MA 214: Introduction to Numerical Analysis, Spring 2014

Mid-Semester Examination

Wednessday,	14:00 - 16:00
February 19, 2014	Weightage 30%

- Write roll number, division and tutorial batch on the main answer book and on the supplements. Complete the index on the top page of your answer book. Failure to do so will invite a penalty of 2 marks.
- **Use of calculators is not permitted.**

---

1. (a) Let $f : [a, b] \to \mathbb{R}$ and $x_0, x_1, \cdots, x_n$ be $n+1$ distinct points in $[a, b]$. Show that there exists a polynomial p_n of degree $\leq n$ such that

$$p_n(x_i) = f(x_i),\ i = 0, 1, \cdots, n,$$

and that it is unique. (2 marks)

(b) Define the divided difference $f[x_0, x_1, \cdots, x_n]$ as the coefficient of x^n in p_n in part (a). Prove the following recurrence formula.

$$f[x_0, x_1, \cdots, x_n] = \frac{f[x_1, \cdots, x_n] - f[x_0, \cdots, x_{n-1}]}{x_n - x_0}.$$

(3 marks)

2. Let $f : [a, b] \to \mathbb{R}$ be 4 times continuously differentiable function.

(a) Write down the formula for the cubic polynomial $p_3(x)$ which interpolates f and f' at a and at b and for the error associated with this cubic Hermite interpolation.

(2 marks)

(b) Let

$$a = t_0 < t_1 < \cdots < t_n = b$$

be an uniform partition of $[a, b]$ and let $h = t_{i+1} - t_i = \dfrac{b-a}{n},\ i = 0, 1, \ldots, n-1.$
Let $g : [a, b] \to \mathbb{R}$ be a piecewise cubic polynomial with respect to the above partition and be such that

$$g(t_i) = f(t_i),\ g'(t_i) = f'(t_i),\ i = 0, 1, \ldots, n.$$

Show that the maximum error $\|f - g\|_\infty = \max\limits_{x \in [a,b]} |f(x) - g(x)|$ is of the order of h^4.

(3 marks)

3. (a) Consider the quadrature rule

$$\int_0^1 f(x)dx \approx \frac{1}{8}\Big(f(0)+3f(\frac{1}{3})+3f(\frac{2}{3})+f(1)\Big).$$

Determine the degree of precision of this rule. (2 marks)

(b) Derive the Trapezoidal rule along with the error formula. (3 marks)

4. (a) Let $f : [a, b] \to \mathbb{R}$ be thrice differentiable function and $x_0 \in [a, b]$. Show that

$$\frac{d}{dx} f[x_0,\ x] = f[x_0,\ x,\ x]. \quad \text{(2 marks)}$$

(b) Let $A = [a_{ij}]$ be an invertible tridiagonal matrix, that is $a_{ij} = 0$ if $|i - j| > 1$. Compute the number of operations needed to solve the system $Ax = b$ by Gauss elimination without partial pivoting. (3 marks)

5. Let

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 5 & 8 & 11 \\ 3 & 8 & 14 & 20 \\ 4 & 11 & 20 & 30 \end{bmatrix}.$$

Find a lower triangular matrix L with all the diagonal entries equal to 1 and an upper triangular matrix U such that $A = LU$. Also, find the determinant of A. (5 marks)

6. Let $q_0, q_1, \ldots$ be the orthonormal polynomials with respect to the usual inner product

$$\langle f,\ g\rangle = \int_a^b f(x)g(x)dx$$

obtained by applying an orthonormalization process to $\{1, x, x^2, \ldots\}$. Note that

$$span\{q_0, q_1, \ldots, q_{n+1}\} = span\{1, x, \ldots, x^{n+1}\}.$$

Let

$$q_{n+1}(x) = \alpha_{n+1}(x - x_0)(x - x_1)\cdots(x - x_n).$$

For $i = 0, 1, \ldots, n$, let

$$l_i(x) = \prod_{\substack{p=0 \\ p\neq i}}^{n} \frac{(x - x_p)}{(x_i - x_p)}$$

denote the Lagrange polynomial. Show that

(a) $$\sum_{i=0}^{n} l_i(x) = 1, \quad \text{(1 mark)}$$

(b) $$\int_a^b l_i(x)l_j(x)dx = 0, \quad \text{if} \quad i \neq j, \quad \text{(2 marks)}$$

(c) $$\int_a^b l_i(x)dx > 0, \quad i = 0, 1, \ldots, n. \quad \text{(2 marks)}$$