AE 230 - **Modeling and Simulation Laboratory**

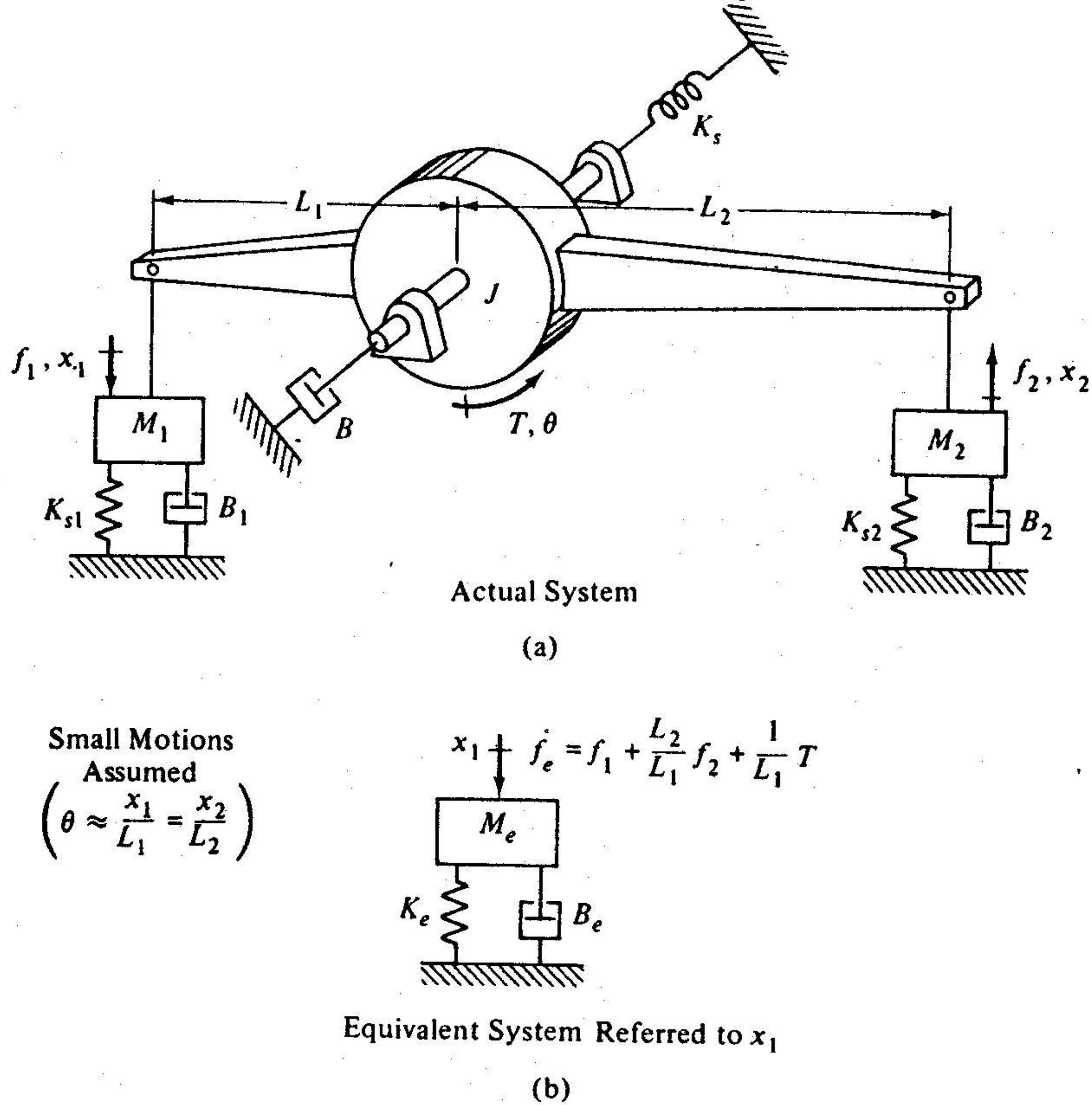
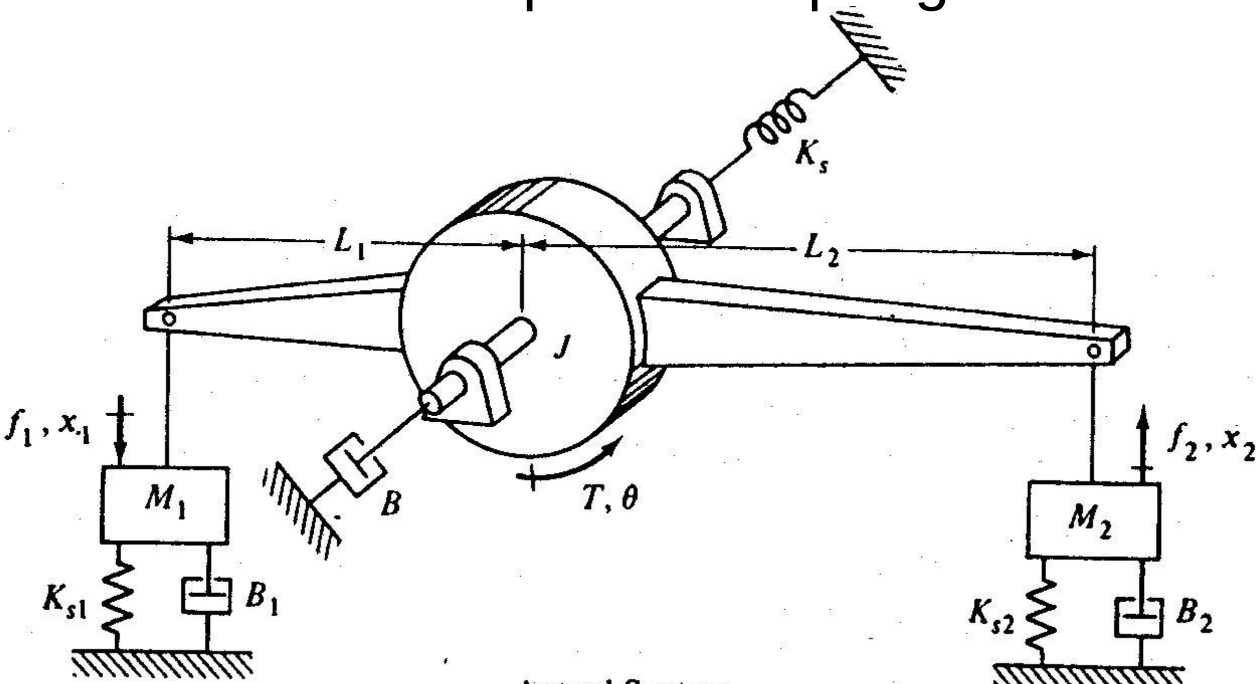


Figure Translational equivalent for complex system.

