# AE 639 - Continuum Mechanics

## Assignment 1

1. (a) $\vec{u}\cdot\vec{v} = u_i v_i$

LHS $= u_1 v_1 + u_2 v_2 + u_3 v_3 =$ RHS

(b) $\vec{u}\times\vec{v} = \varepsilon_{ijk} u_i v_j e_k = \varepsilon_{ijk} e_i u_j v_k$ (open LHS & RHS)

(c) $(\vec{u}\times\vec{v})\cdot\vec{w} = \vec{u}\cdot(\vec{v}\times\vec{w})$

LHS : $(\varepsilon_{ijk} u_i v_j \hat{e}_k)\cdot(w_l \hat{e}_l) = \varepsilon_{ijk}\delta_{kl} u_i v_j w_l = \varepsilon_{ijl} u_i v_j w_l$

RHS : $(u_i \hat{e}_i)\cdot(\varepsilon_{jkl} v_j w_k \hat{e}_l) = \varepsilon_{jkl}\delta_{il} u_i v_j w_k$

$= \varepsilon_{jki} u_i v_j w_k = \varepsilon_{ijk} u_i v_j w_k$

$= \varepsilon_{ijl} u_i v_j w_l =$ LHS

(d) $(\vec{u}\times\vec{v})\times\vec{w} = (\vec{u}\cdot\vec{w})\vec{v} - (\vec{v}\cdot\vec{w})\vec{u}$

$= (\varepsilon_{ijk} u_i v_j \hat{e}_k)\times(w_m \hat{e}_m) = \varepsilon_{ijk} u_i v_j w_m(\hat{e}_k\times\hat{e}_m)$

$= u_i v_j w_m(\varepsilon_{ijk}\varepsilon_{kmn}\hat{e}_n)$

$= \hat{e}_n u_i v_j w_m[\varepsilon_{ijk}\varepsilon_{mnk}] = \hat{e}_n[u_i v_j w_m][\delta_{im}\delta_{jn} - \delta_{in}\delta_{jm}]$

$= (u_i w_m \delta_{im})(\delta_{jn}\hat{e}_n v_j) - (v_j w_m \delta_{jm})(\delta_{in}\hat{e}_n u_i)$

$= (u_i w_i)(v_j \hat{e}_j) - (v_j w_j)(u_i \hat{e}_i) = (\vec{u}\cdot\vec{w})\vec{v} - (\vec{v}\cdot\vec{w})(\vec{u})$

(e) $(\vec{u}\times\vec{v})\cdot(\vec{u}\times\vec{v})$

$= (\varepsilon_{ijk} u_i v_j \hat{e}_k)\cdot(\varepsilon_{mnp} u_m v_n \hat{e}_p)$

$= (u_i u_m v_j v_n)(\hat{e}_k\cdot\hat{e}_p)(\varepsilon_{ijk}\varepsilon_{mnp}) = (u_i u_m v_j v_n)(\delta_{kp})(\varepsilon_{ijk}\varepsilon_{mnp})$

$= (u_i u_m v_j v_n)(\varepsilon_{ijk}\varepsilon_{mnk}) = (u_i u_m v_j v_n)(\delta_{im}\delta_{jn} - \delta_{in}\delta_{mj})$

$= \cancel{(u_i u_i)(v_j v_j)}$

$= (u_i u_m \delta_{im})(v_j v_n \delta_{jn}) - (u_i v_n \delta_{in})(u_m v_j \delta_{mj})$

$= (\vec{u}\cdot\vec{u})(\vec{v}\cdot\vec{v}) - (\vec{u}\cdot\vec{v})(\vec{u}\cdot\vec{v})$

$=$

2. (a) $\text{Det}[A] = \epsilon_{ijk} A_{1i} A_{2j} A_{3k}$

Just expand both side

(b) Take 3 cases → cyclic, anti-cyclic, non-cyclic

(c) from (b)

$$\epsilon_{lmn} \text{Det}[A] = \epsilon_{ijk} A_{il} A_{jm} A_{kn}$$

multiply $\epsilon_{lmn}$ on both side :

$$\underbrace{\epsilon_{lmn} \epsilon_{lmn}}_{=6} \text{Det}[A] = \epsilon_{lmn} \epsilon_{ijk} A_{il} A_{jm} A_{kn}$$

(try to Prove)

$$\therefore \quad \text{Det}[A] = \frac{1}{6} \epsilon_{lmn} \epsilon_{ijk} A_{il} A_{jm} A_{kn}$$

3. $A_{ii} = A_{11} + A_{22} + A_{33}$

$$B_{ijj} = \begin{cases} B_{111} + B_{122} + B_{133} \\ B_{211} + B_{222} + B_{233} \\ B_{311} + B_{322} + B_{333} \end{cases}$$

$$a_i T_{ij} = \begin{cases} a_1 T_{11} + a_2 T_{21} + a_3 T_{31} \\ a_1 T_{12} + a_2 T_{22} + a_3 T_{32} \\ a_1 T_{13} + a_2 T_{23} + a_3 T_{33} \end{cases}$$

$$a_i b_j S_{ij} = a_1 b_1 S_{11} + a_1 b_2 S_{12} + a_1 b_3 S_{13} + a_2 b_1 S_{21} + a_2 b_2 S_{22} + a_2 b_3 S_{23} + a_3 b_1 S_{31} + a_3 b_2 S_{32} + a_3 b_3 S_{33}$$

4. $B_{ij} = -B_{ji}$ (as B is skew-symmetric)

