Lecture 27

Last time we did multistep formula for

$$\frac{dy}{dx} = f(x,y)$$

$$y(x_0) = y_0 \qquad a = x_0 \le x \le b$$

Adam-Bashforth method of order 4

$$y_{n+1} = y_n + \frac{h}{24}\left(55 f_n - 59 f_{n-1} + 37 f_{n-2} - 9 f_{n-3}\right)$$

$h \to$ step size

$f_i = f(x_i, y_i)$

Local error $= h^5 y^{(5)}(\xi) \dfrac{251}{720}$

Note that AB method requires that y_0, y_1, y_2, y_3 to be known.

Usually y_1, y_2, y_3 are computed by RK-method of order 4.

Predictor-Corrector method

$$\frac{dy}{dx} = f(x,y)$$

$$\int_{x_n}^{x_{n+1}} \frac{dy}{dx} = \int_{x_n}^{x_{n+1}} f(x,y)\,dx$$

$$y_{n+1} = y_n + \int_{x_n}^{x_{n+1}} f(x,y)\,dx$$

Approximating the integral by Trapezoidal rule

$$y_{n+1} = y_n + \frac{h}{2}\left[f(x_n, y_n) + f(x_{n+1}, y_{n+1})\right]$$

$n = 0, 1, 2, \ldots$

$$\text{Error} = -\frac{h^3}{12} y'''(\xi)$$

This is used as corrector to Euler's method.

Algorithm (Second order predictor corrector method)

$$\frac{dy}{dx} = f(x, y) \qquad y(x_0) = y_0$$

$$x_i = x_0 + ih \qquad i = 0, 1, \ldots$$

1) compute $y_{n+1}^{(0)}$ from Euler method

$$y_{n+1}^{(0)} = y_n + h f(x_n, y_n)$$

2) Compute $y_{n+1}^{(k)}$ until satisfied by

$$y_{n+1}^{(k)} = y_n + \frac{h}{2}\left[f(x_n, y_n) + f(x_{n+1}, y_{n+1}^{(k-1)}) \right]$$

for $k = 1, 2, \ldots$

3) If $y_{n+1}^{(k)}$ is satisfactory then set

$$y_{n+1} = y_{n+1}^{(k)}$$

and repeat 1, 2, 3

Today we do

Adam - Moultan method

(This is used as a corrector for Adam - Bashforth method)

$$\frac{dy}{dx} = f(x,y) \qquad y(x_0) = y_0$$

$$\int_{x_n}^{x_{n+1}} \frac{dy}{dx} = \int_{x_n}^{x_{n+1}} f(x,y)\,dx$$

$$y_{n+1} = y_n + \int_{x_n}^{x_{n+1}} f(x,y)\,dx$$

we use polynomial which interpolates $f(x_i, y(x_i))$ at the points $x_{n+1}, x_n, x_{n-1}, \ldots, x_{n-m}$ for an integer $m > 0$

The Newton backward diff formula which interpolates at these $m+2$ points in terms of

$s = \frac{x - x_n}{h}$ is

$$P_{m+1}(s) = \sum_{k=0}^{m+1} (-1)^k \binom{1-s}{k} \nabla^k f_{n+1}$$

$$s = \frac{x - x_n}{h} \qquad \text{so} \qquad ds = \frac{dx}{h}$$

$$y_{n+1} = y_n + h \int_0^1 \left(\sum_{k=0}^{m+1} (-1)^k \binom{1-s}{k} \nabla^k f_{k+1} \right) ds$$

$$y_{n+1} = y_n + h \left(\gamma_0' f_{n+1} + \gamma_1' \nabla f_{n+1} + \cdots + \gamma_{m+1}' \nabla^{m+1} f_{n+1} \right)$$

$$\gamma_k' = (-1)^k \int_0^1 \binom{1-s}{k} ds$$

$$k = 0, 1, \ldots, m+1$$

The first few values of γ_k' are

$$\gamma_0' = 1, \quad \gamma_1' = \frac{-1}{2}, \quad \gamma_2' = \frac{-1}{12}, \quad \gamma_3' = \frac{-1}{24}$$

$$\gamma_4' = \frac{-19}{720}.$$ (in your textbook $\gamma_4' = \frac{-10}{720}$ is wrong.)

