Tutorial Sheet 0

Mathematical Preliminaries

0.1 Continuity

(1) Study the continuity of f in each of the following cases:

(i) $f(x) = \begin{cases} x^2 & \text{if } x < 1 \\ \sqrt{x} & \text{if } x \geq 1 \end{cases}$

(ii) $f(x) = \begin{cases} -x & \text{if } x < 1 \\ x & \text{if } x \geq 1 \end{cases}$

(iii) $f(x) = \begin{cases} 0 & \text{if } x \text{ is rational} \\ 1 & \text{if } x \text{ is irrational} \end{cases}$

(2) Let P and Q be polynomials. Find

$$\lim_{x\to\infty} \frac{P(x)}{Q(x)} \quad \text{and} \quad \lim_{x\to 0} \frac{P(x)}{Q(x)}$$

in each of the following cases.

(i) The degree of P is less than the degree of Q.

(ii) The degree of P is greater than the degree of Q.

(3) Let f be defined on an interval (a, b) and suppose that f is continuous at $c \in (a, b)$ and $f(c) \neq 0$. Then, show that there exists a $\delta > 0$ such that f has the same sign as $f(c)$ in the interval $(c - \delta, c + \delta)$.

(4) Show that the equation

$$\sin x + x^2 = 1$$

has at least one solution in the interval $[0, 1]$.

(5) Show that $f(x) = (x - a)^2(x - b)^2 + x$ takes on the value $(a + b)/2$ for some $x \in (a, b)$.

(6) Let $f(x)$ be continuous on $[a, b]$, let $x_1, \cdots, x_n$ be points in $[a, b]$, and let $g_1, \cdots, g_n$ be real numbers having same sign. Show that

$$\sum_{i=1}^{n} f(x_i)g_i = f(\xi)\sum_{i=1}^{n} g_i, \quad \text{for some } \xi \in [a, b].$$

(7) Let $I = [0, 1]$ be the closed unit interval. Suppose f is a continuous function from I onto I. Prove that $f(x) = x$ for at least one $x \in I$. [Note: A solution of this equation is called a **fixed point** of the function f]

(8) Show that the equation $f(x) = x$, where

$$f(x) = \sin\left(\frac{\pi x + 1}{2}\right), \quad x \in [-1, 1]$$

has at least one solution in $[-1, 1]$.

0.2 Differentiation

(9) Let $c \in (a,b)$ and $f : (a,b) \to \mathbb{R}$ be differentiable at c. If c is a local extremum (maximum or minimum) of f, then show that $f'(c) = 0$.

(10) Let $f(x) = 1 - x^{2/3}$. Show that $f(1) = f(-1) = 0$, but that $f'(x)$ is never zero in the interval $[-1, 1]$. Explain how this is possible, in view of Rolle's theorem.

(11) Suppose f is differentiable in an open interval (a,b). Prove the following statements
(a) If $f'(x) \geq 0$ for all $x \in (a,b)$, then f is non-decreasing.
(b) If $f'(x) = 0$ for all $x \in (a,b)$, then f is constant.
(c) If $f'(x) \leq 0$ for all $x \in (a,b)$, then f is non-increasing.

(12) Let $f : [a,b] \to \mathbb{R}$ be given by $f(x) = x^2$. Find a point ξ specified by the mean-value theorem for derivatives. Verify that this point lies in the interval (a,b).

0.3 Integration

(13) In the mean-value theorem for integrals, let $f(x) = e^x$, $g(x) = x$, $[a,b] = [0,1]$. Find the point ξ specified by the theorem and verify that this point lies in the interval $(0,1)$.

(14) If n is a positive integer, show that

$$\int_{\sqrt{n\pi}}^{\sqrt{(n+1)\pi}} \sin(t^2)\, dt = \frac{(-1)^n}{c},$$

where $\sqrt{n\pi} \leq c \leq \sqrt{(n+1)\pi}$.

0.4 Taylor's Formula

(15) Find the Taylor's expansion for $f(x) = \sqrt{x+1}$ upto $n = 2$ (ie. the Taylor's polynomial of order 2) with remainder $R_2(x)$ about $c = 0$.

(16) Use Taylor's formula about $c = 0$ to evaluate approximately the value of the function $f(x) = e^x$ at $x = 0.5$ using three terms (ie., $n = 2$) in the formula. Find the value of the remainder $R_2(0.5)$. Add these two values and compare with the exact value.

0.5 Big Oh, Little oh, and Orders of convergence

(17) Prove or disprove:
(i) $2n^2 + 3n + 4 = o(n)$ as $n \to \infty$.
(ii) $\frac{n+1}{n^2} = o(\frac{1}{n})$ as $n \to \infty$.
(iii) $\frac{n+1}{n^2} = O(\frac{1}{n})$ as $n \to \infty$.
(iv) $\frac{n+1}{\sqrt{n}} = o(1)$ as $n \to \infty$.
(v) $\frac{1}{\ln n} = o(\frac{1}{n})$ as $n \to \infty$.
(vi) $\frac{1}{n \ln n} = o(\frac{1}{n})$ as $n \to \infty$.
(vii) $\frac{e^n}{n^5} = O(\frac{1}{n})$ as $n \to \infty$.

(18) Prove or disprove:
(i) $e^x - 1 = O(x^2)$ as $x \to 0$.
(ii) $x^{-2} = O(\cot x)$ as $x \to 0$.
(iii) $\cot x = o(x^{-1})$ as $x \to 0$.
(iv) For $r > 0$, $x^r = O(e^x)$ as $x \to \infty$.
(v) For $r > 0$, $\ln x = O(x^r)$ as $x \to \infty$.

Tutorial Sheet 1

Error Analysis

1.1 Floating-Point Approximation

(1) Consider the real numbers $x = 123.45678$ and $y = 98.7654$. Using 5-digit rounding, verify

(i) $\text{fl}(x + y) = \text{fl}\,(\text{fl}(x) + \text{fl}(y))$

(ii) $\text{fl}(x - y) = \text{fl}\,(\text{fl}(x) - \text{fl}(y))$

(iii) $\text{fl}(x \times y) = \text{fl}\,(\text{fl}(x) \times \text{fl}(y))$

(iv) $\text{fl}(x \div y) = \text{fl}\,(\text{fl}(x) \div \text{fl}(y))$

1.2 Types of Errors

(2) For rounded arithmetic on a binary machine, show that $\delta = 2^{-n}$ is the machine epsilon, where n is the number of digits in the mantissa.

(3) If $\text{fl}(x)$ is the machine approximated number of a real number x and ϵ is the corresponding relative error, then show that $\text{fl}(x) = (1 - \epsilon)x$.

(4) Let x, y and z are the given machine approximated numbers. Show that the relative error in computing $x(y+z)$ is $\epsilon_1 + \epsilon_2 - \epsilon_1\epsilon_2$, where $\epsilon_1 = E_r(\text{fl}(y+z))$ and $\epsilon_2 = E_r(\text{fl}(x\text{fl}(y+z)))$.

(5) Let $\epsilon = E_r(\text{fl}(x))$. Show that

(i) $|\epsilon| \leq 10^{-n+1}$ for n-digit (decimal) chopping.

(ii) $|\epsilon| \leq \frac{1}{2}10^{-n+1}$ for n-digit (decimal) rounding.

(6) Find the truncation error around $x = 0$ for the following functions, when their Taylor series are truncated after the term involving x^n

(i) $f(x) = \sin x$, (ii) $f(x) = \cos x$.

