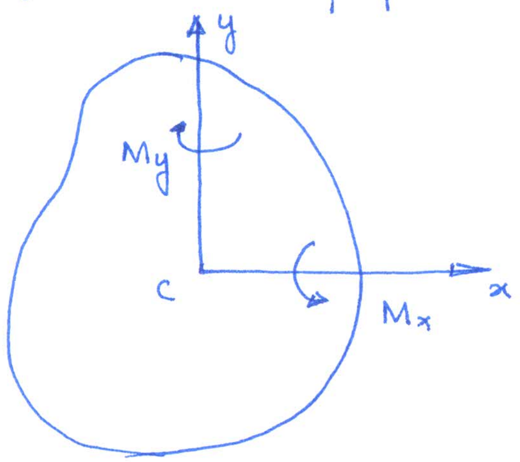
Bi-directional pure bending

Euler-Bernoulli beam theory

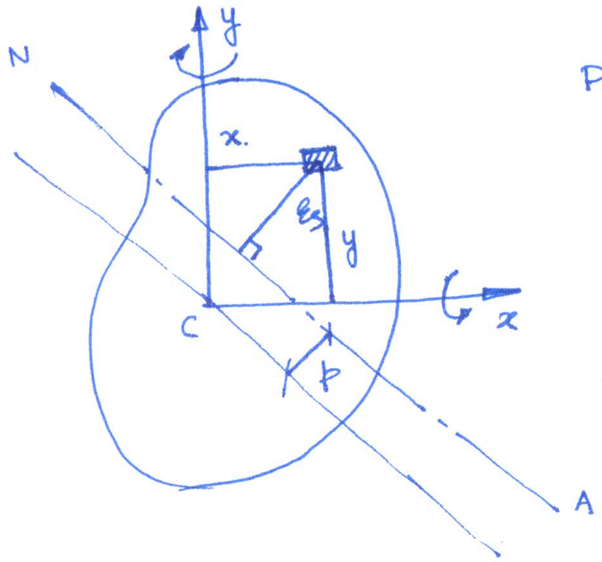
(1) plane section remains plane after bending

(2) c/s plane remains perpendicular to NA after bending



Sign convention

Bending moment producing tension in the first quadrant is taken positive.



Pure bending produces only direct stress

$$\sigma_z = E\epsilon_z \quad \text{at } dA$$

$$\epsilon_z = \frac{\xi}{R} \quad \text{at } dA$$

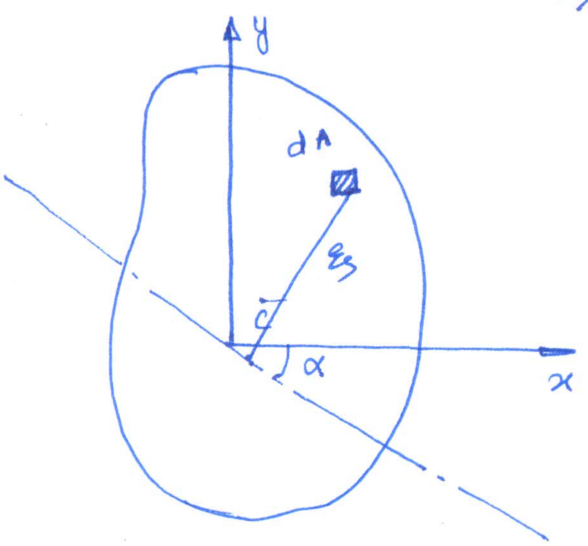
$R \rightarrow$ radius of curvature.

