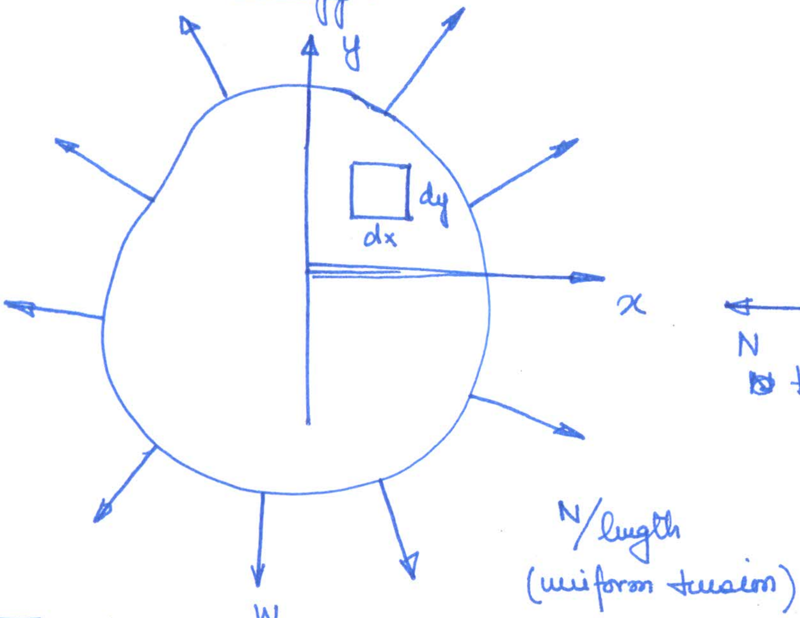
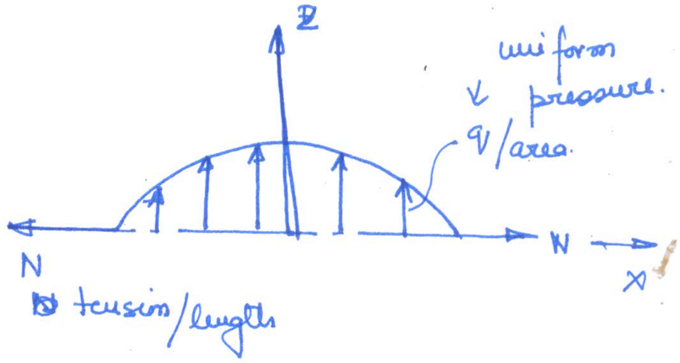
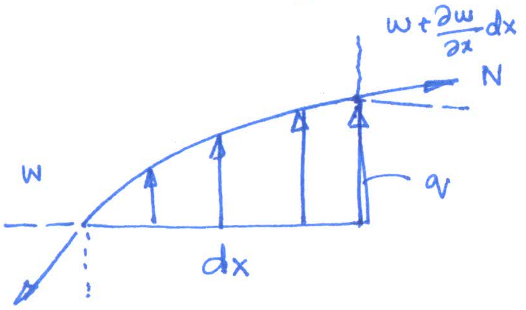
Membrane Analogy:





The deflection w of a membrane supported along the boundary ($w=0$) is of the shape of the c/s of the shaft analogous to the stress function φ for the shaft. A uniform tensile force N/length is applied to the membrane. In addition, a vertical (z-direc) ~~pres~~ pressure (q/area) is applied uniformly to the membrane.

Force equillibrium of an elemental area



$$-N dy \frac{\partial w}{\partial x} + N\left(\frac{\partial w}{\partial x} + \frac{\partial^2 w}{\partial x^2}dx\right)dy$$

$$-N dx \frac{\partial w}{\partial y} + N\left(\frac{\partial w}{\partial y} + \frac{\partial^2 w}{\partial y^2}dy\right)dx$$

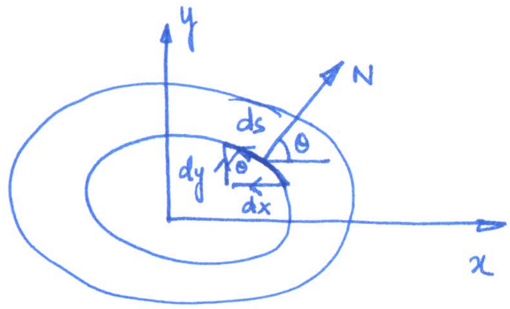
$$+ q\,dx\,dy = 0$$

$$\Rightarrow N\frac{\partial^2 w}{\partial x^2} + N\frac{\partial^2 w}{\partial y^2} = -q$$

$$\Rightarrow \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} = -\frac{q}{N} \quad \text{and } w = 0$$

$\therefore$ φ is analogous to w when $\frac{q}{N}$ is replaced by $2G\theta$.