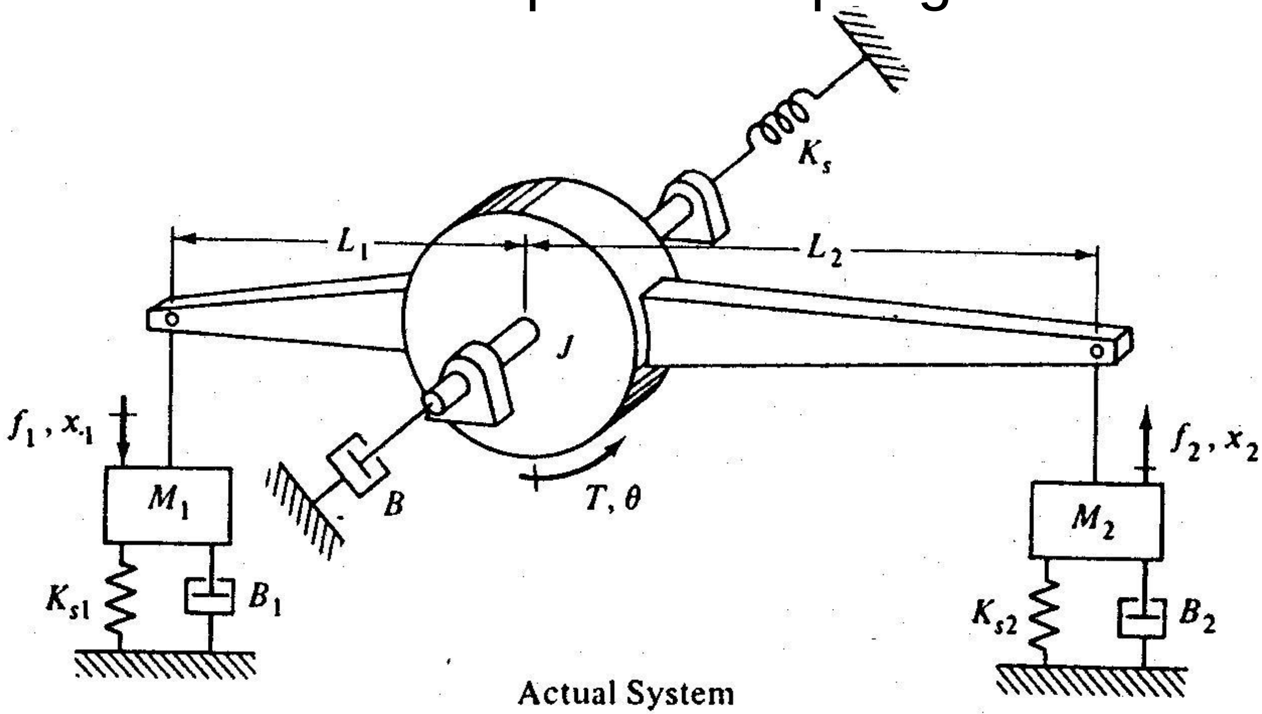
Equivalent spring



Actual System

Single equivalent spring, which will have the same effects as three actual springs. Springs are considered to be ideal. i.e. no damping and mass, remove the dashpot and mass to have only spring effect i.e. only static equivalence (zeroth order)

Equivalent spring



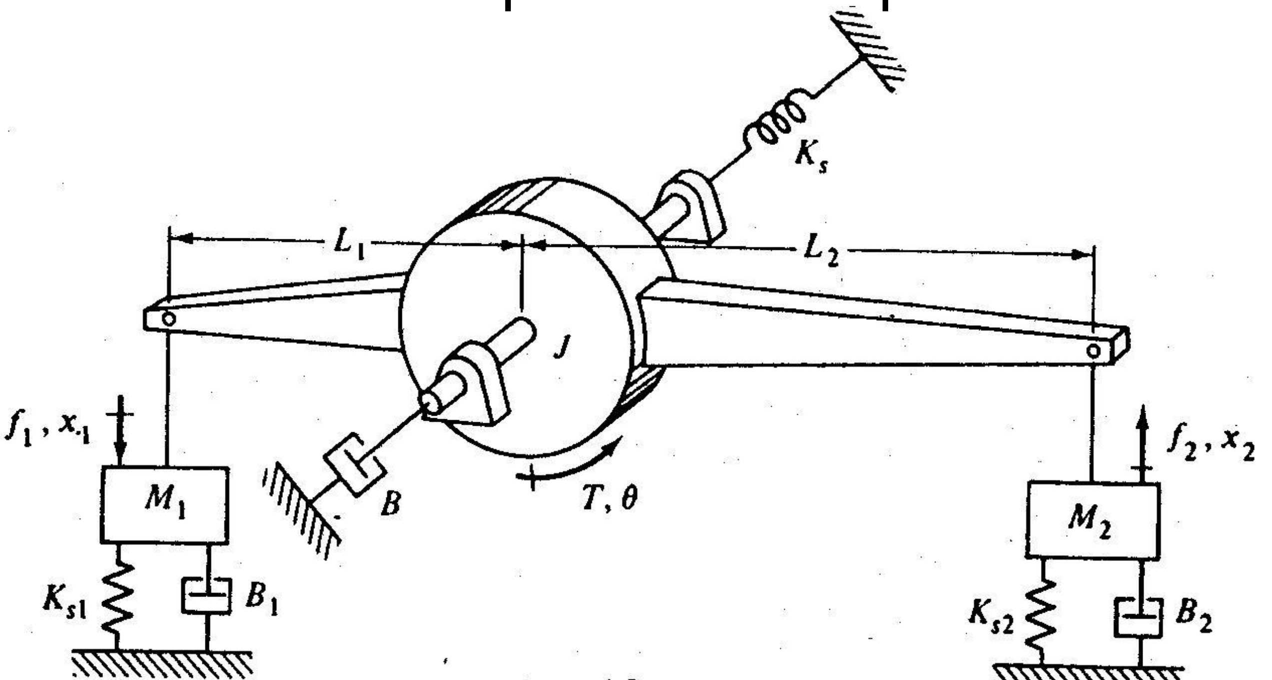
Applying a static force f_1 at location x_1 and writing torque balance equation

$$f_1 L_1 = (K_{s1} x_1) L_1 + \left(\frac{L_2}{L_1} x_1 K_{s2} \right) L_2 + \frac{x_1 K_s}{L_1}$$

