 1. \end{aligned}$$

From this explicit formula, the monk requires $2^{64} - 1 = 18{,}446{,}744{,}073{,}709{,}551{,}615$ moves to transfer the disks. Making one move per second, it will take them more than 500 billion years to complete the transfer.

**Example:** Find the Recurrence Relation and the initial condition for the sequence

$$1,5,17,53,161,484, \ldots .$$

**Solution:** Finding the Recurrence Relation would be easier if we had some context for problem (like Tower of Hanoi). The Recurrence Relation tells us how to get from previous terms to future terms. What is going on here? We could look at the differences between the terms of the sequences, that are: $4,12,36,108, \ldots$ . Notice that these are growing by a factor of 3. Now the terms $a_0, a_1, a_2, a_3$ ...can be written as:

$$\begin{aligned} a_0 &= 1 \\ a_1 &= 5 = 1 \times 3 + 2 \\ a_2 &= 17 = 5 \times 3 + 2 \\ a_3 &= 53 = 17 \times 3 + 2 \end{aligned}$$

So, observing the pattern the $n^{th}$ term of the sequence can be written as $a_n = 3a_{n-1} + 2$, with $a_0 = 1$, which is the required Recurrence Relation.

**Example:** Find a Recurrence Relation with initial conditions for the number of bit strings of length $n$ that do not have two consecutive $0s$. How many such strings are there of length five?

Solution: Let $a_n$ be the number of bit strings of length $n$ that do not have two consecutive $0$s. Then, $a_1 = 2$**,** because both bit strings of length one, $0$ and $1$, do not have two consecutive $0$s. Also $a_2 = 3$, because the valid bit strings of length two that do not have two consecutive $0$s are $01$,$10$ and $11$**.**

To obtain a recurrence relation for $\{a_n\}$, we need to consider the bit strings of length $n$ that do not have consecutive $0s$ equals the number of such strings ending with a $0$ plus the number of such bit strings ending with a $1$**.** We will assume that $n \geq 3$, so that the bit string has at least three bits. The bit strings of length $n$ ending with $1$ that do not have two consecutive $0$s are precisely the bit strings of length $n-1$ with no consecutive $0$s with a $1$ added at the end. Consequently, there are $a_{n-1}$ such bit strings. Bit strings of length $n$ ending with a $0$ that do not have two consecutive $0$s must have 1as their $(n-1)^{\text{th}}$ bit; otherwise they would end with pair of $0$s.It follows that bit strings of length $n$ ending with $0$ that have no two consecutive $0$s are precisely the bit strings of length $n-2$ with no two consecutive $0$s with $10$ at the end. Consequently, there are $a_{n-2}$ such bit strings. Thus, we conclude that recurrence relation can be defined as

$$a_n = a_{n-1} + a_{n-2}\text{ ; for } n \geq 3.$$

The initial conditions are $a_1 = 2$, and $a_2 = 3$.

Finally, to find the number of bit strings of length $5$ that do not have two consecutive $0s$ we have to find $a_5$. Using the above recurrence relation, we have,

$$a_3 = a_2 + a_1 = 5$$

$$a_4 = a_3 + a_2 = 8$$

$$a_5 = a_4 + a_3 = 13$$

Therefore, the number of bit strings of length $5$ that do not have two consecutive $0s$ 13.

## Linear Recurrence Relation

A Recurrence Relation of the form

$$c_0(n)a_n + c_1(n)a_{n-1} + \cdots + c_k(n)a_{n-k} = f(n), n \geq k$$

where, $c_0(n), c_1(n), \ldots, c_k(n)$ and $f(n)$ are functions of $n$, is called a linear recurrence relation.

**Note 1:** If $c_0(n), c_k(n)$ are not identically equal to zero, then the recurrence relation is said to be **recurrence relation of degree $\boldsymbol{k}$.** In other words, a recurrence relation is said to be of degree $k$ if $a_n$ is expressed as function of $a_{n-1}, a_{n-2}, \dots, a_{n-k}$ that appears in the relation.

**Note 2:** If $c_0(n), c_1(n), \dots, c_k(n)$ are constants, then the Recurrence Relation is known as **Linear Recurrence Relation with constant coefficients.**

**Note3:** If $f(n) = 0$, then the Recurrence Relation is said to be **Homogenous**; otherwise, it is **Non-homogenous**.

**Example:** Consider the following recurrence relation:

(i) $a_n = a_{n-1} + a_{n-2}, n \geq 2$

(ii) $a_n = n + a_{n-1}, n \geq 1$

(iii) $a_n - 3a_{n-1} + 2a_{n-2} = 0, n \geq 2$

(iv) $a_n - 3a_{n-1} + 2a_{n-2} = n^2 - 1, n \geq 2$

(v) $a_n - (n-1)a_{n-1} - (n-2)a_{n-2} = 0, n \geq 2$

(vi) $a_n - 9a_{n-1} + 26a_{n-2} - 24a_{n-3} = 4^n, n \geq 3$

(vii) $a_n - 3a_{n-1}^2 + 2a_{n-2} = n^2, n \geq 2$

(viii) $a_n = a_0 a_{n-1} + a_1 a_{n-2} + \cdots + a_{n-1} a_0, n \geq 1$

(ix) $a_n^2 + a_{n-1}^2 = -1, n \geq 1$

(x) $a_n = 3a_{n-1}, n \geq 1$

Clearly, All the above examples are linear recurrence relations except (vii), (viii) and (ix); the relation (vii) is not linear because of the squared term $a_{n-1}^2$. The relations (i), (ii), (iii), (iv), (vi),and (x) are linear with constant coefficients. Relations (ii) and (x) have degree 1; (iii), (iv) and (v) have degree 2; and (vi) has degree 3. Relations (i), (iii),(v) and (x) are homogenous.

## Solving Linear Recurrence Relation with constant coefficients

A wide variety of recurrence relations occur in models. Some of these recurrence relations can be solved using iteration or some other ad hoc technique. It is not possible to solve all Recurrence Relations. Also, there is no general technique to solve all Recurrence Relation. However, one important class of recurrence relations can be explicitly solved in a systematic way. These are recurrence relations that express the terms of a sequence as linear combinations of previous terms, i.e. linear Recurrence Relations with constant coefficients. Nonlinear Recurrence Relations can be solved by converting them into linear Recurrence Relations.

We are going to discuss two methods of solving Linear Recurrence Relation with constant coefficients. They are

(i) By Characteristic roots.

