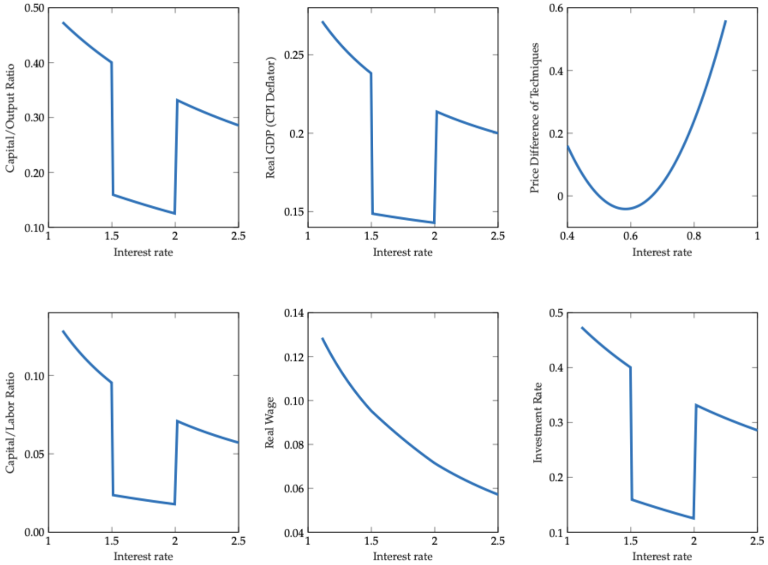
Lecture 3:
The Aggregate Production Function, continued

Kunal Sangani

ECON 416-1

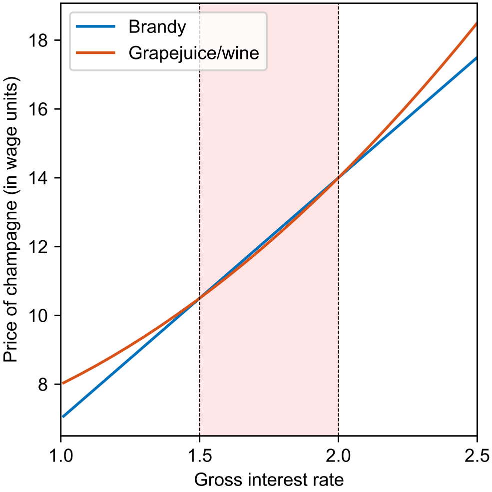
Re-switching example

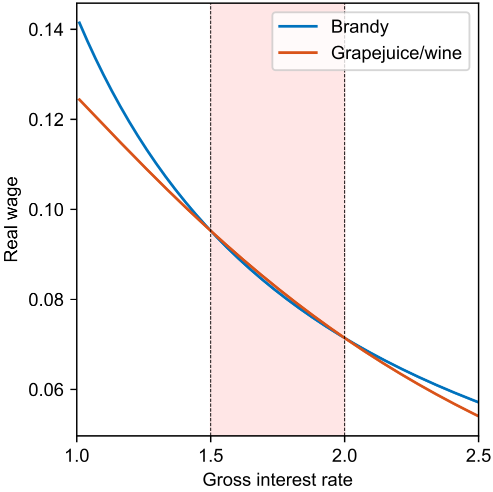


Re-switching example: Detail

- In steady state with $L_t = L$, given preferences $Y = \sum_t \beta^t c_t$, we must have $w_t = \beta w_{t-1}$.
- In steady state, consumption fixed by technology:
  - $c_t = c = L/7$ if you make brandy.
  - $c_t = c = L/8$ if you make grapejuice $\Rightarrow$ wine.
- Prices:
  - $p_t = w_t(7/\beta)$ if you make brandy.
  - $p_t = w_t(6 + 2/\beta^2)$ if you make grapejuice / wine.

Reswitching example: Why is the real wage continuous?





Re-switching example: Smoothed

- How do you create “smooth” switching between technologies?

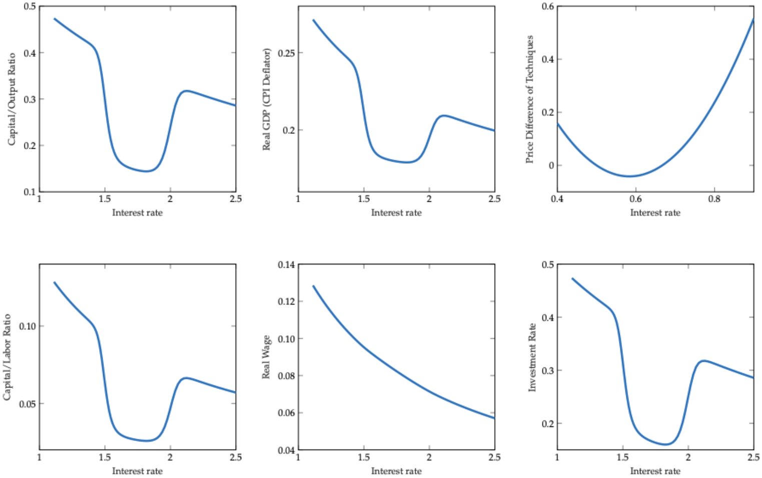


FIGURE 4. Samuelson’s smoothed-out reswitching example

Agenda

1. Aggregate production functions.
   (Baqaee & Farhi, JEEA 2019)
2. Disaggregation and Hulten's Theorem.
   (Hulten 1973)
3. Applications of Hulten's Theorem.
   (Gabaix 2011; Acemoglu et al. 2012)
4. Nonlinearities.
   (Baqaee & Farhi, ECMA 2019)

Recall from last time

- We showed that in an efficient economy, for any producer i,

$$\frac{\partial \log Y}{\partial \log A_i} = \frac{p_i y_i}{PY} = \lambda_i.$$

  Sales are a sufficient statistic for effect of micro shocks.

- Disaggregating our "measure of ignorance":

$$d \log Y = d \log A + \sum_f \Lambda_f d \log L_f = \sum_i \lambda_i d \log A_i + \sum_f \Lambda_f d \log L_f.$$

- Even i.i.d. shocks $d \log A_i$ need not wash out when λ_i is fat-tailed.
  - Can be fat-tailed due to distribution of firms sizes (Gabaix 2011).
  - Can be fat-tailed due to network structure (Acemoglu et al. 2012, 2017).

