# The Persistent Topology of Dynamic Data

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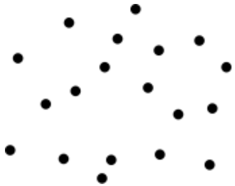
# Outline

1. Persistent Homology
2. Persistent Homology for **dynamic** data

# Persistent homology

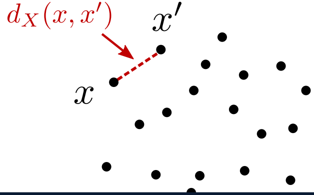
Data is often represented as a **point cloud** or a **finite metric space**.



**Q.** How can we quantify the topological features of this data set?

**A.** Let's place higher dimensional structures on top of the data set.

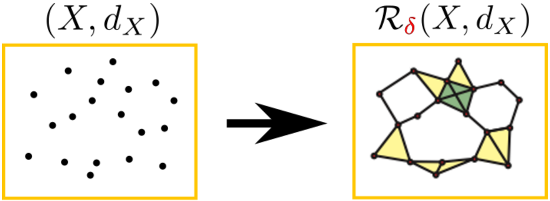
Let $(X, d_X)$ be a metric space.




## Definition ($\delta$-Rips complex of $(X, d_X)$) [Vie27]

For $\delta \in [0, \infty)$,

$$\mathcal{R}_\delta(X, d_X) := \{A \subset X : \text{for all } x, x' \in A,\ d_X(x, x') \leq \delta\}.$$




**Good theorems** (e.g. Hausmann, Latschev) and **abundant applications**.

## But, in general

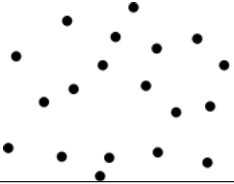
There is no rationale for a certain choice of $\delta$.

## The gist of PH

Instead of choosing a certain $\delta$,

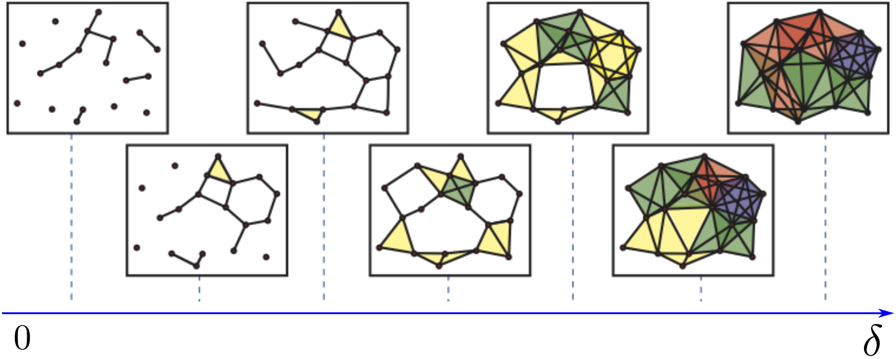
**TRACK the evolution of $\mathcal{R}_\delta(X, d_X)$ as $\delta$ increases.**

**Input** A metric space $(X, d_X)$



**Output** Filtered simplicial complexes

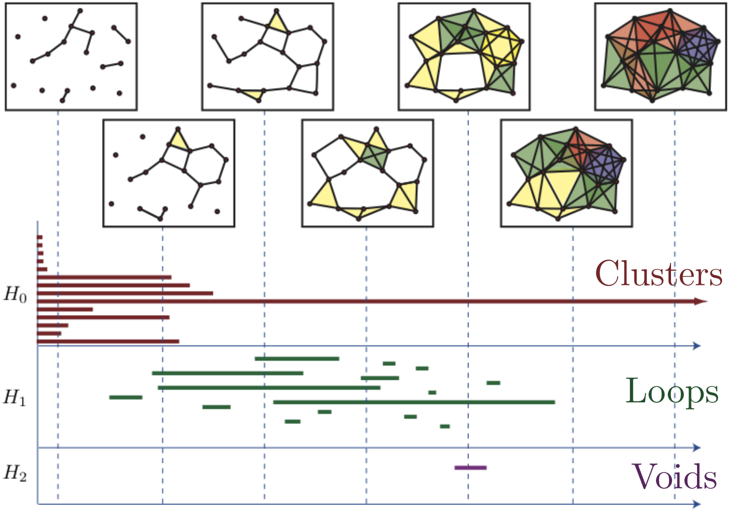
Image: Robert Ghrist (2008)




