Lecture-2 part 2

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Nested form of a polynomial

$$p(x) = a_0 + a_1(x-c) + a_2(x-c)^2 + a_3(x-c)^3 + \cdots$$
$$\cdots + a_n(x-c)^n$$

computing directly requires more multiplications and additions

better to compute it as

$$p(x) = a_0 + (x-c)\left\{a_1 + a_2(x-c) + a_3(x-c)^2 + \cdots + a_n(x-c)^{n-1}\right\}$$
$$= a_0 + (x-c)\left\{a_1 + (x-c)\left\{a_2 + a_3(x-c) + \cdots\right\}\right\}$$
$$= a_0 + (x-c)\left\{a_1 + (x-c)\left\{a_2 + (x-c)\left\{a_3 + \cdots\right\}\right\}\right\}$$

Bonus of nesting
preservation of sig. digits

Example

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$p(x) = x^3 - 6.1x^2 + 3.2x + 1.5$

4 sig-digits

$p(4.71) = -14.26$ (correct upto 4 sig-digits

However if you directly compute

$p(4.71) = (4.71)^3 - 6.1(4.7)^2 + 3.2(4.7) + 1.5$

$= -14.23$ (so correct upto 3 sig digits

In nested form

$p(x) = x(x^2 - 6.1x + 3.2) + 1.5$

$= x(x(x-6.1) + 3.2) + 1.5$

$p(4.71) \approx -14.26$

More on Instability

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Instability = sensitivity of the numerical process for the calculation of $f(x)$ from x to the inevitable rounding errors commited in a calculator or a computer

Example

$$f(x) = \sqrt{x+1} - \sqrt{x}$$

$$\text{Condition} = \left| \frac{f'(x)\, x}{f(x)} \right| = \frac{1}{2} \frac{x}{\sqrt{x+1}\sqrt{x}} \approx \frac{1}{2}$$

So conditioning is good

$$f(12345) = 111.113 - 111.08$$

$$= 0.005$$

actual value $= 0.0045000$

ans off by 10%

So we analyze what goes wrong

$$x_0 = 12345$$

$$x_1 = f_1(x_0) = x_0 + 1 = 12346$$

$$x_2 = f_2(x_1) = \sqrt{x_1}$$

$$x_3 = f_3(x_0) = \sqrt{x_0}$$

$$f_4(t) = x_2 - t$$

$$x_4 = f_4(x_3) = f(x)$$

Let us analyse

condition of f_4 is

$$\left| \frac{f_4'(t)\, t}{f_4(t)} \right| = \left| \frac{t}{x_2 - t} \right|$$

f_4 is well conditioned except when $t \approx x_2$

In our example $x_2 - x_3 \approx 0.005$

$x_3 = t \approx 111.11$

So condition of $f_4 \approx 22{,}222$

condition of $f_4 \approx 40{,}000$ times condition of f

<u>What to do</u>

$$f(x) = \sqrt{x+1} - \sqrt{x} = \frac{1}{\sqrt{x+1} + \sqrt{x}}$$

$$f(12345) = \frac{1}{111.113 + 111.108}$$

$$= 4.50002 \text{ E} -3$$

correct upto 6 sig digits

<u>Note</u>

It is possible to estimate the effects of instability by considering the rounding errors one at a time