# Lecture 2

## Recall

1) Suppose $x^*$ is approximation to exact value $x$

Then

$|x - x^*|$ absolute error

$\dfrac{|x - x^*|}{|x|}$ relative error $(x \neq 0)$

2) $x^*$ is said to approximate $x$ to $t$ significant digits if

$$\frac{|x - x^*|}{|x|} \leq 5 \times 10^{-t}$$

3) Thing which create loss in significant digits

1) Subtracting nearly equal quantities

2) dividing by a very small number.

4) Once an error is committed, it contaminates subsequent results

error propagation

condition

instability

5) condition $\longrightarrow$ sensitivity of $f(x)$ to changes in $x$

$$= \max\left\{ \frac{\left|\frac{f(x)-f(x^*)}{f(x)}\right|}{\left|\frac{x-x^*}{x}\right|} : |x-x^*| \text{ is small} \right\}$$

$$\simeq \left| \frac{f'(x)\, x}{f(x)} \right|$$

i) $f(x) = \sqrt{x}$ is well conditioned

ii) $f(x) = \dfrac{10}{1-x^2}$ is ill-conditioned near 1.

Today I first give an example of instability

$$\begin{cases} \dfrac{du_1}{dt} = 9u_1 + 24u_2 + 5\cos t - \dfrac{1}{3}\sin t \\ \dfrac{du_2}{dt} = -24u_1 - 51u_2 - 9\cos t + \dfrac{1}{3}\sin t \end{cases}$$

$$\left\{ u_1(0) = \frac{4}{3} \qquad u_2(0) = 2/3, \right.$$

exact solution

$$u_1(t) = 2e^{-3t} - e^{-39t} + \frac{1}{3}\cos t$$

$$u_2(t) = -e^{-3t} + 2e^{-39t} - \frac{1}{3}\cos t$$

RK method of order 4

$\tilde{u}_1$ approx to $u_1$

$\tilde{u}_2$ " " $u_2$

| $t$ | $u_1(t)$ | $\tilde{u}_1(t)$ |
|---|---|---|
| 0.1 | 1.793061 | −2.645169 |
| 0.2 | 1.423901 | −18.45158 |
| ⋮ | ⋮ | ⋮ |
| 0.9 | 0.3416143 | −695332.0 |
| 1.0 | 0.2796748 | −3099671 |

| $t$ | $u_2(t)$ | $\tilde{u}_2(t)$ |
|---|---|---|
| 0.1 | −1.032001 | 7.844527 |
| 0.2 | −0.8746809 | 38.87631 |
| ⋮ | ⋮ | ⋮ |
| 0.9 | −0.2744088 | 1390664 |
| 1.0 | −0.2298877 | 6199352 |

# Mathematical Preliminaries

1) Interemediate-value theorem for continuous function.

$f : [a,b] \longrightarrow \mathbb{R}$ continuous function

$x_1, x_2 \in [a,b)$ and

say $f(x_1) < \alpha < f(x_2)$

Then $\exists\, c \in [a,b]$ such that

$$f(c) = \alpha.$$

Corollary 1

Let $f : [a,b] \to \mathbb{R}$ be continuous.

Let $x_1, \ldots, x_n \in [a,b]$ and let

$g_1, \ldots, g_n$ be real numbers all

of one sign. Then

$$\sum_{i=1}^{n} f(x_i)\, g_i = f(\xi) \sum_{i=1}^{n} g_i \quad \text{for some } \xi \in [a,b]$$

Proof (sketch)

without loss of generality

$$f(x_1) = \min\{f(x_i) : i=1,\dots,n\}$$

$$f(x_n) = \max\{f(x_i) : i=1,\dots,n\}$$

$$f(x_1)\sum_{i=1}^{n} g_i \le \sum_{i=1}^{n} f(x_i) g_i \le f(x_n)\sum_{i=1}^{n} g_i$$

$$h(x) = f(x)\sum_{i=1}^{n} g_i \qquad \text{is cts}$$

$$h(x_1) \le \sum_{i=1}^{n} f(x_i) g_i \le h(x_n)$$

So $\exists\, \xi$

$$h(\xi) = \sum_{i=1}^{n} f(x_i) g_i$$

i.e

$$f(\xi)\sum_{i=1}^{n} g_i = \sum_{i=1}^{n} f(x_i) g_i$$

Similary one has the following

**Cor 2** Let $g : [a,b] \to \mathbb{R}$ be a non-negative (or non-positive) integrable function.
Let $f : [a,b] \to \mathbb{R}$ be continuous.

Then

$$\int_a^b f(x) g(x)\, dx = f(\xi) \int_a^b g(x)\, dx$$

for some $\xi \in [a,b]$.

**Remark** The assumption $g(x)$ is of one sign is essential

**example** $f(x) = g(x) = x \qquad x \in [-1,1]$

$\int_{-1}^{1} f \cdot g\, dx = \int_{-1}^{1} x^2 dx = 2/3$. But $\int_{-1}^{1} g(x)\, dx = 0$.

