Lecture 21

Last time we discussed some methods to find root of non-linear equation

$$f(x) = 0$$

We discussed primarily the following methods:-

1) <u>Bisection method</u>

If $f(a_n) f(b_n) < 0$ (so f has a root in $[a_n, b_n]$)

$$m = \frac{a_n + b_n}{2}$$

If $f(a_n) f(m) < 0$ set $a_{n+1} = a_n$ and $b_{n+1} = m$

Otherwise set $a_{n+1} = m$, $b_{n+1} = b$

So root is in $[a_{n+1}, b_{n+1}]$

Convergence is slow in bisection method

Regula-falsi method

$f(a_n) f(b_n) < 0$

$$w = \frac{f(b_n)a_n - f(a_n) b_n}{f(b_n) - f(a_n)}$$

If $f(a) f(w) < 0$ then set $a_{n+1} = a_n$, $b_{n+1} = w$

otherwise set $a_{n+1} = w$, $b_{n+1} = b_n$

Regula-falsi method produces a pt x^* for which $|f(x^*)|$ is small but many times it fails completely to give a "small interval" in which a zero is known to lie

Two improvements of Regula-falsi method

1) modified regula-falsi method

2) Secant method.

Secant-method

Given a function $f(x)$ and two pts x_{-1}, x_0

for $n = 0, 1, 2, \ldots$ until satisfied do

calculate $$x_{n+1} = \frac{f(x_n)x_{n-1} - f(x_{n-1})x_n}{f(x_n) - f(x_{n-1})}$$

- $f(x_n)$ and $f(x_{n-1})$ need not be of opp sign so above expression is prone to round-off error.

Better to compute

$$x_{n+1} = x_n - f(x_n) \cdot \frac{x_n - x_{n-1}}{f(x_n) - f(x_{n-1})}$$

i.e

$$x_{n+1} = x_n - \frac{f(x_n)}{[f(x_n), f(x_{n-1})]}$$

Newton's method

for $n = 0, 1, 2, \ldots$ until satisfied do

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

Newton's method gives fast convergence to the root.

However x_0 has to be "close" to the root for convergence

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fixed pt iteration

$$g(x) = x - \frac{f(x)}{f'(x)}$$

In Newton's-method we are essentially trying to find fixed pt of $g(x)$
i.e a pt ξ s/t

$$g(\xi) = \xi.$$

Fixed-pt iteration

Our goal is to find root of

$$f(x) = 0 \qquad (1)$$

One derives from (1) an equation of the form

$$x = g(x) \qquad (2)$$

So any solution of (2) (i.e. a fixed pt of $g(x)$) is a solution for (1)

Usually there are many choices of $g(x)$

Example

$$f(x) = x^3 - x - 1$$

choices of $g(x)$ are

(1) $g(x) = x^3 - 1$

(2) $g(x) = \sqrt[3]{1+x}$

Algorithm for fixed pt iteration

Given an iteration function $g(x)$ and a starting pt x_0

For $n = 0, 1, 2, \ldots$ until satisfied do

Calculate $x_{n+1} = g(x_n)$

We have not yet discussed conditions for convergence

example $\quad g(x) = x^3 - 1$

$X_0 = 1$

$$x_1 = 0$$
$$x_2 = -1$$
$$x_3 = -2$$
$$x_4 = -9$$
$$x_5 = -730$$
$$x_6 = -3.89 \times 10^8$$

this seq diverges

On the other hand

$$g(x) = (1+x)^{1/3}$$

$$x_0 = 0$$

$$x_1 = 1$$
$$x_2 = 1.25992$$
$$x_3 = 1.31229$$

$$x_4 = 1.32235$$
$$x_5 = 1.32427$$
$$x_6 = 1.32463$$

$$x_7 = 1.32470$$
$$x_8 = 1.32471$$
$$x_9 = 1.32472$$

$$x_{10} = 1.32472$$
$$x_j = x_{10} \quad \text{for } j \geq 10$$

$$f(x) = x^3 - x - 1$$
$$f(x_{10}) = -9.001 \quad E-8$$

Conditions for convergence of fixed pt iteration

Condition 1 There is an interval $I = [a,b]$ such that for all $x \in I$, $g(x)$ is defined and $g(x) \in I$, i.e the function $g(x)$ maps I into itself.

Condition 2 The iteration function $g(x)$ is continuous on $I = [a, b]$

Theorem Let $g : I \to I$ be cts, $I = [a,b]$
Then g has a fixed pt

Proof If $g(a) = a$ or $g(b) = b$ then g obviously has a fixed pt.

