Nonlinear Dynamical Systems: Lecture 11

Shriram C Jugade (today)
Control & Computing group
Department of Electrical Engineering
Indian Institute of Technology Bombay

(IIT Bombay)

Outline

- **Bendixson criterion and Poincaré-Bendixson criterion**
- **van der Pol oscillator**

Bendixson criterion

Consider

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} (25 - x_1^2 - x_2^2) & 1 \\ -1 & (25 - x_1^2 - x_2^2) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

This can be written in matrix form as

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} \epsilon(r) & 1 \\ -1 & \epsilon(r) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

We get $\frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2} = 2(25 - 2r^2)$. $(\frac{5}{\sqrt{2}} = 3.53)$

We get $\frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2} = 2(25 - 2r^2)$. $(\frac{5}{\sqrt{2}} = 3.53)$
We apply Bendixson criterion for the two regions

We get $\frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2} = 2(25 - 2r^2)$. $(\frac{5}{\sqrt{2}} = 3.53)$
We apply Bendixson criterion for the two regions

1. When $r < 3.53$, $\frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2} > 0$.

We get $\frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2} = 2(25 - 2r^2)$. $(\frac{5}{\sqrt{2}} = 3.53)$
We apply Bendixson criterion for the two regions

1. When $r < 3.53$, $\frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2} > 0$.
2. When $r > 3.54$, $\frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2} < 0$.

We get $\frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2} = 2(25 - 2r^2)$. $(\frac{5}{\sqrt{2}} = 3.53)$
We apply Bendixson criterion for the two regions

1. When $r < 3.53$, $\frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2} > 0$.
2. When $r > 3.54$, $\frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2} < 0$.

No periodic orbit within $r < 3.53$ (by Bendixson criteria).

We get $\frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2} = 2(25 - 2r^2)$. $(\frac{5}{\sqrt{2}} = 3.53)$
We apply Bendixson criterion for the two regions

1. When $r < 3.53$, $\frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2} > 0$.
2. When $r > 3.54$, $\frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2} < 0$.

No periodic orbit within $r < 3.53$ (by Bendixson criteria).
No periodic orbit in $r > 3.54$?

We get $\frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2} = 2(25 - 2r^2)$. $(\frac{5}{\sqrt{2}} = 3.53)$
We apply Bendixson criterion for the two regions

1. When $r < 3.53$, $\frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2} > 0$.
2. When $r > 3.54$, $\frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2} < 0$.

No periodic orbit within $r < 3.53$ (by Bendixson criteria).
No periodic orbit in $r > 3.54$?
Cannot conclude: Bendixson criteria requires

We get $\frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2} = 2(25 - 2r^2)$. $(\frac{5}{\sqrt{2}} = 3.53)$
We apply Bendixson criterion for the two regions

1. When $r < 3.53$, $\frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2} > 0$.
2. When $r > 3.54$, $\frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2} < 0$.

No periodic orbit within $r < 3.53$ (by Bendixson criteria).
No periodic orbit in $r > 3.54$?
Cannot conclude: Bendixson criteria requires simply connected region.
(No 'holes' in the region.

We get $\frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2} = 2(25 - 2r^2)$. $(\frac{5}{\sqrt{2}} = 3.53)$
We apply Bendixson criterion for the two regions

1. When $r < 3.53$, $\frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2} > 0$.
2. When $r > 3.54$, $\frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2} < 0$.

No periodic orbit within $r < 3.53$ (by Bendixson criteria).
No periodic orbit in $r > 3.54$?
Cannot conclude: Bendixson criteria requires simply connected region.
(No 'holes' in the region. Every simple closed curve can be shrunk to a point, being within the region.)

