Lecture 5

Last time we did error in interpolating polynomial.

$P_n(x)$ interpolates $f(x)$ at $x_0, x_1, \ldots, x_n$

$e_n(x) = f(x) - P_n(x)$ error

If $\bar{x}$ distinct from $x_0, x_1, \ldots, x_n$

then $e_n(\bar{x}) = f[x_0, x_1, \ldots, x_n, \bar{x}] \prod_{j=0}^{n} (x - x_j)$

Theorem

If $x_0, x_1, \ldots, x_k$ are $k+1$ distinct pts in $[a,b]$

then $f[x_0, x_1, \ldots, x_k] = \dfrac{f^{(k)}(\xi)}{k!}$ for some $\xi \in (a,b)$

Osculatary interpolation

Example

In some problems we have

$x_0, x_1, \ldots, x_n$

$f(x_0), f(x_1), \ldots, f(x_n)$

$f'(x_0), f'(x_1), \ldots, f'(x_n)$

We desire a polynomial of degree $\leq 2n+1$ such that

$$p(x_i) = f(x_i)$$

and $p'(x_i) = f'(x_i)$

Algorithim :-

define $y_0, y_1, \ldots, y_{2n}$ as

$y_0 = x_0, \; y_1 = x_0$

$y_2 = x_1, \; y_3 = x_1$

$y_4 = x_2, \; y_5 = x_2$

$\vdots$

$y_{2i} = x_i, \; y_{2i+1} = x_i$

$\vdots$

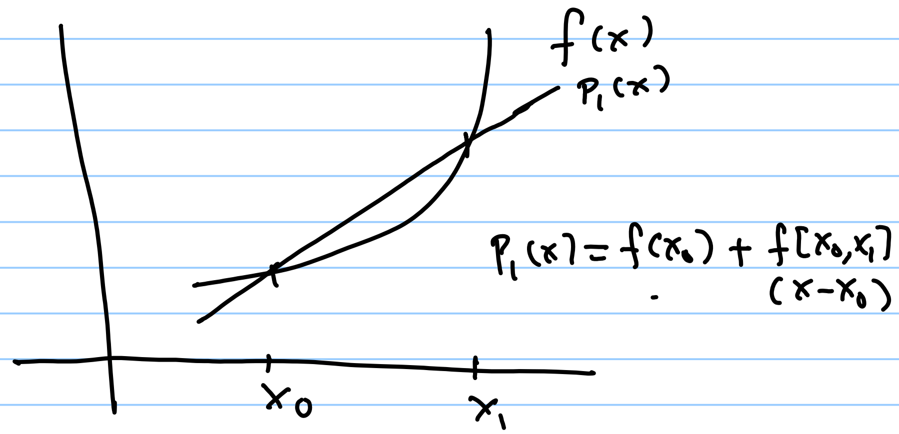
$y_{2n} = x_n, \; y_{2n+1} = x_n$

Form divided diff table using

$(*)$ $\quad f[a,a] = f'(a)$

motivation of $(*)$ $\quad f[x_0,x_1] = \dfrac{f(x_1)-f(x_0)}{x_1-x_0} \longrightarrow f'(x_0)$ as $x_1 \to x_0$

Geometric reasoning



as $x_1 \to x_0$ $\quad P_1(x) = f(x_0) + f'(x_0)(x-x_0)$

Example (repeat of last time)

Example :-

$f(1) = 0 \qquad\qquad f'(1) = 1$

$f(2) = 6.931\text{E}-1 \qquad f'(2) = 0.5$

need cubic polynomial such that

$P_3(1) = f(1), \quad P_3'(1) = f'(1)$

$P_3(2) = f(2), \quad P_3'(2) = f'(2)$

Solution

Set $y_0 = y_1 = 1$

$$y_2 = y_3 = 2$$

$$f[y_0, y_0] = f'(y_0) = 1$$

$$f[y_1, y_2] = \frac{f(y_2) - f(1)}{y_2 - y_1} = 0.6931$$

$$f[y_2, y_3] = f'(y_2) = 0.5$$

$$f[y_0, y_1, y_2] = \frac{f[y_1, y_2] - f[y_0, y_1]}{y_2 - y_0} = -0.3069$$

$$f[y_1, y_2, y_3] = \frac{f[y_2, y_3] - f[y_1, y_2]}{y_3 - y_1} = -0.1931$$

$$f[y_0, y_1, y_2, y_3] = \frac{f[y_1, y_2, y_3] - f[y_0, y_1, y_2]}{y_3 - y_0} = 0.1137$$

