Unit-II

Block diagram representation of control system

A control system may consist of a number of components. The function of each component in the system is represented by using a diagram called block diagram.

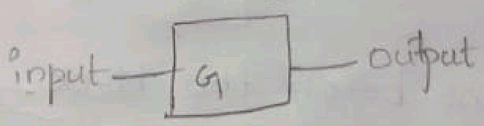
(or)

A block diagram of a system is a pictorial representation of the functions performed by each component & the flow of signals.

The elements of a block diagram are

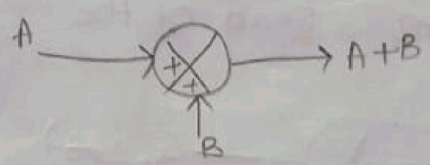
1. Block
2. Summing point
3. Branch point.

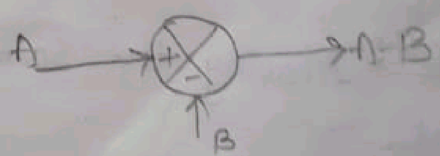
1. **Block:** A block is a symbol for mathematical operation of the input signal to the block that produces the output.



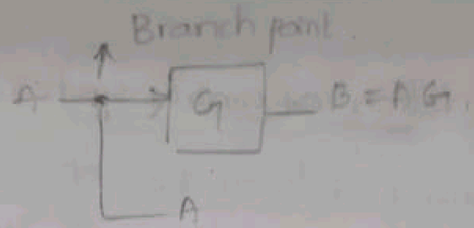
2. **Summing point:** Summing points are used to add two or more signals in the system.

It is a circle with a cross is the symbol for summing point operation.



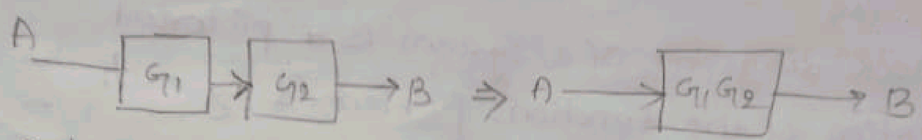


3. **Branch Point:** It is a point from which the signal from a block goes concurrently to other blocks or summing points.

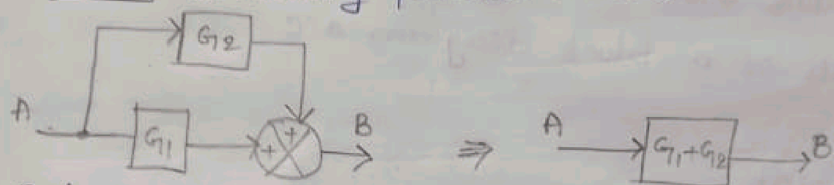


Rules for Block diagram reduction

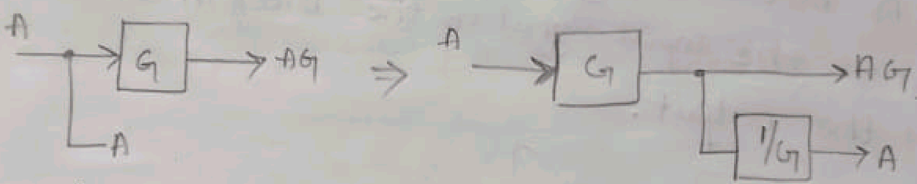
Rule 1: combinding the blocks in cascate.



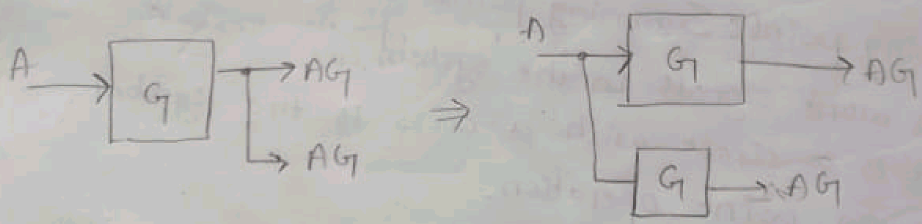
Rule 2: Combining parallel blocks.



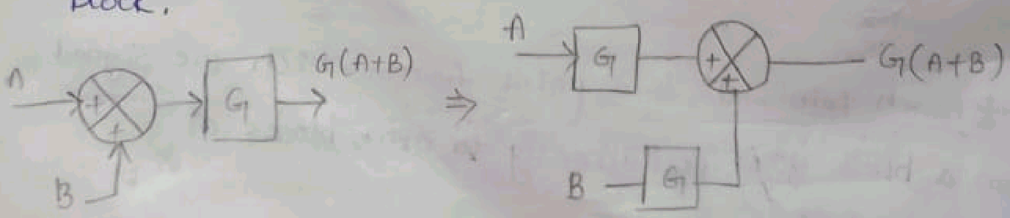
Rule 3: Move the branch point a head of a block.



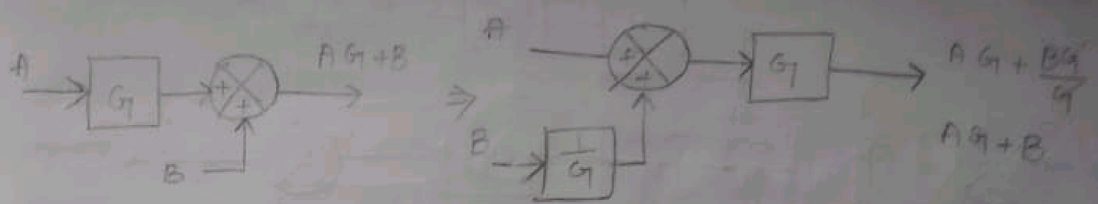
Rule 4: Moving branch point before the block.



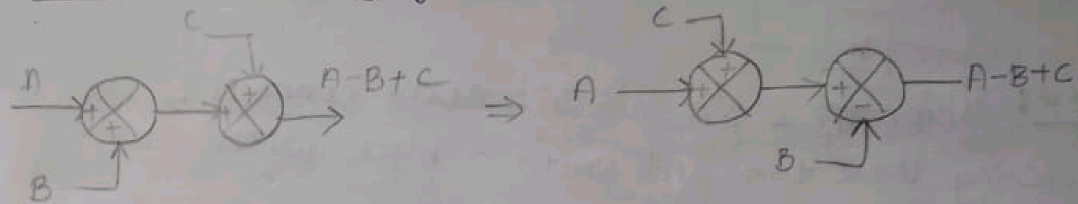
Rule 5: Moving the Summing point a head of the block.



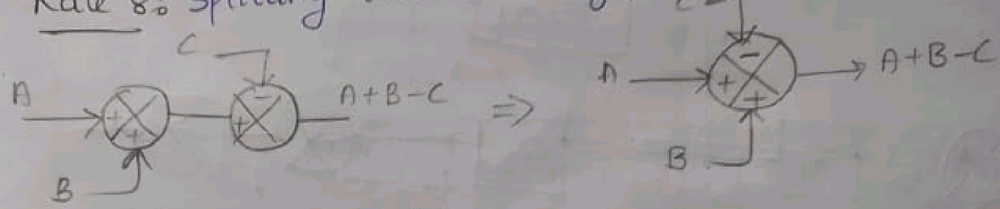
Rule 6: Moving the summing point before the block



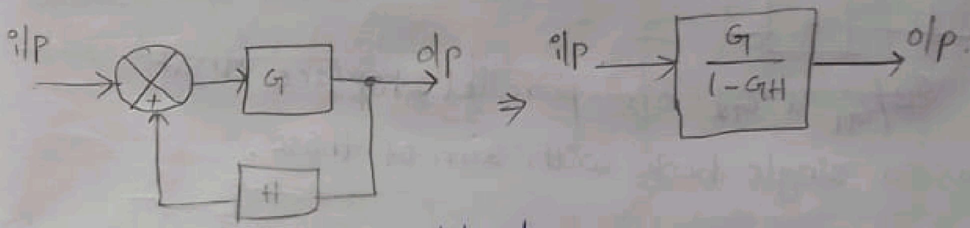
Rule 7: Interchanging the summing point.



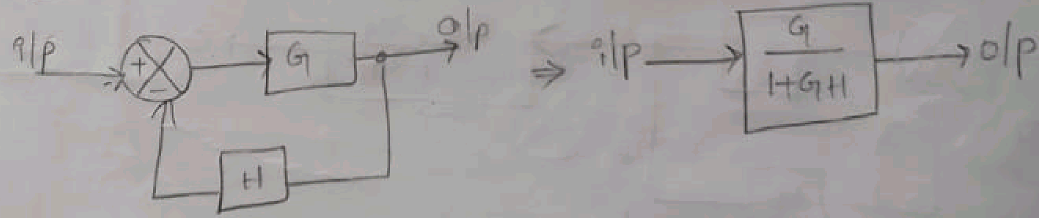
Rule 8: Splitting the summing point.



Rule 10: Positive feed back.

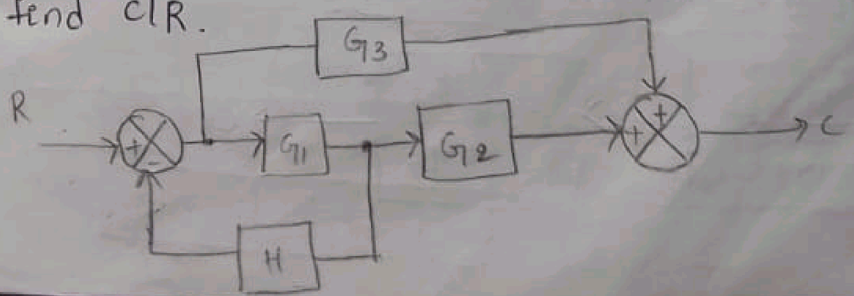


Rule 11: negative feed back.

