Lecture 20

The solution of Non-linear equations

One of the most frequently occuring problems in scientific work is to find the roots of equations of the form

$$f(x) = 0$$

In general we can only hope for "approximate solutions" i.e a pt x^* for which $|f(x^*)|$ is "small".

Some iterative methods of root finding

1) Bisection method

Recall intermediate value theorem

If f is cts and $f(a)f(b) < 0$ then f has a zero in (a,b).

Suppose $f(a)f(b) < 0$

set $a_0 = a$ $\quad b_0 = b$

$$m = \frac{a_0 + b_0}{2}$$

* If $f(a_0)f(m) < 0$ then set

$b_1 = m$ $\quad a_1 = a_0$

Otherwise set $a_1 = m$ $\quad b_1 = b$

so root lies in $[a_1, b_1)$

Example $\quad f(x) = x^2 - 2$

$a_0 = 1 \qquad f(a_0) = -1$

$b_0 = 2 \qquad f(b_0) = 2$

$$m = \frac{1+2}{2} = 1.5$$

$$f(1.5) = 2.25 - 2 = 0.25$$

So $a_1 = 1 \qquad b_1 = 1.5$

root lies in $[1, 1.5]$

Algorithm for Bisection method

Given a function continuous in $[a_0, b_0]$ and such that $f(a_0) f(b_0) < 0$

For $n = 0, 1, 2, \ldots$ until satisfied do

Set $m = \dfrac{a_n + b_n}{2}$

If $f(a_n) f(m) < 0$ then set $a_{n+1} = a_n$
$b_{n+1} = m$

Otherwise set $a_{n+1} = m, \; b_{n+1} = b_n$

Then $f(x)$ has a root in the interval

$[a_{n+1}, b_{n+1}]$

Example

$f(x) = x^2 - 2$

(4 sig digits)

n	a_n	b_n	$f(a_n)$	$f(b_n)$
0	1	2	-1	2
1	1	1.5	-1	2.5 E-1
2	1.25	1.5	-4.375 E-1	2.5 E-1
3	1.375	1.5	-1.094 E-1	2.5 E-1
4	1.375	1.438	-1.094 E-1	6.641 E-1
5	1.407	1.438	-2.035 E-2	6.641 E-1
6	1.407	1.423	-2.035 E-2	2.493 E-2
7	1.407	1.415	-2.035 E-2	2.225 E-3
8	1.411	1.415	-9.079 E-3	2.225 E-3
	1.413	1.415	-3.431 E-3	2.225 E-3
10	1.414	1.415	-6.040 E-4	2.225 E-3

$$\frac{1.414 + 1.415}{2} = 1.415 \quad \text{in 4 sig digits}$$

∴ So algorithm ends.

root lies in [1.414, 1.415]

- Bisection method always converges to the root
- Convergence is slow

One can hope to get to root faster by using fully the information about $f(x)$ available at each step.

In our example

$$f(x) = x^2 - 2$$

$$f(1) = -1 \quad \text{and} \quad f(2) = 2$$

Since $|f(1)|$ is closer to zero than $|f(2)|$ the root ξ is likely to be closer to 1 than 2.

Hence rather than check the midpt or avg value of 1 and 2, we check f(x) at the weighted average

$$w = \frac{|f(2)|\,1 + |f(1)|\,2}{|f(2)| + |f(1)|}$$

Since f(1), f(2) have opposite sign

$$w = \frac{f(2)\cdot 1 - f(1)\cdot 2}{f(2) - f(1)}$$

In our example

$$w = \frac{2 + 2}{3} = 1.333$$

$f(w) < 0$

So the root lies in $[1.333, 2]$

repeating the process we get

$w = 1.400$ and so-on

This algorithim is known as regula-falsi or false-position method.