$$f_1 = K_{se} x_1$$

$$K_{se} = K_{s1} + \left(\frac{L_2}{L_1} \right)^2 K_{s2} + \frac{K_s}{L_1^2}$$

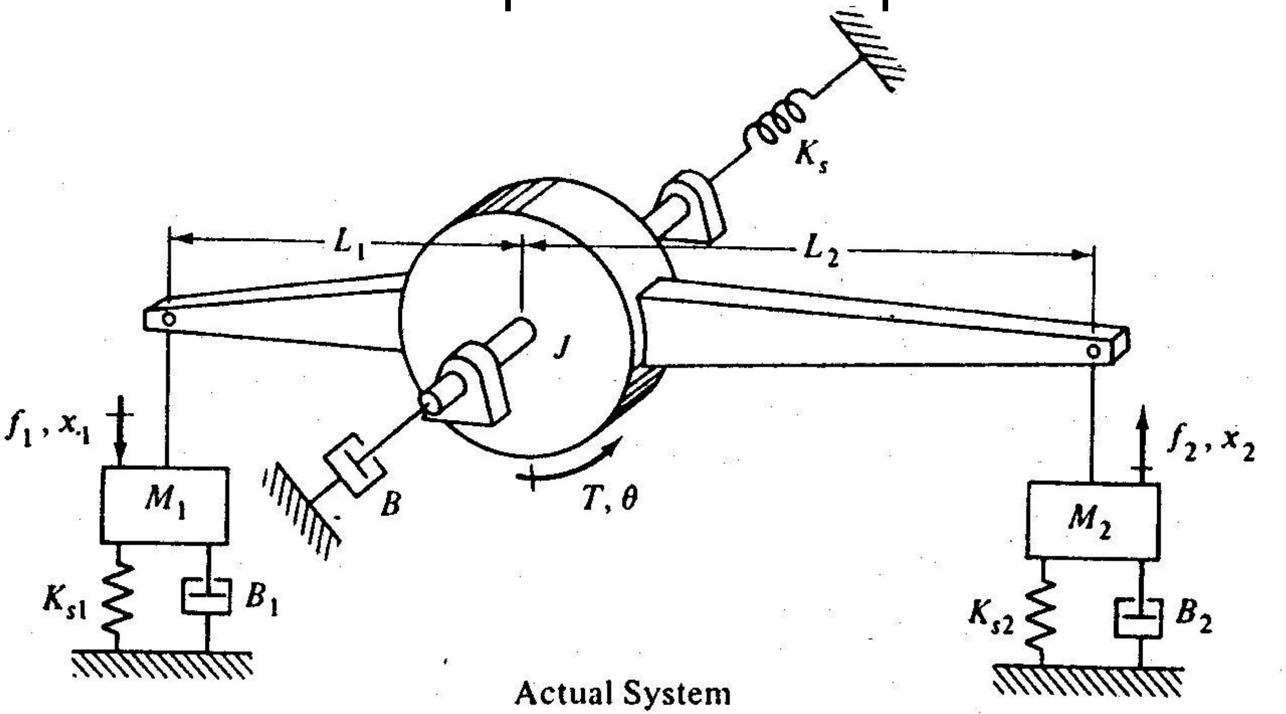
Equivalent damper



Actual System

Single equivalent damper, which will have the same effects as three actual damper. dampers are considered to be ideal. i.e. no spring and mass, remove the spring and mass from the system to have only damper effects i.e. only velocity equivalence, no inertia and no energy storage

Equivalent damper



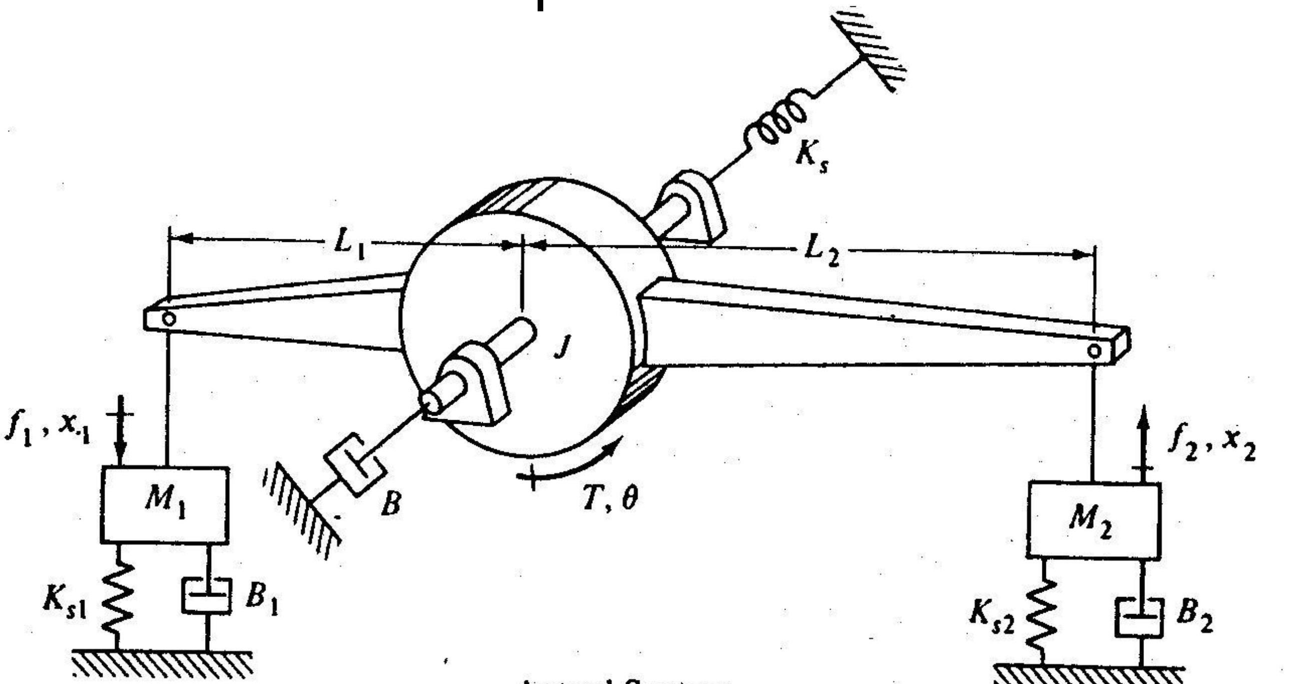
Applying a force f_1 at location x_1 and writing torque balance equation

$$f_1 L_1 = (B_1 \dot{x}_1) L_1 + (B_2 \dot{x}_2) L_2 + (B\dot{\theta})$$

$$f_1 L_1 = (B_1 \dot{x}_1) L_1 + (B_2 \dot{x}_1) \frac{L_2^2}{L_1} + \frac{\dot{x}_1}{L_1} B$$

$$f_1 = B_e \dot{x}_1 \qquad B_e = B_1 + \left(\frac{L_2}{L_1}\right)^2 B_2 + \frac{B}{L_1^2}$$