(7) For small values of x, the approximation $\sin x \approx x$ is often used. Estimate the error in using this formula with the aid of Taylor's theorem. For what range of values of x will this approximation give an absolute error of at most $\frac{1}{2}10^{-6}$?

(8) Let $x_A = 3.14$ and $y_A = 2.651$ be obtained from x_T and y_T using 4-digit rounding. Find the smallest interval that contains

(i) x_T

(ii) y_T

(iii) $x_T + y_T$

(iv) $x_T - y_T$

(v) $x_T \times y_T$

(vi) x_T / y_T.

1.3 Loss of Significance and Propagation of Error

(9) Let s denote the number of significant digits in x_A with respect to x. Fill in the following table for the given tuples of x and x_A.

x	x_A	s
451.01	451.023	
-0.04518	-0.045113	
23.4604	23.4213	

(10) Let x_A and y_A be approximations to x and y respectively, and be such that the relative errors $E_r(x_A)$ and $E_r(y_A)$ are very much smaller than 1. Then show that (i) $E_r(x_A y_A) \approx E_r(x_A) + E_r(y_A)$ and (ii) $E_r(x_A/y_A) \approx E_r(x_A) - E_r(y_A)$. (This shows that relative errors propagate slowly with multiplication and division).

1.4 Stable and Unstable Computations

(11) Show that the function

$$f(x) = \frac{1 - \cos x}{x^2}$$

leads to unstable computation when $x \approx 0$. Suggest an alternate formula for this function to avoid loss of significant digits when $x \approx 0$. Further check the stablity of this new formula for f when $x \approx 0$.

(12) The ideal gas law is given by $PV = nRT$, where R is a gas constant given (in MKS system) by $R = 8.3143 + \epsilon$, with $|\epsilon| \leq 0.12 \times 10^{-2}$. By taking $P = V = n = 1$, find a bound for the relative error in computing the temperature T.

(13) Find the condition number at a point $x = c$ for the following functions

(i) $f(x) = x^2$

(ii) $g(x) = \pi^x$

(iii) $h(x) = b^x$

(14) Given an approximate value $x_A = 2.5$ of x_T with an absolute error at most 0.01. Estimate the resulting absolute error in the value of the function $f(x) = x^3$ at $x = x_T$.

(15) Compute and interpret (find whether the funtions are well or ill-conditioned) the condition number for

(i) $f(x) = \tan x$, at $x = \frac{\pi}{2} + 0.1\left(\frac{\pi}{2}\right)$.

(ii) $f(x) = \tan x$, at $x = \frac{\pi}{2} + 0.01\left(\frac{\pi}{2}\right)$.

(16) Let $f(x) = (x-1)(x-2)\cdots(x-8)$. Obtain an approximate value of $f(1 + 10^{-4})$ using mean-value theorem and $f'(1)$.

Tutorial Sheet 2

Interpolation

2.1 Basic Problem of Interpolation

(1) Let $x_0, x_1, \cdots, x_n$ be distinct nodes. If $p(x)$ is a polynomial of degree less than or equal to n, then show that

$$p(x) = \sum_{i=0}^{n} p(x_i) l_i(x),$$

where $l_i(x)$ is the i^{th} Lagrange polynomial.

(2) Show that the polynomial $1 + x + 2x^2$ is an interpolating polynomial for the data

x	0	1	2
y	1	4	11

.

Find an interpolating polynomial for the new data

x	0	1	2	3
y	1	4	11	-2

.

Does there exist a quadratic polynomial that satisfies the new data? Justify your answer.

(3) Let $p(x)$, $q(x)$, and $r(x)$ be interpolating polymials for the three sets of data

x	0	1
y	y_0	y_1

,

x	1	2
y	y_1	y_2

, and

x	1	2	3
y	y_1	y_2	y_3

respectively. If $p(x) = 1 + x$, $q(x) = 3x - 1$, and $r(1.5) = 4$, then find the values of y_0, y_1, y_2, y_3.

2.2 Lagrange Form of Interpolating Polynomial

(4) Obtain Lagrange interpolation formula for equally spaced nodes.

(5) Using Lagrange form of interpolating polynomial for the function $g(x) = 3x^2 + x + 1$, express the rational function

$$f(x) = \frac{3x^2 + x + 1}{(x-1)(x-2)(x-3)}$$

as a sum of partial fractions.

(6) Find the Largrange form of interpolating polynomial for the data:

x	-2	-1	1	3
y	-1	3	-1	19

(7) Find the Lagrange form of interpolating polynomial $p_2(x)$ that interpolates the function $f(x) = e^{-x^2}$ at the nodes $x_0 = -1$, $x_1 = 0$ and $x_2 = 1$. Further, find the value of $p_2(-0.9)$ (use 6-digit rounding). Compare the value with the true value $f(-0.9)$ (use 6-digit rounding). Find the percentage error in this calculation.

2.3 Newton Form of Interpolating Polynomial

(8) Find the Newton form of interpolating polynomial for the data

x	-3	-1	0	3	5
y	-30	-22	-12	330	3458

(9) Prove that if $f(x)$ is a polynomial of degree k, then for $n > k$,

$$f[x_0, x_1, \cdots, x_n] = 0$$

for any set of nodes $x_0, x_1, \cdots, x_n$.

(10) For the particular function $f(x) = x^m$ $(m \in \mathbb{N})$, show that

$$f[x_0, x_1, \cdots, x_n] = \begin{cases} 1 & \text{if } n = m \\ 0 & \text{if } n > m \end{cases}$$

(11) Calculate the n^{th} divided difference $f[x_0, x_1, \cdots, x_n]$ of $f(x) = \dfrac{1}{x}$.

(12) Let $x_0, x_1, \cdots, x_n$ be nodes, and f be a given function. Define $w(x) = \prod_{i=0}^{n}(x - x_i)$. Prove that

$$f[x_0, x_1, \cdots, x_n] = \sum_{i=0}^{n} \frac{f(x_i)}{w'(x_i)}.$$

Tutorial - 2

Answers for Selected Problems

(2) $P_3(x) = -4x^3 + 14x^2 - 7x + 1$.

(3) $y_0 = 1, \quad y_1 = 2, \quad y_2 = 5, \quad y_3 = -4$.

(4) $$P_n(x) = \frac{1}{\Delta x^n} \sum_{k=0}^{n} \frac{(-1)^{k-n} f(x_k) \prod_{\substack{i=0 \\ k \neq i}}^{n} (x - x_i)}{k!\,(n-k)!}.$$

(5) $$f(x) = \frac{5}{2(x-1)} - \frac{15}{x-2} + \frac{31}{2(x-3)}.$$

(6) $P_3(x) = x^3 - 3x + 1$

(7) $P_2(x) = 1 - 0.632121\,x^2, \quad P_2(-0.9) \approx 0.487982$

Percentage error $= 9.69\,\%$.

(8) $P_4(x) = 5x^4 + 9x^3 - 27x^2 - 21x - 12$.

(11) $f[x_0, x_1, \ldots, x_n] = (-1)^n \big/ (x_0 x_1 \cdots x_n)$

Tutorial Sheet 3

Interpolation (contd.)

