Lecture 24

Last time we studied iterative methods to solve system of linear equations

$$Ax = b$$

The method used is based on the idea of an approximate inverse to A, i.e., a matric C such that

$$\| I - CA \| < 1 \quad \text{for some matrix norm}$$

The iteration function g is given by

$$g(\bar{x}) = C\bar{b} + (\underline{I} - CA)\bar{x}$$
$$= \bar{x} + C(\bar{b} - A\bar{x})$$

Thus the iteration is

x_0 any

$$x^{(m+1)} = x^{(m)} + C(b - Ax^{(m)})$$
for $m = 0, 1, 2, \ldots$

Then $x^{(m+1)} \to \xi$ fixed pt of g

also $A\xi = b$.

Two of the most common method are

the
- Jacobi-Method
- Gauss-Siedel method.

Gauss-Siedel method is usually faster than Jacobi-Method.

Gauss-Siedel method converges for strictly row-dominant matrices and positive definite matrices.

Last time we proved that the Jacobi method converges for strictly row-dominant matrices.

Iterative methods are usually used when we have large system of linear equation with a <u>sparse</u> coefficient matrix.

For the next few lectures we will be learning numerical methods to solve initial value problems.

$$\frac{dy}{dx} = f(x, y) \qquad x_0 \le x \le x_n$$

$$y(x_0) = y_0$$

We assume that all partial derivatives to f exist.

By theory there exists a unique solution to the above initial value problem. (in a neighborhood of (x_0, y_0)).

However one cannot hope for exact solutions

Example

$$\frac{dy}{dx} = \sin(x^2 + y^2)$$

$$0 \le x \le 1$$

$$y(0) = 1$$

We cannot find exact answer to above equation

Numerical integration by Taylor Series

$$(*) \qquad \frac{dy}{dx} = f(x,y) \qquad y(x_0) = y_0$$

If $y(x)$ is exact solution of $(*)$ then we can expand $y(x)$ into a Taylor series about the pt x_0

$$(**) \qquad y(x) = y_0 + (x-x_0)\, y'(x_0) + \frac{(x-x_0)^2}{2!}\, y''(x_0) + \cdots$$

The derivatives in this expansion are not known explicitly since the solution is unknown. However if f is sufficiently differentiable they can be obtained by taking the total derivative of $(*)$ with respect to x, keeping in mind that y itself is a function of x.

$$y' = f(x,y)$$
$$y'' = f' = f_x + f_y f$$
$$y''' = f'' = f_{xx} + f_{xy} f + f_{yx} f + f_{yy} f^2 + f_y f_x + f_y^2 f$$

Continuing in this manner, we can express any derivative of y in terms of f and its partial derivatives. It is already clear, however, that unless f is a very simple function, the higher total derivatives become increasingly complex.

For practical reasons then, one must limit the number of terms in the expansion (**) to a reasonable number.

If we assume that the truncated series (**) is a good approximation for a step of length h, i.e., for $x - x_0 = h$, we can then evaluate y at $x_0 + h$, reevaluate the derivatives y', y'' etc and then use (**) to proceed to the next step.

Thus we obtain a discrete set of value y_n which are approximation to the true solution at the points $x_n = x_0 + nh$ $(n = 0, 1, 2, \ldots)$.

Remk We will always denote the value of the exact solution at a pt x_n by $y(x_n)$ and of an approximate solution by y_n.

define

$$T_k(x,y) = f(x,y) + \frac{h}{2!} f'(x,y) + \frac{h^2}{3!} f''(x,y) + \cdots + \frac{h^{k-1}}{k!} f^{(k-1)}(x,y)$$

$$k = 1, 2, \ldots$$

Algorithm 1 (Taylor algorithm of order k)