Example: (about Poincare Bendixson criterion)

Consider the case $\dot{x} = Ax$ where $A = \begin{bmatrix} 0 & 2 \\ -2 & \epsilon \end{bmatrix}$

for the cases $\epsilon > 0$,

Example: (about Poincare Bendixson criterion)

Consider the case $\dot{x} = Ax$ where $A = \begin{bmatrix} 0 & 2 \\ -2 & \epsilon \end{bmatrix}$
for the cases $\epsilon > 0$, $\epsilon = 0$,

Example: (about Poincare Bendixson criterion)

Consider the case $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$ where $\mathbf{A} = \begin{bmatrix} 0 & 2 \\ -2 & \epsilon \end{bmatrix}$
for the cases $\epsilon > 0$, $\epsilon = 0$, $\epsilon < 0$,

Example: (about Poincare Bendixson criterion)

Consider the case $\dot{x} = Ax$ where $A = \begin{bmatrix} 0 & 2 \\ -2 & \epsilon \end{bmatrix}$
for the cases $\epsilon > 0$, $\epsilon = 0$, $\epsilon < 0$,

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & (25 - x_1^2 - x_2^2) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Converting to polar coordinates, we get

Example: (about Poincare Bendixson criterion)

Consider the case $\dot{x} = Ax$ where $A = \begin{bmatrix} 0 & 2 \\ -2 & \epsilon \end{bmatrix}$
for the cases $\epsilon > 0$, $\epsilon = 0$, $\epsilon < 0$,

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & (25 - x_1^2 - x_2^2) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Converting to polar coordinates, we get

$$x_1 = r\cos\theta$$

$$x_2 = r\sin\theta$$

Example (continued)

$$x_1^2 + x_2^2 = r^2$$

Example (continued)

$$x_1^2 + x_2^2 = r^2$$

$$x_1\dot{x}_1 + x_2\dot{x}_2 = r\dot{r}$$

Example (continued)

$x_1^2 + x_2^2 = r^2$

$x_1\dot{x}_1 + x_2\dot{x}_2 = r\dot{r}$

Substituting $x_1, x_2, \dot{x}_1, \dot{x}_2$ using r, $\dot{r}$ and θ, we get $r\dot{r} =$

Example (continued)

$x_1^2 + x_2^2 = r^2$

$x_1\dot{x}_1 + x_2\dot{x}_2 = r\dot{r}$

Substituting $x_1, x_2, \dot{x}_1, \dot{x}_2$ using r, $\dot{r}$ and θ, we get $r\dot{r} =$

$$(r\cos\theta)(r\sin\theta) + (r\sin\theta)(-r\cos\theta + (25 - r^2)r\sin\theta)$$

This reduces to $\dot{r} = (25 - r^2)r\sin^2\theta$

Except when $\theta = 0^o$, or 180^o, $\sin^2\theta$ is positive.

Therefore at $r = 5$ we get $\dot{r} = 0$

Example (continued)

$x_1^2 + x_2^2 = r^2$
$x_1\dot{x}_1 + x_2\dot{x}_2 = r\dot{r}$
Substituting $x_1, x_2, \dot{x}_1, \dot{x}_2$ using r, $\dot{r}$ and θ, we get $r\dot{r} =$

$$(r\cos\theta)(r\sin\theta) + (r\sin\theta)(-r\cos\theta + (25 - r^2)r\sin\theta)$$

This reduces to $\dot{r} = (25 - r^2)r\sin^2\theta$
Except when $\theta = 0^o$, or 180^o, $\sin^2\theta$ is positive.
Therefore at $r = 5$ we get $\dot{r} = 0$
i.e circle with radius 5 is a periodic orbit.

$\sin^2\theta = 0$ means $x_2 = 0$ (i.e. along x_1 axis).

This implies that $\begin{bmatrix} \overset{\bullet}{x}_1 \\ \overset{\bullet}{x}_2 \end{bmatrix} = \begin{bmatrix} 0 \\ -x_1 \end{bmatrix}$

(Vector is perpendicular to the x_1 axis.)

$\sin^2\theta = 0$ means $x_2 = 0$ (i.e. along x_1 axis).

This implies that $\begin{bmatrix} \overset{\bullet}{x}_1 \\ \overset{\bullet}{x}_2 \end{bmatrix} = \begin{bmatrix} 0 \\ -x_1 \end{bmatrix}$

(Vector is perpendicular to the x_1 axis.)

