Lecture 22

Last time we did <u>fixed pt iteration</u>

Let $g : I \to I$ be cts. (here $I = [a,b]$)
Then g has a fixed pt ξ
i.e. $g(\xi) = \xi$.

<u>Algorithm</u>

$x_0 \in I$ any.

for $n \geq 1$ $\quad x_n = g(x_{n-1})$

For convergence we also assume that there is non-negative $K < 1$ such that

$$|g'(x)| \leq K \quad \forall x \in I$$

with this condition the function g has a unique fixed pt $\xi \in I$ and the fixed pt iteration

$x_n = g(x_{n-1})$ converges to ξ.

Fixed pt itearation converges linearly

$e_n = \xi - x_n$ error at the n^{th} stage

$e_n \approx g'(\xi)\, e_{n-1}$

assuming $g'(\xi) \neq 0$, fixed pt iteration converges linearly. So it is kind of slow.

Aitken's Δ-process

$$\hat{x}_n = x_{n+1} - \frac{(\Delta x_n)^2}{\Delta^2 x_{n-1}}$$

here $\Delta x_n = x_{n+1} - x_n$

$$\Delta^2 x_{n-1} = \Delta x_n - \Delta x_{n-1}$$
$$= x_{n+1} - 2x_n + x_{n-1}$$

$\hat{x}_n$ is closer to ξ than x_n or x_{n+1}.

Convergence of Newton and Secant method

Suppose $f(x)$ is continuously twice differentiable and $f(\xi) = 0$, $f'(\xi) \neq 0$

Iteration function of Newton's Method

$$g(x) = x - \frac{f(x)}{f'(x)}$$

$$g'(x) = \frac{f(x) f''(x)}{(f'(x))^2}$$

$$g'(\xi) = 0$$

note $\exists\, \varepsilon > 0$ s.t. on $|x - \xi| \leq \varepsilon$

$$|g'(x)| \leq K < 1$$

$I = [\xi - \varepsilon, \xi + \varepsilon]$

$|x - \xi| \leq \varepsilon$

$$|g(x) - \xi| = |g(x) - g(\xi)| = |g'(\eta)||x - \xi| \leq K|x - \xi| \leq \varepsilon$$

So $g(x) \in I$

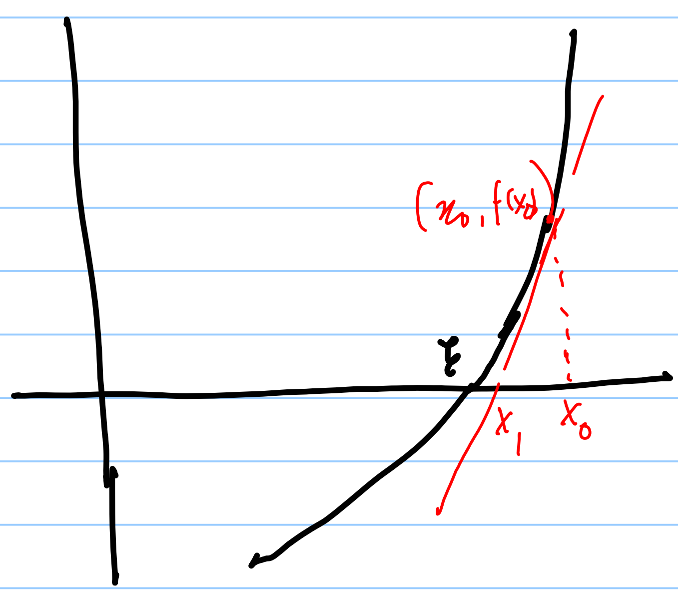
Thus $g: I \to I$; $I = [\xi - \varepsilon, \xi + \varepsilon]$

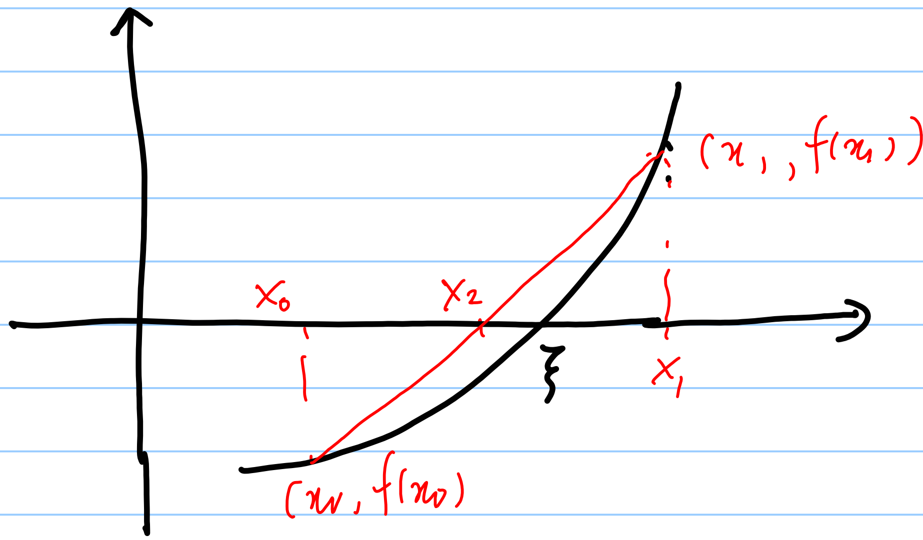
$$|g'(x)| \leq K < 1 \quad \forall x \in I$$