The beam supports pure bending moment so that the resultant normal load on any section is zero

$$\int \sigma_z \, dA = 0$$

$\Rightarrow \int \xi \, dA = 0 \Rightarrow$ first moment of area of the c/s is zero

$\Rightarrow$ the NA passes through the centroid of the section



$$\xi = x\sin\alpha + y\cos\alpha$$

$$\sigma_z = \frac{E}{R}(x\sin\alpha + y\cos\alpha)$$

$$M_x = \int \sigma_z y \, dA = \frac{E}{R}\int (x \sin\alpha + y \cos\alpha) y \, dA$$

$$= \frac{E}{R}\int xy \sin\alpha \, dA + \frac{E}{R}\int y^2 \cos\alpha \, dA$$

$$= \frac{E}{R} I_{xy} \sin\alpha + \frac{E}{R} I_{xx} \cos\alpha.$$

Similarly,

$$M_y = \int \sigma_z x \, dA = \frac{E}{R}\int (x \sin\alpha + y \cos\alpha) x \, dA$$

$$= \frac{E}{R}\int x^2 \sin\alpha \, dA + \frac{E}{R}\int xy \cos\alpha \, dA$$

$$= \frac{E}{R} I_{yy} \sin\alpha + \frac{E}{R} I_{xy} \cos\alpha.$$

$$\therefore \begin{Bmatrix} M_x \\ M_y \end{Bmatrix} = \frac{E}{R}\begin{bmatrix} I_{xy} & I_{xx} \\ I_{yy} & I_{xy} \end{bmatrix}\begin{Bmatrix} \sin\alpha \\ \cos\alpha \end{Bmatrix}$$

$$\Rightarrow \frac{E}{R}\begin{Bmatrix} \sin\alpha \\ \cos\alpha \end{Bmatrix} = \begin{bmatrix} I_{xy} & I_{xx} \\ I_{yy} & I_{xy} \end{bmatrix}^{-1}\begin{Bmatrix} M_x \\ M_y \end{Bmatrix} = \frac{1}{I_{xx}I_{yy} - I_{xy}^2}\begin{bmatrix} -I_{xy} & I_{xx} \\ I_{yy} & -I_{xy} \end{bmatrix}\begin{Bmatrix} M_x \\ M_y \end{Bmatrix}$$

$$\therefore \sigma_z = \frac{E}{R}(x \sin\alpha + y \cos\alpha)$$

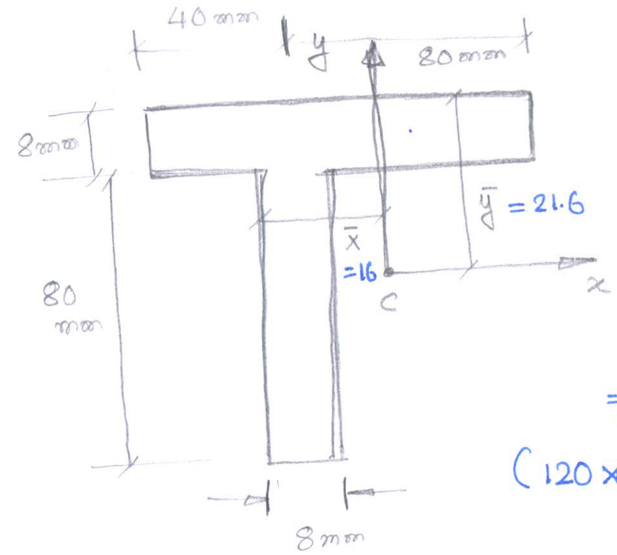
$$= \left[\frac{M_y I_{xx} - M_x I_{xy}}{I_{xx}I_{yy} - I_{xy}^2}\right] x + \left[\frac{M_x I_{yy} - M_y I_{xy}}{I_{xx}I_{yy} - I_{xy}^2}\right] y.$$

or

$$\sigma_z = \frac{M_x (I_{yy} y - I_{xy} x)}{I_{xx}I_{yy} - I_{xy}^2} + \frac{M_y (I_{xx} x - I_{xy} y)}{I_{xx}I_{yy} - I_{xy}^2}$$

$$\tan\alpha = \frac{y_{NA}}{x_{NA}} = \frac{M_y I_{xx} - M_x I_{xy}}{M_x I_{yy} - M_y I_{xy}}.$$

