MA 214 : Mid-Sem Exam, Model Solutions

Q.1 a) Let $l_i(x) = \prod_{\substack{j=0 \\ j\neq i}}^{n} \frac{(x-x_j)}{(x_i-x_j)}$, $j=0,1,\cdots,n$

... ($\frac{1}{2}$ mark)

Then $l_i(x_i) = 1$, $l_i(x_j) = 0$, $i \neq j$

Let $p_n(x) = \sum_{i=0}^{n} f(x_i)\, l_i(x)$, $x \in [a,b]$

... ($\frac{1}{2}$ mark)

Then degree of $p_n \leq n$ and $p_n(x_i) = f(x_i)$, $i=0,1,\cdots,n$

If q_n is a polynomial of degree $\leq n$ and $q_n(x_i) = f(x_i)$, $i=0,1,\cdots,n$, then

$(p_n - q_n)(x_i) = 0$, $i=0,1,\cdots,n \Rightarrow p_n(x) = q_n(x)$ ... ($1/2$ mark)

... ($\frac{1}{2}$ mark)

Q.1 b) Let p_{n-1} and q_{n-1} be polynomials of degree $\le n-1$ such that

... ($\frac{1}{2}$ mark)

$p_{n-1}(x_i) = f(x_i)$, $i = 0, 1, \cdots, n-1$ and

$q_{n-1}(x_i) = f(x_i)$, $i = 1, 2, \cdots, n$ ... ($\frac{1}{2}$ mark)

Then $p_n(x) = \dfrac{(x - x_0)\, q_{n-1}(x) + (x_n - x)\, p_{n-1}(x)}{x_n - x_0}$

... (1 mark)

is a polynomial of degree $\le n$ and

(1 mark)

$p_n(x_i) = f(x_i)$, $i = 0, 1, \cdots, n$ ...

Coeff. of x^n in p_n $= \dfrac{\text{Coeff. of } x^{n-1} \text{ in } q_{n-1} - \text{Coeff. of } x^{n-1} \text{ in } p_{n-1}}{x_n - x_0}$

Q.2 a) $p_3(x) = f(a) + f[a,a](x-a) + f[a,a,b](x-a)^2$

$+ f[a,a,b,b](x-a)^2(x-b)$ ... (1 mark)

$f(x) - p_3(x) = f[a,a,b,b,x](x-a)^2(x-b)^2$

... (1 mark)

b) Note that

$$\Rightarrow |f(x) - p_3(x)| \le \frac{f^{(4)}(c_x)}{4!} |(x-a)(x-b)|^2$$

$$\Rightarrow \|f - p_3\|_\infty \le \frac{\|f^{(4)}\|_\infty}{24} \left(\frac{b-a}{2}\right)^4$$ ... (1 mark)

$$\max_{x \in [t_i, t_{i+1}]} |f(x) - g(x)| \leq \frac{\max_{x \in [t_i, t_{i+1}]} |f^{(4)}(x)|}{24} \left(\frac{h}{2}\right)^4$$

... (1 mark)

$$\leq \frac{\|f^{(4)}\|_\infty}{24} \left(\frac{h}{2}\right)^4$$

Hence

$$\|f - g\|_\infty \leq \frac{\|f^{(4)}\|_\infty}{24} \left(\frac{h}{2}\right)^4$$

... (1 mark)

Q.3 a) $\int_0^1 f(x)\,dx \simeq \frac{1}{8}\left(f(0) + 3f\left(\frac{1}{3}\right) + 3f\left(\frac{2}{3}\right) + f(1)\right)$.

$f(x) = 1$. $LHS = 1$, $RHS = \frac{1}{8}(1+3+3+1) = 1$

$f(x) = x$. $LHS = \frac{1}{2}$, $RHS = \frac{1}{8}(1+2+1) = \frac{1}{2}$ ... ($\frac{1}{2}$)

$f(x) = x^2$. $LHS = \frac{1}{3}$, $RHS = \frac{1}{8}\left(\frac{1}{3}+\frac{4}{3}+1\right) = \frac{1}{3}$ ... ($\frac{1}{2}$)

$f(x) = x^3$. $LHS = \frac{1}{4}$, $RHS = \frac{1}{8}\left(\frac{1}{9}+\frac{8}{9}+1\right) = \frac{1}{4}$ ... ($\frac{1}{2}$)

$f(x) = x^4$. $LHS = \frac{1}{5}$, $RHS = \frac{1}{8}\left(\frac{1}{27}+\frac{16}{27}+1\right) = \frac{11}{54}$

$LHS \neq RHS$ ... ($\frac{1}{2}$ mark)

The rule is exact for polynomials of degree ≤ 3.