So fixed pt iteration converges

Thus Newtons method converges for pts suff close to the root.

Error in Newton's Method & Secant Method





Both methods interpolates $f(x)$ at two pts α and β by a st-line

$$p(x) = f(\alpha) + f[\alpha,\beta](x-\alpha)$$

whose zero $\hat{\xi} = \alpha - \dfrac{f(\alpha)}{f[\alpha,\beta]}$

is taken as the next approximation to the actual zero of $f(x)$

In Secant method we take $\alpha = x_n$, $\beta = x_{n-1}$ and then produce

$$\hat{\xi} = x_{n+1}$$

In Newton's Method we take

$$\alpha = \beta = x_n$$

and so $\hat{\xi} = x_{n+1}$

We know that

$$f(x) = f(\alpha) + f[\alpha,\beta](x-\alpha) + f[\alpha,\beta,x](x-\alpha)(x-\beta)$$

This holds for all x

So for $x = \xi$ $\qquad f(\xi) = 0$

Thus

$$0 = f(\xi) = f(\alpha) + f[\alpha,\beta](\xi-\alpha) + f[\alpha,\beta,\xi](\xi-\alpha)(\xi-\beta)$$

So

$$f[\alpha,\beta](\xi-\alpha) = -f(\alpha) - f[\alpha,\beta,\xi](\xi-\alpha)(\xi-\beta)$$

Solving for ξ we obtain

$$\xi = \alpha - \frac{f(\alpha)}{f[\alpha,\beta]} - \frac{f[\alpha,\beta,\xi]}{f[\alpha,\beta]}(\xi-\alpha)(\xi-\beta)$$

But $\hat{\xi} = \alpha - \dfrac{f(\alpha)}{f[\alpha, \beta]}$

Thus

$$(*) \quad \boxed{\xi = \hat{\xi} - \frac{f[\alpha, \beta, \xi]}{f[\alpha, \beta]} (\xi - \alpha)(\xi - \beta)}$$

This eqn can now be used to obtain error bounds for Newton and Secant method

for Newton's Method

$\alpha = \beta = x_n \qquad \hat{\xi} = x_{n+1}$

$e_j = \xi - x_j$

We obtain from $*$

$$e_{n+1} = - \frac{f[x_n, x_n, \xi]}{f[x_n, x_n]} e_n^2$$

Recall $f[x_n, x_n] = f'(x_n)$

and $f[x_n, x_n, \xi] = \frac{1}{2} f''(\eta_n)$ for

some η_n between x_n and ξ.

then
$$\boxed{e_{n+1} = -\frac{1}{2} \frac{f''(\eta_n)}{f'(x_n)} e_n^2}$$

This shows that Newton's method converges quadratically.

Error bound for secant method

Set $\alpha = x_n$ $\beta = x_{n-1}$ $\hat{\xi} = x_{n+1}$ in (*)

we obtain
$$e_{n+1} = - \frac{f[x_{n-1}, x_n, \xi]}{f[x_{n-1}, x_n]} e_n e_{n-1}$$

error in $(n+1)$-stage is proportional to product of errors in the n^{th} and $(n-1)^{st}$ stage

$$f[x_{n-1}, x_n, \xi] = \frac{1}{2} f''(\delta_n)$$

$$f[x_{n-1}, x_n] = f'(\eta_n)$$

for some η_n, δ_n between x_n, x_{n-1} & ξ

So for n - large we get

$$e_{n+1} \approx -\frac{1}{2} \frac{f''(\xi)}{f'(\xi)} e_n e_{n-1}$$

<u>Def^n</u> (order of convergence). Let $x_0, x_1, x_2, \ldots$ be a sequence which converges to a number ξ and set $e_n = \xi - x_n$. If there exist a number p and a constant $C \neq 0$ such that

$$\lim_{n \to \infty} \frac{|e_{n+1}|}{|e_n|^p} = C$$

then p is called <u>order of convergence</u> and C is called the <u>asymptotic error constant</u>.