3.1 Error in Polynomial Interpolation

(1) Prove that if we take any set of 23 nodes in the interval $[-1,1]$ and interpolate the function $f(x) = \cosh x$ with a polynomial p_{22} of degree less than or equal to 22, then at each $x \in [-1,1]$ the relative error satisfies the bound

$$\frac{|f(x) - p_{22}(x)|}{|f(x)|} \leq 5 \times 10^{-16}.$$

(2) Let $p_n(x)$ be a polynomial of degree less than or equal to n that interpolates a function f at a set of distinct nodes $x_0, x_1, \cdots, x_n$. If $x \notin \{ x_0, x_1, \cdots, x_n \}$, then show that

$$f(x) - p_n(x) = f[x_0, x_1, \cdots, x_n, x] \prod_{i=0}^{n} (x - x_i)$$

(3) If $f \in C^{n+1}[a,b]$ and if $x_0, x_1, \cdots, x_n$ are distinct nodes in $[a,b]$, then show that there exists a point $\xi_x \in (a,b)$ such that

$$f[x_0, x_1, \cdots, x_n, x] = \frac{f^{(n+1)}(\xi_x)}{(n+1)\,!}$$

(4) Let x_0, x_1, $\cdots$, x_n be $n+1$ distinct nodes, and f be a function. For each $i = 0, 1, \cdots, n$, let $\mathrm{fl}\,(f(x_i))$ denote the floating point approximation of $f(x_i)$ using 5-digit rounding. Assume that $0.1 \leq f(x_i) < 1$ for all $i = 0, 1, \cdots, n$. Let $p_n(x)$ denote the Lagrange form of interpolating polynomial corresponding to the data $\{(x_i, f(x_i)) \,:\, i = 0, 1, \cdots, n\}$. Let $\tilde{p}_n(x)$ denote the Lagrange form of interpolating polynomial corresponding to the data $\{(x_i, \mathrm{fl}\,(f(x_i))) \,:\, i = 0, 1, \cdots, n\}$. Show that the arithmetic error at a point $\tilde{x}$ satisfies the inequality

$$|p_n(\tilde{x}) - \tilde{p}_n(\tilde{x})| \leq \frac{1}{2} 10^{-5} \sum_{k=0}^{n} |l_k(\tilde{x})|.$$

(5) Let N be a natural number. Let $p_1(x)$ denote the linear interpolating polynomial on the interval $[N, N+1]$ interpolating the function $f(x) = x^2$ at the nodes N and $N+1$. Find an upper bound for the mathematical error ME_1 using the infinity norm on the interval $[N, N+1]$ (*i.e.*, $\|\mathrm{ME}_1\|_{\infty,\,[N,N+1]}$).

(6) Let $p_3(x)$ denote a polynomail of degree less than or equal to 3 that interpolates the function $f(x) = \ln x$ at the nodes $x_0 = 1$, $x_1 = \frac{4}{3}$, $x_2 = \frac{5}{3}$, $x_3 = 2$. Find a lower bound on the absolute value of mathematical error $|\mathrm{ME}_3(x)|$ at the point $x = \frac{3}{2}$.

(7) Let $f : [a, b] \to \mathbb{R}$ be a given function. **Quadratic interpolation in a table of function values**

x	x_0	x_1	$\cdots$	x_N
$f(x)$	$f(x_0)$	$f(x_1)$	$\cdots$	$f(x_N)$

means the following:
The values of $f(x)$ are tabulated for a select number of points x in $[a, b]$, say at x_i for $i = 0, 1, \cdots, N$. For an $\bar{x} \in [a, b]$ at which the function value $f(\bar{x})$ is not tabulated, the value of $f(\bar{x})$ is taken to be the value of $p_2(\bar{x})$, where $p_2(x)$ is the polynomial of degree less than or equal to 2 that interpolates f at the nodes x_i, x_{i+1}, x_{i+2} where i is the least index such that $x \in [x_i, x_{i+2}]$.

A table of values of $\sqrt{x}$ is required so that the quadratic interpolation will yield a seven-decimal-place accuracy (*i.e.*, error is less than $\frac{1}{2} \times 10^{-7}$) for any value of x in $[1, 2]$. Assume that the tabular values $x_0, x_1, \cdots, x_N$ are equally spaced. Determine the minimum number of entries needed in this table, and the corresponding spacing between x_i and x_{i+1}.

3.2 Spline Interpolation

(8) Find a natural cubic spline interpolating function for the data

x	-1	0	1
y	5	7	9

.

(9) Obtain the natural cubic spline interpolating function for the data

x	0	1	2	3
y	1	2	33	244

.

(10) Determine whether the natural cubic spline function that interpolates the table

x	-1	0	1
y	-3	-1	-1

is or is not the function

$$S(x) = \begin{cases} x^3 + x - 1 & x \in [-1, 0], \\ x^3 + x - 1 & x \in [0, 1]. \end{cases}$$

(11) Determine whether the coefficients a, b, c, d exist so that the function

$$S(x) = \begin{cases} 1 - 2x & x \in [-4, -3] \\ a + bx + cx^2 + dx^3 & x \in [-3, 4] \\ 157 - 32x & x \in [4, 5] \end{cases}$$

is a natural cubic spline interpolating function on the interval $[-4, 5]$ for the data

x	-4	-3	4	5
y	9	7	29	-3

.

(12) Does there exist real numbers a and b so that the function

$$S(x) = \begin{cases} (x-2)^3 + a(x-1)^2 & x \in [-1, 2] \\ (x-2)^3 - (x-3)^2 & x \in [2, 3] \\ (x-3)^3 + b(x-2)^2 & x \in [3, 5] \end{cases}$$

is a natural cubic spline interpolating function on the interval $[-1, 5]$ for the data

x	-1	2	3	5
y	-31	-1	1	17

?

Tutorial Sheet 4

Numerical Integration and Differentiation

4.1 Numerical Integration

(1) Apply Rectangle, Trapezoidal, Simpson and Gaussian methods to evaluate

(a) $I = \displaystyle\int_0^{\pi/2} \frac{\cos x}{1+\cos^2 x}\, dx$ (exact value ≈ 0.623225)

(b) $I = \displaystyle\int_0^{\pi} \frac{dx}{5+4\cos x}\, dx$ (exact value ≈ 1.047198)

(c) $I = \displaystyle\int_0^{1} e^{-x^2} dx$ (exact value ≈ 0.746824),

(d) $I = \displaystyle\int_0^{\pi} \sin^3 x \ \cos^4 x \, dx$ (exact value ≈ 0.114286)

(e) $I = \int_0^1 (1+e^{-x}\sin(4x))\, dx$. (exact value ≈ 1.308250)

Compute the relative error in each method.

(2) Write down the errors in the approximation of

$$\int_0^1 x^4 dx \text{ and } \int_0^1 x^5 dx$$

by the Trapezoidal rule and Simpson's rule. Find the value of the constant C for which the Trapezoidal rule gives the exact result for the calculation of $\int_0^1 (x^5 - Cx^4)dx$.

(3) Obtain expressions for the arithmetic error in approximating the integral $\int_a^b f(x)\, dx$ using Trapezoidal and Simpson's rules. Also obtain upper bounds.

(4) A function f has the values shown below:

x	1	1.25	1.5	1.75	2
$f(x)$	10	8	7	6	5

(i) Use Simpson's rule and the function values at $x = 1, 1.5$, and 2 to approximate $\int_1^2 f(x)\, dx$.

(ii) Use composite Simpson's rule to approximate $\int_1^2 f(x)\, dx$.

(iii) Compare the relative errors in the above two methods.

(5) We want to approximate $\int_1^2 f(x)\, dx$ given the table of values

x	1	5/4	3/2	7/4	2
$f(x)$	10	8	7	6	5

Compute an approximation by the composite trapezoidal rule.

(6) Obtain error formula for the composite Trapezoidal and composite Simpson's rules.