**Note: functoriality**

$$\delta \mapsto \mathcal{R}_\delta(X, d_X), \qquad [\delta \leq \delta'] \mapsto [\mathcal{R}_\delta(X, d_X) \hookrightarrow \mathcal{R}_{\delta'}(X, d_X)].$$

# Theorem: Barcode Representation of $\mathrm{PH}_\bullet^{\mathrm{Rips}}(X, d_X)$




**Theorem:** [Edelsbrunner et al. 02], [Carlsson et al. 05] **Image:** [Ghrist 08]

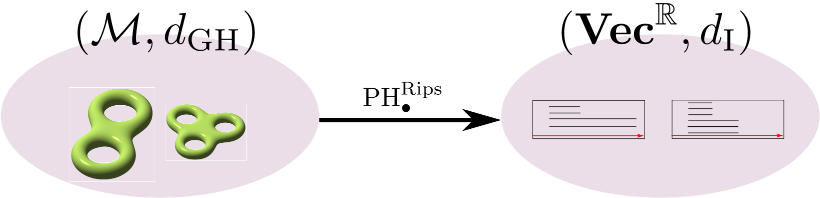
## Remark

**Functoriality** is a key ingredient of this theorem (and further development in TDA)

**Theorem** (Chazal, Cohen-Steiner, de Silva, Guibas, Mémoli, Oudot, 2009-2012)

$(X, d_X), (Y, d_Y)$: totally bounded metric spaces.

$$\boxed{d_{\mathrm{I}}(PH_\bullet^{\mathrm{Rips}}(X, d_X), PH_\bullet^{\mathrm{Rips}}(Y, d_Y))} \leq 2 \cdot d_{\mathrm{GH}}(X, Y).$$



[1]

**Remarks.**

1. $PH_\bullet^{\mathrm{Rips}}$ is stable under perturbations of the input.
2. The $\boxed{\mathrm{LHS}}$ can be obtained in poly-time $\rightarrow$ practical for the classification of metric spaces.

[1] Processed images from Wikipedia

# Motivation: Topological study of collective behaviors

(**Image sources**: Left: [Ter16], Right: Wikipedia)

Watch also: https://youtu.be/dDsmbwOrHJs?t=9

# Topological study of dynamic data (incomplete list)

1. Munch. **Applications of Persistent Homology to Time Varying Systems** (2013), Ph.D. Thesis, Duke.
2. Topaz, Zeigelmeire, Halverson. **Topological Data Analysis of Biological Aggregation Models** (2015).
3. Ulmer, Topaz, Zeigelmeire. **A topological approach to selecting models of biological experiments** (2019).

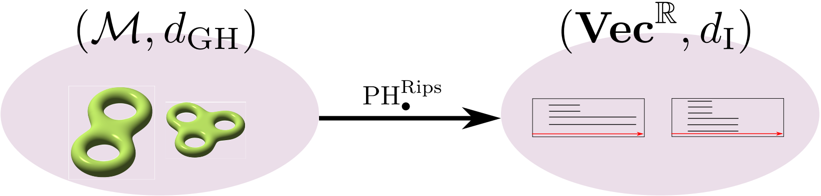
# **Goal:** Build a TDA-framework for dynamic metric spaces

**Flocking of birds = a dynamic metric space (DMS)**

**Q1.** How to quantify the difference between DMSs?

**Q2.** How to summarize topological features of a DMS?

**Q1** and **Q2** have been addressed for **static** metric spaces.



## **Goal:** Extend

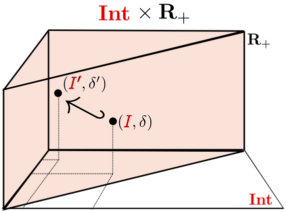
**Theorem (2009-2012)**

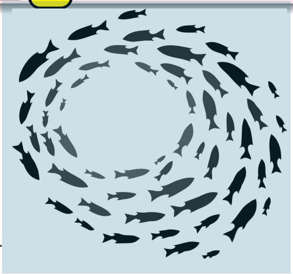
$$d_{\mathrm{I}}(\mathrm{PH}_\bullet(X, d_X), \mathrm{PH}_\bullet(Y, d_Y)) \leq 2 \cdot d_{\mathrm{GH}}((X, d_X), (Y, d_Y)).$$

