Lecture 15

Last time we studied numerical methods to solve system of linear equation

$$a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1$$
$$a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2$$
$$\vdots \qquad \vdots \qquad \qquad \vdots$$
$$a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n = b_n$$

$$A = (a_{ij}) \qquad x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \qquad b = \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix}$$

$$Ax = b$$

We have a two step procedure to solve this

1) (Gauss Elimination, GE)

We convert $Ax = b$ into equivalent system $Ux = \tilde{b}$, where U is upper-triangular.

$U = (u_{ij})$ then $u_{ij} = 0$ for $i < j$

(2) We solve $Ux = \tilde{b}$ by back subsitution

GE

To convert $Ax = b$ into equivalent system $Ux = \tilde{b}$

we do the following

1) we interchang rows
2) we subtract one row with a multiply of another row

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operations required to do GE

$O\left(\frac{n^3}{3}\right)$ multiplication / division

$O\left(\frac{n^3}{3}\right)$ addition / subtractions

operation required for back substitut

$O\left(\frac{n^2}{2}\right)$ multiplication / division

$O\left(\frac{n^2}{2}\right)$ addition / subtraction

Today we learn

LU factorization

steps used to solve a system $Ax=b$ can be used to factor the matrix A. The factorization is particularly useful when it has the form $A = LU$ where L is lower triangular and U is upper triangular.
Although not all matrices have this type of representation, many do that occur in the study of numerical techniques.

Application :- We would want to solve $Ax=b$ for many diff values of b.

If we do GE each time

then we would need $O(n^3/3)$ operation

each time we solve $Ax = b$

On the other hand once $A = LU$

Then we can solve $Ax = b$

as follows

Set $y = Ux$

we solve $LUx = b$

$$Ly = b$$

L is triangular. So determining y requires

$O(n^2)$ operation

Then solving $Ux = y$

requires only $O(n^2)$ operations

Thus the number of operation needed to solve the system $Ax=b$ is reduced from $O(n^3/3)$ to $O(2n^2)$

when $n \geq 100$ this reduces number of computations by more than 99%.

Construction of LU factorization

Suppose $Ax=b$ can be solved without row interchanges

$$Ax=b \qquad A = A^{(1)} = (a_{ij}^{(1)})$$

First step in GE process consists of performing for each $j = 2, 3, \dots, n$ the operations

$$R_j - m_{j1} R_1 \qquad \text{where } m_{j1} = \frac{a_{j1}^{(1)}}{a_{11}^{(1)}}$$

equivalently one can multiply the original matrix A on the left by the matrix

$$M^{(1)} = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ -m_{21} & 1 & 0 & & 0 \\ -m_{31} & 0 & 1 & 0 \cdots & 0 \\ \vdots & & & & \\ -m_{n1} & 0 & 0 & 0 \cdots & 1 \end{bmatrix}$$

Set $A = A^{(1)}$ $\quad b = b^{(1)}$

$$A^{(1)}x = b^{(1)}$$

$$M^{(1)}A^{(1)}x = M^{(1)}b^{(1)}$$

Set $A^{(2)} = M^{(1)}A^{(1)}$ $\quad b^{(2)} = M^{(1)}b^{(1)}$

So we have system

$$A^{(2)}x = b^{(2)}$$

$A^{(2)}$ has $a^{(2)}_{i1} = 0$ for $i \geq 2$.

In a similar manner we construct $M^{(2)}$, the identity matrix with entries below the diagonal in the second column replaced by the negatives of the multipliers

$$m_{j,2} = \frac{a_{j2}^{(2)}}{a_{22}^{(2)}}$$

$A^{(3)} = M^{(2)} A^{(2)}$ has zeros below the diagonal in first 2 columns

$b^{(3)} = M^{(2)} b^{(2)}$

So we have $A^{(3)} x = b^{(3)}$

In general with $A^{(k)} x = b^{(k)}$ already formed, multiply both sides by

$$M^{(k)} = \begin{array}{c} \begin{array}{ccccc} & & (k) & & \end{array} \\ \begin{bmatrix} 1 & & & & \\ & 1 & & & \\ & & 1 & & \\ & & -m_{k+1,k} & & \\ 0 & & \vdots & 0 & \\ & & -m_{n,k} & & 1 \end{bmatrix} \end{array} \begin{array}{c} \\ \\ \\ (k) \\ \\ \\ \\ \end{array}$$

$$A^{(k+1)} = M^{(k)} A^{(k)} = M^{(k)} M^{(k-1)} \cdots M^{(1)} A$$

$$b^{(k+1)} = M^{(k)} b^{(k)} = M^{(k)} M^{(k-1)} \cdots M^{(1)} b$$

So we have $A^{(k+1)} x = b^{(k+1)}$

The process ends with $A^{(n)} x = b^{(n)}$

where

$$A^{(n)} = \begin{pmatrix} a_{11}^{(1)} & a_{12}^{(1)} & \cdots & a_{1n}^{(1)} \\ 0 & a_{22}^{(2)} & \cdots & a_{2n}^{(2)} \\ \vdots & & & \\ 0 & 0 & \cdots & a_{nn}^{(n)} \end{pmatrix}$$

is upper triangular

Set $U = A^{(n)}$

Therefore $U = \underline{M^{(n-1)} M^{(n-2)} \cdots M^{(1)}} A$

Set $L = \left[M^{(n-1)} M^{(n-2)} \cdots M^{(1)} \right]^{-1}$

$= M^{(1)^{-1}} M^{(2)^{-1}} \cdots M^{(n-1)^{-1}}$

$$M^{(1)} = \begin{bmatrix} 1 & & & \\ -m_{21} & 1 & & \\ -m_{31} & & \ddots & \\ & & & 1 \\ -m_{n1} & & & \end{bmatrix}$$

$$M^{(1)^{-1}} = \begin{bmatrix} 1 & & & \\ m_{21} & 1 & & \\ m_{31} & & \ddots & \\ & & & 1 \\ m_{n1} & & & \end{bmatrix}$$

and so on.

