Lecture 6

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Last time we had done osculatory interpolation.

$x_0, x_1, \ldots, x_m$ not necc. distinct pts

we say $f(x)$ and $g(x)$ agree on $x_0, \ldots x_m$ if

$$f^{(j)}(z) = g^{(j)}(z) \quad \text{for } j = 0, 1, \ldots, r-1$$

for any z which appears r times in $x_0, x_1, \ldots, x_m$

Theorem If $f(x)$ has r continuous derivatives and no point in the sequence $x_0, x_1, \ldots, x_m$ occurs more than r times, then there exists exactly one polynomial $P_m(x)$ of degree $\leq m$ which agrees with $f(x)$ at $x_0, x_1, \ldots x_m$ . .

$$f[x_0, x_1, \dots, x_m] = \text{coeff of } x^m \text{ in } P_m(x)$$

$$P_m(x) = P_{m-1}(x) + f[x_0, \dots x_m] \prod_{i=0}^{m-1} (x - x_i)$$

Special cases

1) $x_0 = x_1 = x_2 = \dots = x_m$

$P_m(x) =$ Taylor polynomial with center x_0

$$= f(x_0) + f'(x_0)(x - x_0) + \frac{f''(x_0)}{2}(x - x_0)^2 + \dots + \frac{f^{(m)}(x_0)}{m!}(x - x_0)^m$$

Thus $f[\underbrace{x_0, x_0, \dots, x_0}_{m+1 \text{ times}}] = \dfrac{f^{(m)}(x_0)}{m!}$

This case is $\underbrace{x_0, x_0, \dots x_0}_{m+1 \text{ times}}$

Special case 2

$x_0, x_1, \ldots, x_k$ $\quad$ $k+1$ distinct pts

$f(x_0), f(x_1), \ldots, f(x_k)$

$f'(x_0), f'(x_1), \ldots, f'(x_k)$ is also given

need to find polynomial $p(x)$ of degree $\leq 2k+1$ such that $p(x_i) = f(x_i)$ & $p'(x_i) = f'(x_i)$ $\forall i$

Algorithm

Let $y_0 = x_0$, $y_1 = x_0$

$y_2 = x_1$, $y_3 = x_1$

$\vdots$

$y_{2i} = x_i$, $y_{2i+1} = x_i$

$\vdots$

$y_{2k} = x_k$, $y_{2k+1} = x_k$

We are looking for a polynomial which agrees with $f(x)$ at $y_0, y_1, \ldots, y_{2k}, y_{2k+1}$

Construct divided diff table

"use $f[a,a] = f'(a)$ "

$$p(x) = f[y_0] + \sum_{i=1}^{2k+1} f[y_0, y_1, \ldots y_i] \prod_{j=0}^{i-1} (x - y_j)$$

Example a, b distinct pts

$f(a), f(b)$ $\qquad f'(a), f'(b)$

$$P_3(x) = f(a) + f[a,a](x-a) + f[a,a,b](x-a)^2 + f[a,a,b,b](x-a)^2(x-b)$$

We prove by direct computation that $P_3(x)$ agree with $f(x)$ at a, a, b, b

$$f[a,a] = f'(a)$$

clearly $P_3(a) = f(a)$, $\quad P_3'(a) = f[a,a] = f'(a)$

$$f[a,b] = \frac{f(b)-f(a)}{b-a}$$

$$f[a,a,b] = \frac{f[a,b]-f[a,a]}{b-a}$$

$$= \frac{\frac{f(b)-f(a)}{b-a} - f'(a)}{b-a}$$

$$= \frac{f(b)-f(a)-(b-a)f'(a)}{(b-a)^2}$$

$$P_3(b) = f(a) + f'(a)(b-a) + $$
$$+ \frac{f(b)-f(a)-(b-a)f'(a)}{\cancel{(b-a)^2}} \cdot \cancel{(b-a)^2}$$
$$= f(b)$$