Algorithim (Regula-falsi)

Given a function $f(x)$ continuous on the interval $[a_0, b_0]$ and such that $f(a_0)f(b_0) < 0$

For $n = 0, 1, 2, \ldots$, until satisfied do

Calculate $$w = \frac{f(b_n)a_n - f(a_n)b_n}{f(b_n) - f(a_n)}$$

If $f(a_n)f(w) \leq 0$, set $a_{n+1} = a_n$, $b_{n+1} = w$

Otherwise set $a_{n+1} = w$, $b_{n+1} = b_n$

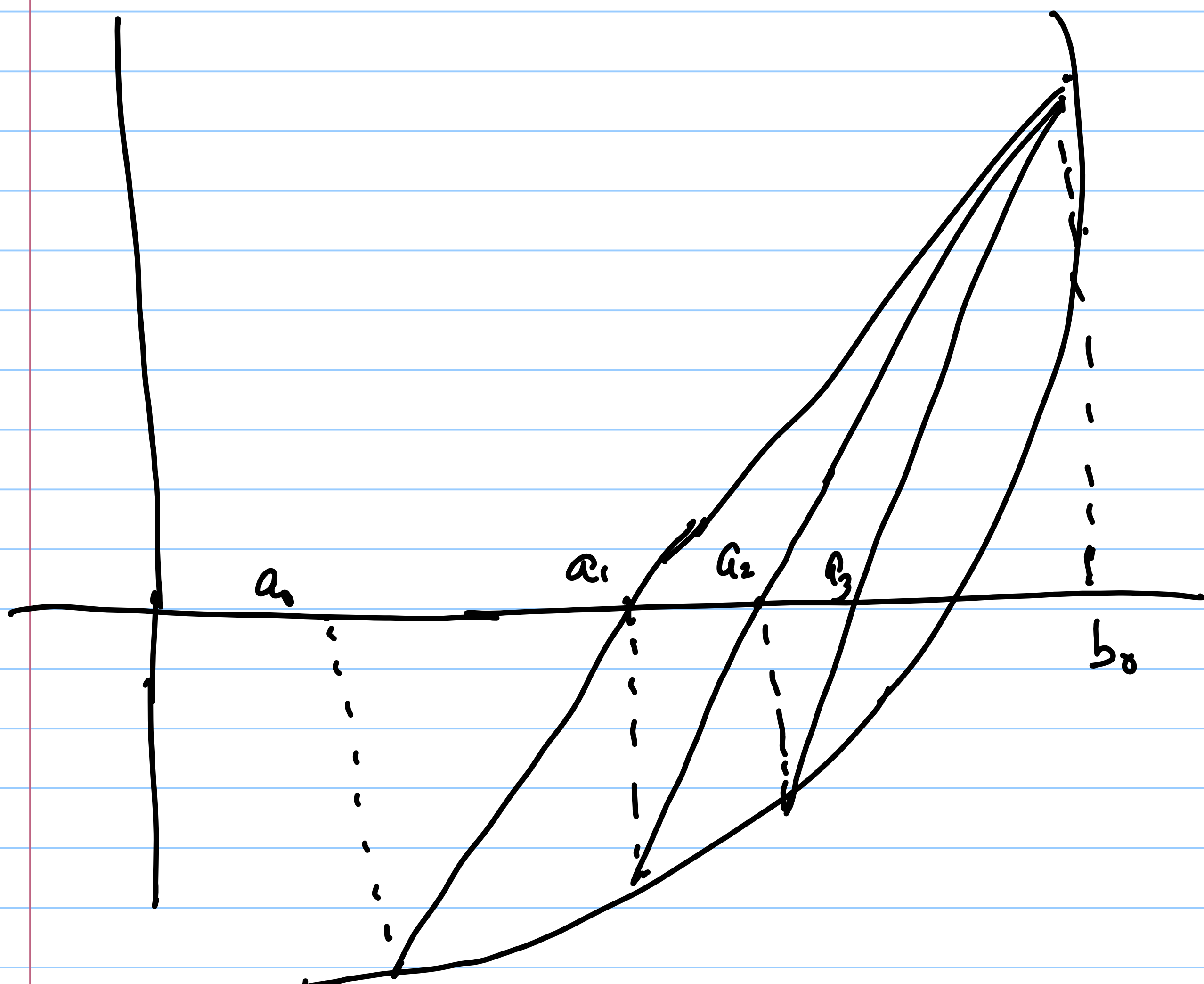
Example

$$f(x) = x^2 - 2$$

4 sig digits

n	a_n	b_n	$f(a_n)$	$f(b_n)$	w_n
0	1	2	-1	2	1.333
1	1.333	2	$-2.222E{-1}$	2	1.400
2	1.400	2	$-4.086E{-2}$	2	1.412
3	1.412	2	$-6.256E{-2}$	2	1.414
4	1.414	2	— —	— — —	

