Lecture 12

Today we do

Richardson Extrapolation.

Extrapolation $\equiv$ using two lower order approximation to obtain higher order approximation

This concept was invented by Richardson in 1927.

It was first used for weather forecasting.

Extrapolation can be applied whenever it is known that an approximation technique has an error term with a predictable form, one that depends on a "step size" h

Suppose that for each $h \neq 0$ we have a formula $N(h)$ which approximates an unknown value M and the truncation error involved has the form

$$(*) \quad M = N(h) + K_1 h + K_2 h^2 + K_3 h^3 + \cdots$$

for some collection of unknown constants $K_1, K_2, K_3, \ldots$

$M \approx N(h)$ is $O(h)$ approximation.

Extrapolation $\equiv$ combine $O(h)$ approximations to produce formulas with higher truncation error

(1) $M = N(h) + K_1 h + K_2 h^2 + K_3 h^3 + \cdots$

This formula holds for all h (suff small)

$$\lim_{h \to 0} N(h) = M.$$

(2) $M = N\left(\frac{h}{2}\right) + K_1 \frac{h}{2} + K_2 \frac{h^2}{4} + K_3 \frac{h^3}{8} + \cdots$

2.(2) − (1) gives

$$M = 2M - M = 2N\left(\tfrac{h}{2}\right) - N(h) + K_2\left(\tfrac{h^2}{2} - h^2\right) + K_3\left(\tfrac{h^3}{4} - h^3\right) + \cdots$$

$$M = \underline{N\left(\tfrac{h}{2}\right) + \left(N\left(\tfrac{h}{2}\right) - N(h)\right)} - \frac{K_2}{2} h^2 - \frac{3}{4} K_3 h^3 - \cdots$$

$$N_1(h) = N(h)$$

$$N_2(h) = N_1\left(\tfrac{h}{2}\right) + \left[N_1\left(\tfrac{h}{2}\right) - N_1(h)\right]$$

(3) $M = N_2(h) - \frac{K_2}{2} h^2 - \frac{3K_3}{4} h^3 \cdots$

this is an $O(h^2)$ approximation

If we replace h by $h/2$ in formula (3) we obtain

(4) $$M = N_2\left(\frac{h}{2}\right) - \frac{K_2}{8}h^2 - \frac{3K_3}{32}h^3 \cdots\cdots$$

$4 \times$ (4) $-$ (3) gives

$$3M = 4N_2\left(\frac{h}{2}\right) - N_2(h) + \frac{3K_3}{8}h^3 + \cdots$$

$$M = N_2\left(\frac{h}{2}\right) + \frac{N_2\left(\frac{h}{2}\right) - N_2(h)}{3} + \frac{K_3}{8}h^3 + \cdots$$

Set $N_3(h) = N_2\left(\frac{h}{2}\right) + \dfrac{N_2\left(\frac{h}{2}\right) - N_2(h)}{3}$

(5) $$M = N_3(h) + \frac{K_3}{8}h^3 + \cdots$$

This is order h^3 formula for approximating M

Ily

$$N_4(h) = N_3\left(\frac{h}{2}\right) + \frac{N_3(h/2) - N_3(h)}{7}$$

will be $O(h^4)$ approximation to M

In general

$$N_j(h) = N_{j-1}\left(\frac{h}{2}\right) + \frac{N_{j-1}\left(\frac{h}{2}\right) - N_{j-1}(h)}{2^{j-1} - 1}$$

will give $O(h^j)$ approximation to M

h	$N_1(h)$			...
$h/2$	$N_1(h/2)$	$N_2(h)$		
$h/4$	$N_1(h/4)$	$N_2(h/2)$	$N_3(h)$	

Example

$$e = \lim_{h \to 0} (1+h)^{\frac{1}{h}}$$

One can show

$$e = (1+h)^{\frac{1}{h}} + K_1 h + K_2 h^2 + \cdots$$

$$N_1(h) = (1+h)^{1/h}$$

(all comp in 6 sig digits)

h	$N_1(h)$	$N_2(h)$	$N_3(h)$	$N_4(h)$
0.4	2.31910			
0.2	2.48832	2.65754		
0.1	2.59374	2.69916	2.71303	
0.05	2.65330	2.71286	2.71743	2.71806

exact value $e = 2.71828$
upto 6 sig digits

Sometimes we have the following

(1) $M = N(h) + K_2 h^2 + K_4 h^4 + K_6 h^6 + \cdots$

$N(h)$ is $O(h^2)$ approximation to M

Set $N_1(h) = N(h)$

Since (1) is valid for all h (suff small) we have

(2) $M = N_1(\frac{h}{2}) + K_2 \frac{h^2}{4} + K_4 \frac{h^4}{16} + K_6 \frac{h^6}{64} + \cdots$

