Department of Mathematics, Indian Institute of Technology Bombay
MA 214: Introduction to Numerical Analysis
Quiz 1

Time: 1 hour — Marks: 10
Instructors: S. Baskar and S. Sivaji-Ganesh — Date: 31-01-2012

Instructions:

1. Write your Name and Roll Number clearly on your answer book as well as every supplement you may use.
2. Number the pages of your answer book and make a question-page index on the front page.
3. The answer to each question should start on a new page. If the answer for a question is split into two parts and written in two different places, the first part alone will be corrected.
4. Only scientific calculators are allowed. Any kind of programing device is not allowed.
5. Formulas used need not be proved but **needs to be stated clearly**.
6. The question paper contains 5 questions each carries 2 marks. **Answer all the questions**.

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(1) Let $x = 12.4568$ and $y = 216.71092$. Compute $x(x + y)$ using 5-digit rounding arithmetic. Show all the steps of the computation. Obtain the total error.

(2) For small values of x, the approximation $\sin x \approx x$ is often used. Obtain a range of values of x for which the approximation gives an absolute error of at most $\dfrac{1}{2} \times 10^{-6}$.

(3) The quadratic polynomial $p_2(x) = \frac{3}{4}x^2 + \frac{1}{4}x + \frac{1}{2}$ interpolates the data

x	-1	0	1
y	1	$\frac{1}{2}$	$\frac{3}{2}$

Can we find a node x_3 ($x_3 \notin \{-1, 0, 1\}$), and a real number y_3 such that the polynomial $p_3(x)$ interpolating the data

x	-1	0	1	x_3
y	1	$\frac{1}{2}$	$\frac{3}{2}$	y_3

is a polynomial of degree less than or equal to 2?
If your answer is 'yes', then give one such pair of values x_3 and y_3 and also give the corresponding interpolating polynomial with full details.
If your answer is 'no', then give a formula for the coefficient of x^3 in the interpolating polynomial and prove that this coefficient is non-zero for all values of x_3 and y_3.

(P.T.O.)

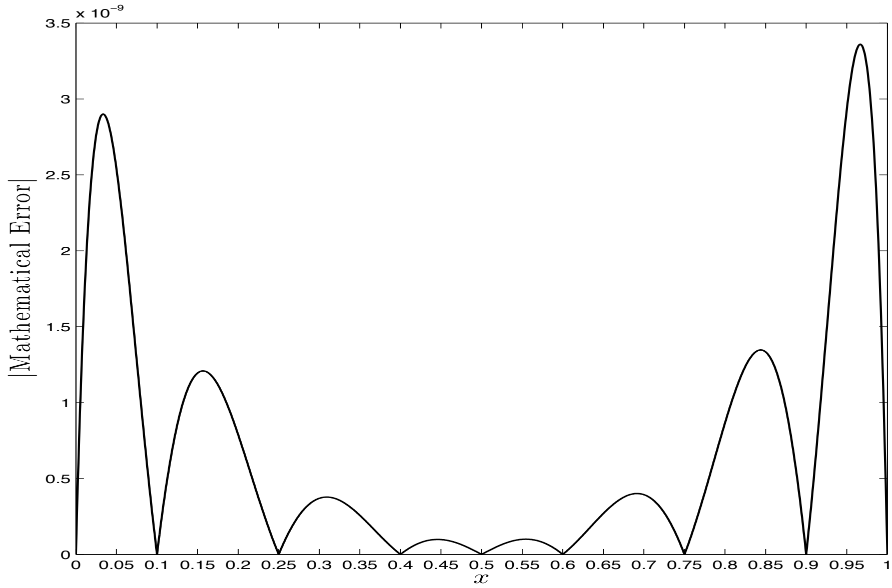

Figure 1:

(4) Let $f : [0,1] \to \mathbb{R}$ be a function, and $p(x)$ be an interpolating polynomial for the data

x	x_0	x_1	x_2	x_3	$\cdots$	x_n
y	$f(x_0)$	$f(x_1)$	$f(x_2)$	$f(x_3)$	$\cdots$	$f(x_n)$

The Figure 1 depicts the absolute value of the mathematical error $|f(x) - p(x)|$. Answer the following questions with reference to Figure 1.