Vector is nonzero if $x_1 \neq 0$.

$\sin^2\theta = 0$ means $x_2 = 0$ (i.e. along x_1 axis).

This implies that $\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 \\ -x_1 \end{bmatrix}$

(Vector is perpendicular to the x_1 axis.)

Vector is nonzero if $x_1 \neq 0$.

Only $x_1 = 0, x_2 = 0$ is equilibrium point.

Moreover, other trajectories 'converge' to this limit cycle:

$\sin^2\theta = 0$ means $x_2 = 0$ (i.e. along x_1 axis).

This implies that $\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 \\ -x_1 \end{bmatrix}$

(Vector is perpendicular to the x_1 axis.)
Vector is nonzero if $x_1 \neq 0$.
Only $x_1 = 0, x_2 = 0$ is equilibrium point.
Moreover, other trajectories 'converge' to this limit cycle: 'isolated stable limit cycle'.

$\sin^2\theta = 0$ means $x_2 = 0$ (i.e. along x_1 axis).

This implies that $\begin{bmatrix} \overset{\bullet}{x}_1 \\ \overset{\bullet}{x}_2 \end{bmatrix} = \begin{bmatrix} 0 \\ -x_1 \end{bmatrix}$

(Vector is perpendicular to the x_1 axis.)

Vector is nonzero if $x_1 \neq 0$.

Only $x_1 = 0, x_2 = 0$ is equilibrium point.

Moreover, other trajectories 'converge' to this limit cycle: 'isolated stable limit cycle'.

$\sin^2\theta = 0$ means $x_2 = 0$ (i.e. along x_1 axis).

This implies that $\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 \\ -x_1 \end{bmatrix}$

(Vector is perpendicular to the x_1 axis.)

Vector is nonzero if $x_1 \neq 0$.

Only $x_1 = 0, x_2 = 0$ is equilibrium point.

Moreover, other trajectories 'converge' to this limit cycle: 'isolated stable limit cycle'.

van der Pol oscillator

Now consider:
$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$ **where** $\mathbf{A} = \begin{bmatrix} 0 & 1 \\ -1 & \epsilon(1 - \mathbf{x}_1^2) \end{bmatrix}$ **where** ϵ **is a positive constant.**

van der Pol oscillator

Now consider:
$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$ **where** $\mathbf{A} = \begin{bmatrix} 0 & 1 \\ -1 & \epsilon(1 - x_1^2) \end{bmatrix}$ **where** ϵ **is a positive constant.**
(Now, $(2, 2)$ **element of A:** $\epsilon(1 - x_1^2)$ **depends only on** x_1 **and not radius.)**

van der Pol oscillator

Now consider:
$\dot{x} = Ax$ **where** $A = \begin{bmatrix} 0 & 1 \\ -1 & \epsilon(1 - x_1^2) \end{bmatrix}$ **where** ϵ **is a positive constant.**
(Now, $(2, 2)$ **element of A:** $\epsilon(1 - x_1^2)$ **depends only on** x_1 **and not radius.)**

This is called van der Pol oscillator.
van der Pol oscillator is a special case of Lienard's equation

Lienard's equation

Let f and g be continuously differentiable.

Lienard's equation

Let f and g be continuously differentiable.

f is even function: i.e. $f(-x) = f(x)$

g is odd function: i.e. $g(-x) = -g(x)$

Lienard's equation

Let f and g be continuously differentiable.

f is even function: i.e. $f(-x) = f(x)$

g is odd function: i.e. $g(-x) = -g(x)$

Then the second order differential equation of the form

Lienard's equation

Let f and g be continuously differentiable.

f is even function: i.e. $f(-x) = f(x)$

g is odd function: i.e. $g(-x) = -g(x)$

Then the second order differential equation of the form

$$\ddot{x} + f(x)\dot{x} + g(x) = 0$$

is called the Lienard equation.

