1 Solving the Heat Equation using Finite Difference Method

The heat equation in one dimension is given by:

$$\frac{\partial u}{\partial t} = \alpha \frac{\partial^2 u}{\partial x^2} \tag{1}$$

where:

- $u(x,t)$ is the temperature at position x and time t
- α is the thermal diffusivity constant

2 Finite Difference Method

We'll discretize both space and time to solve this equation numerically.

1. Discretize space into N points with spacing Δx.
2. Discretize time into steps with spacing Δt.
3. Use finite differences to approximate derivatives:

$$\frac{\partial u}{\partial t} \approx \frac{u_i^{n+1} - u_i^n}{\Delta t} \tag{2}$$

$$\frac{\partial^2 u}{\partial x^2} \approx \frac{u_{i+1}^n - 2u_i^n + u_{i-1}^n}{(\Delta x)^2} \tag{3}$$

4. Rearrange to get the update rule for u:

$$u_i^{n+1} = u_i^n + \frac{\alpha \Delta t}{(\Delta x)^2}(u_{i+1}^n - 2u_i^n + u_{i-1}^n) \tag{4}$$

Condition of Convergence: For the finite difference method to converge and produce a stable solution, the time step Δt and spatial step Δx must satisfy the following condition, known as the CFL (Courant-Friedrichs-Lewy) condition:

$$\frac{\alpha \Delta t}{(\Delta x)^2} \leq \frac{1}{2} \tag{5}$$

3 Condition of Convergence

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4 Justification of the CFL Condition

To justify the CFL condition, we perform a stability analysis on the discretized heat equation.
Stability Analysis: Assume the solution can be expressed as a sum of Fourier modes:

$$u_i^n = \sum_k \hat{u}_k^n e^{ikx_i} \tag{7}$$

Substituting this into the update rule, we get:

$$\hat{u}_k^{n+1} = \hat{u}_k^n + \frac{\alpha \Delta t}{(\Delta x)^2}(\hat{u}_k^n e^{ikx_{i+1}} + \hat{u}_k^n e^{ikx_{i-1}} - 2\hat{u}_k^n e^{ikx_i}) \tag{8}$$

Using the properties of the exponential function, this simplifies to:

$$\hat{u}_k^{n+1} = \hat{u}_k^n \left[1 + \frac{\alpha \Delta t}{(\Delta x)^2}(e^{ik\Delta x} + e^{-ik\Delta x} - 2)\right] \tag{9}$$

Since $e^{ik\Delta x} + e^{-ik\Delta x} = 2\cos(k\Delta x)$, we get:

$$\hat{u}_k^{n+1} = \hat{u}_k^n \left[1 + 2\frac{\alpha \Delta t}{(\Delta x)^2}(\cos(k\Delta x) - 1)\right] \tag{10}$$

Define the amplification factor G as:

$$G = 1 + 2\frac{\alpha \Delta t}{(\Delta x)^2}(\cos(k\Delta x) - 1) \tag{11}$$

For stability, the magnitude of G must be less than or equal to 1:

$$|G| \leq 1 \tag{12}$$

Since $\cos(k\Delta x)$ ranges between -1 and 1, the most stringent condition occurs when $\cos(k\Delta x) = -1$:

$$G = 1 - 4\frac{\alpha \Delta t}{(\Delta x)^2} \tag{13}$$

For $|G| \leq 1$:

$$-1 \leq 1 - 4\frac{\alpha \Delta t}{(\Delta x)^2} \leq 1 \tag{14}$$

The lower bound gives the condition:

$$-1 \leq 1 - 4\frac{\alpha \Delta t}{(\Delta x)^2} \tag{15}$$

$$-2 \leq -4\frac{\alpha \Delta t}{(\Delta x)^2} \tag{16}$$

$$\frac{\alpha \Delta t}{(\Delta x)^2} \leq \frac{1}{2} \tag{17}$$

This is the CFL condition for the heat equation:

$$\frac{\alpha \Delta t}{(\Delta x)^2} \leq \frac{1}{2} \tag{18}$$