Lecture 3.

Recall

1) Last time we discussed notion of condition and instability of numerical methods.

2) We also discussed some mathematical preliminaries that we need: notable,
the intermediate value theorem of continuous functions,
the mean value theorem,
Taylors theorem,
The fundamental theorem of algebra.

Polynomials

→ Power form

$$p(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n.$$

This form may lead to loss of significant digits

→ Remedy use shifted power form

$$p(x) = b_0 + b_1 (x-c) + b_2 (x-c)^2 + \cdots + b_n (x-c)^n$$

to compute $p(x)$ near c.

Textbook: Elementary Numerical Analysis
– an algorithmic approach

by S. D. Conte and C. de Boor

p(x) near c

Suppose you have to calculate
p(x) near c

$$p(x) = a_0 + a_1 x + \cdots + a_n x^n$$

$$p(x) = b_0 + b_1(x-c) + b_2(x-c)^2 + \cdots + b_n(x-c)^n$$

$$p(x) = b_0 + (x-c)\left\{ b_1 + b_2(x-c) + \cdots + b_n(x-c)^{n-1} \right\}$$

$$= b_0 + (x-c)\left\{ b_1 + (x-c)\left[b_2 + \cdots + b_n(x-c)^{n-2} \right] \right\}$$

Last time I had told you to compute polynomials by the "nested form"

<u>Example</u> :-

$$p(x) = x^3 - 6.1x^2 + 3.2x + 1.5$$

(4 sig digits)

$$p(4.71) = -14.26$$ (correct upto 4 sig digits)

However if you directly compute

$$p(4.71) = 4.71^3 - 6.1(4.71)^2 + 3.2 \times 4.71 + 1.5$$

$$= -14.23$$

In nested form

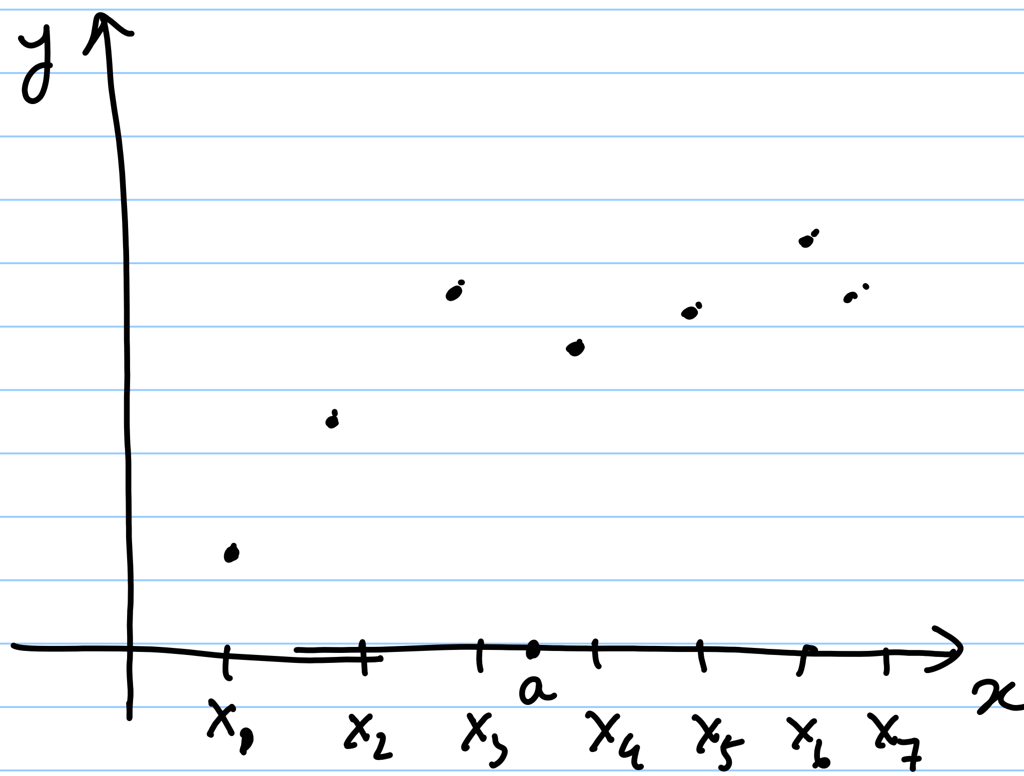
$$p(x) = x(x^2 - 6.1x + 3.2) + 1.5$$

$$= x(x(x-6.1) + 3.2) + 1.5$$

$$p(4.71) = -14.26$$

Today we discuss

"Interpolation"



We have $f(x_1), f(x_2), \ldots, f(x_7)$.

We need to approximate $f(a)$.

idea is to fit a curve passing through $(x_1, f(x_1)), (x_2, f(x_2)), \ldots, (x_n, f(x_n))$ and then approximate $f(a)$.

Question:- Which curve to fit?