(ii) By Generating function method.

## Solving of Linear Homogenous Recurrence Relations with constant coefficients

Recurrence relations may be difficult to solve, but fortunately this is not the case for linear homogenous recurrence relations with constant coefficients. We already said that a recurrence relation of the form

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k}$$

where, $c_1, c_2, \ldots, c_k$, are real numbers and $c_k \neq 0$ is called a **linear homogenous Recurrence Relation of degree $\boldsymbol{k}$** with constant coefficients.

The above recurrence relation is linear since each $a_i$ has power 1 and no terms of the type $a_i a_j$ occurred. The degree of the Recurrence Relation is $k$, since $a_n$ is expressed in terms of the previous $k$ terms of the sequence i.e., degree is the difference between the greatest and lowest subscripts of the members of the sequence occurring in the Recurrence Relation. The coefficients of the terms of the expression are all constants, not functions of $n$. The recurrence relation is **homogeneous** because no terms occur that are not multiples of the $a_j$s.

Solution of Recurrence Relation by Characteristic Polynomial: We can use two key ideas to find all their solutions. First, these recurrence relations have solutions of the form $a_n = r^n$, where $r$ is a constant. The other key observation is that a linear combination of two solutions of a linear homogeneous recurrence relation is also a solution.

Let $a_n = r^n; r \neq 0$, be a solution of the recurrence relation

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k}$$
$$\Rightarrow r^n = c_1 r^{n-1} + c_2 r^{n-2} + \cdots + c_k r^{n-k}$$

$$\Rightarrow r^n - c_1 r^{n-1} - c_2 r^{n-2} - \cdots - c_k r^{n-k} = 0$$
$$\Rightarrow r^{n-k}(r^k - c_1 r^{k-1} - c_2 r^{k-2} - \cdots - c_k) = 0$$
$$\Rightarrow r^k - c_1 r^{k-1} - c_2 r^{k-2} - \cdots - c_k r^k = 0, \text{ since } \quad r^{n-k} \neq 0$$

Consequently, the sequence $\{a_n\}$ with $a_n = r^n$ where $r \neq 0$ is a solution if and only if $r$ is a solution of the last equation. We call this the **characteristic equation** of the recurrence relation. That is, the characteristic equation of the recurrence relation is

$$r^k - c_1 r^{k-1} - c_2 r^{k-2} - \cdots - c_{k-1} r - c_k = 0$$

The solutions of the characteristic equation are called the **characteristic roots** of the recurrence relation. We will now state the general result about the solution of linear homogeneous recurrence relations with constant coefficients, under the assumption that the characteristic equation has distinct roots.

**Theorem:** Let $c_{1,}c_2, \ldots, c_k$ be real numbers. Suppose that the characteristic equation

$$r^k - c_1 r^{k-1} - c_2 r^{k-2} - \cdots - c_{k-1} r - c_k = 0$$

has $k$ distinct roots $r_1, r_2, \ldots, r_k$, then a sequence $\{a_n\}$ is a solution of the recurrence relation

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k}$$

if and only if

$$a_n = \alpha_1 r_1^n + \alpha_2 r_2^n + \cdots + \alpha_k r_k^n$$

for $n = 0,1,2, \ldots$,where $\alpha_1, \alpha_2, \ldots, \alpha_k$ are constants.

**Example**: Find the solution of the recurrence relation

$$a_n = 6a_{n-1} - 11a_{n-2} + 6a_{n-3}$$

with initial conditions $a_0 = 2, a_1 = 5, a_2 = 15.$

**Solution:** The characteristic polynomial of this recurrence relation is

$$r^3 - 6r^2 + 11r - 6.$$

The characteristic roots are $r = 1, r = 2, r = 3$ and they are distinct.

Thus, the solution is of the form

$$a_n = \alpha_1 . 1^n + \alpha_2 . 2^n + \alpha_3 . 3^n.$$

To find the constants $\alpha_1, \alpha_2$and $\alpha_3$, we use the given initial conditions. This gives

$$a_0 = 2 = \alpha_1 + \alpha_2 + \alpha_3,$$
$$a_1 = 5 = \alpha_1 + 2\alpha_2 + 3\alpha_3$$
$$a_2 = 15 = \alpha_1 + 4\alpha_2 + 9\alpha_3$$

After solving three equations, the values of the constants are as follows.

$$\alpha_1 = 1, \alpha_2 = -1, \alpha_3 = 2.$$

Then

$$a_n = 1 - 2^n + 2 \times 3^n.$$

Hence, the unique solution to this recurrence relation and the given initial conditions is the sequence $\{a_n\}$ with $a_n = 1 - 2^n + 2 \times 3^n$.

We now state the most general result about linear homogeneous recurrence relations with constant coefficients, allowing the characteristic equation to have multiple roots. The key point is that for each root $r$ of the characteristic equation, the general solution is of the form $P(n)\, r^n$, where $P(n)$ is a polynomial of degree $m - 1$, with $m$ the multiplicity of this root $r$.

**Theorem:** Let $c_{1,}c_2, \ldots, c_k$ be real numbers. Suppose that the characteristic equation

$$r^k - c_1 r^{k-1} - c_2 r^{k-2} - \cdots - c_{k-1} r - c_k = 0$$

has $t$ distinct roots $r_1, r_2, \ldots, r_t$ with multiplicities $m_1, m_2, \ldots, m_t$, respectively, so that $m_i \geq 1$for $i = 1,2, \ldots, t$ and $m_1 + m_2 + \cdots + m_t = k$.Then a sequence$\{a_n\}$ is a solution of the recurrence relation

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k}$$

if and only if

$$a_n = \left(\alpha_{1,0} + \alpha_{1,1} n + \cdots + \alpha_{1,m_1-1}\, n^{m_1-1}\right) r_1^n + \left(\alpha_{2,0} + \alpha_{2,1} n + \cdots + \alpha n^{m_2-1}\right) r_2^n + \cdots$$
$$+\left(\alpha_{t,0} + \alpha_{t,1} n + \cdots + \alpha_{t,m_t-1}\, n^{m_t-1}\right) r_t^n$$

for $n = 0,1,2 \ldots$, where $\alpha_{i,j}$ are constants for $1 \leq i \leq t$ and $0 \leq j \leq m_i - 1$.

**Example**: Find the solution of the recurrence relation

$$a_n = -3a_{n-1} - 3a_{n-2} - a_{n-3},$$

with initial conditions $a_0 = 1, a_1 = -2$ and $a_2 = -1$.

