Mathematical Model of the Servo System

Linear Model

A DC motor (Figure 1) is described by two classical equations:

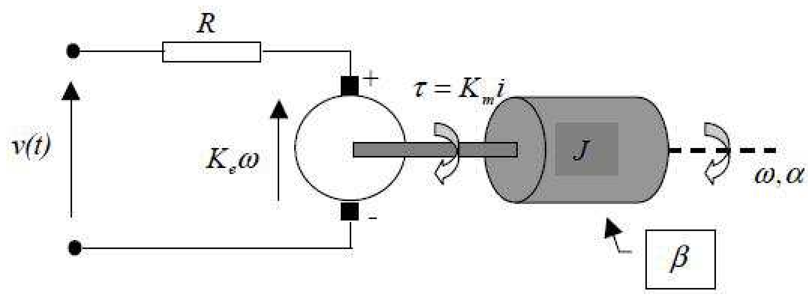

Figure 1: Diagram of the DC Motor

Electrical: Using KVL

$$v(t) = Ri(t) + L\frac{di(i)}{dt} + K_e\omega(t)$$

Considering negligible armature inductance(L) voltage equation is given by

$$v(t) = Ri(t) + K_e\omega(t) \tag{1}$$

Mechanical: Using Newton's law of motion

$$J\dot{\omega}(t) = K_m i(t) - \beta\omega(t) \tag{2}$$

where:
$v(t)$ is the input voltage,
$i(t)$ is the armature current,
$\omega(t)$ is the angular velocity of the rotor,
R is the resistance of armature winding,
L is the inductance of armature,
J is the moment of inertia of the moving parts,
β is the damping coefficient due to viscous friction,
K_e is back EMF constant,
K_m is torque constant,
$K_e\omega(t)$ is the back EMF,
and $\tau = K_m i(t)$ is the electromechanical torque.

This model is linear because static and dry kinetic friction, as well as saturation are neglected.
Substituting value of $i(t)$ from (1) in (2)

$$J\dot{\omega}(t) = K_m \frac{(v(t) - K_e\omega(t))}{R} - \beta\omega(t)$$

On rearranging above equation we obtain the equation of a first order inertial system

$$\frac{RJ}{K_eK_m + \beta R}\dot{\omega}(t) = -\omega(t) + \frac{K_m}{K_eK_m + \beta R}v(t)$$

$$\Rightarrow T_s\dot{\omega}(t) = -\omega(t) + K_{sm}v(t)$$

where the motor time constant T_s and motor gain K_{sm} are given by

$$T_s = \frac{RJ}{\beta R + K_eK_m}, \qquad K_{sm} = \frac{K_m}{\beta R + K_eK_m}.$$

The transfer function has form

$$G(s) = \frac{\omega(s)}{v(s)} = \frac{K_{sm}}{T_sS + 1}.$$

The transfer function for the motor position has form:

$$G(s) = \frac{\alpha(s)}{v(s)} = \frac{K_{sm}}{s(T_sS + 1)}.$$

Because of the control applied in the system, is a PWM signal we assume the dimensionless control signal as the scaled input voltage, $u(t) = v(t)/v_{max}$. The admissible controls satisfy $|u(t)| \leq 1$.

Respecting also $K_s = K_{sm}v_{max}$ we obtain transfer functions in the forms:

Velocity transfer function	Angle transfer function
$G(s) = \frac{\omega(s)}{u(s)} = \frac{K_s}{T_s+1}$	$G(s) = \frac{\alpha(s)}{u(s)} = \frac{K_s}{s(T_sS+1)}$

The model can be written using a state space notation. Let $x = col(x_1, x_2)$ be the state vector where x_1 is the angle α (in[rad]) determining the position of the motor shaft, and $x_2 = \omega$ is the respective angular velocity (in[rad/s]). Time t is measured in [s].
The state equations read

$$\begin{aligned}\dot{x}_1 &= x_2 \\ \dot{x}_2 &= ax_2 + bu\end{aligned}$$

where
$a = -\frac{1}{T_s} < 0, b = \frac{K_s}{T_s} > 0.$

The equivalent classical matrix state space notation has the form

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx \end{aligned}$$

where

$$A = \begin{bmatrix} 0 & 1 \\ 0 & -\frac{1}{T_s} \end{bmatrix}, \qquad B = \begin{bmatrix} 0 \\ \frac{K_s}{T_S} \end{bmatrix}, \qquad C = I.$$

The system can be classified as a multi-variable (SIMO) because it has two measurable state variables and one control variable. The parameters T_s and K_s must be identified by a user.

Observations

Calibration Curve

Motor Input	Encoder Output	Techogenerator Output

Observations at different motor inputs for various combinations:

1. motor without mechanical load

Motor Input	Encoder Output	Techo output	Time Constant	Gain

2. Motor with mechanical load

Motor Input	Encoder Output	Techo output	Time Constant	Gain

3. Motor with eddy current load without mechanical load

Motor Input	Encoder Output	Techo output	Time Constant	Gain

4. Motor with eddy current load with mechanical load

Motor Input	Encoder Output	Techo output	Time Constant	Gain