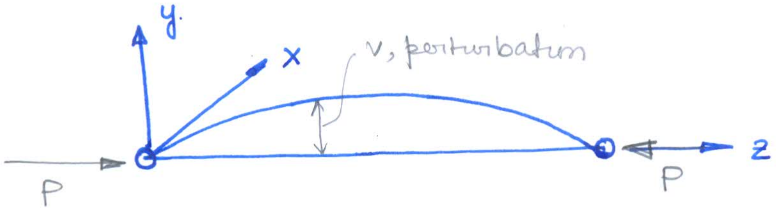
Euler theory of buckling



pinned - pinned support condn.

Moment balance equation:

$$EI\frac{d^2v}{dz^2} = -Pv \Rightarrow EI\frac{d^2v}{dz^2} + Pv = 0$$

$$\Rightarrow \frac{d^2v}{dz^2} + \mu^2 v = 0 \quad \text{where } \mu^2 = \frac{P}{EI}$$

$$V(z) = A\cos\mu z + B\sin\mu z$$

Boundary Conditions

@ $z=0$, $V=0$ $\quad A=0$

@ $z=l$, $V=0$ $\quad B\sin\mu l = 0$

$$\Rightarrow \mu l = n\pi \Rightarrow \frac{P}{EI} = \frac{n^2\pi^2}{l^2}$$

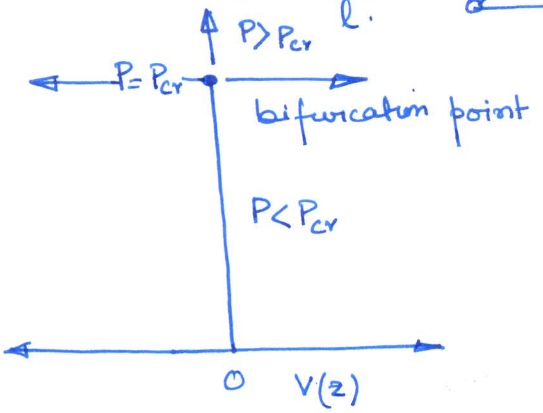
$$\Rightarrow \boxed{P_{cr} = \frac{n^2\pi^2 EI}{l^2}}$$

$P_{cr} = \dfrac{\pi^2 EI}{l^2}$ (first buckling mode)

$V(z) = A_1 \sin\dfrac{\pi z}{l}$

$P_{cr} = \dfrac{4\pi^2 EI}{l^2}$ (second buckling mode)

$V(z) = A_2 \sin\dfrac{2\pi z}{l}$



$$\sigma_{cr} = \frac{\pi^2 E}{(l/r)^2}$$

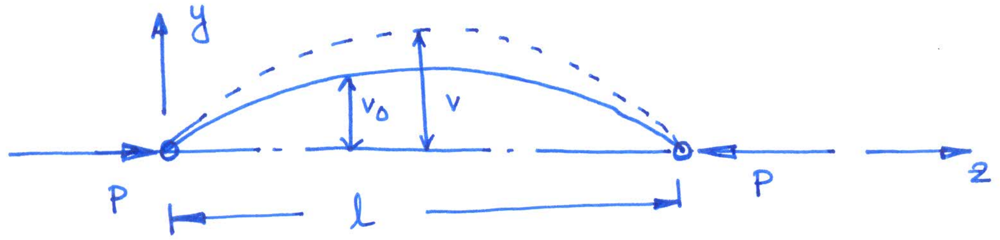
r is the radius of gyration.

For other support condns:

$P_{cr} = \dfrac{4\pi^2 EI}{l^2}$ (fixed-fixed support).

$P_{cr} = \dfrac{\pi^2 EI}{4l^2}$ (fixed - free support)

Effect of initial imperfection



Moment balance equation:

$$EI\frac{d^2v}{dz^2} - EI\frac{d^2v_0}{dz^2} = -Pv$$

Assuming

$$v_0(z) = \sum_{n=1}^{\infty} A_n \sin\frac{n\pi z}{l} \qquad \text{——— (2)}$$

$$\Rightarrow \quad EI\frac{d^2v}{dz^2} + Pv = EI\frac{d^2v_0}{dz^2}$$

$$\Rightarrow \quad \frac{d^2v}{dz^2} + \mu^2 v = \frac{d^2v_0}{dz^2} \qquad \text{where } \mu^2 = \frac{P}{EI} \qquad \text{——— (1)}$$

Substituting (2) in (1), we get,

$$\frac{d^2v}{dz^2} + \mu^2 v = -\sum \frac{A_n n^2\pi^2}{l^2}\sin\frac{n\pi z}{l}$$

$$\Rightarrow \quad v(z) = A\cos\mu z + B\sin\mu z + \sum \frac{n^2 A_n}{n^2-\alpha^2}\sin\frac{n\pi z}{l} \qquad \alpha^2 = \frac{\mu^2 l^2}{\pi^2} = \frac{Pl^2}{\pi^2 EI} = \frac{P}{P_{cr}}$$