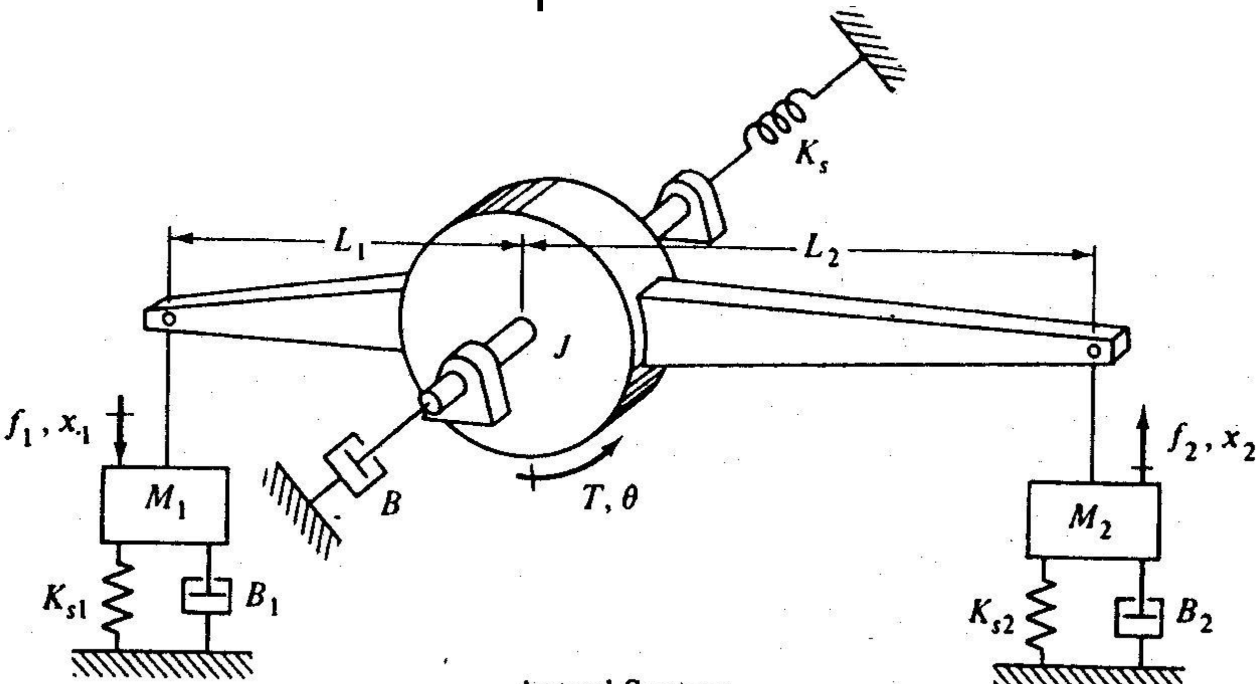
Equivalent mass



Actual System

Single equivalent mass, which will have the same effects as three actual mass. Masses are considered to be ideal. i.e. no spring and damper, remove the spring and damper from the system to have only mass effects i.e. only acceleration equivalence, no dissipation and no energy storage in spring

Equivalent mass

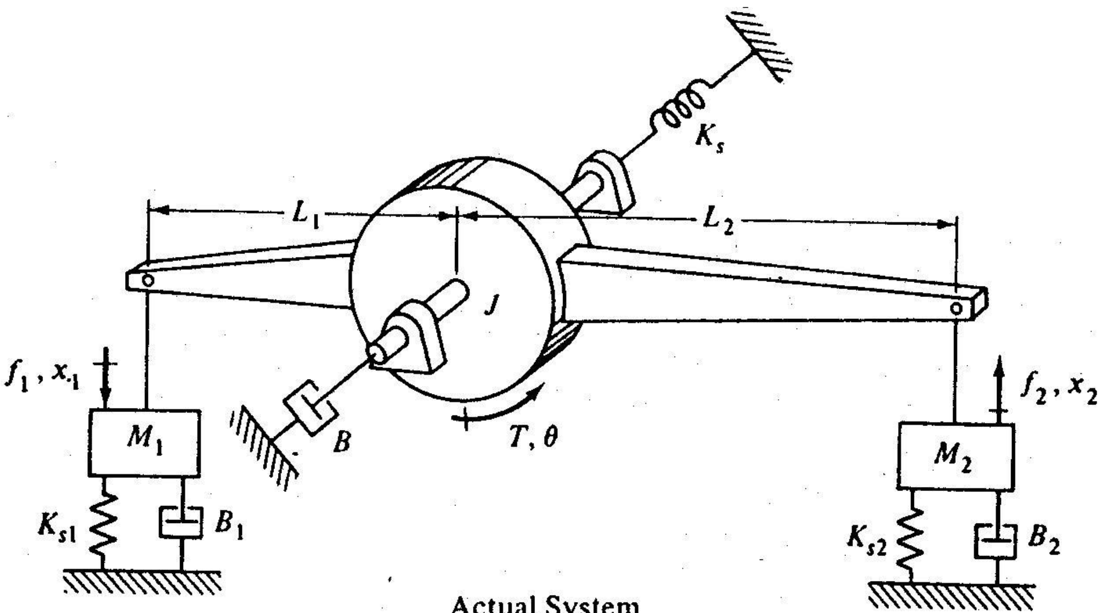


Actual System

Applying a force f_1 at location x_1 and writing torque balance equation

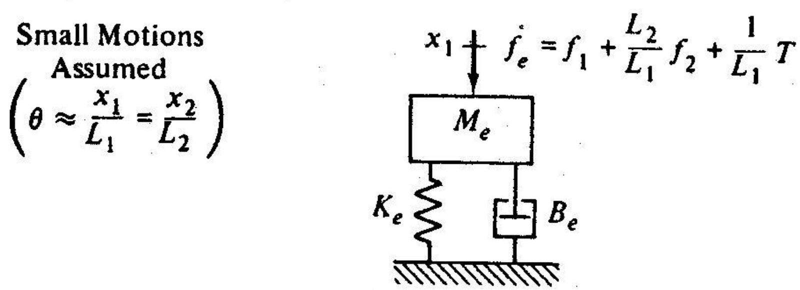
$$f_1 L_1 \approx (M_1 L_1^2)\frac{\ddot{x}_1}{L_1} + (M_2 L_2^2)\frac{\ddot{x}_1}{L_1} + (J)\frac{\ddot{x}_1}{L_1}$$

$$f_1 \approx M_e \ddot{x}_1 \qquad M_e \approx M_1 + M_2\left(\frac{L_2}{L_1}\right)^2 + \frac{J}{L_1^2}$$



