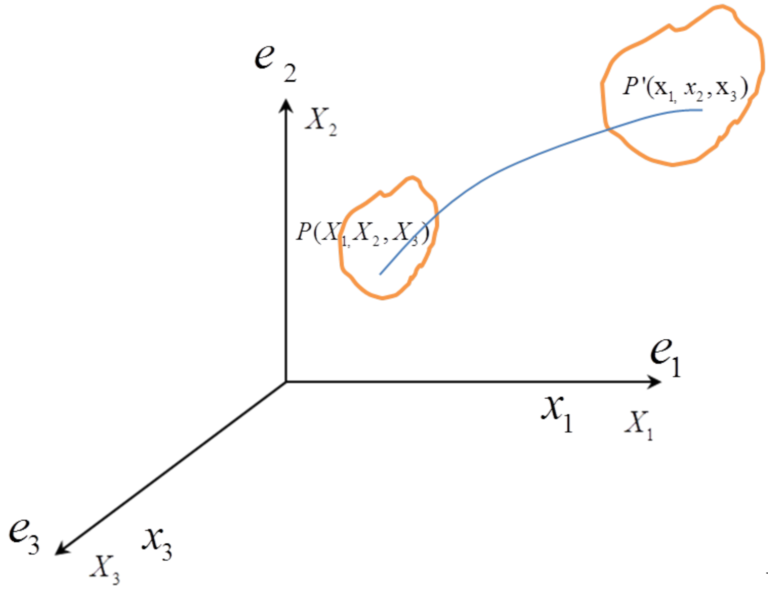
Kinematics

Description of motion of a continuum

Rectangular Cartesian Coordinate system

Let a body occupy a system in space at time $t = t_0$.What are we interested in?
a. Motion of this body
b. Deformation of this body



What do we need? to identify infinitely many points of particles.Material particles are identified as X_i with respect to a fixed rectangular Cartesian coordinate system at $t = t_0$.

Under motion of the body material point P moves to P' whose coordinates w.r.t. fixed rectangular Cartesian coordinate are x_i.Then the equation

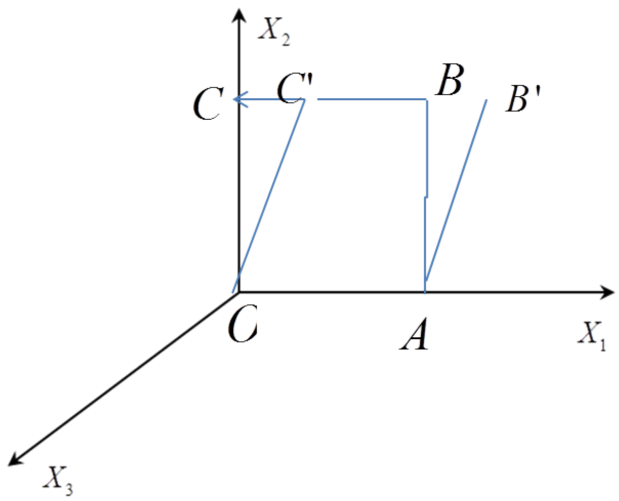
$$x_i = X_i\ (X_1, X_2, X_3, t) \tag{1}$$

describes the motion of the particle.
Configuration at
a. $t = t_0$——Reference configuration.
b. $t = t$——Present configuration. Eq. (1) gives the path line of the material point$P$

$$X_i = x_i(x_1, x_2, x_3, t_0) \tag{2}$$

—-verify that the material point P occupied the place X_i at $t = t_0$. In continuum mechanics it is common to use the same symbol for a function and its value.