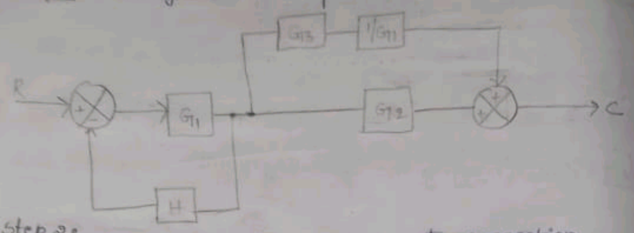


Examples:

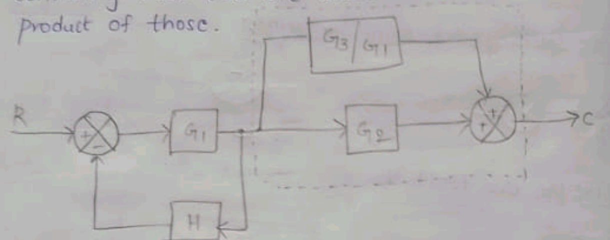
Reduce the block diagram shown in the diagram & find C/R.



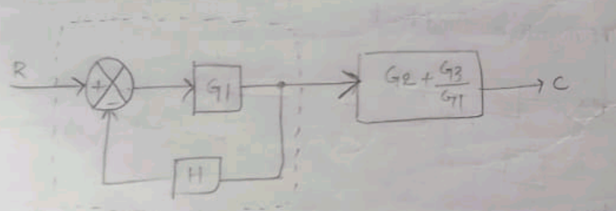
Step 1: Moving the branch point ahead of the block G_1.



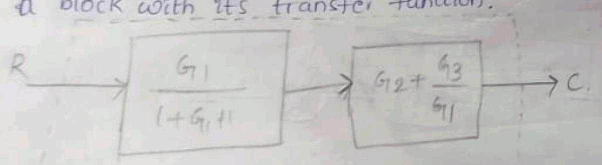
Step 2: Here G_3 & $1/G_1$ are in cascate connection combining these two and write a block with product of those.



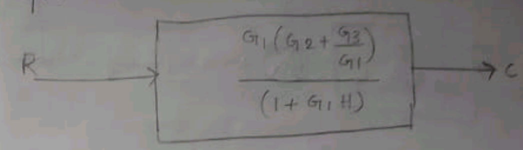
Step 3: G_3/G_1 & G_2 are parallel, replace that blocks as a single bock with sum of those.



Step 4: Eliminate the negative feed back & replace a block with its transfer function.



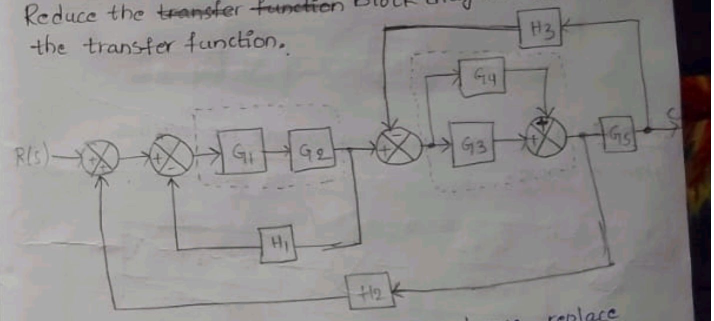
Step 5: Remaining two are in cascate connection hence replace the block with a product of those.



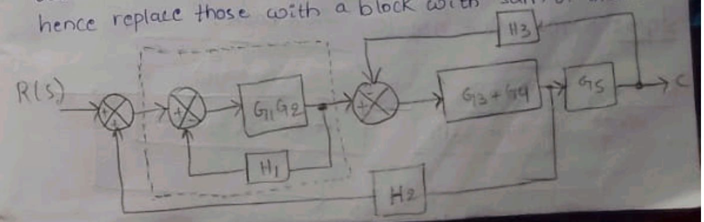
$$\frac{C}{R} = \frac{G_1\left(\frac{G_2G_1+G_3}{G_1}\right)}{(1+G_1H)}$$

$$\boxed{\frac{C}{R} = \frac{\cancel{G_1}(G_2G_1+G_3)}{\cancel{G_1}(1+G_1H)}}$$

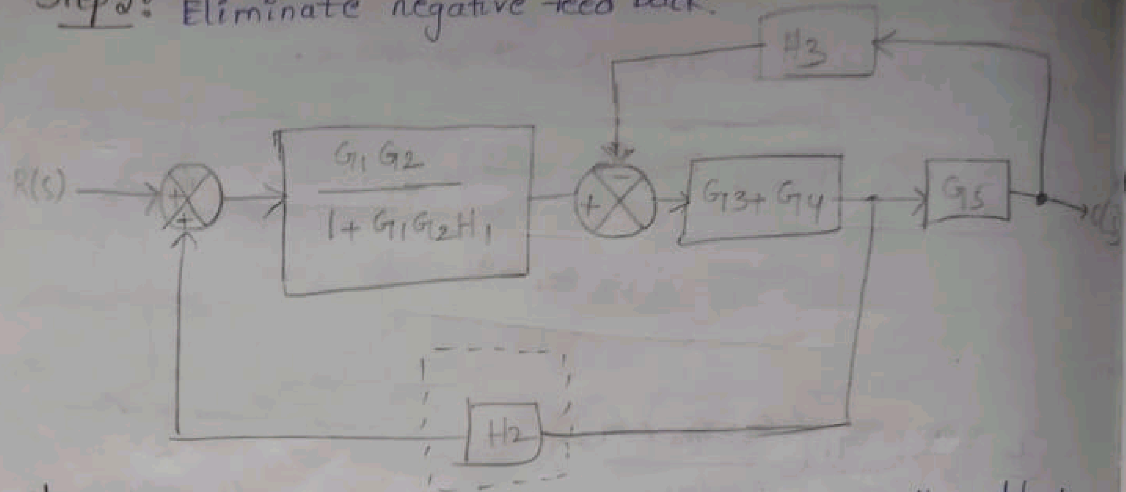
Reduce the ~~transfer function~~ block diagram obtain the transfer function.



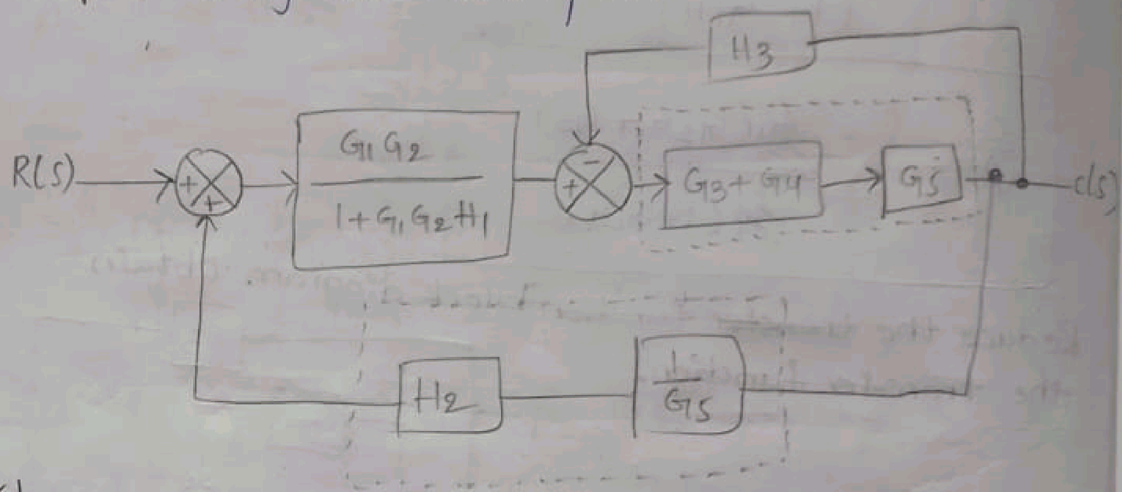
Step 1: Here G_1 & G_2 are in series hence replace that block with product of those. G_3, G_4 are parallel hence replace those with a block with sum of those.



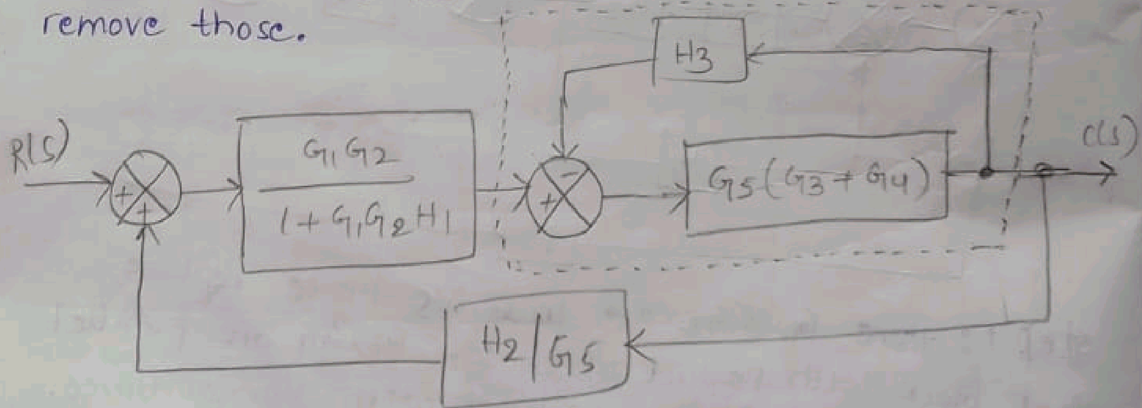
Step 2: Eliminate negative feed back.