Actual System

(a)



Equivalent System Referred to x_1

(b)

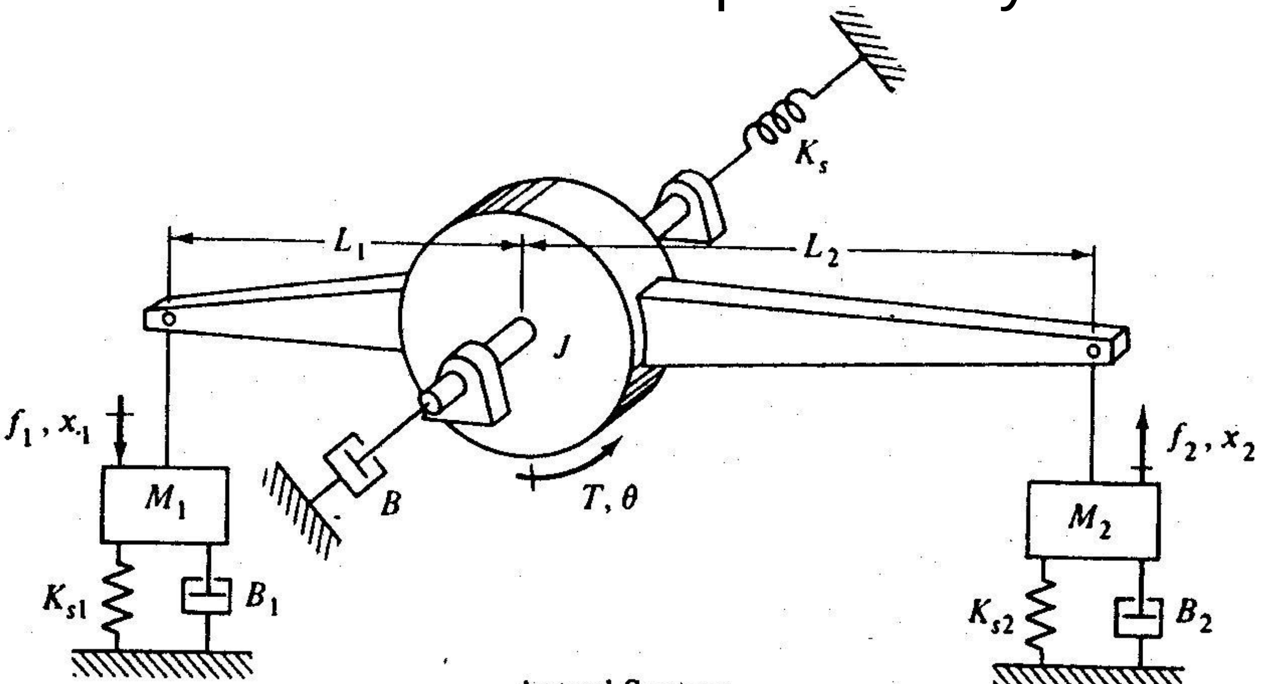
Figure Translational equivalent for complex system.

$$K_{se} = K_{s1} + \left(\frac{L_2}{L_1}\right)^2 K_{s2} + \frac{K_s}{L_1^2}$$

$$B_e = B_1 + \left(\frac{L_2}{L_1}\right)^2 B_2 + \frac{B}{L_1^2}$$

$$M_e \approx M_1 + M_2\left(\frac{L_2}{L_1}\right)^2 + \frac{J}{L_1^2}$$

Home work - Equivalent system



Actual System

Equivalent damper, spring and inertia @ x_2

Linearity

Let a system $\dot{x} = f(x(t), r(t), t)$

Solution of above differential equation for initial condition as $x(t_0)$ at $t = t_0$ and input $r(t)$, $t > t_0$

$$x(t) = \phi(x(t_0), r(t))$$

Homogeneity: $x(t)$ with input $\alpha r(t)$ is equal to α times $x(t)$ with input $r(t)$

$$\phi(x(t_0), \alpha r(t)) = \alpha \phi(x(t_0), r(t))$$

Linearity

Superposition: $x(t)$ with input $(r_1(t)+r_2(t))$ is equal to the sum of $x(t)$ with input $r_1(t)$ and $x(t)$ with input $r_2(t)$

$$\phi(x(t_0),(r_1(t)+r_2(t))) = \phi(x(t_0),r_1(t)) + \phi(x(t_0),r_2(t))$$

Both can be combined

$$\phi(x(t_0),(\alpha r_1(t)+\beta r_2(t))) = \alpha\phi(x(t_0),r_1(t)) + \beta\phi(x(t_0),r_2(t))$$

Linearity

Decomposition: $$x'(t) = \phi(0,(r(t))$$

Above equation is the solution for when the system is in zero state for all input r(t)

Let $x''(t) = \phi(x(t_0),0)$

Above equation is the solution when for all the states $x(t_0)$, the input r(t) is zero

The system is said to have decomposition property if

$$x = x'(t) + x''(t)$$

$$x = \phi(0, r(t)) + \phi(x(t_0),0)$$

Linearity

A system is said to be linear if it satisfies the decomposition property and has zero input linearity and zero state linearity

Zero input linearity mean $r(t)=0$

Zero state linearity mean $x(t_0)=0$

Both the conditions must satisfy homogeneity and superposition properties with respect to various initial conditions

Linearization

Practical systems have various type of non-linearities and most of the analysis requires linear, models.

While understanding system behaviour, model can be linearised about various operating points. Operating points are chosen such that the behaviour between them is close to real system. Relationship could be a one output and one input (line) or one output and multiple input or multiple output and multiple input (surface or a hyper plane) .