Lienard's equation

Let f and g be continuously differentiable.

f is even function: i.e. $f(-x) = f(x)$

g is odd function: i.e. $g(-x) = -g(x)$

Then the second order differential equation of the form

$$\ddot{x} + f(x)\dot{x} + g(x) = 0$$

is called the Lienard equation.

Lienard's Theorem

Define $F(x) := \int_0^x f(\xi)d\xi$.
If for a Lienard system:

Lienard's Theorem

Define $F(x) := \int_0^x f(\xi)d\xi$.
If for a Lienard system:

1. **$g(x) > 0$ for all $x > 0$.**

Lienard's Theorem

Define $F(x) := \int_0^x f(\xi)d\xi$.
If for a Lienard system:

1. **$g(x) > 0$ for all $x > 0$.**
2. **$F(x) \to \infty$ as $x \to \infty$**

Lienard's Theorem

Define $F(x) := \int_0^x f(\xi)d\xi$.
If for a Lienard system:

1. **$g(x) > 0$ for all $x > 0$.**
2. **$F(x) \to \infty$ as $x \to \infty$**

   For some $p > 0$, we have

3. **F satisfies $F(x) < 0$ for $0 < x < p$.**

Lienard's Theorem

Define $F(x) := \int_0^x f(\xi)d\xi$.
If for a Lienard system:

1. $g(x) > 0$ for all $x > 0$.
2. $F(x) \to \infty$ as $x \to \infty$

   For some $p > 0$, we have

3. F satisfies $F(x) < 0$ for $0 < x < p$.
4. F satisfies $F(x) > 0$ and F is monotonic for $x > p$.

Lienard's Theorem

Define $F(x) := \int_0^x f(\xi)d\xi$.
If for a Lienard system:

1. $g(x) > 0$ for all $x > 0$.
2. $F(x) \to \infty$ as $x \to \infty$

   For some $p > 0$, we have

3. F satisfies $F(x) < 0$ for $0 < x < p$.
4. F satisfies $F(x) > 0$ and F is monotonic for $x > p$.

then the Lienard system has **unique** and **stable limit cycle**.
(See: Nonlinear Oscillations-Nicholas Minorsky).

van der Pol Oscillator

Let $f(x) = -\epsilon(1 - x^2)$ where $\epsilon > 0$ and $g(x)=x$ then the system is called van der Pol oscillator.

van der Pol Oscillator

Let $f(x) = -\epsilon(1 - x^2)$ where $\epsilon > 0$ and g(x)=x then the system is called van der Pol oscillator.
The differential equation becomes:

$$\ddot{x} - \epsilon(1 - x^2)\dot{x} + x = 0$$

van der Pol Oscillator

Let $f(x) = -\epsilon(1 - x^2)$ where $\epsilon > 0$ and g(x)=x then the system is called van der Pol oscillator.
The differential equation becomes:

$$\ddot{x} - \epsilon(1 - x^2)\dot{x} + x = 0$$

Stability of oscillations.

Existence of closed orbit

Consider the differential equation

$$\ddot{\mathbf{v}} + \epsilon \mathbf{h}(\mathbf{v})\dot{\mathbf{v}} + \mathbf{v} = \mathbf{0}$$

Existence of closed orbit

Consider the differential equation

$$\ddot{\mathbf{v}} + \epsilon \mathbf{h}(\mathbf{v})\dot{\mathbf{v}} + \mathbf{v} = \mathbf{0}$$

where $\mathbf{h}(\mathbf{v}) = -\mathbf{1} + \mathbf{v}^2$.

Existence of closed orbit

Consider the differential equation

$$\ddot{v} + \epsilon h(v)\dot{v} + v = 0$$

where $h(v) = -1 + v^2$.
Choose the state variables as $x_1 = v$ **and** $x_2 = \dot{v} + \epsilon H(v)$, **where** $H(v)$ **is such that** $\frac{d}{dv}H(v) = h(v)$ **and** $H(0) = 0$.