**Theorem (K, Mémoli, 2018)**

For any two DMSs $\gamma_X$ and $\gamma_Y$:

$$d_{\mathrm{I}}(\mathrm{PH}_\bullet(\gamma_X), \mathrm{PH}_\bullet(\gamma_Y)) \leq 2 \cdot d_{\mathrm{dyn}}(\gamma_X, \gamma_Y).$$



Processed images from Vectorstock.com

**Theorem (K, Mémoli, 2018)**

$$d_{\mathrm{I}}(\mathrm{PH}_\bullet(\gamma_X), \mathrm{PH}_\bullet(\gamma_Y)) \leq 2 \cdot d_{\mathrm{dyn}}(\boxed{\gamma_X}, \gamma_Y).$$

# Dynamic Metric Spaces (DMSs)

A **dynamic metric space** is a pair $\gamma_X = (X, d_X(\cdot))$,
$X$ is a finite set and $d_X(\cdot) : \mathbb{R} \times X \times X \to \mathbb{R}_{\geq 0}$ is such that

- $\forall t \in \mathbb{R}$, $d_X(t)$ is a (pseudo-)metric on $X$.
- $\forall x, x' \in X$, $t \mapsto d_X(t)(x, x')$ is continuous.
- $\forall x, x' \in X$ with $x \neq x'$, $d_X(\cdot)(x, x') : \mathbb{R} \to \mathbb{R}_{\geq 0}$ is not identically zero.

**Remark.** Each $x \in X$ keeps its own identity over time.

**Examples.**

1. (Labelled) dynamic point clouds in $\mathbb{R}^n$.
2. Constant DMSs.

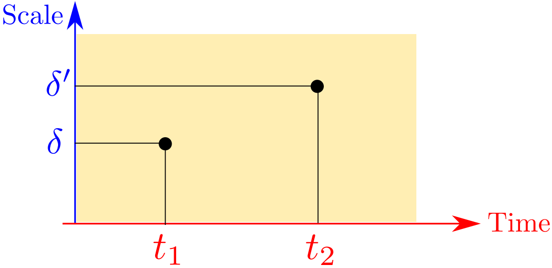
## Theorem (K, Mémoli, 2018)

$$d_{\mathrm{I}}\left(\boxed{\mathrm{PH}_\bullet(\gamma_X)}, \mathrm{PH}_\bullet(\gamma_Y)\right) \leq 2 \cdot d_{\mathrm{dyn}}(\gamma_X, \gamma_Y).$$

**Note.** Now we have two variables: **time** and **scale**.

# Rips filtration of a DMS?

Fix $\gamma_X = (X, d_X(\cdot))$.



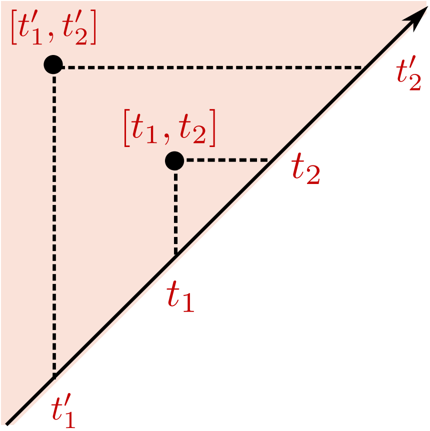

$$\mathcal{R}_{\delta}(X, d_X(t_1)) \overset{?}{\subset} \mathcal{R}_{\delta'}(X, d_X(t_2))$$

No functoriality with respect to the time parameter.