Linearization of static system

Consider a input-output continuous relationship as

$$y(t) = \psi(r(t)) \quad \text{or} \quad y = \psi(r)$$

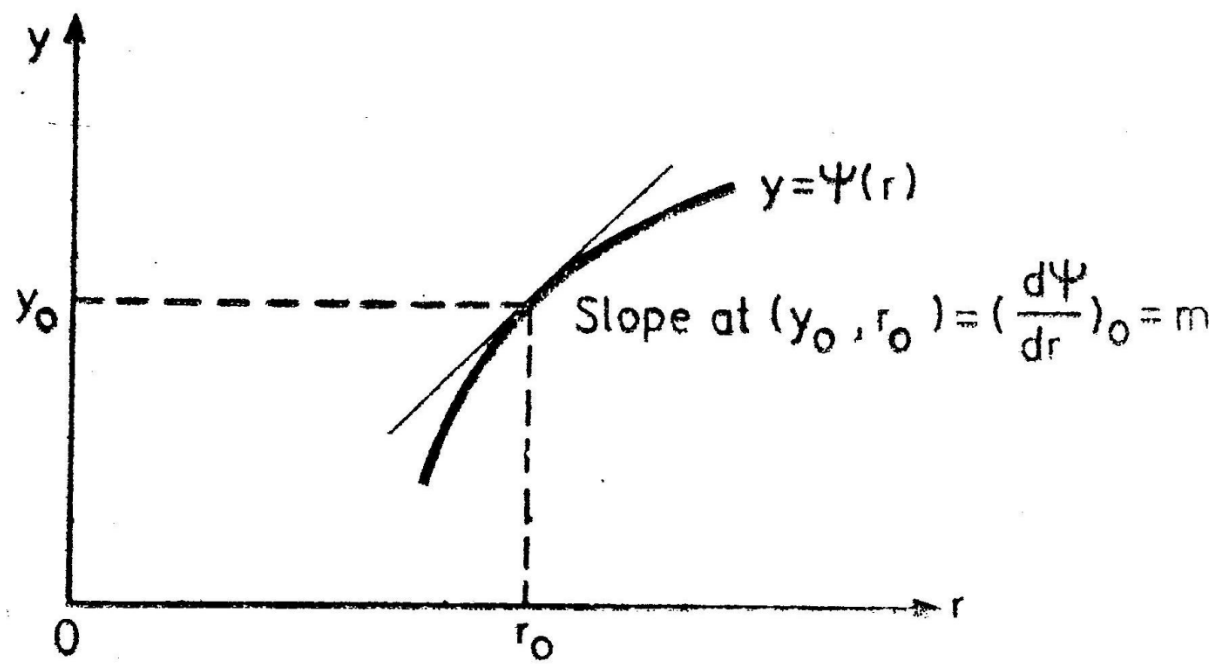


Fig. 2.23 Linearization of a nonlinear system.

Linearization of static system

Expansion of equation into Taylor's series around the operating point (r_0, y_0)

$$y = \psi(r) = \psi(r_0) + \left(\frac{d\psi}{dr}\right)_0 \frac{(r - r_0)}{1!} + \left(\frac{d^2\psi}{dr^2}\right)_0 \frac{(r - r_0)^2}{2!} + \dots$$

If the variation is small about the operating point (r_0, y_0), then higher order terms can be neglected.

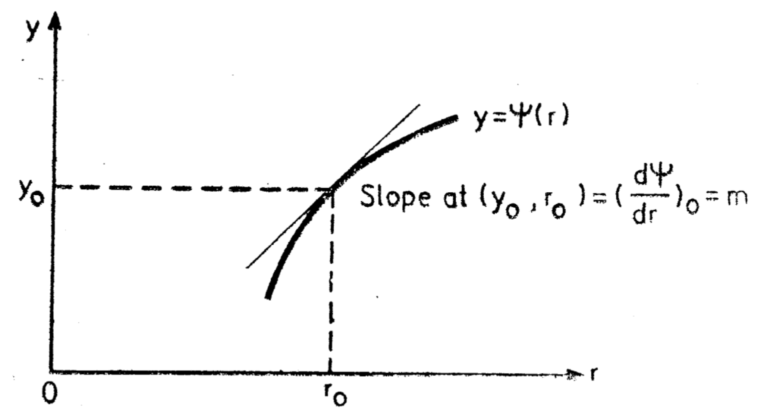


Fig. 2.23 Linearization of a nonlinear system.

$$y = \psi(r) \cong \psi(r_0) + \left(\frac{d\psi}{dr}\right)_0 \frac{(r - r_0)}{1!}$$

Linearization of static system

$$y = \psi(r) \cong \psi(r_0) + \left(\frac{d\psi}{dr}\right)_0 \frac{(r - r_0)}{1!}$$

$$y = y_0 + m(r - r_0) \quad \text{Where, } y_0 = \psi(r_0) \text{ and } \mathrm{m} = \left(\frac{d\psi}{dr}\right)_0$$

$$y - y_0 = m(r - r_0)$$

$$\tilde{y} = m\tilde{r}$$

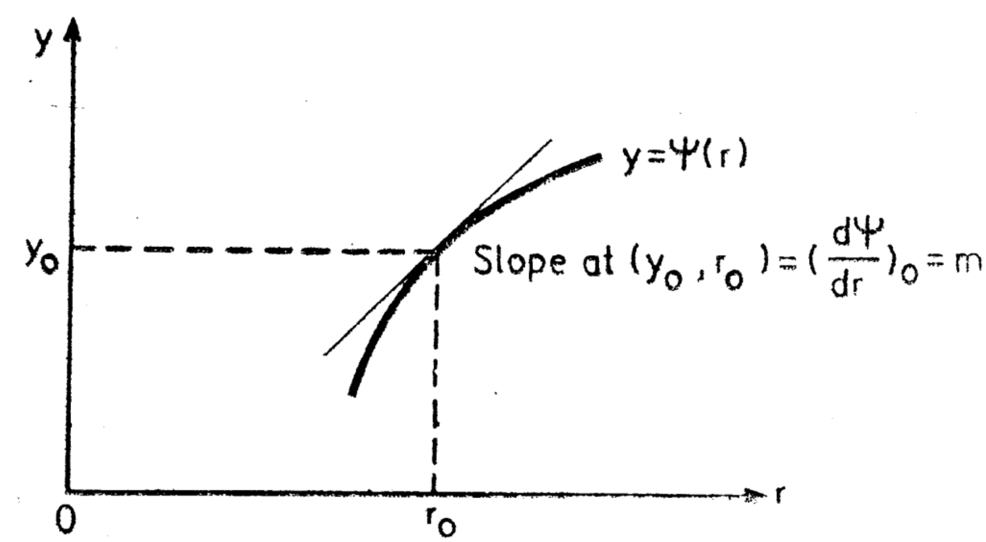


Fig. 2.23 Linearization of a nonlinear system.