Q.3 b) $f : [a, b] \longrightarrow \mathbb{R}$ twice continuously differentiable.

$$f(x) = f(a) + f[a, b](x-a) + f[a\ b\ x](x-a)(x-b) \quad \cdots (1)$$

$$\int_a^b f(x)\,dx = f(a)(b-a) + f[a\ b]\frac{(b-a)^2}{2}$$

$$+ \int_a^b f[a\ b\ x]\,(x-a)(x-b)\,dx. \quad \cdots \left(\tfrac{1}{2}\right)$$

$$= \frac{b-a}{2}\left(f(a) + f(b)\right) + f[a\ b\ c]\int_a^b (x-a)(x-b)\,dx \quad \cdots \left(\tfrac{1}{2}\right)$$

(since $f[a\ b\ x]$ is continuous and $(x-a)(x-b) \leq 0$) $\cdots \left(\tfrac{1}{2}\right)$

$$= \underbrace{\frac{b-a}{2}\left(f(a) + f(b)\right)}_{\text{Rule}} + \underbrace{\frac{f''(\alpha)}{2}\left(-\frac{(b-a)^3}{6}\right)}_{\text{Error}}, \quad \cdots \left(\tfrac{1}{2}\right)$$

for some $\alpha \in (a, b)$

6. $g(x) = f[x_0\ x] = \begin{cases} \dfrac{f(x)-f(x_0)}{x-x_0}, & x \neq x_0 \\ f'(x_0), & x = x_0 \end{cases}$.

For $x \neq x_0$,

$$g'(x) = \frac{(x-x_0)f'(x) - [f(x)-f(x_0)]}{(x-x_0)^2} \underset{\cdots\ (\frac{1}{2})}{\overset{\downarrow}{=}} \frac{f[x\ x] - f[x_0\ x]}{x-x_0} = f[x_0\ x\ x]. \quad \cdots\ (\tfrac{1}{2}).$$

$$g'(x_0) = \lim_{h\to 0} \frac{f[x_0,\ x_0+h] - f'(x_0)}{h} \quad \cdots\ (\tfrac{1}{2})$$

$$= \lim_{h\to 0} \frac{f(x_0+h) - f(x_0) - h f'(x_0)}{h^2} \quad \cdots\ (\tfrac{1}{2}) \quad = \lim_{h\to 0} \frac{\frac{h^2}{2} f''(c)}{h^2}, \quad \cdots\ (\tfrac{1}{2})$$

by the extended MVT, $c \in [x_0\ x_0+h]$

$$= \frac{f''(x_0)}{2} = f[x_0\ x_0\ x_0] \quad \cdots\ (\tfrac{1}{2})$$

7. $[A : b] = \left[\begin{array}{cccccc|c} a_{11} & a_{12} & 0 & \cdots & & 0 & b_1 \\ a_{21} & a_{22} & a_{23} & & & & b_2 \\ 0 & a_{32} & a_{33} & a_{34} & & & \vdots \\ \vdots & & & & & & \vdots \\ & 0 & & & a_{n,n-1} & a_{nn} & b_n \end{array}\right]$.

1st step: $\cdots (1)$

$$m_{21} = \frac{a_{21}}{a_{11}}, \quad R_2 - m_{21} R_1 : \left.\begin{array}{l} a_{22}^{(1)} = a_{22} - m_{21} a_{12} \\ b_2^{(1)} = b_2 - m_{21} b_1 \end{array}\right\} \begin{array}{l} 3 \text{ mult./div.} \\ +2 \text{ subtractions.} \end{array}$$

$$[A : b] = \left[\begin{array}{cccccc|c} a_{11} & a_{12} & 0 & \cdots & & 0 & b_1 \\ 0 & a_{22}^{(1)} & a_{23}^{(1)} & 0 & \cdots & 0 & b_2^{(1)} \\ \vdots & & & & & & \\ \vdots & & & & & & \\ 0 & 0 & \cdots & & a_{n,n-1} & a_{nn} & b_n \end{array}\right]$$

Each step: 3 mult./div. + 2 sub.

