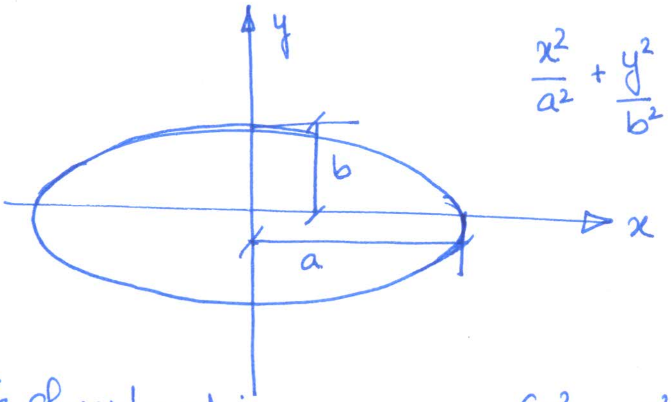
Torsion of elliptical c/s



$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 = 0$$

Prandtl's stress function $\qquad \varphi = m\left(\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1\right)$

$$\frac{\partial^2\varphi}{\partial x^2} + \frac{\partial^2\varphi}{\partial y^2} = -2G\theta$$

$$\Rightarrow \quad \frac{2m}{a^2} + \frac{2m}{b^2} = -2G\theta \Rightarrow m = \frac{-G\theta a^2 b^2}{a^2+b^2}.$$

$$\therefore \quad \varphi = \frac{-G\theta a^2 b^2}{(a^2+b^2)}\left[\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1\right]$$

$\varphi = 0$ on the boundary and hence $\frac{d\varphi}{ds} = 0$

Shear stresses $\qquad \tau_{xz} = \frac{\partial\varphi}{\partial y} = \frac{-2G\theta a^2 y}{(a^2+b^2)}$

$$\tau_{yz} = -\frac{\partial\varphi}{\partial x} = \frac{2G\theta b^2 x}{(a^2+b^2)}$$

Torsional moment $\qquad M_t = 2\iint \varphi \, dx\, dy$

$$= \frac{-2G\theta a^2 b^2}{(a^2+b^2)} \iint \left(\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1\right) dx\, dy$$

$$= \frac{-2G\theta b^2}{(a^2+b^2)} \iint x^2 dx\, dy - \frac{2G\theta a^2}{(a^2+b^2)} \iint y^2 dx\, dy + \frac{2G\theta a^2 b^2}{(a^2+b^2)} \iint dx\, dy$$

$$= \frac{-2G\theta b^2}{(a^2+b^2)} I_y - \frac{2G\theta a^2}{(a^2+b^2)} I_x + \frac{2G\theta a^2 b^2}{(a^2+b^2)} A$$

$$= \frac{G\theta \pi a^3 b^3}{a^2+b^2}$$

$$\therefore \quad M_t = \frac{G\theta \pi a^3 b^3}{a^2 + b^2}$$

$\therefore$ Torsional rigidity $= \dfrac{G\pi a^3 b^3}{a^2 + b^2}$

Shear stresses in terms of M_t

$$\tau_{xz} = \frac{-2M_t y}{\pi a b^3}$$

$$\tau_{yz} = \frac{2M_t x}{\pi a^3 b}$$

Maximum shear stress $\tau_{max} = \dfrac{-2M_t}{\pi a b^2}$

<u>Displacements</u>

$$u = -\theta y z$$

$$v = \theta x z$$

$$w = \frac{M_t (b^2 - a^2) x y}{\pi a^3 b^3 G}$$