Regula falsi method produces a point at which $|f(x)|$ is "small" somewhat faster than the bisection method, it fails completely to give a "small interval" where the root is known to lie.



$$w = \frac{f(b_n)\,a_n - f(a_n)\,b_n}{f(b_n) - f(a_n)}$$

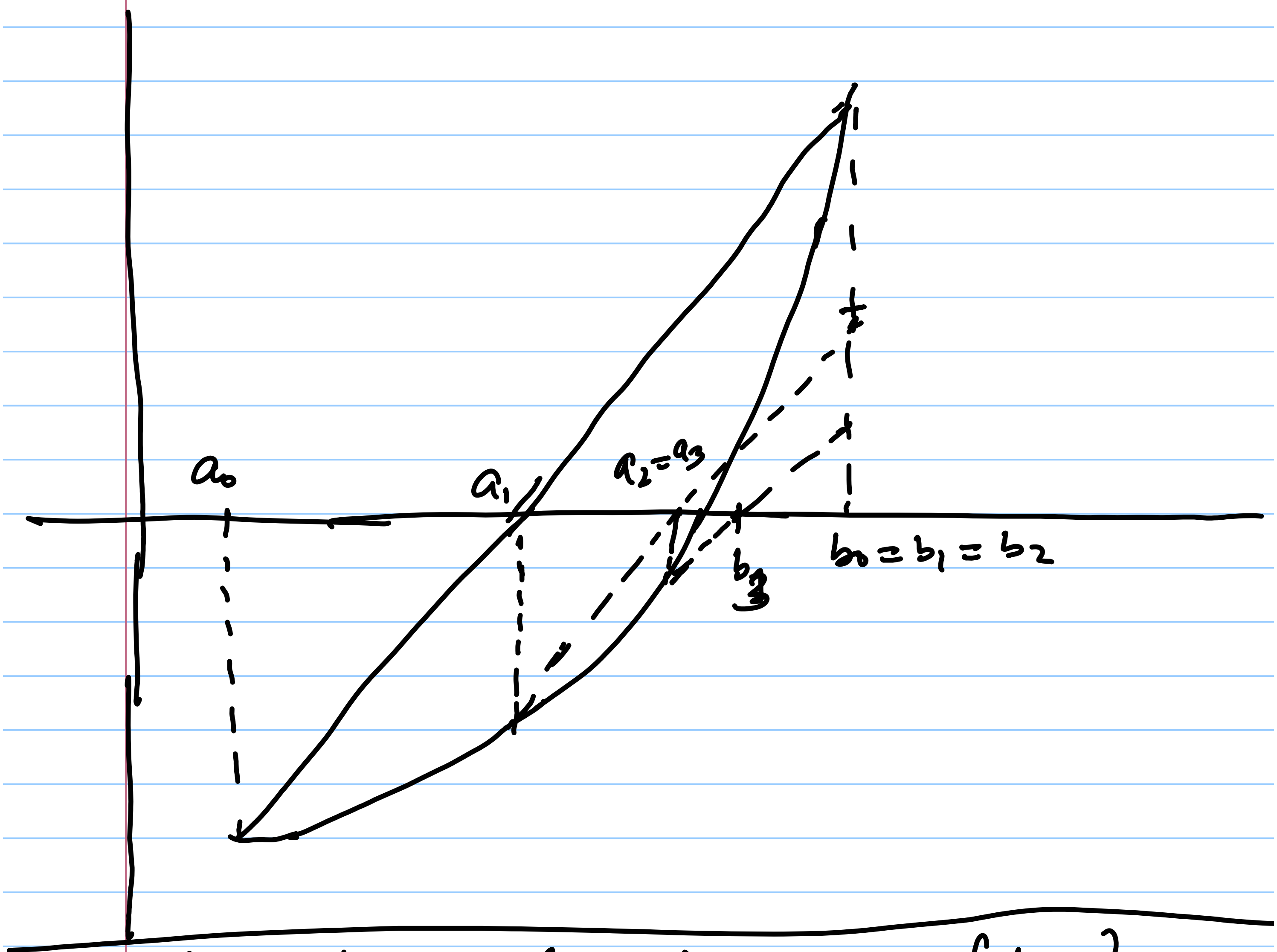
is the pt at which the st-line passing through $(a_n, f(a_n))$, $(b_n, f(b_n))$ intersects the x-axis

Such a st-line is secant to $f(x)$ and in our example $f(x)$ is concave upward and increasing (in the interval $[1,2]$ of interest), hence the secant is always above the graph of $f(x)$. Consequently w always lies to the left of the zero (in our example).

