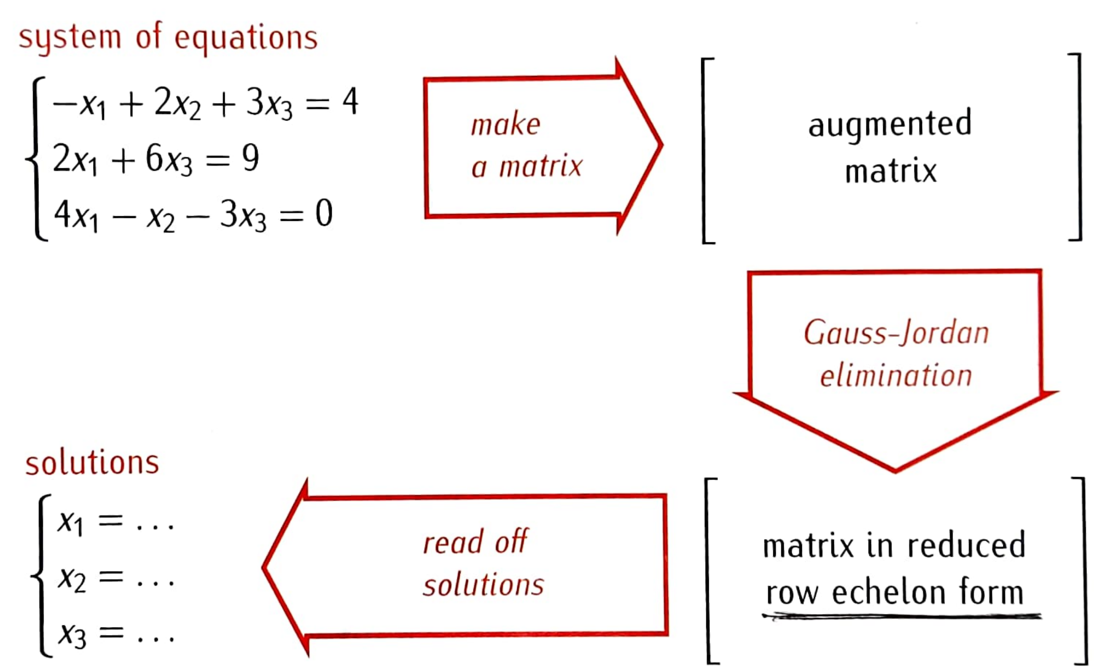
Recall:

How to solve a system of linear equations



- Every system of linear equations can be represented by a matrix
- Elementary row operations:
  - interchange of two rows
  - multiplication of a row by a non-zero number
  - addition of a multiple of one row to another row.
- Elementary row operations do not change solutions of systems of linear equations.

Definition

A matrix is in the *row echelon form* if:

1) the first non-zero entry of each row is a 1 ("a leading one");
2) the leading one in each row is to the right of the leading one in the row above it.

A matrix is in the *reduced row echelon form* if in addition it satisfies:

3) all entries above each leading one are 0.

$$\begin{bmatrix} 1 & * & * & * & 0 & 0 & * & * & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & * & * & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & * & * & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

($* =$ any number)

Example

$$\begin{bmatrix} 1 & 0 & 4 & 0 & 7 & 0 \\ 0 & \boxed{1} & 5 & 0 & 1 & 0 \\ 0 & 0 & 0 & \boxed{1} & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 & \boxed{1} \end{bmatrix} \quad \begin{bmatrix} 1 & 2 & 4 & \boxed{6} & 7 & 0 \\ 0 & \boxed{1} & 5 & 0 & 1 & 2 \\ 0 & 0 & 0 & \boxed{1} & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad \begin{bmatrix} 1 & 0 & 4 & 0 & 7 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 3 & 6 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Fact

If a system of linear equations is represented by a matrix in the reduced row echelon form then it is easy to solve the system.

Example

$$\left[\begin{array}{cccc|c} 1 & 0 & 3 & 0 & 0 \\ 0 & 1 & 7 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{array}\right]$$

$x_1 + 3x_3 = 0$

$x_2 + 7x_3 = 0$

$x_4 = 0$

$\boxed{0 = 1}$

Proposition

A matrix in the reduced row echelon form represents an inconsistent system if and only if it contains a row of the form

$$[0 \quad 0 \quad 0 \ \ldots \quad 0 \quad 1]$$

i.e. with the leading one in the last column.

Example

$$\begin{array}{ccccc} x_1 & x_2 & x_3 & x_4 & \\ \end{array}$$

$$\left[\begin{array}{cccc|c} \boxed{1} & 0 & 3 & 0 & 0 \\ 0 & \boxed{1} & 7 & 0 & 0 \\ 0 & 0 & 0 & \boxed{1} & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

free variable

$$\begin{cases} x_1 + 3x_3 = 0 \\ x_2 + 7x_3 = 0 \\ x_4 = 0 \end{cases}$$

$$x_1 = -3x_3$$
$$x_2 = -7x_3$$

$x_3 = c \in \mathbb{R}$.

$$\begin{cases} x_1 = -3c \\ x_2 = -7c \\ x_3 = c \\ x_4 = 0 \end{cases}$$

$c \in \mathbb{R}$.

Note

In an augmented matrix in the reduced row echelon form free variables correspond to columns of the coefficient matrix that do not contain leading ones.