Imposing boundary conditions, @ $z=0$, $v=0$ and @ $z=l$, $v=0$

$A = 0$ and $B = 0$

$$\Rightarrow \quad v(z) = \sum \frac{n^2 A_n}{n^2-\alpha^2}\sin\frac{n\pi z}{l}$$

as $P \to P_{cr}$, $\alpha \to 1$

$\therefore$ The first term in the series corresponding to $n=1$ will dominate.

$$\therefore \quad v(z) \simeq \frac{A_1}{1-\alpha^2}\sin\frac{\pi z}{l} \quad \text{as } P \to P_{cr}$$

Deflection at the center of the column $v(z=l/2) = \dfrac{A_1}{1-\alpha^2}$.

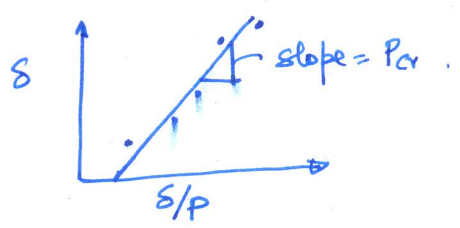
Denoting

$$\delta = v(z=l/2) - A_1$$

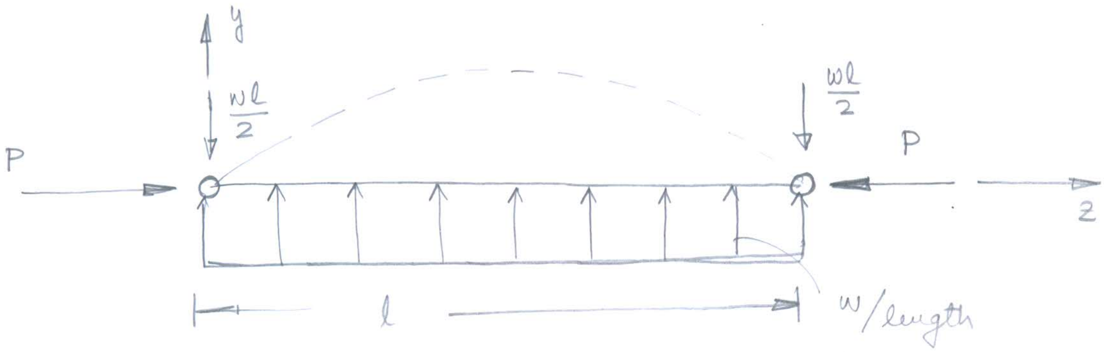
$$= \frac{A_1}{1-P/P_{cr}} - A_1$$

$$\Rightarrow \quad \delta = \frac{\delta P_{cr}}{P} - A_1 \qquad \text{——— (3)}$$

Plotting eqn (3) when $P \to P_{cr}$, we get Southwell plot



Stability of beams under transverse and axial loads



Moment balance eqn:

$$Pv + \frac{wlz}{2} - \frac{wz^2}{2} = -EI\frac{d^2v}{dz^2}.$$

$$\Rightarrow \quad \frac{d^2v}{dz^2} + \mu^2 v = \frac{w}{2EI}\left(z^2 - lz\right) \quad \text{where } \mu^2 = \frac{P}{EI}$$

$$v(z) = A\cos\mu z + B\sin\mu z + \frac{w}{2P}\left(z^2 - lz - \frac{2}{\mu^2}\right)$$

Imposing the BCs: @ $z=0$, $v=0$ and @ $z=l$, $v=0$, gives

$$A = \frac{w}{\mu^2 P} \quad \text{and} \quad B = \frac{w}{\mu^2 P\sin\mu l}\left(1-\cos\mu l\right)$$

$$v(z) = \frac{w}{\mu^2 P}\left[\cos\mu z + \left(\frac{1-\cos\mu l}{\sin\mu l}\right)\sin\mu z\right] + \frac{w}{2P}\left(z^2 - lz - \frac{2}{\mu^2}\right).$$

Max^m deflection occurs at the mid-point of the beam

$$V_{max} = \frac{w}{\mu^2 P}\left(\sec\frac{\mu l}{2} - 1\right) - \frac{wl^2}{8P}.$$

$$M_{max} = \frac{w}{\mu^2}\left(1 - \sec\frac{\mu l}{2}\right)$$

$$= \frac{wl^2}{\pi^2}\,\frac{P_{cr}}{P}\left(1 - \sec\frac{\pi}{2}\sqrt{\frac{P}{P_{cr}}}\right) \qquad P_{cr} = \frac{\pi^2 EI}{l^2}.$$