Error $E = \gamma'_{m+2} h^{m+3} y^{(m+3)}(\xi)$

The case $m=2$ is frequently used (Adam-Moulton formula)

(*) $y_{n+1} = y_n + \frac{h}{24}\left(9f_{n+1} + 19f_n - 5f_{n-1} + f_{n-2}\right)$

$$E = -\frac{19}{720} h^5 y^{(5)}(\xi)$$

This is a corrector formula of closed type since $f_{n+1} = f(x_{n+1}, y_{n+1})$ involves the unknown quantity y_{n+1}

Predictor used is Adam-Bashforth fourth order formula

$\boxed{\text{Algorithm}}$ (The Adam-Moulton predictor-corrector method)

For the diff equation $y' = f(x, y)$ with h fixed and $x_n = x_0 + nh$ and with $(y_0, f_0), (y_1, f_1), (y_2, f_2), (y_3, f_3)$ given for each fixed $n = 3, 4, \ldots$.

1. Compute $y_{n+1}^{(0)}$ using the formula

$$y_{n+1}^{(0)} = y_n + \frac{h}{24}\left[55 f_n - 59 f_{n-1} + 37 f_{n-2} - 9 f_{n-3}\right]$$

here $f_i = f(x_i, y_i)$

2. Compute until satisfied

(*) $$y_{n+1}^{(k)} = y_n + \frac{h}{24}\left[9 f(x_{n+1}, y_{n+1}^{(k-1)}) + 19 f_n - 5 f_{n-1} + f_{n-2}\right]$$

$k = 1, 2, \ldots$

3) If satisfied with $y_{n+1}^{(k)}$ then

set $y_{n+1} = y_{n+1}^{(k)}$.

Exercise :- Show that the iteration (*) converges if $\frac{9h}{24}\left|\frac{\partial f}{\partial y}\right| < 1$

Step-size control

Besides yielding improved accuracy the corrector formula serves another useful function. It provides an estimate of the local error which can be used to decide whether the step h is adequate for the required accuracy.

Local error of A-B method $\quad E_{AB} = \frac{251}{720} h^5 y^{(5)}(\xi_1)$

Local error of A-M method $\quad E_{AM} = \frac{-19}{720} h^5 y^{(5)}(\xi_2)$

Let $y_{n+1}^{(0)}$ represent value of y_{n+1} obtained by A-B method

and $y_{n+1}^{(1)}$ the result obtained by AM method (see $*$)

Thus

$$(**)\quad \begin{cases} y(x_{n+1}) - y_{n+1}^{(0)} = \dfrac{251}{720} h^5 y^{(5)}(\xi_1) \\ y(x_{n+1}) - y_{n+1}^{(1)} = \dfrac{-19}{720} h^5 y^{(5)}(\xi_2) \end{cases}$$

In general $\xi_1 \neq \xi_2$. But we assume that for x_{n+1} close to x_n, $\xi_1 = \xi_2$

By $(**)$ we obtain $h^5 y^{(5)}(\xi) = \dfrac{720}{270}\left(y_{n+1}^{(1)} - y_{n+1}^{(0)}\right)$

Substituting this in second equation above we get

$$y(x_{n+1}) - y_{n+1}^{(1)} = \frac{-19}{270}\left(y_{n+1}^{(1)} - y_{n+1}^{(0)}\right)$$

$$\approx -\frac{1}{14}\left(y_{n+1}^{(1)} - y_{n+1}^{(0)}\right) = D_{n+1}$$

Thus the error of the corrected value is approximately $-\frac{1}{14}$ of the difference between the corrected and predicted value

It is advisable to use corrector only once. If accuracy of D_{n+1} is not sufficient, it is better to reduce the step size than to correct more than once

The error estimate is used in the following manner.

Let us assume that we wish to keep the local error per unit step bounded as

$$E_1 \le \frac{|D_{n+1}|}{h} \le E_2$$

we proceed as follows

1) Use A-B formula to compute $y_{n+1}^{(0)}$

2) Use A-M formula to compute $y_{n+1}^{(1)}$

3) Compute $D_{n+1} = -\frac{1}{14}\left(y_{n+1}^{(1)} - y_{n+1}^{(0)} \right)$

4) If $E_1 \le \frac{|D_{n+1}|}{h} \le E_2$ then proceed to the next integration step, using the same value of h

5) If $\frac{|D_{n+1}|}{h} > E_2$ then the step size h is too large and should be reduced to $\frac{h}{2}$.

(6.) If $\frac{|D_{n+1}|}{h} < E_1$, more accuracy is being obtained than necessary. Hence we can save computer time by replacing h by $2h$.