Ex 1.
$x_1 = X_1 + .2 * t * X_2$
$x_2 = X_2$
$x_3 = X_3$ or equivalently $x_i = X_i + 0.2tX_2\delta_{i1}$
X_1, X_2 *and* X_3 gives value/position of material particle at $t = 0$.Sketch the configuration at time $t = 2$ for a body, which at $t = 0$ has the shape of a cube of unit sides with one corner at the origin.



at origin $x_i = 0 \ \forall t$
at $(X_1, 0, 0)$ $x_i = X_1\delta_{1i}$
⇒particles on line OA do not move.
at $(X_1, 1, 0)$ on line CB (horizontal movement)
at $t = 2;\ x_i = [X_1 + (0.2)(2)(1)]\delta_{1i} + (1)\delta_{2i}$.

For particle (0,X_2,0)

$$x_i = [0 + (0.2)(2)X_2]\delta_{1i} + X_2\delta_{2i} = .04X_2\delta_{1i} + X_2\delta_{2i}; \tag{3}$$

–horizontal movement to right
Simple shearing motion

$X_1 = x_1 - 0.2tx_2$ location @$t = 0$ $X_2 = x_2; X_3 = x_3$

Motion $\rightarrow$ two parallel flat plate with the bottom one fixed and the upper one moved only along the X_1-axis.Viscous fluid flow between two parallel plates.

Referential and spatial description

quantities associated with specific material points change with time
Example: $\theta = \theta(X_1, X_2, X_3, t)$; $v_i = v_i(X_1, X_2, X_3, t)$;
Two ways to describe it:

0.1 Lagrangian description

Follow the material particle i.e. express the quantities as functions of the coordinate of a material particle in a fixed reference configuration.Also known as material description.

0.2 Eulerian description

Observe quantities at fixed locations in space.
$\theta = \theta(x_1, x_2, x_3, t)$; $v_i = v_i(x_1, x_2, x_3, t)$— this are the spatial descriptions and no information of a particle material point.

Example: Given motion $x_i = X_i + .2tX_2\delta_{1i}$ and Temperature field is $\theta = 2x_i + x_i^2$
a. Obtain referential description of temperature
b. Rate of change of temperature of the material particle at a time $t = 0$ at the point (0,1,0)

$\theta = 2(X_1 + 0.2tx_2) + (x_2)^2 = 2X_1 + (X_2 + 0.4t)X_2$ at $t = 0$; (0,1,0) $\theta = 1 + 0.4t$ and $\frac{d\theta}{dt} = 0.4$
spatial $\theta \rightarrow$ independent of time and Referential $\Rightarrow$shows that in actually θ changes from one spatial position from another with time.

Displacement Vector

By definition, the displacement vector of a material particle is the difference between its position vectors at time t and at time $t = t_0$ (or 0) :
$u_i = x_i - X_i$

In Lagrangian description of motion, the displacement u_i is specified as a function of X_i and t.
For example,
$x_1 = X_1t^2 + 2X_2t + X_1$
$x_2 = 2X_1t^2 + X_2t + X_2$
$x_3 = X_3t/2 + x_3$
during the time interval $0 \leq t \leq 1$. The corresponding displacement components are given by:
$u_1 = x_1 - X_1 = X_1t^2 + 2X_2t$ and so on.

In Eulerian description, u_i will be expressed as a function of x_i and t :
$u_1 = x_1 - X_1 = x_1 - \{ \qquad \}$
Continuous deformation of a deformable body
at $t = 1$ and $x_1 = 2(X_1 + X_2); x_2 = 2(X_1 + X_2); X_3 = 3X_3/2$
material particle at all occupy (0,0,0)—-Which is not possible in continuum mechanics.
Collision of particles not allowed
different particles occupy distinct places
Thus, $x_i = x_i(X_1, X_2, X_3, t)$ has one to one mapping from reference to present configuration. $\Rightarrow$ Mapping is continuously differentiable and continuously differentiable inverse. $X_i = X_i(x_1, x_2, x_3, t)$ if and only if,
$J=$ det $[\frac{\partial x_i}{\partial x_j}]=det[\delta_{ij} + \frac{\partial u_i}{\partial x_j}$

$J = (X_1, X_2, X_3, 0) = 1$ and J is continuous function of t.Hence, J must be positive for every t.
$\Rightarrow J > 0$ for continuous deformation to be physical admissible.