Hulten's Theorem vs. "Evidence of the eyes"

- Summers (2013): *"Now, think about the period after the financial crisis. I always like to think of these crises as analogous to a power failure or analogous to what would happen if all the telephones were shut off for some time. Consider such an event. The network would collapse. The connections would go away. And output would, of course, drop very rapidly. There would be a set of economists who would sit around explaining that electricity was only 4% of the economy, and so if you lost 80% of electricity, you couldn't possibly have lost more than 3% of the economy. Perhaps in Minnesota or Chicago there would be people writing such a paper, but most others would recognize this as a case where the evidence of the eyes trumped the logic of straightforward microeconomic theory.*

  And we would understand that somehow, even if we didn't exactly understand it in the model, that when there wasn't any electricity, there wasn't really going to be much economy. Something similar was true with respect to financial flows and financial interconnection. And that's why it is so important to get the lights back on, and that's why it's so important to contain the financial system."

Table of Contents

Nonlinearities

- Second-order Taylor approximation in $\Delta \log A_i$:

$$\Delta \log Y \approx \frac{d \log Y}{d \log A_i} \Delta \log A_i + \frac{1}{2} \frac{d^2 \log Y}{d \log A_i^2} (\Delta \log A_i)^2.$$

Nonlinearities

- Second-order Taylor approximation in $\Delta \log A_i$:

$$\Delta \log Y \approx \frac{d \log Y}{d \log A_i} \Delta \log A_i + \frac{1}{2} \frac{d^2 \log Y}{d \log A_i^2} (\Delta \log A_i)^2.$$

- Hulten's theorem tells us

$$\frac{d \log Y}{d \log A_i} = \lambda_i.$$

- Differentiating further,

$$\frac{d^2 \log Y}{d \log A_i^2} = \frac{d\lambda_i}{d \log A_i}.$$

- So to a second order in $\Delta \log A_i$,

$$\Delta \log Y = \lambda_i \Delta \log A_i + \frac{1}{2} \frac{d\lambda_i}{d \log A_i} (\Delta \log A_i)^2.$$

- E.g., if λ_i rises with negative shock, more costly than Hulten's theorem implies.

Nonlinearities: Interactions

- Likewise,

$$\frac{d\log Y}{d\log A_i} = \lambda_i. \qquad \Rightarrow \qquad \frac{d^2\log Y}{d\log A_i d\log A_j} = \frac{d\lambda_i}{d\log A_j}.$$

- Since order of derivatives can be interchanged,

$$\frac{d\lambda_i}{d\log A_j} = \frac{d\lambda_j}{d\log A_i}.$$

- Interactions between shocks matter to second-order,

$$\Delta\log Y \approx \sum_i \lambda_i \Delta\log A_i + \frac{1}{2}\sum_i\sum_j \frac{d\lambda_i}{d\log A_j}(\Delta\log A_i)(\Delta\log A_j).$$

Nonlinearities

- How do we estimate nonlinearities?

- If we observe a shock, we can read nonlinearities off the data:

$$\begin{aligned}
\Delta \log Y &\approx \lambda_i \Delta \log A_i + \frac{1}{2} \frac{d\lambda_i}{d \log A_i} \left(\Delta \log A_i\right)^2 \\
&\approx \lambda_i \Delta \log A_i + \frac{1}{2} \left(\Delta \lambda_i\right) \left(\Delta \log A_i\right) \\
&\approx \frac{1}{2} \left(\lambda_i + \lambda_i'\right) \Delta \log A_i .
\end{aligned}$$

Nonlinearities: Reduced form

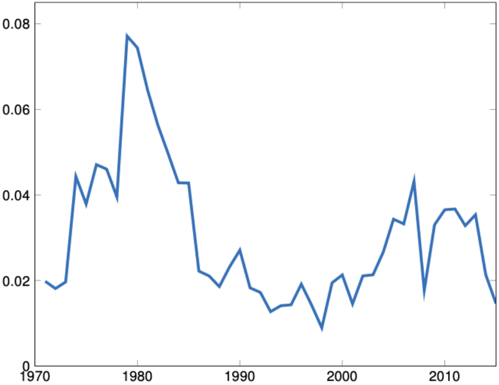


Figure: Global expenditures on crude oil as fraction of world GDP (Baqaee & Farhi 2019).

- First-order: $1.8\% \times -13\% = -0.2\%$.
- Second-order: $0.5(1.8\% + 7.6\%) \times -13\% = -0.6\%$.

Table of Contents

Nonlinearities: Using network structure

- To predict nonlinearities ex ante, we need the network structure. Recall:
- Arbitrary number of factors F in fixed supply L_f.
- Arbitrary number of producers N with neoclassical production functions,

$$y_i = A_i F_i(x_{i1}, \ldots, x_{iN}, L_{i1}, \ldots, L_{iF}).$$

  - New: Impose that F_i is constant returns to scale and CES with elasticity θ_i.
- Representative consumer re-labeled node zero:

$$U(c_1, \ldots, c_N) = F_0(x_{01}, \ldots, x_{0N}).$$

Network structure: Generality

1. Decreasing returns to scale.

2. Input-augmenting technologies.

3. Richer production functions.

4. Elastic factor supply.

Network structure: Generality

1. Decreasing returns to scale.
   - $y_i = A_i K^{\alpha_i}$.
2. Input-augmenting technologies.
3. Richer production functions.
4. Elastic factor supply.

Network structure: Generality

1. Decreasing returns to scale.
   - $y_i = A_i K^{\alpha_i}$.
2. Input-augmenting technologies.
   - $y_i = A_i F_i(A_{iL} L_i, K_i)$.
3. Richer production functions.
4. Elastic factor supply.

Network structure: Generality

1. Decreasing returns to scale.
   - $y_i = A_i K^{\alpha_i}$.
2. Input-augmenting technologies.
   - $y_i = A_i F_i(A_{iL} L_i, K_i)$.
3. Richer production functions.
   - $y_i = A_i \left(K_i^{\frac{\theta-1}{\theta}} + \left(H_i^{\frac{\rho-1}{\rho}} + L_i^{\frac{\rho-1}{\rho}} \right)^{\frac{\rho}{\rho-1}\frac{\theta-1}{\theta}} \right)^{\frac{\theta}{\theta-1}}$.
4. Elastic factor supply.

