# Lecture 16

Last time we did LU factorization of a matrix A

Suppose A can be reduced via Gauss Elimination to an upper-triangular matrix U without any row changes

$$L = \begin{pmatrix} 1 & & & & 0 \\ m_{21} & 1 & & & \\ m_{31} & m_{32} & 1 & & \\ \vdots & \vdots & & \ddots & \\ m_{n1} & m_{3n} & & & 1 \end{pmatrix}$$

$m_{ij}$ multiplier in GE

Then $A = LU$

The system $Ax = b$ can be solved in two steps

<u>Step 1</u> Set $y = Ux$

So $Ax = b$

$\Rightarrow LUx = b$

$\Rightarrow Ly = b$

we solve for $y$ by forward substitution

<u>Step 2</u> We solve $Ux = y$

by backward substitution

— x —

LU factorization is very useful if one wants to solve $Ax = b$ for many different values of $b$

Special classes of matrices where GE can be done without row changes

1) <u>strictly diagonally dominant matrix</u>

i.e, $A = (a_{ij})$ where

$$|a_{ii}| > \sum_{\substack{j=1 \\ j \neq i}}^{n} |a_{ij}|$$

2) <u>Positive definite matrics</u>

A matrix $A$ is positive definite if it is symmetric i.e $A = A^t$ and

$x^t A x > 0$ for every non-zero $x$.

Today

## Cholesky's Algorithm

Given a positive definite $n \times n$ matrix $A$ it factors $A$ into $L L^t$ where $L$ is Lower triangular.

**Example**

$$A = \begin{bmatrix} 4 & 2 & 14 \\ 2 & 17 & -5 \\ 14 & -5 & 83 \end{bmatrix}$$

$$A = L L^t$$

$$\begin{bmatrix} 4 & 2 & 14 \\ 2 & 17 & -5 \\ 14 & -5 & 83 \end{bmatrix} = \begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix} \begin{bmatrix} l_{11} & l_{21} & l_{31} \\ 0 & l_{22} & l_{32} \\ 0 & 0 & l_{33} \end{bmatrix}$$

$$l_{11}^2 = 4 \implies l_{11} = 2$$
$$l_{21} l_{11} = 2 \implies l_{21} = 1$$
$$l_{31} l_{11} = 14 \implies l_{31} = 7$$

$$l_{21}^2 + l_{22}^2 = 17$$

$$l_{22}^2 = 16 \Rightarrow l_{22} = 4$$

$$l_{31} l_{21} + l_{32} l_{22} = -5$$

$$7 + l_{32} \cdot 4 = -5 \Rightarrow l_{32} = -3$$

$$l_{31}^2 + l_{32}^2 + l_{33}^2 = 83$$

$$49 + 9 + l_{33}^2 = 83$$

$$l_{33}^2 = 25 \Rightarrow l_{33} = 5$$

Thus

$$L = \begin{pmatrix} 2 & 0 & 0 \\ 1 & 4 & 0 \\ 7 & -3 & 5 \end{pmatrix}$$

# Cholesky's Algorithm

To factor $A$ into $LL^t$ where $L$ is lower triangular. (Here $A$ is positive definite)

$L = (l_{ij})$

**Step 1** set $l_{11} = \sqrt{a_{11}}$

**Step 2** for $j = 2, \ldots, n$ set

$$l_{j1} = \frac{a_{j1}}{l_{11}}$$

**Step 3** For $i = 2, \ldots, n-1$ do steps a, b

**Step a** set $l_{ii} = \left( a_{ii} - \sum_{k=1}^{i-1} l_{ik}^2 \right)^{\frac{1}{2}}$

**Step b** for $j = i+1, \ldots, n$

set $l_{ji} = \left( a_{ji} - \sum_{k=1}^{i-1} l_{jk} l_{ik} \right) / l_{ii}$

<u>Step 7</u> Set

$$l_{nn} = \left( a_{nn} - \sum_{k=1}^{n-1} l_{nk}^2 \right)^{\frac{1}{2}}$$

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<u>Why do Cholesky factorization</u>

LU factorization, requires $O(n^3/3)$ multiplication/division and $O(n^3/3)$ addition / subtraction

The $LL^t$ Cholesky factorization requires $O(n^3/6)$ multiplication/division and $O(n^3/6)$ addition/subtraction