(a) Give the largest possible set of nodes x_0, x_1, $\cdots$, x_n. Justify the choice of the nodes.

(b) Give the maximum possible degree of the interpolating polynomial $p(x)$.

(c) If $f(x)$ used in generating the given data is a polynomial of degree 4, then will the Figure 1 still depict the absolute value of the mathematical error $|f(x) - p(x)|$? Justify your answer.

(5) Let $p(x)$, $q(x)$, and $r(x)$ be interpolating polynomials for the three sets of data

x	0	1
y	y_0	y_1

,

x	0	2
y	y_0	y_2

, and

x	1	2	3
y	y_1	y_2	y_3

respectively. Let $s(x)$ be the the interpolating polynomial for the data

x	0	1	2	3
y	y_0	y_1	y_2	y_3

If $p(x) = 1 + 2x$, $q(x) = 1 + x$, and $r(2.5) = 3$, then find the value of $s(2.5)$.

End of the question paper.

ALL THE BEST

Department of Mathematics, Indian Institute of Technology Bombay
MA 214: Introduction to Numerical Analysis
Midsemester Examination

Time: 2 hours — Marks: 30
Instructors: S. Baskar and S. Sivaji-Ganesh — Date: 22-02-2012

Instructions:

1. Write your Name and Roll Number clearly on your answer book as well as every supplement you may use.
2. Number the pages of your answer book and make a question-page index on the front page.
3. The answer to each question should start on a new page. If the answer for a question is split into two parts and written in two different places, the first part alone will be corrected.
4. Only scientific calculators are allowed. Any kind of programing device is not allowed.
5. Formulas used need not be proved but **needs to be stated clearly**.
6. The question paper contains 5 questions each carries 6 marks. **Answer all the questions**.

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(1) (a) Let $x < 0 < y$ be such that the approximate numbers x_A and y_A has seven and nine significant digits with x and y respectively. Show that $z_A := x_A - y_A$ has at least six significant digits when compared to $z := x - y$.

(b) Let $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ be continuously differentiable functions such that

- there exists constant $M > 0$ such that $|f'(x)| \geq M$ and $|g'(x)| \leq M$ for all $x \in \mathbb{R}$,
- the process of evaluating f is well-conditioned, and
- the process of evaluating g is ill-conditioned.

Show that $|g(x)| < |f(x)|$ for all $x \in \mathbb{R}$.

(2) Obtain the natural cubic spline interpolating function for the data

x	0	1	2	3
y	1	2	33	244

.

(P.T.O.)

(3) (a) Let f be a continuous function on the interval $[0,2]$ such that

$$f(x) = \begin{cases} ax+b, & \text{if } \ 0 \le x \le 1, \\ -x+5, & \text{if } \ 1 \le x \le 2. \end{cases}$$

Obtain the values of a and b so that the Simpson's rule is exact for f.

(b) Let x_0, x_1, x_2, x_3, and x_4 be nodes symmetrically placed about the origin. Let A_0, A_1, A_2, A_3, A_4 be such that the formula

$$\int_{-1}^{1} f(x)\,dx \approx \sum_{i=0}^{4} A_i\, f(x_i)$$

is exact for all polynomials of degree less than or equal to 4. Show that this formula is exact for all polynomials of degree 5.

(4) Let $f : \mathbb{R} \to \mathbb{R}$ be such that $f^{(4)}$ is a continuous function on $\mathbb{R}$. Consider the formula for the second derivative of a function f given by

$$D_h^{(2)} f(x) = \frac{f(x+h) - 2f(x) + f(x-h)}{h^2}.$$

(a) State the mathematical error involved in the formula.

(b) Derive the total error involved in the formula.

(c) Assuming that there exists an $M > 0$ such that $|f^{(4)}(x)| \le M$ for all $x \in \mathbb{R}$, obtain an upper bound for the total error.

(d) If $f(x) = \cos x$, and the function values are provided with six significant digits when compared to the exact values, then obtain the value of h for which the upper bound obtained above for the total error is minimum.