If $f(x)$ were concave downward and increasing w would always lie to the right of the zero.

Two ways of improving regula-falsi method.

The first one, called "modified regula-falsi" replaces secants by st-lines of ever-smaller slope until w falls to the opp side of the root.



Algorithms (Modified regula falsi)

Given $f(x)$ continuous on $[a_0, b_0]$ and such that $f(a_0) f(b_0) < 0$

Set $F = f(a_0)$ $G = f(b_0)$ $w_0 = a_0$

For $n = 0, 1, 2, \ldots$ until satisfied do

Calculate $w_{n+1} = \dfrac{G a_n - F b_n}{G - F}$

If $f(a_n) f(w_{n+1}) \leq 0$ set $a_{n+1} = a_n$

$b_{n+1} = w_{n+1}$

$G = f(w_{n+1})$

If also $f(w_n) f(w_{n+1}) > 0$ set $F = F/2$.

Otherwise set $a_{n+1} = w_{n+1}$, $F = f(w_{n+1})$

$b_{n+1} = b_n$

If also $f(w_n) f(w_{n+1}) > 0$ set $G = G/2$

Then $f(x)$ has a zero in the interval

$[a_{n+1}, b_{n+1}]$

Example

$$f(x) = x^2 - 2$$

$a_0 = 1$ $b_0 = 2$ $w_0 = 1$

n	a_n	b_n	F	G	w_n
0	1	2	−1	2	1
1	1.333	2	−2.222G	1	1.333
2.	1.333	1.454	−2.222 E−1	1.141 E−1	1.454
3.	1.413	1.454	−3.431 E−3	1.141 E−1	1.413
4.	1.414	1.454	−6.040 E−4	5.705 E−2	1.414
5.	1.414	1.454	−6.040 E−4	2.853 E−2	1.414
6	1.414	1.415	−6.040 E−4	1.742 E−2	1.415

root lies in [1.414, 1.415]

A second, very popular modification of the regula-falsi, called the secant method, retains the use of secants throughout, but may give up the bracketing of the root

Algorithim (secant method)

Given a function $f(x)$ and two points x_{-1} and x_0

For $n = 0, 1, 2, \ldots$ until satisfied do

Calculate

$$x_{n+1} = \frac{f(x_n)\, x_{n+1} - f(x_{n-1})\, x_n}{f(x_n) - f(x_{n-1})}$$

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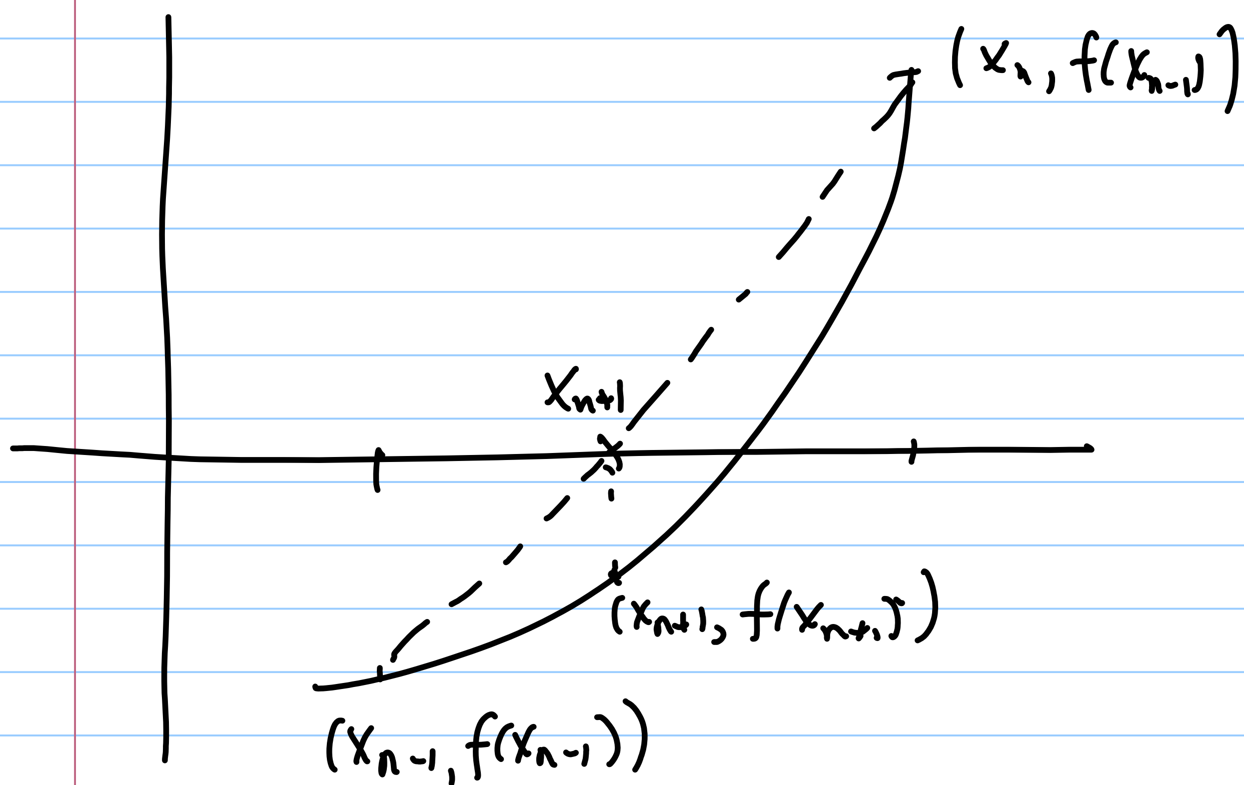

