Nonlinear Dynamical Systems: Lecture 10

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Outline

- **Bendixson criterion and Poincaré-Bendixson criterion**
- **Examples: Lotka-Volterra predator prey model**
- **van der Pol oscillator**

Lotka Volterra Predator Prey Model

Predator –> Hunter
Population dynamics of two species: prey and hunter.
Equations of the model

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x_h is the amount of hunter specimen in the model
x_p is the amount of prey specimen in the model
More generally,

$$\dot{x}_h = -a x_h + b x_h x_p$$
$$\dot{x}_p = c x_p - d x_h x_p$$

(parameters a, b, c and d are positive.)

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Periodic orbits?
Bendixson criterion and Poincaré Bendixson criterion?

Another example

Consider the following:

$$\begin{aligned}\dot{x}_1 &= x_2 + (x_1 x_2^2)\\ \dot{x}_2 &= -x_1 + (x_2 x_1^2)\end{aligned}$$

$$\nabla \cdot f(x) = \frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2} = (x_2^2 + x_1^2)$$

$\nabla \cdot f(x)$ is always positive,

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$\nabla \cdot f(x)$ is always positive, except at equilibrium point. Hence, by Bendixson criteria, there are no periodic orbits.

Another example

Consider

$$\dot{x}_1 = x_2 + x_1 x_2^2$$
$$\dot{x}_2 = -x_1 - x_2 x_1^2$$

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$\nabla \cdot f(x)$ is zero for $x_1 = x_2$, and changes sign.

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Bendixson criterion ?

Periodic orbits

Linear case:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & a \\ -2 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = Ax$$

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for $\epsilon > 0$?

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for $\epsilon > 0$? for $\epsilon < 0$

What if ϵ 'changes' sign ?

Consider

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} (25 - x_1^2 - x_2^2) & 1 \\ -1 & (25 - x_1^2 - x_2^2) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

This can be written in matrix form as

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} \epsilon(r) & 1 \\ -1 & \epsilon(r) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

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Consider the cases

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Consider the cases

1. When $r = 5$, we have $\epsilon(r) = 0$.
2. When $r < 5$, we have $\epsilon(r) > 0$.

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Consider the cases

1. **When $r = 5$, we have $\epsilon(r) = 0$.**
2. **When $r < 5$, we have $\epsilon(r) > 0$.**
3. **When $r > 5$, we have $\epsilon(r) < 0$.**

Hence, the trajectories will approach the periodic orbit.

Example: (about Poincare Bendixson criterion)

Consider the case $\dot{x} = Ax$ where $A = \begin{bmatrix} 0 & 2 \\ -2 & \epsilon \end{bmatrix}$

for the cases $\epsilon > 0$,

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Consider the case $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$ where $\mathbf{A} = \begin{bmatrix} 0 & 2 \\ -2 & \epsilon \end{bmatrix}$ for the cases $\epsilon > 0$, $\epsilon = 0$,

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Converting to polar coordinates, we get

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Converting to polar coordinates, we get

$$x_1 = r\cos\theta$$

$$x_2 = r\sin\theta$$

Example (continued)

$$\mathbf{x}_1^2 + \mathbf{x}_2^2 = \mathbf{r}^2$$

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$$x_1\dot{x}_1 + x_2\dot{x}_2 = r\dot{r}$$

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Substituting $x_1, x_2, \dot{x}_1, \dot{x}_2$ using r, $\dot{r}$ and θ, we get $r\dot{r} =$

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Substituting $x_1, x_2, \dot{x}_1, \dot{x}_2$ using r, $\dot{r}$ and θ, we get $r\dot{r} =$

$$(r\cos\theta)(r\sin\theta) + (r\sin\theta)(-r\cos\theta + (25 - r^2)r\sin\theta)$$

This reduces to $\dot{r} = (25 - r^2)r\sin^2\theta$
Therefore at $r = 5$ we get $\dot{r} = 0$ i.e circle with radius 5 is a periodic orbit.
Moreover, other trajectories 'converge' to this limit cycle:

Example (continued)

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$x_1\dot{x}_1 + x_2\dot{x}_2 = r\dot{r}$
Substituting $x_1, x_2, \dot{x}_1, \dot{x}_2$ using r, $\dot{r}$ and θ, we get $r\dot{r} =$

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This reduces to $\dot{r} = (25 - r^2)r\sin^2\theta$
Therefore at $r = 5$ we get $\dot{r} = 0$ i.e circle with radius 5 is a periodic orbit.
Moreover, other trajectories 'converge' to this limit cycle: 'isolated stable limit cycle'.

van der Pol oscillator

This oscillator has a 'stable limit cycle'.
Differential equation

$$\ddot{x} - \epsilon(1 - x^2)\dot{x} + x = 0$$

where $\epsilon > 0$
Let $\dot{x} = y$ and $\dot{y} = -x + \epsilon(1 - x^2)y$