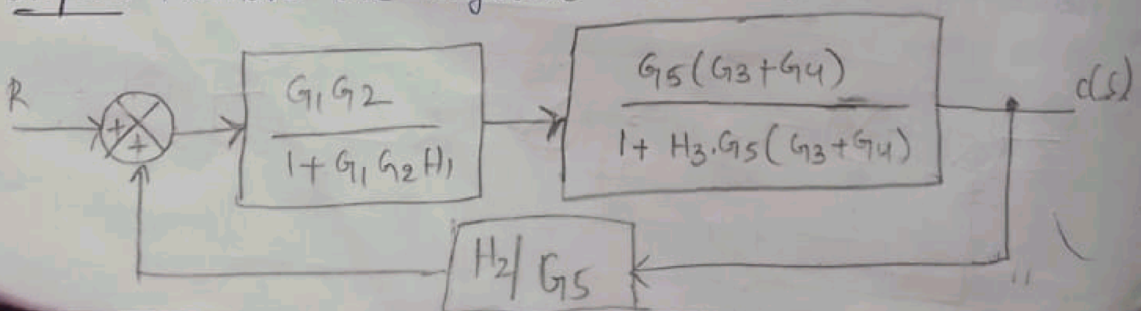
Step 3: Moving the branch point a head of the block.



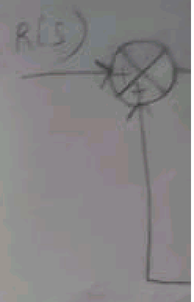
Step 4: Here, we have two cascate connections remove those.



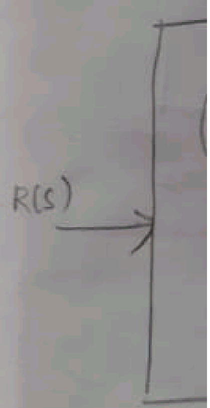
step 5: Remove the negative feed back.



step 6: T



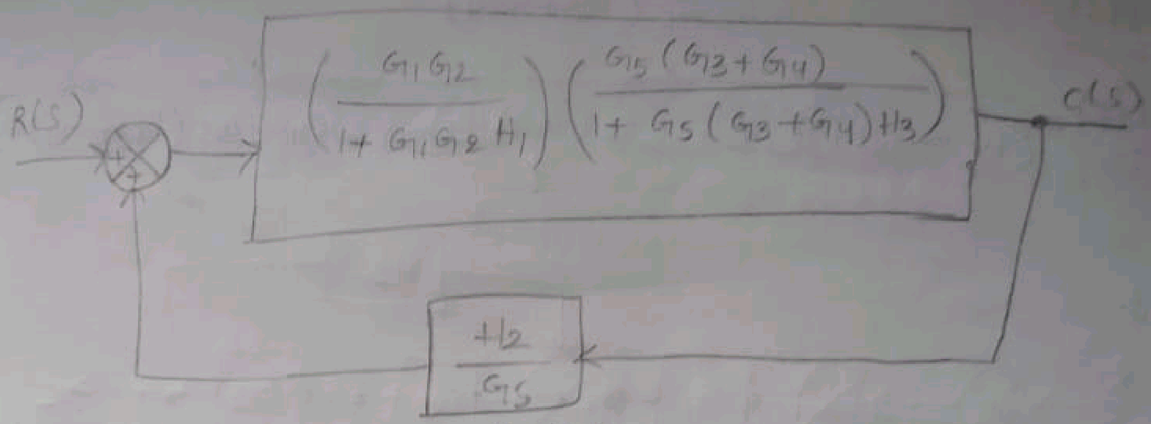
step 7:



$\frac{C(s)}{R(s)} =$

$= \frac{\quad}{(1+G}$

step 6: Two blocks are in cascate connection.



step 7: Remove the ~~negative~~ Postive feed back.

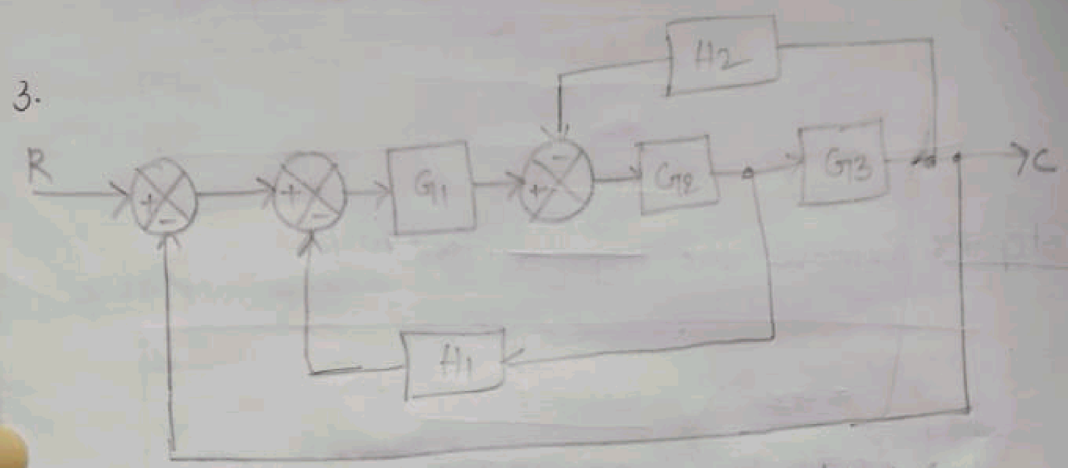
$$R(s) \rightarrow \boxed{\dfrac{\left(\dfrac{G_1G_2}{1+G_1G_2H_1}\right)\left(\dfrac{G_5(G_3+G_4)}{1+G_5(G_3+G_4)H_3}\right)}{1-\left(\dfrac{G_1G_2}{1+G_1G_2H_1}\right)\left(\dfrac{G_5(G_3+G_4)}{1+G_5(G_3+G_4)H_3}\right)\dfrac{H_2}{G_5}}} \rightarrow c(s)$$

$$\frac{C(s)}{R(s)} = \frac{\left(\dfrac{G_1G_2}{1+G_1G_2H_1}\right)\left(\dfrac{G_5(G_3+G_4)}{1+G_5(G_3+G_4)H_3}\right)}{1-\left(\dfrac{G_1G_2}{1+G_1G_2H_1}\right)\left(\dfrac{G_5(G_3+G_4)}{1+G_5(G_3+G_4)H_3}\right)\dfrac{H_2}{G_5}}$$

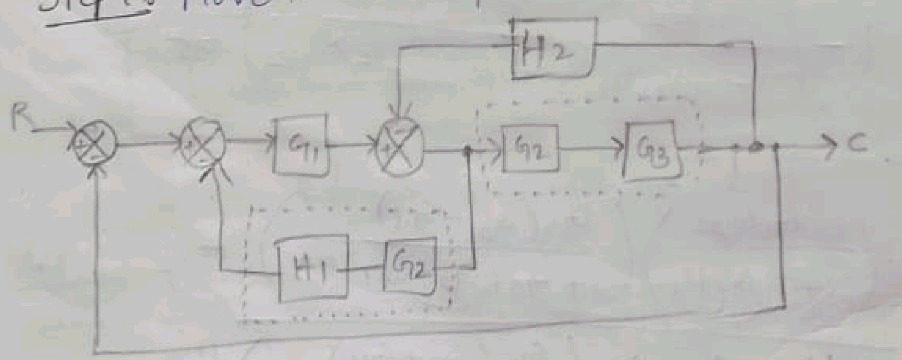
$$= \frac{\dfrac{(G_1G_2)G_5(G_3+G_4)}{(1+G_1G_2H_1)(1+G_5(G_3+G_4)H_3)}}{\dfrac{(1+G_1G_2H_1)(1+G_5(G_3+G_4)H_3)G_5-(G_1G_2)(G_5(G_3+G_4))H_2}{(1+G_1G_2H_1)(1+G_5(G_3+G_4)H_3)G_5}}$$

$$\frac{C(s)}{R(s)} = \frac{G_1 G_2 G_5 (G_3 + G_4)}{(1+G_1G_2H_1)(1+G_5(G_3+G_4)H_3)G_5 - G_1G_2G_5(G_3+G_4)H_2}$$

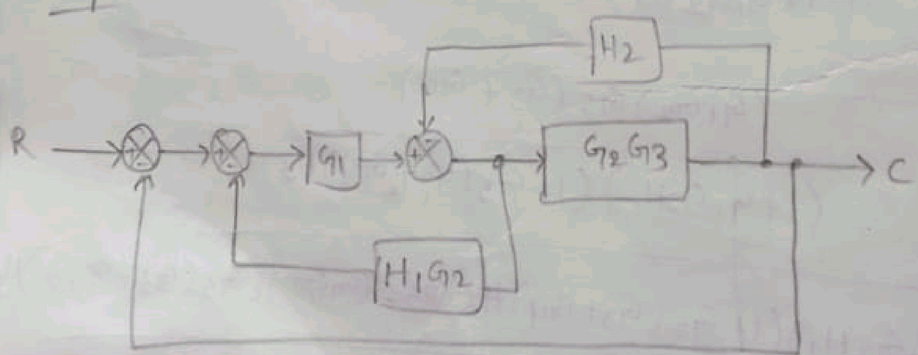
3.