$$P_3(x) = 0 + 1\,(x-1) + (-0.3069)\,(x-1)^2 + 0.1137\,(x-1)^2(x-2)$$

Example :- Suppose we are given the following problem

x_0 $\quad x_1$

$f(x_0)$ $\quad f(x_1)$

$f'(x_0)$

$f''(x_0)$

need $P_3(x)$

s/t $P_3(x_0) = f(x_0)$, $P_3(x_1) = f(x_1)$

$P_3'(x_0) = f'(x_0)$

$P_3''(x_0) = f''(x_0)$

Algorithm

$y_0 = x_0 \quad y_1 = x_0 \quad y_2 = x_0 \quad y_3 = x_1$

Calculate divided differences

using $f[a,a] = f'(a)$

$$f[a,a,a] = \frac{f''(a)}{2!}$$

$\cdot$

$\vdots$

Example 2

$f(0) = 1 \qquad f'(0) = 0 \qquad f''(0) = 1$

$f(0.1) = 9.95\text{ E}{-1}$

approximate $f(0.05)$

Solution

$y_0 = 0 \qquad\qquad y_3 = 0.1$

$y_1 = 0$

$y_2 = 0$

$f[y_0, y_1] = f'(0) = 0$

$f[y_1, y_2] = f'(0) = 0$

$$f[y_2, y_3] = \frac{f(y_3) - f(y_2)}{y_3 - y_2} = -5\text{ E}{-2}$$

$$f[y_0, y_1, y_2] = \frac{f[y_1, y_2] - f[y_0, y_1]}{y_2 - y_0}$$

$$= f''(y_0)/2 \qquad \text{if } \begin{array}{l} y_1 \to y_0 \\ y_2 \to y_0 \end{array}$$

$$= 0.5$$

$$f[y_1, y_2, y_3] = \frac{f[y_2, y_3] - f[y_1, y_2]}{y_3 - y_1}$$

$$= \frac{-5\,E{-2} \; - 0}{.1}$$

$$= -5\,E{-1}$$

$$f[y_0, y_1, y_2, y_3] = \frac{f[y_1, y_2, y_3] - f[y_0, y_1, y_2]}{y_3 - y_0}$$

$$= \frac{-5\,E{-1} - 5\,E{-1}}{0.1}$$

$$= -10$$

$$p(x) = 1 + 0\,(x-0) + 1(x-0)^2 + 15\,(x-0)^3$$

$$= 1 + x^2 - 10x^3$$

$p(0.05) = 1.00$[illegible]

Theory of Osculatory interpolation

Convention

$x_0, x_1, \ldots, x_m$ not necc distinct pts

We say two functions $f(x)$ and $g(x)$ agree at the pts $x_0, \ldots, x_m$ if

$$f^{(j)}(z) = g^{(j)}(z) \quad \text{for } j = 0, 1, \ldots, k-1$$

for every pt $\underline{\underline{z}}$ which occurs k times in the sequence $x_0, \ldots, x_m$

Example

$x = 1, 2, 2, 1, 3, 2$

$f(x), g(x)$ agree at $1, 2, 2, 1, 3, 2$ if

$f(1) = g(1)$
$f'(1) = g'(1)$

$f(2) = g(2)$
$f'(2) = g'(2)$
$f''(2) = g''(2)$

$f(3) = g(3)$

Problem Given $x_0, x_1, \dots, x_m$ not necc distinct pts
and $f : [a, b] \to \mathbb{R}$.

We need a polynomial $p(x)$ of degree $\le m$ such that
$p(x)$ & $f(x)$ agree at $x_0, x_1, \dots, x_m$

Remark

Two polynomials of degree $\le m$ which agree at $x_0, x_1, -, x_m$ are equal.

So it makes sense to talk about the polynomial of degree $\le m$ which agrees with $f(x)$ at $m+1$ pts. $x_0, \ldots, x_m$.

Theorem If $f(x)$ has r continuous derivatives and no point in the sequence $x_0, x_1, \ldots, x_m$ occurs more than r times, then there exists exactly one polynomial $P_m(x)$ of degree $\le m$ which agrees with $f(x)$ at $x_0, x_1, \ldots x_m$.

Proof

Uniquen already taken care of.

Proof of Existence

Assume $x_0 \le x_1 \le x_2 \le \cdots \le x_n$

for $m=0$ nothing to show.