To find an approximate solution of the diff equation

$$y' = f(x,y)$$
$$y(a) = y_0$$

over an interval $[a,b]$

1. Choose a step $h = \frac{b-a}{N}$.

   Set $x_n = a + nh \qquad n = 0, 1, \ldots, N$

2. Generate approximation y_n to $y(x_n)$ from the recursion

$$y_{n+1} = y_n + h\,T_k(x_n, y_n) \qquad n = 0, 1, \ldots, N-1$$

Taylor's theorem with remainder shows that the local error of Taylor's algorithm of order k is

$$E = \frac{h^{(k+1)} f^{(k)}(\xi, y(\xi))}{(k+1)!} \qquad x_n < \xi < x_n + h$$

$$E = \frac{h^{k+1}}{(k+1)!} y^{(k+1)}(\xi)$$

Example Use Taylor's method of order 2 to solve

$$\frac{dy}{dx} = y - x^2 + 1 \qquad 0 \le x \le 2$$

$$y(0) = 0.5$$

$$f' = \frac{d}{dx}(y - x^2 + 1)$$

$$= \frac{dy}{dx} - 2x$$

$$= y - x^2 + 1 - 2x$$

$$T_2(x_i, y_i) = f(x_i, y_i) + \frac{h}{2} f'(x_i, y_i)$$

$$= y_i - x_i^2 + 1 + \frac{h}{2}(y_i - x_i^2 - 2x_i + 1)$$

$$= \left(1 + \frac{h}{2}\right)(y_i - x_i^2 + 1) + \frac{h}{2}$$

Taylor method of order 2

$$y_0 = 0.5$$

$$y_{i+1} = y_i + h\left[\left(1+\frac{h}{2}\right)(y_i - x_i^2 + 1) - h x_i\right]$$

Exact solution to above d.e is

$$y(x) = (x+1)^2 - 0.5e^x$$

x_i	Exact $y(x_i)$	Taylor order 2 y_i	Error
0.0	0.5	0.5	0
0.2	0.8292986	0.830000	0.0007014
0.4	1.2140877	1.2158000	0.0017123
0.6	1.6489406	1.6520760	0.0031354
0.8	2.1272295	2.1323327	0.0051032
1.0	2.6408591	2.6486459	0.0077812
1.2			
1.4			
1.6			
1.8			
2.0	5.3054720	5.3476843	0.042123

On setting $k=1$ in our Algorithm we obtain <u>Euler's method</u>

$$\boxed{y_{n+1} = y_n + h\, f(x_n, y_n)}$$

local error $\quad E = \frac{h^2}{2} y''(\xi) \quad x_n \le \xi \le x_{n+1}$

<u>Example</u>

$$\frac{dy}{dx} = y - x^2 + 1 \qquad 0 \le x \le 2$$

$$y(0) = 0.5$$

$$h = 0.2 \qquad x_n = 0.2n$$

$$y_{n+1} = y_n + 0.2\left(y_n - 0.04n^2 + 1\right)$$

<u>Exact sol</u> $\quad y(x) = (x+1)^2 - 0.5e^x$

x_i	Eulers method y_i	Exact $y(x_i)$	Error $\lvert y_i - y(x_i)\rvert$
0	0.5	0.5	0
0.2	0.8	0.8292986	0.0292981
0.4	1.152	1.2140877	0.0620877
0.6	1.5504	1.6489406	0.0985406
0.8	1.9884800	2.1272295	0.1387495
1.0	2.4581760	2.6408591	0.1826831
1.2	|	|	|
1.4	|	|	|
1.6			|
1.8	|	|	
2.0	4.8657845	5.3054720	0.4396854

<u>Nottice</u> error grows slightly as the value of x increases.

This controlled error growth is a consequence of the stability of Euler's method, which implies that the error is expected to grow in no worse than a linear manner.

Error estimates and convergence of Euler's method

To solve the d.e

$$y' = f(x, y)$$
$$y(x_0) = y_0$$

by Euler's method, we choose a constant step size h and we apply the formula

(*) $$y_{n+1} = y_n + h f(x_n, y_n)$$
$$n = 0, 1, 2, \ldots$$

where $x_n = x_0 + nh$.

We denote the true solution of the d.e. at $x = x_n$ by $y(x_n)$ and the approximate solⁿ obtained by applying (*) as y_n.