Examples

(1) For fixed pt iteration in general

$$e_{n+1} \approx e_n \, g'(\xi) \qquad \text{and} \quad g'(\xi) \neq 0$$

$$\lim_{n\to\infty} \frac{|e_{n+1}|}{|e_n|} = |g'(\xi)|$$

So order of convergence is 1 and the asymptotic error constant is $|g'(\xi)|$.

(2) For Newton's Method

$$\lim_{n\to\infty} \frac{|e_{n+1}|}{|e_n|^2} = \frac{1}{2} \frac{|f''(\xi)|}{|f'(\xi)|}$$

provided $f'(\xi) \neq 0$.

So order of convergence is 2 and the asymptotic error constant is

$$\frac{|f''(\xi)|}{2|f'(\xi)|}$$

③ Secant method

We get from our previous calculation

(*) $$|e_{n+1}| = c_n |e_n| |e_{n-1}|$$

with $\lim_{n\to\infty} c_n = c_\infty = \frac{1}{2} \frac{|f''(\xi)|}{|f'(\xi)|}$

(we are assuming $f'(\xi) \neq 0$)

We seek a number p such that

$$\lim_{n\to\infty} \frac{|e_{n+1}|}{|e_n|^p} = c$$

for some non-zero constant c.

From (*)

(**) $$\frac{|e_{n+1}|}{|e_n|^p} = c_n |e_n|^{1-p} |e_{n-1}| = c_n \left(\frac{|e_n|}{|e_{n-1}|^p}\right)^\alpha$$

provided $\alpha = 1-p$ and so $\alpha p = -1$

i.e., provided

$$p - p^2 = -1$$

The equation $p^2 - p - 1$ has the simple positive root $p = \frac{1+\sqrt{5}}{2} = 1.618\ldots$

With this choice of p and of $\alpha = 1-p$ (**) defines a "fixed-point like iteration"

$$y_{n+1} = c_n \, y_n^{-\frac{1}{p}}$$

$$y_{n+1} = \frac{|e_{n+1}|}{|e_n|^p} \quad \text{and} \quad \lim_{n\to\infty} c_n = c_\infty .$$

It follows that y_n converges to the fixed pt of the equation

$$x = c_\infty \, x^{-\frac{1}{p}}$$

whose sol^n is $c_\infty^{\frac{1}{p}}$ since $1 + \frac{1}{p} = p$

Thus $$\lim_{n\to\infty} \frac{|e_{n+1}|}{|e_n|^p} = \left| \frac{1}{2} \frac{f''(\xi)}{f'(\xi)} \right|^{\frac{1}{p}}$$

Thus
Order of convergence of secant method

$$= p = \text{positive root of } p^2 - p - 1$$

$$= 1.618\ldots$$

asymptotic error constant is

$$\left| \frac{f''(\xi)}{2 f'(\xi)} \right|^{\frac{1}{p}}.$$

Recall that Newton's method will converge if the initial approximation x_0 is "close enough" to the root ξ.

The phrase "close enough" is vague and many times Newton's iteration will not converge or it will converge to a another zero than the one being sought.

It would be desirable to establish condition which guarantee convergence (of Newton's method) for any choice of the initial iterate in a given interval

One such set of conditions is contained in the following theorem

__Theorem :-__ Let $f(x)$ be twice continuously diff on the interval $[a, b]$ and let the following condition be satisfied

(i) $f(a) f(b) < 0$

(ii) $f'(x) \neq 0 \quad \forall x \in [a, b]$

(iii) $f''(x)$ is either ≥ 0 or ≤ 0 for all $x \in [a, b]$

(iv) At the endpts a, b

$$\frac{|f(a)|}{|f'(a)|} \leq b-a \quad , \quad \frac{|f(b)|}{|f'(b)|} \leq b-a$$

Then the Newton's method converges to the unique solution ξ of $f(x) = 0$ in $[a, b]$ for __any__ choice $x_0 \in [a, b]$.

Comments on these conditions

Condition (i) and (ii) guarantee that there is one and only one solution in $[a, b]$

Condition (iii) states that the graph of $f(x)$ is either concave from above or concave from below. Furthermore together with (i) implies that $f'(x)$ is monotone.

Condition (iv) states that tgt to the curve at either end pt intersect the x-axis within the interval $[a, b]$.