Network structure: Generality

1. Decreasing returns to scale.
    - $y_i = A_i K^{\alpha_i}$.
2. Input-augmenting technologies.
    - $y_i = A_i F_i(A_{iL} L_i, K_i)$.
3. Richer production functions.
    - $$y_i = A_i \left(K_i^{\frac{\theta-1}{\theta}} + \left(H_i^{\frac{\rho-1}{\rho}} + L_i^{\frac{\rho-1}{\rho}} \right)^{\frac{\rho}{\rho-1}\frac{\theta-1}{\theta}} \right)^{\frac{\theta}{\theta-1}}.$$
4. Elastic factor supply.
    - Let $L_f(w_f Y, Y)$ be supply of factor f as function of real price and aggregate income.

Network structure

- Define the input-output matrix Ω as

$$\Omega_{ij} = \frac{p_j x_{ij}}{p_i y_i}.$$

Ω_{ij} captures i's direct reliance on j. Use $\Omega^{(i)}$ and $\Omega_{(j)}$ to refer to i'th row, j'th column.

- Leontief inverse

$$\Psi = (I - \Omega)^{-1} = I + \Omega + \Omega^2 + \Omega^3 + \ldots$$

Ψ_{ij} captures i's total reliance on j (including indirect links through network).

- Domar weights are simply first row of Ψ, i.e., $\lambda = \Psi^{(0)}$.
    - As before, we will use λ_i for producers $1, \ldots, N$, and Λ_f for factors $1, \ldots, F$.

Network structure

- Define the input-output matrix Ω as

$$\Omega_{ij} = \frac{p_j x_{ij}}{p_i y_i}.$$

Ω_{ij} captures i's direct reliance on j. Use $\Omega^{(i)}$ and $\Omega_{(j)}$ to refer to i'th row, j'th column.

- Leontief inverse

$$\Psi = (I - \Omega)^{-1} = I + \Omega + \Omega^2 + \Omega^3 + \ldots$$

Ψ_{ij} captures i's total reliance on j (including indirect links through network).

- Domar weights are simply first row of Ψ, i.e., $\lambda = \Psi^{(0)}$.
  - As before, we will use λ_i for producers $1, \ldots, N$, and Λ_f for factors $1, \ldots, F$.

- If we can characterize $d\lambda_i / d \log A_i$, we're done!

Network structure: Forward propagation

- Start with how prices for i respond to productivity shock and changes in suppliers' prices:

$$d\log p_i = -d\log A_i + \sum_j \Omega_{ij} d\log p_j + \sum_f \Omega_{if} d\log w_f.$$

- Stacking, and using $d\log\Lambda_f = d\log w_f + d\log L_f = d\log w_f$, we have:

$$d\log p = -d\log A + \Omega d\log p + \Omega_{(f)} d\log\Lambda.$$

$$\Rightarrow d\log p = \Psi\left(-d\log A + \Omega_{(f)} d\log\Lambda\right).$$

Network structure: Forward propagation

- Start with how prices for i respond to productivity shock and changes in suppliers' prices:

$$d\log p_i = -d\log A_i + \sum_j \Omega_{ij} d\log p_j + \sum_f \Omega_{if} d\log w_f.$$

- Stacking, and using $d\log \Lambda_f = d\log w_f + d\log L_f = d\log w_f$, we have:

$$d\log p = -d\log A + \Omega d\log p + \Omega_{(f)} d\log \Lambda.$$

$$\Rightarrow d\log p = \Psi\left(-d\log A + \Omega_{(f)} d\log \Lambda\right).$$

Aside: dual proof for Hulten's theorem given vector of productivity shocks $d\log A$,

$$d\log Y = -d\log p_0 = -\Psi^{(0)}(-d\log A) = \lambda' d\log A.$$

- Forward propagation: We have $d\log p$ in terms of shocks and factor share changes.

Network structure: Backward propagation

- How do input shares change given price changes?

$$d \log \Omega_{ij} = d \log \frac{p_j x_{ij}}{p_i y_i} = -(\theta_i - 1) \left[d \log p_j - \sum_k \Omega_{ik} d \log p_k \right].$$

Network structure: Backward propagation

- How do input shares change given price changes?

$$d \log \Omega_{ij} = d \log \frac{p_j x_{ij}}{p_i y_i} = -(\theta_i - 1)\left[d \log p_j - \sum_k \Omega_{ik} d \log p_k\right].$$

- Given input share changes, we can write

$$d\lambda_i = \sum_{j=0}^{N} d\left(\frac{p_i x_{ji}}{PY}\right) = \sum_{j=0}^{N} d\left(\Omega_{ji}\lambda_j\right) = \sum_{j=0}^{N} \lambda_j d\Omega_{ji} + \sum_{j=0}^{N} \Omega_{ji} d\lambda_j.$$

Network structure: Backward propagation

- How do input shares change given price changes?

$$d\log\Omega_{ij} = d\log\frac{p_j x_{ij}}{p_i y_i} = -(\theta_i - 1)\left[d\log p_j - \sum_k \Omega_{ik} d\log p_k\right].$$

- Given input share changes, we can write

$$d\lambda_i = \sum_{j=0}^{N} d\left(\frac{p_i x_{ji}}{PY}\right) = \sum_{j=0}^{N} d\left(\Omega_{ji}\lambda_j\right) = \sum_{j=0}^{N} \lambda_j d\Omega_{ji} + \sum_{j=0}^{N} \Omega_{ji} d\lambda_j.$$

- Define the covariance operator

$$Cov_{\Omega^{(j)}}(\Psi_{(i)}, d\log p) = \sum_k \Omega_{jk}\Psi_{ki} d\log p - \left(\sum_k \Omega_{jk}\Psi_{ki}\right)\left(\sum_k \Omega_{jk} d\log p_k\right).$$

- Backward propagation: with some algebra, get $d\lambda$ in terms of $d\log p$,

$$d\lambda_i = -\sum_{j=0}^{N} (\theta_j - 1)\lambda_j Cov_{\Omega^{(j)}}(\Psi_{(i)}, d\log p).$$

Nonlinearities: One factor

$$d \log p_i = -\sum_{j=0}^{N} \Psi_{ij} d \log A_j + \sum_{f=1}^{F} \Psi_{if} d \log \Lambda_f. \qquad \text{(Forward propagation)}$$

$$d\lambda_i = -\sum_{j=0}^{N} (\theta_j - 1) \lambda_j Cov_{\Omega^{(j)}} (\Psi_{(i)}, d \log p). \qquad \text{(Backward propagation)}$$

- We are done... almost! Still need to know changes in factor shares, $d \log \Lambda_f$.

