2D Heat Equation - Finite Difference Method

The 2D heat equation describes how heat diffuses through a given region over time. The equation is given by:

$$\frac{\partial u}{\partial t} = \alpha \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right)$$

where $u(x, y, t)$ is the temperature at point (x, y) and time t, and α is the thermal diffusivity of the material. To solve this equation numerically using finite differences, we can discretize both the spatial and time domains. Here is an outline of the process:

Discretization

1. **Spatial Grid**: Divide the region into a grid of points (x_i, y_j) with spacing Δx and Δy.
2. **Time Steps**: Discretize time into steps of size Δt.
3. **Finite Difference Approximation**: Use finite differences to approximate the derivatives. For example:

- $\frac{\partial u}{\partial t}$ at time step n is approximated by $\frac{u_{i,j}^{n+1} - u_{i,j}^{n}}{\Delta t}$.
- $\frac{\partial^2 u}{\partial x^2}$ is approximated by $\frac{u_{i+1,j}^{n} - 2u_{i,j}^{n} + u_{i-1,j}^{n}}{(\Delta x)^2}$.
- $\frac{\partial^2 u}{\partial y^2}$ is approximated by $\frac{u_{i,j+1}^{n} - 2u_{i,j}^{n} + u_{i,j-1}^{n}}{(\Delta y)^2}$.

Combining these, the heat equation at point (x_i, y_j) and time step $n + 1$ is approximated as:

$$\frac{u_{i,j}^{n+1} - u_{i,j}^{n}}{\Delta t} = \alpha \left(\frac{u_{i+1,j}^{n} - 2u_{i,j}^{n} + u_{i-1,j}^{n}}{(\Delta x)^2} + \frac{u_{i,j+1}^{n} - 2u_{i,j}^{n} + u_{i,j-1}^{n}}{(\Delta y)^2} \right)$$

Rearranging this to solve for $u_{i,j}^{n+1}$:

$$u_{i,j}^{n+1} = u_{i,j}^{n} + \alpha \Delta t \left(\frac{u_{i+1,j}^{n} - 2u_{i,j}^{n} + u_{i-1,j}^{n}}{(\Delta x)^2} + \frac{u_{i,j+1}^{n} - 2u_{i,j}^{n} + u_{i,j-1}^{n}}{(\Delta y)^2} \right)$$

Algorithm

1. **Initialize** the temperature distribution $u(x_i, y_j, 0)$ at time $t = 0$.
2. **Iterate** over time steps:

- For each grid point (x_i, y_j), update $u_{i,j}^{n+1}$ using the finite difference approximation.
- Apply boundary conditions as necessary.

3. **Continue** until the desired final time is reached.

Stability Condition for the 2D Heat Equation

The 2D heat equation is given by:

$$\frac{\partial u}{\partial t} = \alpha \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right) \tag{1}$$

The finite difference approximation for the equation is:

$$u_{i,j}^{n+1} = u_{i,j}^{n} + \alpha \Delta t \left(\frac{u_{i+1,j}^{n} - 2u_{i,j}^{n} + u_{i-1,j}^{n}}{(\Delta x)^2} + \frac{u_{i,j+1}^{n} - 2u_{i,j}^{n} + u_{i,j-1}^{n}}{(\Delta y)^2} \right) \tag{2}$$

To analyze the stability, we assume a solution of the form:

$$u_{i,j}^{n} = G^n e^{i(k_x x_i + k_y y_j)} \tag{3}$$

where G is the amplification factor, and k_x and k_y are the wave numbers in the x and y directions, respectively. Substituting this into the discretized equation, we get:

$$Ge^{i(k_x x_i + k_y y_j)} = e^{i(k_x x_i + k_y y_j)} + \alpha \Delta t \left(\frac{e^{i(k_x(x_i+\Delta x))} - 2e^{i(k_x x_i)} + e^{i(k_x(x_i-\Delta x))}}{(\Delta x)^2} + \frac{e^{i(k_y(y_j+\Delta y))} - 2e^{i(k_y y_j)} + e^{i(k_y(y_j-\Delta y))}}{(\Delta y)^2} \right) \tag{4}$$

Simplifying, we use the fact that:

$$e^{ik_x(x_i+\Delta x)} = e^{ik_x x_i} e^{ik_x \Delta x} \tag{5}$$

$$e^{ik_x(x_i-\Delta x)} = e^{ik_x x_i} e^{-ik_x \Delta x} \tag{6}$$

Thus:

$$G = 1 + \alpha \Delta t \left(\frac{e^{ik_x \Delta x} + e^{-ik_x \Delta x} - 2}{(\Delta x)^2} + \frac{e^{ik_y \Delta y} + e^{-ik_y \Delta y} - 2}{(\Delta y)^2} \right) \tag{7}$$

Using Euler's formula $e^{i\theta} = \cos(\theta) + i\sin(\theta)$, we get:

$$e^{ik_x \Delta x} + e^{-ik_x \Delta x} = 2\cos(k_x \Delta x) \tag{8}$$

$$e^{ik_y \Delta y} + e^{-ik_y \Delta y} = 2\cos(k_y \Delta y) \tag{9}$$

Therefore:

$$G = 1 + \alpha \Delta t \left(\frac{2\cos(k_x \Delta x) - 2}{(\Delta x)^2} + \frac{2\cos(k_y \Delta y) - 2}{(\Delta y)^2} \right) \tag{10}$$

$$G = 1 + \alpha \Delta t \left(\frac{2(\cos(k_x \Delta x) - 1)}{(\Delta x)^2} + \frac{2(\cos(k_y \Delta y) - 1)}{(\Delta y)^2} \right) \tag{11}$$

To ensure stability, the magnitude of the amplification factor G must be less than or equal to 1:

$$|G| \leq 1 \tag{12}$$

Considering the worst-case scenario where $\cos(k_x \Delta x)$ and $\cos(k_y \Delta y)$ reach their minimum value of -1:

$$\left| 1 + \alpha \Delta t \left(\frac{2(-1-1)}{(\Delta x)^2} + \frac{2(-1-1)}{(\Delta y)^2} \right) \right| \leq 1 \tag{13}$$

$$\left| 1 - \alpha \Delta t \left(\frac{4}{(\Delta x)^2} + \frac{4}{(\Delta y)^2} \right) \right| \leq 1 \tag{14}$$

For stability:

$$1 - \alpha \Delta t \left(\frac{4}{(\Delta x)^2} + \frac{4}{(\Delta y)^2} \right) \geq -1 \tag{15}$$

$$2 \geq \alpha \Delta t \left(\frac{4}{(\Delta x)^2} + \frac{4}{(\Delta y)^2} \right) \tag{16}$$

$$\frac{1}{\alpha} \geq 2\Delta t \left(\frac{1}{(\Delta x)^2} + \frac{1}{(\Delta y)^2} \right) \tag{17}$$

$$\Delta t \leq \frac{(\Delta x)^2 (\Delta y)^2}{2\alpha((\Delta y)^2 + (\Delta x)^2)} \tag{18}$$

Therefore, the stability condition for the 2D heat equation using the finite difference method is:

$$\Delta t \leq \frac{(\Delta x)^2 (\Delta y)^2}{2\alpha((\Delta y)^2 + (\Delta x)^2)} \tag{19}$$

which implies for $\Delta x = \Delta y = h$

$$\Delta t \leq \frac{1}{2\alpha} \left(\frac{1}{(\Delta x)^2} + \frac{1}{(\Delta y)^2} \right)^{-1} = \frac{h^2}{4\alpha} \tag{20}$$

This ensures that the numerical solution remains stable over time.