**Solution**: The characteristic polynomial of this recurrence relation is

$$r^3 + 3r^2 + 3r + 1 = 0.$$

The roots are$-1, -1, -1$. Then $r = -1$ with multiplicity 3. Thus, the solutions of the recurrence relation are of the form

$$a_n = (\alpha_{1,0} + \alpha_{1,1} n + \alpha_{1,2} n^2)(-1)^n.$$

To find the constants $\alpha_{1,0}$, $\alpha_{1,1}$and $\alpha_{1,2}$, use the initial conditions. This gives

$$a_0 = 1 = \alpha_{1,0}$$

$$a_1 = -2 = -\alpha_{1,0} - \alpha_{1,1} - \alpha_{1,2}$$

$$a_2 = -1 = \alpha_{1,0} + 2\alpha_{1,1} + 4\alpha_{1,2}$$

The simultaneous solution of these three equations is $\alpha_{1,0} = 1, \alpha_{1,1} = 3$ and $\alpha_{1,2} = -2$. Hence, the unique solution to this Recurrence Relation and the given initial conditions is the sequence $\{a_n\}$ with

$$a_n = (1 + 3n - 2n^2)(-1)^n.$$

## Solving Linear Non- Homogenous Recurrence Relations with Constant Coefficients

Linear non- homogenous recurrence relations with constant coefficients is a recurrence relation of the form

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k} + f(n)$$

where, $c_{1,} c_2, \ldots, c_k$ are real numbers and $f(n)$ is a function not identically zero depending only on $n$. The Recurrence Relation

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k}$$

is called the associated homogenous recurrence relation. It plays an important role in solving the given non-homogenous recurrence relation with constant coefficients.

**Examples:** The following are examples of non-homogenous recurrence relation with constant coefficients.

(i) $a_n = a_{n-1} + 2^n$

(ii) $a_n = a_{n-1} + a_{n-2} + n^2 + n + 1$

(iii) $a_n = 3a_{n-1} + n3^n$

The key fact about linear nonhomogeneous recurrence relations with constant coefficients is that every solution is the sum of a particular solution and a solution of the associated linear homogeneous recurrence relation.

**Theorem:** If $\left\{a_n^{(p)}\right\}$ is a particular solution of the non-homogenous linear recurrence relation with constant coefficients

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k} + f(n),$$

then every solution is of the form $\left\{a_n^{(p)} + a_n^{(h)}\right\}$, where $a_n^{(h)}$ is a solution of the associated homogenous recurrence relation of the form

$$a_n = c_1a_{n-1} + c_2a_{n-2} + \cdots + c_ka_{n-k}.$$

There is no general procedure for finding the particular solution of a recurrence relation. However, if $f(n)$ has any one of the forms (i) polynomials in $n$, (ii) a constant or power of a constant, then we may guess the forms of particular solution and exactly find out it by the method of undetermined coefficients.

| $\boldsymbol{f(n)}$ | **Form of a particular solution** |
|---|---|
| A constant, $c$ | A constant , $d$ |
| A linear function, $c_0 + c_1n$ | A linear function, $d_0 + d_1n$ |
| $n^2$ | $d_0 + d_1n + d_2n^2$ |
| An $m^{th}$ degree polynomial<br>$c_0 + c_1n + c_2n^2 + \cdots + c_mn^m$ | An $m^{th}$ degree polynomial<br>$d_0 + d_1n + d_2n^2 + \cdots + d_mn^m$ |
| power of a constant<br>$c^n$ | For a constant $d$<br>$dc^n$ |

**Example**: Solve the recurrence relation $a_n = 3a_{n-1} + 2^n, a_0 = 1$

Solution: The associated homogenous recurrence relation is $a_n - 3a_{n-1} = 0$. The characteristic equation is

$$r - 3 = 0 \Rightarrow r = 3.$$

∴ The homogenous solution is

$$a_n^{(h)} = \alpha 3^n$$

where $\alpha$ is a constant. Since $f(n)$ of the recurrence relation is $2^n$, the particular solution of the recurrence relation is

$$a_n^{(p)} = a_n = d2^n.$$

Using this equation in the given recurrence relation, we get

$$d2^n - 3d2^{n-1} = 2^n \Longrightarrow d - \frac{3}{2}d = 1 \Longrightarrow 2d - 3d = 2 \Longrightarrow d = -2.$$

∴ $a_n^{(p)} = -2(2)^n = -2^{n+1}$.

Hence, the general solution is $a_n = a_n^{(h)} + a_n^{(p)}$

$$\Rightarrow a_n = \alpha 3^n - 2^{n+1}$$

Using the condition$a_0 = 1$, we get $a_0 = \alpha 3^0 - 2^1 = 1 \Rightarrow \alpha = 3$.

The required solution is

$$a_n = 3(3)^n - 2^{n+1} = 3^{n+1} - 2^{n+1}.$$

**Example:** Solve the recurrence relation

$$a_n - 7a_{n-1} + 10a_{n-2} = 6 + 8n, a_0 = 1, a_1 = 2.$$

Solution: The associated homogenous recurrence relation is $a_n - 7a_{n-1} + 10a_{n-2} = 0$. Then the characteristic equation is $r^2 - 7r + 10 = 0 \Rightarrow (r-5)(r-2) = 0 \Rightarrow r = 2, 5$. Therefore, homogenous solution is $a_n^{(h)} = c_1 2^n + c_2 3^n$.

Let $a_n^{(p)} = d_0 + d_1 n$ be the particular solution, since $f(n)$ is a linear polynomial in $n$**.** Using this equation in the given recurrence relation, we get

$$(d_0 + d_1 n) - 7\big(d_0 + d_1(n-1)\big) + 10\big(d_0 + d_1(n-2)\big) = 6 + 8n.$$

Equating the corresponding coefficients on both sides, we get

$4d_0 - 13d_1 = 6$ and $4d_1 = 8. \Rightarrow d_1 = 2$ and $d_0 = 8$.

Thus,

$$a_k^{(p)} = 8 + 2n.$$

Hence, the general solution is $a_n = a_n^{(h)} + a_n^{(p)}$

$$\Rightarrow a_n = c_1 2^n + c_2 5^n + 8 + 2n$$

Given that,$a_0 = 1, a_1 = 2$. Thus,

$$a_0 = 1 \Rightarrow c_1 + c_2 + 8 = 1 \text{ and } a_1 = 2 \Rightarrow 2c_1 + 5c_2 + 8 + 2 = 2.$$

Solving the equations, we get

$$c_1 = -9,\ c_2 = 2.$$

The required solution is $a_n = a_n^{(h)} + a_n^{(p)} = -9(2^n) + 2(5^n) + 8 + 2n$.

