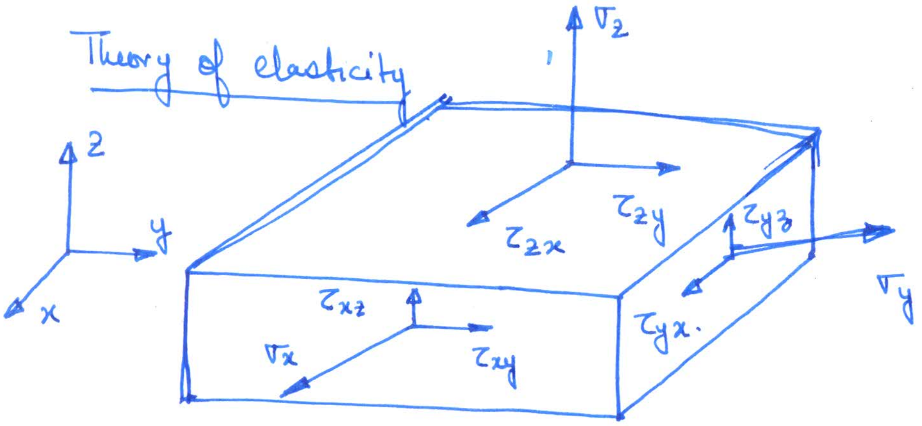
Lecture 3

Theory of elasticity



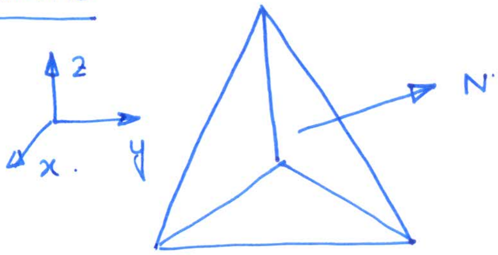
Eqn. of equilbm

component of body force / volume

(1) ——

$$\left\{\begin{aligned}
&\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} + X = 0 \\
&\cancel{\frac{\partial \sigma_y}{\partial y} + \partial \tau_{xy}}\ \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} + \frac{\partial \tau_{yz}}{\partial z} + Y = 0 \\
&\frac{\partial \tau_{xz}}{\partial x} + \cancel{\partial \tau_{xy}}\ \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \sigma_z}{\partial z} + Z = 0
\end{aligned}\right.$$

Boundary condns



$$\cos(Nx) = l$$
$$\cos(Ny) = m$$
$$\cos(Nz) = n$$

component of surface force / area

$$\hat{X} = \sigma_x l + \tau_{yx} n + \tau_{xz} m$$
$$\hat{Y} = \tau_{xy} l + \sigma_y n + \tau_{yz} m$$
$$\hat{Z} = \tau_{xz} l + \tau_{yz} n + \sigma_z n.$$

Strain – displacement relation

$$\epsilon_x = \frac{\partial u}{\partial x} \qquad \epsilon_y = \frac{\partial v}{\partial y} \qquad \epsilon_z = \frac{\partial w}{\partial z}.$$

$$\gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \qquad \gamma_{yz} = \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} \qquad \gamma_{xz} = \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}.$$

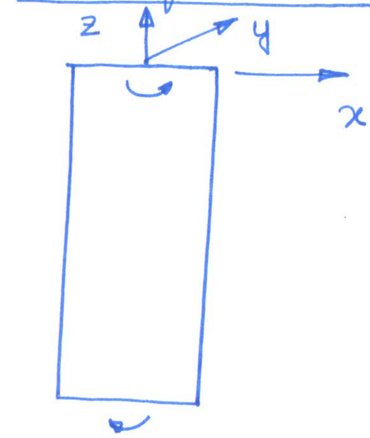
Stress - strain relation

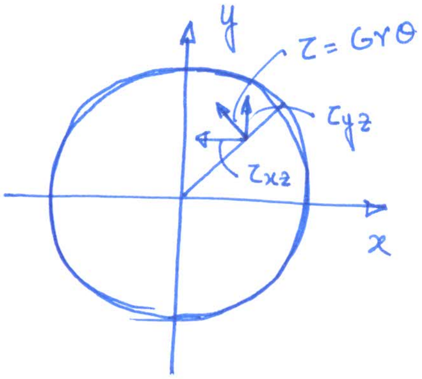
$$\epsilon_x = \frac{1}{E}\left[\sigma_x - \nu(\sigma_y + \sigma_z)\right] \qquad \gamma_{xy} = \frac{1}{G}\tau_{xy}$$

$$\epsilon_y = \frac{1}{E}\left[\sigma_y - \nu(\sigma_x + \sigma_z)\right] \qquad \gamma_{yz} = \frac{1}{G}\tau_{yz}$$

$$\epsilon_z = \frac{1}{E}\left[\sigma_z - \nu(\sigma_x + \sigma_y)\right] \qquad \gamma_{xz} = \frac{1}{G}\tau_{xz}.$$

Torsion of circular shaft





(1) circular c/s of the shaft remains circular during the twist

(2) It is in pure shear.

