Lecture 4

Recall,
Last time we introduced interpolation polynomial

Given distinct points $x_0, x_1, \ldots, x_n$
and function values $f(x_0), f(x_1), \ldots, f(x_n)$
there exists a <u>Unique</u> polynomial
$P_n(x)$ of degree $\leq n$ such that
$$P_n(x_i) = f(x_i) \text{ for } i = 0, 1, \ldots, n.$$

2 forms to write $P_n(x)$

1) Lagrange's form
2) Newton's divided difference form

Lagrange's form

$$l_k(x) = \prod_{\substack{i=0 \\ i \neq k}}^{n} \frac{x - x_i}{x_k - x_i}$$

$k = 0, 1, \ldots, n$

- $l_k(x)$ polynomial of degree $\leq n$
- $l_k(x_k) = 1$
- $l_k(x_i) = 0$ for $i \neq k$.

$$P_n(x) = \sum_{k=0}^{n} f(x_k)\, l(x_k)$$

$P_n(x_k) = f(x_k)$ for $k = 0, 1, \ldots, n$.

Newton's divided difference form

$*$ $P_n(x) = a_0 + a_1(x-x_0) + a_2(x-x_0)(x-x_1)$
$+ \cdots$
$+ a_n(x-x_0)(x-x_1)\cdots(x-x_{n-1})$

(note any polynomial of degree $\leq n$ can be written as above for suitable $a_0, \dots, a_n$)

$P_{n-1}(x)$ polynomial which interpolate $f(x)$ at $x_0, \dots, x_{n-1}$ ($\deg P_{n-1}(x) \leq n-1$)

$$P_n(x) = P_{n-1}(x) + f[x_0, \dots x_n](x-x_0)\cdots(x-x_{n-1})$$

$f[x_0, \dots, x_n]$ = coefficient of x^n in $P_n(x)$

$f[x_0, \dots, x_i] = a_i$ in $*$

$$P_n(x) = f[x_0] + f[x_0, x_1](x - x_0)$$
$$+ f[x_0, x_1, x_2](x - x_0)(x - x_1)$$
$$+ \cdots$$
$$\vdots$$
$$+ f[x_0, \ldots, x_n](x - x_0)(x - x_1) \cdots (x - x_{n-1})$$

$$f[x_0] = f(x_0)$$

$$f[x_0, x_1] = \frac{f(x_1) - f(x_0)}{x_1 - x_0}$$

$$\vdots$$

$$f[x_0, \ldots x_k] = \frac{f[x_1, \ldots x_k] - f[x_0, \ldots x_{k-1}]}{x_k - x_0}$$

Always remember

$$P_n(x) = P_{n-1}(x) + f[x_0, x_1, \ldots x_n](x - x_0) \cdots (x - x_{n-1})$$

i.e., $f[x_0, \ldots, x_n]$ = coeff of x^n in $P_n(x)$.

Example 2

$$f(x) = \int_0^x \sin(t^2)\,dt$$

x	$f(x)$	$f[\,,\,]$	$f[\,,\,,\,]$	$f[\,,\,,\,,\,]$
		$f[0.8, 0.9]$	$f[0.8, 0.9, 1.0]$	
0.8	1.657 E-1	6.62 E-1	6.1 E-1	-2.5 E-1
0.9	2.319 E-1	7.84 E-1	5.35 E-1	$f[0.8, 0.9, 1, 1.1]$
			$\parallel$ $f[0.9, 1.0, 1.1]$	
1.0	3.103 E-1	8.91 E-1		
		$\parallel$ $f[1.0, 1.1]$		
1.1	3.994 E-1			

$$f[0.8, 0.9] = \frac{f(0.9) - f(0.8)}{0.9 - 0.8}$$

$$f[0.9, 1] = \frac{f(1) - f(0.9)}{1 - 0.9}$$

$$f[0.8, 0.9, 1] = \frac{f[0.9, 1] - f[0.8, 0.9]}{1 - 0.8}$$

$$P_2(x) = f[0.8] + f[0.8, 0.9](x - 0.8) + f[0.8, 0.9, 1](x - 0.8)(x - 0.9)$$

$$= 1.657\text{ E-1} + 6.62\text{ E-1}\,(x - 0.8) + 6.1\text{ E-1}\,(x - 0.8)(x - 0.9)$$