In both Examples we were able to find the particular solutions. Now we have to select the particular solution in more general way. When $f(n)$ is the product of a polynomial in $n$ and the $n^{th}$ power of a constant, we have to select the particular solution which is stated in the below theorem.

**Theorem:** Suppose that $\{a_n\}$ satisfies the linear non-homogenous recurrence relation

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k} + f(n),$$

where $c_{1,} c_2, \dots, c_k$ are real numbers, and

$$f(n) = (b_t n^t + b_{t-1} n^{t-1} + \cdots + b_1 n + b_0) s^n,$$

where $b_0, b_1, \dots, b_t$ and $s$ are real numbers.

(i) When $s$ is not root of the characteristic equation of the associated homogenous linear recurrence relation, there is a particular solution of the form

$$(p_t n^t + p_{t-1} n^{t-1} n + \cdots + p_1 n + p_0) s^n.$$

(ii) When $s$ is a root of this associated homogenous characteristic equation and its multiplicity is $m$, there is a particular solution of the form

$$n^m (p_t n^t + p_{t-1} n^{t-1} n + \cdots + p_1 n + p_0) s^n.$$

**Example**: Solve the recurrence relation

$$a_n = 4a_{n-1} - 4a_{n-2} + (n+1)2^n.$$

Solution: The associated homogenous recurrence relation is

$$a_n - 4a_{n-1} + 4a_{n-2} = 0.$$

The characteristic equation is $r^2 - 4r + 4 = 0$.

i.e, $(r-2)^2 = 0 \;\; \Rightarrow r = 2, 2$.

∴ The homogenous solution is

$$a_n^{(h)} = (d_1 + d_2 n)2^n.$$

Since, the $f(n)$ of the recurrence relation is $(n+1)2^n$and the characteristic root 2 is repeated twice, we assume the particular solution of the recurrence relation to be

$$a_n^{(p)} = (c_1 + c_2 n)n^2 2^n.$$

Using this equation in the given recurrence relation, we get

$$(c_1 + c_2 n)n^2 2^n - 4(c_1 + c_2(n-1)(n-1)^2 2^{n-1}) + 4\big(c_1 + c_2(n-2)\big)(n-2)^2 2^{n-2}$$
$$= (n+1)2^n.$$

$$\Rightarrow (c_1 + c_2 n)n^2 - 2\big(c_1 + c_2(n-1)\big)(n-1)^2\big) + \big(c_1 + c_2(n-2)\big)(n-2)^2 = (n+1).$$

Putting $n = 0$, we get

$$-2(c_1 - c_2) + 4(c_1 - 2c_2) = 1 \Rightarrow 2c_1 - 6c_2 = 4 \Rightarrow c_1 - 3c_2 = \tfrac{1}{2}.$$

Putting $n = 1$,

$$(c_1 + c_2) + (c_1 - c_2) = 2 \Rightarrow 2c_1 = 2 \Rightarrow c_1 = 1.$$

Thus, $c_2 = \frac{1}{6}$. And

$$a_n^{(p)} = \left(1 + \tfrac{1}{6}n\right) n^2 2^n = \left(n^2 + \tfrac{n^3}{6}\right) 2^n.$$

Thus, the general solution of the recurrence relation is

$$a_n = a_n^{(h)} + a_n^{(p)} = \left(d_1 + d_2 n + n^2 + \tfrac{n^3}{6}\right) 2^n.$$

**Example:** Solve the recurrence relation

$$a_n = 4a_{n-1} - 4a_{n-2} + 3n + 2^n,\ \ a_0 = 1, a_1 = 1.$$

Solution: The associated homogenous recurrence relation is

$$a_n - 4a_{n-1} + 4a_{n-2} = 0.$$

The characteristic equation is

$$r^2 - 4r + 4 = 0.$$

i.e., $(r-2)^2 = 0 \ \Rightarrow r = 2, 2$.

∴ The homogenous solution is

$$a_n^{(h)} = (c_1 + c_2 n)2^n.$$

Since $f(n) = 3n + 2^n$, the particular solution is of the form

$$a_n^{(p)} = a_n^{(p_1)} + a_n^{(p_2)}.$$

Where, $a_n^{(p_1)} = d_0 + d_1 n$ and $a_n^{(p_2)} = dn^2 2^n$.

Using the solution $a_n^{(p_1)}$ in the recurrence relation, we get

$$(d_0 + d_1 n) - 4\big(d_0 + d_1(n-1)\big) + 4\big(d_0 + d_1(n-2)\big) = 3n.$$

$$\Rightarrow (d_0 - 4d_1) + d_1 n = 3n.$$

Equating the coefficient of $n$ on both the sides, we get

$$d_1 = 3.$$

Equating the constant terms on both the sides, we get

$$d_0 - 4d_1 = 0 \Rightarrow d_0 = 12.$$

Therefore, the particular solution corresponding to $3n$ is

$$a_n^{(p_1)} = 12 + 3n$$

Let $= dn^2 2^n$

Using the solution $a_n^{(p_2)}$in the recurrence relation, we get

$$dn^2 2^n - 4d(n-1)^2 2^{n-1} + 4d(n-2)^2 2^{n-2} = 2^n.$$

Putting $n = 0$, we get

$$-2d + 4d = 1.$$

$$\Rightarrow d = \tfrac{1}{2}.$$

Therefore, $a_n^{(p_2)} = \frac{1}{2} n^2 2^n = n^2 2^{n-1}$.

Therefore, the particular solution is $a_n^{(p)} = a_n^{(p_1)} + a_n^{(p_2)} = 12 + 3n + n^2 2^{n-1}$.

Hence, the general solution is

$$a_n = (c_1 + c_2 n)2^n + 12 + 3n + n^2 2^{n-1}.$$

Given that, $a_0 = 1, a_1 = 1$.

Now, $a_0 = 1 \Rightarrow c_1 + 12 = 1 \Rightarrow c_1 = -11$.

Also, $a_1 = 1 \Rightarrow (c_1 + c_2)2 + 12 + 3 + 2^2 = 1 \Rightarrow 2c_1 + 2c_2 = -18 \Rightarrow c_1 + c_2 = -9 \Rightarrow c_2 = 2$.

