# Lecture 7

Last time I introduced

piecewise-polynomial interpolation

We did

- piecewise linear interpolation
- piecewise cubic interpolation

$\begin{cases} \text{piecewise-cubic Hermite} \checkmark \\ \text{piecewise cubic spline} \quad \text{(we will do today)} \end{cases}$

**Piecewise-cubic Hermite**

$a = x_1 < x_2 < \cdots < x_{N+1} = b$

in $[x_i, x_{i+1}]$ $g_i(x)$ is given by polynomial $P_i(x)$ which interpolates $f$ at

$x_i, x_i, x_{i+1}, x_{i+1}$

i.e.,

$$P_i(x_i) = f(x_i), \quad P_i'(x_i) = f'(x_i)$$

$$P_i(x_{i+1}) = f(x_{i+1}), \quad P_i'(x_{i+1}) = f'(x_{i+1})$$

So

$$P_i(x) = f(x_i) + f[x_i, x_i](x - x_i) + f[x_i, x_i, x_{i+1}](x - x_i)^2 + f[x_i, x_i, x_{i+1}, x_{i+1}](x - x_i)^2 (x - x_{i+1})$$

$x \in [x_i, x_{i+1}]$

The piecewise-cubic Hermite polynomial $g_3(x)$ is continuously diff in $[a, b]$

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piecewise cubic-spline interpolation is twice continuously differentiable on $[a, b]$

# Cubic spline interpolation

**Def$^n$** Given a function $f$ defined on $[a,b]$ and a set of nodes

$a = x_0 < x_1 < \cdots < x_n = b$ a cubic spline interpolant $S$ for $f$ is a function that satisfies the following conditions:

1) $S(x)$ is a cubic polynomial, denoted by $S_j(x)$ on the subinterval $[x_j, x_{j+1}]$ for each $j = 0, 1, \ldots, n-1$

2) $S_j(x_j) = f(x_j)$, $S_j(x_{j+1}) = f(x_{j+1})$ for each $j = 0, 1, \ldots, n-1$

3) $S_{j+1}(x_{j+1}) = S_j(x_{j+1})$ for each $j = 0, 1, \ldots, n-2$.

4) $S'_{j+1}(x_{j+1}) = S'_j(x_{j+1})$ for $j = 0, 1, \ldots, n-2$

5) $S''_{j+1}(x_{j+1}) = S''_j(x_{j+1})$ for $j = 0, 1, \ldots, n-2$

(6) One of the following sets of boundary condition is satisfied

(i) $S''(x_0) = S''(x_n) = 0$ (free boundary)

(ii) $S'(x_0) = f'(x_0)$, $S'(x_n) = f'(x_n)$ (clamped boundary)

## Construction of cubic spline interpolate

on $[x_j, x_{j+1}]$

$$S_j(x) = a_j + b_j(x-x_j) + c_j(x-x_j)^2 + d_j(x-x_j)^3$$

$$S_j(x_j) = f(x_j). \quad \text{So} \quad a_j = f(x_j).$$

$$a_{j+1} = S_j(x_{j+1}) = a_j + b_j(x_{j+1}-x_j) + c_j(x_{j+1}-x_j)^2 + d_j(x_{j+1}-x_j)^3$$

set $h_j = x_{j+1} - x_j$

$$\boxed{a_{j+1} = a_j + b_j h_j + c_j h_j^2 + d_j h_j^3} \longrightarrow (1)$$

$$S_j'(x) = b_j + 2c_j(x-x_j) + 3d_j(x-x_j)^2$$

$$S_j'(x_j) = b_j$$

So $b_{j+1} = S_{j+1}'(x_{j+1}) = S_j'(x_{j+1})$

$$\boxed{b_{j+1} = b_j + 2c_j h_j + 3d_j h_j^2} \longrightarrow (2)$$

define $b_n = S'(x_n)$.

$$S_j''(x) = 2c_j + 6d_j(x-x_j)$$

$$c_j = S_j''(x_j)/2$$

$$c_{j+1} = S_{j+1}''(x_{j+1})/2 = \frac{S_j''(x_{j+1})}{2}$$

So $\boxed{c_{j+1} = c_j + 3d_jh_j} \longrightarrow (3)$

define $c_n = S''(x_n)/2$

By 3 $\quad d_j = \frac{1}{3h_j}(c_{j+1} - c_j)$

By 1

$$a_{j+1} = a_j + b_jh_j + c_jh_j^2 + d_jh_j^3$$

$$= a_j + b_jh_j + c_jh_j^2 + \frac{1}{3}(c_{j+1} - c_j)h_j^2$$

$$\boxed{a_{j+1} = a_j + b_jh_j + \frac{1}{3}(2c_j + c_{j+1})h_j^2} \rightarrow (4)$$

