1 Laplace: Finite Difference Approximation

The finite difference approximation for the second derivatives is given by:

$$\frac{\partial^2 \phi}{\partial x^2} \approx \frac{\phi_{i+1,j} - 2\phi_{i,j} + \phi_{i-1,j}}{(\Delta x)^2}$$

$$\frac{\partial^2 \phi}{\partial y^2} \approx \frac{\phi_{i,j+1} - 2\phi_{i,j} + \phi_{i,j-1}}{(\Delta y)^2}$$

Substituting these into the Laplace equation:

$$\frac{\phi_{i+1,j} - 2\phi_{i,j} + \phi_{i-1,j}}{(\Delta x)^2} + \frac{\phi_{i,j+1} - 2\phi_{i,j} + \phi_{i,j-1}}{(\Delta y)^2} = 0$$

For a uniform grid ($\Delta x = \Delta y = h$), this simplifies to:

$$\phi_{i+1,j} + \phi_{i-1,j} + \phi_{i,j+1} + \phi_{i,j-1} - 4\phi_{i,j} = 0$$

Rearranging to solve for the value at each grid point:

$$\phi_{i,j} = \frac{1}{4}(\phi_{i+1,j} + \phi_{i-1,j} + \phi_{i,j+1} + \phi_{i,j-1})$$

1.1 Iterative Methods

Jacobi Method

$$\phi_{i,j}^{(n+1)} = \frac{1}{4}(\phi_{i+1,j}^{(n)} + \phi_{i-1,j}^{(n)} + \phi_{i,j+1}^{(n)} + \phi_{i,j-1}^{(n)})$$

Gauss-Seidel Method

$$\phi_{i,j}^{(n+1)} = \frac{1}{4}(\phi_{i+1,j}^{(n+1)} + \phi_{i-1,j}^{(n+1)} + \phi_{i,j+1}^{(n+1)} + \phi_{i,j-1}^{(n)})$$

Successive Over-Relaxation (SOR)

$$\phi_{i,j}^{(n+1)} = (1-\omega)\phi_{i,j}^{(n)} + \frac{\omega}{4}(\phi_{i+1,j}^{(n+1)} + \phi_{i-1,j}^{(n+1)} + \phi_{i,j+1}^{(n+1)} + \phi_{i,j-1}^{(n)})$$

Convergence Criteria Stop the iteration when the maximum change in ϕ values between iterations is less than a specified tolerance.