Lecture 19

Last time we studied about size of error in solving linear system $Ax=b$

If $\hat{x}$ is computed solution of $Ax=b$

Then error $e = x - \hat{x}$ is not known to us. However we can calculate the residual

$$\bar{r} = A\bar{e} = Ax - A\hat{x} = b - A\hat{x}$$

Condition number

$$\text{cond}(A) = \|A\| \, \|A^{-1}\|$$

Then

$$\frac{1}{\text{cond}(A)} \frac{\|r\|}{\|b\|} \leq \frac{\|e\|}{\|x\|} \leq \text{cond}(A) \frac{\|r\|}{\|b\|}$$

If Condition number is high then we say that the system $Ax=b$ is ill-conditioned

note that if condition number is high we cannot compute A^{-1} (and so $\|A^{-1}\|$) accurately.

So we need indirect method to compute condition number

Thm

$$\frac{1}{\mathrm{cond}(A)} = \min\left\{ \frac{\|A-B\|}{\|A\|} \;\middle|\; B \text{ is not invertible} \right\}$$

we proved $\leq$

i.e we proved $\frac{1}{\|A^{-1}\|} \leq \|A-B\|$ if B is not invertible

Then we studied "perturbed linear systems".

In applications the coeff matrix A is known upto some error.

So instead of solving $Ax = b$ we are ineffect solving $\hat{A}\hat{x} = b$

$$A = \hat{A} + E$$

E = matrix containing errors in coefficient

$$\frac{\|x - \hat{x}\|}{\|\hat{x}\|} \le \operatorname{cond}(A) \frac{\|E\|}{\|\hat{A}\|}$$

We also did iterative improvement of a solution to a linear system

$Ax = b$ has app sol^n $\hat{x}^{(1)}$

$e = x - \hat{x}^{(1)}$ is unknown

we solve $Ae = r = b - A\hat{x}^{(1)}$

to find $\hat{e}^{(1)}$ approximation to e

$\hat{x}^{(2)} = \hat{x}^{(1)} + \hat{e}^{(1)}$ is usually a better approximation to x than $\hat{x}^{(1)}$.

If necessary we can compute the new residual $r = b - A\hat{x}^{(2)}$

and solve $Ae = r$ again to obtain a new correction $\hat{e}^{(2)}$

$$\hat{x}^{(3)} = \hat{x}^{(2)} + \hat{e}^{(2)}$$

and so on.

Today we study determinants.

If $A = (a_{ij})$ is a $n \times n$ matrix

$$\det(A) = \sum_{p} \sigma_p \, a_{1,p_1} a_{2,p_2} \cdots a_{n,p_n}$$

where the sum is taken over all $n!$ permutation of degree n and $\sigma_p = 1$ or -1 depending on whether p is even or odd

This def^n is not good for practical purposes as $n!$ grows very large with n

However one can derive some rules from this def^n and then get a simpler rule to calculate determinants.

The determinant of a matrix is of importance because of the following thm.

Theorem :- Let A be an $n \times n$ matrix.

Then A is invertible if and only if $\det(A) \neq 0$

Rule 1 If A is an upper (or lower) triangular matrix then

$$\det(A) = a_{11} a_{22} \cdots a_{nn}$$

i.e the det is just the product of the diagonal entries

Pf Assume A is upper-triangular.

If p is any permutation other than the identity then for some i $p_i < i$

$\Rightarrow a_{i,p_i} = 0$

$\therefore a_{1,p_1} \cdots a_{n,p_n} = 0$

$\therefore \det A = \sum \sigma_p a_{1,p_1} \cdots a_{n,p_n} = a_{11} a_{22} \cdots a_{nn}$

Rule 2)

If P is a $n\times n$- permutation matrix given by $P_{ij} = i_{p_j}$ $j = 1, 2, \ldots, n$ for some permutation p

Then $\det(P) = \begin{cases} 1 & \text{if } p \text{ is even} \\ -1 & \text{if } p \text{ is odd} \end{cases}$

Rule 3

If the matrix B results from the matrix A by the interchange of two columns (rows) of A then

$$\det(B) = -\det(A)$$

Rule 4 If the matrix B is formed by multiplying all entries of one column (row) of A by the same number α then

$$\det B = \alpha \det A$$

$\boxed{\text{Rule 5}}$

Suppose 3 matrices A_1, A_2, A_3 differ only in one column (row) say the j^{th} and the j^{th} column (row) of A_3 is the vector sum of the j^{th} columns of A_1 and A_2

then $\det(A_1) + \det(A_2) = \det(A_3)$

Theorem If A and B are $n \times n$ matrices then

$$\det(AB) = \det A \cdot \det B$$

Practical method to find determinant

factor A into PLU

P permutation matrix

L Lower triangular with 1's in the diagonal

U upper triangular matrix

$$A = PLU$$

$$\det A = \det P \cdot \det L \cdot \det U$$

$$= (-1)^{i} \cdot 1 \cdot u_{11} \cdots u_{nn}$$

i = # number of row-interchanges in GE

<u>Exercise</u> Let A be a $n \times n$ invertible matrix such that $A = LU$ where L is unit lower triangular matrix and U is upper triangular matrix. Let A_k denote the principal submatrix of A formed by the first k-rows and the first k columns of A

