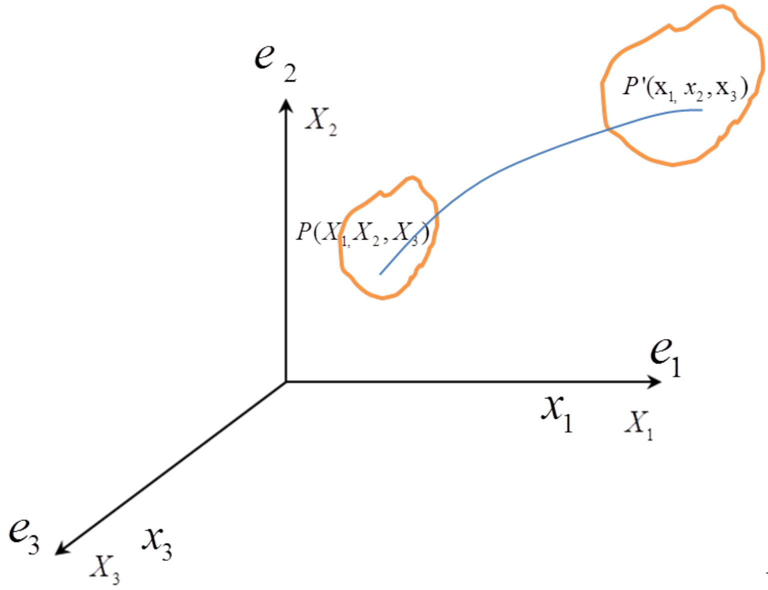
Kinematics

Description of motion of a continuum

Rectangular Cartesian Coordinate system

Let a body occupy a system in space at time $t = t_0$.What are we interested in?
a. Motion of this body
b. Deformation of this body



What do we need? to identify infinitely many points of particles.Material particles are identified as X_i with respect to a fixed rectangular Cartesian coordinate system at $t = t_0$.

Under motion of the body material point P moves to P' whose coordinates w.r.t. fixed rectangular Cartesian coordinate are x_i.Then the equation

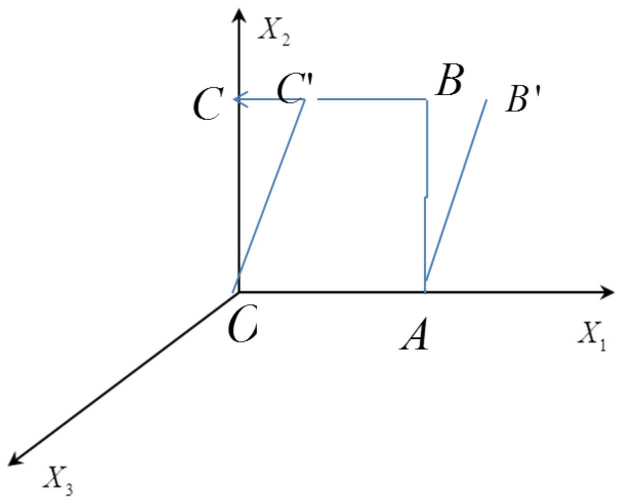
$$x_i = X_i \left(X_1, X_2, X_3, t\right) \tag{1}$$

describes the motion of the particle.
Configuration at
a. $t = t_0$——Reference configuration.
b. $t = t$——Present configuration. Eq. (1) gives the path line of the material point$P$

$$X_i = x_i(x_1, x_2, x_3, t_0) \tag{2}$$

—-verify that the material point P occupied the place X_i at $t = t_0$. In continuum mechanics it is common to use the same symbol for a function and its value.

Ex 1.
$x_1 = X_1 + .2 * t * X_2$
$x_2 = X_2$
$x_3 = X_3$ or equivalently $x_i = X_i + 0.2tX_2\delta_{i1}$
X_1, X_2 *and* X_3 gives value/position of material particle at $t = 0$.Sketch the configuration at time $t = 2$ for a body, which at $t = 0$ has the shape of a cube of unit sides with one corner at the origin.



at origin $x_i = 0\ \forall t$
at $(X_1, 0, 0)$ $x_i = X_1\delta_{1i}$
⇒particles on line OA do not move.
at $(X_1, 1, 0)$ on line CB (horizontal movement)
at $t = 2$; $x_i = [X_1 + (0.2)(2)(1)]\delta_{1i} + (1)\delta_{2i}$.

For particle (0,X_2,0)

$$x_i = [0 + (0.2)(2)X_2]\delta_{1i} + X_2\delta_{2i} = .04X_2\delta_{1i} + X_2\delta_{2i}; \tag{3}$$

–horizontal movement to right
Simple shearing motion

$X_1 = x_1 - 0.2tx_2$ location @$t = 0$ $X_2 = x_2; X_3 = x_3$

Motion $\rightarrow$ two parallel flat plate with the bottom one fixed and the upper one moved only along the X_1-axis.Viscous fluid flow between two parallel plates.

Referential and spatial description

quantities associated with specific material points change with time
Example: $\theta = \theta(X_1, X_2, X_3, t)$; $v_i = v_i(X_1, X_2, X_3, t)$;
Two ways to describe it:

0.1 Lagrangian description

Follow the material particle i.e. express the quantities as functions of the coordinate of a material particle in a fixed reference configuration.Also known as material description.

0.2 Eulerian description

Observe quantities at fixed locations in space.
$\theta = \theta(x_1, x_2, x_3, t)$; $v_i = v_i(x_1, x_2, x_3, t)$— this are the initial descriptions and no information of a particle material point.

