Lecture 1

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MA 214 Numerical Analysis

Lectures

Monday 10:35 – 11:30

Tuesday 11:35 – 12:30

Thursday 8:30 – 9:25

Prof. Tony J. Puthenpurakal

Textbook

Elementary Numerical Analysis
by Conte and de Boor

80% Attendence Required

Introduction

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1) $\int_a^b f(x)\,dx$ exists for example

when $f:[a,b] \longrightarrow \mathbb{R}$ is continuous.

However in most cases it is impossible to compute it.

Examples

1) $I_1 = \int_0^1 \sin(x^2)\,dx$

2) $I_2 = \int_0^1 e^{-x^2}\,dx$

3) $I_3 = \int_0^{1/4} \frac{1}{\sqrt{1-x^3}}\,dx$

Not only do we have to approximate the integrals, we also have to do it with pre-assigned accuracy.

For example

Approximate I_1 upto 10^{-6}

i.e. $|I_1 - \text{approx}| \le 10^{-6}$

Interpolation

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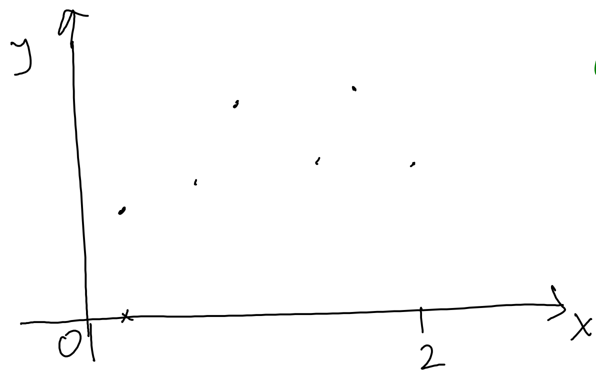
Suppose you are given a function

$f : [0,2] \to \mathbb{R}$ at some values $x_0, x_1, \ldots, x_n$

<u>Problem</u> :- Approximate $f(t)$ at

$$t \in [0,2] \setminus \{x_0, x_1, \ldots x_n\}$$

Graphically



<u>Goal</u>

To find a curve passing through these points

There are two standard methods to do this job

1) Lagrange interpolation

{ if points are equi spaced can use Newton-Raphson method

2) Piecewise methods

a) piecewise linear

b) cubic Spline interpolation

Initial value differential equations

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$$\begin{cases} \dfrac{dy}{dx} = f(x,y) \\ y(x_0) = y_0 \end{cases}$$

In general not possible to find y exactly.
However for many applications approximate value is enough.

<u>Example</u> $\quad \dfrac{dy}{dx} = \sin(x+y^2)$

$y(0) = 1$

<u>Approx</u> $\quad y(1)$

All methods will first approximate inbetween pts first and then find $y(1)$

for example $y(0.1)$ is approximated first.
Then using approx value of $y(0.1)$
$y(0.2)$ is approximated.

Then $y(0.3)$ is app. using $y(0.2)$
$y(0.4)$ " " " $y(0.3)$
$\vdots$
$y(1.0)$ " " " $y(0.9)$

This creates an additional issue and that is of Error propogation.

Eigenvalue and Eigenvectors.

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Let A be a $n \times n$ matrix.

Recall $\lambda \in \mathbb{R}$ is said to be an eigenvalue if there exists $\bar{x} \neq 0$ such that

$$A\bar{x} = \lambda \bar{x}$$

$\bar{x}$ is called eigenvector corresponding to λ

<u>Question</u> :- How do we find eigenvalues and eigenvectors?

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In applications the size of the matrix is large

$n \geq 10{,}000$ is common

$n \geq 1$ million for significant % of cases.

So usual method of finding

$p(t) = |tI - A|$

and then finding roots of $p(t)$ is not feasible.

In practise A will have a dominant eigen value

i.e $\exists\ \lambda_0$ s.t $|\lambda_0| > |\lambda_i|$ for all other eigen values λ_i

and it is enough for applications to find λ_0.

Floating point arithmetic

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n- digit floating point number in base β has the form

$$x = \pm (.d_1 d_2 \ldots d_n)_\beta \, \beta^e$$

$(0.d_1 d_2 \ldots d_n)_\beta$ is called <u>mantissa</u>

$e \longrightarrow$ exponent

for most computers $\beta = 2$
for calculators $\beta = 10$

$x \leftarrow$ real number
$fl(x) \leftarrow$ floating pt representation of x

Example
$x = \sqrt{3}$ $\qquad fl(x) = 0.1732 \; 10^1$ in 4 sig digits

Suppose x^* — approximation to x

$$|x - x^*| = \text{Absolute error}$$

$$\frac{|x - x^*|}{|x|} := \text{relative error (provided } x \neq 0\text{)}$$

Problem we dont know x
So how do we find
relative error

$$\alpha = \frac{x - x^*}{x}$$

Then $\dfrac{x - x^*}{x^*} = \dfrac{\alpha}{1-\alpha} \approx \alpha$ if α is small

Definition :- x^* is said to approximate x to t significant digits if

$$\left|\frac{x - x^*}{x}\right| \leq 5 \times 10^{-t} \quad \left(\begin{array}{c}\text{we assume} \\ x \neq 0\end{array}\right)$$

Loss of significant digits

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It is not true that if $x \sim x^*$ 4 sig digits
$y \sim y^*$ 4 sig digits

$\Rightarrow x \pm y \sim x^* \pm y^*$ 4 sig digits

or $xy \sim x^* y^*$ 4 sig digits

$\frac{x}{y} \sim \frac{x^*}{y^*}$ 4 sig digits

Things which create loss of significant digits

1) Subtraction of nearly equal quantities

2) division by a number which is close to zero

Examples.