One classical (and non-trivial) result is the following

<u>Theorem</u> (Weierstrass Approximation theorem)

Suppose $f:[a,b] \to \mathbb{R}$ is continuous. For each $\varepsilon > 0$, there exists a polynomial $P(x)$ such that

$$|f(t) - P(t)| < \varepsilon \quad \text{for all } t \text{ in } [a,b]$$

However, the polynomial constructed for proving this theorem has slow convergence. So ineffective in practice

- We might be tempted to use Taylor polynomials

Example $\quad f(x) = \frac{1}{x} \quad x \in [1, 4]$

$$f'(x) = -\frac{1}{x^2} \qquad f''(x) = (-1)^2 \frac{2}{x^3}$$

$$f^{(k)}(x) = (-1)^k \frac{k!}{x^{k+1}}$$

So n^{th} Taylor polynomial

$$T_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(1)}{k!} (x-1)^k$$

$$= \sum_{k=0}^{n} (-1)^k (x-1)^k$$

n	2	3	4	5	6	7
$T_n(3)$	3	-5	11	-21	43	-85

Problem

Given $n+1$ distinct points $x_0, x_1, \ldots, x_n$ in $[a,b]$, and a function $f:[a,b] \to \mathbb{R}$ does, there exists a polynomial $p(x)$ of degree $\le n$ which interpolates $f(x)$ at the points $x_0, x_1, \ldots, x_n$ i.e., $p(x)$ satisfies

$$p(x_i) = f(x_i) \quad \text{for } i = 0, 1, \ldots, n$$

We prove that there exist a unique polynomial which does the job.

Lagrange polynomials

Given $x_0, x_1, \ldots, x_n$ distinct pts

$$l_k(x) = \prod_{\substack{i=0 \\ i \neq k}}^{n} \frac{x - x_i}{x_k - x_i} \qquad k = 0, 1, \ldots, n$$

for example

$n=1$ $\qquad x_0, x_1, x_2$ distinct pts

$$l_0(x) = \frac{(x - x_1)(x - x_2)}{(x_0 - x_1)(x_0 - x_2)}$$

$$l_1(x) = \frac{(x - x_0)(x - x_2)}{(x_1 - x_0)(x_1 - x_2)}$$

$$l_2(x) = \frac{(x - x_0)(x - x_1)}{(x_2 - x_0)(x_2 - x_1)}$$

$x_0, x_1, \ldots, x_n$ $\quad$ $n+1$ distinct pts

$$l_k(x) = \prod_{\substack{i=0 \\ i \neq k}}^{n} \frac{x - x_i}{x_k - x_i}$$

$k = 0, 1, \ldots, n$

1) $l_k(x)$ is a degeree n-polynomial $\forall k = 0, 1, \ldots, n$

2) $l_k(x_k) = 1$

3) $l_k(x_i) = 0$ for $i \neq k$

notice

$$P_n(x) = \sum_{i=0}^{n} f(x_i)\, l_i(x)$$

has the property $p(x_k) = f(x_k)$ for $k = 0, 1, \ldots, n$

Thus $p_n(x) = \sum_{K=0}^{n} f(x_k) l_k(x)$

is an interpolating polynomial

(Uniquen)

If $p(x), q(x)$ interpolate $f(x)$ in $x_0, \ldots, x_n$ then $p(x) = q(x)$

Proof $\deg p(x) \leq n$

$\deg q(x) \leq n$

$p(x) - q(x)$ has $n+1$ zeros

so is identically zero.

i.e $p(x) = q(x)$.

Example

x	$f(x)$
2	6.931 E−1
3	1.099
4	1.386

approximate $f(3.2)$

Ans $P_2(x) = f(2)\ell_0(x) + f(3)\ell_1(x) + f(4)\ell_2(x)$ interpolates $f(x)$.

Recall

$$\ell_0(x) = \frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)}$$

Thus.

$$\ell_0(3.2) = \frac{(3.2-3)(3.2-4)}{(2-3)(2-4)} = -8\ E{-2}$$

Similarly

$$\ell_1(3.2) = \frac{(3.2-2)(3.2-4)}{(3-2)(3-4)} = 9.6\ E{-1}$$

$$\ell_2(3.2) = \frac{(3.2-2)(3.2-3)}{(4-2)(4-3)} = 1.2\ E{-1}$$

$$f(3.2) \approx (6.931\ E{-1})(-8\ E{-2}) + (9.6\ E{-1})(1.099) + 1.386 \times 1.2\ E{-1}$$

$$= 1.166$$

$f(x) = \ln(x)$ $\qquad$ $f(3.2) = \ln(3.2) = 1.163$

$$p_n(x) = \sum_{k=0}^{n} f(x_k)\, l_k(x)$$

is called "Lagrange" form of interpolating polynomial.