Example: Given motion $x_i = X_i + .2tX_2\delta_{1i}$ and Temperature field is $\theta = 2x_i + x_i^2$
a. Obtain referential description of temperature
b. Rate of change of temperature of the material particle at a time $t = 0$ at the place (0,1,0)

$\theta = 2(X_1 + 0.2tx_2) + (x_2)^2 = 2X_1 + (X_2 + 0.4t)X_2$ at $t = 0$; (0,1,0) $\theta = 1 + 0.4t$ and $\frac{d\theta}{dt} = 0.4$
spatial $\theta \rightarrow$ independent of time and Referential $\Rightarrow$shows that in actually θ changes from one spatial position from another with time.

Displacement Vector

By definition, the displacement vector of a material particle is the difference between its position vectors at time t and at time $t = t_0$ (or 0) :
$u_i = x_i - X_i$
In Lagrangian description of motion, the displacement u_i is specified as a function of X_i and t.
For example,
$x_1 = X_1t^2 + 2X_2t + X_1$
$x_2 = 2X_1t^2 + X_2t + X_2$
$x_3 = X_3t/2 + x_3$

during the time interval $0 \leq t \geq 1$. The corresponding displacement components are given by:
$u_1 = x_1 - X_1 = X_1 t^2 + 2X_2 t$ and so on.

In Eulerian description, u_i will be expressed as a function of x_i and t :
$u_1 = x_1 - X_1 = x_1 - \{ \qquad \}$
Continuous deformation of a deformable body
at $t = 1$ and $x_1 = 2(X_1 + X_2); x_2 = 2(X_1 + X_2); X_3 = 3X_3/2$
material particle at all occupy (0,0,0)—-Which is not possible in continuum mechanics.
Collision of particles not allowed
different particles occupy distinct places
Thus, $x_i = x_i(X_1, X_2, X_3, t)$ has one to one mapping from reference to present configuration. $\Rightarrow$ Mapping is continuously differentiable and continuously differentiable inverse. $X_i = X_i(x_1, x_2, x_3, t)$ if and only if,
J= det $[\frac{\partial x_i}{\partial x_j}]$=$det[\delta_{ij} + \frac{\partial u_i}{\partial x_j}$

$J = (X_1, X_2, X_3, 0) = 1$ and J is continuous function of t.Hence, J must be positive for every t.
$\Rightarrow J > 0$ for continuous deformation to be physical admissible.

Material derivative

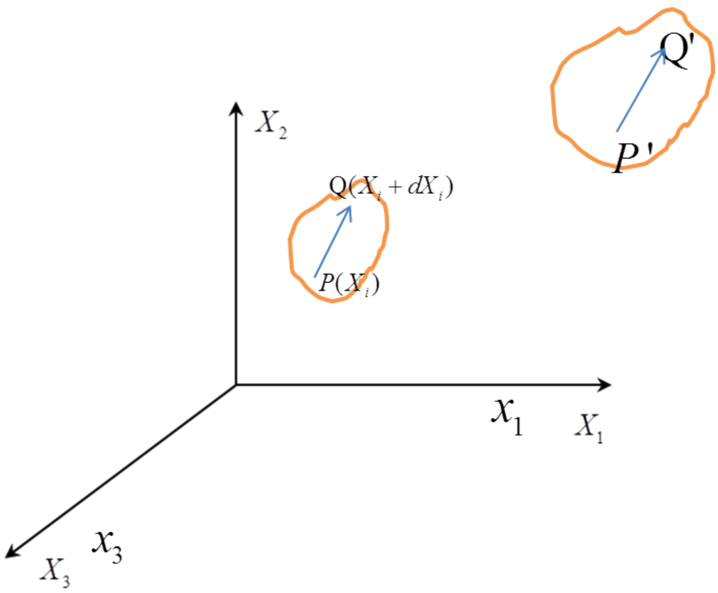
Time rate of change of a quantity of a material particle.
Ex:$\frac{D(\theta)}{Dt}$. Depends on Lagrange or Eulerian description.
1.Lagrangian description:
$\theta = \theta(X_1, X_2, X_3, t)$
$\dot{\theta} = \frac{D(\theta)}{Dt} = \frac{\partial \theta}{\partial t}\Big|_{x_i fixed}$ 2.Spatial or Eulerian description:
$\theta = \tilde{\theta}(X_1, X_2, X_3, t)$; $x_i = x_i(X_1, X_2, X_3, t)$
so, $\dot{\theta} = \frac{D(\tilde{\theta})}{Dt} = \frac{\partial \tilde{\theta}}{\partial t}\Big|_{x_i fixed} + \frac{\partial \tilde{\theta}}{\partial x_j} \frac{\partial x_j}{\partial t}\Big|_{x_i fixed}$

Example: Given the motion, $x_i = X_i(1 + t) 0 \leq t \geq 1$
Find the spatial description of the velocity field.[$v_i = \dot{x}_i = X_i = x_i/(1 + t)$]

Given temperature field $\theta = 2(x_1^2 + x_2^2$ where $x_i = X_i(1 + t)$. Find at $t = 1$, the rate of change of temperature of the particle, which in reference configuration was at (1,1,1)
Express θ as function of X_i find $\dot{\theta}$ and substitute X and θ at $t = 1$
$\dot{\theta}$ as function of x_i and t find x_i at $t = 1$ and then substitute
$\dot{\theta} = \frac{\partial \theta}{\partial t} + \frac{\partial \theta}{\partial x_j} \partial x_j \partial t = 0 + 4x_1 x_1/(1 + t) + 4x_2 x_2/(1 + t)$
now finding out the value at $t = 1 \quad (1, 1, 1)$