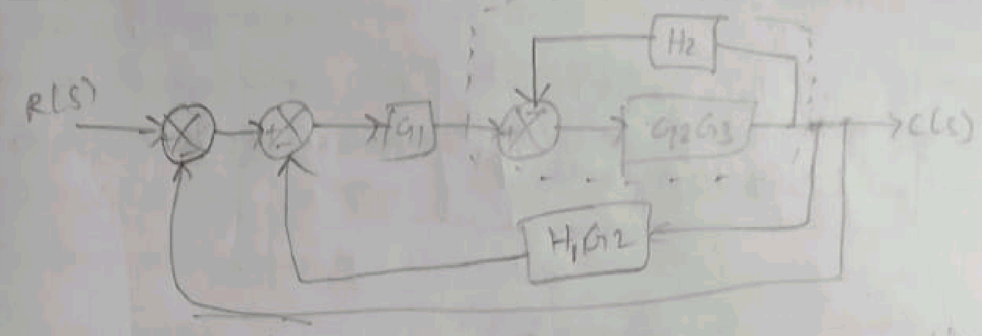
Step 1: Move the branch point before the block G_2.



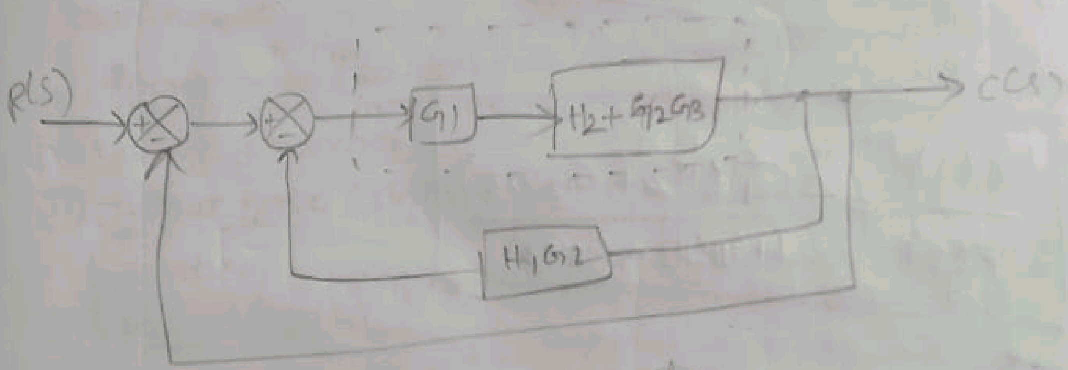
Step 2: Here H_1, G_2 and G_2, G_3 are in cascade connection.



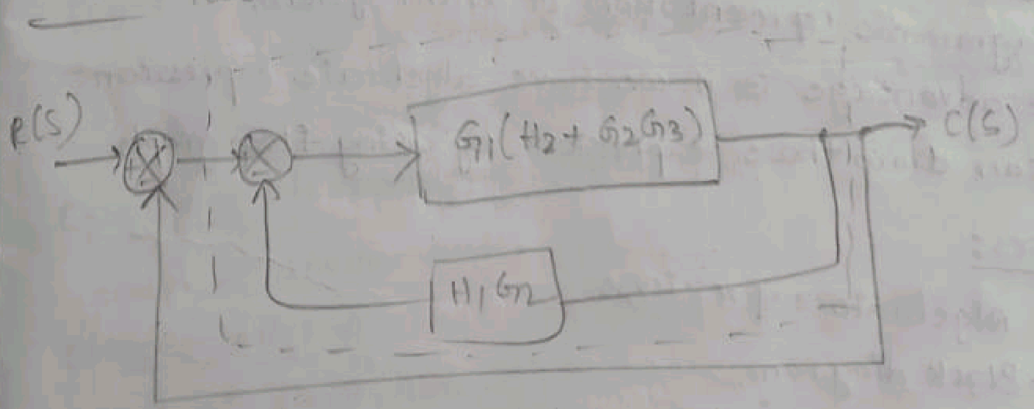
step ③ : move the branch point ahead of the block



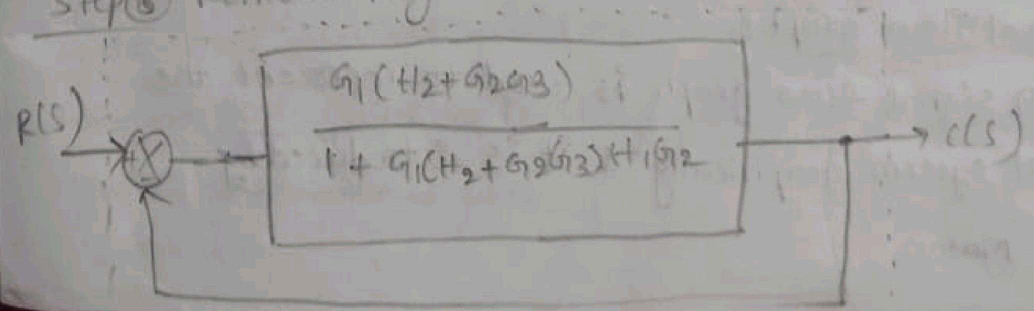
step ④ Here two blocks are parallel replace those with a single block.



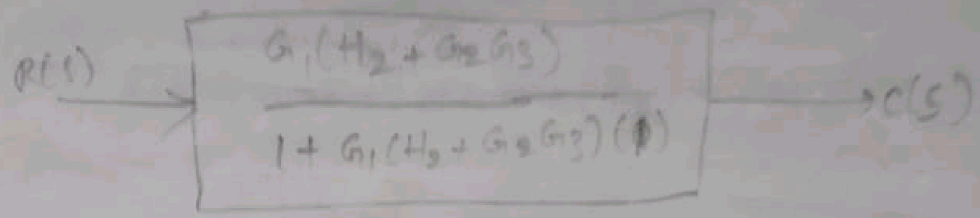
step ⑤ Two blocks are in cascade

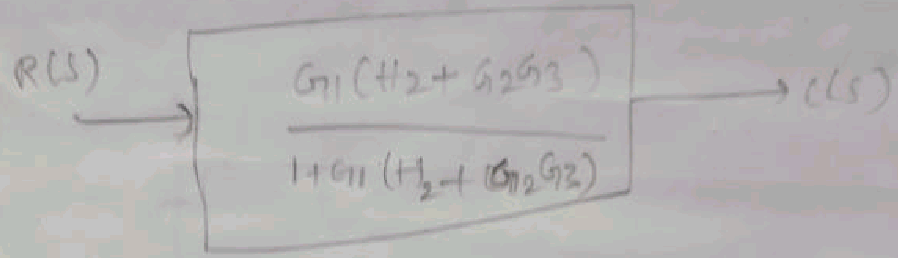


step ⑥ Remove negative feed feedback



Step 4 Remove negative feed back





$$\boxed{\frac{C(s)}{R(s)} = \frac{G_1(H_2+G_2G_3)}{1+G_1(H_2+G_2G_3)}}$$

*****
Signal flow graph

⇒ Diagramatic representation of control system.

⇒ The advantage is, if we have algebraic expression we can draw those expressions by using flow graph.

Types:

1. Algebraic expressions
2. Block diagram.
3. Direct signal flow graph.

Signal flow graph:

⇒ A signal flow graph is used to represent the control system graphically & it is introduced by SJ Mason.

⇒ Signal flow graph is a diagram that represents a set of linear algebraic expressions.

⇒ In a signal flow graph the signal flows in only one direction. The direction of the signal is indicated by an arrow placed on the branch & the gain is indicated along the branch.

Important terms in signal flow graph

1. Node: A node is a point representing a variable or a signal.

2. Branch: A branch is directed line segment joining two nodes.

(i) Input node (source): It is a node with only outgoing branches.

(ii) Output node (sink): It is a node with incoming branches.

(iii) Mixed node: It is a node with both incoming & outgoing branches.

Path: A path is traversal of connected branches in the direction of branch arrows.

The path should not cross a node more than ones.

Open path: A open path starts at one node & ends at another node.

Closed path: A closed path starts at one node & ends at same node.

Forward path: It is a path from the input to the o/p node that does not cross any node more than ones.

Forward path gain: It is the product of the branch gains of a forward path.

Individual loops: It is a closed path starting from a node & after passing through a certain path of a graph arrives at same node without crossing any node more than ones.

loop gain: It is the product of the branch gains of a loop.

Non-touching loops: If the loops doesn't have a common node then they are said to non-touching loops.

Signal flow graph reduction:

Mason's gain formula: This law is used to determine the transfer function of the system from the signal flow graph.

=) The Mason's gain formula states that overall gain of the system.

$$T = \frac{1}{\Delta} \sum_{K} P_K \Delta_K$$

where,

K = number of forward paths

T = transfer function of overall gain.

P_k = forward path gain of overall k^{th} forward path.

$$\Delta = 1 - \begin{pmatrix}\text{Sum of the}\\ \text{gains of}\\ \text{individual}\\ \text{loops}\end{pmatrix} + \begin{pmatrix}\text{Sum of gain of}\\ \text{Products of}\\ \text{all possible}\\ \text{two non-touching}\\ \text{loops}\end{pmatrix} - \begin{pmatrix}\text{Sum of gain}\\ \text{Products of}\\ \text{all possible}\\ \text{three non-}\\ \text{touching loops}\end{pmatrix} + \cdots\cdots$$

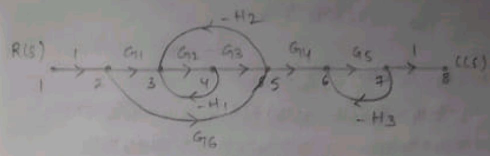
$\Delta_K = \Delta$ of forward path K.