Material derivative

Time rate of change of a quantity of a material particle.
Ex:$\frac{D(\theta)}{Dt}$. Depends on Lagrange or Eulerian description.
1.Lagrangian description:
$\theta = \theta(X_1, X_2, X_3, t)$
$\dot{\theta} = \frac{D(\theta)}{Dt} = \frac{\partial\theta}{\partial t}\Big|_{x_i fixed}$ 2.Spatial or Eulerian description:
$\theta = \tilde{\theta}(X_1, X_2, X_3, t)$; $x_i = x_i(X_1, X_2, X_3, t)$
so, $\dot{\theta} = \frac{D(\tilde{\theta})}{Dt} = \frac{\partial\tilde{\theta}}{\partial t}\Bigg|_{x_i fixed} + \frac{\partial\tilde{\theta}}{\partial x_j}\frac{\partial x_j}{\partial t}\Bigg|_{x_i fixed}$

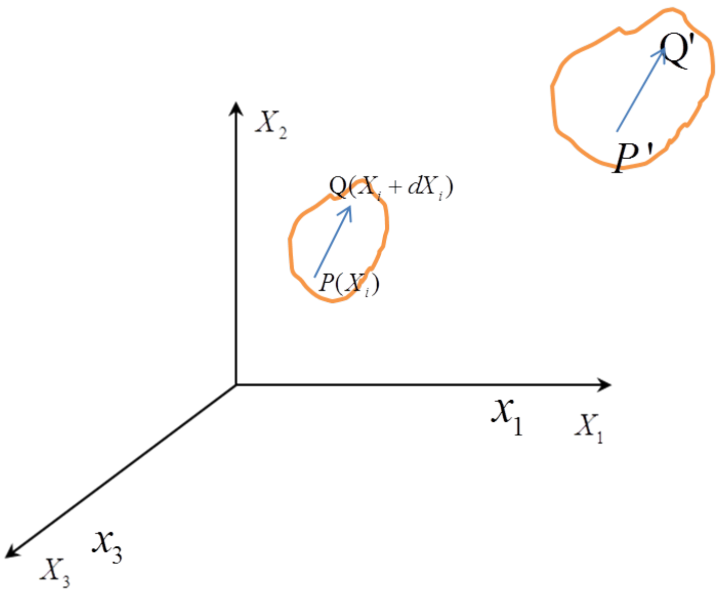
Example: Given the motion, $x_i = X_i(1 + t)0 \leq t \geq 1$
Find the spatial description of the velocity field.$[v_i = \dot{x_i} = X_i = x_i/(1 + t)]$

Given temperature field $\theta = 2(x_1^2 + x_2^2$ where $x_i = X_i(1 + t)$. Find at

$t=1$, the rate of change of temperature of the particle, which in reference configuration was at (1,1,1)
Express θ as function of X_i find $\dot{\theta}$ and substitute X and θ at $t=1$
$\dot{\theta}$ as function of x_i and t find x_i at $t=1$ and then substitute
$\dot{\theta}=\frac{\partial\theta}{\partial t}+\frac{\partial\theta}{\partial x_j}\partial x_j\partial t=0+4x_1x_1/(1+t)+4x_2x_2/(1+t)$
now finding out the value at $t=1 \quad (1,1,1)$