(7) Let $a = x_0 < x_1 < \cdots < x_n = b$ be equally spaced nodes ($i.e.,\ x_k = x_0 + kh$ for $k = 1, 2, \cdots, n$) in the interval $[a, b]$. Note that $h = \frac{b-a}{n}$. Let α_n denote the approximate value of $\int_a^b f(x)\,dx$ by the composite Trapezoidal rule and let e_n denote the corresponding error. Assuming that f is twice continuously differentiable on $[a, b]$, prove that $e_n \to 0$ as $n \to \infty$ (one uses the terminology **Composite trapezoidal rule is convergent** in such a case).

(8) Use composite Simpson's and composite Trapezoidal rules to obtain an approximate value for the improper integral

$$\int_1^\infty \frac{1}{x^2+9}dx, \quad \text{with} \quad n = 4.$$

(9) Determine the minimum number of subintervals and the corresponding step size h so that the error for the composite trapezoidal rule is less than 5×10^{-9} for approximating the integral $\int_2^7 dx/x$.

(10) Determine the coefficients in the quadrature formula

$$\int_0^{2h} x^{-1/2} f(x)\,dx \approx (2h)^{1/2}(w_0 f(0) + w_1 f(h) + w_2 f(2h))$$

such that the formula is exact for all polynomials of degree as high as possible. What is the degree of precision?

(11) Use the two-point Gaussian quadrature rule to approximate

$$\int_{-1}^{1} \frac{dx}{x+2}$$

and compare the result with the trapezoidal and Simpson's rules.

(12) Assume that $x_k = x_0 + kh$ are equally spaced nodes. The quadrature formula

$$\int_{x_0}^{x_3} f(x)dx \approx \frac{3h}{8}\left(f(x_0) + 3f(x_1) + 3f(x_2) + f(x_3)\right)$$

is called the Simpson's $\frac{3}{8}$ rule. Determine the degree of precision of Simpson's $\frac{3}{8}$ rule.

Tutorial Sheet 5

Numerical Integration and Differentiation (contd.)

5.1 Numerical Differentiation

(1) In this problem, perform the calculations using 6-digit rounding arithmetic.

  (i) Find the value of the derivative of the function $f(x) = \sin x$ at $x = 1$ using the forward, backward, and central difference formulae with $h_1 = 0.015625$, and $h_2 = 0.000015$.

  (ii) Find $f'(1)$ directly and compare with the values obtained for each h_i $(i = 1, 2)$.

(2) Obtain the central difference formula for $f'(x)$ using polynomial interpolation with nodes at $x - h$, x, $x + h$, where $h > 0$.

(3) Given the values of the function $f(x) = \ln x$ at $x_0 = 2.0$, $x_1 = 2.2$ and $x_2 = 2.6$, find the approximate value of $f'(2.0)$ using the method based on quadratic interpolation. Obtain an error bound.

(4) The following data corresponds to the function $f(x) = \sin x$.

x	0.5	0.6	0.7
$f(x)$	0.4794	0.5646	0.6442

Obtain the approximate value of $f'(0.5)$, $f'(0.6)$, $f'(0.7)$ using forward, backward, and central difference formulae whichever are applicable. Compute the relative error in all the three cases.

(5) The following data corresponds to the function $f(x) = e^x - 2x^2 + 3x + 1$.

x	0.0	0.2	0.4
$f(x)$	0.0	0.7414	1.3718

Obtain the approximate value of $f'(0.0)$, $f'(0.2)$, $f'(0.4)$ using forward, backward, and central difference formulae whichever are applicable. Compute the relative error in all the three cases.

(6) Obtain expressions for the arithmetic error in approximating the first derivative of a function using the forward, backward, and central difference formulae.

(7) Find an approximation to $f'(x)$ as a formula involving $f(x)$, $f(x+h)$, and $f(x+2h)$. Obtain an expression for the mathematical error involved in this approximation.

(8) Let $h > 0$. Use the method of undetermined coefficients to find a numerical differentiation formula for approximating $f''(x)$ such that the formula uses values of the function f at each of the following sets of points:

(a) $x + 2h$, $x + h$ and x.

(b) $x + 3h$, $x + 2h$ $x + h$ and x.

Obtain expressions for mathematical error in both the cases.

(9) For the method

$$f'(x) \approx \frac{4f(x + h) - f(x + 2h) - 3f(x)}{2h},$$

obtain an expression for mathematical error, arithmetic error, and hence total error. Find a bound on the absolute value of the total error as function of h. Determine the optimal value of h for which the bound obtained is minimum.

(10) Repeat the previous problem when central difference formula is used for numerical differentiation.

Tutorial Sheet 6

Numerical Ordinary Differential Equations

(1) Consider the initial value problem $y'(x) = f(x, y)$, $\;y(x_0) = y_0$, with

$$\frac{\partial f(x,y)}{\partial y} \leq 0,$$

for all $x_0 \leq x \leq x_n$ and for all y, where $x_j = x_0 + jh$, $j = 1, 2, \cdots, n$ are nodes.

(i) Using error analysis of Euler method, show that there exists an $h > 0$ such that

$$|e_n| \leq |e_{n-1}| + \frac{h^2}{2} f''(\xi) \quad \text{for some}\, \xi \in (x_{n-1}, x_n),$$

where $e_n = y(x_n) - y_n$ with y_n obtained using Euler method.

(ii) Applying the conclusion of (i) above recursively, prove that

$$|e_n| \leq |e_0| + n\, h^2\, Y \quad \text{where } Y = \frac{1}{2} \max_{x_0 \leq x \leq x_n} |y''(x)|. \tag{6.1}$$

(2) The solution of the initial value problem

$$y'(x) = \lambda y(x) + \cos x - \lambda \sin x, \quad y(0) = 0$$

is $y(x) = \sin x$. For $\lambda = -20$, find the approximate value of $y(3)$ using Euler method with $h = 0.5$. Compute the error bound given in (6.1), and compare it with the actual absolute error. Reason why the actual absolute error exceeds the computed error bound using the formula (6.1).

(3) Derive an Euler-type method for finding the approximate value of $y(x_n)$ for some $x_n < 0$, where y satisfies the initial value problem $y'(x) = f(x, y)$, $\;y(0) = y_0$.

(4) Consider the initial value problem $y' = xy$, $y(0) = 1$. Estimate the error at $x = 1$ when Euler method (with infinite precision) is used to find an approximate solution to this problem with step size $h = 0.01$.

(5) Find an upper bound for the propagated error in Euler method (with infinite precision) with $h = 0.1$ for solving the initial value problem $y' = y$, $\;y(0) = 1$, in the interval (i) [0,1] and (ii) [0,5].

(6) For each $n \in \mathbb{N}$, write down the Euler method for the solution of the initial value problem $y' = y$, $\;y(0) = 1$ on the interval $[0, 1]$ with step size $h = 1/n$. Let the resulting approximation to $y(1)$ be denoted by α_n. Show using limiting argument (without using the error bound) that $\alpha_n \to y(1)$ as $n \to \infty$.

(7) Consider the initial value problem $y' = -2y,\ 0 \le x \le 1,\ y(0) = 1$.

(i) Find an upper bound on the error in Euler method at $x = 1$ in terms of the step size h.

(ii) For each h, solve the difference equation which results from Euler's method, and obtain an approximate value of $y(1)$.

(iii) Find the error involved in the approximate value of $y(1)$ obtained in (ii) above by comparing with the exact solution.

(iv) Compare the error bound obtained in (i) with the actual error obtained in (iii) for $h = 0.1$, and for $h = 0.01$.

(v) If we want the absolute value of the error obtained in (iii) to be at most 0.5×10^{-6}, then how small the step size h should be?