Corollary:



Contour lines on the deformed membrane $\frac{\partial w}{\partial s} = 0$

Resultant of shear stress at any point on the contour line in the normal direction,

$$\tau_{xz} \cos\theta + \tau_{yz} \sin\theta.$$

$$\frac{\partial \varphi}{\partial y} \cdot \frac{dy}{ds} + \frac{\partial \varphi}{\partial x} \frac{dx}{ds} = \frac{\partial \varphi}{\partial s} = 0$$

(φ is ~~zero~~ constant along the contour line since φ is analogous to w)

Resultant shear stress along the tangential dirc of the contour.

$$-\tau_{xz} \sin\theta + \tau_{yz} \cos\theta = \tau$$

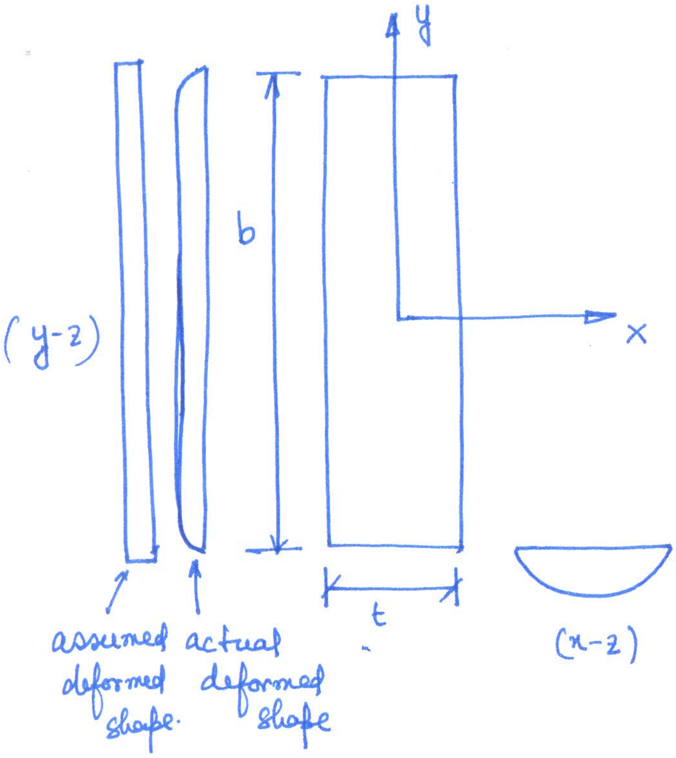
$$\Rightarrow \tau = -\frac{\partial \varphi}{\partial y} \frac{dy}{dn} - \frac{\partial \varphi}{\partial x} \cdot \frac{dx}{dn}$$

$$= -\frac{\partial \varphi}{\partial n} \quad \left(\text{analogous to } -\frac{\partial w}{\partial n}\right)$$

= slope of the deformed shape at the point

∴ Maximum shear stress will occur at a point where the membrane will have maximum slope.

Torsion of a bar with narrow rectangular c/s



Neglecting the end effects, the stress function φ is assumed to be independent of y.

$$\Rightarrow \frac{d^2\varphi}{dx^2} = -2G\theta$$

$$\Rightarrow \varphi = -G\theta x^2 + C_1 x + C_2.$$

Boundary condns @ $\pm\frac{t}{2}$ $\varphi = 0$

$$0 = -G\theta\frac{t^2}{4} + C_1\frac{t}{2} + C_2$$

$$0 = -G\theta\frac{t^2}{4} - C_1\frac{t}{2} + C_2.$$

$$\Rightarrow \varphi = -G\theta\left[x^2 - \frac{t^2}{4}\right]$$

Shear stress

$$\tau_{yz} = -\frac{\partial\varphi}{\partial x} = 2G\theta x.$$

Maxm shear stress $\tau_{yz}@\pm\frac{t}{2} = \boxed{\pm G\theta t}$

Torque $M_t = 2\iint \varphi\, dx\, dy$

$$= -2G\theta\int_{-t/2}^{t/2}\int_{-b/2}^{b/2}\left[x^2 - \frac{t^2}{4}\right]dx\,dy = -2G\theta b\int_{-t/2}^{t/2}\left[x^2 - \frac{t^2}{4}\right]dx$$

$$= -2G\theta b\left[\frac{x^3}{3} - \frac{t^2 x}{4}\right]_{-t/2}^{t/2}$$

$$= -2G\theta b\left[\frac{t^3}{12} - \frac{t^3}{4}\right]$$

$$= \frac{G\theta b t^3}{3}$$

$\therefore$ Torsional rigidity

$$\boxed{J = \frac{bt^3}{3}}$$