plugging value of $d_j$ in (2) we get

$$b_{j+1} = b_j + 2c_jh_j + h_j(c_{j+1} - c_j)$$

$$\boxed{b_{j+1} = b_j + h_j(c_j + c_{j+1})} \longrightarrow (5)$$

by (4) we get

$$b_j = \frac{1}{h_j}(a_{j+1} - a_j) - \frac{h_j}{3}(2c_j + c_{j+1}) \longrightarrow (6)$$

$$b_{j-1} = \frac{1}{h_{j-1}}(a_j - a_{j-1}) - \frac{h_{j-1}}{3}(2c_{j-1} + c_j)$$

by 5 $\quad b_j = b_{j-1} + h_{j-1}(c_{j-1} + c_j)$

so

$$\frac{1}{h_j}(a_{j+1} - a_j) - \frac{h_j}{3}(2c_j + c_{j+1})$$

$$= \frac{1}{h_{j-1}}(a_j - a_{j-1}) - \frac{h_{j-1}}{3}(2c_{j-1} + c_j)$$

$$+ h_{j-1}(c_{j-1} + c_j)$$

$$\frac{1}{3}h_{j-1}c_{j-1} + \frac{1}{3}(2h_{j-1} + 2h_j)c_j + \frac{h_j}{3}c_{j+1}$$

$$= \frac{1}{h_j}(a_{j+1} - a_j) - \frac{1}{h_{j-1}}(a_j - a_{j-1})$$

$$\boxed{h_{j-1}c_{j-1} + 2(h_{j-1} + h_j)c_j + h_j c_{j+1} = \frac{3}{h_j}(a_{j+1} - a_j) - \frac{3}{h_{j-1}}(a_{j-1} - a_{j-1})}$$

$$[\quad j = 1, 2, \dots, n-1 \quad]$$

$n-1$ equations and $n+1$ unknowns $c_0, \dots, c_n$

Case 1 free boundary

$$S''(x_0) = S''(x_n) = 0$$

$$0 = 2c_0 + 6d_0(x_0 - x_0) \quad \Big\| \quad S''(x_0) = 0$$

So $c_0 = 0$

$$c_n = \frac{S''(x_n)}{2} = 0$$

So we have a system

$$Ax = b$$

where $A$ is the $(n+1) \times (n+1)$ matrix

$$\begin{bmatrix} 1 & 0 & 0 & \cdots & & & 0 \\ h_0 & 2(h_0+h_1) & h_1 & 0 & \cdots & & 0 \\ 0 & h_1 & 2(h_1+h_2) & h_2 & \cdots & & 0 \\ & & & & & & \vdots \\ & & & & h_{n-2} & 2(h_{n-2}+h_{n-1}) & h_{n-1} \\ 0 & 0 & 0 & 0 & \cdots & & 1 \end{bmatrix}$$

$$X = \begin{bmatrix} c_0 \\ c_1 \\ \vdots \\ c_n \end{bmatrix}$$

$$b = \begin{bmatrix} 0 \\ \frac{3}{h_1}(a_2 - a_1) - \frac{3}{h_0}(a_1 - a_0) \\ \vdots \\ \frac{3}{h_{n-1}}(a_n - a_{n-1}) - \frac{3}{h_{n-2}}(a_{n-1} - a_{n-2}) \\ 0 \end{bmatrix}$$

A is strictly diagonally dominant

A matric $T = (t_{ij})_{n \times n}$ is called strictly diagonally dominant if

$$|t_{ii}| > \sum_{\substack{j=1 \\ j \neq i}}^{n} |t_{ij}|$$

So A is invertible.

So can solve $c_0, c_1, \dots, c_n$

and then obtain $d_0, d_1, \dots, d_{n-1}$

and $b_0, b_1, \dots, b_n$

## Example :-

We approximate $f(x) = e^x$ in the interval $[0, 3]$

$x_0 = 0 \quad x_1 = 1 \quad x_2 = 2 \quad x_3 = 3$

$f(x_0) = 1 \quad f(x_1) = e \quad f(x_2) = e^2 \quad f(x_3) = e^3$

Find cubic-spline with free boundary

Ans $n = 3$

$h_0 = h_1 = h_2 = 1$

$a_0 = 1 \quad a_1 = e \quad a_2 = e^2 \quad a_3 = e^3$

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 4 & 1 & 0 \\ 0 & 1 & 4 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\underline{b} = \begin{bmatrix} 0 \\ 3(e^2 - 2e + 1) \\ 3(e^3 - 2e^2 + e) \\ 0 \end{bmatrix}$$