Ex



$M_x = 1500$ Nm

$M_y = 0$

$(120 \times 8 + 80 \times 8)\bar{y} = 120 \times 8 \times 4 + 80 \times 8 \times 48$

$\Rightarrow \bar{y} = 21.6$ mm

$(120 \times 8 + 80 \times 8)\bar{x} = 120 \times 8 \times 24 + 80 \times 8 \times 4$

$\bar{x} = 16$ mm.

$$I_{xx} = \frac{1}{12} \times 120 \times (8)^3 + 120 \times 8 \times (21.6 - 4)^2 + \frac{1}{12} \times (80)^3 \times (8) + 80 \times 8 \times (48 - 21.6)^2 = 1.09 \times 10^6 \text{ mm}^4$$

$$I_{yy} = \frac{1}{12}(120)^3 \times 8 + 120 \times 8\, (24-16)^2 + \frac{1}{12} \times 80 \times (8)^3 + 80 \times 8 \times (16-4)^2$$

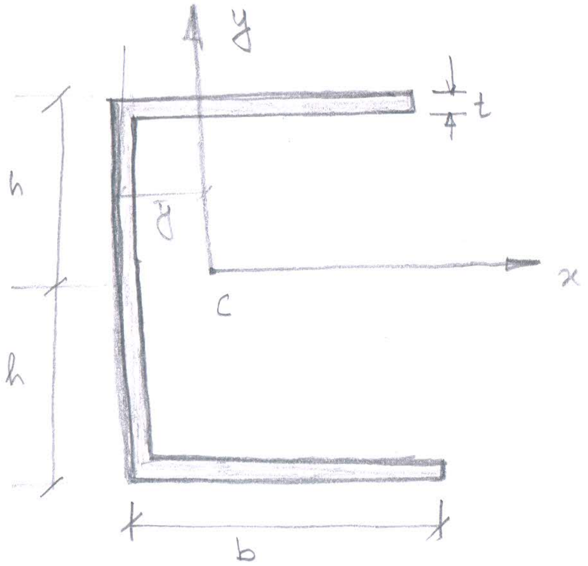
$$= 1.31 \times 10^6 \text{ mm}^4$$

$$I_{xy} = 120 \times 8 \times 8 \times 17.6 + 80 \times 8 \times (-12)(-26.4) = 0.34 \times 10^6 \text{ mm}^4$$

$$\therefore \sigma_z = \frac{M_x\,(1.31 \times 10^6 y - 0.34 \times 10^6 x)}{1.09 \times 1.31 \times 10^{12} - 0.34^2 \times 10^{12}} = 1.5y - 0.39x$$

By inspection we see that max^m σ_z at $x = -8$ mm and $y = -66.4$ mm

$$\sigma_z^{max} = -96 \text{ N/mm}^2.$$



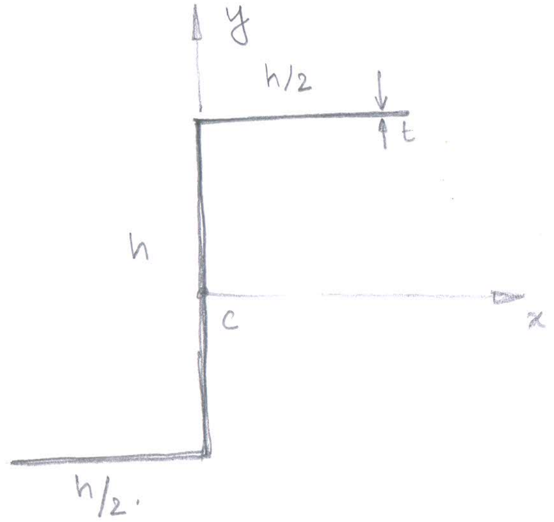
Approximation of sectional parameters for a thin-walled cross-section

$$I_{xx} = \frac{1}{12}t(2h)^3 + 2bth^2 + \underbrace{2\times\frac{1}{12}bt^3}_{\approx 0 \text{ neglected}}$$