Δ_k is the Δ for the part of the graph which is not touching k^{th} forward path.

Procedure for signal flow graph.

1. Determine the forward path & calculate the forward path gain.
2. Identify the individual loops & calculate the loop gain.
3. Determine the gain product of two-non touching loops.
4. Calculate Δ & Δ_K.
5. Using the Mason's formula obtain the transfer function.

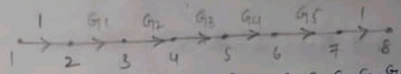
Find the overall transfer function of the system whose signal flow graph is shown in the diagram.



k = no. of forward paths = 2.

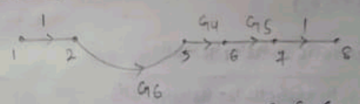
Step ①:-

Forward path 1:



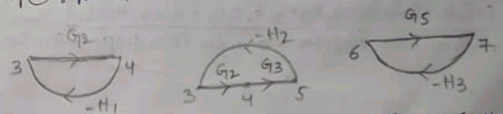
forward path 1 gain $P_1 = G_1G_2G_3G_4G_5$

Forward path 2:



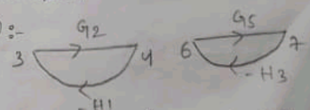
Forward path 2 gain $P_2 = G_4G_5G_6$.

Step② : No. of individual loops are 3.

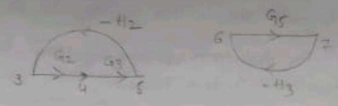


$P_{11} = -G_2H_1$ $P_{21} = -G_2G_3H_2$ $P_{31} = -G_5H_3$

Step③ :-



$P_{12} = G_2G_5H_1H_3$



step④:

$$P_{13} = G_2G_3G_5H_2H_3$$

$$\Delta = 1-(P_{11}+P_{21}+P_{31})+(P_{12}+P_{13})$$
$$= 1-(-G_2H_1-G_2G_3H_2-G_5H_3)+(G_2G_5H_1H_3+G_2G_3G_5H_2H_3$$
$$= 1+G_2H_1+G_2G_3H_2+G_5H_3+G_2G_5H_1H_3+G_2G_3G_5H_2H_3$$

$$\Delta_1 = 1-(0)+(0)$$
$$=1$$
$$\Delta_2 = 1-(-G_2H_1)$$
$$= 1+G_2H_1$$

step⑤:

We know that, Mason's formula is

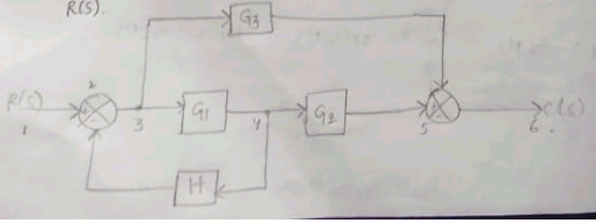
$$T = \frac{1}{\Delta}\sum_{K=2} P_K\Delta_K$$

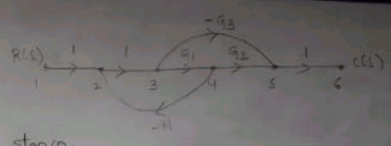
$$= \frac{1}{\Delta}(P_1\Delta_1+P_2\Delta_2)$$

$$= \frac{G_1G_2G_3G_4G_5+G_4G_5G_6+G_2G_4G_5G_6H_1}{1+G_2H_1+G_2G_3H_2+G_5H_3+G_2G_5H_1H_3+G_2G_3G_5H_2H_3}$$

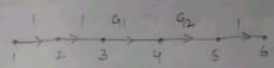
$$\boxed{\frac{C(s)}{R(s)} = \frac{G_1G_2G_3G_4G_5+G_4G_5G_6+G_2G_4G_5G_6H_1}{1+G_2H_1+G_2G_3H_2+G_5H_3+G_2G_5H_1H_3+G_2G_3G_5H_2H_3}}$$

convert the given block diagram into signal flow graph determine $\frac{C(s)}{R(s)}$.





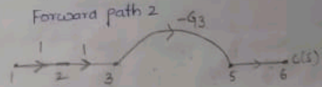
step①: Forward path 1



gain of forward path 1

$$P_1 = G_1G_2$$

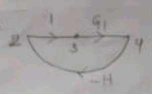
Forward path 2



gain of forward path 2

$$P_2 = -G_3$$

step②:



$$P_{11} = -HG_1$$

step③: There is no two non touching loops.

step④:

$$\Delta = 1-(P_{11})+0 = 1-(-HG_1)$$
$$= 1+HG_1$$
$$\Delta_1 = 1-0 = 1$$
$$\Delta_2 = 1-0 = 1$$

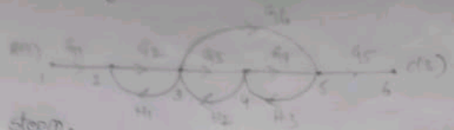
step⑤:

$$T = \frac{1}{\Delta}\sum_{K=2} P_K\Delta_K$$

$$= \frac{1}{\Delta}(P_1\Delta_1+P_2\Delta_2)$$

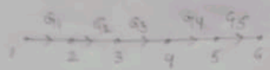
$$\boxed{\frac{C(s)}{R(s)} = \frac{G_1G_2-G_3}{1+G_1H}}$$

The signal flow graph for a feed back control system is shown in the figure. determine the closed loop transfer, function $\frac{C(s)}{R(s)}$



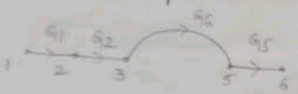
step① :

Forward path 1.



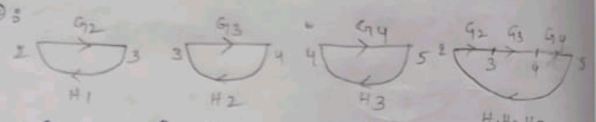
$P_1 = G_1G_2G_3G_4G_5$

Forward path 2



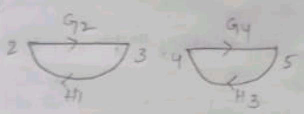
$P_2 = G_1G_2G_5G_6$

step② :



$P_{12} = G_2H_1$ $\quad P_{13} = G_3H_2$ $\quad P_{14} = G_4H_3$ $\quad P_{15} = G_2G_3G_4H_1H_2H_3$

step③ :



$P_{22} = G_2G_4H_1H_3$

step④ :

$\Delta = 1-(P_{12}+P_{13}+P_{14}+P_{15}) + P_{22}$

$= 1-(G_2H_1 + G_3H_2 + G_4H_3 + G_2G_3G_4H_1H_2H_3) + G_2G_4H_1H_3$

$= 1-G_2H_1 - G_3H_2 - G_4H_3 - G_2G_3G_4H_1H_2H_3 + G_2G_4H_1H_3$

$\Delta_1 = 1-0 = 1$

$\Delta_2 = 1-0 = 1$

step⑤ :

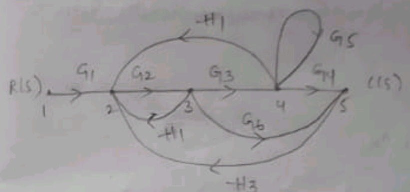
$$T = \frac{1}{\Delta}\sum_{K=2} P_K\Delta_K$$

$$= \frac{P_1\Delta_1 + P_2\Delta_2}{\Delta}$$

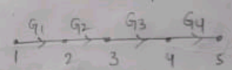
$$\frac{C(s)}{R(s)} = \frac{G_1G_2G_3G_4G_5 + G_1G_2G_5G_6}{1-G_2H_1 - H_3G_4 - G_3H_2 - G_2G_3G_4H_1H_2H_3 + G_2G_4H_1H_3}$$

$$\boxed{\frac{C(s)}{R(s)} = \frac{G_1G_2G_5(G_3G_4+G_6)}{1-H_1G_2(1+G_3G_4H_2H_3) - H_3G_4(1+G_2H_1) - G_3H_2}}$$

Find the overall gain $\frac{C(s)}{R(s)}$ for the signal flow graph shown in the diagram.



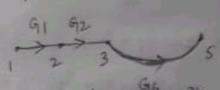
Step①: Forward path 1



Forward path 1 gain

$P_1 = G_1G_2G_3G_4$

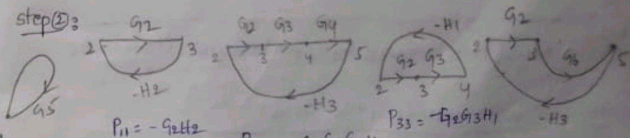
Forward path 2



Forward path 2 gain

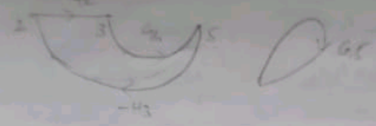
$P_2 = G_1G_2G_6$

step② :

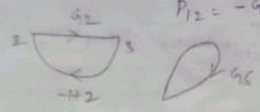


$P_{55} = G_5$ $\quad P_{11} = -G_2H_2$ $\quad P_{22} = -G_2G_3G_4H_3$ $\quad P_{33} = -G_2G_3H_1$ $\quad P_{44} = -G_2G_6H_3$

step ③ :



$P_{12} = -G_2G_5G_6H_3$



$P_{13} = -G_2G_5H_2$

step ④ : $\Delta = 1 - [P_{11} + P_{22} + P_{33} + P_{44} + P_{55}] + [P_{12} + P_{13}]$