Example: A fluid rotates as a rigid body with a constant angular velocity $\omega = \omega e_3$.Write explicit component of velocity of a material point in the Eulerian description of motion.
Note: $v = \omega \times r = \omega e_3 \times r = \epsilon_{i3k}\omega x_k$
$\Rightarrow \tilde{v}_i = \epsilon_{ijk}\omega_i r_j e_k = \epsilon_{i3k}\omega x_k$
$a_i = \frac{\partial \tilde{v}_i}{\partial t} + \frac{\partial \tilde{v}_i}{\partial x_j}\frac{\partial x_j}{\partial t}$
$= 0 + \epsilon_{i3k}\omega \frac{\partial x_k}{\partial x_j}\tilde{v}_j$
$= \omega^2\{\epsilon_{i3k}\epsilon_{3mj}x_m\}$
$= \omega^2\{\delta_{i3}\delta_{3m} - \delta_{im}\delta_{33}\}x_m$
$a_i = \omega^2\{\delta_{i3}x_3 - x_i\}$
Example:
Find the acceleration for the case of simple shear deformation.$x_i = X_i + 0.2tX_2\delta_{i1}[v_i = \dot{x}_i = 0.2(X_2\delta_{i1}]$
$a_i = \dot{v}_i = 0.$

Deformation Gradient



Let the motion of the body be given by, $x_i = x_i(X_1, X_2, X_3, t)$
continuously differentiable function of its arguments and $J > 0$.
$P'Q' = \{x_i(X_1 + dX_1, X_2 + dX_2, X_3 + dX_3, t) - x_i(X_1, X_2, X_3, t)\}e_i$;
following Taylor expansion for the first term on the right-hand side,
$P'Q' = [\frac{\partial x_i}{\partial X_1}dX_1 + \frac{\partial x_i}{\partial x_2}dX_2 + \frac{\partial x_i}{\partial X_3}dX_3]e_i + 0(|dX|^2)$
$P'Q' = [\frac{\partial x_i}{\partial X_j}\Big|_P dX_j]e_i$+neglect higher order terms

$= [\frac{\partial x_i}{\partial X_A} dX_A] e_i$
component form,
$(P'Q')_j = \frac{\partial x_i}{\partial X_A} dX_A e_i . e_j$
$= \frac{\partial x_i}{\partial X_A} dX_A \delta_{ij}$
$= \frac{\partial x_j}{\partial X_A} dX_A$
$(P'Q') = F_{jA}|_P (PQ)_A$ relates components of vectors PQ in reference configuration to the components of the vectors $(P'Q')$.
$u_i = x_i - X_A \delta_{iA}$
$\frac{\partial u_i}{\partial X_A} = \frac{\partial x_i}{\partial X_A} - \delta_{iA}$
$\Rightarrow F_{iA} = u_{i,A} + \delta_{iA}$

Example: The deformation of a body is given by $u_1 = (3X_1^2 + X_2), u_2 = (2X_2^2 + X_3), u_3 = (4X_3^2 + X_1)$
Compute the vectors into which the vectors $\epsilon(1/3, 1/3, 1/3)$ passing through the material point $(1, 1, 1)$ in the reference configuration is deformed. $\epsilon \rightarrow$ infinitesimal real.[Ans. (1/3,1/3,1/3) components of a vector PQ]

$$F_{iA} = \begin{bmatrix} 1+6X_1 & 1 & 0 \\ 0 & 1+4X_2 & 1 \\ 1 & 0 & 1+8X_3 \end{bmatrix}$$

$$F_{iA}|_P = \begin{bmatrix} 7 & 1 & 0 \\ 0 & 5 & 1 \\ 1 & 0 & 9 \end{bmatrix}$$

$$\text{hence,} \{P'Q\}_j = \begin{bmatrix} 7 & 1 & 0 \\ 0 & 5 & 1 \\ 1 & 0 & 9 \end{bmatrix} \begin{Bmatrix} \epsilon/3 \\ \epsilon/3 \\ \epsilon/3 \end{Bmatrix}$$

$$= \epsilon/3 \begin{Bmatrix} \epsilon/3 \\ \epsilon/3 \\ \epsilon/3 \end{Bmatrix}.$$

Example: Simple Extension
$x_1 = \alpha(t)X_1; x_2 = \beta(t)X_2; x_3 = \gamma X_3$

$$F_{iA} = \begin{bmatrix} \alpha & 0 & 0 \\ 0 & \beta & 0 \\ 0 & 0 & \gamma \end{bmatrix}$$

Given, $u_1 = 0.1X_2^2; u_2 = u_3 = 0;$
a. Is this deformation possible? Prove your answers.
b.Find vectors into which material vectors $0.01e_1$ and $0.015e_2$ passing through the material point $P(1, 1, 0)$in the reference configuration, are deformed.

a. $F_{iA} = \begin{bmatrix} 1 & 0.2X_2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ and det$[F_{iA}]$=1¿0

at (1,1,0) $\begin{bmatrix} 1 & 0.2(1)^2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} 0 \\ 0.015 \\ 0 \end{Bmatrix} = \begin{Bmatrix} 0.003 \\ 0.015 \\ 0 \end{Bmatrix}$

Notes:

1) Non-singular-tensor F depends on X which denotes a so-called inhomogeneous deformation

2)A deformation of a body in question is said to be homogeneous if F does not depend on the space co-ordinates. F_{iA} depends only on time.
Associated motion is called affine.

3) Rigid-body translation $\Rightarrow$ displacement field is independent of X

4) No motion $F = I \rightarrow x = X$

1. Ration of the length of the vector $P'Q'$ to that of the vector PQ is called the stretch at the material point P in the direction of the vector PQ.