# Spatiotemporal Rips filtration

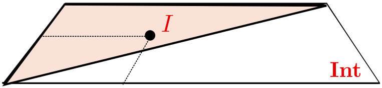
Consider the collection of **time-intervals**.

$$\mathbf{Int} := \{[t_1, t_2] : t_1 \leq t_2 \text{ in } \mathbb{R}\}.$$

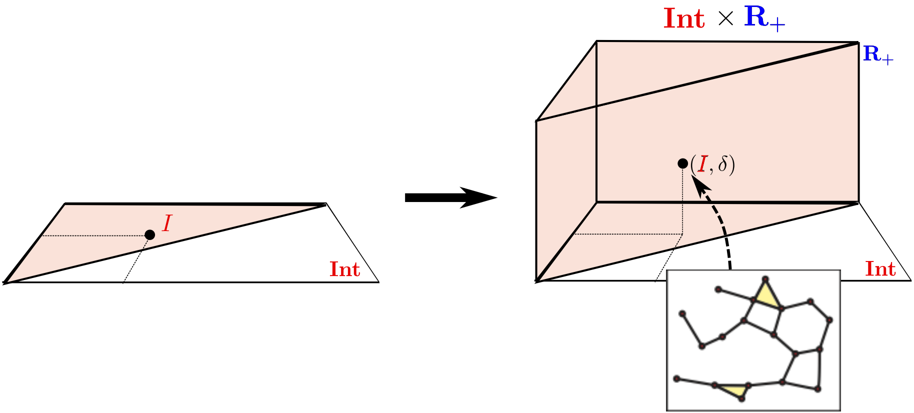



$$[t_1, t_2] \subset [t'_1, t'_2]$$

# Spatiotemporal Rips filtration



# Spatiotemporal Rips filtration

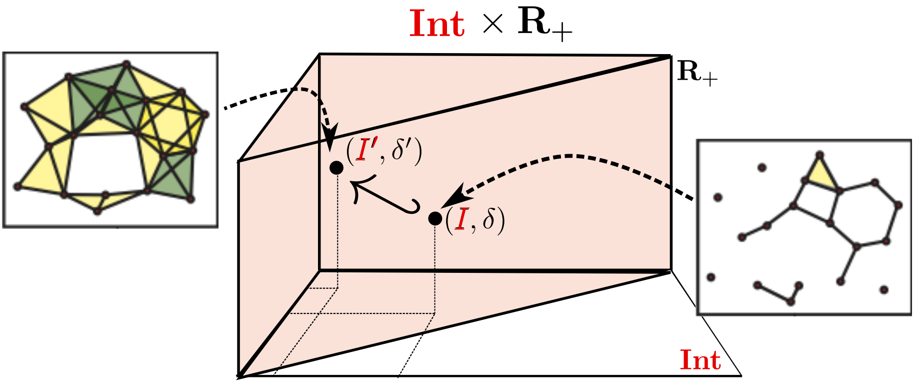



$$\mathcal{R}_{\delta}\left(X, \min_{I} d_X\right) := \{A \subset X : \forall x, x' \in A,\ \min_{t \in I} d_X(t)(x, x') \leq \delta\}$$

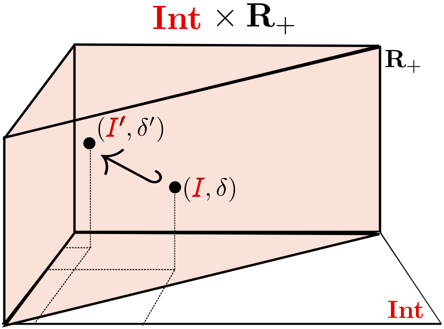
# Spatiotemporal Rips filtration

If $I \subset I'$ in **Int** and $\delta \leq \delta'$, then

$$\mathcal{R}_\delta\left(X, \min_I d_X\right) \subset \mathcal{R}_{\delta'}\left(X, \min_{I'} d_X\right).$$



$$\boxed{\mathrm{PH}_\bullet(\gamma_X) : \mathbf{Int} \times \mathbf{R}_+ \to \mathbf{Vec}_k}$$




**Remarks.**

- If $\gamma_X \equiv (X, d_X)$, then $\mathrm{PH}_\bullet(\gamma_X)$ is constant with respect to the factor **Int**.
- If $\gamma_X \equiv (X, d_X), \gamma_Y \equiv (Y, d_Y)$, then

$$d_{\mathrm{I}}(\mathrm{PH}_\bullet(\gamma_X), \mathrm{PH}_\bullet(\gamma_Y)) = d_{\mathrm{I}}(\mathrm{PH}_\bullet^{\mathrm{Rips}}(X, d_X), \mathrm{PH}_\bullet^{\mathrm{Rips}}(Y, d_Y)).$$

# Q. Which features of a DMS are significant?

We have no barcode or persistence diagram anymore!