Nonlinearities: One factor

$$d\log p_i = -\sum_{j=0}^{N} \Psi_{ij} d\log A_j + \sum_{f=1}^{F} \Psi_{if} d\log \Lambda_f. \qquad \text{(Forward propagation)}$$

$$d\lambda_i = -\sum_{j=0}^{N} (\theta_j - 1)\lambda_j Cov_{\Omega^{(j)}}(\Psi_{(i)}, d\log p). \qquad \text{(Backward propagation)}$$

- We are done... almost! Still need to know changes in factor shares, $d\log \Lambda_f$.
- We must have $\sum_f \Lambda_f = 1$. So if we have a single factor, $d\log \Lambda_f = 0$:

$$\frac{d^2 \log Y}{d\log A_i d\log A_k} = \frac{d\lambda_i}{d\log A_k} = \sum_{j=0}^{N} (\theta_j - 1)\lambda_j Cov_{\Omega^{(j)}}(\Psi_{(i)}, \Psi_{(k)}).$$

$$\frac{d^2 \log Y}{d\log A_i^2} = \frac{d\lambda_i}{d\log A_i} = \sum_{j=0}^{N} (\theta_j - 1)\lambda_j Var_{\Omega^{(j)}}(\Psi_{(i)}).$$

Example 1: Horizontal economy

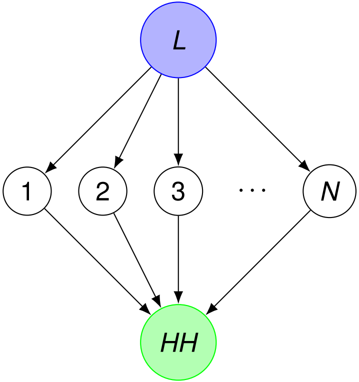


Figure: Horizontal economy.

- Consider a horizontal economy with N firms,

$$y_i = A_i L_i,$$

$$Y = \left(\sum_{i=1}^{N} y_i^{\frac{\theta-1}{\theta}} \right)^{\frac{\theta}{\theta-1}},$$

$$L = \sum_{i=1}^{N} L_i.$$

- What is $d^2 \log Y / (d \log A_i)(d \log A_j)$?

Example 1: Horizontal economy – By hand

- Demand curve,

$$\frac{y_i}{Y} = \left(\frac{p_i}{P}\right)^{-\theta}.$$

- Supply curve,

$$p_i = \frac{w}{A_i}.$$

- So we get:

$$\Rightarrow L_i = \frac{A_i^{\theta-1}}{\sum_k A_k^{\theta-1}} L. \qquad \lambda_i = \frac{A_i^{\theta-1}}{\sum_k A_k^{\theta-1}}.$$

- Differentiating,

$$d\lambda_i = \lambda_i (\theta - 1) \left[d\log A_i - \sum_k \lambda_k d\log A_k \right].$$

Example 1: Horizontal economy – BF (2019) formula

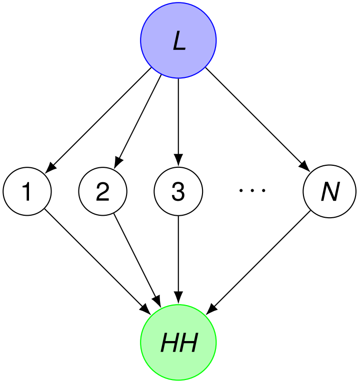


Figure: Horizontal economy.

$$\frac{d\lambda_i}{d\log A_k} = \sum_{j=0}^{N} (\theta_j - 1)\lambda_j Cov_{\Omega^{(j)}}(\Psi_{(i)}, \Psi_{(k)}).$$

- Any guesses?

Example 1: Horizontal economy – BF (2019) formula

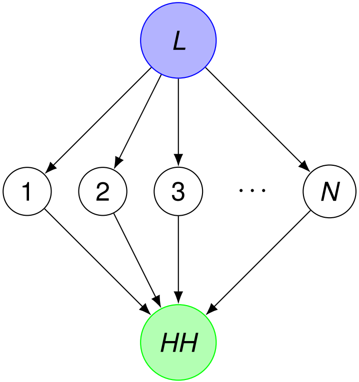


Figure: Horizontal economy.

$$\frac{d\lambda_i}{d\log A_k} = \sum_{j=0}^{N} (\theta_j - 1)\lambda_j Cov_{\Omega^{(j)}}(\Psi_{(i)}, \Psi_{(k)}).$$

- We get:

$$\frac{d\lambda_i}{d\log A_k} = (\theta_0 - 1) Cov_{\lambda}(1_{(i)}, 1_{(k)})$$

$$= (\theta_0 - 1)\left[\sum_i \lambda_i 1\{i = k\} - \lambda_i \lambda_k\right].$$

Example 1: Horizontal economy

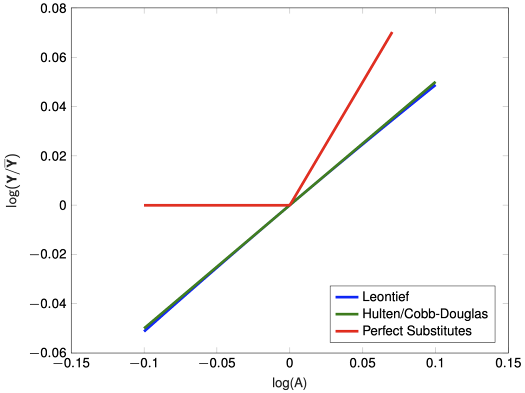


Figure: Nonlinearities in horizontal economy (Baqaee & Farhi 2019).

- Nonlinearities given by:

$$\frac{d^2 \log Y}{d \log A_i^2} = (\theta_0 - 1)\lambda_i(1 - \lambda_i).$$

- Intuition: When $\theta_0 > 1$, economy substitutes toward i as price of i falls. $\Rightarrow$ Convexity.

- Taleb (2012): *"Fragility implies more to lose than to gain, equals more downside than upside, equals (unfavourable) asymmetry. Antifragility implies more to gain than lose, equals more upside than downside, equals (favourable) asymmetry."*

Example 2: Round-about economy

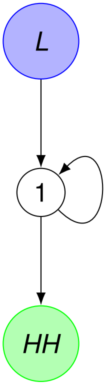


Figure: Horizontal economy.