Otherwise

$g(a) \neq a, \quad g(b) \neq b$

$g(a), g(b) \in I$. So

$g(a) > a$ and $g(b) < b$

$h(x) = g(x) - x$ is cts

$h(a) > 0$ and $h(b) < 0$

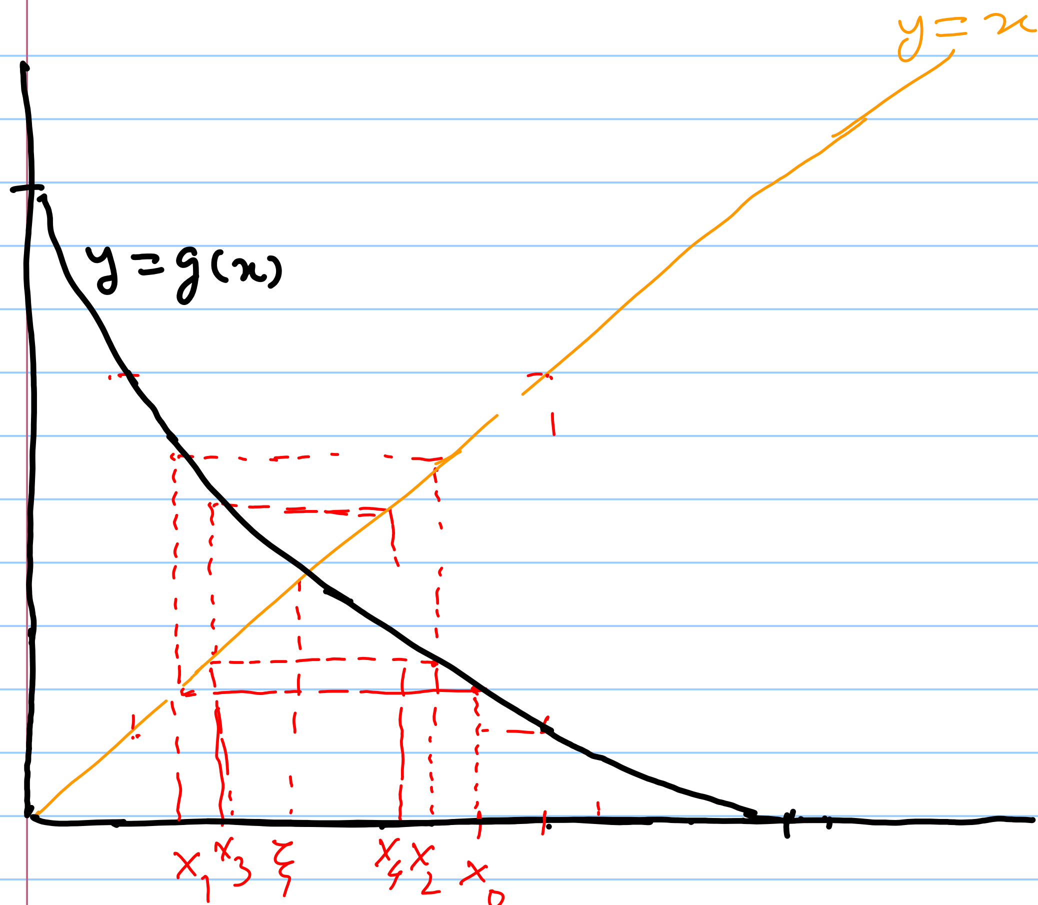
So by intermediate value theorem

$h(x_0) = 0$ for some x_0.