Problem with Lagrange form

Suppose we have found $p_n(x)$ interpolating $f(x)$ at points $x_0, x_1, \ldots, x_n$

Suppose we also know $f(x_{n+1})$

Then we can form $p_{n+1}(x)$ interpolating $f(x)$ at $x_0, x_1, \ldots, x_n, x_{n+1}$

There is no obvious relation between $p_n(x)$ and $p_{n+1}(x)$.

We write

$$P_n(x) = a_0 + a_1(x-x_0) + a_2(x-x_0)(x-x_1)$$
$$+ \cdots + a_n(x-x_0)(x-x_1)\cdots(x-x_{n-1})$$

Set

$$q(x) = a_0 + a_1(x-x_0) + \cdots + a_{n-1}(x-x_0)\cdots(x-x_{n-2})$$

Thus $P_n(x) = q(x) + a_n(x-x_0)\cdots(x-x_{n-1})$

Note $q(x_i) = P_n(x_i) = f(x_i)$
for $i = 0, 1, \ldots, n-1$

So by uniquen of interpolating polynomial

$$q(x) = P_{n-1}(x)$$

Thus

$$\underline{P_n(x) = P_{n-1}(x) + a_n \boxed{(x-x_0)(x-x_1)\cdots(x-x_{n-1})}}$$

Note a_n = coefficient of x^n in $P_n(x)$.

$$f[x_0, x_1, \ldots, x_n] =: a_n \quad .$$

$f[x_0, \ldots, x_n]$ is called the n^{th} divided difference of $f(x)$ at the point $x_0, x_1, \ldots, x_n$.

We write

$$P_n(x) = f[x_0] + f[x_0, x_1](x-x_0) + f[x_0, x_1, x_2](x-x_0)(x-x_1) + \cdots$$
$$\cdots + f[x_0, \ldots x_n](x-x_0)(x-x_1)\cdots(x-x_{n-1})$$

$$f[x_0] = f(x_0)$$

$$P_1(x) = f[x_0] + f[x_0, x_1](x - x_1)$$

$$P_1(x_1) = f(x_1)$$

$$\frac{f(x_1) - f(x_0)}{x_1 - x_0} = f[x_0, x_1]$$

Claim

$$\boxed{f[x_0, \ldots x_k] = \frac{f[x_1, \ldots x_k] - f[x_0, \ldots, x_{k-1}]}{x_k - x_0}}$$

<u>Proof of claim.</u>

Let $P_i(x)$ = polynomial of deg $\leq i$ which agrees with $f(x)$ at the pts $x_0, x_1, \ldots, x_i$

Let $q_{k-1}(x)$ be the polynomial of degree $\le k-1$ which agrees with $f(x)$ at the points $x_1, \dots, x_k$.

Set

$$p(x) = \frac{x - x_0}{x_k - x_0}\, q_{k-1}(x) + \frac{x_k - x}{x_k - x_0}\, P_{k-1}(x)$$

note $p(x_i) = f(x_i)$ for $i = 0, 1, \dots, k$

by uniquen of interpolating polynomial

$$p(x) = P_k(x)$$

$$f[x_0, \dots, x_k] = \text{coeff of } x^k \text{ in } P_k(x)$$

$$= \frac{\text{coeff of } x^{k-1} \text{ in } q_{k-1}(x)}{x_k - x_0} - \frac{\text{coeff of } x^{k-1} \text{ in } P_{k-1}(x)}{x_k - x_0}$$

$$= \frac{f[x_1, \dots, x_k] - f[x_0, \dots, x_{k-1}]}{x_k - x_0}$$

$$f[2,3] = \frac{f(3) - f(2)}{3 - 2}$$

Example :-

x	$f(x)$	$f[\,,\,]$	$f[\,,\,,\,]$	$f[\,,\,,\,,\,]$
2	6.931 E-01	4.059 E-1 $f[2,3]$	-5.945 E-2 $f[2,3,4]$	9.15 E-3 $f[2,3,4,5]$
3	1.099	2.87 E-1 $f[3,4]$	-3.2 E-2 $f[3,4,5]$	
4	1.386	2.230 E-1 $f[4,5]$		
5	1.609			

Approximate $f(3.2)$

$$P_2(x) = f[2] + f[2,3](x-2) + f[2,3,4](x-2)(x-3)$$

$$P_2(x) = 6.931\text{ E-01} + 4.059\text{ E-1}\,(x-1)$$
$$- 5.945\text{ E-2}\,(x-2)(x-3)$$

So $P_2(3.2) = 1.166$

Exact value $f(3.2) = \ln(3.2) = 1.163$

$$P_3(x) = P_2(x) + 9.15\text{ E-3}\,(x-2)(x-3)(x-4)$$

$$P_3(3.2) = 1.166 + 9.15\text{ E-3}\,(1.2)(0.2)(-0.8)$$

$$P_3(3.2) = 1.164$$

$$f[x_0, x_1] = \frac{f(x_1) - f(x_0)}{x_1 - x_0}$$

what happens when $x_1 \to x_0$?

$$\lim_{x_1 \to x_0} f[x_0, x_1] = f'(x_0)$$