Assume the statement correct for

$n=k-1$ and consider it for $n=k$

There are two cases.

Case 1 $\quad x_0 = x_k$

Then $x_0 = x_1 = \cdots = x_k$. So $r \geq m$

by assumption, i.e., f has at least k

continuous derivatives

Then the Taylor polynomial $P_k(x)$ for $f(x)$

around the center $c = x_0$ does the job

Note that its leading coefficient is the number $\dfrac{f^{(k)}(x_0)}{k!}$.

<u>Case</u> $x_0 < x_k$.

Then by induction hypothesis we can find polynomial $p_{k-1}(x)$ of degree $\leq k-1$ which agrees with $f(x)$ at $x_0, x_1, \ldots, x_{k-1}$ and polynomial $q_{k-1}(x)$, of degree $\leq k-1$ which agree with $f(x)$ at $x_1, x_2, \ldots, x_k$.

Verify
$$\boxed{p_k(x) = \frac{x - x_0}{x_k - x_0}\, q_{k-1}(x) + \frac{x_k - x}{x_k - x_0}\, p_{k-1}(x)}$$

(slightly tricky
see textbook pg 64)

Convention

$x_0, x_1, \ldots, x_n$ not necc distinct pts

$P_n(x)$ unique polynomial which agrees with $f(x)$ at $x_0, x_1, \ldots, x_n$

$f[x_0, x_1, \ldots, x_n] =$ leading coefficient of $P_n(x)$.

$=$ coeff of x^n in $P_n(x)$

We also have

$$P_n(x) = P_{n-1}(x) + f[x_0, \ldots, x_n] \prod_{i=0}^{n-1} (x - x_i)$$

Proof $P_n(x) - f[x_0, \ldots, x_n] \prod_{i=0}^{n-1} (x - x_i)$ has degree $\leq n-1$ and agrees with $f(x)$ at $x_0, x_1, \ldots, x_{n-1}$. So by uniquen of interpolating poly result follows

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Thus we can write $P_n(x)$ as

$$P_n(x) = f[x_0] + f[x_0, x_1](x - x_0) + f[x_0, x_1, x_2](x - x_0)(x - x_1) + \cdots + f[x_0, \ldots, x_n](x - x_0)(x - x_1) \cdots (x - x_{n-1})$$

Case 1 $x_0 = x_1 = \cdots = x_n$

$P_n(x) =$ Taylor polynomial with center x_0

$$= f(x_0) + \frac{f'(x_0)}{1}(x - x_0) + \frac{f''(x_0)}{2!}(x - x_0)^2 + \cdots + \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n$$

$$f[\underbrace{x_0, x_0, \ldots, x_0}_{n+1 \text{ times}}] = \frac{f^{(n)}(x_0)}{n!}$$

Otherwise say $x_n \neq x_0$

$$f[x_0, x_1, \ldots, x_n] = \frac{f[x_1, \ldots x_n] - f[x_0, \ldots, x_{n-1}]}{x_n - x_0}$$

<u>Theorem</u> :- $f[x_0, x_1, \ldots, x_n]$ is a cts function of $x_0, x_1, \ldots, x_n$.

i.e if for each r $x_0^{(r)}, \ldots, x_n^{(r)}$ are $n+1$ pts in $[a,b]$ and $\lim_{r \to \infty} x_i^{(r)} = y_i$ for $i = 0, \ldots, n$

then

$$\lim_{r \to \infty} f[x_0^{(r)}, \ldots, x_n^{(r)}] = f[y_0, \ldots, y_n]$$

<u>Pf</u> See text pg 65.

Problem

$f(0) = 1$

$f(1) = 1.9$

$f'(0) = 2$

$f'(1) = 2.5$

approximate $f(0.4)$

Sol^n $y_0 = 0,\ y_1 = 0,\ y_2 = 1,\ y_3 = 1$

y	$f(y)$	$f[\ ,\ ]$	$f[\ ,\ ,\ ]$	$f[\ ,\ ,\ ,\ ]$
0	1	2	-1.1	$f[0,0,1,1] = \frac{f[0,1,1] - f[0,0,1]}{1-0} = 2.7$
0	1	0.9	$f[0,1,1] = 1.6$	
1	1.9	2.5		
1	1.9			

$$P_3(x) = 1 + 2(x-0) + (-1.1)(x-0)^2 + 2.7\,(x-0)^2(x-1)$$

Find $P_3(0.4)$