Existence of closed orbit

Consider the differential equation

$$\ddot{v} + \epsilon h(v)\dot{v} + v = 0$$

where $h(v) = -1 + v^2$.
Choose the state variables as $x_1 = v$ and $x_2 = \dot{v} + \epsilon H(v)$, where $H(v)$ is such that $\frac{d}{dv}H(v) = h(v)$ and $H(0) = 0$.
Therefore,

$$\begin{aligned}\dot{x}_1 &= x_2 - \epsilon H(x_1)\\ \dot{x}_2 &= -x_1\end{aligned}$$

It has a unique equilibrium point, which is at the origin.

The state plane is divided into four regions by following two curves (Figure 1)

$$\dot{x}_1 = x_2 - \epsilon H(x_1) = 0$$
$$\dot{x}_2 = -x_1 = 0$$

Each curve separates $\dot{x}_i > 0$ from $\dot{x}_i < 0$

The state plane is divided into four regions by following two curves (Figure 1)

$$\dot{x}_1 = x_2 - \epsilon H(x_1) = 0$$
$$\dot{x}_2 = -x_1 = 0$$

Each curve separates $\dot{x}_i > 0$ from $\dot{x}_i < 0$
Four regions:

1. $\dot{x}_1 > 0$ and $\dot{x}_2 < 0$
2. $\dot{x}_1 < 0$ and $\dot{x}_2 < 0$
3. $\dot{x}_1 < 0$ and $\dot{x}_2 > 0$
4. $\dot{x}_1 > 0$ and $\dot{x}_2 > 0$

For initial condition $(0, k)$, with $k > 0$, trajectory describes an arc intersecting x_2 axis again,

For initial condition $(0, k)$, with $k > 0$, trajectory describes an arc intersecting x_2 axis again, at say, $(0, -\alpha)$ and $\alpha > 0$.

For initial condition $(0, k)$, with $k > 0$, trajectory describes an arc intersecting x_2 axis again, at say, $(0, -\alpha)$ and $\alpha > 0$.
α depends on k: $\alpha(k)$.
Now if k is large enough then $\alpha(k) < k$.

For initial condition $(0, k)$, with $k > 0$, trajectory describes an arc intersecting x_2 axis again, at say, $(0, -\alpha)$ and $\alpha > 0$.
α depends on k: $\alpha(k)$.
Now if k is large enough then $\alpha(k) < k$.
Why?

For initial condition $(0, k)$, with $k > 0$, trajectory describes an arc intersecting x_2 axis again, at say, $(0, -\alpha)$ and $\alpha > 0$.
α depends on k: $\alpha(k)$.
Now if k is large enough then $\alpha(k) < k$.
Why?

- Consider the function $V(x) = (x_1^2 + x_2^2)/2$ ($V(x)$ is equivalent to system energy).
- Hence, $\dot{V}(x) = -\epsilon x_1 H(x_1)$. (Easy to verify.)
- For $x_1 > p$, $\dot{V}(x) < 0$ and for $x_1 < p$, $\dot{V}(x) > 0$, where p is the root of $H(x_1)$ for $x > 0$.
- Let $\delta(k) = V(E) - V(A) = \int_{AE} \dot{V}(x)dt = \int_{AB} \dot{V}(x)dt + \int_{BCD} \dot{V}(x)dt + \int_{DE} \dot{V}(x)dt$.
  $\delta(k) = \delta_1(k) + \delta_2(k) + \delta_3(k)$.

- $\delta_1(k)$ is positive because $x_1 > 0$ & $H(x_1) < 0$.
- $\delta_2(k)$ is negative because $x_1 > 0$ & $H(x_1) > 0$.
- As k increase $\delta_2(k)$ decreases and therefore $\lim_{k\to\infty} \delta_2(k) = -\infty$.
- $\delta_3(k)$ is positive because $x_1 > 0$ & $H(x_1) < 0$.
- $\delta(k) < 0$ for large k.Therefore, $V(E) < V(A)$. Energy at E<Energy at A. Therefore $\alpha(k) < k$.
- Let $x_1(t), x_2(t)$ be a solution so is $-x_1(t), -x_2(t)$ because $H(x)$ and x are odd functions.
- Consider another arc symmetric to the above arc about origin.

