Lecture 10

Last time we studied Gaussian Quadrature.

$$P_n(x) = \sum_{i=0}^{n} f(x_i)\, l_i(x)$$ interpolates $f(x)$ at $x_0, \ldots, x_n$

$$l_i(x) = \prod_{\substack{j=0 \\ j \neq i}}^{n} \frac{x - x_j}{x_i - x_j}$$

$$I = \int_a^b f(x)\,dx \approx \int_a^b P_n(x)\,dx$$

$$= \sum_{i=0}^{n} f(x_i)\, c_i$$

where $c_i = \int_a^b l_i(x)\,dx$ is independent of f

Gaussian quadrature is to choose pts $x_0, x_1, \ldots x_n$ such that error is small

In $[-1, 1]$ the choice of nodes is given by zeros of Legendre's polynomials

Introduction to Legendre's polynomials

$\{Q_0, Q_1, \ldots Q_n, Q_{n+1}, \ldots\}$ is the set of Legendre polynomials. It has the following properties

- $Q_0(x) = 1$
- $Q_n(x)$ is monic has degree n for $n = 1, 2, \ldots$.

* - $\int_{-1}^{1} P(x) Q_n(x)\, dx = 0$ whenever $P(x)$ is a polynomial of degree $< n$.

$Q_1(x) = x$

$Q_2(x) = x^2 - \frac{1}{3}$

$Q_3(x) = x^3 - \frac{3}{5}x \qquad Q_4(x) = x^4 - \frac{6}{7}x^2 + \frac{3}{35}$

Important property of Legendre polynomials

- $Q_n(x)$ has n distinct roots in $(-1,1)$
- Furthermore the roots are symmetric w.r.t the origin

Notation

$x_0, x_1, \ldots x_n$ zero's of $Q_{n+1}(x)$

$$c_i = \int_{-1}^{1} \left(\prod_{\substack{j=0 \\ j\neq i}}^{n} \frac{x - x_j}{x_i - x_j} \right) dx.$$

$i = 0, 1, \ldots, n$

Theorem If $P(x)$ is any polynomial of degree $\leq 2n+1$ then

$$\int_{-1}^{1} P(x)\,dx = \sum_{i=0}^{n} P(x_i)\, c_i$$

Today we do Composite rules

To compute $\int_a^b f(x)\,dx$ we developed some simple rules

These rules do not give good estimate for I when $[a,b]$ is resonably large.

$\boxed{\text{Example}}$

$$I = \int_0^5 e^x\,dx = 1.474\ E2$$ (correct upto 4 sig digits)

by Trapezoidal rule

$$I \approx \frac{5}{2}\left[e^0 + e^5\right] = 3.735\ E2$$

by Simpson's rule

$$I \approx \frac{5}{6}\left[e^0 + 4e^{2.5} + e^5\right] = 1.347\ E2$$

So idea is to divide $[a, b]$ into N smaller intervals and to apply quadrature rule to each of these subintervals.

$$a = x_0 < x_1 < x_2 < \cdots < x_N = b$$

and we apply quadrature rules for each $[x_{i-1}, x_i]$

We choose x_i's to be equally spaced

$$x_i = a + ih \qquad i = 0, 1, \ldots, N$$

$$h = \frac{b-a}{N}$$

Composite Trapezoidal rule

$$\int_{x_{i-1}}^{x_i} f(x)\,dx \approx \frac{h}{2}\left[f(x_{i-1}) + f(x_i)\right]$$

$$\int_a^b f(x)\,dx = \sum_{i=1}^{N} \int_{x_{i-1}}^{x_i} f(x)\,dx$$

$$\approx \sum_{i=1}^{N} \frac{h}{2}\left[f(x_{i-1}) + f(x_i)\right]$$

$$T_N = \frac{h}{2}\left[f(x_0) + 2\sum_{i=1}^{N-1} f(x_i) + f(x_N)\right]$$

$I \approx T_N$ "composite Trapezoidal rule".

$$\text{Error} = \sum_N \text{error at } [x_{i-1}, x_i]$$

$$= \sum_{i=1}^{N} -\frac{f''(\eta_i)\,h^3}{12} \qquad \eta_i \in (x_{i-1}, x_i)$$

$$= f''(\xi) \sum_{i=1}^{N} -\frac{h^3}{12} \qquad \xi \in (a, b)$$

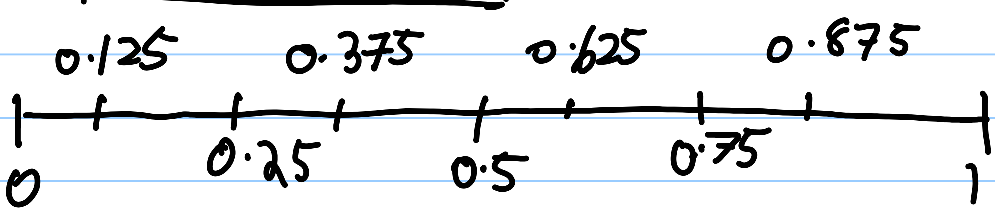
$$\text{Error} = -\frac{f''(\xi)\, h^3 N}{12}$$