(8) In each of the following initial value problems, use Euler method, Runge-Kutta method of order 2 and 4 to find the solution at the specified point with specified step size h:

(i) $y' = x + y;\quad y(0) = 1$. Find $y(0.2)$ (For Euler method take $h = 0.1$ and for other methods, take $h = 0.2$) Exact Solution: $y(x) = -1 - x + 2e^x$.

(ii) $y' = 2\cos x - y,\ y(0) = 1$. Find $y(0.6)$ (For Euler method take $h = 0.1$ and for other methods, take $h = 0.2$) Exact Solution: $y(x) = \sin x + \cos x$.

(9) Use Euler, Runge-Kutta methods of order 2 and 4 to solve the IVP $y' = 0.5(x - y)$ for all $x \in [0, 3]$ with initial condition $y(0) = 1$. Compare the solutions for $h = 1, 0.5, 0.25, 0.125$ along with the exact solution $y(x) = 3e^{-x/2} + x - 2$.

(10) Show that the Euler and Runge-Kutta methods fail to determine an approximation to the non-trivial solution of the initial-value problem $y' = y^\alpha,\ 0 < \alpha < 1,\ y(0) = 0$. Note that the ODE is in separable form and hence a non-trivial soluton to initial value problem can be found analytically.

(11) Write the formula using Euler method for approximating the solution to the initial value problem

$$y' = x^2 + y^2, \quad y(x_0) = y_0$$

at the point $x = x_1$ with $h = x_0 - x_1 > 0$. Find the approximate value y_1 of the solution to this initial value problem at the point x_1 when $x_0 = 0$, $y_0 = 1$, and $h = 0.25$. Find a bound for the truncation error in obtaining y_1.

Tutorial - 3

(5) $\|ME_1\|_{\infty,[N,N+1]} \leq 0.25$

(6) $\left|ME_3(3/2)\right| \geq \frac{1}{9216}$

$\approx 0.10851 \times 10^{-3}$

(8) $S(x) = \begin{cases} 2x+7, & x \in [-1,0] \\ 2x+7, & x \in [0,1] \end{cases}$

(9)

$$S(x) = \begin{cases} -4x^3+5x+1, & x \in [0,1] \\ 50x^3-162x^2+167x-53, & x \in [1,2] \\ -46x^3+414x^2-985x+715, & x \in [2,3] \end{cases}$$

(11) Such an a, b, c, d do not exist.

(12) NO.

Tutorial - 4

(1) (a)

$I_R(f) \approx 0.785398,$

$I_T(f) \approx 0.392699$

$I_S(f) \approx 0.624553,$

$I_G(f) \approx 0.623458$

(b)

$I_R(f) \approx 0.349066$

$I_T(f) \approx 1.745329$

$I_S(f) \approx 1.000656$

$I_G(f) \approx 1.041984$

(c)

$I_R(f) \approx 1.0$

$I_T(f) \approx 0.68394$

$I_S(f) \approx 0.747180$

$I_G(f) \approx 0.746595$

(d)

$I_R(f) \approx 0.0$

$I_T(f) \approx 0.0$

$I_S(f) \approx 0.0$

$I_G(f) \approx 0.282821$

(e)

$I_R(f) \approx 1.0$

$I_T(f) \approx 0.860794$

$I_S(f) \approx 1.321276$

$I_G(f) \approx 1.299848$

(2) $E_T(x^4) = -\frac{3}{10}$

$E_T(x^5) = -\frac{1}{3}$

$E_S(x^4) = -\frac{1}{120}$

$E_S(x^5) = -\frac{1}{48}$

$C = \frac{10}{9}$

(3) let $f(a) = f_a + \varepsilon_a$,

$f\left(\frac{a+b}{2}\right) = f_{1/2} + \varepsilon_{1/2}$,

$f(b) = f_b + \varepsilon_b$.

Then $I_T(f) = \left(\frac{b-a}{2}\right)\left(f(a) + f(b)\right)$

$= \frac{b-a}{2}\left[f_a + f_b + \varepsilon_a + f_b\right]$

$\therefore AE_T = I_T(f) - \left(\frac{b-a}{2}\right)(f_a + f_b)$

$= \left(\frac{b-a}{2}\right)(\varepsilon_a + \varepsilon_b)$

$\therefore |AE_T| \leq \frac{b-a}{2} \times \varepsilon$, where

$\varepsilon = \max\{|\varepsilon_a|, |\varepsilon_b|\}$

Similarly, (for Simpson rule),

$$AE_S = I_S(f) - \frac{(b-a)}{6}\left[f_a + 4f_{1/2} + f_b\right]$$

$$= \frac{b-a}{6}\left[\varepsilon_a + 4\varepsilon_{1/2} + \varepsilon_b\right]$$

$$\therefore |AE_S| \le \frac{b-a}{6}\left\{|\varepsilon_a| + 4|\varepsilon_{1/2}| + |\varepsilon_b|\right\}$$

Denote by $\eta = \max\{|\varepsilon_a|, |\varepsilon_{1/2}|, |\varepsilon_b|\}$,

We get

$$|AE_S| \le (b-a)\,\eta .$$

(4) (i) $\frac{43}{6}$, (ii) $\frac{57}{12}$

(6) Error in Composite Trapezoidal rule is

$$E_T^n(f) = \frac{-(b-a)}{12} h^2 f''(\xi)$$

Error in Composite Simpson's rule is

$$E_S^n(f) = -\frac{(b-a)h^4}{2880} f^{(4)}(\xi).$$

(8) use the transformation $t = 1/x$ to get $\displaystyle\int_1^\infty \frac{dx}{x^2+9} = \frac{1}{9}\int_0^1 \frac{dt}{t^2+\left(\frac{1}{3}\right)^2}$.

Now use the composite Simpson's rule and the composite Trapezoidal rule.

(9) $h < 0.2191 \times 10^{-3}$

(10) $w_0 = \frac{12}{15}$, $w_1 = \frac{16}{15}$, $w_2 = \frac{2}{15}$.

Degree of precision is 2.

(12) Degree of precision of Simpson's 3/8 rule is 3.

Tutorial - 5

(1) (i) $h = 0.015625$

$$D_h^- f(1) = 0.54688$$

$$D_h^0 f(1) = 0.540288$$

$$D_h^+ f(1) = 0.533696$$

(ii) $h = 0.000015$

$$D_h^- f(1) = 0.533333$$

$$D_h^0 f(1) = 0.533333$$

$$D_h^+ f(1) = 0.533333$$

(2) $$f'(x) \approx \frac{f(x+h) - f(x-h)}{2h}$$

(6) let $x_0 = x-h$, $x_1 = x$, $x_2 = x+h$;

$f(x_0) = f_0 + \varepsilon_0$, $f(x_1) = f_1 + \varepsilon_1$,

$f(x_2) = f_2 + \varepsilon_2$.

Then $\displaystyle D_h^- f(x) = \frac{f(x_1) - f(x_0)}{h}$

$$= \frac{f_1 - f_0}{h} + \frac{\varepsilon_1 - \varepsilon_0}{h}$$

$\therefore$ Arithmetic error in Backward difference formula is

$$D_h^- f(x) - \frac{f_1 - f_0}{h} = \frac{\varepsilon_1 - \varepsilon_0}{h}.$$

$$\therefore \left| D_h^- f(x) - \frac{f_1 - f_0}{h} \right| \le \frac{|\varepsilon_0| + |\varepsilon_1|}{h}$$

$$\le \frac{2\varepsilon}{h},$$

where $\varepsilon = \max\{|\varepsilon_0|, |\varepsilon_1|\}$.