One can prove that

$$L = \begin{pmatrix} 1 & & & \\ m_{21} & 1 & & \\ m_{31} & m_{32} & & \\ \vdots & & & \\ m_{n1} & m_{n2} & \cdots & 1 \end{pmatrix}$$

Example

$$\begin{aligned} x_1 + x_2 + 0x_3 + 3x_4 &= 4 \\ 2x_1 + x_2 - x_3 + x_4 &= 1 \\ 3x_1 - x_2 - x_3 + 2x_4 &= -3 \\ -x_1 + 2x_2 + 3x_3 - x_4 &= 4 \end{aligned}$$

Step 1 $\quad R_2 - 2R_1, \quad R_3 - 3R_1, \quad R_4 + R_1$

given $A^{(2)} = \begin{pmatrix} 1 & 1 & 0 & 3 \\ 0 & -1 & -1 & -5 \\ 0 & -4 & -1 & -7 \\ 0 & 3 & 3 & 2 \end{pmatrix}$

$$L = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 3 & & 1 & 0 \\ -1 & & & 1 \end{pmatrix}$$

<u>Step 2</u> $R_3 - 4R_2 \qquad R_4 + 3R_2$

$$A^{(3)} = \begin{pmatrix} 1 & 1 & 0 & 3 \\ 0 & -1 & -1 & -5 \\ 0 & 0 & 3 & 13 \\ 0 & 0 & 0 & -13 \end{pmatrix} = U$$

$$L = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 3 & 4 & 1 & 0 \\ -1 & -3 & \underline{\underline{0}} & 1 \end{pmatrix}$$ note

solve

$$Ax = b = \begin{pmatrix} 8 \\ 7 \\ 14 \\ -7 \end{pmatrix}$$

Set $y = Ux$

$$Ly = b$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 3 & 4 & 1 & 0 \\ -1 & -3 & 0 & 1 \end{bmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{pmatrix} = \begin{pmatrix} 8 \\ 7 \\ 14 \\ -7 \end{pmatrix}$$

$$y_1 = 8$$

$2y_1 + y_2 = 7$ $\quad y_2 = 7 - 16 = -9$

$3y_1 + 4y_2 + y_3 = 14$ $\quad$ so $y_3 = 26$

$-y_1 - 3y_2 + y_4 = -7$ $\quad$ so $y_4 = -26$

We then solve $Ux = y$

$$\begin{bmatrix} 1 & 1 & 0 & 3 \\ 0 & -1 & -1 & -5 \\ 0 & 0 & 3 & 13 \\ 0 & 0 & 0 & -13 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{pmatrix} 8 \\ -9 \\ 26 \\ -26 \end{pmatrix}$$

We use "back substitution

$$-13x_4 = -26 \qquad \text{so } x_4 = 2$$

$$3x_3 + 13x_4 = 26 \implies x_3 = 0$$

$$-x_2 - x_3 - 5x_4 = -9 \implies x_2 = -1$$

$$x_1 + x_2 + 3x_4 = 8 \implies x_1 = 3$$

Two classes of matrices for which Gauss elimination can be performed effectively without row interchanges

(1) strictly diagonally dominant matrix

(2) positive definite matrices

Recall

An $n \times n$ matrix $A = (a_{ij})$ is said to be strictly diagonally dominant if

$$|a_{ii}| > \sum_{\substack{j=1 \\ j \neq i}}^{n} |a_{ij}|$$

for each $i = 1, 2, \dots, n$

Example

$$A = \begin{bmatrix} 7 & 2 & 0 \\ 3 & 5 & -1 \\ 0 & 5 & -6 \end{bmatrix}$$

A is strictly diagonally dominant

Note $$A^t = \begin{bmatrix} 7 & 3 & 0 \\ 2 & 5 & 5 \\ 0 & -1 & -6 \end{bmatrix}$$

A^t is not strictly diagonally dominant

Theorem A strictly diagonally dominant matrix A is non-singular.

Proof We prove by contradiction

Suppose A is singular.

So there exists $x \neq 0$ such that

$$Ax = 0$$

$$x = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$$

Let k be index for which

$$0 < |x_k| = \max_{1 \le j \le n} |x_j|$$

Since $\sum_{j=1}^{n} a_{ij} x_j = 0$ for each $i = 1, \dots, n$

we have when $i = k$

$$\sum_{j=1}^{n} a_{kj} x_j = 0$$

So $a_{kk} x_k = -\sum_{\substack{j=1 \\ j \neq k}}^{n} a_{kj} x_j$

So

$$|a_{kk}|\,|x_k| \leq \sum_{\substack{j=1 \\ j \neq k}}^{n} |a_{kj}|\,|x_j|$$

or $|a_{kk}| \leq \sum_{\substack{j=1 \\ j \neq k}}^{n} |a_{kj}| \frac{|x_j|}{|x_k|} \leq \sum_{\substack{j=1 \\ j \neq k}}^{n} |a_{kj}|$

This inequality contradicts the strict diagonal dominance of A

Positive definite matrices

A matrix A is positive definite if

1) it is symmetric, i.e $A^t = A$
2) $x^t A x > 0$ for every n-dim vector $x \neq 0$

Remark A positive definite matrix is non-singular

for $Ax = 0$

$\Rightarrow x^t A x = 0$

$\Rightarrow x = 0$