2.Different unit vectors through the point P are stretched differently, therefore, the stretch λ at the point P varies with direction of vector PQ.

3. It is assumed that PQ is infinitesimal. However, no assumption was made as to the magnitude of the gradient F_{iA}. Hence valid for small and large gradients. $\Rightarrow$ applicable for small nd large deformations.

C. Determine the stretches of the point $(1,1,0)$ in the X_1 and X_2 direction.

D.Determine the change in the angle between lines passing the point $P(1,1,0)$ that was parallel to the X_1 and X_2 axes in the reference configuration.
Stretch at the point $(1,1,0)$ in the X_1-direction $= \frac{0.01}{0.01} = 1.$
Stretch at the point (1,1,0) in the X_2 direction$=\frac{\sqrt{0.003^2+0.015^2}}{0.015} = 1.02$
Angle between the vectors into which vectors $0.01e_1$ and $0.015e_2$ through the point (1,1,0) are deformed
$=\cos^{-1}\left\{\frac{(0.01)(0.003)+0+0}{(0.01)\sqrt{0.003^2+0.015^2}}\right\} = \pm 78.7^o$
change in angle$=11.3^o$.

Given the following displacement components $u_1 = 2X_1^2 + X_1X_2$ and $u_2 = X_2^2$ and $u_3 = 0$ and that for the points in reference configuration of the body $X_1 \geq 0, X_2 \geq 0$.
a. Find the vector in the reference configuration that is deformed into a vector parallel the the x_1. through the point (1,1,0) in the present configuration.
b.Find the stretch of a line element that is deformed into a vector parallel to the x_1 axis through the point(1,1,0) in the present configuration.
$\Rightarrow x_1 = X_1 + 2x_1 + X_1X_2$
$x_2 = X_2 + X_2^2$
$x_3 = 0$ and from these equations $\Rightarrow X_1 = X_2, X_2 = 0, X_3 = 0$ for (1/2,0,0).

$$F_{iA} = \begin{bmatrix} 1+4X_1+X_2 & X_1 & 0 \\ 0 & 1+2X_2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

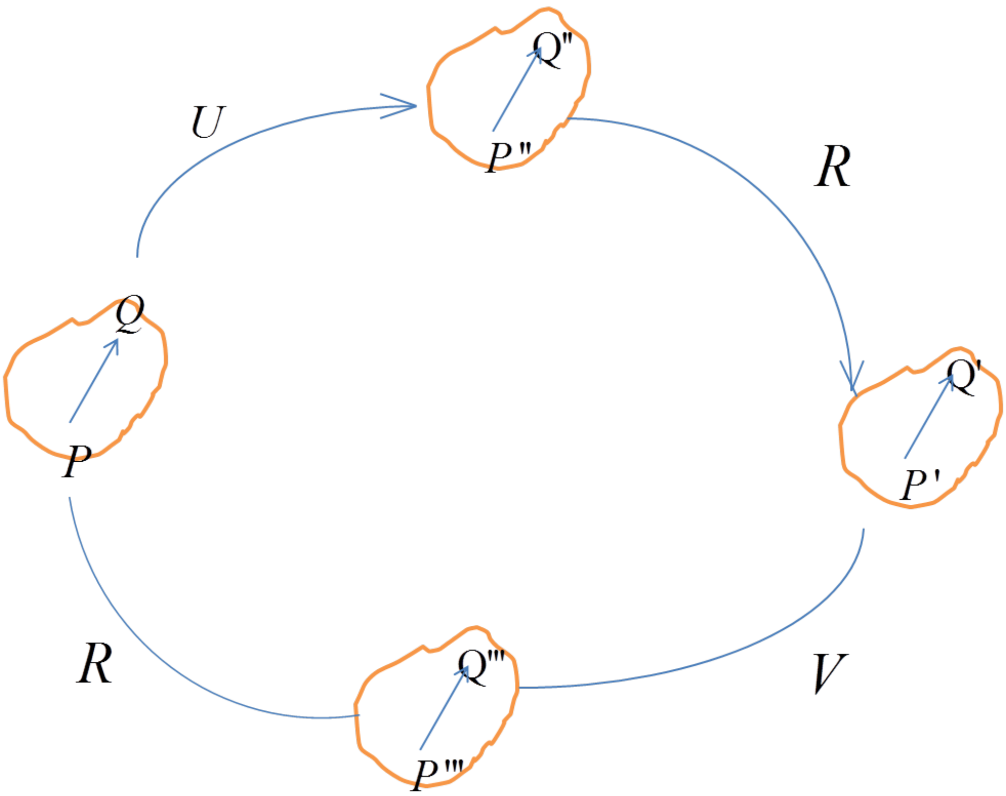
$$F_{iA}|_P = \begin{bmatrix} 3 & 0.5 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Hence, $\begin{Bmatrix} 1 \\ 0 \\ 0 \end{Bmatrix} = \begin{bmatrix} 3 & 0.5 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} dX_1 \\ dX_2 \\ dX_3 \end{Bmatrix}$
$\Rightarrow (1/3, 0, 0)$; and stretch$= \frac{\sqrt{1^2+0+0}}{\sqrt{3^2+0+0}} = 3.$

Kinematics Continued

Excluding translation (rigid body motion) that does not induce any stretch, a generalized deformation can be thought of as rigid body rotation followed by stretch and local line vector rotation passing through points differently $\Rightarrow F_{iA} = R_{ij}U_{jA} = V_{ij}R_{jA}$ where R is an orthogonal matrix which only rotates a line and does not change its length.
U, V being symmetric positive definite matrices rotate as well as change line length.