Thus, the required solution is

$$a_n = (2n - 11)2^n + 12 + 3n + n^2 2^{n+1}.$$

**Example**: Solve the recurrence relation

$a_n - 2a_{n-1} + a_{n-2} = 2, a_0 = 25, a_1 = 16.$

The associated homogenous recurrence relation is

$a_n - 2a_{n-1} + a_{n-2} = 0.$

The characteristic equation is $r^2 - 2r + 1 = 0 \Rightarrow r = 1,1$.

∴ The homogenous solution is $a_n^{(h)} = (c_1 + c_2 n)1^n = c_1 + c_2 n$.

Since $f(n) = 2 = 2(1)^n$, 1 is the root of the characteristic equation of multiplicity 2,

So, the particular solution is $a_n^{(p)} = Dn^2$.

Using this solution in the recurrence relation, we get

$Dn^2 - 2D(n-1)^2 + D(n-2)^2 = 2.$

$\Rightarrow Dn^2 - 2D(n^2 + 1 - 2n) + D(n^2 + 4 - 4n) = 2.$

Comparing the like coefficients of $n$ on both the sides, we get

$2D = 2 \Rightarrow D = 1.$

So, $a_n^{(p)} = n^2$. And hence,

$a_n = a_n^{(h)} + a_n^{(p)} = c_1 + c_2 n + n^2.$

Now, $a_0 = 25 \Rightarrow c_1 = 25$ and $a_1 = 16 \Rightarrow c_1 + c_2 + 1 \Rightarrow c_2 = -10$.

So,$a_n = 25 - 10n + n^2$.

## Generating Functions

**Definition**: The generating function for the sequence $\{a_n\}$ of real numbers is the infinite series

$$G(x) = a_0 + a_1 x + \cdots + a_n x^n + \cdots = \sum_{n=0}^{\infty} a_n x^n.$$

The generating function of $\{a_n\}$ given in this definition is sometimes called the ordinary generating function of $\{a_n\}$ to distinguish it from other types of generating functions for the sequence.

**Example:** The generating function for the sequences $\{a_n\}$ with $a_n = 3$ is given by

$$G(x) = 3 + 3.x + 3.x^2 + 3.x^3 + \cdots + 3.x^n + \cdots + .. = \sum_{n=0}^{\infty} 3x^n = \frac{3}{1-x}$$

when $|x| < 1$.

Similarly, if $a_n = n + 1$,

$$G(x) = \sum_{n=0}^{\infty}(n+1)x^n = \frac{1}{(1-x)^2}\ ;\ \ |x| < 1.$$

We can define generating functions for finite sequence of real numbers by extending a finite sequence $a_0, a_1, \ldots, a_n$ into an infinite sequence by setting $a_{n+1} = 0,\ a_{n+2} = 0$, and so on. The generating function $G(x)$ of this infinite sequence $\{a_n\}$ is a polynomial of degree $n$ because no terms of the form $a_j x^j, j > n$ occur, that is, $G(x) = a_0 + a_1 x + \cdots + a_n x^n$.

**Example:** What is the generating function for the sequence 1,1,1,1,1,1?

**Solution:** The generating function for the sequence1,1,1,1,1,1is

$$G(x) = 1 + x + x^2 + x^3 + x^4 + x^5.$$

**Example:** Find the closed form expression of the generating function for the sequence $1, a, a^2, \ldots$.

**Solution:** The closed form expression of the generating function for the sequence $1, a, a^2, ..$ can be written as

$$G(x) = 1 + ax + a^2x^2 + \cdots = 1 + ax + (ax)^2 + (ax)^3 + \cdots = \frac{1}{1-ax}$$

when $|ax| < 1$ .

**Example:** Find the closed form expression of the generating function for the Fibonacci sequence

$$F_n = F_{n-1} + F_{n-2}, n \geq 2, F_0 = 0, F_1 = 1.$$

The generating function of a Fibonacci sequence $\{F_n\}$ is given by

$$F(z) = F_0 + F_1 z + F_2 z^2 + F_3 z^3 + \cdots = \sum_{n=0}^{\infty} F_n z^n.$$

Multiplying both sides of above equation by $z^n$and summing over all $n \geq 2$, we get

$$\sum_{n=2}^{\infty} F_n z^n = \sum_{n=2}^{\infty} F_{n-1} z^n + \sum_{n=2}^{\infty} F_{n-2} z^n.$$

$$\Rightarrow \sum_{n=2}^{\infty} F_n z^n = z\sum_{n=2}^{\infty} F_{n-1}z^{n-1} + z^2\sum_{n=2}^{\infty} F_{n-2}z^{n-2}.$$

$$\Rightarrow F(z) - F_0 - F_1 z = z[F(z) - F_0] + z^2F(z).$$

Since $F_0 = 0, F_1 = 1$, we have

$$F(z) - 0 - z = z[F(z) - 0] + z^2F(z).$$

$$\Rightarrow (1 - z - z^2)F(z) = z.$$

$$\Rightarrow F(z) = \frac{z}{1 - z - z^2}.$$

## Properties of Generating functions

Let $\{a_n\}$ and $\{b_n\}$ be two sequences and $G(z)$ and $F(z)$ be the corresponding generating functions. That is,

$$G(z) = \sum_{n=0}^{\infty} a_n z^n \quad \text{and} \quad F(z) = \sum_{n=0}^{\infty} b_n z^n.$$

**1.** The sum of two generating functions is a generating function.

The sum of the generating functions $G(z)$ and $F(z)$ is defined as

$$H(z) = G(z) + F(z) = \sum_{n=0}^{\infty} a_n z^n + \sum_{n=0}^{\infty} b_n z^n = \sum_{n=0}^{\infty} (a_n + b_n)z^n = \sum_{n=0}^{\infty} c_n z^n$$

where $c_n = a_n + b_n$.

**2.** The scalar product of any generating function, i.e., if $\lambda$ is any scalar, then $\lambda G(z)$is a generating function.

$$H(z) = \lambda G(z) = \lambda\sum_{n=0}^{\infty} a_n z^n = \sum_{n=0}^{\infty} (\lambda a_n)z^n = \sum_{n=0}^{\infty} c_n z^n$$

Where $c_n = \lambda a_n$.