Nevertheless,

**A:** **Large** homological features which **persist for a long time** are considered significant.

**Reasoning:**

Given a DMS $\gamma_X$, consider $d_{\mathrm{I}}(\mathrm{PH}_1(\{*\}), \mathrm{PH}_1(\gamma_X))$. This value is large when $\gamma_X$ has a **persistent** and **large** circular configuration.

The same principle holds for other orders of homology.

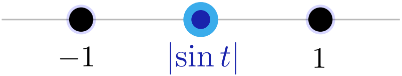
**Theorem (K, Mémoli, 2018)**

$$d_{\mathrm{I}}(\mathrm{PH}_\bullet(\gamma_X), \mathrm{PH}_\bullet(\gamma_Y)) \leq 2 \cdot \boxed{d_{\mathrm{dyn}}}(\gamma_X, \gamma_Y).$$

# Metric on DMSs

Why don't we integrate $d_{\mathrm{GH}}$?

**Example.**


**Observation.**

1. For each $t \in \mathbb{R}$, $\gamma_X(t)$ is **isometric** to $\gamma_Y(t)$.

$$\int_{t\in\mathbb{R}} d_{\mathrm{GH}}(\gamma_X(t), \gamma_Y(t))\, dt = 0.$$

2. Also, $$\int_{t\in\mathbb{R}} d_{\mathrm{I}}(\mathrm{PH}_\bullet^{\mathrm{Rips}}(\gamma_X(t)), \mathrm{PH}_\bullet^{\mathrm{Rips}}(\gamma_Y(t))\, dt = 0.$$

## Definition

$\gamma_X = (X, d_X(\cdot))$ and $\gamma_Y = (Y, d_Y(\cdot))$ are **strongly isomorphic** if there exists a bijection $\phi : X \to Y$ such that for all $t \in \mathbb{R}$, $\phi$ is an isometry between $(X, d_X(t))$ and $(Y, d_Y(t))$, i.e.

$$d_X(t)(x, x') = d_Y(t)(\phi(x), \phi(x')).$$

**Non-example.**


**We want a metric which discriminates weakly isomorphic DMSs.**

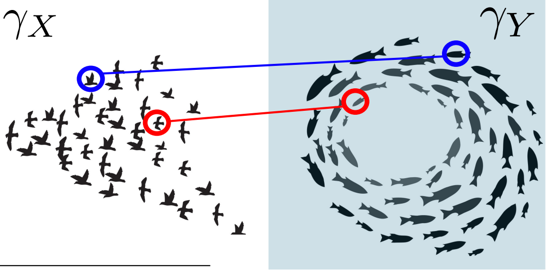
$$d_{\mathrm{dyn}}(\gamma_X, \gamma_Y) := \inf\{\varepsilon \in [0,\infty] : \gamma_X \cong_\varepsilon \gamma_Y\}.$$

$\gamma_X \cong_\varepsilon \gamma_Y$ means:

$\exists$ a correspondence $R \subset X \times Y$,

$\forall\ t \in \mathbb{R}$, $\forall (x,y),(x',y') \in R$,

$$\min_{s \in [t-\varepsilon, t+\varepsilon]} d_X(s)(x,x') \leq d_Y(t)(y,y') + 2\varepsilon, \text{ and sym.}$$




[2]

[2] Processed images from VectorStock.com

# Interleaving distance between DMSs

**Remark.** For $\varepsilon \leq \varepsilon'$, if $\gamma_X \cong_\varepsilon \gamma_Y$, then, $\gamma_X \cong_{\varepsilon'} \gamma_Y$.

$$d_{\mathrm{dyn}}(\gamma_X, \gamma_Y) := \inf\{\varepsilon \in [0, \infty] : \gamma_X \cong_\varepsilon \gamma_Y\}.$$