$\Delta = 1 - [-G_2H_2 - G_2G_3G_4H_3 - G_2G_3H_1 - G_2G_6H_3 + G_5] + [-G_2G_5G_6H_3 - G_2G_5H_2]$

$= 1 + G_2H_2 + G_2G_3G_4H_3 + G_2G_3H_1 + G_2G_6H_3 - G_5 - G_2G_5G_6H_3 - G_2G_5H_2$

$\Delta_1 = 1 - (0) + (0) = 1$

$\Delta_2 = 1 - (G_5) + 0 = 1 - G_5$

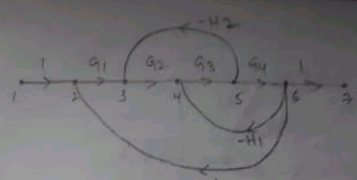
step ⑤ : $T = \frac{1}{\Delta} \sum_{k=2} P_K \Delta_K$

$= \frac{P_1\Delta_1 + P_2\Delta_2}{\Delta}$

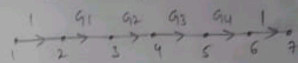
$= \frac{G_1G_2G_3G_4 + G_1G_2G_6 - G_1G_2G_5G_6}{1 + H_2G_2 + G_2G_3G_4H_3 + G_2G_3H_1 + G_2G_6H_3 - G_5 - G_2G_6G_5H_3 - G_2G_5H_2}$

$\frac{C(s)}{R(s)} = \frac{G_1G_2(G_3G_4 + G_6 - G_5G_6)}{1 + G_2H_2(1 - G_5) + G_2G_3(G_4H_3 + H_1) + G_2G_6H_3(1 - G_5) - G_5}$

$\frac{C(s)}{R(s)} = \frac{G_1G_2(G_3G_4 + G_6 - G_5G_6)}{1 + (G_2H_2 + G_2G_6H_3)(1 - G_5) + G_2G_3(G_4H_3 + H_1) - G_5}$

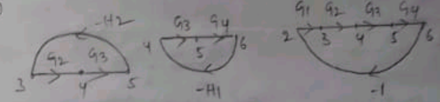


step ① Forward path 1



$P_1 = G_1G_2G_3G_4$

step ②



$P_{11} = -G_2G_3H_2$ $P_{22} = -G_3G_4H_1$ $P_{33} = -G_1G_2G_3G_4$

step ③ 0

step ④ $\Delta = 1 - [P_{11} + P_{22} + P_{33}] + 0$

$= 1 - \{-G_2G_3H_2 - G_3G_4H_1 - G_1G_2G_3G_4\}$

$= 1 + G_2G_3H_2 + G_3G_4H_1 + G_1G_2G_3G_4$

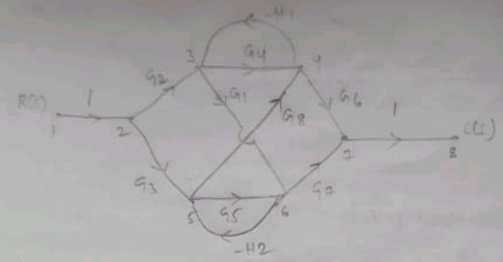
$\Delta_1 = 1 - (0) + (0) = 1$

step ⑤ : $T = \frac{1}{\Delta} \sum_{k=1} P_K \Delta_K$

$= \frac{P_1\Delta_1}{\Delta}$

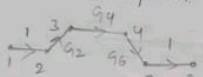
$= \frac{G_1G_2G_3G_4}{1 + G_2G_3H_2 + G_3G_4H_1 + G_1G_2G_3G_4}$

$\frac{C(s)}{R(s)} = \frac{G_1G_2G_3G_4}{1 + G_2G_3(H_2 + G_1G_4) + G_3G_4H_1}$



step ①:

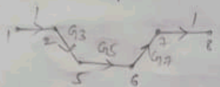
Forward path 1.



Forward path 1 gain

$P_1 = G_2G_4G_6$

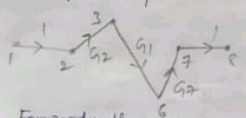
Forward path 2



Forward path 2 gain

$P_2 = G_3G_5G_7$

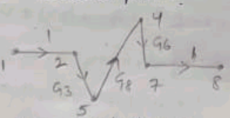
Forward path 3.



Forward path 3 gain

$P_3 = G_1G_2G_7$

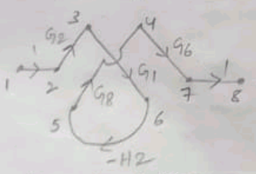
Forward path 4.



Forward path 4 gain

$P_4 = G_3G_6G_8$

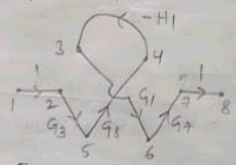
Forward path 5



Forward path 5 gain

$P_5 = -G_1G_2G_6G_8H_2$

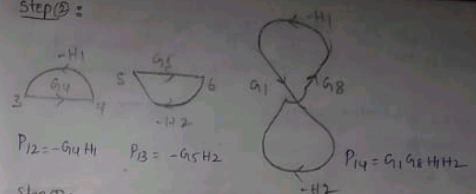
Forward path 6.



Forward path 6 gain

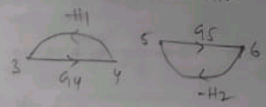
$P_6 = -G_1G_3G_7G_8H_1$

step ②:



$P_{12} = -G_4H_1$ $\quad P_{13} = -G_5H_2$ $\quad P_{14} = G_1G_8H_1H_2$

step ③:



$P_{11} = G_4G_5H_1H_2$

step ④: $\Delta = 1-[P_{12}+P_{13}+P_{14}]+[P_{11}]$

$= 1-[-G_4H_1-G_5H_2+G_1G_8H_1H_2]+G_4G_5H_1H_2$

$= 1+G_4H_1+G_5H_2-G_1G_8H_1H_2+G_4G_5H_1H_2$

$\Delta_1 = 1-(-G_5H_2) = 1+G_5H_2$

$\Delta_2 = 1-(-G_4H_1) = 1+G_4H_1$

$\Delta_3 = 1-(0) = 1$, $\Delta_5 = 1-(0) = 1$

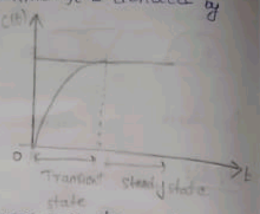
$\Delta_4 = 1-(0) = 1$, $\Delta_6 = 1-(0) = 1$

step ⑤: $T = \frac{1}{\Delta}\sum_{K=6} P_K\Delta_K$

$$= \frac{P_1\Delta_1+P_2\Delta_2+P_3\Delta_3+P_4\Delta_4+P_5\Delta_5+P_6\Delta_6}{\Delta}$$

$$\frac{C(s)}{R(s)} = \frac{G_2G_4G_6(1+G_5H_2)+G_3G_5G_7(1+G_4H_1)+G_1G_2G_7+G_3G_6G_8-G_1G_2G_6G_8H_2-G_1G_3G_7G_8H_1}{1+G_4H_1+G_5H_2-G_1G_8H_1H_2+G_4G_5H_1H_2}$$

Time response: The time response of the system is the output of the control system as a function of time. It is denoted by "$c(t)$".



The time response of a control system consists of two parts
1. Transient state
2. Steady state.

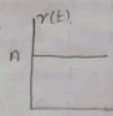
1. Transient state: the temporary behavior as it changes from initial to final state
2. Steady state: The long term behavior after transients fade, reflecting its final, stable operating point.

Test signals:

The commonly used test signals are
1. Impulse signal 2. Step signal 3. Ramp signal
4. Parabolic signal 5. Sinusoidal signal.

Step signal: the step signal is a signal whose values changes from 0 to A at $t=0$ & remains constant for $t>0$. The mathematical representation is

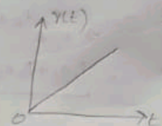
$$r(t) = \begin{cases} A & \text{for } t \geq 0 \\ 0 & \text{for } t < 0 \end{cases}$$



Ramp signal: The ramp signal is the signal whose values increases linearly with time from 0 at $t=0$.

Mathematical expression is

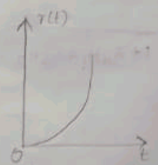
$$r(t) = \begin{cases} At & \text{for } t \geq 0 \\ 0 & \text{for } t < 0 \end{cases}$$



Parabolic signal: In parabolic signal the instantaneous value varies as square of the time from initial value of 0 at $t=0$.

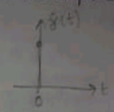
The mathematical expression is

$$p(t) = \begin{cases} \frac{At^2}{2} & \text{for } t \geq 0 \\ 0 & \text{for } t < 0 \end{cases}$$



Impulse signal: A signal of very large magnitude which is available for short duration is called impulse signal. The impulse signal is mathematically expressed as

$$\delta(t) = \begin{cases} \infty & \text{for } t = 0 \\ 0 & \text{for } t \neq 0 \end{cases}$$



Name of a signal	Time domain eq^n of a signal	Laplace transform of a signal.
Unit step	1	$\frac{1}{s}$
Step	A	$\frac{A}{s}$
Ramp	At	$\frac{A}{s^2}$
unit Ramp	t	$1/s^2$
Parabolic	$At^2/2$	A/s^3
unit parabolic	$t^2/2$	$1/s^3$
Impulse	$\delta(t)$	1

Order of a system: The input & output relationship of a control system can be expressed by n^{th} order differential eq^n as shown in the eq^n below.

$$a_0 \frac{d^n}{dt^n} p(t) + a_1 \frac{d^{n-1}}{dt^{n-1}} p(t) + a_2 \frac{d^{n-2}}{dt^{n-2}} p(t) + \cdots + a_{n-1} \frac{d}{dt} p(t) + a_n p(t)$$

$$= b_0 \frac{d^m}{dt^m} q(t) + b_1 \frac{d^{m-1}}{dt^{m-1}} q(t) + \cdots + b_{m-1} \frac{d}{dt} q(t) + b_m q(t)$$

where $p(t)$ is the output & $q(t)$ is the input.