**3.** The product of two generating function is again generating function.

$$H(z) = G(z)F(z) = \sum_{n=0}^{\infty}\left(\sum_{j=0}^{n} a_j b_{n-j}\right)x^n$$

**4.** If $p$ is a positive integer, then $z^pG(z)$ is a generating function.

**5.** Differentiation of generating function is again generating function.

Differentiating $G(z)$ term by term we get,

$$G'(z) = a_1 + 2a_2z + 3a_3z^2 + \cdots = \sum_{n=0}^{\infty} na_n z^{n-1}.$$

Thus, $G'(z)$ is generating function of the sequence $\{na_n\}$.

## Some Useful Generating Functions

| $G(x)$ | $a_k$ |
|---|---|
| $(1+x)^n=\sum_{k=0}^{n} C(n,k)x^k=1+C(n,1)x+C(n,2)x^2+\cdots+x^n$ | $C(n,k)=\frac{n!}{k!\,(n-k)!}$ |
| $(1+ax)^n=\sum_{k=0}^{n} C(n,k)a^kx^k$ $=1+C(n,1)ax+C(n,2)a^2x^2+\cdots+a^nx^n$ | $C(n,k)a^k$ |
| $(1+x^r)^n=\sum_{k=0}^{n} C(n,k)x^{rk}=1+C(n,1)x^r+C(n,2)x^{2r}+\cdots+x^{rn}$ | $\begin{cases} C\left(n,\frac{k}{r}\right) & if\ r\mid k \\ 0 & \text{otherwisre} \end{cases}$ |
| $\frac{1-x^{n+1}}{1-x}=\sum_{k=0}^{n} x^k=1+x+x^2+\cdots+x^n$ | $\begin{cases} 1 & \text{if } k\le n \\ 0 & \text{otherwisre} \end{cases}$ |
| $\frac{1}{1-x}=\sum_{k=0}^{\infty} x^k=1+x+x^2+\cdots$ | 1 |
| $\frac{1}{1-ax}=1+ax+a^2x^2+\cdots$ | $a^k$ |
| $\frac{1}{1-x^r}=\sum_{k=0}^{\infty} x^{rk}=1+x^r+x^{2r}+\cdots$ | $\begin{cases} 1 & if\ r\mid k \\ 0 & \text{otherwisre} \end{cases}$ |
| $\frac{1}{(1-x)^2}=\sum_{k=0}^{\infty}(k+1)x^k=1+2x+3x^2+\cdots$ | $k+1$ |
| $\frac{1}{(1-x)^n}=\sum_{k=0}^{\infty} C(n+k-1,k)x^k=1+C(n,1)x+C(n+1,2)x^2+\cdots$ | $C(n+k-1,k)$ $C(n+k-1,n-1)$ |
| $\frac{1}{(1+x)^n}=\sum_{k=0}^{\infty} C(n+k-1,k)(-1)^kx^k$ $=1-C(n,1)x+C(n+1,2)x^2-\cdots$ | $(-1)^kC(n+k-1,k)$ $(-1)^kC(n+k-1,n-1)$ |
| $\frac{1}{(1-ax)^n}=\sum_{k=0}^{\infty}(n+k-1,k)a^kx^k$ $=1+C(n,1)ax+C(n+1,2)a^2x^2+\cdots$ | $C(n+k-1,k)a^k$ $C(n+k-1,n-1)a^k$ |
| $e^x=\sum_{k=0}^{\infty}\frac{x^k}{k!}=1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\cdots$ | $\frac{1}{k!}$ |

| | |
|---|---|
| $\ln(1+x)=\sum_{k=0}^{\infty}\frac{(-1)^{k+1}}{k}x^k = x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+\cdots$ | $\frac{(-1)^{k+1}}{k}$ |

## Solving of Recurrence Relation using generating function

**Example**: Use generating function to solve the recurrence relation $a_n = 3a_{n-1}+2, n \geq 1$ with $a_0 = 1$.

Solution: Let the generating function of the sequence $\{a_n\}$ be $G(z)=\sum_{n=0}^{\infty} a_n z^n$. Given the recurrence relation is

$$a_n = 3a_{n-1}+2.$$

Multiplying both the sides by $z^n$ and summing over all $n \geq 1$, we have

$$\sum_{n=1}^{\infty} a_n z^n = 3\sum_{n=1}^{\infty} a_{n-1} z^n + 2\sum_{n=1}^{\infty} z^n.$$

$$= 3z\sum_{n=1}^{\infty} a_{n-1} z^{n-1} + 2z\sum_{n=1}^{\infty} z^{n-1}.$$

$$\Rightarrow G(z) - a_0 = 3zG(z) + \frac{2z}{1-z}.$$

$$\Rightarrow G(z) - 1 = 3zG(z) + \frac{2z}{1-z}.$$

$$\Rightarrow (1-3z)G(z) = 1+\frac{2z}{1-z} = \frac{1+z}{1-z}.$$

$$\Rightarrow G(z) = \frac{(1+z)}{(1-z)(1-3z)}.$$

Let

$$\frac{(1+z)}{(1-z)(1-3z)} = \frac{A}{(1-z)} + \frac{B}{(1-3z)}.$$

$$\Rightarrow (1+z) = A(1-3z) + B(1-z).$$

$$\Rightarrow A = -1, B = 2.$$

$$\therefore G(z) = \frac{2}{1-3z} - \frac{1}{1-z}.$$

$$a_n = 2(3^n) - 1.$$

**Example**: Given $a_0 = 2, a_1 = 7$, solve the recurrence relation

$$a_n = 5a_{n-1} - 6a_{n-2}; \quad \text{for all } n \geq 2$$

by using generating function.

**Solution:** Let $G(t)$ be the generating function of the sequence $\{a_n\}$ . Then

$$G(t) = \sum_{n=0}^{\infty} a_n t^n .$$

Given that,

$$a_n = 5a_{n-1} - 6a_{n-2}; \text{ for all } n \geq 2.$$

Multiplying $t^n$ in both sides, we get

$$a_n t^n = 5a_{n-1}t^n - 6a_{n-2}t^n.$$

Now taking the sum over $n$ from 2 to $\infty$, we have

$$\sum_{n=2}^{\infty} a_n t^n = \sum_{n=2}^{\infty} 5a_{n-1}t^n - \sum_{n=2}^{\infty} 6a_{n-2}t^n.$$