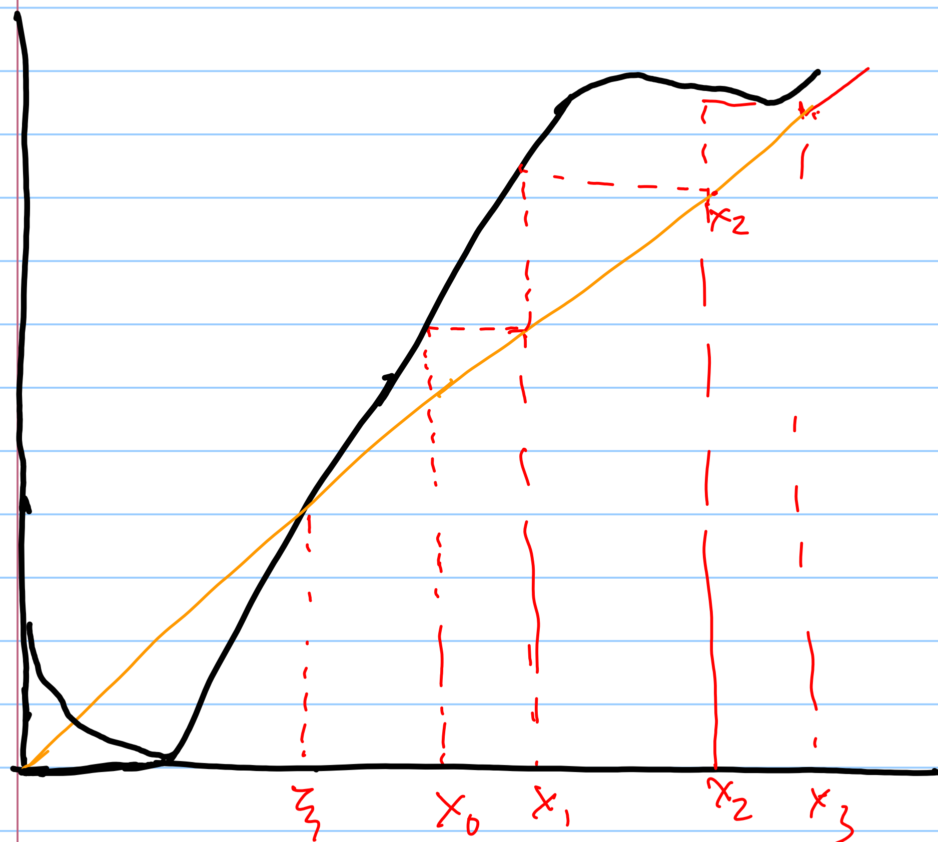
So $g(x_0) = x_0$.

Still we have not found condition guaranteeing convergence.

It is better to look at iteration graphically



Here $x_i \longrightarrow \xi \leftarrow$ fixed pt



$x_i \not\to \xi$

slope of $g(x)$ is too large in absolute value, near ξ.

<u>Condition 3</u> The iteration function $g(x)$ is differentiable on $I = [a, b]$. Further there exist a non-negative constant $K < 1$ such that

for all $x \in I$ $\quad |g'(x)| \leq K$

<u>Theorem</u> Let $g(x)$ be a function satisfying condition 1, 2, 3. Then g has a unique fixed pt ξ on $I = [a, b]$. Starting with any pt $x_0 \in I$, the seq $x_1, \ldots x_n = g(x_{n-1}), \ldots$ gen by fixed pt iteration converges to ξ.

<u>Proof</u>:- By earlier result there exist a fixed pt ξ of $g(x)$ in I.

Let $x_0 \in I$ be any.

for $n \geq 1$, set $x_n = g(x_{n-1})$.

Set $e_n = \xi - x_n$ the error in the n^{th} iterate

$$\xi = g(\xi) \ \& \ x_n = g(x_{n-1})$$

$$e_n = g(\xi) - g(x_{n-1})$$

$$= g'(\eta_n)(\xi - x_{n-1})$$ by MVT

η_n between ξ and x_{n-1}

$$e_n = g'(\eta_n)\, e_{n-1}$$

$$|e_n| = |g'(\eta_{n-1})|\,|e_{n-1}| \leq K\,|e_{n-1}|$$

By induction on n we get

$$|e_n| \leq K\,|e_{n-1}| \leq K^2\,|e_{n-2}| \leq \cdots \leq K^n\,|e_0|$$

Since $0 \leq K < 1$ we have

$$\lim_{n\to\infty} K^n = 0.$$

Therefore $\lim_{n\to\infty} |e_n| \leq \lim_{n\to\infty} |K^n||e_0| = 0$

Thus $\lim_{n\to\infty} e_n = 0$

i.e. $x_n \to \xi$.

<u>Uniquen of fixed pt</u>

Say α is another fixed pt, $\alpha \neq \xi$

$\alpha = g(\alpha)$, $\xi = g(\xi)$

$$\alpha - \xi = g(\alpha) - g(\xi)$$
$$= g'(u)(\alpha - \xi)$$

u between α and ξ

$|\alpha - \xi| \leq K |\alpha - \xi|$ when $K < 1$

$\times$ contradiction

Example

find root of $f(x) = e^x - 4x^2$ between 0 and 1.

$$e^x - 4x^2 = 0$$
$$4x^2 = e^x$$
$$x = \frac{1}{2} e^{x/2}$$

$$g(x) = \frac{1}{2} e^{x/2}$$

$$g(0) = \frac{1}{2} \qquad g(1) = \frac{1}{2} e^{1/2} = 0.82$$

$$g'(x) = \frac{1}{4} e^{x/2}$$

$$|g'(x)| \leq |g(1)| = 0.41$$

Thus fixed pt iteration of g will converge on interval $I = [0,1]$

Calc in 6 sig digits

$x_0 = 0$

$x_1 = 0.5$

$x_2 = 0.642013$

$x_3 = 0.689257$

$\vdots$

$x_{13} = 0.714805$

$x_{14} = 0.714806$

$x_n = x_{14}$ for $n \geq 14$

$f(0.714806) = -3.2\text{E}-7$

Thus 0.714806 is approximately a root of $f(x)$.