Pictorial Representation

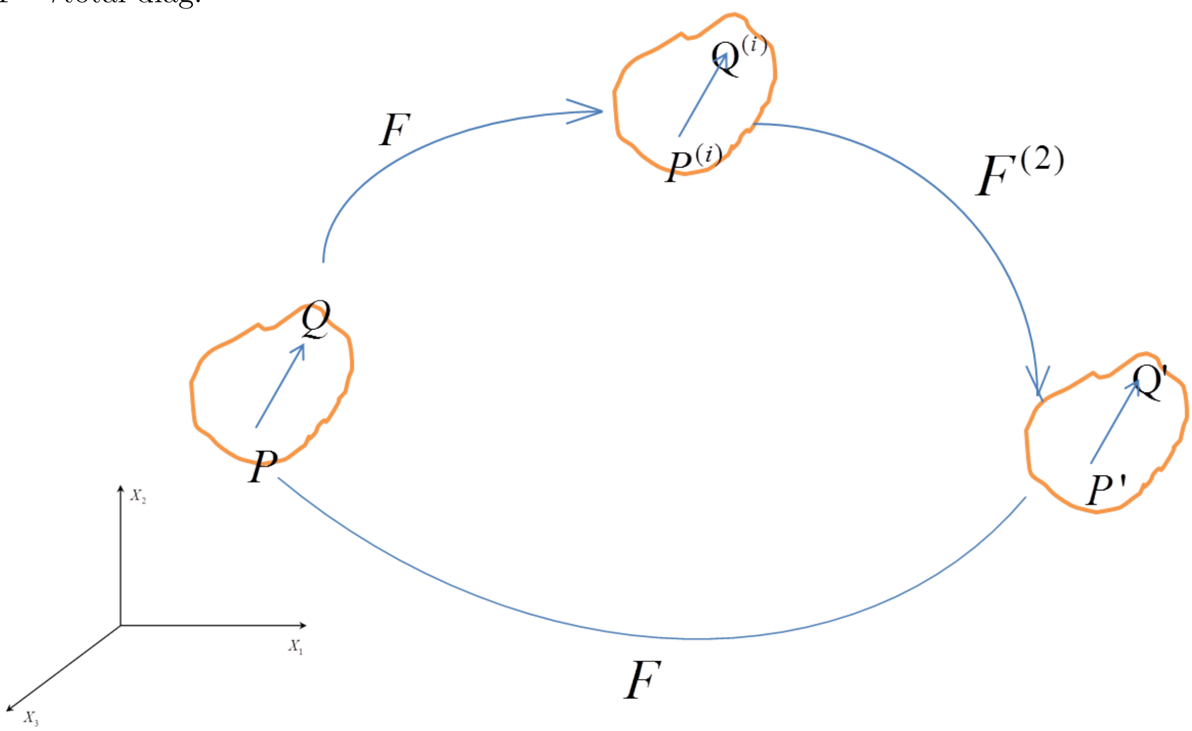


what happens to a line element dX?

$dx_i = R_{ij}U_{jA}dX_A = V_{ij}R_{jA}dX_A$

During homogeneous deformation R, U, V are independent of X. $\lambda \rightarrow$stretch depends on direction of line element PQ. Let e_i and E_A represent unit vector/base vectors of Cartesian system and they are parallel then
$R_{ij} = e_i.Re_j$ and $R_iA = e_i.RE_A$
This decomposition of F into RU and VR is called *polar decomposition*.

What happens when a body undergoes sequential deformation?
EX: A body undergoes tension and then undergoes twist
$F^{(1)} \rightarrow$ diag. from reference to intermediate
$F^{(2)} \rightarrow$ diag. from intermediate to present
$F \rightarrow$total diag.



$x^{(i)} = x^{(i)}(X, t); x = x(x^{(i)}, t)$
$F^{(}_{jA}i) = \frac{\partial x_j^{(i)}}{\partial X_A}; x = x(x(i), t)$
$F^{(i)}_{jA} = \frac{\partial x_j^{(i)}}{\partial X_A}; F^{(2)}_{jk} = \frac{\partial x_j}{\partial x_k^{(i)}}; F_{kA} = F^{(2)}_{kj}F^{(2)}_{jA} \rightarrow$ useful to study small configuration superposed on large deformations.
$F = RU = R^{(2)}U^{(2)}R^{(1)}U^{(1)} = v^{(2)}R^{(2)}V^{(1)}U^{(1)}$
Example: Find the deformation gradient for a circular cylinder first subjected to simple external deformation and then by torsional deformation.