Similarly,

Arithmetic error in forward difference formula is $\dfrac{\varepsilon_2 - \varepsilon_1}{h}$

and in central difference formula is $\dfrac{\varepsilon_2 - \varepsilon_0}{2h}$

and estimates can be obtained similarly.

(7)

$$f'(x) \simeq \frac{-f(x+2h)+4f(x+h)-3f(x)}{2h}$$

mathematical error is of $O(h^2)$.

(8)(i) $$f''(x) \simeq \frac{f(x+2h)-2f(x+h)+f(x)}{2h^2},$$

mathematical error is of $O(h)$.

(ii)

$$f''(x) \simeq \frac{2f(x)-5f(x+h)+4f(x+2h)-f(x+3h)}{h^2}$$

(9) mathematical error is $\frac{h^2}{3} f'''(\xi)$, for some $\xi \in (x, x+2h)$.

let $f(x) = f_0 + \varepsilon_0$,

$f(x+h) = f_1 + \varepsilon_1$,

$f(x+2h) = f_2 + \varepsilon_2$ and

f' be computed as $\dfrac{4f_1 - f_2 - 3f_0}{2h}$,

then arithmetic error is given by

$$\frac{4\varepsilon_1 - \varepsilon_2 - 3\varepsilon_0}{2h}.$$

Denoting by $\varepsilon = \max\{|\varepsilon_0|, |\varepsilon_1|, |\varepsilon_2|\}$,

we get the following bound for arithmetic error:

$$\frac{4\varepsilon}{h}.$$

Thus
$$|\text{Total error}| \le \frac{4\varepsilon}{h} + \frac{h^2}{3} M,$$

where $M = \|f'''\|_{\infty, [x, x+2h]}$.

The bound is minimum

if $h = \left(\frac{6\varepsilon}{m}\right)^{1/3}$.

(10) $h = \left(\frac{3\varepsilon}{m}\right)^{1/3}$.

Tutorial - 6

(3) $y' \approx \frac{1}{h}\left(y(x) - y(x-h)\right)$,

The required method is

$$y_{n-1} = y_n - h f(x_n, y_n).$$

(4) $|\text{Error}| \le \frac{(0.01) \times 2e^{1/2}}{2}(e-1)$

≈ 0.03

(5) (i) error bound $= (0.05) \times e(e-1)$

≈ 0.23354

(ii) error bound $= (0.05) \times e^5(e^5 - 1)$

≈ 1093.90623

(6) Hint: $y_h(1) = (1+h)^n$

& $\lim_{n \to \infty} \left(1 + \frac{1}{n}\right)^n = e$

(7) (i) $h(e^2 - 1) \approx 6.389056 \times h$.

(ii) $y_{n+1} = (1-2h)\, y_n$

(iii) $y(1) \approx 0.135335$

$y_h(1) \approx 0.107374$ (with $h = 0.1$)

$\therefore$ the actual error is 0.027961.

(v) $h \approx 0.000\,000\,078$.

(8) (i) Euler method : 1.22

R-K of order 4 : 1.2428

Exact solution = 1.2428

(ii) Euler method : 1.419069

R-K of order 4 : 1.389968

Exact solution : 1.389978

Tutorial Sheet 7

Numerical Linear Algebra

7.1 Gaussian Elimination Methods

(1) Given the linear system $2x_1 - 6\alpha x_2 = 3,\ \ 3\alpha x_1 - x_2 = \frac{3}{2}$.
(a) Find value(s) of α for which the system has no solution. (b) Find value(s) of α for which the system has infinitely many solutions. (c) Find value(s) of α for which the system has a unique solution. Also find the solution.

(2) Use Gaussian elimination method (both with and without pivoting) to find the solution of the following systems:
(i) $6x_1 + 2x_2 + 2x_3 = -2,\ 2x_1 + 0.6667x_2 + 0.3333x_3 = 1,\ x_1 + 2x_2 - x_3 = 0$
(ii) $0.729x_1 + 0.81x_2 + 0.9x_3 = 0.6867,\ x_1 + x_2 + x_3 = 0.8338,\ 1.331x_1 + 1.21x_2 + 1.1x_3 = 1$
(iii) $x_1 - x_2 + 3x_3 = 2,\ \ 3x_1 - 3x_2 + x_3 = -1,\ \ x_1 + x_2 = 3.$

(3) Solve the system $0.001x_1 + x_2 = 1,\ \ x_1 + x_2 = 2$ (i) exactly, (ii) by Gaussian elimination using 2-digit rounding, and (iii) by modified Gaussian elimination method with partial pivoting using 2-digit rounding.

(4) Solve the following system by Gaussian elimination, and by modified Gaussian elimination with partial pivoting, using 4-digit rounding.

$$x + 592y = 437,\quad 592x + 4308y = 2251.$$

(5) Solve the system $0.5x_1 - x_2 = -9.5,\ \ 1.02x_1 - 2x_2 = -18.8$ using Gaussian elimination method. Solve the system obtained by changing the coefficient of x_1 in the first equation from 0.5 to 0.52 using Gaussian elimination method. In both the cases, use 5-digit rounding. Obtain the residual error in each case.

(6) Let ϵ be such that $0 < \epsilon \ll 1$. Solve the linear system

$$x_1 + x_2 + x_3 = 6,\quad 3x_1 + (3+\epsilon)x_2 + 4x_3 = 20,\quad 2x_1 + x_2 + 3x_3 = 13$$

using Gaussian elimination method, and using modified Gaussian elimination method with partial pivoting. Obtain the residual error in each case on a computer for which the ϵ is the machine epsilon.

(7) Consider the $n \times n$ system of linear equations given by

$$a_{11}x_1 + \cdots + a_{1n}x_n = b_1, \qquad \cdots \quad , \quad a_{n1}x_1 + \cdots + a_{nn}x_n = b_n,$$

where $a_{ij} = 0$ whenever $i - j \geq 2$. Write the general form of this system. Use Gaussian elimination method to solve it, taking advantage of the elements that are known to be zero. Count the number of operations involved in this computation.

7.2 LU Decomposition

(8) Obtain the Doolittle factorization of the matrix

$$\begin{bmatrix} 4 & 1 & 1 \\ 1 & 4 & -2 \\ 3 & 2 & -4 \end{bmatrix}$$

Use this factorization to solve the system with $\mathbf{b} = (4, 4, 6)^T$.

(9) Show that the matrix $\begin{bmatrix} 1 & 2 & 6 \\ 4 & 8 & -1 \\ -2 & 3 & 5 \end{bmatrix}$ does not have an LU factorization.

(10) Show that the matrix

$$\begin{bmatrix} 2 & 2 & 1 \\ 1 & 1 & 1 \\ 3 & 2 & 1 \end{bmatrix}$$

is invertible but has no LU factorization. Do a suitable interchange of rows to get an invertible matrix, which has an LU factorization.

(11) Prove that the matrix $A = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$ does not have an LU-factorization. (**Hint:** Assume A has an LU-factorization and arrive at a contradiction).

(12) Prove that if an invertible matrix A has an LU-factorization, then all principal minors of A are invertible.

(13) Consider

$$A = \begin{pmatrix} 2 & 6 & -4 \\ 6 & 17 & -17 \\ -4 & -17 & -20 \end{pmatrix}.$$

Determine directly (without using Gaussian elimination) the factorization $A = LDL^T$, where D is diagonal and L is a lower triangular matrix with 1s on its diagonal.

(14) Find the Doolittle factorization of the matrix

$$A = \begin{pmatrix} 3 & 0 & 1 \\ 0 & -1 & 3 \\ 1 & 3 & 0 \end{pmatrix}.$$

(15) Factor the matrix $A = \begin{pmatrix} 1 & 2 \\ 2 & 5 \end{pmatrix}$ so that $A = LL^T$, where L is lower triangular.