$$= -\frac{f''(\xi)\, h^2 (b-a)}{12} \qquad \xi \in (a,b)$$

<u>Example</u>

$$I = \int_0^1 \sin(x^2)\, dx$$

N	T_N
1	0.4208
2	0.3341
4	0.3159
8	0.3117



We compute Number of subdivisions required to get error $< 10^{-5}$

$$f(x) = \sin(x^2)$$

$$f^{(2)}(x) = -4x^2 \sin(x^2) + 2\cos(x^2)$$

$$|f^{(2)}(x)| \le 6$$

$$E = -\frac{f''(\xi)\, h^2 \cdot 1}{12}$$

$$|E| = \left| \frac{-f''(\xi)\, h^2 \cdot 1}{12} \right| \le \frac{6h^2}{12} < 10^{-5}$$

$$\frac{1}{2N^2} < 10^{-5}$$

$$N^2 > 5 \cdot 10^4$$

$$N > \sqrt{5} \cdot 10^2 \simeq 224$$

Composite Simpson's rule

$$\int_{x_{i-1}}^{x_i} f(x)\,dx \simeq \frac{h}{6}\left[f(x_{i-1}) + 4f(x_{i-1}+\tfrac{h}{2}) + f(x_i) \right]$$

So
$$\int_a^b f(x)\,dx = \sum_{i=1}^{N} \int_{x_{i-1}}^{x_i} f(x)\,dx$$
$$\simeq \sum_{i=1}^{N} \frac{h}{6}\left[f(x_{i-1}) + 4f(x_{i-1}+\tfrac{h}{2}) + f(x_i) \right]$$

$$S_N = \frac{h}{6}\left[f(x_0) + 2\sum_{i=1}^{N-1} f(x_i) + 4\sum_{i=1}^{N} f(x_{i-1}+\tfrac{h}{2}) + f(x_N) \right]$$

composite Simpson's rule with N nodes

$$\text{Error} = \sum_{i=1}^{N} \text{Error at } [x_{i-1}, x_i]$$

$$= \sum_{i=1}^{N} -\frac{f^{iv}(\eta_i)}{90}\left(\frac{h}{2}\right)^5 \qquad \eta_i \in (x_{i-1}, x_i)$$

$$= -f^{(iv)}(\xi) \sum_{i=1}^{N} \frac{(h/2)^5}{90} \qquad \xi \in (a, b)$$

$$= -f^{(iv)}(\xi)\, \frac{(h/2)^5 \cdot N}{90}$$

$$= -\frac{f^{(iv)}(\xi)\,(h/2)^4\,(b-a)}{180}$$

Example

$$I = \int_0^1 \sin(x^2)\,dx$$

N	S_N
1	0.3052
2	0.3099
4	0.3103 ---- correct upto 4 sig digits

We compute Number of subdivisions required to get error $< 10^{-5}$

$$f^{(iv)}(x) = \sin(x^2).\,16x^4 - 48x^2\cos x^2 - 12\sin x^2$$

$$|f^{(iv)}(x)| \leq 76$$

$$|E_N^S| \leq \frac{f^{(iv)}(\xi)\left(\frac{h}{2}\right)^4 \cdot 1}{180}$$

$$E_N^S \leq \frac{76}{180 \times 16 \times N^4} \leq 10^{-5}$$

$$N^4 \geq \frac{76 \times 10^5}{180 \times 16}$$

$$N \geq 7.1$$

Thus $N = 8$ will give answer correct upto 10^{-5}.

(Note that $N \approx 224$ for Trapezoidal rule to give acc. upto 10^{-5})

Example

$$I = \int_0^4 e^x \, dx$$

by Trapezoid rule

N	T_N
1	111.2
2	70.37
4	57.99

Exact $= e^4 - 1$
$= 53.59$

by Simpson's rule

N	S_N
1	56.76
2	53.86
4	53.61

$$E_N^T = -\frac{f''(\eta)\, h^2\, (b-a)}{12}$$

$$E_N^S = -\frac{f^{(iv)}(\xi)\, \left(\frac{h}{2}\right)^4 (b-a)}{180}$$

find N such that $\quad N=4$

$$|E_N^T| \leq 10^{-5}$$

$$|E_N^S| \leq 10^{-5}$$

1) $f^{(ii)}(x) = e^x$

$$|f^{(ii)}(x)| \leq e^4 \quad \text{in } [0,4]$$

$$|E_N^T| \leq \frac{e^4}{12} \frac{4^2}{N^2} \cdot 4 < 10^{-5}$$

$$N^2 > \frac{4^2 e^4}{3} 10^5$$

$$N > 4 e^2 \cdot 10^2 \cdot \sqrt{\frac{10}{3}}$$