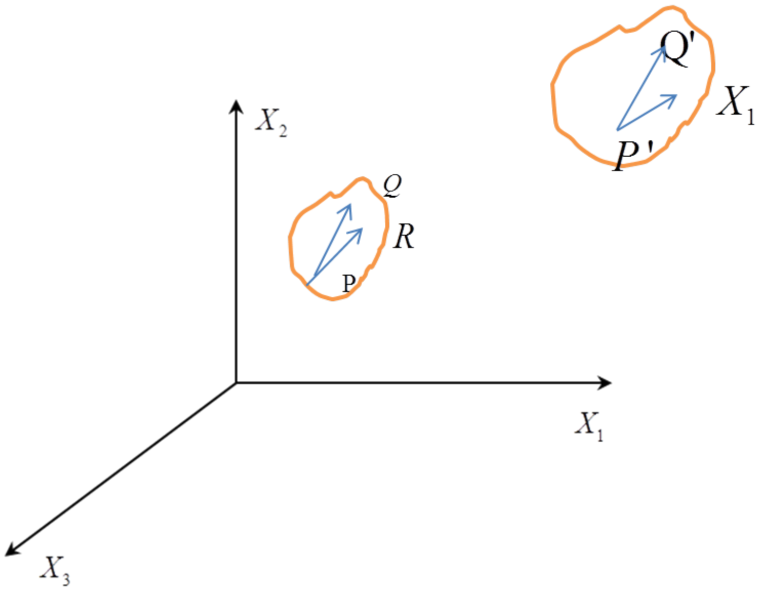
Extension takes the point $P \to P^{(i)}$
$x_1^{(i)} = \alpha X_1; x_2^{(i)} = \beta X_2; x_3^{(i)} = \gamma X_3^{(3)}$
Torsional deformation takes the point from $P^{(i)} \to P'$
$x_1 = x_1^{(i)} cos(\theta x_3^{(i)}) - x_2^{(i)} sin(\theta x_3^{(i)})$
$x_2 = x_1^{(i)} sin(\theta x_3^{(i)}) + x_2^{(i)} cos(\theta x_3^{(i)})$
–substituting these into above equations we get

calculate $F_{(iA)} = \begin{bmatrix} \alpha cos(\theta\gamma X_3) & -\beta sin(\theta\gamma X_3) & -\theta x_2 \\ \alpha sin(\theta\gamma X_3) & -\beta cos(\theta\gamma X_3) & \theta\gamma x_1 \\ 0 & 0 & \gamma \end{bmatrix}$

case(b), you get
$x_1 = \alpha(X_1 cos(\theta X_3) - X_2 sin(\theta X_2)$
$x_2 = \beta(X_2 sin(\theta X_3) + X_2 cos(\theta X_3))$
$x_3 = \gamma X_3$

$$F_{(iA)} = \begin{bmatrix} \alpha cos(\theta\gamma X_3) & -\alpha sin(\theta\gamma X_3) & -\alpha/\beta\theta x_2 \\ \beta sin(\theta\gamma X_3) & \beta cos(\theta\gamma X_3) & \alpha/\beta\theta x_1 \\ 0 & 0 & \gamma \end{bmatrix}$$

Strain Tensor



stretched + rotation $\Rightarrow$ body is in strained condition.
$(P'Q')_j = F_{jA}|_P (PQ)_A; (P'R')_j = F_{jA}F_{jB}(PQ)_A(PR)_B$
$(P'R')_j = F_{jA}|_P (PR)_A$

consider, $P'Q'.P'R' = F_{jA}F_{jB}(PQ)_A(PR)_B = (PQ)_A C_{AB}(PR)_B$
$C = F^T F = F_{jA}F_{jB} \qquad C_{AB} = C_{BA}$—$C$ is symmetric.

Physical Interpretation of C

$PQ = \xi(1,1,0) \qquad PR = \xi(1,0,0)$
$P'Q'.P'Q' = \xi^2 C_{11}$
So, $|P'Q'| = \xi\sqrt{C_{11}}$ or $\frac{|P'Q'|}{|PQ|} = \sqrt{C_{11}}$
$\Rightarrow C_{11}$—square of stretch $PQ = \xi_1(1,0,0) \qquad PR = \xi_2(0,0,1)$

$P'Q'.P'R' = \xi_1\xi_2 C_{12} \qquad C_{12}$ measures of change in angle between two material lines passing through the point P.
Hence, $\frac{P'Q'.P'R'}{|P'Q'||P'R'|} = \frac{C_{12}}{\sqrt{C_{11}C_{22}}}$
In terms of displacements, $C_{AB} = F_{iA}F_{iB} = (\delta_{iA} + u_{i,B})$
$=\delta_{AB} + u_{B,A} + u_{A,B} + u_{i,A}u_{i,B} \rightarrow$ right-Cauchy Green Tensor
$E_{AB} = (C_{AB} - \delta_{AB})/2 \rightarrow$ Green-St. Venant strain tensor.
$P'Q'P'R' - (PQ)(PR) = (PQ)_A(C_{AB} - \delta_{AB}(PR)_B = 2(PQ)_A E_{AB}(PR)_B \rightarrow$ measure of strain.

Or, $(PQ)(PR) = (F^{-1})_{A_i}(F^{-1})_{A_j}(P'Q')_i(PR)_j$
$=(B^{-1})_{ij}(P'Q')_i(P'R')_j \qquad B_{ij} \rightarrow$ left-Couchy Green tensor
$\epsilon_{ij} = (\delta_{ij} - (B^{-1})_{ij})/2 = \{u_{i,j} + u_{j,i} - u_{A,i}u_{A,j}\}/2$—Almansi-Hamel Strain tensor.
Strain measure in spatial description.
A theory based on E and ϵ is called geometrically non-linear theory.

Principal Strains

$PQ = \epsilon(N_1, N_2, N_3)$;$N_i$s are components of unit vectors N along PQ.

Goal is to find N such that the stretch at the material point P along N is maximum or minimum.
$(P'Q')_j = F_{jA}(PQ)_A = \epsilon F_{jA}N_A$

So, $\frac{|P'Q'|^2}{|PQ|^2} = F_{jA}F_{jB}N_AN_B = C_{AB}N_AN_B$

Find unit vector N such that $C_{AB}N_AN_B$ has an extreme value:
$C_{AB}N_AN_B - \lambda(N_AN_B - 1)$
$\frac{\partial}{\partial N_i}\{\text{-do-}\} = 0; \frac{\partial}{\partial \lambda}\{\text{-do-}\} = 0.$ $C_{AB}N_A - \lambda N_A = 0$ or $N_AN_A = 1$——Eigen value problem.