(5) (a) Using the Gaussian rule

$$\int_{-1}^{1} f(x)dx \approx f\left(-\frac{1}{\sqrt{3}}\right) + f\left(\frac{1}{\sqrt{3}}\right),$$

determine a formula for solving the initial value problem

$$y' = e^{-x^2},$$
$$y(x_0) = y_0.$$

in the form

$$y_{j+1} = y_{j-1} + h\left(e^{-(x_j - \frac{h}{\sqrt{3}})^2} + e^{-(x_j + \frac{h}{\sqrt{3}})^2}\right)$$

when the nodes are **equally spaced** with spacing $h = x_{j+1} - x_j$, $j \in \mathbb{Z}$. $(h > 0)$

(b) Let $x_0 = 0$, and $y_0 = 1$. Using the method derived above, obtain approximate values of $y(-0.1)$ and $y(0.1)$.

End of the question paper.

ALL THE BEST

Department of Mathematics, Indian Institute of Technology Bombay
MA 214: Introduction to Numerical Analysis
Quiz 2

Time: 1 hour | Marks: 10
Instructors: S. Baskar and S. Sivaji-Ganesh | Date: 03-04-2012

Instructions:

1. Write your Name and Roll Number clearly on your answer book as well as every supplement you may use.
2. Number the pages of your answer book and make a question-page index on the front page.
3. The answer to each question should start on a new page. If the answer for a question is split into two parts and written in two different places, the first part alone will be corrected.
4. Only scientific calculators are allowed. Any kind of programing device is not allowed.
5. Formulas used need not be proved but **needs to be stated clearly**.
6. The question paper contains 5 questions each carries 2 marks. **Answer all the questions**.

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(1) Let the system $A\mathbf{x} = \mathbf{b}$ be such that

$$A = \begin{pmatrix} 3 & 1 & -1 \\ 0 & 2 & 1 \\ 0 & -1 & 2 \end{pmatrix}, \quad \text{and} \quad \mathbf{b} = \begin{pmatrix} 1 \\ -3 \\ 2 \end{pmatrix}.$$

Let $\mathbf{x}_a = \begin{pmatrix} 0.9167 \\ -1.625 \\ 0.1875 \end{pmatrix}$ be an approximation to the solution of this system.

(i) Find the residual error vector of $\mathbf{x}_a$.

(ii) Without using the exact solution $\mathbf{x}_t$, obtain the error $\mathbf{e} = \mathbf{x}_t - \mathbf{x}_a$.

You may use

$$A^{-1} \approx \begin{pmatrix} 0.3333 & -0.0667 & 0.2 \\ 0 & 0.4 & -0.2 \\ 0 & 0.2 & 0.4 \end{pmatrix}$$

(2) Let D denote the set of all $n \times n$ diagonal matrices. Determine all the elements of D having condition number equal to 1. Here use the maximum norm for vectors in $\mathbb{R}^n$ and the matrix norm to be subordinate to this vector norm.

(P.T.O.)

(3) Show that the matrix A given by

$$A = \begin{pmatrix} 1 & 2 & 1 \\ 3 & 6 & 4 \\ 0 & 1 & 2 \end{pmatrix}$$

has no Doolittle factorization. Interchange suitably the rows of A to get a matrix that has a Doolittle factorization. Justify your answer. (There is no need to compute the factorization).

(4) Decide if each of the followings statements is true or false. If a statement is true, then prove it. If a statement is false, give an example (usually called, a counter-example) to support your case.

(i) If a 3×3 matrix A is invertible, then none of the Gerschgorin disks associated to the matrix A contains zero.

(ii) If a 3×3 matrix B is not invertible, then the real number zero belongs to every Gerschgorin disk associated to the matrix B.

(5) Count the number of arithmetic operations (addition/subtraction, multiplication, and divison) that are needed to reduce the matrix

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 7 & 4 & 1 \\ 0 & 6 & 2 & 1 \\ 0 & 0 & 3 & 1 \end{pmatrix}$$

to an upper triangular matrix using Modified Gaussian elimination method with partial pivoting. Note that you should not count operations like taking maxima, taking modulus, interchanging of rows as we do not call them arithmetic operations.

