Nonlinear Dynamical Systems: Lecture 11

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Linearization

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- Behaviour of the system if often of interest in a limited state region and small set of initial conditions and input functions.

Therefore, the nonlinear system can be substituted by valid linear system by linearization.

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- Therefore

$$\dot{x}_n(t) = f(x_n(t), u_n(t))$$

- Consider a perturbation or motion in the neighborhood of this trajectory, that is

$$x(t) = x_n(t) + \Delta x(t)$$

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- RHS can be expanded into Taylor series about the nominal system trajectory and input and we get

$$\text{RHS} = f(x_n, u_n) + \frac{\partial f}{\partial x}(x_n, u_n)\Delta x(t) + \frac{\partial f}{\partial u}(x_n, u_n)\Delta u(t)$$

$+$ higher order terms

- Higher order terms contain very small quantities and can be neglected.

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- The linearized system is time varying.
- Initial condition for the linearized system is

$$\Delta x0 = x(0) - x_n(0)$$

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- **Let the state equation for the $\mathbf{n^{th}}$ order system be**

$$\dot{\mathbf{x}}(\mathbf{t}) = \mathbf{f}(\mathbf{x}(\mathbf{t}), \mathbf{u}(\mathbf{t}))$$

where x(t) is n-dimensional state space vector and u is m-dimensional input vector.

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  $\Delta x(t)$ = small perturbation in system trajectory.
  $\Delta u(t)$ = small change in system input.
- Following the same procedure we get

$$\Delta \dot{x}(t) = \frac{\partial f}{\partial x}_{|x_n(t), u_n(t)} \Delta x(t) + \frac{\partial f}{\partial u}_{|x_n(t), u_n(t)} \Delta u(t)$$

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$\frac{\partial f}{\partial x}|_{x_n(t),u_n(t)} = A_{n\times n} =$

$$\begin{bmatrix} \frac{\partial f_1}{\partial x_1} & . & . & . & \frac{\partial f_1}{\partial x_n} \\ . & . & . & . & . \\ . & . & . & . & . \\ . & . & . & . & . \\ \frac{\partial f_n}{\partial x_1} & . & . & . & \frac{\partial f_n}{\partial x_n} \end{bmatrix}_{|x_n(t),u_n(t)}$$

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- Similarly $\frac{\partial f}{\partial u}_{|x_n(t),u_n(t)} = B_{n\times m} =$

$$\begin{bmatrix} \frac{\partial f_1}{\partial u_1} & \cdot & \cdot & \cdot & \frac{\partial f_1}{\partial u_m} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \frac{\partial f_n}{\partial u_1} & \cdot & \cdot & \cdot & \frac{\partial f_n}{\partial u_m} \end{bmatrix}_{|x_n(t),u_n(t)}$$

- **Therefore linearized system is given by**

$$\Delta\dot{x}(t) = A(t)\Delta x(t) + B(t)\Delta u(t)$$