Thus it requires only 50%

disadvantage of Cholesky algorithm is that it is only valid for positive definite matrices

Note that LU decomposition is possible if GE can be done without row changes

<u>what to do when GE has row changes?</u>

An $n \times n$ permutation matrix $P = (p_{ij})$ is obtained by rearranging the rows of $I$

<u>Example</u> $P = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$ is a $3 \times 3$ permutation matrix

Then $PA = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$

$= \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{31} & a_{32} & a_{33} \\ a_{21} & a_{22} & a_{23} \end{pmatrix}$ $R_2 \leftrightarrow R_3$ in $A$

Two useful properties of permutation matrices

→ Suppose $k_1, \ldots, k_n$ is a permutation of $1, \ldots, n$. and the permutation matrix $P = (p_{ij})$ is defined by

$$p_{ij} = \begin{cases} 1 & \text{if } j = k_i \\ 0 & \text{otherwise} \end{cases}$$

Then

(i) $PA$ permutes the rows of $A$

(ii) $P^{-1}$ exists and $P^{-1} = P^t$

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## PLU factorization of a matrix

Let $A$ be a matrix.

Suppose if possible we have done some row changes while doing GE on $A$

This implies that there exists a permutation matrix $P$ such that GE can be done on $PA$ without any row changes

Thus $PA = LU$ so $A = P^t LU$

<u>Solving</u> $Ax = b$

$$PAx = Pb = b'$$

$$LUx = b'$$

$$y = Ux$$

first <u>Solve</u> $Ly = b'$

Then solve $Ux = y$.

example

$$A = \begin{bmatrix} 0 & 1 & -1 & 1 \\ 1 & 1 & -1 & 2 \\ -1 & -1 & 1 & 0 \\ 1 & 2 & 0 & 2 \end{bmatrix}$$

$R_1 \leftrightarrow R_2$

$$\begin{bmatrix} 1 & 1 & -1 & 2 \\ 0 & 1 & -1 & 1 \\ -1 & -1 & 1 & 0 \\ 1 & 2 & 0 & 2 \end{bmatrix}$$

$R_3 + R_1$ $\qquad$ $R_4 - R_1$ gives

$$\begin{bmatrix} 1 & 1 & -1 & 2 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 2 \\ 0 & 1 & 1 & 0 \end{bmatrix}$$

$R_3 \leftrightarrow R_4$

$$\begin{bmatrix} 1 & 1 & -1 & 2 \\ 0 & 1 & -1 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}$$

$R_3 - R_2$

$$\begin{bmatrix} 1 & 1 & -1 & 2 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 2 & -1 \\ 0 & 0 & 0 & 2 \end{bmatrix} = U$$

$P = \begin{matrix} R_1 \leftrightarrow R_2 \\ R_3 \leftrightarrow R_4 \end{matrix}$ done on identity matrix.

$$P = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$$PA = \begin{bmatrix} 1 & 1 & -1 & 2 \\ 0 & 1 & -1 & 1 \\ 1 & 2 & 0 & 2 \\ -1 & -1 & 1 & 0 \end{bmatrix}$$

$R_3 - R_1 \qquad R_4 + R_1$ gives

$$\begin{bmatrix} 1 & 1 & -1 & 2 \\ 0 & 1 & -1 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}$$

$R_3 - R_2$ gives

$$\begin{bmatrix} 1 & 1 & -1 & 2 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 2 & -1 \\ 0 & 0 & 0 & 2 \end{bmatrix} = U$$

$$L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ -1 & 0 & 0 & 1 \end{bmatrix}$$

# Errors associated with Gauss Elimination

## Example

$$0.0003 x_1 + 1.566 x_2 = 1.569$$
$$0.3454 x_1 - 2.436 x_2 = 1.018$$

exact answer : $x_1 = 10, \quad x_2 = 1$

4 decimal floating arithmetic

$$\left[\begin{array}{cc|c} 0.0003 & 1.566 & 1.569 \\ 0.3454 & -2.436 & 1.018 \end{array}\right]$$

$$m_{21} = \frac{0.3454}{0.0003} = 1,151$$

$$a_{22}^{(2)} = -2.436 - (1,151)(1.566)$$
$$= -1,804$$

$$b_2^{(2)} = 1.018 - (1,152)(1.569)$$
$$= -1805$$

$$\left[\begin{array}{cc|c} 0.0003 & 1.566 & 1.569 \\ 0 & -1804 & -1805 \end{array}\right]$$

$$x_2 = \frac{-1805}{-1804} = 1.001$$

Hence from the first equation

$$x_1 = \frac{1.569 - (1.566)(1.001)}{0.0003}$$

$$= 3.333 \qquad \text{(exact is 10)}$$

$x_1$ has lot of error.