End of the question paper.

ALL THE BEST

Department of Mathematics, Indian Institute of Technology Bombay
MA 214: Introduction to Numerical Analysis
Make-up Examination

Time: 2 hours — Marks: 30
Instructors: S. Baskar and S. Sivaji-Ganesh — Date: 17-04-2012

Instructions:

1. Write your Name and Roll Number clearly on your answer book as well as every supplement you may use.
2. Number the pages of your answer book and make a question-page index on the front page.
3. The answer to each question should start on a new page. If the answer for a question is split into two parts and written in two different places, the first part alone will be corrected.
4. Only scientific calculators are allowed. Any kind of programing device is not allowed.
5. Formulas used need not be proved but **needs to be stated clearly**.
6. The question paper contains 5 questions each carries 6 marks. **Answer all the questions**.

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(1) Apply the modified Gaussian elimination method with partial pivoting to solve the system of linear equations

$$\begin{pmatrix} \frac{1}{6} & \frac{1}{3} & \frac{1}{2} \\ \frac{1}{3} & \frac{1}{2} & \frac{1}{6} \\ \frac{1}{2} & \frac{1}{6} & \frac{1}{3} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

using 4-digit rounding.

(2) Let p be the polynomial interpolating a function f at the nodes x_0, x_1, x_2 lying in the interval $[a, b]$, where $f \in C^3[a, b]$. Prove that for each $x \in [a, b]$, there exists a $\xi_x \in (a, b)$ such that

$$f(x) - p(x) = \frac{f^{(3)}(\xi_x)}{6}(x - x_0)(x - x_1)(x - x_2).$$

(3) Let $h > 0$.

(i) Use the method of undetermined coefficients to find a numerical differentiation formula for approximating $f'(x)$ such that the formula uses values of the function f at $x - h$, x, and $x + h$.

(ii) Show that the mathematical error in the formula derived above is of order two (in other words, the formula derived above is of order two).

(P.T.O.)

(4) (i) Find a formula for the sequence of iterates (x_n) when Newton-Raphson method is used for approximating a root of the equation $\sin x = 0$ in the interval $(-\pi/2, \pi/2)$.

(ii) Let $\alpha \in (-\pi/2, \pi/2)$ with $\alpha \neq 0$ be such that the sequence of iterates x_n is a cycle of length two *i.e.*, $x_0 = x_2 = x_4 = \cdots$, and $x_1 = x_3 = x_5 = \cdots$ if $x_0 = \alpha$. Find a function g such that α satisfies $g(\alpha) = 0$.

(iii) Starting with the initial guess 1, perform five iterations using Newton-Raphson method for the equation $g(x) = 0$ to find an approximate value of α.

(iv) Using the approximate value of α obtained in the fifth iteration above as the initial guess in the Newton-Raphson method for solving $\sin x = 0$, compute x_1, x_2, x_3, x_4, x_5.

(5) Solve the initial value problem $y' = \sin 2x + \cos 3x$, $0 \leq x \leq 1$, $y(0) = 1$, with $h = 0.5$. Find the value of $y(0.5)$ using Euler method. Using this value of $y(0.5)$, find the value of $y(1)$ by using the Runge-Kutta method of order 2. (x in radians)

End of the question paper.

ALL THE BEST

Department of Mathematics, Indian Institute of Technology Bombay
MA 214: Introduction to Numerical Analysis
End-Semester Examination

Time: 3 hours — Marks: 50
Instructors: S. Baskar and S. Sivaji-Ganesh — Date: 25-04-2012

Instructions:

1. Write your Name and Roll Number clearly on your answer book as well as every supplement you may use.
2. Number the pages of your answer book and make a question-page index on the front page.
3. The answer to each question should start on a new page. If the answer for a question is split into two parts and written in two different places, the first part alone will be corrected.
4. Only scientific calculators are allowed. Any kind of programing device is not allowed.
5. Formulas used need not be proved but **needs to be stated clearly**.
6. The question paper contains 5 questions, each question carries 10 marks. **Answer all the questions**.