Sketch of proof of theorem

We assume without loss of generality $f(a) < 0$

We then distinguish two cases

Case (i) $f''(x) \geq 0$

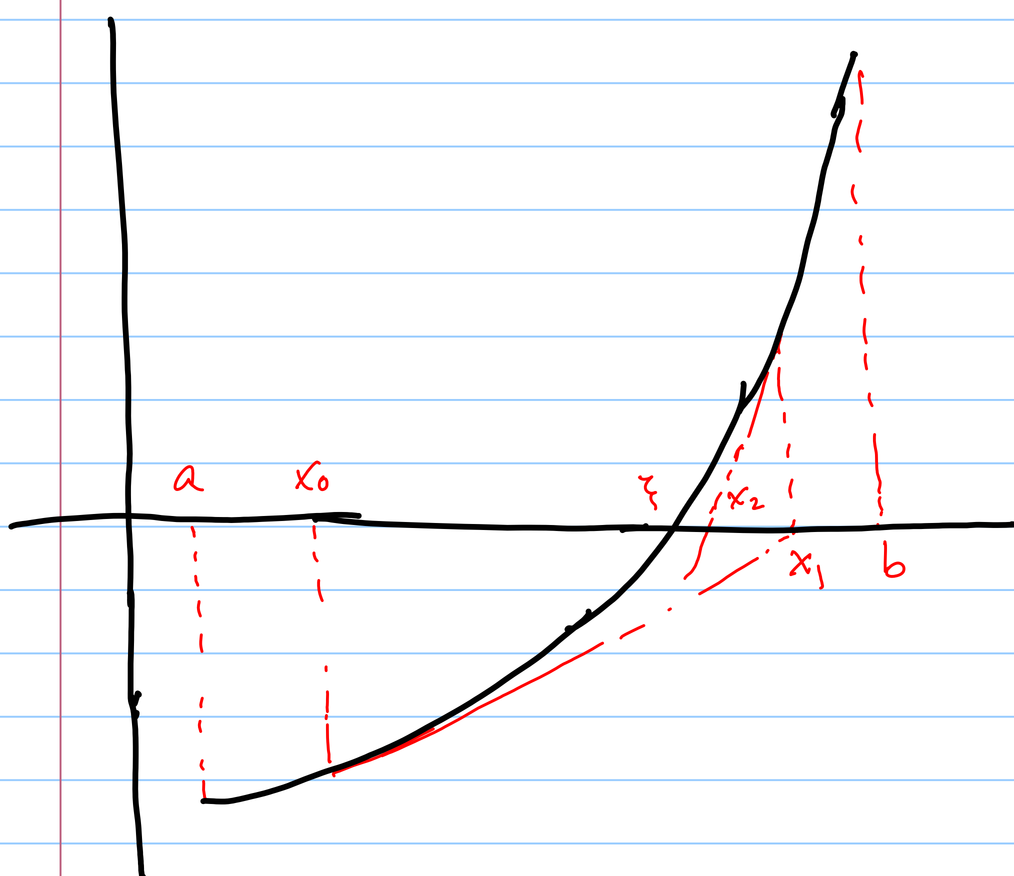
Case (ii) $f''(x) \leq 0$

Case (ii) reduces to case (i) if we replace

f by $-f$.

It therefore suffices to consider case (i).

The graph of $f(x)$ has the appearance given in the following figure



From the graph it is evident that for $x_0 > \xi$ the resulting iterates decrease monotonely to ξ while for $a \le x_0 < \xi$

x_1 falls between ξ and b. Then the subsequent iterates converge monotonely to ξ.

Recall that when we said that Newton's method converges quadratically we assume $f'(\xi) \neq 0$. (i.e ξ is a simple root of $f(x)$).

What happens if f is a double root?

i.e. $f(\xi) = 0 = f'(\xi)$, $f''(\xi) \neq 0$.

Let $g(x) = x - \dfrac{f(x)}{f'(x)}$ be the iteration function of Newton's method

Then

$$g'(x) = \frac{f(x)\, f''(x)}{f'(x)^2}$$

$$\lim_{x\to\xi} g'(x) = \lim_{x\to\xi} \frac{f(x)}{f'(x)^2} \lim_{x\to\xi} f''(x)$$

$$= \left(\lim_{x\to\xi} \frac{f(x)}{f'(x)^2}\right) \cdot f''(\xi)$$

$$= \left(\lim_{x\to\xi} \frac{\cancel{f'(x)}}{2\cancel{f'(x)} f''(x)}\right) \cdot f''(\xi)$$

(by L'Hopital' rule

$$= \left(\lim_{x\to\xi} \frac{1}{2 f''(x)}\right) \cdot f''(\xi)$$

$$= \frac{1}{2 f''(\xi)} \cdot f''(\xi)$$

$$g'(\xi) = \frac{1}{2}.$$

Thus the convergence is linear