It requires 14 iterations to get fixed pt of $g(x)$ above.

Usually it takes many more iterations.

Convergence acceleration for fixed-pt iteration

$g : I \to I$ $\qquad\qquad I = [a, b]$

$x_0 \in I$

$x_1, x_2, \ldots \qquad\qquad x_n = g(x_{n-1})$

$\xi \to$ fixed pt

$$\left.\begin{aligned} e_{n+1} &= \xi - x_{n+1} \\ &= g(\xi) - g(x_n) \\ &= g'(\eta_n)(\xi - x_n) \end{aligned}\right\} (*)$$

η_n between ξ and x_n

$$e_{n+1} = g'(\eta_n)\, e_n$$

$\lim_{n\to\infty} x_n = \xi.$ $\quad$ So $\lim_{n\to\infty} \eta_n = \xi$

So $\lim_{n\to\infty} g'(\eta_n) = g'(\xi)$

$$e_{n+1} = g'(\xi)\, e_n + \varepsilon_n e_n$$

$\lim \varepsilon_n = 0$

Hence if $g'(\xi) \neq 0$

$$\boxed{e_{n+1} \approx g'(\xi)\, e_n}$$

i.e., the error in the $(n+1)$st iterate depends (more or less) linearly on the error e_n in the n^{th} iterate

We say that $x_0, x_1, x_2, \ldots$ converges linearly to ξ

We can solve $(*)$ for ξ. For

$$e_{n+1} = \xi - x_{n+1} = g'(\eta_n)(\xi - x_n)$$

$$\xi(1 - g'(\eta_n)) = x_{n+1} - g'(\eta_n)\, x_n$$

$$= (1 - g'(\eta_n))\, x_{n+1} + g'(\eta_n)(x_{n+1} - x_n)$$

Therefore

$$\xi = x_{n+1} + \frac{g'(\eta_n)\,(x_{n+1} - x_n)}{1 - g'(\eta_n)}.$$

$$\xi = x_{n+1} + \frac{x_{n+1} - x_n}{g'(\eta_n)^{-1} - 1}$$

We do not know the number η_n

But we know that the ratio

$$r_n := \frac{x_n - x_{n-1}}{x_{n+1} - x_n} = \frac{x_n - x_{n-1}}{g(x_n) - g(x_{n-1})}$$

$$= \frac{1}{g'(\xi_n)}$$

for some ξ_n between x_n and x_{n-1}

$$r_n = \frac{1}{g'(\xi_n)} \approx \frac{1}{g'(\xi)} \approx \frac{1}{g'(\eta_n)}$$

So the pt

$$\hat{x}_n = x_{n+1} + \frac{x_{n+1} - x_n}{r_n - 1}$$

with $r_n = \dfrac{x_n - x_{n-1}}{x_{n+1} - x_n}$

will be a better approximation to ξ than is x_n or x_{n+1}

note

$$\hat{x}_n = x_{n+1} - \frac{(x_{n+1} - x_n)^2}{x_{n+1} - 2x_n + x_{n-1}}$$

$$\Delta x_k = x_{k+1} - x_k$$

$$\Delta^2 x_k = \Delta x_{k+1} - \Delta x_k$$

$$= x_{k+2} - 2x_{k+1} + x_k$$

Thus $$\boxed{\hat{x}_n = x_{n+1} - \frac{(\Delta x_n)^2}{\Delta^2 x_{n-1}}}$$

Aitken's Δ^2-process.