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(1) Let $f : [0, \frac{\pi}{6}] \to \mathbb{R}$ be a given function. **Cubic interpolation in a table of function values**

x	x_0	x_1	$\cdots$	x_N
$f(x)$	$f(x_0)$	$f(x_1)$	$\cdots$	$f(x_N)$

means the following:
The values of $f(x)$ are tabulated for a set of equally spaced points in $[a, b]$, say x_i for $i = 0, 1, \cdots, N$ with $x_0 = 0, x_N = \frac{\pi}{6}$, and $h = x_{i+1} - x_i > 0$ for every $i = 0, 1, \cdots, N-1$. For an $\bar{x} \in [0, \frac{\pi}{6}]$ at which the function value $f(\bar{x})$ is not tabulated, the value of $f(\bar{x})$ is taken to be the value of $p_3(\bar{x})$, where $p_3(x)$ is the polynomial of degree less than or equal to 3 that interpolates f at the nodes $x_i, x_{i+1}, x_{i+2}, x_{i+3}$ where i is the least index such that $x \in [x_i, x_{i+3}]$.
Take $f(x) = \sin x$ for $x \in [0, \frac{\pi}{6}]$; and answer the following questions.

(i) Show that
$$|f(\bar{x}) - p_3(\bar{x})| \le \frac{h^4}{48},$$
where $\bar{x}$ and p_3 are as described above.

(ii) If $h = 0.005$, then show that **cubic interpolation in the table of function values** yields the value of $f(\bar{x})$ with at least 10 decimal-place accuracy.

(2) For this question, we take the matrix

$$A = \begin{pmatrix} -2.7083 & -2.6824 & 0.4543 \\ 0.1913 & 0.7629 & 0.1007 \\ -0.3235 & -0.4052 & 5.0453 \end{pmatrix}.$$

Let $\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3$ denote eigenvectors corresponding to the eigenvalues $\lambda_1, \lambda_2, \lambda_3$ respectively of the matrix A which are given by

$$\lambda_1 \approx 5.0187, \lambda_2 \approx -2.5313, \lambda_3 \approx 0.6125$$

and

$$\boldsymbol{v}_1 \approx (0.25, 0.13, 5.02)^T, \boldsymbol{v}_2 \approx (2.53, -0.15, 0.1)^T, \boldsymbol{v}_3 \approx (-0.49, 0.61, 0.02)^T.$$

Answer the following questions:

(i) Obtain the Gerschgorin disks associated to the matrix A and pictorially represent them in the complex plane.

(ii) Can Gerschgorin theorem be used to show that the matrix A has a unique dominant eigenvalue? Justify your answer.

(iii) Define the iterative sequences $\{\mu_k\}$ and $\boldsymbol{x}^{(k)}$ using power method that converge to λ_1 and $\alpha \boldsymbol{v}_1$ for some constant α when the initial guess is $\boldsymbol{x}^{(0)} = (1, 1, 1)^T$. Perform one iteration.

(iv) If we take the initial guess $\boldsymbol{x}^{(0)} = (0, 1, 1)^T$, then show that the iterative sequence obtained by power method converges to λ_j and $K\boldsymbol{v}_j$ for some $j \in \{1, 2, 3\}$ and for some constant K. What is the value of j and possible values of K?
(**Hint:** $\boldsymbol{x}^{(0)} \approx 0.3060\boldsymbol{v}_1 + 0.1864\boldsymbol{v}_2 + 1.6748\boldsymbol{v}_3$.)

(v) Give all possible initial guesses for which the sequence $\{\mu_k\}$ obtained using power method converges to λ_2. Justify your answer.

(vi) Give all possible initial guesses for which the sequence $\{\mu_k\}$ obtained using power method converges to λ_3. Justify your answer.

(3) Using Newton's method, obtain the iterative sequence for finding a point of local extremum of the function

$$f(x_1, x_2) = x_1^2 + x_2 - \sin(x_1 - x_2).$$

By starting the iteration with initial guess $(x_1, x_2) = (1, 0)$, perform two iterations and obtain the residual error in the computed value at each iteration.