Show $\det(A_k) \neq 0$ for $k = 1, 2, \ldots, n$

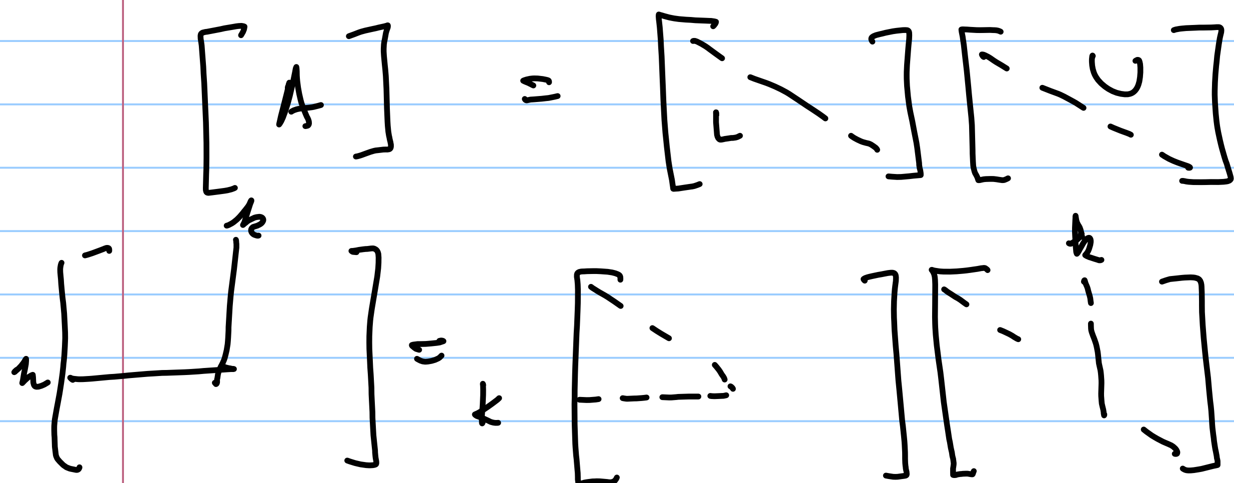
Ans $A = LU$

$A_k = L_k U_k$

note that L_k and U_k are invertible

so A_k is invertible

so $\det(A) \neq 0$



Remark A +ve definite matrix can be factored into LU.

One can prove a symmetric matrix A is +ve definite iff $\det(A_k) > 0$ for $k = 1, 2, \dots, n$.

Recall Cholesky decomposition

$A = LL^t$ if A is +ve definite.

<u>Exercise</u> Let M be a non-singular matrix.

Let $A = MM^t$. Show A is +ve definite

<u>Pf</u>

$$A = MM^t$$

$$\begin{aligned} A^t &= (MM^t)^t \\ &= (M^t)^t M^t \\ &= MM^t \\ &= A \end{aligned}$$

Ans Let $\bar{x}$ be non-zero

$$\bar{x}^t A \bar{x} = \bar{x}^t M M^t \bar{x}$$

Let $\bar{y} = M^t \bar{x} \Rightarrow \bar{y}^t = \bar{x}^t M$

note that $\bar{y} \neq 0 \because M^t$ is non-singular

then $\bar{x}^t A \bar{x} = \bar{y}^t y = \|y\|_2^2 > 0$

Computing A^{-1}

There is usually no good reason for ever calculating the inverse

However if one has to calculate A^{-1} then one does the following

- Factor A into PLU
- Solve $Ax = e_j$

  e_j j^{th} unit column vector

  to obtain j^{th} column of the inverse

  (by back substitution)

Example

$$A = \begin{bmatrix} 2 & 1 & 0 & 0 \\ -1 & 3 & 3 & 0 \\ 2 & -2 & 1 & 4 \\ -2 & 2 & 2 & 5 \end{bmatrix}$$

Find LU dec. of A and then find A^{-1}

$R_2 + \frac{1}{2} R_1$, $R_3 - R_1$, $R_4 + R_1$ gives

$$\begin{bmatrix} 2 & 1 & 0 & 0 \\ 0 & 3\frac{1}{2} & 3 & 0 \\ 0 & -3 & 1 & 4 \\ 0 & 3 & 2 & 5 \end{bmatrix}$$

$R_3 + \frac{6}{7} R_2$ $\qquad$ $R_3 - \frac{6}{7} R_2$

$$\begin{bmatrix} 2 & 1 & 0 & 0 \\ 0 & 3\frac{1}{2} & 3 & 0 \\ 0 & 0 & \frac{25}{7} & 4 \\ 0 & 0 & -\frac{4}{7} & 5 \end{bmatrix}$$

$R_4 + \frac{4}{25} R_3$ yields

$$\begin{bmatrix} 2 & 1 & 0 & 0 \\ 0 & \frac{3}{2} & 3 & 0 \\ 0 & 0 & \frac{25}{7} & 4 \\ 0 & 0 & 0 & \frac{141}{25} \end{bmatrix} = U$$

$$L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -\frac{1}{2} & 1 & 0 & 0 \\ 1 & -6/7 & 1 & 0 \\ -1 & 6/7 & -\frac{4}{25} & 1 \end{bmatrix}$$

$$LUx = e_1$$

$$Ux = y$$

$$Ly = e_1$$

$1y_1 = 1 \qquad \Rightarrow y_1 = 1$

$-\frac{1}{2}y_1 + y_2 = 0 \qquad y_2 = \frac{1}{2}$

$y_1 - \frac{6}{7}y_2 + y_3 = 0 \qquad y_3 = -1 + \frac{\cancel{6}^{\,3}}{7}\cdot\frac{1}{\cancel{2}} = \frac{-4}{7}$

$-y_1 + \frac{6}{7}y_2 - \frac{y_3}{25} + y_4 = 0 \qquad y_4 = +1 * \frac{3}{7} - \frac{4}{175}$

$$y_4 = \frac{175 - 75 - 4}{175} = \frac{96}{175}$$

$$Ux = y$$

$$\frac{141}{25} x_4 = y_4 = \frac{96}{175}$$

$$x_4 = \frac{96}{7} \frac{1}{141} = \frac{32}{7 \cdot 47} = \frac{32}{327}$$