Simple difference equations

A difference equation of order N is a relation between the differences

$$y_n = \Delta^0 y_n, \quad \Delta^1 y_n, \quad \Delta^2 y_n, \ldots, \Delta^N y_n$$

of a number sequence i.e.,

$$(*) \qquad \Delta^N y_n = f(n, y_n, \Delta y_n, \ldots, \Delta^{N-1} y_n)$$

Recall $\quad \Delta y_n = y_{n+1} - y_n$

$$\Delta^k y_n = \Delta(\Delta^{k-1} y_n) \quad \text{for } k \geq 2.$$

A solution to $(*)$ is a sequence $y_m, y_{m+1}, \ldots$ of numbers such that $(*)$ holds for $n = m, m+1, m+2, \ldots$.

If $(*)$ is a linear difference equation then $(*)$ can be written explicitly in terms of y's as

$$y_{n+N} + a_{n,N-1}\, y_{n+N-1} + \cdots + a_{n,0}\, y_n = b_n$$

Examples of linear difference equations

$$y_{n+1} - y_n = 1 \qquad \text{all } n$$

$$y_{n+1} - y_n = n \qquad \text{all } n \geq 0$$

$$y_{n+1} - (n+1)\, y_n = 0 \qquad \text{all } n \geq 0$$

$$y_{n+2} - (2\cos\gamma)\, y_{n+1} + y_n = 0 \qquad \text{all } n$$

By direct substitution, these eqn's have solutions

$$y_n = n + c \qquad \text{all } n$$

$$y_n = \frac{n(n-1)}{2} + c \qquad \text{all } n \geq 0$$

$$y_n = c\, n! \qquad \text{all } n \geq 0$$

$$y_n = c \cos \gamma n \qquad \text{all } n$$

with c an arbitrary constant.

We consider in detail a homogeneous linear difference equation with constant coefficients

(*) $$y_{n+N} + a_{N-1} y_{n+N-1} + \cdots + a_0 y_n = 0$$

We seek solutions of the form $y_n = \beta^n$ for all n.

Substituting in (*) we get

$$\beta^{n+N} + a_{N-1} \beta^{n+N-1} + \cdots + a_0 \beta^n = 0$$

dividing by β^n, we obtain the characteristic equation

$$\rho(\beta) = \beta^N + a_{N-1} \beta^{N-1} + \cdots + a_0$$

The characteristic polynomial is of degree N.

We first assume that its zeros $\beta_1, \beta_2, \ldots, \beta_N$ are distinct. Then $\beta_1^n, \beta_2^n, \ldots, \beta_N^n$ are all solution of (*), and by linearity it follows the

(**) $$y_n = c_1 \beta_1^n + c_2 \beta_2^n + \cdots + c_N \beta_N^n \quad \text{all } n$$

for arbitrary constant c_i, is also a solution of (*)

It can be shown that (**) is the general solution of (*)

Example

$$y_{n+3} - 2y_{n+2} - y_{n+1} + 2y_n = 0$$

char. polynomial

$$\beta^3 - 2\beta^2 - \beta + 2 = 0$$

roots of this polynomial equation is $1, -1, 2$

So general solution is

$$y_n = c_1 (1)^n + c_2 (-1)^n + c_3 . (2^n)$$
$$= c_1 + c_2 (-1)^n + 2^n c_3$$

If the first $N-1$ values of y_n are given then we can solve (*) explicitly for all succeeding values of n.

Example $y_{n+3} - 2y_{n+2} - y_{n+1} + 2y_n = 0$

If $y_0 = 0,\ y_1 = 1,\ y_2 = 1$

Then $y_3 = 2\cdot(1) + 1 - 0 = 3$

$y_4 = 5, \quad y_5 = 11$ etc

This does not give closed formula for y_n.