## Plausible explanation

$a_{11} = 0.0003$ is very small.

So the algorithm performs badly for $a_{11}$ "near zero".

However, consider the system in Example, but with first equation multiplied by $10^m$ where $m$ is some integer

$$0.0003 \cdot 10^m x_1 + 1.566 \cdot 10^m x_2 = 1.569 \cdot 10^m$$

$$0.3454 x_1 - 2.436 x_2 = 1.018$$

$$m_{21} = \frac{0.3454}{0.0003 \cdot 10^m} = 1{,}151 \cdot 10^{-m}$$

$$a_{22}^{(2)} = -2.436 - (1{,}151 \cdot 10^{-m})(1.566 \, 10^m)$$

$$= -1804$$

Ily $b_{22}^{(2)} = -1805$

we get $x_2 = 1.001$

and finally $x_1 = 3.333$

Explanation of the error

$|a_{11}|$ is small compared with $|a_{12}|$
thus a small error in computed value of $X_2$ leads to a large error in $x_1$

$$\left|\frac{a_{12}}{a_{11}}\right| \simeq 5220$$

$$\left|\frac{a_{22}}{a_{21}}\right| \simeq 6$$

So we do $R_1 \leftrightarrow R_2$

We get $m_{11} = \frac{0.0003}{0.3454} = 0.0008686$

So now "new" second eqn becomes

$$1.568\,x_2 = 1.568$$

$$x_2 = 1$$

and from "new" first equation

we get $x_1 = 10$

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## Scaled partial pivoting

Step 1 $\quad s_i = \max_{1 \le j \le n} |a_{ij}|$

scale factor for row $i$

$s_i \neq 0$ since otherwise all entries in row $i$ is zero $\Rightarrow$ A is singular $\ast$

$$\frac{|a_{p1}|}{s_p} = \max_{1 \le k \le n} \frac{|a_{k1}|}{s_k}$$

perform $R_1 \Leftrightarrow R_p$ (if $p \neq 1$)

In a similar manner before eliminating variable $x_i$ from rows $i+1, \ldots, n$

we select the smallest integer $p \geq i$ such that

$$\frac{|a_{pi}|}{s_p} = \max_{i \le k \le n} \frac{|a_{ki}|}{s_k}$$

and perform $R_i \Leftrightarrow R_p$ if $p \neq i$

Example (Calculation in 3 sig digit)

$$2.11x_1 - 4.21x_2 + 0.921x_3 = 2.01$$
$$4.01x_1 + 10.2x_2 - 1.12x_3 = -3.09$$
$$1.09x_1 + 0.987x_2 + 0.832x_3 = 4.21$$

$s_1 = 4.21 \qquad s_2 = 10.2 \qquad s_3 = 1.09$

$$\frac{|a_{11}|}{s_1} = \frac{2.11}{4.21} = 0.501$$

$$\frac{|a_{21}|}{s_2} = \frac{4.01}{10.2} = 0.393$$

$$\frac{|a_{31}|}{s_3} = \frac{1.09}{1.09} = 1$$

So we do $R_1 \leftrightarrow R_3$

$$\left[\begin{array}{ccc|c} 1.09 & 0.987 & 0.832 & 4.21 \\ 4.01 & 10.2 & -1.12 & -3.09 \\ 2.11 & -4.21 & 0.921 & 2.01 \end{array}\right]$$

$$R_2 - \frac{4.09}{1.09} R_1 \qquad R_3 - \frac{2.11}{1.09} R_1$$

gives

$$\left[\begin{array}{ccc|c} 1.09 & 0.987 & 0.832 & 4.21 \\ 0 & 6.57 & -4.18 & -18.6 \\ 0 & -6.12 & -0.689 & -6.16 \end{array}\right]$$

$$\frac{|a_{22}|}{s_2} = \frac{6.57}{10.2} = 0.644$$

$$\frac{|a_{32}|}{s_3} = \frac{6.12}{4.21} = 1.45$$

so do $R_3 \leftrightarrow R_2$

and do further calculations