(4) Consider the linear system $A\boldsymbol{x} = \boldsymbol{b}$, where

$$A = \begin{pmatrix} -2 & 4 & 1 \\ 7 & 2 & -4 \\ -1 & 4 & 6 \end{pmatrix}$$

and

$$\boldsymbol{b} = \begin{pmatrix} \int_0^1 \sin t \, dt \\ \int_0^1 \cos t \, dt \\ \int_0^1 (\cos t + \sin t) \, dt \end{pmatrix}.$$

(Sine and Cosine are evaluated in radians). Answer the following questions:

(i) Use composite Simpson's method with partition $[0, 1] = [0, \frac{1}{2}] \cup [\frac{1}{2}, 1]$ to obtain approximate values of the components of the vector $\boldsymbol{b}$ using 4-digit rounding. Let us denote the resulting vector by $\tilde{\boldsymbol{b}}$.

(ii) Let $\tilde{\boldsymbol{x}}$ be the exact solution of the linear system $A\boldsymbol{x} = \tilde{\boldsymbol{b}}$, where $\tilde{\boldsymbol{b}}$ is the approximation to the RHS vector $\boldsymbol{b}$ as obtained in (i) above. Give formula for a Gauss-Seidel iterative sequence $\{\tilde{\boldsymbol{x}}^{(k)}\}$ that converges to $\tilde{\boldsymbol{x}}$. Justify that the sequence $\{\tilde{\boldsymbol{x}}^{(k)}\}$ indeed converges by showing that

$$\|\tilde{\boldsymbol{e}}^{(k+1)}\|_\infty \leq \left(\frac{6}{7}\right)^k \|\tilde{\boldsymbol{e}}^{(0)}\|_\infty, \; k = 0, 1, 2, \cdots,$$

where $\tilde{\boldsymbol{e}}^{(k)} = \tilde{\boldsymbol{x}} - \tilde{\boldsymbol{x}}^{(k)}$.

(iii) Let $\boldsymbol{x}^*$ be the exact solution of the system $A\boldsymbol{x} = \boldsymbol{b}$ where A and $\boldsymbol{b}$ are as given above. If

$$\boldsymbol{e}^{(k)} = \boldsymbol{x}^* - \tilde{\boldsymbol{x}}^{(k)}$$

is the error in the k^{th} itereation obtained in (ii), then show that

$$\|\boldsymbol{e}^{(k)}\|_\infty \leq \left(\frac{6}{7}\right)^{k-1} \|\tilde{\boldsymbol{e}}^{(0)}\|_\infty + \|A^{-1}\|_\infty \left(\frac{1}{23040} + 10^{-3}\right).$$

(5) Consider the initial value problem

$$y' = f(x, y),$$
$$y(0) = y_0.$$

(i) Derive the numerical scheme

$$y_{j+1} = y_j + \frac{h}{2} \left(f(x_j, y_j) + f(x_{j+1}, y_{j+1}) \right),$$

where $h = x_{j+1} - x_j > 0$, for approximating solution of the above initial value problem.

(ii) Use Euler method to obtain an approximate value $y_1^{(0)}$ of $y(0.1)$ when

$$f(x, y) = x^2 + y^2, \; y_0 = 1$$

with $h = 0.1$.

(iii) Show that the sequence $\{y_1^{(k)}\}$ defined by

$$y_1^{(k+1)} = y_0 + \frac{h}{2}\left(f(0, y_0) + f(0.1, y_1^{(k)})\right), \; k = 0, 1, 2, \cdots \tag{1}$$

converges to the root $y^* \in [1, 2]$ of the equation

$$y = y_0 + \frac{h}{2}\left(f(0, y_0) + f(0.1, y)\right), \tag{2}$$

where $f(x, y) = x^2 + y^2$, $y_0 = 1$.

(iv) Let $y_1^{(0)}$ obtained in (ii) above be the initial guess for the iterative sequence given by the formula (1). How many iterations l are needed so that $y_1^{(l)}$ has 5 significant digits when compared to the root y^* of the equation (2).

End of the question paper.

ALL THE BEST