Total: $3(n-1)$ mult./div. $+ 2(n-1)$ sub. $\cdots \left(\frac{1}{2}\right)$

Back Substitution:

$$u_{11}\, x_1 + u_{12}\, x_2 = y_1 .$$

$$u_{22}\, x_2 + u_{23}\, x_3 = y_2$$

$$\vdots$$

$$u_{n-1,n-1}\, x_{n-1} + u_{n-1,n}\, x_n = y_{n-1}$$

$$u_{n,n}\, x_n = y_n .$$

$$x_n = \frac{y_n}{u_{nn}} , \quad x_i = \frac{y_i - u_{i,i+1}\, x_{i+1}}{u_{ii}} , \quad i = n-1, \cdots, 1 \;\dots (1)$$

Total : $n-1$ mult. + $n-1$ subtractions + n divisions $\cdots \left(\frac{1}{2}\right)$

Q.5 $\begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 5 & 8 & 11 \\ 3 & 8 & 14 & 20 \\ 4 & 11 & 20 & 30 \end{bmatrix}$ $m_{21} = 2$, $m_{31} = 3$, $m_{41} = 4$

$R_2 - 2R_1$, $R_3 - 3R_1$, $R_4 - 4R_1$

$\downarrow$ ($\frac{1}{2}$ mark)

$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 3 \\ 0 & 2 & 5 & 8 \\ 0 & 3 & 8 & 14 \end{bmatrix}$ $m_{32} = 2$, $m_{42} = 3$

$R_3 - 2R_2$, $R_4 - 3R_2$

$\downarrow$ ($\frac{1}{2}$ mark)

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 2 & 5 \end{bmatrix} \xrightarrow[R_4 - 2R_3]{m_{43} = 2} \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix} = U$$

($\frac{1}{2}$ mark)

$$L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 3 & 2 & 1 & 0 \\ 4 & 3 & 2 & 1 \end{bmatrix} \quad \cdots \text{(1½ marks)}$$

$$\det(A) = \det(U) = 1 \quad \cdots \text{(2 marks)}$$

Q.6 a) Let $f(x) = 1$, $x \in [a, b]$

Then $p_n(x) = \sum_{i=0}^{n} f(x_i)\, l_i(x)$: interpolating polynomial of degree $\le n$

$= \sum_{i=0}^{n} l_i(x) = f(x) = 1$.

... (1 mark)

b) Note that

$$l_i(x)\, l_j(x) = \left[\prod_{\substack{p=0 \\ p \neq i}}^{n} \frac{(x - x_p)}{(x_i - x_p)}\right] \left[\prod_{\substack{q=0 \\ q \neq j}}^{n} \frac{(x - x_q)}{(x_j - x_q)}\right]$$

$$= \prod_{p=0}^{n} (x - x_p)\; r(x),$$ where

$$r(x) = \frac{1}{\prod\limits_{\substack{p=0 \\ p\neq i}}^{n} (x_i - x_p)} \cdot \frac{1}{\prod\limits_{\substack{q=0 \\ j\neq q}}^{n} (x_j - x_q)} \prod_{\substack{q=0 \\ q\neq j \\ q\neq i}}^{n} (x - x_q),$$

a polynomial of degree $n-1$ ... (1 mark)

Thus $\quad l_i(x)\, l_j(x) = \dfrac{q_{n+1}(x)}{\alpha_{n+1}}\, r(x)$.

Hence $\quad \displaystyle\int_a^b l_i(x)\, l_j(x)\, dx = \frac{1}{\alpha_{n+1}} \underbrace{\langle q_{n+1}, r \rangle}_{=\,0}$

... (1 mark)

Q.6 c) $\int_a^b l_i(x)\,dx$

$$= \int_a^b l_i(x) \sum_{j=0}^{n} l_j(x)\,dx \quad \cdots \text{(1 mark)}$$

$$= \underbrace{\int_a^b l_i(x)^2\,dx}_{>0} + \sum_{\substack{j=0 \\ j\neq i}}^{n} \int_a^b l_i(x)\,\underbrace{l_j(x)}_{=0}\,dx$$

$\cdots$ (1 mark)