(16) Prove or disprove: If a singular matrix has a Doolittle factorization, then that factorization is not unique.

(17) Prove the uniqueness of the factorization $A = LL^T$, where L is a lower triangular matrix all of whose diagonal entries are positive. (**Hint:** Assume that there are lower triangular matrices L_1 and L_2 with positive diagonals. Prove that $L_1 L_2^{-1} = I$.)

Tutorial Sheet 8

Numerical Linear Algebra (contd.)

8.1 Matrix Norms

(1) Show that $\|\mathbf{x}\|_\infty \le \|\mathbf{x}\|_2 \le \|\mathbf{x}\|_1$ for all $\mathbf{x} \in \mathbb{R}^n$, and that equalities can occur, even for nonzero vectors.

(2) Show that the norm defined on the set of all $n \times n$ matrices by

$$\|A\| := \max_{\substack{1\le i\le n\\1\le j\le n}} |a_{ij}|$$

is not subordinate to any vector norm on $\mathbb{R}^n$.

(3) Let A be an invertible matrix. Show that its condition number $\kappa(A)$ satisfies $\kappa(A) \ge 1$.

(4) Let A and B be invertible matrices with condition numbers $\kappa(A)$ and $\kappa(B)$ respectively. Show that $\kappa(AB) \le \kappa(A)\kappa(B)$.

(5) Let $A(\alpha)$ be a matrix depending on a parameter α given by

$$A(\alpha) = \begin{pmatrix} 0.1\alpha & 0.1\alpha \\ 1.0 & 2.5 \end{pmatrix}$$

Determine α such that the condition number of $A(\alpha)$ is minimized. In the computation of condition numbers, use the matrix norm $\|A\|_\infty$ that is subordinate to the l_∞ vector norm on $\mathbb{R}^2$.

(6) What are all the matrices that have condition number equal to 1?

(7) Let A be a matrix such that there exists a non-trivial fixed point $\mathbf{x}$ for it. That is, there exists a non-zero vector $\mathbf{x}$ such that $A\mathbf{x} = \mathbf{x}$. Show that $\|A\| \ge 1$ for any matrix norm that is subordinate to some vector norm.

(8) In solving the system of equations $A\mathbf{x} = \mathbf{b}$ with matrix $A = \begin{pmatrix} 1 & 2 \\ 1 & 2.01 \end{pmatrix}$, predict how slight changes in $\mathbf{b}$ will affect the solution $\mathbf{x}$. Test your prediction in the concrete case when $\mathbf{b} = (4,4)^T$ and $\tilde{\mathbf{b}} = (3,5)^T$. Use l_∞ vector norm for vectors in $\mathbb{R}^2$.

8.2 Iterative Methods

(9) Let A be a diagonally dominant matrix. Show that all the diagonal elements of A are non-zero (*i.e.,* $a_{ii} \ne 0$ for $i = 1, 2, \cdots, n$.).

(10) Find the $n \times n$ matrix B and the n-dimensional vector $\mathbf{c}$ such that the Gauss-Seidal method can be written in the form

$$\mathbf{x}^{(k+1)} = B\mathbf{x}^{(k)} + \mathbf{c}, \qquad k = 0, 1, 2, \cdots$$

(11) Study the iterative sequences of Jacobi and Gauss-Seidel methods for the following systems by starting with $\mathbf{x}^0 = (0,0,0)^T$ and performing three iterations:
(i)$5x_1 + 2x_2 + x_3 = 0.12,\ 1.75x_1 + 7x_2 + 0.5x_3 = 0.1,\ x_1 + 0.2x_2 + 4.5x_3 = 0.5.$
(ii)$x_1 - 2x_2 + 2x_3 = 1,\ \ -x_1 + x_2 - x_3 = 1,\ \ -2x_1 - 2x_2 + x_3 = 1.$
(iii)$x_1 + x_2 + 10x_3 = -1,\ 2x_1 + 3x_2 + 5x_3 = -6,\ 3x_1 + 2x_2 - 3x_3 = 4.$

8.3 Eigenvalue Problems

(12) Use Gerschgorin Circle theorem to prove that any diagonally dominant matrix is invertible.

(13) The matrix

$$A = \begin{pmatrix} 0.7825 & 0.8154 & -0.1897 \\ -0.3676 & 2.2462 & -0.0573 \\ -0.1838 & 0.1231 & 1.9714 \end{pmatrix}$$

has eigenvalues $\lambda_1 = 2$, $\lambda = 2$ and $\lambda_3 = 1$. Does the power method converge for the above matrix? Justify your answer. Perform 5 iterations starting from the initial guess $\mathbf{x}^{(0)} = (1,3,6)$ to verify your answer.

(14) The matrix

$$A = \begin{pmatrix} 2 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 0 & 1 \end{pmatrix}$$

has eigenvalues $\lambda_1 = 2$, $\lambda = 1$ and $\lambda_3 = 1$ and the corresponding eigenvectors may be taken as $\mathbf{v}_1 = (1,2,3)^T$, $\mathbf{v}_2 = (0,1,2)^T$ and $\mathbf{v}_3 = (0,2,1)^T$. Perform 3 iterations to find the eigenvalue and the corresponding eigen vector to which the power method converges when we start the iteration with the initial guess $\mathbf{x}^{(0)} = (0, 0.5, 0.75)^T$. Without performing the iteration, find the eigenvalue and the corresponding eigenvector to which the power method converges when we start the iteration with the initial guess $\mathbf{x}^{(0)} = (0.001, 0.5, 0.75)^T$. Justify your answer.

(15) The matrix

$$A = \begin{pmatrix} 5.4 & 0 & 0 \\ -113.0233 & -0.5388 & -0.6461 \\ -46.0567 & -6.4358 & -0.9612 \end{pmatrix}$$

has eigenvalues $\lambda_1 = 5.4$, $\lambda_2 = 1.3$ and $\lambda_3 = -2.8$ with corresponding eigenvectors $\mathbf{v}_1 = (0.2, -4.1, 2.7)^T$, $\mathbf{v}_2 = (0, 1.3, -3.7)^T$ and $\mathbf{v}_3 = (0, 2.6, 9.1)^T$. To which eigenvalue and the corresponding eigenvector does the power method converge if we start with the initial guess $\mathbf{x}^{(0)} = (0,1,1)$? Justify your answer.

(16) Use Gerschgorin's circle theorem to determine the intervals in which the eigenvalues of the matrix

$$A = \begin{pmatrix} 0.5 & 0 & 0.2 \\ 0 & 3.15 & -1 \\ 0.57 & 0 & -7.43 \end{pmatrix}.$$

lie. Show that power method can be applied for this matrix to find the dominant eigenvalue. Use Power method to compute the dominant eigenvalue and corresponding eigenvector.

Answers for selected problems

Tutorial Sheet 7

(1) (a) $\alpha = -1/3$, (b) $\alpha = 1/3$, (c) $x_1 = \dfrac{3}{2(3\alpha+1)}$,

$x_2 = \dfrac{9\alpha - 3}{2(1-9\alpha^2)}$.

(2) (i) $(2.599928, -3.799904, -4.999880)$

(ii) $(0.224545, 0.281364, 0.327891)$

(iii) $(1.1875, 1.8125, 0.875)$.