$(x_n, f(x_{n-1}))$
x_{n+1}
$(x_{n+1}, f(x_{n+1}))$
$(x_{n-1}, f(x_{n-1}))$

.

Example

$$f(x) = x^2 - 2$$

$$x_{-1} = 1$$

$$x_0 = 2$$

n	x_n	$f(x_n)$
-1	1	-1
0	2	2
1	1.333	-2.222 E-1
2	1.400	-4.000 E-2
3	1.415	2.225 E-3
4	1.414	-6.040 E-4
5.	1.414	-6.040 E-4

frozen ends

" note algorithm ends if $f(x_n) = f(x_{n-1})$

This makes the calculation of x_{n+1} impossible

The expression

$$x_{n+1} = \frac{f(x_n)\,x_{n-1} - f(x_{n-1})\,x_n}{f(x_n) - f(x_{n-1})}$$

is prone to round-off error since $f(x_n)$ and $f(x_{n-1})$ need not be of opposite signs.

It is better to calculate x_{n+1} from the equivalent expression

$$x_{n+1} = x_n - f(x_n) \cdot \frac{x_n - x_{n-1}}{f(x_n) - f(x_{n-1})}$$

in which x_{n+1} is obtained from x_n by adding the correction term

$$\frac{-f(x_n)}{\dfrac{f(x_n) - f(x_{n-1})}{x_n - x_{n-1}}} = \frac{-f(x_n)}{[f(x_n), f(x_{n-1})]}$$

If x_n close to x_{n-1} then we get

$$[f(x_n), f(x_{n-1})] \approx f'(x_n)$$

Newton's method

Given $f(x)$ continuously diff and a pt x_0

For $n = 0, 1, 2, \ldots$ until satisfied do

Calculate $x_{n+1} = x_n - \dfrac{f(x_n)}{f'(x_n)}$

example

$f(x) = x^2 - 2$

$x_0 = 1$

$x_1 = 1.5$

$x_2 = 1.416666667$

$x_3 = 1.414215686$

$x_4 = 1.414213562$

$x_n = x_4$ for $n \geq 4$.

Newton's method is special example of "fixed-pt iteration"

$$g(x) = x - \frac{f(x)}{f'(x)}$$

$$x_{n+1} = g(x_n)$$

$g : I \longrightarrow I$

I closed interval containing root

If the seq $x_1, x_2, \ldots \to \ldots$, so general converges to some pt ξ and $g(x)$ is cts

then
$$\begin{aligned}\xi &= \lim_{n\to\infty} x_{n+1} \\ &= \lim_{n\to\infty} g(x_n) \\ &= g\left(\lim_{n\to\infty} x_n\right) \\ &= g(\xi)\end{aligned}$$

So ξ is a fixed pt of g.