**Remark.** If $\gamma_X \equiv (X, d_X)$ and $\gamma_Y \equiv (Y, d_Y)$,

$$d_{\mathrm{dyn}}(\gamma_X, \gamma_Y) = d_{\mathrm{GH}}((X, d_X), (Y, d_Y)).$$

$\forall\ t \in \mathbb{R}$, $\forall (x, y), (x', y') \in R$,

$$\min_{s \in [t-\varepsilon, t+\varepsilon]} d_X(s)(x, x') \leq d_Y(t)(y, y') + 2\varepsilon, \text{ and sym.}$$

$$\Updownarrow$$

$\forall (x, y), (x', y') \in R$,

$$|d_X(x, x') - d_Y(y, y')| \leq 2\varepsilon.$$

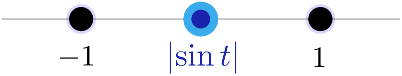
$\mathcal{M}^{\mathrm{dyn}}_{\mathrm{B}}$: the collection of all bounded DMSs.

(A DMS $(X, d_X(\cdot))$ is bounded if there exists $M > 0$ s.t. $d_X(\cdot)(\cdot,\cdot) \leq M$).

**Theorem (K, Mémoli 2017)**

*$d_{\mathrm{dyn}}$ is a metric on $\mathcal{M}^{\mathrm{dyn}}_{\mathrm{B}}$ (up to strong isomorphism).*

**Examples.** $d_{\mathrm{dyn}}$ discriminates

# Remark

**Theorem (K, Mémoli, 2018)**

For any two DMSs $\gamma_X$ and $\gamma_Y$:

$$d_{\mathrm{I}}(\mathrm{PH}_\bullet(\gamma_X), \mathrm{PH}_\bullet(\gamma_Y)) \leq 2 \cdot d_{\mathrm{dyn}}(\gamma_X, \gamma_Y).$$

$\Downarrow$ if $\gamma_X \equiv (X, d_X)$ and $\gamma_Y \equiv (Y, d_Y)$

**Theorem (2009-2012)**

$$d_{\mathrm{I}}(\mathrm{PH}_\bullet(X, d_X), \mathrm{PH}_\bullet(Y, d_Y)) \leq 2 \cdot d_{\mathrm{GH}}((X, d_X), (Y, d_Y)).$$

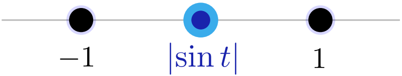
# Computational aspects

**Theorem**

$$d_{\mathrm{I}}(\mathrm{PH}_\bullet(\gamma_X), \mathrm{PH}_\bullet(\gamma_Y)) \leq 2 \cdot d_{\mathrm{dyn}}(\gamma_X, \gamma_Y).$$

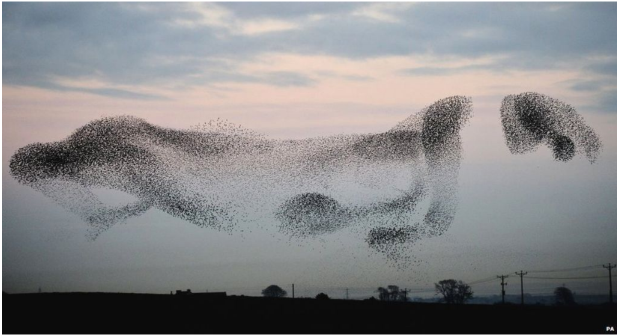
**Caution.** Computing each side is NP-hard.[3]

- We find a lower bound for the LHS, utilizing their **dimension function** and **rank function.** (We use **Patel's erosion distance**).
- This lower bound is poly-time computable and good enough to discriminates:


[3] Bjerkevik, Botnan, Kerber (2019) and Schmiedl (2017).