$$f[a,b,b] = \frac{f[b,b]-f[a,b]}{b-a}$$
$$= \frac{f'(b) - \frac{f(b)-f(a)}{b-a}}{b-a}$$
$$= \frac{f'(b)(b-a) - (f(b)-f(a))}{(b-a)^2}$$

$$f[a,a,b,b] = \frac{f[a,b,b]-f[a,a,b]}{b-a}$$
$$= \frac{f'(b)(b-a) - (f(b)-f(a)) - (f(b)-f(a)) + f'(a)(b-a)}{(b-a)^3}$$
$$= \frac{(f'(b)+f'(a))(b-a) - 2(f(b)-f(a))}{(b-a)^3}$$

$$P_3(x) = f(a) + f[a,a](x-a) + f[a,a,b](x-a)^2 + f[a,a,b,b](x-a)^2(x-b)$$

$$P_3'(b) = f'(a) + 2f[a,a,b](b-a) + f[a,a,b,b](b-a)^2$$

$$= f'(a) + 2\frac{\{f(b) - f(a) - (b-a)f'(a)\}}{b-a} + \frac{\{(f'(b)+f'(a))(b-a) - 2(f(b)-f(a))\}}{b-a}$$

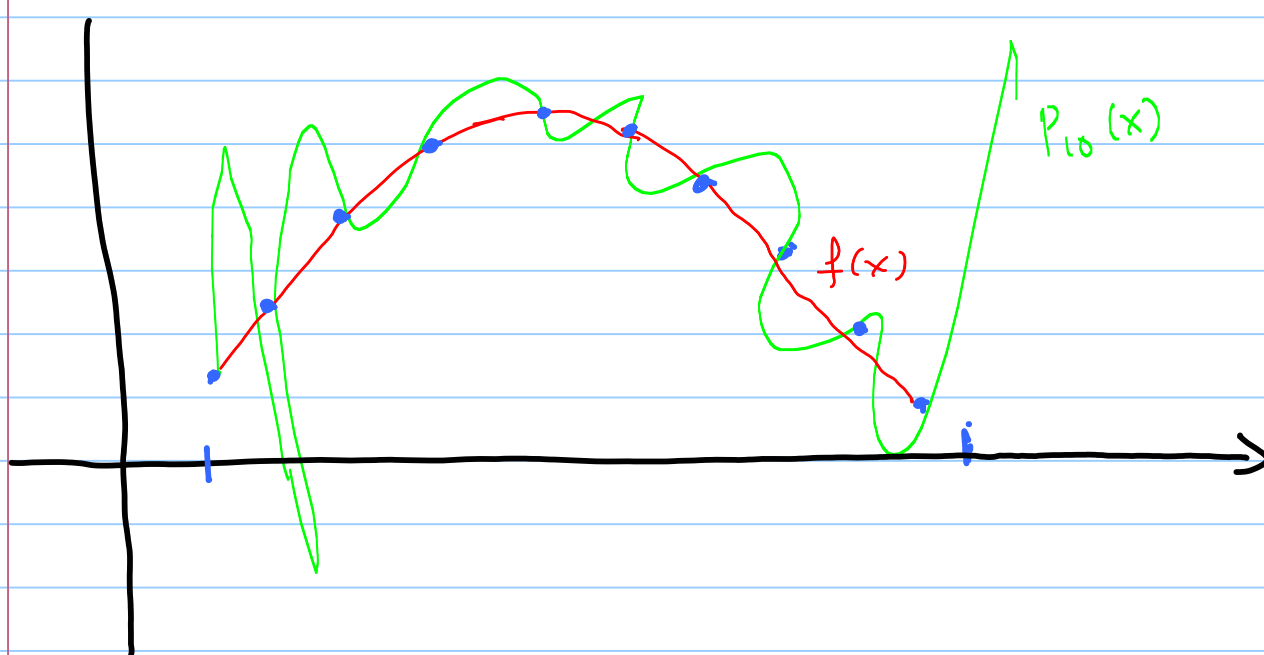
$$= f'(b)$$

Disadvantages of interpolation

Note that if $x_0, x_1, \ldots, x_k$ are pts in $[a,b]$
then interpolating polynomial has degree k.