$$= \frac{8}{12}h^3 t + 2bth^2$$

$$I_{yy} = \underbrace{\frac{1}{12}2ht^3}_{\approx 0} + 2ht\bar{y} + \cdots$$

Ex



$M_y = 0$ $\qquad \sigma_z = \dfrac{M_y(I_{yy}y - I_{xy}x)}{I_{xx}I_{yy} - I_{xy}^2}$

$$I_{xx} = 2\frac{h}{2}t\left(\frac{h}{2}\right)^2 + \frac{1}{12}th^3$$

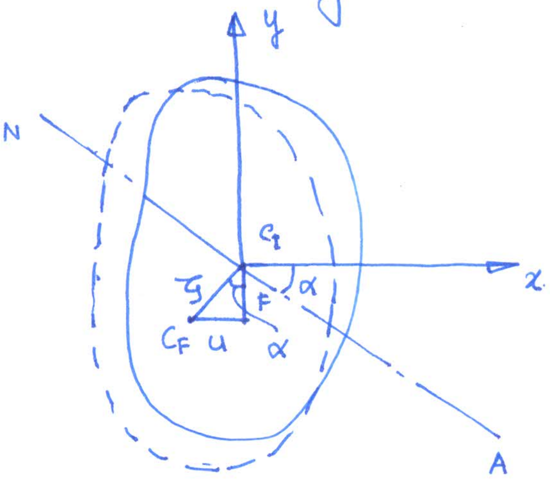
$$= \frac{h^3t}{4} + \frac{h^3t}{12} = \frac{h^3t}{3}$$

$$I_{yy} = 2\times\frac{1}{12}t\left(\frac{h}{2}\right)^3 + 2t\frac{h}{2}\times\left(\frac{h}{4}\right)^2$$

$$= \frac{h^3t}{48} + \frac{h^3t}{16} = \frac{4h^3t}{48} = \frac{h^3t}{12}$$

$$\sigma_z = \frac{M_x}{h^3t}(6.86y - 10.30x) \qquad I_{xy} = 2\frac{h}{2}t\left(+\frac{h}{2}\right)\left(\frac{h}{2}\right) = \frac{h^3t}{8}$$

Deflections due to bending



Let us consider that the centroid has moved by ξ in perpendicular dirc. after deflection

$$\frac{1}{R} = \frac{d^2\xi}{dz^2} \qquad u = -\xi \sin\alpha \qquad v = -\xi \cos\alpha$$

$$\frac{\sin\alpha}{R} = -\frac{d^2u}{dz^2} \qquad \frac{\cos\alpha}{R} = -\frac{d^2v}{dz^2}$$

$$\frac{1}{R}\begin{Bmatrix}\sin\alpha \\ \cos\alpha\end{Bmatrix} = \frac{1}{E(I_{xx}I_{yy} - I_{xy}^2)}\begin{bmatrix}-I_{xy} & I_{xx} \\ I_{yy} & -I_{xy}\end{bmatrix}\begin{Bmatrix}M_x \\ M_y\end{Bmatrix}$$

$$\begin{Bmatrix}\dfrac{d^2u}{dz^2} \\ \dfrac{d^2v}{dz^2}\end{Bmatrix} = \frac{-1}{E(I_{xx}I_{yy} - I_{xy}^2)}\begin{bmatrix}-I_{xy} & I_{xx} \\ I_{yy} & -I_{xy}\end{bmatrix}\begin{Bmatrix}M_x \\ M_y\end{Bmatrix}$$

$$\Rightarrow \quad \begin{Bmatrix}M_x \\ M_y\end{Bmatrix} = -E\begin{bmatrix}I_{xy} & I_{xx} \\ I_{yy} & I_{xy}\end{bmatrix}\begin{Bmatrix}u'' \\ v''\end{Bmatrix}$$

$$M_x = -EI_{xy}u'' - EI_{xx}v''$$

$$M_y = -EI_{yy}u'' - EI_{xy}v''$$