Torsion of non-circular shaft

$$\boxed{\begin{aligned} u &= -\theta y z \\ v &= \theta x z \\ w &= 0 \end{aligned}} \longrightarrow \boxed{\begin{aligned} u &= -\theta y z \\ v &= \theta x z \\ w &= \theta\psi(x,y) \end{aligned}}$$

for circular c/s

non-circular c/s
Saint Venant's solution
undergoes out of plane warping unlike circular c/s

$\psi(x,y)$ → warping function.

$$\therefore \quad \tau_{xz} = G\theta\left(\frac{\partial\psi}{\partial x} - y\right) \qquad \sigma_x = \sigma_y = \sigma_z = 0$$

$$\tau_{yz} = G\theta\left(\frac{\partial\psi}{\partial y} + x\right) \qquad \tau_{xy} = 0$$

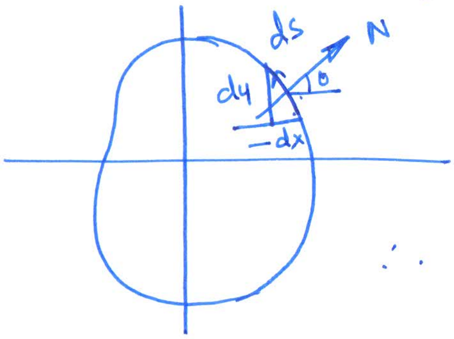
Eqn. of equilm

$$\Rightarrow \boxed{\frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} = 0} \quad \text{——— (1)}$$

Boundary condns on the lateral surfaces

$$\tau_{xz}\, l + \tau_{yz}\, m = 0$$

$$\Rightarrow G\theta\left(\frac{\partial\psi}{\partial x} - y\right) l + G\theta\left(\frac{\partial\psi}{\partial y} + x\right) m = 0$$



$$\cos(Nx) = l = \frac{dy}{ds}$$

$$\cos(Ny) = m = \frac{-dx}{ds}$$

$$\therefore \boxed{G\theta\left(\frac{\partial\psi}{\partial x} - y\right)\frac{dy}{ds} - G\theta\left(\frac{\partial\psi}{\partial y} + x\right)\frac{dx}{ds} = 0} \quad (2)$$

$\therefore$ Saint Venant's solution requires a warping function satisfying eqns. (1) and (2)

Torsion of non-circular shaft

(i) cross-sectional shape is not distorted

(ii) out-of-plane warping governed by $\psi(x,y)$

(iii) warping is constant along the length of the shaft

(iv) $\frac{d\phi}{dz} = \theta$ is constant

(v) it is in pure shear

Prandtl's stress function

$$\tau_{xz} = \frac{\partial \varphi}{\partial y} \leftarrow \text{Prandtl's stress function} \qquad \tau_{yz} = -\frac{\partial \varphi}{\partial x}$$

Eqns. of equilm

$$\frac{\partial \cancel{\sigma_x}^{=0}}{\partial x} + \frac{\partial \cancel{\tau_{xy}}^{=0}}{\partial y} + \cancel{\frac{\partial \tau_{xz}}{\partial z}}^{=0} = 0$$

$$\frac{\partial \cancel{\tau_{xy}}^{=0}}{\partial x} + \frac{\partial \cancel{\sigma_y}^{=0}}{\partial y} + \cancel{\frac{\partial \tau_{yz}}{\partial z}}^{=0} = 0$$

$$\frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \cancel{\frac{\partial \sigma_z}{\partial z}}^{=0} = 0$$

$$\frac{\partial \varphi}{\partial x \partial y} - \frac{\partial \varphi}{\partial x \partial y} = 0$$

Eqns. of equilm are satisfied.

$$\tau_{xz} = \frac{\partial \varphi}{\partial y} = G\theta\left(\frac{\partial \psi}{\partial x} + y\right)$$

$$\tau_{yz} = -\frac{\partial \varphi}{\partial x} = G\theta\left(\frac{\partial \psi}{\partial y} - x\right)$$

$$\therefore \boxed{\frac{\partial^2 \varphi}{\partial y^2} + \frac{\partial^2 \varphi}{\partial x^2} = -2G\theta} \quad \text{——— (1)}$$

Boundary condns on lateral surfaces

$$\tau_{xz} l + \tau_{yz} m = 0 \Rightarrow \frac{\partial \varphi}{\partial y} \cdot \frac{dy}{ds} + \frac{\partial \varphi}{\partial x} \frac{dx}{dx} = 0$$

$$\Rightarrow \boxed{\frac{d\varphi}{ds} = 0} \quad \text{——— (2)}$$

φ is taken as zero on the boundary.

Prandtl's stress function should satisfy eqns. (1) and (2)

BCs on top and bottom surfaces

$$\hat{X} = \tau_{xz} \quad \text{and} \quad \hat{Y} = \tau_{yz}$$

Resultant
force in x-dirc due to BC
on top and bottom surfaces.

$$S_x = \iint_{x\,y} \hat{X}\,dx\,dy = \iint \tau_{xz}\,dx\,dy = \iint \frac{\partial \varphi}{\partial y}\,dx\,dy$$

$$= \int dx \int \frac{\partial \varphi}{\partial y}\,dy = \oint \varphi\,dx \quad \text{[Using Green's theorem]}$$

$$= 0$$

as φ is zero on the boundary.

$$S_y = \iint_{x\,y} \hat{Y}\,dx\,dy = \iint \tau_{yz}\,dx\,dy = -\iint \frac{\partial \varphi}{\partial x}\,dx\,dy$$

$$= -\int dy \int \frac{\partial \varphi}{\partial x}\,dx = -\oint \varphi\,dy$$

$$= 0$$

Torsional moment $\quad M_t = \displaystyle\iint_{x\,y} (\hat{Y}x - \hat{X}y)\,dx\,dy$

$$= \iint_{x\,y} (\tau_{yz}\,x - \tau_{xz}\,y)\,dx\,dy = \iint -\frac{\partial \varphi}{\partial x}\,x\,dx\,dy - \iint \frac{\partial \varphi}{\partial y}\,y\,dx\,dy$$

$$= -\iint \frac{\partial}{\partial x}(\varphi x)\,dx\,dy - \iint \frac{\partial}{\partial y}(\varphi y)\,dx\,dy + 2\iint \varphi\,dx\,dy$$

$$= -\overbrace{\oint \cancel{\varphi x}\,dy}^{=0} - \overbrace{\oint \cancel{\varphi}\,y\,dx}^{=0} + 2\iint \varphi\,dx\,dy = 2\iint \varphi\,dx\,dy$$

$$\therefore \quad \boxed{M_t = 2\iint \varphi\,dx\,dy}$$