In practice k is large

However a polynomial of degree k (k large) oscillates a lot



for example if there are 101 data pts, then it is not advisable to work with a degree 100 interpolating formula as this also creates lot of round-off error

Strategy

Use piecewise-polynomial approximation

Simplest-case

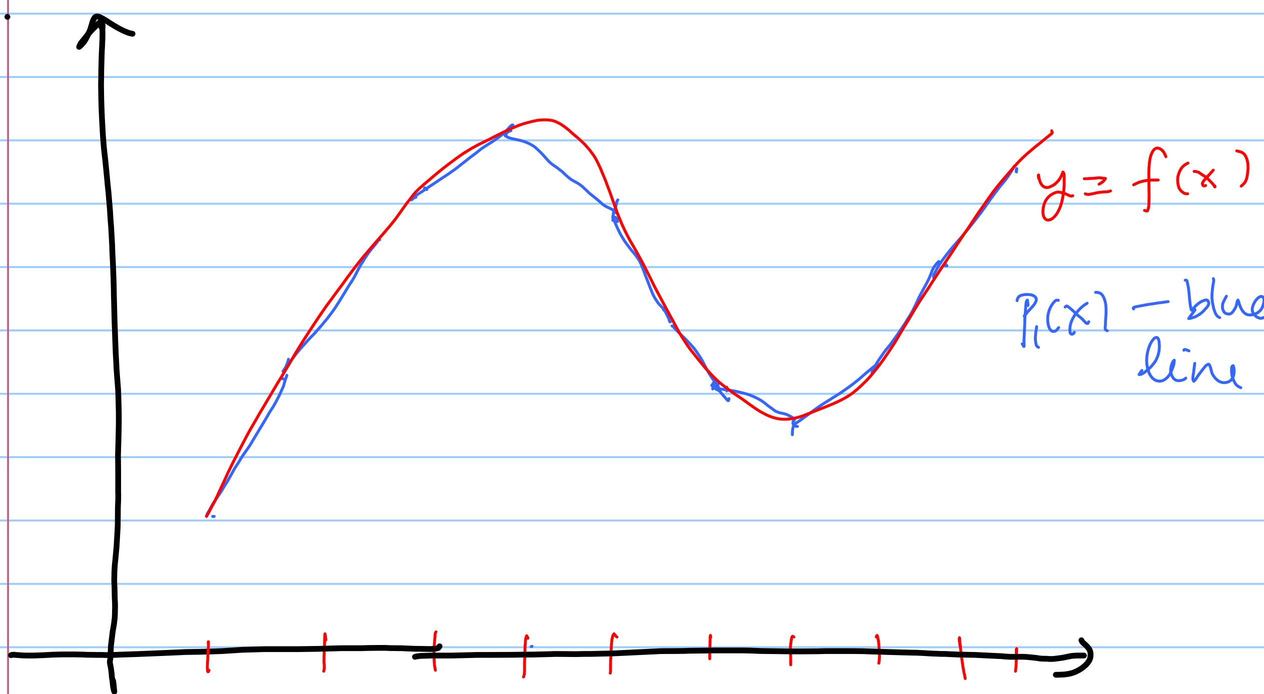
piecewise-linear interpolation

$$a = x_1 < x_2 < x_3 < \dots \quad < x_{N+1} = b$$

$f(\bar{x})$ is approximated at a pt $\bar{x}$ by locating the interval $[x_k, x_{k+1}]$ containing $\bar{x}$ and then taking

$$P_1(\bar{x}) = f(x_k) + f[x_k, x_{k+1}](\bar{x} - x_k)$$

Graphical representation



note that if $|x_i - x_{i+1}|$ is small then $P_1(x)$ approximates $f(x)$ to a a high degree.

If we use higher degree (say cubic) piecewise-polynomial approximation then we get a better approximation.