$C \rightarrow$ symmetric +Positive definite $\Rightarrow$ all roots are positive.

$\lambda_1^2, \lambda_2^2, \lambda_3^2 \rightarrow$ 3 roots.
$\lambda_1, \lambda_2, \lambda_3 \rightarrow$ principal strains.
$N^{(1)}, N^{(2)} and N^{(3)} \rightarrow$ principal areas of stretch
Principal axial strains $(\lambda_1^2-1)/2, (\lambda_2^2-1)/2, (\lambda_3^2-1)/2$——defined in reference configuration.

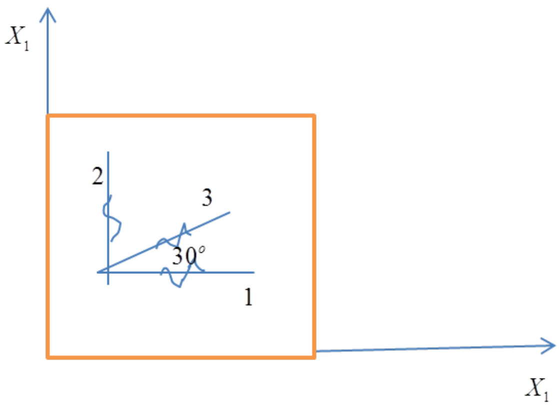
Example-The deformation of a body is given by:
$u_1 = 3X_1^2 + X_2; u_2 = \alpha X_2^2 + X_3; u_3 = 4X_3^2 + X_1$
a. Find the principal axial strains at the material point (1,1,1) in the reference configuration.
b.Find the direction of the maximum strain axial strain through the material point (1,1,1) in the reference configuration. Also, find the direction of the maximum principal strain in the deformed configuration.

Ans. a. At the point, $F_{iA} = \begin{bmatrix} 1+6X_1 & 1 & 0 \\ 0 & 1+4X_2 & 1 \\ 1 & 0 & 1+8X_3 \end{bmatrix}_{(1,1,1)} =$
$\begin{bmatrix} 7 & 1 & 0 \\ 0 & 5 & 1 \\ 1 & 0 & 9 \end{bmatrix}$.
$C = F^T F = \begin{bmatrix} 50 & 7 & 9 \\ 7 & 26 & 5 \\ 9 & 5 & 82 \end{bmatrix}$ thus, $\lambda = 42.06, 23.95, 11.5$

b. To obtain $N^{(1)}$ we need to solve,
$(C_{AB} - \lambda^2 \delta_{AB}) N_B^{(1)} = 0; N_B^{(1)} N_B^{(1)} = 1;$
for $\lambda_1 = 42.06; (0.268, 0.1126, 0.957)$

To find the direction cosines of the line into which this is defined
$PQ = ds(0.268, 0.1128)$
Then, $(P'Q')_j = F_{jA}(PQ)_A = \begin{bmatrix} 7 & 1 & 0 \\ 0 & 5 & 1 \\ 1 & 0 & 9 \end{bmatrix} ds \begin{Bmatrix} 0.268 \\ 0.1126 \\ 0.957 \end{Bmatrix}$
$= ds \begin{Bmatrix} 1.989 \\ 1.52 \\ 8.881 \end{Bmatrix}$.

Example:- In an undefined system configuration three strain gages are glued to the surface of a thick plate as shown in the figure. Assume that a plane strain state of deformation occurs in the plate i.e., $x_1 = x_1(X_1, X_2); x_2 = x_2(X_1, X_2)$ and $x_3 = X_3$.Find principal strains and this direction at the location of the strain gages when the strains in the gages 1,2,3 equal 0.5,0.2 and 0.3, respectively. Recall that a strain gage leads change in length/length in the direction of the gage.



$C_{11} = \lambda_1^2 = (1.5)^2; C_{12} = \lambda_2^2 = (1.3)^2$
$N = (cos30^o, sin30^o, 0) = (\sqrt{3}/2, 1/2, 0)$
$\frac{|P'Q'|}{|PQ|} = 1.2$

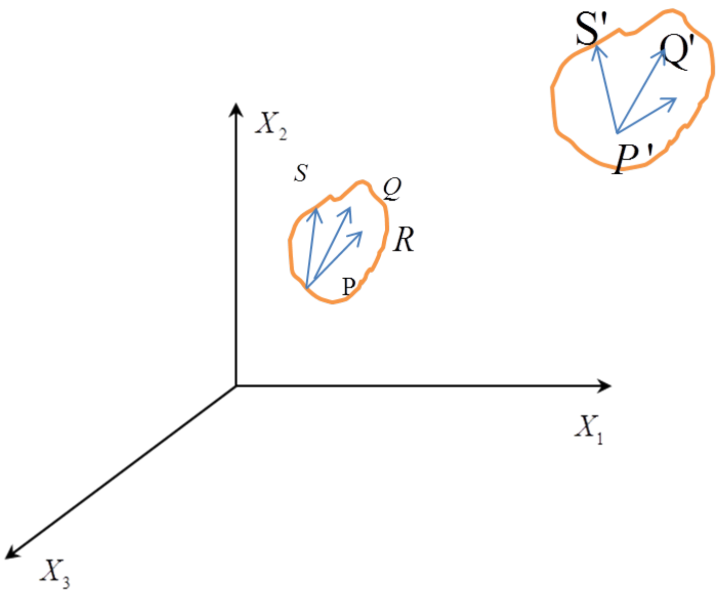
$$\frac{|P'Q'|^2}{|PQ|} = C_{AB}N_AN_B = 3/4C_{11} + 1/4C_{22} + \sqrt{3}/2C_{12}$$

$$\Rightarrow C_{12} = -0.773$$

$$C = \begin{bmatrix} 2.25 & -0.773 & 0 \\ -.773 & 1.69 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Eigenvalues of $C(2.793, 1.147, 1.0)$; eigen vectors (0.819,-0.574,0); (0.574,0.819,0),(0,0,1). So, Principal axial strains are (.8165,0.0737,0) and directions same as above.