Example: A fluid rotates as a rigid body with a constant angular velocity $\omega=\omega e_3$.Write explicit component of velocity of a material point in the Eulerian description of motion.
Note: $v=\omega\times r=\omega e_3\times r=\epsilon_{i3k}\omega x_k$
$\Rightarrow \tilde{v_i}=\epsilon_{ijk}\omega_i r_j e_k=\epsilon_{i3k}\omega x_k$
$a_i=\frac{\partial\tilde{v_i}}{\partial t}+\frac{\partial\tilde{v_i}}{\partial x_j}\frac{\partial x_j}{\partial t}$
$=0+\epsilon_{i3k}\omega\frac{\partial x_k}{\partial x_j}\tilde{v_j}$
$=\omega^2\{\epsilon_{i3k}\epsilon_{3mj}x_m\}$
$=\omega^2\{\delta_{i3}\delta_{3m}-\delta_{im}\delta_{33}\}x_m$
$a_i=\omega^2\{\delta_{i3}x_3-x_i\}$
Example:
Find the acceleration for the case of simple shear deformation.$x_i=X_i+0.2tX_2\delta_{i1}[v_i=\dot{x_i}=0.2(X_2\delta_{i1}]$
$a_i=\dot{v_i}=0.$

Deformation Gradient



Let the motion of the

body be given by, $x_i = x_i(X_1, X_2, X_3, t)$
continuously differentiable function of its arguments and $J > 0$.
$P'Q' = \{x_i(X_1 + dX_1, X_2 + dX_2, X_3 + dX_3, t) - x_i(X_1, X_2, X_3, t)\}e_i$;
following Taylor expansion for the first term on the right-hand side,
$P'Q' = [\frac{\partial x_i}{\partial X_1} dX_1 + \frac{\partial x_i}{\partial x_2} dX_2 + \frac{\partial x_i}{\partial X_3} dX_3]e_i + 0(|dX|^2)$
$P'Q' = [\frac{\partial x_i}{\partial X_j}\Big|_P dX_j]e_i$+neglect higher order terms
$= [\frac{\partial x_i}{\partial X_A} dX_A]e_i$
component form,
$(P'Q')_j = \frac{\partial x_i}{\partial X_A} dX_A e_i.e_j$
$= \frac{\partial x_i}{\partial X_A} dX_A \delta_{ij}$
$= \frac{\partial x_j}{\partial X_A} dX_A$
$(P'Q') = F_{jA}|_P (PQ)_A$ relates components of vectors PQ in reference configuration to the components of the vectors $(P'Q')$.
$u_i = x_i - X_A \delta_{iA}$
$\frac{\partial u_i}{\partial X_A} = \frac{\partial x_i}{\partial X_A} - \delta_{iA}$
$\Rightarrow F_{iA} = u_{i,A} + \delta_{iA}$

Example: The deformation of a body is given by $u_1 = (3X_1^2 + X_2), u_2 = (2X_2^2 + X_3), u_3 = (4X_3^2 + X_1)$
Compute the vectors into which the vectors $\epsilon(1/3, 1/3, 1/3)$ passing through the material point $(1, 1, 1)$ in the reference configuration is deformed. $\epsilon \rightarrow$ infinitesimal real.[Ans. (1/3,1/3,1/3) components of a vector PQ]

$$F_{iA} = \begin{bmatrix} 1+6X_1 & 1 & 0 \\ 0 & 1+4X_2 & 1 \\ 1 & 0 & 1+8X_3 \end{bmatrix}$$
$$F_{iA}|_P = \begin{bmatrix} 7 & 1 & 0 \\ 0 & 5 & 1 \\ 1 & 0 & 9 \end{bmatrix}$$
$$\text{hence,}\{P'Q\}_j = \begin{bmatrix} 7 & 1 & 0 \\ 0 & 5 & 1 \\ 1 & 0 & 9 \end{bmatrix} \begin{Bmatrix} \epsilon/3 \\ \epsilon/3 \\ \epsilon/3 \end{Bmatrix}$$
$$= \epsilon/3 \begin{Bmatrix} \epsilon/3 \\ \epsilon/3 \\ \epsilon/3 \end{Bmatrix}.$$

Example: Simple Extension
$x_1 = \alpha(t)X_1; x_2 = \beta(t)X_2; x_3 = \gamma X_3$

$$F_{iA} = \begin{bmatrix} \alpha & 0 & 0 \\ 0 & \beta & 0 \\ 0 & 0 & \gamma \end{bmatrix}$$