(7) Operation Counting

Elimination part :-

Addition/Subtraction $= (n-1) + (n-2) + \cdots + 1 = \dfrac{n(n-1)}{2}$

Multiplication $= (n-1) + (n-2) + \cdots + 1 = \dfrac{n(n-1)}{2}$

Division $= \underbrace{1 + 1 + \cdots + 1}_{(n-1)\text{-times}} = n-1$

modifying RHS vector :-

Addition / Subtraction $= \underbrace{1 + 1 + \cdots + 1}_{(n-1)\text{ times}} = n-1$

Multiplication $= \underbrace{1 + 1 + \cdots + 1}_{(n-1)\text{ times}} = n-1$

Division $= \underbrace{1 + 1 + \cdots + 1}_{(n-1)\text{-times}} = n-1$

Back Substitution :-

Addition/Subtraction $= 1 + 2 + \cdots + (n-2) + (n-1) = \dfrac{n(n-1)}{2}$

Multiplication/division $= 1 + 2 + \cdots + (n-1) + n = \dfrac{n(n+1)}{2}$.

(8) $L = \begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{4} & 1 & 0 \\ \frac{3}{4} & \frac{1}{3} & 1 \end{bmatrix}$, $U = \begin{bmatrix} 4 & 1 & 1 \\ 0 & \frac{15}{4} & -\frac{9}{4} \\ 0 & 0 & -4 \end{bmatrix}$

Tutorial Sheet - 8

(5) $|\alpha| = 12.5$

(11) (i) Both Jacobi and Gauss-Seidel Converges

(ii) Jacobi converges but Gauss-Seidel does not converge.

(iii) Jacobi diverges but Gauss-Seidel converges.

(13) NO.

(16) $D_1 = \{ z \in \mathbb{C} \mid |z - 0.5| \le 0.2 \}$

$D_2 = \{ z \in \mathbb{C} \mid |z - 3.15| \le 1 \}$

$D_3 = \{ z \in \mathbb{C} \mid |z + 7.43| \le 0.57 \}$

Since D_3 is disjoint from D_1 + D_2 and contains the dominant eigenvalue by G.T, we can use Power method to compute this ~~eo~~ eigenvalue.

Tutorial Sheet 9

Nonlinear Equations

9.1 Fixed-Point Iteration Method

(1) For each of the following equations, find the correct iteration function that converges to the desired solution:
(a) $x - \tan x = 0$, (b) $e^{-x} - \cos x = 0$.
Study geometrically how the iterations behave with different iteration functions.

(2) Show that $g(x) = \pi + \frac{1}{2}\sin(x/2)$ has a unique fixed point on $[0, 2\pi]$. Use fixed-point iteration method with g as the iteration function and $x_0 = 0$ to find an approximate solution for the equaton $\frac{1}{2}\sin(x/2) - x + \pi = 0$. Stop the iteration when the residual error is less than 10^{-4}.

(3) Let α and β be the roots of $x^2 + ax + b = 0$. If the iterations

$$x_{n+1} = -\frac{ax_n + b}{x_n} \text{ and } x_{n+1} = -\frac{b}{x_n + a}$$

converge, then show that they converge to α and β, respectively, if $|\alpha| > |\beta|$.

(4) Let $\{x_n\} \subset [a, b]$ be a sequence generated by a fixed point iteration method with continuous iteration function $g(x)$. If this sequence converges to x^*, then show that

$$|x_{n+1} - x^*| \leq \frac{\lambda}{1-\lambda}|x_{n+1} - x_n|,$$

where $\lambda := \max_{x\in[a,b]} |g'(x)|$. (This estimate helps us to decide when to stop iterating.)

(5) Explain why the sequence of iterates $x_{n+1} = 1 - 0.9x_n^2$, with initial guess $x_0 = 0$, does not converge to any solution of the quadratic equation $0.9x^2 + x - 1 = 0$? [Hint: Observe what happens after 25 iterations]

(6) Let x^* be the smallest positive root of the equation $20x^3 - 20x^2 - 25x + 4 = 0$. If the fixed-point iteration method is used in solving this equation with the iteration function $g(x) = x^3 - x^2 - \frac{x}{4} + \frac{1}{5}$ for all $x \in [0, 1]$ and $x_0 = 0$, then find the number of iterations n required so that $|x^* - x_n| < 10^{-3}$.

9.2 Bisection Method

(7) Find the number of iterations to be performed in the bisection method to obtain a root of the equation

$$2x^6 - 5x^4 + 2 = 0$$

in the interval $[0, 1]$ with absolute error less than or equal to 10^{-3}. Find the approximation solution.

(8) Find the approximate solution of the equation $x \sin x - 1 = 0$ (x is in radians) in the interval $[0, 2]$ using Bisection method. Obtain the number of iterations to be performed to obtain a solution whose absolute error is less than 10^{-3}.

(9) Find the root of the equation $10^x + x - 4 = 0$ correct to four significant digits by the bisection method.

9.3 Secant and Newton-Raphson Method

(10) Give an example of a nonlinear equation $f(x) = 0$ for which the secant method iterating sequence does not exist.

(11) Determine the initial approximations for finding the smallest positive root of the equations: (a) $x^4 - x - 10 = 0$, (b) $x - e^{-x} = 0$.
Use these to find the roots upto a desired accuracy with secant and Newton-Raphson methods.

(12) Find the iterative method based on Newton-Raphson method for finding $\sqrt{N}$ and $N^{1/3}$, where N is a positive real number. Apply the methods to $N = 18$ to obtain the results correct to two significant digits.

(13) Find the iterative method based on the Newton-Raphson method for approximating the root of the equation $\sin x = 0$ in the interval $(-\pi/2, \pi/2)$.
Let $\alpha \in (-\pi/2, \pi/2)$ and $\alpha \neq 0$ be such that if the above iterative process is started with the initial guess $x_0 = \alpha$, then the iteration becomes a cycle in the sense that $x_{n+2} = x_n$, for $n = 0, 1, \cdots$. Find a non-linear equation $g(x) = 0$ whose solution is α.
Starting with the initial guess $x_0 = \alpha$, perform the first five iterations using Newton-Raphson method for the equation $\sin x = 0$.
Starting with the initial guess $x_0 = 1$, perform five iterations using Newton-Raphson method for the equation $g(x) = 0$ to find an approximate value of α.

(14) Consider the equation $x \sin x - 1 = 0$. Choose an initial guess $x_0 > 1$ such that the Newton-Raphson method converges to the solution x^* of this equation such that $-10 < x^* < -9$. Compute four iterations and give an approximate value of this x^*. For the same equation, choose another initial guess $x_0 > 1$ such that the Newton-Raphson method converges to the smallest positive root of this equation. Compute four iterations and give an approximate value of this smallest positive root.

(15) Give an initial guess x_0 for which the Newton-Raphson method does not converge while trying to obtain the real root for the equation $\frac{1}{3}x^3 - x^2 + x + 1 = 0$. Give reasons.

(16) Can Newton-Raphson method be used to solve $f(x) = 0$ where
(i) $f(x) = x^2 - 14x + 50$? (ii) $f(x) = x^{1/3}$? (iii) $f(x) = (x - 3)^{1/2}$ with $x_0 = 4$?
Give reasons.

9.4 Systems of Nonlinear Equations

(17) Using Newton's method, obtain a root for the following nonlinear systems:
(i) $x_1^2 + x_2^2 - 2x_1 - 2x_2 + 1 = 0, \quad x_1 + x_2 - 2x_1x_2 = 0.$
(ii) $4x_1^2 + x_2^2 - 4 = 0, \quad x_1 + x_2 - \sin(x_1 - x_2) = 0.$

(18) Use Newton's method to find the minimum value of the function $f(\boldsymbol{x}) = x_1^4 + x_1x_2 + (1 + x_2)^2$.