- What is the effect of a large shock $\Delta \log A_i$?

$$\frac{d\lambda_i}{d \log A_k} = \sum_{j=0}^{N} (\theta_j - 1)\lambda_j Cov_{\Omega^{(j)}}(\Psi_{(i)}, \Psi_{(k)}).$$

Example 2: Round-about economy

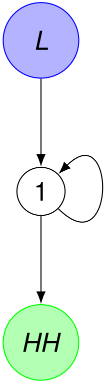


Figure: Horizontal economy.

- What is the effect of a large shock $\Delta \log A_i$?

$$\frac{d\lambda_1}{d\log A_1} = (\theta_1 - 1)\lambda_1 \, Var_{\Omega^{(1)}}(\Psi_{(1)}) \\ = (\theta_1 - 1)\lambda_1 \Psi_{11}^2 \Omega_{11}(1 - \Omega_{11}).$$

Example 2: Round-about economy

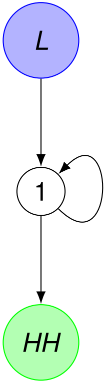


Figure: Horizontal economy.

- What is the effect of a large shock $\Delta \log A_i$?

$$\begin{aligned}\frac{d\lambda_1}{d\log A_1} &= (\theta_1 - 1)\lambda_1 \, Var_{\Omega^{(1)}}(\Psi_{(1)}) \\ &= (\theta_1 - 1)\lambda_1 \Psi_{11}^2 \Omega_{11} (1 - \Omega_{11}).\end{aligned}$$

- What is Ω_{11}? What is Ψ_{11}?

Example 2: Round-about economy

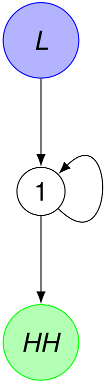


Figure: Horizontal economy.

- What is the effect of a large shock $\Delta \log A_i$?

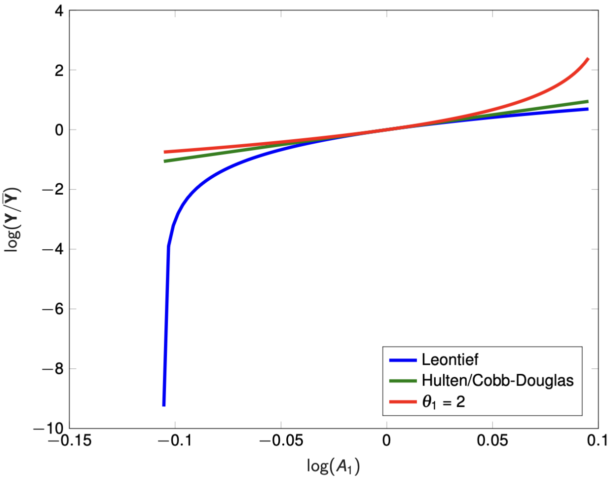
$$\frac{d\lambda_1}{d\log A_1} = (\theta_1 - 1)\lambda_1 \, Var_{\Omega^{(1)}}(\Psi_{(1)}) = (\theta_1 - 1)\lambda_1 \Psi_{11}^2 \Omega_{11} (1 - \Omega_{11}).$$

- What is Ω_{11}? What is Ψ_{11}?

- We find:

$$\Delta \log Y \approx \lambda_1 \Delta \log A_1 + \frac{1}{2}(\theta_1 - 1)\lambda_1 (\lambda_1 - 1)(\Delta \log A_1)^2.$$

Example 2: Round-about economy



$$\frac{d\lambda_1}{d\log A_1} = (\theta_1 - 1)\lambda_1(\lambda_1 - 1).$$

Figure: Round-about economy (Baqaee & Farhi 2019).

Nonlinearities: Multiple factors

- With multiple factors, need to solve system of equations for changes in factor shares,

$$\frac{d^2 \log Y}{d \log A_i d \log A_k} = \sum_{j=0}^{N} (\theta_j - 1)\lambda_j Cov_{\Omega^{(j)}} \left(\Psi_{(i)}, \Psi_{(k)} - \sum_{f=1}^{F} \frac{d \log \Lambda_f}{d \log A_k} \Psi_{(f)} \right),$$

  The changes in factor shares are given by

$$\frac{d \log \Lambda}{d \log A_k} = \Gamma \frac{d \log \Lambda}{d \log A_k} + \delta^{(k)},$$

  where

$$\Gamma_{f,g} = -\frac{1}{\Lambda_f} \sum_{j=0}^{N} (\theta_j - 1)\lambda_j Cov_{\Omega^{(j)}} \left(\Psi_{(f)}, \Psi_{(g)} \right)$$

$$\delta_f^{(k)} = \frac{1}{\Lambda_f} \sum_{j=0}^{N} (\theta_j - 1)\lambda_j Cov_{\Omega^{(j)}} \left(\Psi_{(f)}, \Psi_{(k)} \right).$$

Example: Horizontal Economy – Frictions to reallocation

- Now suppose production has decreasing returns to scale. Model with fixed factors

$$y_i = A_i {L_i}^{\omega_g} L_{s_i}^{1-\omega_g},$$

where s_i is labor specific to production of good i.

- When $\omega_g \to 1$, labor reallocates flexibly.
- When $\omega_g < 1$, some labor is unable to reallocate.

Example: Horizontal Economy – Frictions to reallocation

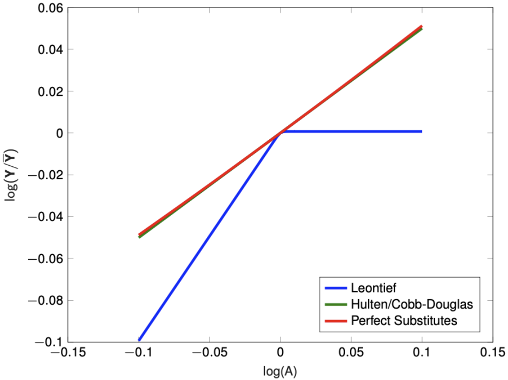


Figure: Nonlinearities in horizontal economy with no reallocation ($\omega_g = 0$) (Baqaee & Farhi 2019).