Given, $u_1 = 0.1X_2^2; u_2 = u_3 = 0;$

a. Is this deformation possible? Prove your answers.

b.Find vectors into which material vectors $0.01e_1$ and $0.015e_2$ passing through the material point $P(1,1,0)$in the reference configuration, are deformed.

a. $F_{iA} = \begin{bmatrix} 1 & 0.2X_2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ and det[F_{iA}]=1¿0

at (1,1,0) $\begin{bmatrix} 1 & 0.2(1)^2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} 0 \\ 0.015 \\ 0 \end{Bmatrix} = \begin{Bmatrix} 0.003 \\ 0.015 \\ 0 \end{Bmatrix}$

Notes:

1) Non-singular-tensor F depends on X which denotes a so-called inhomogeneous deformation

2)A deformation of a body in question is said to be homogeneous if F does not depend on the space co-ordinates. F_{iA} depends only on time.
Associated motion is called affine.

3) Rigid-body translation $\Rightarrow$ displacement field is independent of X

4) No motion $F = I \rightarrow x = X$

1. Ratio of the length of the vector $P'Q'$ to that of the vector PQ is called the stretch at the material point P in the direction of the vector PQ.

2.Different unit vectors through the point P are stretched differently, therefore, the stretch λ at the point P varies with direction of vector PQ.

3. It is assumed that PQ is infinitesimal. However, no assumption was made as to the magnitude of the gradient F_{iA}. Hence valid for small and large gradients. $\Rightarrow$ applicable for small nd large deformations.

C. Determine the stretches of the point $(1,1,0)$ in the X_1 and X_2 direction.

D.Determine the change in the angle between lines passing the point $P(1,1,0)$ that was parallel to the X_1 and X_2 axes in the reference configuration.
Stretch at the point $(1,1,0)$ in the X_1-direction $= \frac{0.01}{0.01} = 1.$
Stretch at the point (1,1,0) in the X_2 direction$=\frac{\sqrt{0.003^2+0.015^2}}{0.015} = 1.02$
Angle between the vectors into which vectors $0.01e_1$ and $0.015e_2$ through the point (1,1,0) are deformed
$=\cos^{-1}\left\{\frac{(0.01)(0.003)+0+0}{(0.01)\sqrt{0.003^2+0.015^2}}\right\} = \pm 78.7^o$
change in angle$=11.3^o$.

Given the following displacement components $u_1 = 2X_1^2 + X_1X_2$ and $u_2 = X_2^2$ and $u_3 = 0$ and that for the points in reference configuration of the body $X_1 \geq 0, X_2 \geq 0$.
a. Find the vector in the reference configuration that is deformed into a vector parallel the the x_1. through the point (1,1,0) in the present configuration.
b.Find the stretch of a line element that is deformed into a vector parallel to the x_1 axis through the point(1,1,0) in the present configuration.
$\Rightarrow x_1 = X_1 + 2x_1 + X_1X_2$
$x_2 = X_2 + X_2^2$
$x_3 = 0$ and from these equations $\Rightarrow X_1 = X_2, X_2 = 0, X_3 = 0$ for (1/2,0,0).

$$F_{iA} = \begin{bmatrix} 1+4X_1+X_2 & X_1 & 0 \\ 0 & 1+2X_2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$F_{iA}|_P = \begin{bmatrix} 3 & 0.5 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Hence, $\left\{\begin{array}{c} 1 \\ 0 \\ 0 \end{array}\right\} = \begin{bmatrix} 3 & 0.5 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \left\{\begin{array}{c} dX_1 \\ dX_2 \\ dX_3 \end{array}\right\}$
$\Rightarrow (1/3, 0, 0)$; and stretch$= \frac{\sqrt{1^2+0+0}}{\sqrt{3^2+0+0}} = 3.$