However general solution is

$$y_n = c_1 + (-1)^n c_2 + 2^n c_3$$

using initial value $y_0 = 0,\ y_1 = 1,\ y_2 = 1$
we get

$$\begin{aligned} 0 &= c_1 + c_2 + c_3 \\ 1 &= c_1 - c_2 + 2c_3 \\ 1 &= c_1 + c_2 + 4c_3 \end{aligned}$$

solution is $c_1 = 0,\ c_2 = -\frac{1}{3},\ c_3 = \frac{1}{3}$. So the closed form of sol^n of initial-value problem is

$$y_n = -\frac{1}{3}(-1)^n + \frac{2^n}{3}$$

If the characteristic polynomial has a pair of conjugate-complex zeros, the solution can still be expressed in real form. Thus if $\beta_1 = \alpha + i\beta$, $\beta_2 = \alpha - i\beta$ we express β_1, β_2 in polar form

$$\beta_1 = r e^{i\theta} \qquad \beta_2 = r e^{-i\theta}$$

where $r = \sqrt{\alpha^2 + \beta^2}$, $\theta = \tan^{-1}(\beta/\alpha)$

Then the solution of (*) corresponding to this pair of zeros is

$$c_1 \beta_1^n + c_2 \beta_2^n = c_1 r^n e^{in\theta} + c_2 r^n e^{-in\theta}$$

$$= r^n \left(c_1 (\cos n\theta + i \sin n\theta) + c_2 (\cos n\theta - i \sin n\theta) \right)$$

$$= r^n \left(c_1' \cos n\theta + c_2' \sin n\theta \right)$$

where $c_1' = c_1 + c_2$, $c_2' = i(c_1 - c_2)$.

Example

$$y_{n+2} - 2y_{n+1} + 2y_n = 0$$

char. polynomial $\quad \beta^2 - 2\beta + 2 = 0$

$\beta_1 = 1 + i \qquad \beta_2 = 1 - i \qquad$ roots

$r = \sqrt{2}, \qquad \theta = \pi/4.$

So general solution is

$$y_n = (\sqrt{2})^n \left(c_1 \cos \frac{n\pi}{4} + c_2 \sin \frac{n\pi}{4} \right)$$

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we are solving

$$(*) \qquad y_{n+N} + a_{N-1} y_{n+N-1} + \cdots + a_0 y_n = 0$$

char. polynomial

$$p(\beta) = \beta^N + a_{N-1} \beta^{N-1} + \cdots + a_0 = 0$$

Suppose β_1 is a double root of $p(\beta)$

Then second solution of (*) is $n\beta_1^n$.

To verify this, we note that if β_1 is a double root of $\rho(\beta)$ then $\rho(\beta_1) = 0$ and $\rho'(\beta_1) = 0$

Substituting $y_n = n\beta_1^n$ in (*) we get

$$(n+N)\beta_1^{n+N} + a_{N-1}(n+N-1)\beta_1^{n+N-1} + \cdots + a_0 n \beta_1^n$$

$$= \beta_1^n \left\{ \begin{array}{l} n(\beta_1^N + a_{N-1}\beta_1^{N-1} + \cdots + a_0) \\ + \beta_1 (N\beta_1^{N-1} + a_{N-1}(N-1)\beta_1^{N-2} + \cdots + a_1) \end{array} \right\}$$

$$= \beta_1^n [n\rho(\beta_1) + \beta_1 \rho'(\beta_1)]$$

$$= 0$$

It can also be shown that the two solutions β_1^n and $n\beta_1^n$ are linearly independent.

$\boxed{\text{Example}}$

$$y_{n+3} - 5y_{n+2} + 8y_{n+1} - 4y_n = 0$$

char poly.

$$\beta^3 - 5\beta^2 + 8\beta - 4 = 0$$

roots $2, 2, 1$. So general solution is

$$y_n = c_1 2^n + c_2 \cdot n 2^n + c_3 \cdot 1^n$$
$$= 2^n (c_1 + n c_2) + c_3$$

Non-homogeneous linear diff equation with constant coefficients

The general solution of the eqn

$$(**) \quad y_{n+N} + a_{N-1} y_{n+N-1} + \cdots + a_0 y_n = b_n$$

can be written in the form

$$y_n = y_n^a + y_n^p$$

where y_n^a is the general solution of the homogeneous system (**) and y_n^P is a particular solution of (**)

Special case $b_n = b$ is a constant

a particular solution of (**) can be obtained by setting $y_n^P = A$ (a constant).

Substitute in (**) we get

$$A = \frac{b_0}{1 + a_{N-1} + \cdots + a_0}$$

(provided denominator $\neq 0$).

Example

$$y_{n+2} - 2y_{n+1} + 2y_n = 1$$

$y_p = A$ gets

$$A = \frac{1}{1 - 2 + 2} = 1.$$

so $y_p = 1$.

general sol^n is $y_n = (\sqrt{2})^n \left(C_1 \cos\frac{\pi}{4}n + C_2 \sin\frac{\pi}{4}n \right) + 1$