Linearization of static system

Consider now jth output variable y_j of a multiple input and multiple output system described by static functional relation

$$y_j = \psi_j(r_1, r_2 \ldots r_m)$$

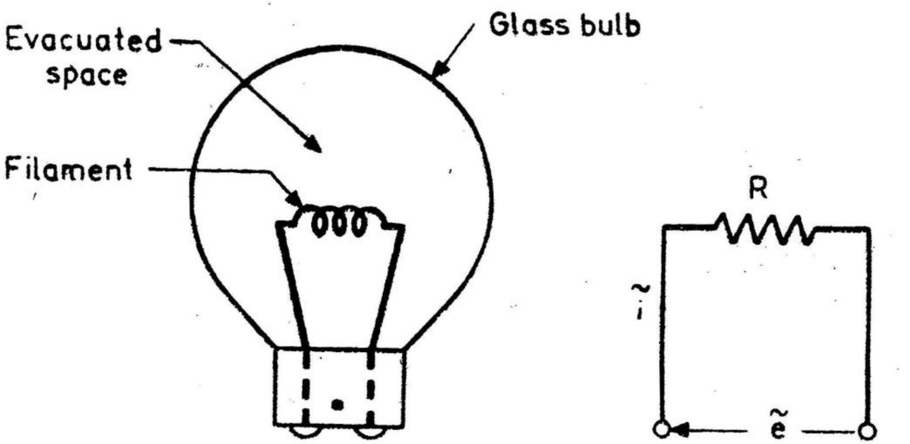
$$y_j - y_0 \cong \left(\frac{d\psi_j}{dr_1}\right)_0 \frac{(r - r_{10})}{1!} + \left(\frac{d\psi_j}{dr_2}\right)_0 \frac{(r - r_{20})}{1!} +$$

$$\ldots \left(\frac{d\psi_j}{dr_m}\right)_0 \frac{(r - r_{m0})}{1!}$$

Linearization of static system

Consider ordinary electric bulb

$i = \psi(e) = ke^2;\ k = \text{const}$



(a) Light bulb (b) Linear-circuit model

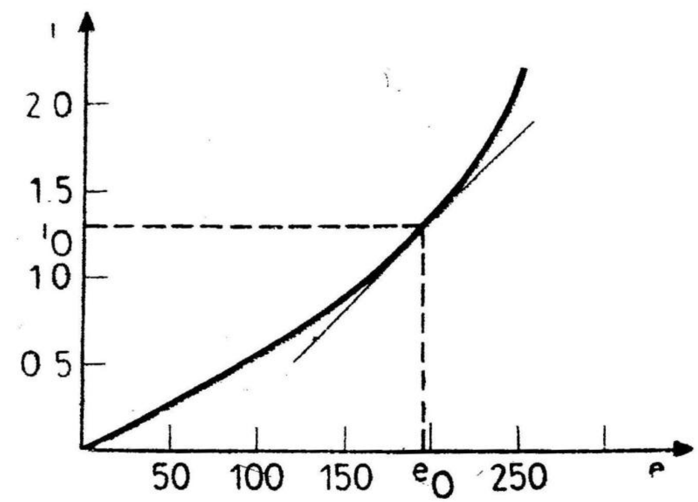


Fig. 2.25 Linearization of nonlinear resistance.

Linearization of static system

If the range of operation is 150 v to 250 V

$$i = \psi(e_0) + \left(\frac{d\psi}{de}\right)_0 (e - e_0) = i_o + \left(\frac{d\psi}{de}\right)_0 (e - e_0)$$

$$i - i_o = 2ke_0(e - e_0) \qquad \tilde{i} = \tilde{e} / R$$

In above equation $\tilde{i}$ and $\tilde{e}$ are excursion of current and voltage about the operating point and *R* is linearised resistance

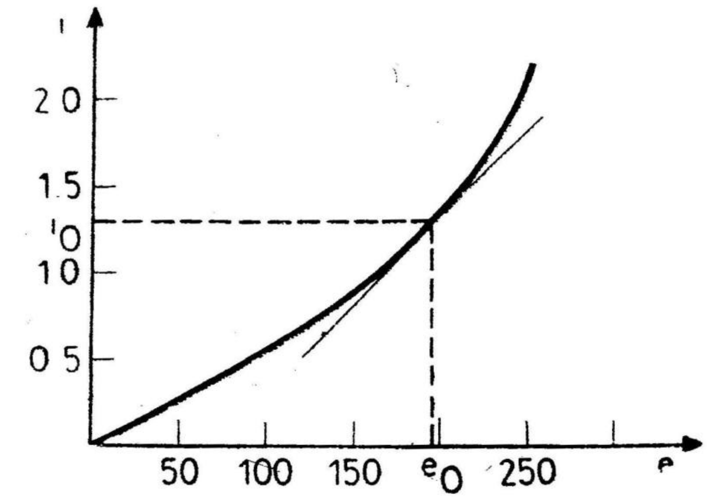


Fig. 2.25 Linearization of nonlinear resistance.

Linearization of a dynamic system

Consider a nonlinear state equation of a continuous dynamic non-linear time-invariant system

$$\dot{x} = f(x(t), r(t))$$

Approximate linear time-invariant system can be represented as

$$\tilde{x} = A\tilde{x}(t) + B\tilde{r}(t)$$

To linearise, first step is to find equilibrium state

$$\frac{dx(t)}{dt} = f(x(t), r_0) = 0$$

Linearization of a dynamic system

For a general n state variable and m inputs, state equation can be written as

$$\dot{x}_1 = f_1(x_1, x_2,..x_n, r_1, r_2,...r_m)$$

$$----$$

$$\dot{x}_n = f_n(x_1, x_2,..x_n, r_1, r_2,...r_m)$$

Linearization of a dynamic system

Taylor series expansion for the state equation about the equilibrium state and retaining first order terms

$$x - x_0 = \tilde{x} \text{ and } \mathrm{r} - \mathrm{r}_0 = \tilde{r}$$

$$\dot{x}_1 = \dot{\tilde{x}}_1 = \left(\frac{\partial f_1}{\partial x_1}\right)_0 \tilde{x}_1 + ..\left(\frac{\partial f_1}{\partial x_n}\right)_0 \tilde{x}_n + \left(\frac{\partial f_1}{\partial r_1}\right)_0 \tilde{r}_1 + ..\left(\frac{\partial f_1}{\partial r_m}\right)_0 \tilde{r}_m$$

$$----$$

$$\dot{x}_n = \dot{\tilde{x}}_n = \left(\frac{\partial f_n}{\partial x_1}\right)_0 \tilde{x}_1 + ..\left(\frac{\partial f_n}{\partial x_n}\right)_0 \tilde{x}_n + \left(\frac{\partial f_n}{\partial r_1}\right)_0 \tilde{r}_1 + ..\left(\frac{\partial f_n}{\partial r_m}\right)_0 \tilde{r}_m$$