This region is invariant.

- $\delta_1(k)$ is positive because $x_1 > 0$ & $H(x_1) < 0$.
- $\delta_2(k)$ is negative because $x_1 > 0$ & $H(x_1) > 0$.
- As k increase $\delta_2(k)$ decreases and therefore $\lim_{k\to\infty} \delta_2(k) = -\infty$.
- $\delta_3(k)$ is positive because $x_1 > 0$ & $H(x_1) < 0$.
- $\delta(k) < 0$ for large k.Therefore, $V(E) < V(A)$. Energy at E<Energy at A. Therefore $\alpha(k) < k$.
- Let $x_1(t), x_2(t)$ be a solution so is $-x_1(t), -x_2(t)$ because $H(x)$ and x are odd functions.
- Consider another arc symmetric to the above arc about origin.

This region is invariant.

Jacobian matrix at origin is given by

$$\mathbf{A} = \frac{\partial \mathbf{f}}{\partial \mathbf{x}}\Big|_{\mathbf{x}=0} = \begin{bmatrix} -\epsilon h(0) & 1 \\ -1 & 0 \end{bmatrix}$$

Jacobian matrix at origin is given by

$$A = \frac{\partial f}{\partial x}\Big|_{x=0} = \begin{bmatrix} -\epsilon h(0) & 1 \\ -1 & 0 \end{bmatrix}$$

$\epsilon > 0$ and $h(0) < 0$. Therefore eigenvalues has positive real parts. By Poincaré-Bendixson criterion we can say that there is a closed orbit in M.

RLC circuit

Consider the RLC circuit

1. Inductance $L > 0$,
2. Capacitance $C > 0$ &
3. Special resistive element : active circuit with v-i characteristic $i = H(v)$.

The function h satisfies the following conditions:

RLC circuit

Consider the RLC circuit

1. **Inductance $L > 0$,**
2. **Capacitance $C > 0$ &**
3. **Special resistive element : active circuit with v-i characteristic $i = H(v)$.**

The function h satisfies the following conditions:

1. **$H(0) = 0$.**

RLC circuit

Consider the RLC circuit

1. **Inductance $L > 0$,**
2. **Capacitance $C > 0$ &**
3. **Special resistive element : active circuit with v-i characteristic $i = H(v)$.**

The function h satisfies the following conditions:

1. $H(0) = 0$.
2. $H'(0) < 0$.

RLC circuit

Consider the RLC circuit

1. **Inductance $L > 0$,**
2. **Capacitance $C > 0$ &**
3. **Special resistive element : active circuit with v-i characteristic $i = H(v)$.**

The function h satisfies the following conditions:

1. $H(0) = 0$.
2. $H'(0) < 0$.
3. $H(v) \to \infty$ as $v \to \infty$.

RLC circuit

Consider the RLC circuit

1. **Inductance $L > 0$,**
2. **Capacitance $C > 0$ &**
3. **Special resistive element : active circuit with v-i characteristic $i = H(v)$.**

The function h satisfies the following conditions:

1. **$H(0) = 0$.**
2. **$H'(0) < 0$.**
3. **$H(v) \to \infty$ as $v \to \infty$.**

(H(v) is like F(x) in Lienard's equation: H and F are both odd functions.)