$4 \times (2) - (1)$ gives

$$3M = 4N_1(\tfrac{h}{2}) - N_1(h) + K_4\left(\frac{h^4}{4} - h^4\right) + K_6\left(\frac{h^6}{16} - h^6\right) + \cdots$$

Set $N_2(h) = N_1(\frac{h}{2}) + \dfrac{N_1(h/2) - N_1(h)}{3}$

So

(3) $M = N_2(h) + \underline{K_4'} h^4 + \underline{K_6'} h^6 \cdots$

So $N_2(h)$ is $O(h^4)$ approximation to M.

replacing h by $h/2$ in (3) we get

$$(4) \quad M = N_2\left(\frac{h}{2}\right) + K_4' \frac{h^4}{16} + K_6' \frac{h^6}{64} + \cdots$$

$16 \times (4) - (3)$ yields

$$15M = 16 N_2\left(\frac{h}{2}\right) - N_2(h) + K_6'\left(\frac{h^6}{4} - h^6\right) + \cdots$$

$$N_3(h) = N_2\left(\frac{h}{2}\right) + \frac{N_2\left(\frac{h}{2}\right) - N_2(h)}{15}$$

$$M = N_3(h) + K_6'' h^6 + \cdots$$

$N_3(h)$ is $O(h^6)$ approximation to M

Continuing this procedure we get

$$N_j(h) = N_{j-1}\left(\frac{h}{2}\right) + \frac{N_{j-1}\left(\frac{h}{2}\right) - N_{j-1}(h)}{4^{j-1} - 1}$$

is $O(h^{2j})$ approx to M

Example

$$e = \lim_{h\to 0} \left(\frac{2+h}{2-h}\right)^{\frac{1}{h}} = 2.7182818$$
in 8 sig digits

One can prove that

$$e = \left(\frac{2+h}{2-h}\right)^{1/h} + K_2 h^2 + K_4 h^4 + K_6 h^6 + \cdots$$

$$N_1(h) = \left(\frac{2+h}{2-h}\right)^{1/h}$$

h	N_1	N_2	N_3	N_4
0.4	2.7556760			
0.2	2.7274128	2.7179917		
0.1	2.7205514	2.7182643	2.7182825	
0.05	2.7188484	2.7182807	2.7182818	2.7182818

Our approximation 2.7182818 is correct upto 8 sig digits

Romberg integration

Extrapolation used in the context of composite Trapezoid rule of integration is called Romberg integration

Recall composite Trapezoidal rule

$$\int_a^b f(x)\,dx \approx T_N$$

$$N = \frac{b-a}{h} \qquad x_i = a + ih \quad i = 0, 1, \ldots, N$$

$$T_N = \frac{h}{2}\left[f(x_0) + 2\sum_{i=1}^{N-1} f(x_i) + f(x_N)\right]$$

$$I = T_N + c_1 h^2 + O(h^4)$$

$$c_1 = \frac{[f'(a) - f'(b)]}{12}$$

(do composite corrected Trap rule

If N is even then note that one use $T_{N/2}$ to compute T_N.

Specifically

$$T_N = \frac{T_{N/2}}{2} + h \sum_{i=1}^{N/2} f(a + (2i-1)h)$$

One can prove that

$$\underline{I} = T_N + c_2 h^2 + c_4 h^4 + c_6 h^6 + \cdots + c_{2k} h^{2k} + \cdots$$

for all <u>sufficiently small</u> values of h

$$T_N' = T_N + \frac{T_N - T_{N/2}}{3} \quad \text{is } O(h^4)$$

approx to $\underline{I}$

$$\underline{I} = T_N' + c_4' h^4 + c_6' h^6 + \cdots$$

So one can do further extrapolation

$$\boxed{T_N^2 = T_N^1 + \frac{T_N^1 - T_{N/2}^1}{15}}$$

is $O(h^6)$ approximation to I

more generally

$$T_N^m = T_N^{m-1} + \frac{T_N^{m-1} - T_{N/2}^{m-1}}{4^m - 1}$$

is $\underline{O(h^{2m+2})}$ approximation to I

<u>note that</u> $N/2^m$ must be an integer for T_N^m to be defined.