In s-domain

$$a_0 s^n p(s) + a_1 s^{n-1} p(s) + a_2 s^{n-2} p(s) + \cdots + a_{n-1} s\, p(s) + a_n p(s)$$

$$= b_0 s^m q(s) + b_1 s^{m-1} q(s) + \cdots + b_{m-1} s\, q(s) + b_m q(s)$$

To obtain transfer function

$$p(s)\left[a_0 s^n + a_1 s^{n-1} + \cdots + a_{n-1}s + a_n\right] = q(s)\left[b_0 s^m + b_1 s^{m-1} + \cdots + b_{m-1}s + b_m\right]$$

$$\boxed{\frac{p(s)}{q(s)} = \frac{b_0 s^m + b_1 s^{m-1} + \cdots + b_{m-1}s + b_m}{a_0 s^n + a_1 s^{n-1} + \cdots + a_{n-1}s + a_n}}$$

The order of the system is nth order.

where, $p(s)$ is the numerator polynomial, $q(s)$ is the denominator polynomial.

* The above transfer function can be expressed in the factorised form as shown below.

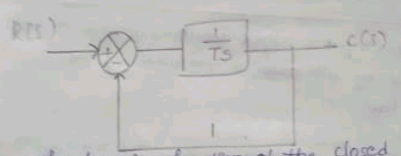
$$\boxed{\frac{P(s)}{Q(s)} = \frac{(s+z_1)(s+z_2)(s+z_3)\cdots\cdots(s+z_m)}{(s+p_1)(s+p_2)(s+p_3)\cdots\cdots(s+p_4)}}$$

where,
$z_1, z_2, z_3, \ldots z_m$ are zeroes
$p_1, p_2, p_3, \ldots p_n$ are poles

Find the response of first order system for unit step input with negative feedback

Consider a closed loop system with unity feedback.

$$\left\{\frac{C(s)}{R(s)} = \frac{G}{1+GH}\right\}$$



The transfer function of the closed loop system is

$$\frac{C(s)}{R(s)} = \frac{\frac{1}{Ts}}{1+\frac{1}{Ts}(1)} = \frac{\frac{1}{Ts}}{\frac{Ts+1}{Ts}}$$

$$= \frac{1}{Ts+1} = \frac{1}{T\left(s+\frac{1}{T}\right)}$$

$$\boxed{\frac{C(s)}{R(s)} = \frac{1/T}{s+\frac{1}{T}}} \longleftarrow (1)$$

For unit step signal,

$r(t) = 1$, $R(s) = 1/s$

From eqⁿ (1)

$$C(s) = R(s)\,\frac{1/T}{s+1/T}$$

$$C(s) = \frac{1/T}{s(s+1/T)} \qquad \left[\because R(s) = \frac{1}{s}\right]$$

By using partial expansion method

$$C(s) = \frac{A}{s} + \frac{B}{s+\frac{1}{T}} \rightarrow (2)$$

$$A = C(s)\times s\Big|_{s=0} = \frac{\frac{1}{T}}{\cancel{s}\left(s+\frac{1}{T}\right)}\times\cancel{s}\Bigg|_{s=0} = \frac{\frac{1}{T}}{s+\frac{1}{T}}\Bigg|_{s=0} = 1$$

$$B = C(s)\times s+\frac{1}{T}\Big|_{s=\frac{-1}{T}} = \frac{\frac{1}{T}}{s\cancel{\left(s+\frac{1}{T}\right)}}\cancel{\left(s+\frac{1}{T}\right)}\Bigg|_{s=\frac{-1}{T}} = \frac{\frac{1}{T}}{s}\Bigg|_{s=\frac{-1}{T}} = -1$$

substitute A, B in eqⁿ (2)

$$C(s) = \frac{1}{s} - \frac{1}{s+\frac{1}{T}}$$

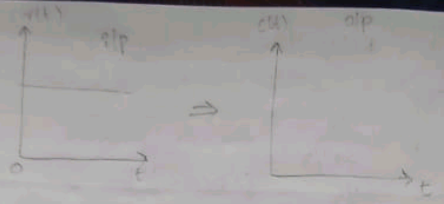
Apply inverse laplace transform on B.S

$$L^{-1}[C(s)] = L^{-1}\left[\frac{1}{s}\right] - L^{-1}\left[\frac{1}{s+\frac{1}{T}}\right]$$

$L^{-1}\left[\frac{1}{s}\right] = 1$
$L^{-1}\left[\frac{1}{s+a}\right] = e^{-at}$

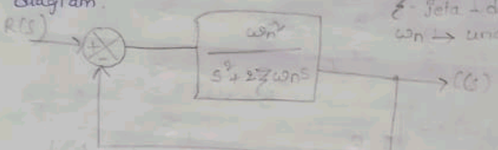
$$\boxed{c(t) = 1 - e^{-\frac{t}{T}}}$$

The above expression $c(t)$ is the time response for the closed loop system with unit step signal as input.



Second order system

Consider the closed loop control system shown in the diagram.



ζ - zeta → damping ratio
ω_n → undamping natural frequency

$$\frac{C(s)}{R(s)} = \frac{\dfrac{\omega_n^2}{s^2+2\zeta\omega_n s}}{1+\dfrac{\omega_n^2}{s^2+2\zeta\omega_n s}}$$

$$= \frac{\dfrac{\omega_n^2}{s^2+2\zeta\omega_n s}}{\dfrac{s^2+2\zeta\omega_n s+\omega_n^2}{s^2+2\zeta\omega_n s}}$$

$$\boxed{\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2}}$$

1. $\zeta = 0$, undamped
2. $0<\zeta<1$, underdamped.
3. $\zeta = 1$, critically damped.
4. $\zeta > 1$, over damped.

characteristic equation

$$s^2+2\zeta\omega_n s+\omega_n^2$$

$$s = \frac{-b\pm\sqrt{b^2-4ac}}{2a} = \frac{-2\zeta\omega_n\pm\sqrt{4\zeta^2\omega_n^2-4\omega_n^2}}{2(1)}$$

$$= \frac{-2\zeta\omega_n\pm\sqrt{4\omega_n^2(\zeta^2-1)}}{2}$$

$$= \frac{-2\zeta\omega_n\pm 2\omega_n\sqrt{\zeta^2-1}}{2}$$

$$\boxed{s_1, s_2 = -\zeta\omega_n\pm\omega_n\sqrt{\zeta^2-1}}$$

i) when $\zeta = 0$.

$$s_1, s_2 = -0(\omega_n)\pm\omega_n\sqrt{0-1}$$

$$= \omega_n\sqrt{-1} = \pm\omega_n j$$

$$\boxed{s_1, s_2 = \pm j\omega_n}$$

roots are purely complex & system is undamped.

ii) when $\zeta = 1$

$$s_1, s_2 = -1(\omega_n)\pm\omega_n\sqrt{1-1}$$

$$s_1, s_2 = -\omega_n$$

roots are real & same & system is critically damped.

(iii) when $\zeta > 1$

$$\boxed{s_1, s_2 = -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2-1}}$$

roots are ~~complex conjugate~~ real & unequal & system is over damped

(iv) when $0 < \zeta < 1$

$$s_1, s_2 = -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2-1}$$

$$= -\zeta\omega_n \pm \omega_n\sqrt{(-1)[1-\zeta^2]}$$

$$= -\zeta\omega_n \pm \omega_n\sqrt{j^2[1-\zeta^2]}$$

$$\boxed{s_1, s_2 = -\zeta\omega_n \pm j\omega_n\sqrt{1-\zeta^2}}$$

roots are complex conjugate & system is underdamped.

Find the time response of undamped second order system for unit step input.

We know that,

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2}$$

For undamped second order system $\zeta = 0$.

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2+\omega_n^2}$$

Here, For unit step signal $R(s) = \frac{1}{s}$

$$C(s) = R(s)\cdot\frac{\omega_n^2}{s^2+\omega_n^2}$$

$$C(s) = \frac{\omega_n^2}{s(s^2+\omega_n^2)}$$

By partial expansion method

$$C(s) = \frac{A}{s} + \frac{B}{s^2+\omega_n^2}$$

$s^2+\omega_n^2=0$

$$A = C(s)\times s\Big|_{s=0} \qquad B = C(s)\times(s^2+\omega_n^2)\Big|_{s=-\omega_n^2}$$

$$= \frac{\omega_n^2}{\cancel{s}(s^2+\omega_n^2)}\times\cancel{s}\Big|_{s=0} \qquad = \frac{\omega_n^2}{s(\cancel{s^2+\omega_n^2})}\times(\cancel{s^2+\omega_n^2})\Big|_{s=j\omega_n}$$

$$= \frac{\omega_n^2}{s^2+\omega_n^2} \qquad = \frac{\omega_n^2}{s}\Big|_{s=j\omega_n}$$

$$= 1 \qquad = \frac{\omega_n^2}{j\omega_n} = -j\omega_n$$

$$C(s) = \frac{1}{s} - \frac{j\omega_n}{s^2+\omega_n^2}$$

$s = j\omega_n$, $B = -j\omega_n = -s$

Apply inverse laplace transform on B.S

$$c(t) = L^{-1}[C(s)] = L^{-1}\left(\frac{1}{s}\right) - L^{-1}\left[\frac{j\omega_n}{s^2+\omega_n^2}\right]$$

$L^{-1}\left(\frac{s}{s^2+a^2}\right) = \cos\omega_n t$

$$= 1 - L^{-1}\left(\frac{+s}{s^2+\omega_n^2}\right)$$

$$\boxed{c(t) = 1 - \cos\omega_n t}$$

Time response of critically damped second order system for unit step signal.