Simplifying,

$$\Rightarrow \sum_{n=0}^{\infty} a_n t^n - a_0 - a_1 t = 5t \sum_{n=2}^{\infty} a_{n-1}t^{n-1} - 6t^2 \sum_{n=2}^{\infty} a_{n-2}t^{n-2}$$

$$\Rightarrow G(t) - 2 - 7t = 5t \sum_{m=1}^{\infty} a_m t^m - 6t^2 \sum_{s=0}^{\infty} a_s t^s$$

$$\Rightarrow G(t) - 2 - 7t = 5t\left(\sum_{m=0}^{\infty} a_m t^m - a_0\right) - 6t^2 G(t)$$

$$\Rightarrow G(t) - 2 - 7t = 5t(G(t) - 2) - 6t^2 G(t)$$

$$\Rightarrow (1 - 5t + 6t^2)G(t) = 2 - 3t$$

$$\Rightarrow G(t) = \frac{2 - 3t}{1 - 5t + 6t^2}.$$

Thus, generating function $G(t)$ of $\{a_n\}$ is $\frac{2-3t}{1-5t+6t^2}$. Now $a_n$ is the coefficient of $t^n$ in the expansion of $G(t)$. Thus,

$$G(t) = \frac{2-3t}{(1-2t)(1-3t)} = \frac{3}{1-3t} - \frac{1}{1-2t} = 3\sum_{n=0}^{\infty}(3t)^n - \sum_{n=0}^{\infty}(2t)^n = \sum_{n=0}^{\infty}(3^{n+1}-2^n)t^n.$$

Therefore, $a_n = 3^{n+1} - 2^n$.

**Example:** Use generating function to solve the recurrence relation $a_n - 2a_{n-1} - 3a_{n-2} = 0, n \geq 2$ with $a_0 = 3, a_1 = 1$.

**Solution:** Let the generating function of the sequence $\{a_n\}$ be $G(z) = \sum_{n=0}^{\infty} a_n z^n$.

Multiplying both the sides of recurrence relation by $z^n$ and summing over all $n \geq 2$, we have

$$\sum_{n=2}^{\infty} a_n z^n - 2\sum_{n=2}^{\infty} a_{n-1} z^n - 3\sum_{n=2}^{\infty} a_{n-2} z^n = 0.$$

$$\Rightarrow \sum_{n=2}^{\infty} a_n z^n - 2z\sum_{n=2}^{\infty} a_{n-1} z^{n-1} - 3z^2\sum_{n=2}^{\infty} a_{n-2} z^{n-2} = 0.$$

$$\Rightarrow [G(z) - a_0 - a_1 z] - 2z[G(z) - a_0] - 3z^2 G(z) = 0.$$

$$\Rightarrow (1 - 2z - 3z^2)G(z) - 3 - z - 2z(-3) = 0.$$

$$\Rightarrow (1 - 2z - 3z^2)G(z) = 3 - 5z.$$

$$\Rightarrow G(z) = \frac{(3-5z)}{(1-2z-3z^2)} = \frac{(3-5z)}{(1-3z)(1+z)}.$$

Let $\frac{(3-5z)}{(1-3z)(1+z)} = \frac{A}{(1-3z)} + \frac{B}{(1+z)}$.

Equating the numerators on both the sides, we get

$$3 - 5z = A(1+z) + B(1-3z).$$

From this, we get $A = 1, B = 2$ and

$$\therefore G(z) = \frac{1}{1+3z} + \frac{2}{1+z} = 1(3)^n + 2(-1)^n.$$

Thus, the required solution is $a_n = 3^n + 2(-1)^n$.

**Example:** Use generating function to solve the recurrence relation $a_n - 4a_{n-1} + 4a_{n-2} = 4^n, n \geq 2$ with $a_0 = 2, a_1 = 8$.

**Solution**: Multiplying both the sides of recurrence relation by $z^n$ and summing over all $n \geq 2$, we have

$$\sum_{n=2}^{\infty} a_n z^n = 4\sum_{n=2}^{\infty} a_{n-1} z^n - 4\sum_{n=2}^{\infty} a_{n-2} z^n + \sum_{n=2}^{\infty} 4^n z^n.$$

$$\Rightarrow \sum_{n=2}^{\infty} a_n z^n = 4z \sum_{n=2}^{\infty} a_{n-1} z^{n-1} - 4z^2 \sum_{n=2}^{\infty} a_{n-2} z^{n-2} + \sum_{n=2}^{\infty} 4^n z^n.$$

$$\Rightarrow [G(z) - a_0 - a_1 z] - 4z[G(z) - a_0] + 4z^2 G(z) = \frac{1}{(1-4z)} - 1 - 4z.$$

$$\Rightarrow [G(z) - 2 - 8z] - 4z[G(z) - 2] + 4z^2 G(z) = \frac{1}{(1-4z)} + 1 - 4z.$$

$$\Rightarrow [1 - 4z + 4z^2]G(z) = \frac{1}{(1-4z)} + 1 - 4z.$$

$$\Rightarrow G(z) = \frac{1 + (1-4z)^2}{(1-4z)(1-2z)^2}.$$

Let

$$\frac{1 + (1-4z)^2}{(1-4z)(1-2z)^2} = \frac{A}{1-4z} + \frac{B}{1-2z} + \frac{C}{(1-2z)^2}$$

Equating the numerators on both the sides, we get

$$1 + (1-4z)^2 = A(1-2z)^2 + B(1-4z)(1-2z) + C(1-4z)$$

Substituting $z = \frac{1}{2}, \frac{1}{4}, 0$, we get $C = -2, A = 4, B = 0$. Thus,

$$G(z) = \frac{4}{1-4z} - \frac{2}{(1-2z)^2}.$$

And

$$a_n = 4(4^n) - 2(n+1)2^n = 4^{n+1} - (n+1)2^{n+1}.$$

The required solution is $a_n = 4^{n+1} - (n+1)2^{n+1}$.

**Example: (Application of Recurrence Relation for Code word Enumeration)** A computer system considers a string of decimal digits a valid codeword if it contains an even number of $0$ digits. Let $a_n$ be the number of valid $n$ −digit codewords. Find a recurrence relation for $a_n$. Use generating function method to find a solution of the recurrence relation.