We wish to estimate the magnitude of the discretization error e_n defined by

$$e_n = y(x_n) - y_n$$

Assuming that appropriate derivatives exist, we can expand $y(x_{n+1})$ about $x = x_n$ using Taylor's theorem with remainder

(**) $$y(x_{n+1}) = y(x_n) + h y'(x_n) + \frac{h^2}{2} y''(\xi_n)$$

$$x_n \le \xi \le x_{n+1}$$

$E_n = \frac{h^2}{2} y''(\xi_n) \rightarrow$ local discretization error

This is the error committed in the single step from x_n to x_{n+1} assuming that y and y' are known exactly at $x = x_n$

(we ignore Round-off errors in this section)

On subtracting (*) from (**) we obtain

$$e_{n+1} = e_n + h\left[f(x_n, y(x_n)) - f(x_n, y_n)\right] + \frac{h^2}{2} y''(\xi_n)$$

By M.V.T

$$f(x_n, y(x_n)) - f(x_n, y_n) = \frac{\partial f}{\partial y}(x_n, \bar{y}_n)\,(y(x_n) - y_n)$$

$$= f_y(x_n, \bar{y}_n) \cdot e_n$$

where $\bar{y}_n$ is between y_n and $y(x_n)$

So we have

$$\boxed{e_{n+1} = e_n + h f_y(x_n, \bar{y}_n)\, e_n + \frac{h^2}{2} y''(\xi_n)}$$

Theorem Let y_n be the approximate sol^n to

$$\frac{dy}{dx} = f(x, y) \qquad y(x_0) = y_0$$

If the exact sol^n $y(x)$ of above d.e has a continuous second derivative on the interval $[x_0, b]$ and if on this interval the inequalities

$$|f_y(x,y)| \le L \quad , \quad |y''(x)| < Y$$

are satisfied for fixed +ve constants L and Y, the error $e_n = y(x_n) - y_n$ of Euler's method at a pt $x_n = x_0 + nh$ is bounded as follows

$$|e_n| \le \frac{hY}{2L}\left(e^{(x_n - x_0)L} - 1\right)$$

Pf

$$e_{n+1} = e_n + h f_y(x_n, \bar{y}_n) e_n + \frac{h^2}{2} y''(\xi_n)$$

$$|e_{n+1}| \le |e_n| + hL|e_n| + \frac{h^2}{2} Y$$

$$= (1+hL)|e_n| + \frac{h^2}{2} Y$$

We show by induction that the solution of the difference equation

$$\xi_{n+1} = (1+hL)\xi_n + \frac{h^3}{2} Y \quad , \quad \xi_0 = 0$$

dominates $|e_n|$, i.e we will show

(*) $$\xi_n \geq |e_n| \quad \text{for } n = 0, 1, \dots$$

Since $e_0 = \xi_0 = 0$, (*) is true for $n = 0$.

assume truth of (*) for n

$$\xi_n \geq |e_n|$$

$$\xi_{n+1} = (1+h\gamma)\xi_n + \frac{h^2}{2}Y$$

$$\geq (1+h\gamma)|e_n| + \frac{h^2}{2}Y$$

$$\geq |e_{n+1}|$$

Thus $\xi_n \geq |e_n| \quad \forall n = 0, 1, 2, \dots$

$$\begin{cases} \xi_{n+1} = (1+h\gamma)\xi_n + \frac{h^2}{2}Y \\ \xi_0 = 0 \end{cases}$$

sol^n $\xi_n = c(1+hL)^n - B$

where $B = \frac{hY}{2L}$ and c is a constant to be determined

$\varepsilon_0 = 0$

so $C = B$

$\varepsilon_n = B(1+hL)^n - B$

$e^x = 1 + x + e^{\varepsilon} \frac{x^2}{2}$

Thus $e^x \geq 1 + x$ for all x

$1 + hL \leq e^{hL}$

so $(1+hL)^n \leq e^{nhL}$

Therefore $\varepsilon_n \leq B(e^{nhL} - 1)$

$= \frac{hY}{2L}(e^{nhL} - 1)$

$= \frac{hY}{2L}(e^{(x_n - x_0)L} - 1)$

note $nh = x_n - x_0$)

Since $|e_n| \leq \varepsilon_n$

we have proved the theorem.