$$\tilde{x} = A\tilde{x}(t) + B\tilde{r}(t)$$

Linearization of a dynamic system

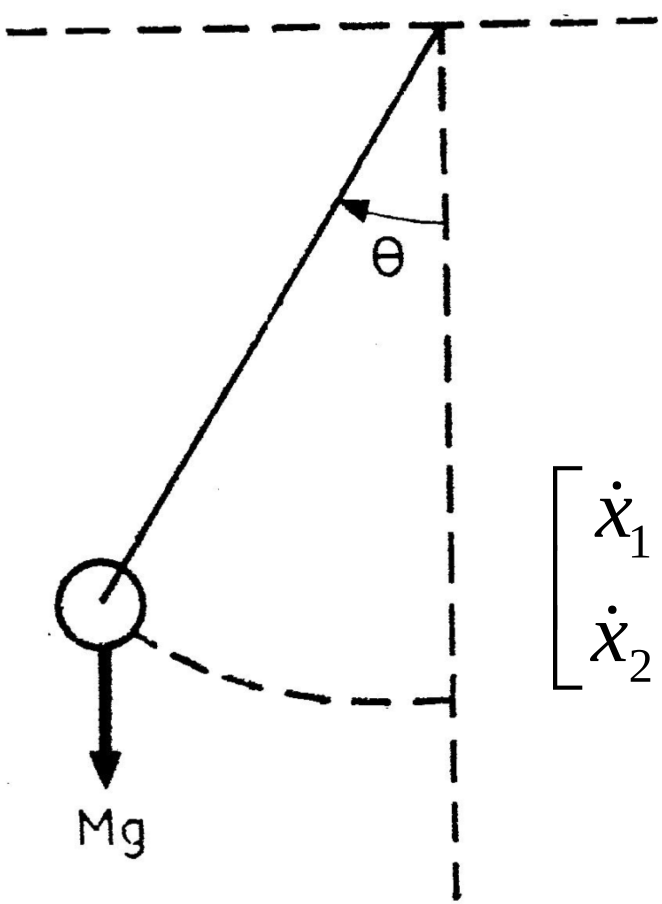
Taylor series expansion for the state equation about the equilibrium state and retaining first order terms

$$\tilde{x} = A\tilde{x}(t) + B\tilde{r}(t)$$

$$A = \begin{bmatrix} \left(\dfrac{\partial f_1}{\partial x_1}\right) & \cdots & \left(\dfrac{\partial f_1}{\partial x_n}\right) \\ . & & . \\ \left(\dfrac{\partial f_n}{\partial x_1}\right) & \cdots & \left(\dfrac{\partial f_n}{\partial x_n}\right) \end{bmatrix}_0 \qquad B = \begin{bmatrix} \left(\dfrac{\partial f_1}{\partial r_1}\right) & \cdots & \left(\dfrac{\partial f_1}{\partial r_m}\right) \\ . & & . \\ \left(\dfrac{\partial f_n}{\partial r_1}\right) & \cdots & \left(\dfrac{\partial f_n}{\partial r_m}\right) \end{bmatrix}_0$$

Linearization of a dynamic system

Simple pendulum example



$$\ddot{\theta} + \frac{g}{l}\sin(\theta) = 0$$

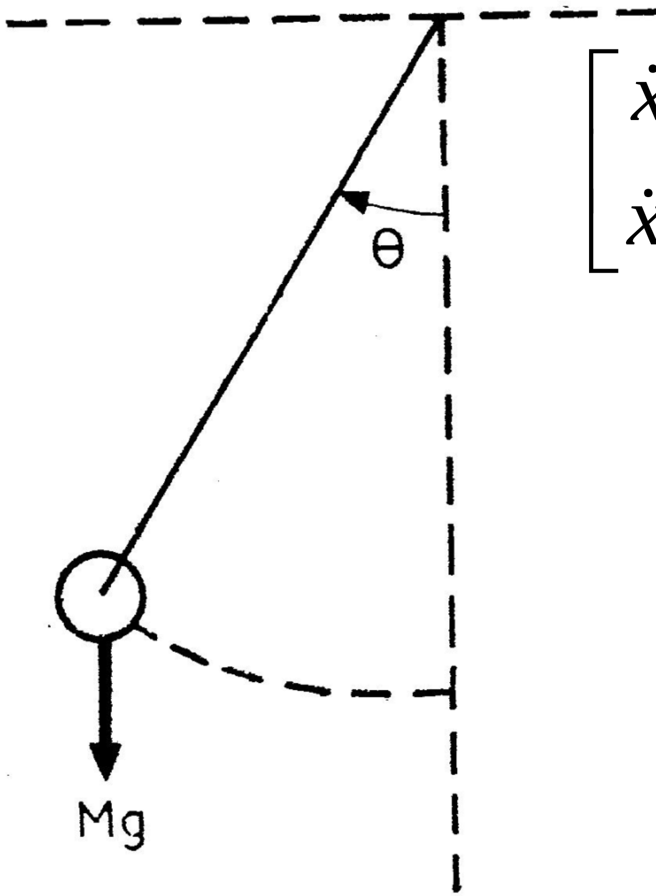
$$x_1 = \theta$$

$$x_2 = \dot{\theta}$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} f_1(x_1, x_2) \\ f_2(x_1, x_2) \end{bmatrix} = \begin{bmatrix} x_2 \\ -g\sin(x_1)/l \end{bmatrix}$$

Linearization of a dynamic system

Simple pendulum example



$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} f_1(x_1, x_2) \\ f_2(x_1, x_2) \end{bmatrix} = \begin{bmatrix} x_2 \\ -g\sin(x_1)/l \end{bmatrix}$$

Equilibrium state

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}_0 = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \& \begin{bmatrix} \pi \\ 0 \end{bmatrix}$$

Linearization of a dynamic system

Simple pendulum example

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} f_1(x_1, x_2) \\ f_2(x_1, x_2) \end{bmatrix} = \begin{bmatrix} x_2 \\ -g\sin(x_1)/l \end{bmatrix}$$

$$x_1 = \theta$$

$$x_2 = \dot{\theta}$$

$$A = \begin{bmatrix} \left(\dfrac{\partial f_1}{\partial x_1}\right) & .... & \left(\dfrac{\partial f_1}{\partial x_n}\right) \\ . & & . \\ \left(\dfrac{\partial f_n}{\partial x_1}\right) & .... & \left(\dfrac{\partial f_n}{\partial x_n}\right) \end{bmatrix}_0$$

$$A = \begin{bmatrix} 0 & 1 \\ -g/l & 0 \end{bmatrix}_0 \qquad \tilde{r}(t) = 0$$

$$\tilde{x} = A\tilde{x}(t) + B\tilde{r}(t)$$