- Nonlinearities become:

$$\frac{d^2 \log Y}{d \log A_i^2} = \lambda_i (1 - \lambda_i) \left(1 - \frac{1}{\theta_0}\right).$$

- Intuition: When $\theta_0 > 1$, economy substitutes toward i as price of i falls, but unable to reallocate any more resources to i.

- When $\theta_0 < 1$, i becomes a bottleneck, cannot be overcome with more resources.

Takeaway

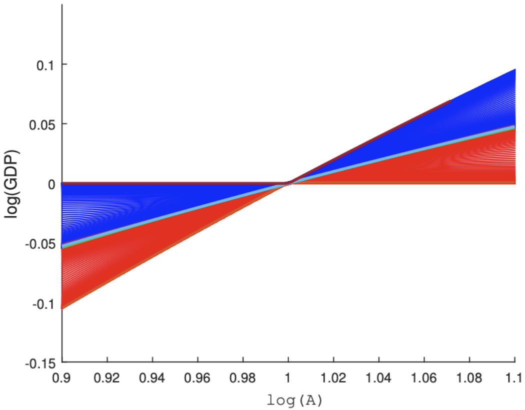


Figure: Possible nonlinear effects in horizontal economy.

- All of these economies are equivalent to a first-order (Hulten's).

Table of Contents

Quantitative examples

1. Oil vs. retail.
2. Skewness and kurtosis of GDP.
3. Baumol's cost disease.

Oil vs. retail

- For each industry, production using labor L_i and composite intermediate input X_i,

$$y_i = A_i \left(\omega_{iL} L_i^{\frac{\theta-1}{\theta}} + (1-\omega_{iL}) X_i^{\frac{\theta-1}{\theta}} \right)^{\frac{\theta}{\theta-1}},$$

  where X_i is given by

$$X_i = \left(\sum_{j=1}^{N} \omega_{ij} x_{ij}^{\frac{\varepsilon-1}{\varepsilon}} \right)^{\frac{\varepsilon}{\varepsilon-1}}.$$

- Calibrate elasticity between labor and materials $\theta = 0.5$, between materials $\varepsilon = 0.001$, and rep. consumer elasticity of substitution $\sigma = 0.9$. (Atalay 2017).

- Calibrate all shares ω_{iL}, ω_{ij} to match input-output table.

Oil vs. retail

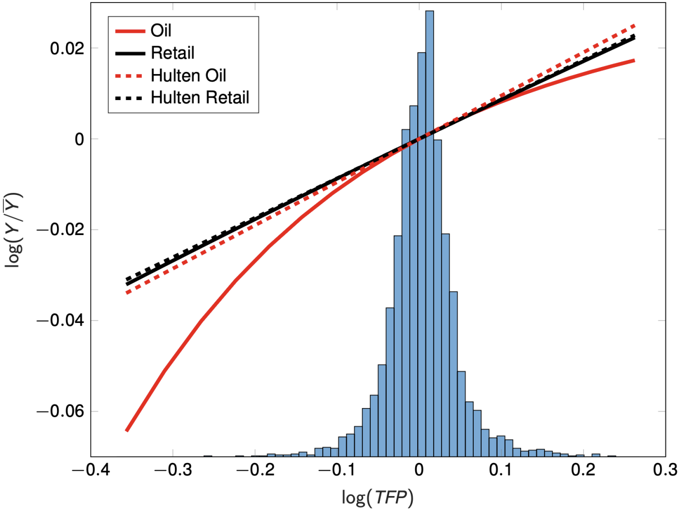


Figure: Nonlinear effects of productivity shock to oil vs. retail sectors (Baqaee & Farhi 2019).

Skewness and kurtosis of GDP

- Simulation of input-output network with elasticity between labor and materials $\theta = 0.5$, between materials $\varepsilon = 0.001$, and rep. consumer elasticity of substitution $\sigma = 0.9$.

- Assume no reallocation (labor to each sector is a different input).

$(\sigma, \theta, \varepsilon)$	Mean	Std	Skew	Ex-Kurtosis	σ_λ
(0.7, 0.3, 0.001)	-0.0045	0.012	-0.31	0.4	0.171
(0.9, 0.5, 0.001)	-0.0034	0.012	-0.18	0.1	0.115
(0.9, 0.6, 0.2)	-0.0024	0.011	-0.11	0.1	0.068
(0.99, 0.99, 0.99)	-0.0011	0.011	0.00	0.0	0.001
Annual Data	-	0.015	-	-	0.13

Skewness and kurtosis of GDP

$(\sigma, \theta, \varepsilon)$	Mean	Std	Skew	Ex-Kurtosis	σ_λ
No Reallocation - Quadrennial					
(0.7, 0.3, 0.001)	-0.0270	0.037	-2.18	12.7	0.404
(0.9, 0.5, 0.001)	-0.0187	0.030	-1.11	3.6	0.267
(0.9, 0.6, 0.2)	-0.0129	0.027	-0.44	0.7	0.154
(0.99, 0.99, 0.99)	-0.0057	0.025	0.00	0.0	0.002
Quadrennial Data	-	0.030	-	-	0.27

Skewness and kurtosis of GDP

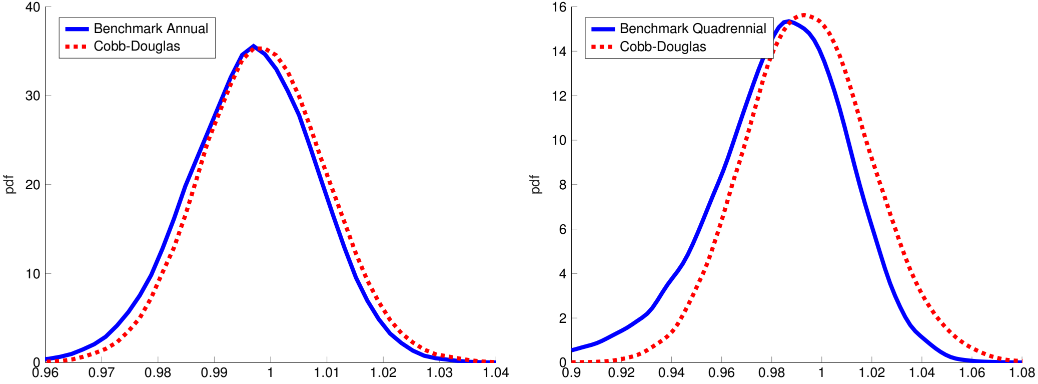


Figure: The left panel shows the distribution of *GDP* for the annual model. The right panel shows these for shocks for quadrennial shocks.