$$P_2(0.85) = 1.973\text{ E-1}$$

$$f(0.85) = 1.974\text{ E-1}$$

$$P_3(x) = P_2(x) + f[0.8, 0.9, 1.0, 1.1](x-0.8)(x-0.9)(x-1)$$

$$P_3(0.85) = P_2(0.85) - 2.5\,E{-1}\,(0.05)(-0.05)(-0.15)$$

$$= 1.973\,E{-1}$$

Remark :- It is possible that interpolation error to increase if we increase number of pts

$$e_n(x) = f(x) - P_n(x) \quad \text{error}$$

it is possible that

$$\max_{x \in [a,b]} |e_{n+1}(x)| > \max_{x \in [a,b]} |e_n(x)|$$

(See example 2.4 pg 44 of your textbook

The error of the interpolating function.

$x_0, x_1, \ldots, x_n$ $n+1$ distinct pts

$f(x_0), f(x_1), \ldots, f(x_n)$ function values

$f : [a,b] \to \mathbb{R}$.

$P_n(x)$ = polyn which interpolates $f(x)$ at $x_0, x_1, \ldots, x_n$

error: $\quad e_n(x) = f(x) - P_n(x)$

Let $\bar{x}$ be distinct from $x_0, x_1, \ldots, x_n$

Need $e_n(\bar{x})$.

Let $P_{n+1}(x)$ be the polynomial which interpolates f at $x_0, x_1, \ldots, x_n, \bar{x}$ ($n+2$ pts)

$$P_{n+1}(x) = P_n(x) + f[x_0, x_1, \ldots, x_n, \bar{x}] \prod_{j=0}^{n} (x - x_j)$$

$$f(\bar{x}) = P_{n+1}(\bar{x}) \qquad \text{by def}^n$$

So $$e_n(\bar{x}) = f(\bar{x}) - P_n(\bar{x})$$
$$= P_{n+1}(\bar{x}) - P_n(\bar{x})$$

$$\boxed{e_n(\bar{x}) = f[x_0, x_1, \ldots, x_n, \bar{x}] \prod_{j=0}^{n} (\bar{x} - x_j)}$$

To estimate error we need to approximate $f[x_0, x_1, \ldots, x_n, \bar{x}]$

and $$\prod_{j=0}^{n} (\bar{x} - x_j)$$

Theorem Let $f(x)$ be a continuous function on $[a,b]$ and k times differentiable in (a,b). If $x_0, x_1, \dots x_k$ are $k+1$ distinct pts in $[a,b]$, then there exists $\xi \in (a,b)$ such that

$$f[x_0, x_1, \dots, x_k] = \frac{f^{(k)}(\xi)}{k!}$$

Proof $k=1$

$$f[x_0, x_1] = \frac{f(x_1) - f(x_0)}{x_1 - x_0} = f'(\xi)$$

by MVT

$$e_k(x) = f(x) - P_k(x)$$

has at least $k+1$ zeros $(x_0, x_1, \dots, x_k)$

by Rolles theorem

$e_k'(x)$ has k zeros

$e_k''(x)$ has $k-1$ zeros

$\vdots$

$e_k^{(k)}(x)$ has atleast one zero in (a, b)

Let ξ be a zero of $e_k^{(k)}(x)$

$$0 = e_k^{(k)}(\xi) = f^{(k)}(\xi) - p_k^{(k)}(\xi)$$

$p_k(x)$ is poly of deg $\leq k$ having $f[x_0, \ldots x_k]$ as coeff of x^k

So $p_k^{(k)}(\xi) = f[x_0, x_1, \ldots, x_k]\, k!$

$$\text{So } f[x_0, \ldots x_k] = \frac{f^{k}(\xi)}{k!}$$

<u>Cor</u>

$$e_n(\bar{x}) = f(\bar{x}) - p_n(\bar{x})$$

$$= \frac{f^{(n+1)}(\xi)}{(n+1)!} \prod_{j=0}^{n} (\bar{x} - x_j)$$

estimating

$$\psi_{n+1}(x) = \prod_{j=0}^{n} (x - x_j)$$

It is possible to choose $x_0, x_1, -, x_n$ in $[a, b]$ such that

$|\psi_{n+1}(x)|$ is as small as possible

This choice of pts are called Chebyshev pts of $[a, b]$

(Unfortunately it is not in syllabus)