Example

$$I = \int_0^1 \sin(x^2)\, dx$$

$$= 0.3103 \quad \text{in 4 sig digits}$$

N	T_N	T'_N		
1	0.4208			
2	0.3341	0.3052		
4	0.3159	0.3098	0.3101	
8	0.3117	0.3103	0.3103	0.3103

Thus by Romberg integration we get accuracy upto 4 sig digits

Example

$$I = \int_0^2 \sqrt{1+\cos^2 x}\, dx$$

$$= 2.352 \quad (\text{correct upto 4 sig digits})$$

$$T_1 = \frac{1}{2} \cdot 2 \, [f(0) + f(2)]$$

$$= 1. \, [1.414 + 1.083]$$

$$= 2.497$$

$$T_2 = \frac{1}{2} \cdot 1 \, [f(0) + 2f(1) + f(2)]$$

$$= \frac{1}{2} [2.497 + 1.137 \times 2]$$

$$= \frac{1}{2} [4.771]$$

$$= 2.386$$

$$T_4 = \frac{1}{2} \cdot 0.5 \, [f(0) + 2f(0.5) + 2f(1) + 2f(1.5) + f(2)]$$

$$= \frac{1}{4}[4.771 + 2f(0.5) + 2f(1.5)]$$

$$= \frac{1}{4}[4.771 + 2 \times 1.330 + 2 \times 1.002]$$

$$= \frac{1}{4}[9.435]$$

$$= 2.359$$

Romberg integration

N	T_N	T_N'	
1	2.497		
2	2.386	2.349	
4	2.359	2.350	2.350

So by Romberg int'g error is reduced

Example 3

$$I = \int_0^1 e^{-x^2} dx$$

Find T_1, T_2, T_4, T_8

$$T_1 = \frac{1}{2} \cdot 1 \left(f(0) + f(1) \right)$$

$$= \frac{1}{2} \left(1 + 3.679 \text{ E-1} \right)$$

$$= \frac{1}{2} \left(1.368 \right)$$

$$= 0.6840$$

$$T_2 = \frac{1}{2} \cdot \frac{1}{2} \left(f(0) + 2 f(0.5) + f(1) \right)$$

$$= \frac{1}{4} \left(1.368 + 2 \times 7.788 \text{ E-1} \right)$$

$$= \frac{1}{4} \left(2.926 \right)$$

$$= 0.7314$$

$$T_4 = \frac{1}{2}\cdot\frac{1}{4}\left[f(0) + 2f(0.25) + 2f(0.5) + 2f(0.75) + f(1)\right]$$

$$= \frac{1}{8}\left[2.926 + 2f(0.25) + 2f(0.75)\right]$$

$$= \frac{1}{8}\left[2.926 + 2\times 9.394E{-1} + 2\times 5.698E{-1}\right]$$

$$= \frac{1}{8}\left[5.944\right]$$

$$= 0.7431$$

$$T_8 = \frac{1}{2}\cdot\frac{1}{8}\left\{f(0) + 2f(0.125) + 2f(0.25) + 2f(0.375) + 2f(0.5) + 2f(0.625) + 2f(0.75) + 2f(0.875) + f(1)\right\}$$

$$= \frac{1}{16}\left[5.944 + 2f(0.125) + 2f(0.375) + 2f(0.625) + 2f(0.875)\right]$$

$$= \frac{1}{16}\left[5.944 + 2\times 9.845E{-1} + 2\times 8.688E{-1} + 2\times 6.766E{-1} + 2\times 4.65E{-1}\right]$$

$$= \frac{1}{16}\left[11.93\right] = 0.7459$$

Romberg integration

N	T_N	T_N^1	T_N^2	T_N^3
1	0.684			
2	0.7314	0.7472		
4	0.7431	0.7470	0.7470	
8	0.7459	0.7468	0.7468	0.7468

$$\int_0^1 e^{-x}\,dx = 0.7468$$

correct upto 4 sig digits