Baumol's cost disease

- Baumol (1967): *"A half hour horn quintet calls for the expenditure of 2 1/2 man hours in its performance, and any attempt to increase productivity here is likely to be viewed with concern by critics and audience alike.*

  If the ratio of the outputs of the two sectors is held constant, more and more of the total labor force must be transferred to the nonprogressive sector."

- (This is the implication of Baumol most often cited.)

Baumol's cost disease

- Nonlinear measure of aggregate TFP growth given by cumulating first-order effects,

$$\Delta \log A = \int_{1948}^{2014} \sum_i \lambda_{i,t} d \log A_{i,t}.$$

  Approximate using discrete left Riemann sums,

$$\Delta \log A \approx \sum_{t=1948}^{2013} \lambda_{i,t} (\log A_{i,t+1} - \log A_{i,t}).$$

- If economy was log-linear, TFP growth would instead be

$$\Delta \log A^{\text{first order}} \approx \sum_i \lambda_{i,1948} (\log A_{i,2014} - \log A_{i,1948}).$$

Baumol’s cost disease

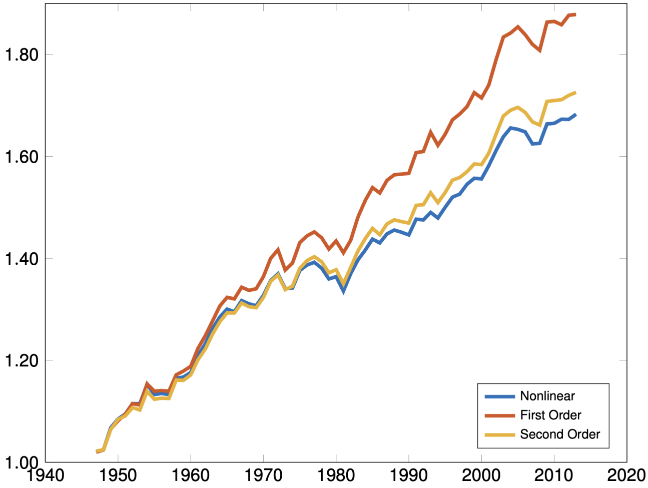


Figure: Baumol’s cost disease (Baqaee & Farhi 2019).

Quantitative examples

- Is oil vs. retail an example of concavity or convexity?
- Is Baumol's cost disease an example of concavity or convexity?
- Where does concavity / convexity in each example come from?

Table of Contents

Elasticities of substitution

- Consider the aggregate production function,

$$Y = Y(A_1, ..., A_N, L_1, ..., L_F).$$

- Define the pseudo-elasticity of substitution ρ_{ji} as,

$$1/\rho_{ji} = \frac{d \log(\partial Y / \partial A_j) - d \log(\partial Y / \partial A_i)}{d \log A_i}.$$

  i.e., change in marginal product of j relative to i as the supply of i increases.

- Likewise, for factors,

$$1/\rho_{ji} = \frac{d \log(\partial Y / \partial L_j) - d \log(\partial Y / \partial L_i)}{d \log L_i}.$$

- Next few slides: A few examples that this is the statistic of interest.

1. For CES, statistic coincides with technological elasticity of substitution

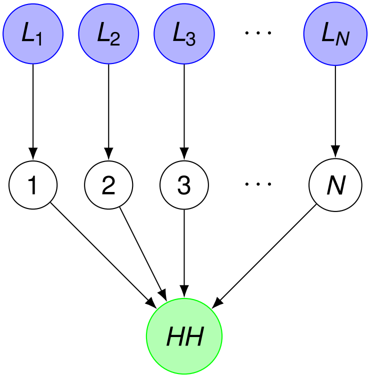


Figure: Horizontal economy.

- Output is

$$Y = \left(\sum_{i=1}^{N} L_i^{\frac{\theta-1}{\theta}} \right)^{\frac{\theta}{\theta-1}}.$$

- Marginal product of L_i is

$$\frac{\partial Y}{\partial L_i} = \left(\frac{L_i}{Y} \right)^{-1/\theta}.$$

- Elasticity is:

$$\frac{1}{\rho_{ji}} = \frac{d \log \frac{\partial Y / \partial L_j}{\partial Y / \partial L_i}}{d \log L_i} = \frac{1}{\theta} \lambda_i + \frac{1}{\theta} (1 - \lambda_i).$$

$$\Rightarrow \quad \rho_{ji} = \theta.$$

2. Relationship to preexisting Morishima elasticity measure

- Suppose $Y = Y(K, L)$ is homogeneous of degree one. Then ρ_{KL} is equal to the Morishima (1967) elasticity of substitution,

$$\frac{1}{\rho_{KL}^{\text{Morishima}}} = \frac{d \log MRS_{KL}}{d \log L/K},$$

  where the marginal rate of substitution $MRS_{KL} = \frac{\partial Y/\partial K}{\partial Y/\partial L}$.

- Why?

- If $Y = Y(K, L)$ homogeneous of degree one, then we also get $\rho_{KL} = \rho_{LK}$. Proof?

3. Reflects payments to factors

- Elasticity ρ_{ji} tells us how the income share of factors change when relative supply changes.

$$\frac{d\log \Lambda_L/\Lambda_K}{d\log L} = \frac{d\log\left(\frac{\partial Y}{\partial L}\frac{L}{Y}\right)/\left(\frac{\partial Y}{\partial K}\frac{K}{Y}\right)}{d\log L}$$
$$= 1 - \frac{d\log\left(\frac{\partial Y}{\partial K}\right)/\left(\frac{\partial Y}{\partial L}\right)}{d\log L}$$
$$= 1 - \frac{1}{\rho_{KL}}.$$

- If $\rho_{ji} \in (0,1)$, factor j is a GE-complement for i.