Osculatory interpolation

Example

Sometime we have the following situation

we have

$$x_0, x_1, \quad , x_n$$

$$f(x_0), f(x_1), \ldots, f(x_n)$$

and

$$f'(x_0), f'(x_2), \ldots, f'(x_n)$$

We need a polynomial $p(x)$ such that

$$p(x_i) = f(x_i) \qquad i = 0, 1, \ldots, n$$

$$p'(x_i) = f'(x_i) \qquad i = 0, 1, \ldots, n.$$

$\deg p(x) \leq 2n+1$

Example where this happens

$$\frac{dy}{dx} = g(x, y)$$

$$y(x_0) = y_0$$

x_0	$y(x_0)$	$y'(x_0) = g(x_0, y_0)$
x_1	$y(x_1)$	$y'(x_1) = g(x_1, y_1)$
x_2	$y(x_2)$	
$\vdots$		
x_n	$y(x_n)$	$y'(x_n) = g(x_n, y_n)$

find using some numerical method

So for $y(\bar{x})$ we have $2n+2$ data pts.

Remark

$$f[x_0, x_1] = \frac{f(x_1) - f(x_0)}{x_1 - x_0}$$

$$\lim_{x_1 \to x_0} f[x_0, x_1] = f'(x_0)$$

def^n $\quad f[x_0, x_0] = f'(x_0)$

Example :-

$f(1) = 0 \qquad f'(1) = 1$

$f(2) = 6.931$ E-1 $\qquad f'(2) = 0.5$

need cubic polynomial

such that $P_3(x)$

$P_3(1) = f(1), \quad P_3'(1) = f'(1)$

$P_3(2) = f(2), \quad P_3'(2) = f'(2)$

Solution

Set $y_0 = y_1 = 1$

$y_2 = y_3 = 2$

$$f[y_0, y_0] = f'(y_0) = 1$$

$$f[y_1, y_2] = \frac{f(y_2) - f(1)}{y_2 - y_1} = 0.6931$$

$$f[y_2, y_3] = f'(y_2) = 0.5$$

$$f[y_0, y_1, y_2] = \frac{f[y_1, y_2] - f[y_0, y_1]}{y_2 - y_0} = -0.3069$$

$$f[y_1, y_2, y_3] = \frac{f[y_2, y_3] - f[y_1, y_2]}{y_3 - y_1} = -0.1931$$

$$f[y_0, y_1, y_2, y_3] = \frac{f[y_1, y_2, y_3] - f[y_0, y_1, y_2]}{y_3 - y_0} = 0.1137$$

$$P_3(x) = 0 + 1\,(x-1) + (-0.3069)(x-1)^2 + 0.1137\,(x-1)^2(x-2)$$

$x_0, x_1, \ldots\ x_n$

$f(x_0), f(x_1), \ldots\ f(x_n) \qquad f'(x_0), f'(x_1), \ldots, f'(x_n)$

$y_0 = x_0 \quad y_2 = x_1 \quad y_4 = x_2 \ \ldots\ y_{2i} = x_i$

$y_1 = x_0 \quad y_3 = x_1 \quad y_5 = x_2 \qquad y_{2i+1} = x_i$

Use $f[a,b] = f'(a)$ if $b = a$.

y_0	$f[y_0, y_1]$
y_1	$f[y_1, y_2]$
y_2	$f[y_2, y_3]$
y_3	

$$
\begin{aligned}
f_{2n+1}(x) = {} & f[y_0] + f[y_0, y_1](x - y_0) \\
& + f[y_0, y_1, y_2](x-y_0)(x-y_1) + \\
& + f[y_0, y_1, y_2, y_3](x-y_0)(x-y_1)(x-y_2) \\
& + \cdots \\
& + f[y_0, y_1, \ldots, y_{2n+2}] \prod_{i=0}^{2n+1} (x - y_i)
\end{aligned}
$$