....Continued

Applying KCL

$$\mathbf{i_C + i_L + i_R = 0.}$$

....Continued

Applying KCL

$$i_C + i_L + i_R = 0.$$

Hence in differential form, equation can be written as

$$\frac{d^2v}{dt^2} + \frac{v}{LC} + \frac{h'(v)}{C}\frac{dv}{dt} = 0$$

....Continued

Applying KCL

$$i_C + i_L + i_R = 0.$$

Hence in differential form, equation can be written as

$$\frac{d^2v}{dt^2} + \frac{v}{LC} + \frac{h'(v)}{C}\frac{dv}{dt} = 0$$

Let $\tau = \frac{t}{\sqrt{LC}}$

Applying KCL

$$i_C + i_L + i_R = 0.$$

Hence in differential form, equation can be written as

$$\frac{d^2v}{dt^2} + \frac{v}{LC} + \frac{h'(v)}{C}\frac{dv}{dt} = 0$$

Let $\tau = \frac{t}{\sqrt{LC}}$ Then differential equation becomes

$$\frac{d^2v}{d\tau^2} + h'(v)\sqrt{\frac{L}{C}}\frac{dv}{d\tau} + v = 0$$

....Continued

The coefficient $h'(v)\sqrt{\frac{L}{C}}$ determines the damping of voltage.

....Continued

The coefficient $h'(v)\sqrt{\frac{L}{C}}$ determines the damping of voltage. This nonlinear damping term determines energy exchange of the circuit. Let the active resistive element be negative resistance tunnel diode circuit as shown in the figure.

....Continued

The coefficient $h'(v)\sqrt{\frac{L}{C}}$ determines the damping of voltage.This nonlinear damping term determines energy exchange of the circuit. Let the active resistive element be negative resistance tunnel diode circuit as shown in the figure.Then $h(v) = -v + \frac{1}{3}v^3$,
$h'(v)\sqrt{\frac{L}{C}} = (v^2 - 1)\sqrt{\frac{L}{C}}$

....Continued

The coefficient $h'(v)\sqrt{\frac{L}{C}}$ determines the damping of voltage.This nonlinear damping term determines energy exchange of the circuit. Let the active resistive element be negative resistance tunnel diode circuit as shown in the figure.Then $h(v) = -v + \frac{1}{3}v^3$,
$h'(v)\sqrt{\frac{L}{C}} = (v^2 - 1)\sqrt{\frac{L}{C}}$Consider the following:

1. When $|v| \ll 1$ damping is negative and energy is fed in the LC tank circuit and $v_R.i_R < 0$.

....Continued

The coefficient $h'(v)\sqrt{\frac{L}{C}}$ determines the damping of voltage.This nonlinear damping term determines energy exchange of the circuit. Let the active resistive element be negative resistance tunnel diode circuit as shown in the figure.Then $h(v) = -v + \frac{1}{3}v^3$,
$h'(v)\sqrt{\frac{L}{C}} = (v^2 - 1)\sqrt{\frac{L}{C}}$Consider the following:

1. When $|v| \ll 1$ damping is negative and energy is fed in the LC tank circuit and $v_R.i_R < 0$.
2. When $|v| \gg 1$ damping is positive and energy is dissipated in the active resistive element and $v_R.i_R > 0$.

....Continued

The coefficient $h'(v)\sqrt{\frac{L}{C}}$ determines the damping of voltage.This nonlinear damping term determines energy exchange of the circuit. Let the active resistive element be negative resistance tunnel diode circuit as shown in the figure.Then $h(v) = -v + \frac{1}{3}v^3$,

$h'(v)\sqrt{\frac{L}{C}} = (v^2 - 1)\sqrt{\frac{L}{C}}$Consider the following:

1. When $|v| \ll 1$ damping is negative and energy is fed in the LC tank circuit and $v_R.i_R < 0$.
2. When $|v| \gg 1$ damping is positive and energy is dissipated in the active resistive element and $v_R.i_R > 0$.

Main features

For each initial condition:

1. trajectories remains bounded.
2. trajectories encircle the origin: voltage and current changes sign repeatedly.
3. For a given initial condition (e.g. initial voltage of capacitor and inductor current), after sufficient time, trajectories are 'almost periodic'.
4. For an oscillation, i.e. along a periodic orbit, the active resistor
   - feeds energy into LC tank for some time,
   - absorbs energy from LC tank for remaining time.
5. Along a periodic orbit, energy fed = energy absorbed.
6. For initial conditions much outside periodic orbit ($v \gg 1$) , This is the condition for stable oscillations or the trajectory to be a stable limit