Using ρ_{ji} for second-order approximation

- Define the input-output multiplier $\xi = \sum_i \lambda_i$. Then we can show:

$$\frac{d\lambda_i}{d\log A_i} = \frac{\lambda_i}{\xi} \sum_{k \neq i} \lambda_k \left(1 - \frac{1}{\rho_{ki}}\right) + \lambda_i \frac{d\log \xi}{d\log A_i}.$$

- Proof:

$$d\xi_i = \sum_i d\lambda_i.$$

$$d\log \xi = \sum_k \frac{\lambda_k}{\xi} d\log(\lambda_k/\lambda_i) + d\log \lambda_i.$$

$$\frac{d\log \xi}{d\log A_i} = -\sum_{k \neq i} \frac{\lambda_k}{\xi} \left(1 - \frac{1}{\rho_{ki}}\right) + \frac{1}{\lambda_i} \frac{d\lambda_i}{d\log A_i}.$$

- Thus, a complementary way to understand second-order effects is through elasticities of substitution and change in input-output multiplier.

Using ρ_{ji} for second-order approximation

- We have:

$$\frac{d\lambda_i}{d\log A_i} = \frac{\lambda_i}{\xi}\sum_{k\neq i}\lambda_k\left(1-\frac{1}{\rho_{ki}}\right)+\lambda_i\frac{d\log\xi}{d\log A_i}.$$

- Under what conditions is $d\lambda_i/d\log A_i = 0$?

Using ρ_{ji} for second-order approximation

- We have:

$$\frac{d\lambda_i}{d\log A_i} = \frac{\lambda_i}{\xi}\sum_{k\neq i}\lambda_k\left(1-\frac{1}{\rho_{ki}}\right)+\lambda_i\frac{d\log\xi}{d\log A_i}.$$

- Under what conditions is $d\lambda_i/d\log A_i = 0$?

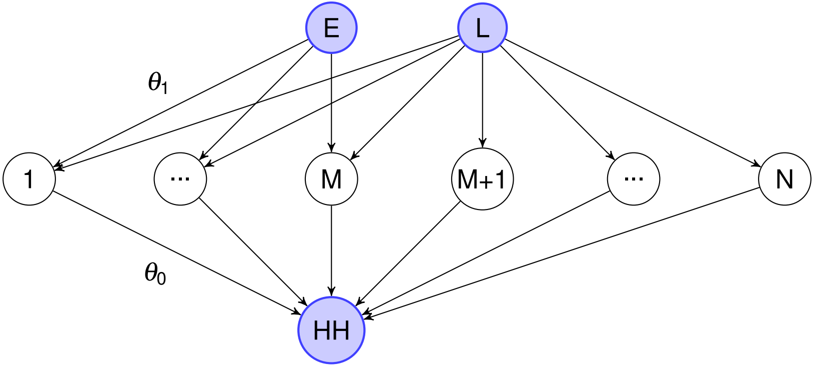
- Suppose many other goods are GE-complements with *i*. Does this make negative productivity shocks to *i* more or less costly?

Using ρ_{ji} for second-order approximation

- We have:
$$\frac{d\lambda_i}{d\log A_i} = \frac{\lambda_i}{\xi}\sum_{k\neq i}\lambda_k\left(1-\frac{1}{\rho_{ki}}\right)+\lambda_i\frac{d\log\xi}{d\log A_i}.$$
- Under what conditions is $d\lambda_i/d\log A_i = 0$?
- Suppose many other goods are GE-complements with i. Does this make negative productivity shocks to i more or less costly?
    - $\rho_{ki}\in(0,1)$ makes $d\lambda_i/d\log A_i$ more negative.
    - Thus a decrease in A_i increases λ_i more.
    - $\Rightarrow$ Nonlinearities amplify losses.

Example: Universal energy input

- Two factors: electricity and labor. Sectors use energy and labor with elasticity $\theta_1 < 1$.
- Final demand uses downstreams sectors with elasticity $\theta_0 \gg \theta_1$.



$$\frac{d^2 \log Y}{d \log A_E^2} = \frac{d\Lambda_E}{d \log A_E^2} = \frac{(\theta_0 - 1)\Lambda_E(1 - \Lambda_E) - (\theta_0 - \theta_1)\Lambda_E\left(1 - \frac{N}{M}\Lambda_E\right)}{\theta_0 - (\theta_0 - \theta_1)\frac{\left(1 - \frac{N}{M}\Lambda_E\right)}{1 - \Lambda_E}}.$$

Example: Universal energy input

$$\frac{d^2 \log Y}{d \log A_E^2} = \frac{(\theta_0 - 1)\Lambda_E(1-\Lambda_E) - (\theta_0 - \theta_1)\Lambda_E\left(1 - \frac{N}{M}\Lambda_E\right)}{\theta_0 - (\theta_0 - \theta_1)\frac{\left(1-\frac{N}{M}\Lambda_E\right)}{1-\Lambda_E}}.$$

- Suppose $\theta_0 = \theta_1 = \theta$. Then:

$$\frac{d^2 \log Y}{d \log A_E^2} = \frac{\theta - 1}{\theta}\Lambda_E(1-\Lambda_E).$$

  $\theta < 1$ implies output is concave; $\theta > 1$ implies output is convex.

- Suppose $M = N$ (energy is “universal”). Then:

$$\frac{d^2 \log Y}{d \log A_E^2} = \frac{\theta_1 - 1}{\theta_1}\Lambda_E(1-\Lambda_E).$$

  Downstream elasticity θ_0 doesn’t help! Intuition: Energy input is GE-complement.

Recap

- Summers (2013): *"Consider [a power failure]. The network would collapse. The connections would go away. And output would, of course, drop very rapidly. [...] If you lost 80% of electricity, you [would lose] more than 3% of the economy."*

- We've given one reason:

$$\Delta \log Y \approx \lambda_i \Delta \log A_i + \frac{d\lambda_i}{d \log A_i} (\Delta \log A_i)^2.$$

- If you observe the event, can read nonlinearities off the data.

- If you don't, you can back out nonlinearities from input–output structure.

- Shocks to inputs that are GE-complements with other goods are very costly.

Mind the Gap

- Are we done?

Mind the Gap

- Are we done?
- So far, shock transmission has been very mechanical.
- Increase price of i by 10%, any k that uses i increases price by $\Psi_{ki} \times 10\%$. Realistic?

Mind the Gap

- Are we done?
- So far, shock transmission has been very mechanical.
- Increase price of i by 10%, any k that uses i increases price by $\Psi_{ki} \times 10\%$. Realistic?
- Another angle: Large gaps in output per worker in rich and poor countries.
- In our setup, Solow residual is composed solely of microeconomic productivities.
- But certainly there is a role for policy that allows capital to flow to better firms, matches workers to best opportunities, allows land to be traded...