1 Introduction

The **heat equation** in one dimension (1D) is a partial differential equation (PDE) that describes how heat (or temperature) distributes over time in a given medium. It is given by:

$$\frac{\partial u}{\partial t} = \alpha \frac{\partial^2 u}{\partial x^2} \tag{1}$$

where:

- $u(x,t)$ represents the temperature at position x and time t.
- α is the **thermal diffusivity** of the material, given by $\alpha = \frac{k}{\rho c}$, where k is the thermal conductivity, ρ is the density, and c is the specific heat capacity.
- $\frac{\partial u}{\partial t}$ represents the **rate of change of temperature** with respect to time.
- $\frac{\partial^2 u}{\partial x^2}$ represents the **spatial curvature of the temperature distribution**, which determines how heat diffuses.

2 Boundary and Initial Conditions

To solve the heat equation, we need boundary and initial conditions.

2.1 Initial Condition

Specifies the temperature distribution at $t = 0$:

$$u(x,0) = f(x) \tag{2}$$

where $f(x)$ is a given function.

2.2 Boundary Conditions

Define how heat behaves at the boundaries. Common types include:

- **Dirichlet condition**: Fixed temperature at boundaries,

$$u(0,t) = T_1, \quad u(L,t) = T_2. \tag{3}$$

- **Neumann condition**: Specifies the heat flux (insulated or heat flow),

$$\left.\frac{\partial u}{\partial x}\right|_{x=0} = q_1, \quad \left.\frac{\partial u}{\partial x}\right|_{x=L} = q_2. \tag{4}$$

- **Mixed condition**: A combination of Dirichlet and Neumann conditions.

3 Solution Methods

There are several methods to solve the heat equation:

- **Separation of Variables**: Used for simple cases with homogeneous boundary conditions.
- **Fourier Series**: Expands the solution in sine or cosine series.
- **Fourier Transform**: Used for problems on infinite domains.
- **Numerical Methods**: Finite difference, finite element, and finite volume methods for complex geometries and varying properties.

4 Numerical Algorithms

There are several numerical algorithms to solve the 1D heat equation.

4.1 Finite Difference Method (FDM)

The finite difference method discretizes the heat equation using numerical approximations for derivatives.

4.1.1 Explicit Method (FTCS)

Using a forward difference for time and a centered difference for space, the heat equation:

$$\frac{\partial u}{\partial t} = \alpha \frac{\partial^2 u}{\partial x^2} \tag{5}$$

is discretized as:

$$u_i^{n+1} = u_i^n + \lambda(u_{i+1}^n - 2u_i^n + u_{i-1}^n) \tag{6}$$

where $\lambda = (\alpha \Delta t)/\Delta x^2$ is the stability parameter. The explicit method is conditionally stable if:

$$\lambda \leq \frac{1}{2} \tag{7}$$

4.1.2 Implicit Method (BTCS)

Using a backward difference for time:

$$u_i^{n+1} - \lambda(u_{i+1}^{n+1} - 2u_i^{n+1} + u_{i-1}^{n+1}) = u_i^n \tag{8}$$

This leads to a system of linear equations that can be solved using matrix inversion. The implicit method is unconditionally stable.

4.1.3 Crank-Nicolson Method (CN)

A combination of explicit and implicit methods:

$$u_i^{n+1} - \frac{\lambda}{2}(u_{i+1}^{n+1} - 2u_i^{n+1} + u_{i-1}^{n+1}) = u_i^n + \frac{\lambda}{2}(u_{i+1}^n - 2u_i^n + u_{i-1}^n) \tag{9}$$

This method is unconditionally stable and second-order accurate.

4.2 Finite Element Method (FEM)

The finite element method divides the domain into elements and approximates $u(x,t)$ using basis functions.

4.3 Finite Volume Method (FVM)

The finite volume method applies the integral form of the heat equation to control volumes and approximates heat fluxes across volume boundaries.

5 Comparison of Methods

Method	Stability	Accuracy	Complexity	Application
Explicit (FTCS)	Conditional ($\lambda \leq 1/2$)	First-order	Simple	Fast, but unstable for large time steps
Implicit (BTCS)	Unconditional	First-order	Medium	More stable, requires solving a system
Crank-Nicolson	Unconditional	Second-order	High	Best accuracy, but computationally expensive
Finite Element	Unconditional	Second-order+	High	Best for complex geometries
Finite Volume	Unconditional	Second-order+	High	Best for conservation laws

Table 1: Comparison of Numerical Methods