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Example -1. $f(x) = 1 - \cos x$

(in 4 sig digits)

$$f(0.01) = 1 - 1 = 0$$

Actual answer $f(0.01) = 5\ E{-5}$ (5×10^{-5})

Loss of sig digit arises since $\cos(0.01) = 1$ in 4 sig digits

To avoid this

$$f(x) = 1 - \cos x$$

$$= \frac{1 - \cos^2 x}{1 + \cos x}$$

$$= \frac{\sin^2 x}{1 + \cos x}$$

$$f(0.01) = \frac{(1\ E{-2})^2}{1+1} = 5\ E{-5}$$

Example 2

$$x^2 + 111.11x + 1.2121 = 0 \qquad \left(\begin{matrix}4 \text{ sig} \\ \text{digits}\end{matrix}\right)$$

$$b^2 = 1.235\ E{+4}$$

$$b^2 - 4ac = 1.234\ E4$$

$$\sqrt{b^2 - 4ac} = 1.111\ E2$$

$= b$ in 4 sig digits

$$x_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a} = 0$$

Again loss of significant digits occur because we are subtracting nearly equal quantities so use

$$x_1 = \frac{-b + \sqrt{b^2-4ac}}{2a} \times \frac{-b-\sqrt{b^2-4ac}}{-b-\sqrt{b^2-4ac}}$$

$$= \frac{-2c}{b+\sqrt{b^2-4ac}}$$

$$x_1 = \frac{1.212}{222.1} = 1.091\ E{-2}$$

correct upto 4 sig digits

Example 3

$$f(x) = \frac{x - \sin x}{\tan x}$$ (4 sig digits

$$f(0.01) = \frac{0.01 - 1\,E{-2}}{1\ \ E{-2}}$$

$$= 0$$

Rewrite $f(x) = \frac{(x - \sin x)(x + \sin x)}{\tan x \ (x + \sin x)}$

$$= \frac{x^2 - (\sin x)^2}{\tan x \ (x + \sin x)}$$

again $f(0.01) = \frac{1\,E{-4} - 1\,E{-4}}{1\,E{-2}\ (\times)} = 0$

Actual value $f(0.01) \approx 1.667\ E{-5}$ 4 sig digits

So one has to use Taylor expansion

$$\sin x \approx x - \frac{x^3}{6}$$

$$\tan x \approx x - \frac{x^3}{3}$$

$$f(x) \approx \frac{x^3/6}{x - \frac{x^3}{3}} \approx \frac{x^2/6}{1 - x^2/3} = 1.667\ E{-5}$$

Error propogation

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Once an error is commited it contaminates subsequent results

Error propagation through subsequent calulations is studied in terms of two related concepts

1. condition
2. instability

Today we study condition

condition $\longleftrightarrow$ sensitivity of $f(x)$ to changes in x.

$$= \max \left\{ \frac{\frac{|f(x)-f(x^*)|}{|f(x)|}}{\frac{|x-x^*|}{|x|}} : |x-x^*| \text{ is small} \right\}$$

$$\approx \left| \frac{f'(x)\, x}{f(x)} \right|$$

The larger the condition, the more ill-conditioned the function is said to be

Examples

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Example 1 :-

$$f(x) = \sqrt{x}$$

$$f'(x) = \frac{1}{2\sqrt{x}}$$

$$\left| \frac{f'(x)\, x}{f(x)} \right| = \left| \frac{\frac{1}{2\sqrt{x}}\, x}{\sqrt{x}} \right| = \frac{1}{2}$$

So taking square-roots is well conditioned since it actually reduces the relative Error

Example 2

$$f(x) = \frac{10}{1-x^2}$$

$$\left| \frac{f'(x)\, x}{f(x)} \right| = \frac{2x^2}{|1-x^2|}$$ large when $|x|$ is close to 1

What to do?

put $x = 1-y$ $\qquad$ $y \sim 0$ if $x \sim 1$

$$f(x) = \frac{10}{1-(1-y)^2} = \frac{10}{2y+y^2} \underset{y \sim 0}{\approx} \frac{10}{2y} = \frac{5}{y}$$

$$g(y) = 5/y$$

$$\left| \frac{g'(y)\, y}{g(y)} \right| = \left| \frac{-\frac{5}{y^2}\, y}{5/y} \right| = 1$$ much better than before