Construction of piecewise-cubic function $g_3(x)$ which interpolates

$f(x)$ at the pts $x_1, \ldots, x_{N+1}$ where

$$a = x_1 < x_2 < \cdots < x_{N+1} = b$$

on each $[x_i, x_{i+1}]$ we construct $g_3(x)$ as a cubic polynomial $P_i(x)$ $i=1, \ldots, N$

$$P_i(x) = c_{1,i} + c_{2,i}(x-x_i) + c_{3,i}(x-x_i)^2 + c_{4,i}(x-x_i)^3$$

$$i=1, \ldots, N$$

Since $g_3(x_i) = f(x_i)$ for $i=1, \ldots, N+1$

We have

$$P_i(x_i) = f(x_i) \quad \& \quad P_i(x_{i+1}) = f(x_{i+1})$$

$$i=1, \ldots, N$$

In particular

$$P_{i-1}(x_i) = P_i(x_i) = f(x_i)$$

$$i = 2, \ldots, N$$

So $g_3(x)$ is continuous on $[a,b]$.

Only constraint for $P_i(x)$

is $P_i(x_i) = f(x_i)$

and $P_i(x_{i+1}) = f(x_{i+1})$

So we have some freedom in choosing the $P_i(x)$

We study 2 cases

1) Piecewise - cubic Hermite interpolation

2) Cubic - spline interpolation

1) piecewise-cubic Hermite interpolation

One determines $p_i(x)$ so as to interpolate $f(x)$ at $x_i, x_i, x_{i+1}, x_{i+1}$

i.e.,
$$p_i'(x_i) = f'(x_i) \quad \& \quad p_i'(x_{i+1}) = f'(x_{i+1})$$

By Newton's formula

$$p_i(x) = f(x_i) + f[x_i, x_i](x - x_i) + f[x_i, x_i, x_{i+1}](x - x_i)^2 + f[x_i, x_i, x_{i+1}, x_{i+1}](x - x_i)^2 (x - x_{i+1})$$

Note $(x - x_{i+1}) = (x - x_i) + (x_i - x_{i+1})$

So
$$p_i(x) = f(x_i) + f'(x_i)(x - x_i) + \left(f[x_i, x_i, x_{i+1}] - f[x_i, x_i, x_{i+1}, x_{i+1}]\Delta x_i\right)(x - x_i)^2 + f[x_i, x_i, x_{i+1}, x_{i+1}](x - x_i)^3$$

Algorithm

$$f_i = f(x_i) \qquad \Delta x_i = x_{i+1} - x_i$$

$$s_i = f'(x_i)$$

$$i = 1, \ldots, N+1$$

$$c_{1i} = f_i \qquad c_{2i} = s_i$$

$$c_{3,i} = f[x_i, x_i, x_{i+1}] - f[x_i, x_i, x_{i+1}, x_{i+1}]\Delta x_i$$

$$= \frac{f[x_i, x_{i+1}] - s_i}{\Delta x_i} - c_{4i}\,\Delta x_i$$

$$c_{4i} = f[x_i, x_i, x_{i+1}, x_{i+1}]$$

$$= \frac{f[x_i, x_{i+1}, x_{i+1}] - f[x_i, x_i, x_{i+1}]}{\Delta x_i}$$

$$= \frac{s_{i+1} + s_i - 2f[x_i, x_{i+1}]}{\Delta x_i}$$

Example

$a = x_1$	x_2	x_3	$x_4 = b$
$f(x_1)$	$f(x_2)$	$f(x_3)$	$f(x_4)$
$f'(x_1)$	$f'(x_2)$	$f'(x_3)$	$f'(x_4)$

Construction of $g_3(x)$

in $[x_1, x_2]$

$$g_3(x) = P_1(x) = f[x_1] + f[x_1, x_1](x - x_1) + f[x_1, x_1, x_2](x - x_1)^2 + f[x_1, x_1, x_2, x_2](x - x_1)^2(x - x_2)$$

In $[x_2, x_3]$

$$g_3(x) = P_2(x) = f[x_2] + f[x_2, x_2](x - x_2) + f[x_2, x_2, x_3](x - x_2)^2 + f[x_2, x_2, x_3, x_3](x - x_2)^2(x - x_3)$$

In $[x_3, x_4]$

$$g_3(x) = P_3(x) = f[x_3] + f[x_3, x_3](x - x_3) + f[x_3, x_3, x_4](x - x_3)^2 + f[x_3, x_3, x_4, x_4](x - x_3)^2(x - x_4)$$

This method is used in numerical solution to first order differential equations.

