1 Numerical Solution of the 1D Laplace Equation

The 1D Laplace equation is given by:

$$\frac{d^2u}{dx^2} = 0, \tag{1}$$

with boundary conditions:

$$u(0) = A, \quad u(L) = B. \tag{2}$$

1.1 Finite Difference Approximation

Using the central difference method, the second derivative is approximated as:

$$\frac{u_{i-1} - 2u_i + u_{i+1}}{\Delta x^2} = 0. \tag{3}$$

Rearranging:

$$u_i = \frac{u_{i-1} + u_{i+1}}{2}. \tag{4}$$

1.2 Matrix Formulation

For N interior points, define:

$$\mathbf{u} = [u_1, u_2, \dots, u_{N-1}]^T$$

(unknown values) Boundary conditions:

$$u_0 = A, \quad u_N = B$$

Grid spacing:

$$\Delta x = \frac{L}{N+1}$$

The finite difference equations for all interior points lead to a system of N-2 equations:

$$-2u_1 + u_2 = -A$$

$$-u_1 - 2u_2 + u_3 = 0$$

$$u_2 - 2u_3 + u_4 = 0$$

$$\vdots$$

$$u_{N-2} - 2u_{N-1} = -B$$

For N interior points, we construct the system:

$$A\mathbf{u} = \mathbf{b}, \tag{5}$$

where:

$$A = \begin{bmatrix} -2 & 1 & 0 & 0 & \cdots & 0 \\ 1 & -2 & 1 & 0 & \cdots & 0 \\ 0 & 1 & -2 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 1 & -2 & 1 \\ 0 & 0 & 0 & 0 & 1 & -2 \end{bmatrix}, \tag{6}$$

The matrix A has dim: $(N-2) \times (N-2)$

$$
\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ \vdots \\ u_{N-2} \\ u_{N-1} \end{bmatrix}, \tag{7}
$$

Note the dimension of $\mathbf{u}$: $= N-2$ and same for $\mathbf{b}$.

$$
\mathbf{b} = \begin{bmatrix} -A \\ 0 \\ 0 \\ \vdots \\ 0 \\ -B \end{bmatrix}. \tag{8}
$$

After soilving the equation one needs to add the elements u(0) and u(L) to obtain the full vector.

1.3 Conclusion

The numerical solution approximates the exact solution, which is a linear function:

$$
u(x) = A + (B-A)\frac{x}{L}. \tag{9}
$$

This method can be extended to higher dimensions using iterative solvers like Gauss-Seidel.

2 Numerical Solution of the 2D Laplace Equation: Finite Difference Approximation

The finite difference approximation for the second derivatives is given by:

$$\frac{\partial^2 \phi}{\partial x^2} \approx \frac{\phi_{i+1,j} - 2\phi_{i,j} + \phi_{i-1,j}}{(\Delta x)^2}$$
$$\frac{\partial^2 \phi}{\partial y^2} \approx \frac{\phi_{i,j+1} - 2\phi_{i,j} + \phi_{i,j-1}}{(\Delta y)^2}$$

Substituting these into the Laplace equation:

$$\frac{\phi_{i+1,j} - 2\phi_{i,j} + \phi_{i-1,j}}{(\Delta x)^2} + \frac{\phi_{i,j+1} - 2\phi_{i,j} + \phi_{i,j-1}}{(\Delta y)^2} = 0$$

For a uniform grid ($\Delta x = \Delta y = h$), this simplifies to:

$$\phi_{i+1,j} + \phi_{i-1,j} + \phi_{i,j+1} + \phi_{i,j-1} - 4\phi_{i,j} = 0$$

Rearranging to solve for the value at each grid point:

$$\phi_{i,j} = \frac{1}{4}(\phi_{i+1,j} + \phi_{i-1,j} + \phi_{i,j+1} + \phi_{i,j-1})$$

2.1 Iterative Methods

Jacobi Method

$$\phi_{i,j}^{(n+1)} = \frac{1}{4}(\phi_{i+1,j}^{(n)} + \phi_{i-1,j}^{(n)} + \phi_{i,j+1}^{(n)} + \phi_{i,j-1}^{(n)})$$

Gauss-Seidel Method

$$\phi_{i,j}^{(n+1)} = \frac{1}{4}(\phi_{i+1,j}^{(n+1)} + \phi_{i-1,j}^{(n+1)} + \phi_{i,j+1}^{(n+1)} + \phi_{i,j-1}^{(n)})$$

Successive Over-Relaxation (SOR)

$$\phi_{i,j}^{(n+1)} = (1-\omega)\phi_{i,j}^{(n)} + \frac{\omega}{4}(\phi_{i+1,j}^{(n+1)} + \phi_{i-1,j}^{(n+1)} + \phi_{i,j+1}^{(n+1)} + \phi_{i,j-1}^{(n)})$$

Convergence Criteria Stop the iteration when the maximum change in ϕ values between iterations is less than a specified tolerance.