# Application: classifying simulated flocking behaviors



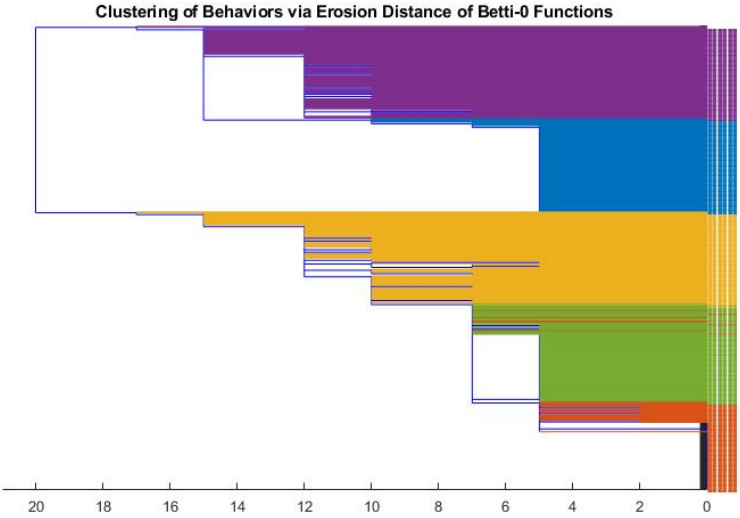

(Image: BBC)

# Experiments with Nate Clause (Ohio State) and Zane Smith (U of Minnesota)

- A variation of **Boids** simulation (parameters: Cohesion, Alignment, Separation, etc.)
- Identify 5 distinct behaviors by tuning the parameters.
- Repeat simulation 100 times for each behavior with random initial configurations.
- Construct 500×500-erosion-distance-matrix. Carry out hierarchical clustering on it or MDS-plot it.

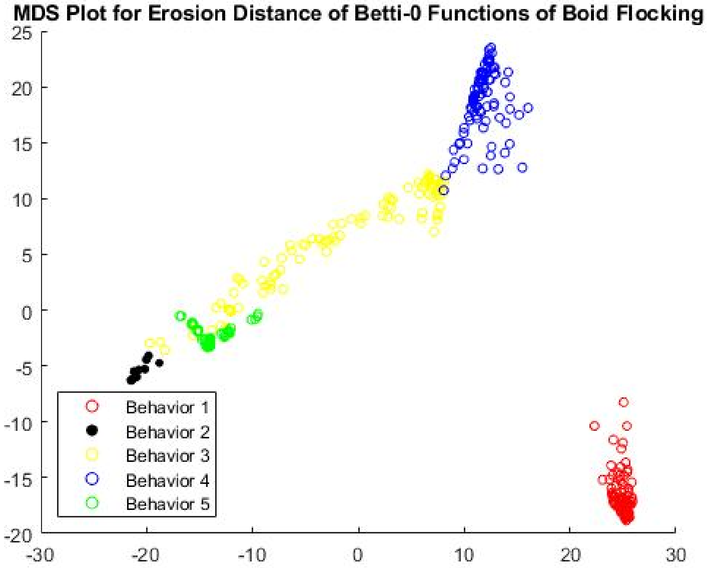
- `https://research.math.osu.edu/networks/formigrams/`
- `https://github.com/ndag/PHoDMSs`

# Single-Linkage Hierarchical Clustering result




B1: Blue/ B2: Orange / B3: Yellow/ B4: Purple/ B5: Green.

# Multidimensional Scaling result



# Summary

- We defined Spatiotemporal Rips filtration for DMSs (3-D) which is stable with respect to the metric $d_{\mathrm{dyn}}$ on DMSs.
- In particular, on the collection of constant DMSs, our stability theorem reduces to the standard stability.
- Classification experiment.

# References

- **Spatiotemporal Persistent Homology for Dynamic Metric Spaces**, Kim and Mémoli, *Discrete & Computational Geometry (2020)*.

## Computation tools

- https://research.math.osu.edu/networks/formigrams/
- https://github.com/ndag/PHoDMSs

## Image/video sources

- **Barcodes: The persistent topology of data**, R. Ghrist, Bulletin of the American Mathematical Society (2008).
- **Vision-based Measurement Methods for Schools of Fish and Analysis of their Behaviors**, Kei Terayama, Ph.D. Thesis (2016).
- **Can A Thousand Tiny Swarming Robots Outsmart Nature?** https://youtu.be/QXNVZJ3KUsA

**Thank you for paying attention!**

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Vision-based measurement methods for schools of fish and analysis of their behaviors.
2016.

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Über den höheren zusammenhang kompakter räume und eine klasse von zusammenhangstreuen abbildungen.
*Mathematische Annalen*, 97(1):454–472, 1927.