$$\underline{X} = \begin{bmatrix} c_0 \\ c_1 \\ c_2 \\ c_3 \end{bmatrix}$$

$c_0 = 0 \qquad c_3 = 0$

$$4c_1 + c_2 = 3(e^2 - 2e + 1)$$

$$c_1 + 4c_2 = 3(e^3 - 2e^2 + e)$$

$c_1 = 0.7569 \qquad c_2 = 5.83$

$$b_0 = \frac{1}{h_0}(a_1 - a_0) - \frac{h_0}{3}(2c_0 + c_1)$$

$$= a_1 - a_0 - \frac{1}{3}c_1$$

$$= 1.466$$

$$b_1 = \frac{1}{h_1}(a_2 - a_1) - \frac{h_1}{3}(c_2 + 2c_1)$$

$$= 2.223$$

$$b_2 = \frac{1}{h_2}(a_3 - a_2) - \frac{h_2}{3}(c_3 + 2c_2)$$

$$= 8.81$$

$$d_0 = \frac{1}{3h_0}(c_1 - c_0) = 0.2523$$

$$d_1 = \frac{1}{3h_0}(c_2 - c_1) = 1.691$$

$$d_2 = \frac{1}{3h_2}(c_3 - c_2) = -1.943$$

$$s(x) = \begin{cases} 1 + 1.466x + 0x^2 + 0.2523x^3 & 0 \le x \le 1 \\ 2.718 + 2.223(x-1) + 0.7569(x-1)^2 + 1.691(x-1)^3 & 1 \le x \le 2 \\ 7.389 + 8.80(x-2) + 5.83(x-2)^2 - 1.943(x-2)^3 & 2 \le x \le 3 \end{cases}$$

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Case 2 clamped boundary

$$s'(x_0) = f'(x_0) \quad \& \quad s'(x_n) = f'(x_n)$$

$$f'(a) = s'(a) = b_0$$

Put this in eqn 6

$$f'(a) = \frac{1}{h_0}(a_1 - a_0) - \frac{h_0}{3}(2c_0 + c_1)$$

So $$\boxed{2h_0 c_0 + h_0 c_1 = \frac{3}{h_0}(a_1 - a_0) - 3f'(a)}$$

Similarly

$$f'(b) = b_n = b_{n-1} + h_{n-1}(c_{n-1} + c_n)$$

So eqn 6 with $j = n-1$

gives

$$f'(b) = \frac{1}{h_{n-1}}(a_n - a_{n-1}) - \frac{h_{n-1}}{3}(2c_{n-1} + c_n) + h_{n-1}(c_{n-1} + c_n)$$

Simplify to get

$$\boxed{h_{n-1}c_{n-1} + 2h_{n-1}c_n = 3f'(b) - \frac{3}{h_{n-1}}(a_n - a_{n-1})}$$

Thus we obtain

$$A\underline{x} = \underline{b} \quad \text{where}$$

$$\underline{x} = \begin{pmatrix} c_0 \\ c_1 \\ \vdots \\ c_n \end{pmatrix}$$

$$A = \begin{bmatrix} 2h_0 & h_0 & 0 & \cdots & & 0 \\ h_0 & 2(h_0+h_1) & h_1 & 0 & \cdots & 0 \\ 0 & h_1 & 2(h_1+h_2) & h_2 & \cdots & 0 \\ & & & & & \\ 0 & \cdots & & h_{n-2} & 2(h_{n-2}+h_{n-1}) & h_{n-1} \\ & & & & h_{n-1} & 2h_{n-1} \end{bmatrix}$$

$$\underset{\sim}{f} = \begin{bmatrix} \frac{3}{h_0}(a_1-a_0) - 3f'(a) \\ \frac{3}{h_1}(a_2-a_1) - \frac{3}{h_0}(a_1-a_0) \\ \vdots \\ \frac{3}{h_{n-1}}(a_n-a_{n-1}) - \frac{3}{h_{n-2}}(a_{n-1}-a_{n-2}) \\ 3f'(b) - \frac{3}{h_{n-1}}(a_n-a_{n-1}) \end{bmatrix} = \begin{bmatrix} 3(e-1) - 3 \\ 3(e^2-e) - 3(e-1) \\ 3(e^3-e^2) - 3(e^2-e) \\ 3e^3 - 3(e^3-e^2) \end{bmatrix}$$

The matrix $A$ is strictly diagonally dominant.

So $A$ is invertible.

Thus we can obtain $c_0, \dots, c_n$ by solving $Ax = \underline{b}$.

Then one can obtain $b_0, \dots, b_{n-1}$ and $d_0, \dots, d_{n-1}$

by using $\underline{c} = c_0, c_1, \dots, c_n$

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<u>Example</u> $f(x) = e^x$

interval $[0, 3]$

$x_0 = 0,\ x_1 = 1,\ x_2 = 2,\ x_3 = 3$

$f(x_0) = 1 \quad f(x_1) = e \quad f(x_2) = e^2 \quad f(x_3) = e^3$

$f'(x_0) = 1 \quad f'(x_3) = e^3$

find clamped cubic spline interpolant of $f$

$$A = \begin{bmatrix} 2 & 1 & 0 & 0 \\ 1 & 4 & 1 & 0 \\ 0 & 1 & 4 & 1 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

$$x = \begin{pmatrix} c_0 \\ c_1 \\ c_2 \\ c_3 \end{pmatrix} \qquad \underline{b} = \begin{bmatrix} 2\cdot 13 \\ \\ \\ \\ \end{bmatrix}$$

Files / Lectures