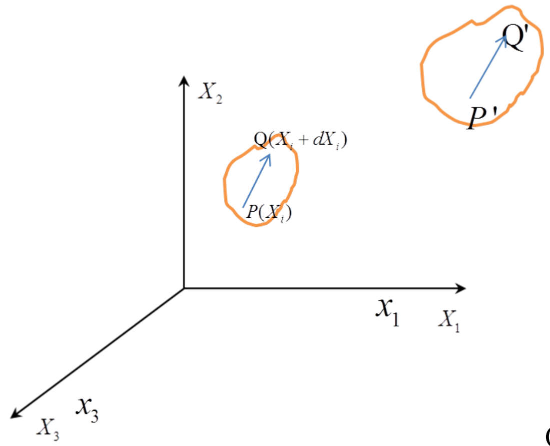
Deformation of area and volume



$dA = PQ \times PR$
$(dA)_B = \epsilon_{BCD}(PQ)_C(PR)_D$
$da = P'Q' \times P'R'$
$(da)_i = \epsilon_{ijk}(P'Q')_j(P'R')_k$
Now, $(P'Q')_j = F_{jA}(PQ)_A$ and $(P'R')_k = F_{kB}(PR)_B$
$(da)_i = \epsilon_{ijk}F_{jC}(PQ)_C F_{kD}(PR)_D$
$=\epsilon_{pjk}F_{jC}F_{kD}(F_{pB}(F^{-1}_{Bi}))(PQ)_C(PR)_D$
$= J(F^{-1})_{Bi}dA_B$
$= J(F^{-1})^T dA$

Volume: $dV = PQ \times PR.PS = \epsilon_{BCD}(PQ)_B(PR)_C(PS)_D$
$(dV') = \epsilon_{ijk}F_{iB}(PQ)_B F_{jC}(PR)_c F_{RD}(PS)_D$
$=J\epsilon_{BCD}(PQ)_B(PR)_C(PS)_D$
$=JdV$

Rate of Deformation



Consider the deformed line vector $P'Q'$.
$(P'Q')_i = x_i(X_A + dX_A, t) - x_i(X_A, t)$
We wish to compute $\frac{D(P'Q')}{Dt}$; rate of change of line and direction of $P'Q'$.
Hence,
$\frac{D(P'Q')_i}{Dt} = V_i(X_A + dX_A, t) - V_i(X_A, t)$
$= \frac{\partial V_i}{\partial X_A} dX_A = \frac{\partial V_i}{\partial x_j}\frac{\partial x_j}{\partial X_A} dX_A = V_{i,j} dx_j = \left\{\frac{V_{i,j}+V_{j,i}}{2} + \frac{V_{i,j}-v_{j,i}}{2}\right\} dx_j$
$= (D_{ij} + W_{ij}) dx_j$

D_{ij}

1.Symmetric part of velocity gradient
2. Strain-rate tensor
3. Geometric meaning (Rate of change of (extension) length/unit length) $\Rightarrow$ stretching.
4.Eigen values of D are called Principal Stretchings
5. In present configuration the principal stretch need not coincide with principal stretching
6.$\frac{1}{ds}\frac{D(ds)}{Dt} = n_i D_{ij} n_j$

$W_{ij} \rightarrow$ 1.Anti-symmetric part of velocity gradient—Spin tensor.
2.Geometric meaning rate of change of eigenvector of D
3.Axial vector $\omega_i = \epsilon_{ijk} = \epsilon_{ijk} V_{k,j} = CurlV$
$\rightarrow$ velocity vectors (ω) and eigen vector of W corresponding to zero eigen values.
–gives the rate of change of eigenvector of D.
If, $W = 0$–it implies **irrotational motion** $\rightarrow CurlV = 0$.

Example

Given the velocity field $V_i = 2x_2\delta_{1i}$
a. the strain rate and the spin tensor
b.the rate of extension per unit length of the line segment $P'Q' = \epsilon(1,2,0)$ and
c.the maximum and the minimum stretchings

Ans.a. $[V_{ij}] = \begin{bmatrix} 0 & 2 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$;

$D_{ij} = (V_{i,j} + V_{j,i})/2 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

$W_{i,j} = (v_{i,j} - vj,i)/2 = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

b.Given $P'Q' = \epsilon(1,2,0) = \epsilon(\sqrt{5})(1/\sqrt{5}, 2/\sqrt{5}, 0)$
So,$n = ((1/\sqrt{5}, 2/\sqrt{5}, 0) = \frac{1}{ds}\frac{D(ds)}{Dt} = n_j D_{ij} n_j$

$= \begin{bmatrix} 1/\sqrt{5} & 2/\sqrt{5} & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} \frac{1}{\sqrt{5}} \\ 2/\sqrt{5} \\ 0 \end{Bmatrix}$

$= 4/5$

c.Determine the eigen value of D
$\det[D_{ij} - \lambda\delta_{ij}] = 0; 0 and \pm 1$; eigen-vectors are $\lambda_1 = 1$
$n^{(1)} = (1/\sqrt{2})(1,1,0); n^{(2)} = (1/\sqrt{2})(1,-1,0); n^{(3)} = (0,0,1)$