We know that,

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2}$$

for critically damped second order $\zeta = 1$

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2+2\omega_n s+\omega_n^2}$$

$$C(s) = R(s) \frac{\omega_n^2}{s^2+2\omega_n s+\omega_n^2}.$$

for a unit step signal, $R(s) = \frac{1}{s}$

$$C(s) = \frac{\omega_n^2}{s(s^2+2\omega_n s+\omega_n^2)}$$

$$= \frac{\omega_n^2}{s(-\omega_n)(-\omega_n)}$$

$$= \frac{\omega_n^2}{s\,\omega_n^2}$$

$$C(s) = \frac{1}{s}$$

Apply inverse laplace transform on B.S

$$c(t) = L^{-1}[C(s)] = L^{-1}\left(\frac{1}{s}\right)$$

$$c(t) = 1.$$

$$C(s) = \frac{\omega_n^2}{s(s+\omega_n)^2}$$

By partial expension method

$$C(s) = \frac{A}{s} + \frac{B}{s+\omega_n} + \frac{C}{(s+\omega_n)^2}$$

$$A = C(s) \times s \Big|_{s=0}$$

$$= \frac{\omega_n^2}{\cancel{s}(s+\omega_n)^2} \cancel{s} \Big|_{s=0}$$

$$= \frac{\omega_n^2}{(s+\omega_n)^2} \Big|_{s=0}$$

$$A = 1.$$

$$B = \frac{d}{ds}\left(C(s) \times (s+\omega_n)^2 \Big|_{s=-\omega_n} \right)$$

$$= \frac{d}{ds}\left(\frac{\omega_n^2}{s\cancel{(s+\omega_n)^2}} \cancel{(s+\omega_n)^2} \Big|_{s=-\omega_n} \right)$$

$$= \frac{d}{ds} \cdot \frac{1}{s} \cdot \omega_n^2 \Big|_{s=-\omega_n}$$

$$= \frac{-\omega_n^2}{s^2} \Big|_{s=-\omega_n}$$

$$= \frac{-\cancel{\omega_n^2}}{\cancel{\omega_n^2}}$$

$$= -1.$$

$$C = C(s) \times (s+\omega_n)^2 \Big|_{s=-\omega_n}$$

$$= \frac{\omega_n^2}{s\cancel{(s+\omega_n)^2}} \cancel{(s+\omega_n)^2} \Big|_{s=-\omega_n}$$

$$= \frac{\omega_n^2}{-\omega_n} = -\omega_n$$

$$C(s) = \frac{1}{s} - \frac{1}{s+\omega_n} - \frac{\omega_n}{(s+\omega_n)^2}$$

Apply inverse Laplacetransform on B.S

$$c(t) = L^{-1}[C(s)] = L^{-1}\left[\frac{1}{s}\right] - L^{-1}\left[\frac{1}{s+\omega_n}\right] - L^{-1}\left[\frac{\omega_n}{(s+\omega_n)^2}\right]$$

$$c(t) = 1 - e^{-\omega_n t} - \omega_n t\, e^{-\omega_n t}$$

$$= 1 - e^{-\omega_n t}[1 + \omega_n t]$$

Time response of under damped second order system for unit step signal.

The roots of the characteristic eq^n $s^2+2\zeta\omega_n s+\omega_n^2=0$ when damping ratio lies b/n 0 & 1 given as

$$s_1, s_2 = -\zeta\omega_n \pm j\omega_n\sqrt{(1-\zeta^2)}$$

Let us consider $\omega_d = \omega_n\sqrt{1-\zeta^2} \rightarrow (1)$.

The transfer function of the 2nd order system is

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2}$$

$$C(s) = R(s)\frac{\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2}$$

For unit step signal $R(s) = 1/s$

$$C(s) = \frac{\omega_n^2}{s(s^2+2\zeta\omega_n s+\omega_n^2)}$$

By partial fraction expansion,

$$C(s) = \frac{A}{s} + \frac{Bs+C}{s^2+2\zeta\omega_n s+\omega_n^2} \rightarrow (2)$$

$$A = C(s)\times s\Big|_{s=0}$$

$$= \frac{\omega_n^2}{s(s^2+2\zeta\omega_n s+\omega_n^2)}\times s\Big|_{s=0}$$

$$= \frac{\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2}\Big|_{s=0}$$

$$= \frac{\omega_n^2}{\omega_n^2}$$

$$A = 1$$

From eq^n (2)

$$C(s) = \frac{A}{s} + \frac{Bs+C}{s^2+2\zeta\omega_n s+\omega_n^2}$$

$$\frac{\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2} = \frac{A(s^2+2\zeta\omega_n s+\omega_n^2)+(Bs+C)s}{s(s^2+2\zeta\omega_n s+\omega_n^2)}$$

$$\omega_n^2 = 1(s^2+2\zeta\omega_n s+\omega_n^2) + Bs^2 + Cs$$

Equate or compare the coefficients of s^2

$$0 = 1 + B$$

$$\boxed{B = -1.}$$

Compare coefficients of s.

$$0 = 2\zeta\omega_n + C$$

$$\boxed{C = -2\zeta\omega_n}$$

Substitute in eq^n (2)

$$C(s) = \frac{1}{s} - \frac{s+2\zeta\omega_n}{s^2+2\zeta\omega_n s+\omega_n^2}$$

Add & subtract $\zeta^2\omega_n^2$ in the 2nd term denominator of above eq^n

$$C(s) = \frac{1}{s} - \frac{s+2\zeta\omega_n}{s^2+2\zeta\omega_n s+\omega_n^2+\zeta^2\omega_n^2-\zeta^2\omega_n^2}$$

$$= \frac{1}{s} - \frac{s+2\zeta\omega_n}{(s+\zeta\omega_n)^2+\omega_n^2(1-\zeta^2)}$$

From eq^n (1) $\omega_d = \omega_n\sqrt{1-\zeta^2}$

$$C(s) = \frac{1}{s} - \frac{s+2\zeta\omega_n}{(s+\zeta\omega_n)^2+\omega_d^2}$$

$$= \frac{1}{s} - \frac{s+2\xi\omega_n}{(s+\xi\omega_n)^2+\omega_d^2}$$

$$= \frac{1}{s} - \frac{s+\xi\omega_n+\xi\omega_n}{(s+\xi\omega_n)^2+\omega_d^2}$$

$$= \frac{1}{s} - \frac{(s+\xi\omega_n)}{(s+\xi\omega_n)^2+\omega_d^2} + \frac{\xi\omega_n}{(s+\xi\omega_n)^2+\omega_d^2}$$

The term of the eq^n is multiplied with ω_d

$$C(s) = \frac{1}{s} - \frac{s+\xi\omega_n}{(s+\xi\omega_n)^2+\omega_d^2} - \frac{\xi\omega_n \times \omega_d}{((s+\xi\omega_n)^2+\omega_d^2)\omega_d}$$

$$= \frac{1}{s} - \frac{s+\xi\omega_n}{(s+\xi\omega_n)^2+\omega_d^2} - \frac{\xi\omega_n}{\omega_d}\left[\frac{\omega_d}{(s+\xi\omega_n)^2+\omega_d^2}\right]$$

Apply inverse laplace transform $L^{-1}\left[\frac{a}{(s+a)^2+b^2}\right] = e^{-at}\sin bt$

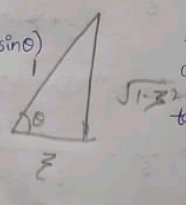
$L^{-1}\left[\frac{s+a}{(s+a)^2+b^2}\right] = e^{-at}\cos bt$

$$c(t) = 1 - e^{-\xi\omega_n t}\cdot\cos\omega_d t - \frac{\xi\omega_n}{\omega_d}\cdot\sin\omega_d t.$$

$$= 1 - e^{-\xi\omega_n t}\cos\omega_d t + \frac{-\xi\omega_n}{\omega_n\sqrt{1-\xi^2}}\sin\omega_d t.$$

$$= 1 - \frac{1}{\sqrt{1-\xi^2}}\Big[\qquad = 1 - \frac{e^{-\xi\omega_n t}}{\sqrt{1-\xi^2}}\left[\sqrt{1-\xi^2}\cos\omega_d t + \xi\sin\omega_d t\right]$$

$$= 1 - \frac{e^{-\xi\omega_n t}}{\sqrt{1-\xi^2}}(\sin\omega_d t\cos\theta + \cos\omega_d t\sin\theta)$$



$\sin\theta = \sqrt{1-\xi^2}$

$\cos\theta = \xi$

$\tan\theta = \frac{\sqrt{1-\xi^2}}{\xi}$

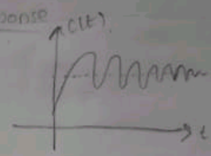
$$= 1 - \frac{e^{-\xi\omega_n t}}{\sqrt{1-\xi^2}}\left[\sin(\omega_d t+\theta)\right]$$

$$c(t) = 1 - \frac{e^{-\xi\omega_n t}}{\sqrt{1-\xi^2}}\left[\sin(\omega_d t+\theta)\right]$$

where $\omega_d = \omega_n\sqrt{1-\xi^2}$

$\theta = \tan^{-1}\left(\frac{\sqrt{1-\xi^2}}{\xi}\right)$

Time response



Find the response of over damped second order system.