$b_i = \frac{1}{2} \epsilon_{ijk} B_{jk}$

$$\epsilon_{pqi} b_i = \frac{1}{2} (\epsilon_{pqi} \epsilon_{ijk}) B_{jk} = \frac{B_{jk}}{2} (\epsilon_{pqi} \epsilon_{jki}) = \frac{B_{jk}}{2} [\delta_{pj} \delta_{qk} - \delta_{pk} \delta_{qj}]$$

$$= \frac{1}{2} [B_{pq} - B_{qp}] = B_{pq}$$

5. $B_{ij} = -B_{ji} \Rightarrow B_{11} = B_{22} = B_{33} = 0$

$A_{ij} = A_{ji}$

Just expand $A_{ij} B_{ij}$. All terms cancel $\Rightarrow A_{ij} B_{ij} = 0$

6.

$$T_{ij} = \lambda \varepsilon_{kk} \delta_{ij} + 2\mu \varepsilon_{ij}$$

Please note that this equation means

$$T_{11} = \lambda(\varepsilon_{11} + \varepsilon_{22} + \varepsilon_{33}) + 2\mu \varepsilon_{11} \quad \text{etc.}$$

Contracting the eq$^n$

$$\delta_{ij} T_{ij} = \lambda \varepsilon_{kk} \delta_{ij} \delta_{ij} + 2\mu \varepsilon_{ij} \delta_{ij}$$

$$\Rightarrow \quad T_{ii} = 3\lambda \varepsilon_{kk} + 2\mu \varepsilon_{ii} \qquad [\varepsilon_{kk} = \varepsilon_{ii}]$$

$$\Rightarrow \quad \varepsilon_{kk} = \frac{T_{mm}}{2\mu + 3\lambda}$$

$$\Rightarrow \quad T_{ij} = \lambda \left(\frac{T_{mm}}{2\mu + 3\lambda}\right) \delta_{ij} + 2\mu \varepsilon_{ij}$$

$$\Rightarrow \quad \varepsilon_{ij} = \frac{1}{2\mu}\left(T_{ij} - \frac{\lambda T_{mm} \delta_{ij}}{2\mu + 3\lambda}\right) \qquad \underline{\text{Ans.}}$$

7. (a) $A = A_{ij}$ (d) $A^2 = A_{ik} A_{kj}$

(b) $AB = A_{ik} B_{kj}$ (e) $BA = B_{ik} A_{kj}$

(c) $A^T B = A_{ki} B_{kj}$ (f) $A^T B A = A_{ki} B_{km} A_{mj}$

8. ~~Let~~ for Kronecker delta:

To prove $\delta_{ij}$ is isotropic 2$^{nd}$ order tensor, we write it tensor notation to transformed co-ordinate system and show it gives $\delta' = \delta$

$$\delta'_{mn} = a_{mi} a_{nj} \delta_{ij} = a_{mj} a_{nj} = \delta_{mn} \quad [\text{proved}]$$

Similarly for $\varepsilon_{ijk}$:

$$\varepsilon'_{\alpha\beta\gamma} = a_{\alpha i}\, a_{\beta j}\, a_{\gamma k}\, \varepsilon_{ijk} = \varepsilon_{\alpha\beta\gamma} \underbrace{\det[A]}_{\to 1} \quad \{\text{see } 2(b)\}$$

$\to 1$ [That's how transf. is defined]

$$= \varepsilon_{\alpha\beta\gamma} \quad \underline{\text{proved}}$$

9. (a) $\delta_{1j}\,\delta_{jk}\,\delta_{kl}\,\delta_{il}$

$$= \delta_{ij}\,\delta_{jk}\,\delta_{ki} = \delta_{ij}\,\delta_{ji} = \delta_{ii} = 3$$

(b) $\varepsilon_{ijk}\,\delta_{jk} = \varepsilon_{ijj} = 0$

(c) $\varepsilon_{jk2}\,\varepsilon_{k2j}$ = ( don't use identity here )

$\downarrow$ It is true only if all indices are free )

$$= \cancel{\varepsilon_{132}\,\varepsilon_{321}}\ \cancel{\varepsilon_{1k2}\,\varepsilon_{k21}} + \varepsilon \quad [\,j = k \Rightarrow 0 \quad j,k = 2 \Rightarrow 0\,]$$

$$= \varepsilon_{132}\,\varepsilon_{321} + \varepsilon_{312}\,\varepsilon_{123} = 2 \quad (\text{and not } \underline{-6} \text{ which identity would have given})$$

(d) $\varepsilon_{23i}\,\varepsilon_{2i3}$

$$= \varepsilon_{231}\,\varepsilon_{213} = \varepsilon_{123}\,\varepsilon_{132} = -1$$

COMMON MISTAKES:

1) Using indices more than 2 times [ $u_{ii}\, v_{ij}$ is wrong ]
2) Incorrectly expanding determinant
3) $B_{ijj}$ is a vector and not a matrix.
4) Incorrectly interpreting $T_{ij} = \lambda\, \varepsilon_{kk}\, \delta_{ij} + 2\mu\, \varepsilon_{ij}$
5) Please see soln of Q.7 carefully. (especially $A^T$ terms)
6) $\delta_{11} = 1$, but $\delta_{ii} = 3$
7) $\varepsilon_{ijk}\,\varepsilon_{pqk} = \delta_{ip}\,\delta_{jq} - \delta_{iq}\,\delta_{jp}$ is valid only if $i, j, k, p$ and $q$ can take all values (i.e. they are free indices).
If one of them is constant (say $k = 1$), then this identity doesn't hold true.