## Theorem

Let $f:[a,b] \longrightarrow \mathbb{R}$ be cts.
Then $\exists\ \alpha, \beta \in [a,b]$ such that

$$f(\alpha) \leq f(x) \leq f(\beta) \quad \forall x \in [a,b]$$

Let us recall

**Theorem** (Rolle's theorem).
Let $f:[a,b] \to \mathbb{R}$ be cts and assume $f:(a,b) \to \mathbb{R}$ is differentiable.
If $f(a) = f(b) = 0$,
then $f'(\xi) = 0$ for some $\xi \in (a,b)$.

Rolle's theorem implies the famous mean-value theorem

<u>Theorem</u> (mean value theorem)

Let $f:[a,b]\to\mathbb{R}$ be cts

& $f:(a,b)\to\mathbb{R}$ be differentiable

Then $\dfrac{f(b)-f(a)}{b-a} = f'(\xi)$ for some $\xi\in(a,b)$.

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Pf apply Rolle's theorem to

$$F(x) = f(x)-f(a) + \frac{f(b)-f(a)}{b-a}(x-a)$$

Theorem (Taylor's formula with integral remainder)

If $f(x)$ has $n+1$ continuous derivatives on $[a,b]$ and $c$ is some pt in $[a,b]$ then for all $x \in [a,b]$

$$f(x) = f(c) + f'(c)(x-c) + \frac{f''(c)}{2}(x-c)^2 + \cdots$$

$$\cdots + \frac{f^{(n)}(c)}{n!}(x-c)^n + R_{n+1}(x)$$

where $R_{n+1}(x) = \frac{1}{n!}\int_c^x (x-s)^n f^{(n+1)}(s)\,ds$

Let $p(x) = a_0 + a_1 x + \cdots + a_n x^n$

$(a_n \neq 0, \quad n \geq 1)$. $\quad a_0, \ldots, a_n \in \mathbb{C}$.

It is easy to see that $p(x)$ has at most $n$ roots.

However the following is non-trivial to prove

Theorem (fundamental theorem of algebra).

$p(x)$ has a root, i.e., there exists $\xi \in \mathbb{C}$ such that $p(\xi) = 0$

# Measuring how fast sequences converge

Note $\frac{1}{n^p} \to 0$ for any $p>0$

Intuitively $\frac{1}{n} \to 0$ more slowly

than $\frac{1}{n^2} \to 0$ and so-on

<u>Defn</u>:- Let $\{\alpha_n\}_{n\geq 1}$ and $\{\beta_n\}_{n\geq 1}$ be sequences. We say $\alpha_n$ is <u>big-oh</u> of $\beta_n$ and write

$$\alpha_n = O(\beta_n)$$

if $|\alpha_n| \leq K|\beta_n|$ for some constant $K$ and for all sufficiently large $n$.

## Examples

$$\left.\begin{array}{l} 1/n \\ \dfrac{1000}{n} \\ \dfrac{10}{n} - \dfrac{40}{n^2} + e^{-n} \\ 1/n^p \quad \text{for } p>1 \end{array}\right\} = O(1/n)$$

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**Def$^n$** Let $\{\alpha_n\}, \{\beta_n\}$ be sequen

We say $\alpha_n$ is <u>little-oh</u> of $\beta_n$ and write

$$\alpha_n = o(\beta_n) \text{ if}$$

$$\lim_{n\to\infty} \frac{\alpha_n}{\beta_n} = 0$$

Example

$$\left.\begin{array}{c} 1/n^2 \\ \\ \dfrac{1}{n\ln(n)} \end{array}\right\} = o\left(\frac{1}{n}\right)$$

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<u>Remark</u>:- A convergence order of rate of $\frac{1}{n}$ is much too slow to be useful in calculations.

<u>Example</u>

$$\frac{\pi}{4} = \sum_{i=0}^{\infty} \frac{(-1)^i}{2i+1} = 1 - \sum_{j=1}^{\infty} \frac{2}{16j^2-1}$$

Set $\alpha_n = 1 - \sum_{j=1}^{n} \frac{2}{16j^2-1}$

The sequence $\{\alpha_n\}$ is monotone decreasing

Moreover

$$0 \le \alpha_n - \frac{\pi}{4} \le \frac{1}{4n+3} \qquad n = 1, 2, \ldots$$

To calculate $\frac{\pi}{4}$ correctly to within $10^{-6}$ we would need $10^{6} \le 4n+3$ or roughly $n = 250{,}000$.

However round of errors in calculating $\alpha_{250{,}000}$ is usually greater than $10^{-6}$.

<u>Here</u> $\alpha_n = \pi/4 + O(1/n)$

$$\alpha_n \ne \pi/4 + o(1/n)$$

## Polynomials

$p(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n$

this is called power form

This may lead to loss of sig-digits

## Example

(4 sig digits) suppose p is st-line

$p(6000) = 1/3$ , $p(6001) = -2/3$

$p(x) = 6000 - x$ in 4 sig digits

$p(6000) \simeq 0$ $p(6001) = -1$

## Remedy

$p(x) = 3.333\,E{-1} - (x - 6000)$

$p(6000) = 3.333\,E{-1}$

$p(6001) = -6.667\,E{-1}$

$$p(x) = x^3 + 2x^2 + x + 1$$
$$= x(x^2 + 2x + 1) + 1$$
$$= x(x(x+2) + 1) + 1$$

Nested multiplication gives less round off error.