Example

$$\frac{dy}{dx} = y - x^2 + 1 \qquad 0 \le x \le 1$$

$$y(0) = 0.5$$

computed by a numerical method

x_i	$y(x_i)$	$y'(x_i)$
0	0.500	1.5
0.2	0.826	1.786
0.4	1.207	2.647
0.6	1.637	2.277
0.8	2.110	2.470
1.0	2.618	2.618

find $y(0.7)$, $y(0.9)$

usual Osculatory interpolating polynomial has degree 11

So we use piecewise Hermite interpolation

x	$y(x)$	$f[\,,\,]$	$f[\,,\,,\,]$	$f[\,,\,,\,,\,]$
0.6	1.637	2.277	4.4 E-1	4.25 E-1
0.6	1.637	2.365	5.25 E-1	
0.8	2.110	2.470		
0.8	2.110			

$$P_4(x) = 1.637 + 2.277\,(x-0.6) + 4.4\,E{-1}\,(x-0.6)^2 + 4.25\,E{-1}\,(x-0.6)^2\,(x-0.8)$$

$P_4(0.7) = 1.869$

$P_4(0.9) = ?$

x	$y(x)$	$f[\,,\,]$	$f[\,,\,,\,]$	$f[\,,\,,\,,\,]$
0.8	2.11	2.47	0.35	0.2
0.8	2.11	2.54	0.39	
1.0	2.618	2.618		
1.0	2.618			

$$P_5(x) = f[0.8] + f[0.8,0.8](x-0.8)$$
$$+ f[0.8,0.8,1.0](x-0.8)^2$$
$$+ f[0.8,0.8,0.9,0.9](x-0.8)^2(x-1)$$

$$P_5(0.9) = 2.3603$$
$$= 2.360 \text{ in 4 sig digits}$$

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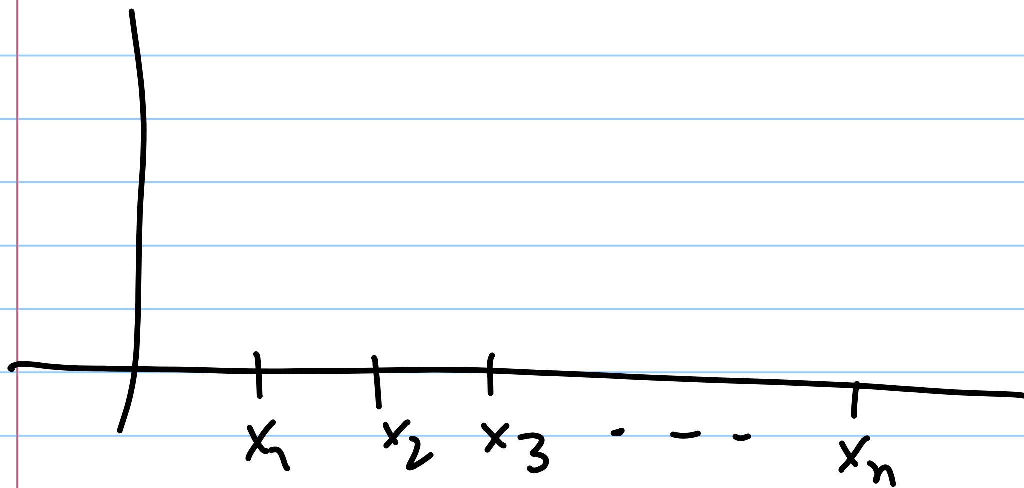
Disadvantages of Hermite interpolation

It requires knowledge of
$f'(x_1), \ldots f'(x_{N+1})$.

This is not always available.



$g_3(x)$ cubic spline

$g_3(x)$ is continuously twice differentiable

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