This process is applicable to any linearly cgt sequence, whether gen by fixed pt iteration or not

Aitken's Δ-process

Given any seq $\{x_n\}_{n\geq 0}$ linearly cgt to ξ

then the seq $\{\hat{x}_n\}_{n\geq 1}$,

$$\hat{x}_n = x_{n+1} - \frac{(\Delta x_n)^2}{\Delta^2 x_{n-1}}$$

$$\xi - x_{n+1} = K(\xi - x_n) + o(\xi - x_n)$$

for some $K \neq 0$

then $\hat{x}_n = \xi + o(\xi - x_n)$

Example

$$g(x) = \frac{1}{2} e^{x/2}$$

$\hat{x}_1 = 0.698349$

$\hat{x}_2 = 0.712809$

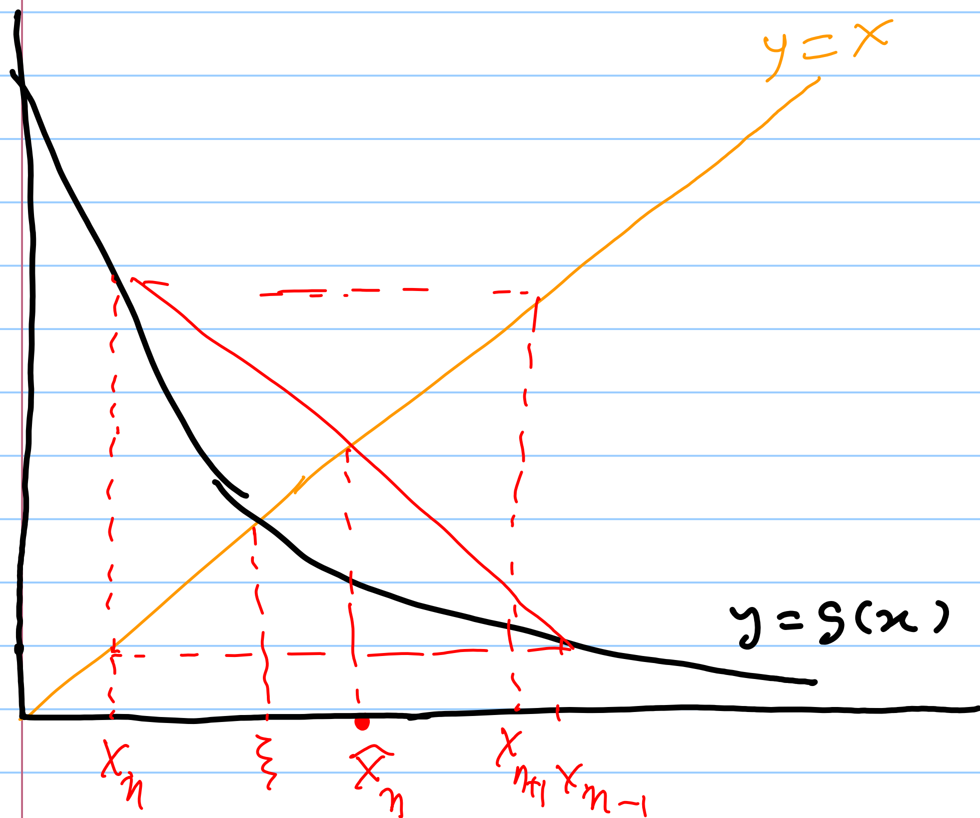
$\hat{x}_3 = 0.714556$

$\hat{x}_4 = 0.714772$

$\hat{x}_5 = 0.714804$

$\hat{x}_6 = 0.714806$

That $\hat{x}_n$ is better approximation to ξ than x_n or x_{n+1} can be seen graphically



Check

$$\hat{x}_n = x_{n+1} - \frac{(\Delta x_n)^2}{\Delta^2 x_{n-1}}$$

For fixed pt iteration.

If $\hat{x}_n$ is better approximation to ξ than x_n, then it is certainly wasteful to continue generating x_{n+2}, x_{n+3}

It seems more reasonable to start fixed pt iteration afresh with $\hat{x}_n$ as the initial guess

Steffensen iteration

Given the iteration function $g(x)$ and a pt y_0

For $n = 0, 1, 2, \dots$ until satisfied do

$$
\begin{cases}
x_0 = y_n \\
\text{Calculate } x_1 = g(x_0), \quad x_2 = g(x_1) \\
\text{Calculate } \Delta x_1 \text{ and } \Delta^2 x_0 \\
\text{Calculate } y_{n+1} = x_2 - \dfrac{(\Delta x_1)^2}{\Delta^2 x_0}
\end{cases}
$$

__Example__

$$g(x) = \frac{1}{2} e^{x/2}$$

$y_0 = 0$

$x_0 = 0$

$x_1 = 0.5$

$x_2 = 0.642013$

$y_1 = 0.698349$

$x_0 = 0.698349$

$x_1 = 0.708948$

$x_2 = 0.712715$

$y_2 = 0.714792$

$x_0 = 0.714792$

$x_1 = 0.714801$

$x_2 = 0.714804$

$y_3 = 0.714806$