(3) $\frac{d\phi}{dx} = \theta$ constant ϕ = angle of twist

(4) c/s remains plane, there is no out-of-plane deformation

(5) Shear strain $\gamma_{@r} = \theta r$

$\tau_{@r} = G\theta r$.

$$\tau_{xz} = -G\theta r \sin\theta = -G\theta r \cdot \frac{y}{r} = -G\theta y$$

$$\tau_{yz} = G\theta r \cos\theta = G\theta r \frac{x}{r} = G\theta x.$$

Eqns. of equilm:

$$\overset{=0}{\cancel{\frac{\partial \sigma_x}{\partial x}}} + \overset{=0}{\cancel{\frac{\partial \tau_{xy}}{\partial y}}} + \overset{=0}{\cancel{\frac{\partial \tau_{xz}}{\partial z}}} = 0$$

$$\overset{=0}{\cancel{\frac{\partial \tau_{xy}}{\partial x}}} + \overset{=0}{\cancel{\frac{\partial \sigma_y}{\partial y}}} + \overset{=0}{\cancel{\frac{\partial \tau_{yz}}{\partial z}}} = 0$$

$$\overset{=0}{\cancel{\frac{\partial \tau_{xz}}{\partial x}}} + \overset{=0}{\cancel{\frac{\partial \tau_{yz}}{\partial y}}} + \overset{=0}{\cancel{\frac{\partial \sigma_z}{\partial z}}} = 0$$

eqns. of equillm are satisfied

Boundary conditions on the lateral surfaces

$$\cancel{\sigma_x}^{=0} l + \cancel{\tau_{yx}}^{=0} m + \tau_{xz} \cancel{n}^{=0} = 0$$

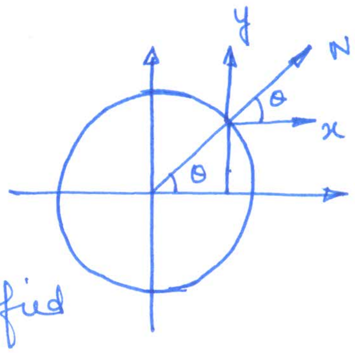
$$\cancel{\tau_{xy}}^{=0} l + \cancel{\sigma_y}^{=0} m + \tau_{yz} \cancel{n}^{=0} = 0$$

$$\tau_{xz} l + \tau_{yz} m + \cancel{\sigma_z}^{=0} n = 0$$

$$\Rightarrow \quad \tau_{xz} l + \tau_{yz} m = 0$$

$$\Rightarrow \quad -G\theta y \cos(Nx) + G\theta x \cos(Ny) = 0$$

$$\Rightarrow \quad -G\theta y \frac{x}{r} + G\theta x \cdot \frac{y}{r} = 0$$



$\therefore$ The BCs on the lateral surfaces are satisfied

Boundary condns on the top and bottom surfaces

$$\cancel{\sigma_x l} + \cancel{\tau_{xy} m}^{=0} + \tau_{xz} \cancel{n}^{=1} = \tau_{xz} = \hat{X}$$

$$\cancel{\tau_{xy}}^{=0} l + \cancel{\sigma_y}^{=0} m + \tau_{yz} \cancel{n}^{=1} = \tau_{yz} = \hat{Y}$$

$$\tau_{xz} l + \tau_{yz} m + \sigma_z n = 0$$

$\therefore$ In order to satisfy the BCs on top and bottom surfaces, the torques should be applied as $\hat{X}$ and $\hat{Y}$ with distribution similar to τ_{xz} & τ_{yz} on top and bottom surfaces.

Strain – displacement relation

$$\frac{\partial u}{\partial x} = 0 \qquad \frac{\partial v}{\partial y} = 0 \qquad \frac{\partial w}{\partial z} = 0 \qquad \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = 0$$

$$\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} = -\theta y \quad \text{and} \quad \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} = \theta z$$

Solving the above eqns and imposing the boundary condns

$$\boxed{\begin{aligned} u &= -\theta y z \\ v &= \theta x z \\ w &= 0 \end{aligned}}$$