**Solution:** Initially $a_1 = 9$ as there are $9$ valid one digit codeword except $0$.Firstly, a valid string of $n$ digits can be made by appending a valid string of $n-1$ digits with a digit other than $0$. This can be done in $9a_{n-1}$ ways. Secondly, a valid string of $n$ digits can be obtained by appending a $0$ to a string of length $n-1$ that is not valid. The number of ways this can be done equals the number of invalid $(n-1)$digit strings. Since, there are $10^{n-1}$strings of length $n-1$ and $a_{n-1}$valid strings , there are $10^{n-1} - a_{n-1}$ valid $n$ −digit strings obtained by appending an invalid string of length $n-1$ with $0$. Because all the valid strings of length $n$ are produced in one of these two ways, it follows that there are total number of valid string $a_n$ of length $n$ are given as

$$a_n = 9a_{n-1} + (10^{n-1} - a_{n-1})$$

i.e.

$$a_n = 8a_{n-1} + 10^{n-1}, \qquad a_1 = 9$$

Let us multiply both the sides of the recurrence relation by $x^n$ to obtain

$$a_n x^n = 8a_{n-1}x^n + 10^{n-1}x^n.$$

Let $G(x) = \sum_{n=0}^{\infty} a_n x^n$ be the generating function of the sequence $a_0, a_1, a_2, \ldots$. We sum both the side of the last equation with $n = 1$, to find that

$$\sum_{n=1}^{\infty} a_n x^n = \sum_{n=1}^{\infty} 8a_{n-1}x^n + \sum_{n=1}^{\infty} 10^{n-1}x^n$$

$$\Rightarrow G(x) - 1 = 8xG(x) + \frac{x}{(1-10x)}.$$

Solving for $G(x)$ shows that

$$G(x) = \frac{1-9x}{(1-8x)(1-10x)}.$$

Expanding the R.H.S of this equation into partial fractions gives

$$G(x) = \frac{1}{2}\left(\frac{1}{1-8x} + \frac{1}{1-10x}\right).$$

$$G(x) = \frac{1}{2}\left(\sum_{n=0}^{\infty} 8^n x^n + \sum_{n=0}^{\infty} 10^n x^n\right).$$

$$= \sum_{n=0}^{\infty} (8^n + 10^n)x^n.$$

Consequently, we have the solution of the Recurrence Relation

$$a_n = \frac{1}{2}(8^n + 10^n).$$

**Example:** A popular puzzle of the late nineteenth century invented by the French mathematician Édouard Lucas, called the Tower of Hanoi, consists of three pegs mounted on a board together with disks of different sizes. Initially these disks are placed on the first peg in order of size, with the largest on the bottom. The rules of the puzzle allow disks to be moved one at a time from one peg to another as long as a disk is never placed on top of a smaller disk. The goal of the puzzle is to have all the disks on the second peg in order of size, with the largest on the bottom.

Let $H_n$denote the number of moves needed to solve the Tower of Hanoi problem with $n$ disks. Set up a recurrence relation for the sequence $H_n$ and solve using generating function.

**Solution:** Let $H_n$denote the number of moves needed to solve the Tower of Hanoi problem with $n$ disks.

Begin with *n* disks on peg 1, we can transfer the top *n* − 1 disks, following the rules of the puzzle, to peg 3 using $H_{n-1}$ moves.

We keep the largest disk fixed during these moves. Then, we use one move to transfer the largest disk to the second peg.

Now we can transfer the *n* − 1 disks on peg 3 to peg 2 using $H_{n-1}$ additional moves, placing them on top of the largest disk, which always stays fixed on the bottom of peg 2.

Moreover, it is easy to see that the puzzle cannot be solved using fewer steps. This shows that

$$H_n = H_{n-1} + 1 + H_{n-1} = 2H_{n-1} + 1.$$

The initial condition is $H_1 = 1$, because one disk can be transferred from peg 1 to peg 2, according to the rules of the puzzle, in one move.

Thus, the recurrence relation for the sequence $H_n$ is

$$H_n = 2H_{n-1} + 1 \text{ for all } n \geq 2$$

with initial condition $H_1 = 1$.

Now we have to solve the recurrence relation using generating function. Let $H_0 = 0$, as zero disk can be transferred from peg 1 to peg 2, according to the rules of the puzzle, in zero move.

As

$$H_n = 2H_{n-1} + 1.$$

Multiplying $t^n$ in both sides

$$H_n t^n = 2H_{n-1} t^n + t^n.$$

Now taking the sum over $n$ from 2 to $\infty$,

$$\sum_{n=2}^{\infty} H_n t^n = \sum_{n=2}^{\infty} 2H_{n-1} t^n + \sum_{n=2}^{\infty} t^n$$

$$\Rightarrow \sum_{n=0}^{\infty} H_n t^n - H_0 - H_1 t = 2t \sum_{n=2}^{\infty} H_{n-1} t^{n-1} + \sum_{n=0}^{\infty} t^n - 1 - t$$

$$\Rightarrow G(t) - t = 2t \sum_{n=2}^{\infty} H_{n-1} t^{n-1} + \frac{1}{1-t} - 1 - t$$

Where $G(t)$ is the generating function of the sequence $\{H_n\}$. i.e. $G(t) = \sum_{n=0}^{\infty} H_n t^n$.

Thus, by shifting of index

$$G(t) = 2t \sum_{m=1}^{\infty} H_m t^m + \frac{1}{1-t} - 1$$

$$\Rightarrow G(t) = 2t\left(\sum_{m=0}^{\infty} H_m t^m - H_0\right) + \frac{1}{1-t} - 1$$

$$\Rightarrow G(t) = 2tG(t) + \frac{1}{1-t} - 1$$

$$\Rightarrow (1-2t)G(t) = \frac{1}{1-t} - 1$$

$$\Rightarrow (1-2t)G(t) = \frac{t}{1-t}$$

$$\Rightarrow G(t) = \frac{t}{(1-t)(1-2t)}$$

Thus, generating function $G(t)$ of $\{H_n\}$ is $\frac{t}{(1-t)(1-2t)}$.

Now $H_n$ is the coefficient of $t^n$ in the expansion of $G(t)$.

Thus,

$$G(t) = \frac{t}{(1-t)(1-2t)}$$

$$= \frac{1}{1-2t} - \frac{1}{1-t}$$

$$= \sum_{n=0}^{\infty} (2t)^n - \sum_{n=0}^{\infty} t^n$$

$$= \sum_{n=0}^{\infty} 2^n t^n - \sum_{n=0}^{\infty} t^n$$

$$= \sum_{n=0}^{\infty} (2^n - 1)\, t^n$$

i.e.

$$G(t) = \textstyle\sum_{n=0}^{\infty} (2^n - 1)\, t^n.$$

Therefore, $H_n = 2^n - 1$. i.e. the number of moves needed to solve the Tower of Hanoi problem with $n$ disks is $2^n - 1$.

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