Example

$$\begin{array}{c} \begin{array}{cccc} x_1 & x_2 & x_3 & x_4 \end{array} \\ \left[\begin{array}{cccc|c} \boxed{1} & 0 & 0 & 0 & 5 \\ 0 & \boxed{1} & 0 & 0 & 6 \\ 0 & 0 & \boxed{1} & 0 & 7 \\ 0 & 0 & 0 & \boxed{1} & 8 \end{array}\right] \end{array} \rightarrow \text{Reduced row echelon form}$$

$x_1 = 5$
$x_2 = 6$
$x_3 = 7$
$x_4 = 8$.

(1) Last column contain a leading one $\longrightarrow$ No solⁿ.

(2) If at least one of the column do not contain leading one $\rightarrow$ many solⁿ.

(3) If all column except last one contain leading one $\rightarrow$ unique solⁿ.

Note

A matrix in the reduced row echelon form represents a system of equations with exactly one solution if and only if it has a leading one in every column except for the last one.

Gauss-Jordan elimination process (= row reduction)

$$\begin{bmatrix} \text{matrix} \end{bmatrix} \xrightarrow{\textit{elementary row operations}} \begin{bmatrix} \text{matrix in the reduced row echelon form} \end{bmatrix}$$

① Interchange rows, if necessary, to bring a non-zero element to the top of the first non-zero column of the matrix.

② Multiply the first row so that its first non-zero entry becomes 1.

③ Add multiples of the first row to eliminate non-zero entries below the leading one.

④ Ignore the first row; apply steps 1-3 to the rest of the matrix.

⑤ Eliminate non-zero entries above all leading ones.

$$\begin{bmatrix} 0 & 1 & 2 & 3 \\ 2 & 5 & 3 & 1 \\ 3 & 1 & 9 & 7 \\ 2 & 5 & 7 & 1 \end{bmatrix} \xrightarrow{R_{12}} \begin{bmatrix} 2 & 5 & 3 & 1 \\ 0 & 1 & 2 & 3 \\ 3 & 1 & 9 & 7 \\ 2 & 5 & 7 & 1 \end{bmatrix} \times \frac{1}{2}.$$

Example.

$$\left[\begin{array}{cccc|c} 0 & 4 & -8 & 0 & 4 \\ 2 & 6 & -6 & -2 & -4 \\ 2 & 7 & -8 & 0 & -1 \end{array}\right]$$

$$\downarrow R_{12}$$

$$\begin{bmatrix} 2 & 6 & -6 & -2 & -4 \\ 0 & 4 & -8 & 0 & 4 \\ 2 & 7 & -8 & 0 & -1 \end{bmatrix} \times \frac{1}{2}$$

$$\downarrow \frac{1}{2} \times R_1$$

$$\begin{bmatrix} 1 & 3 & -3 & -1 & -2 \\ 0 & 4 & -8 & 0 & 4 \\ 2 & 7 & -8 & 0 & -1 \end{bmatrix} \; R_3 - 2R_1$$

$$\downarrow R_3 - 2 \cdot R_1$$

$$\begin{bmatrix} 1 & 3 & -3 & -1 & -2 \\ 0 & 4 & -8 & 0 & 4 \\ 0 & 1 & -2 & 2 & 3 \end{bmatrix} \cdot \frac{1}{4}$$

$$\downarrow \frac{1}{4} \times R_2$$

$$\begin{bmatrix} 1 & 3 & -3 & -1 & -2 \\ 0 & 1 & -2 & 0 & 1 \\ 0 & 1 & -2 & 2 & 3 \end{bmatrix}$$

$$\downarrow R_3 - R_2$$

$$\begin{bmatrix} 1 & 3 & -3 & -1 & -2 \\ 0 & 1 & -2 & 0 & 1 \\ 0 & 0 & 0 & 2 & 2 \end{bmatrix}$$

$$R_1 - 3R_2$$

$$\begin{bmatrix} 1 & 0 & 3 & -1 & -5 \\ 0 & 1 & -2 & 0 & 1 \\ 0 & 0 & 0 & 2 & 2 \end{bmatrix} \times \frac{1}{2}$$

$$\downarrow \frac{1}{2} R_3$$

$$\begin{bmatrix} 1 & 0 & 3 & -1 & -5 \\ 0 & 1 & -2 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}$$

$$\downarrow R_1 + R_3$$

$$\begin{array}{ccccc} x_1 & x_2 & x_3 & x_4 & \\ \end{array}$$
$$\left[\begin{array}{cccc|c} 1 & 0 & 3 & 0 & -4 \\ 0 & 1 & -2 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{array}\right]$$

$$\uparrow$$

Free variable.

$$\begin{cases} x_1 + 3x_3 = -4 \\ x_2 - 2x_3 = 1 \\ x_4 = 1 \end{cases}$$

$$x_1 = -3x_3 - 4$$
$$x_2 = +2x_3 + 1$$
$$x_4 = 1$$

How to solve systems of linear equations: example

$$\begin{cases} 4x_2 - 8x_3 = 4 \\ 2x_1 + 6x_2 - 6x_3 - 2x_4 = -4 \\ 2x_1 + 7x_2 - 8x_3 = -1 \end{cases}$$

Augmented matrix.

$$\left[\begin{array}{cccc|c} 0 & 4 & -8 & 0 & 4 \\ 2 & 6 & -6 & -2 & -4 \\ 2 & 7 & -8 & 0 & -1 \end{array}\right]$$

$$\Big\downarrow \text{ elementary row operation.}$$

$$\left[\begin{array}{ccccc} 1 & 0 & 3 & 0 & -4 \\ 0 & 1 & -2 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{array}\right]$$

$\cancel{x_1 + 3x_3 = -4}$

$$\begin{cases} x_1 = -3c - 4 \\ x_2 = 2c + 1 \\ x_3 = c \\ x_4 = 1 \end{cases}$$

$\underline{c \in \mathbb{R}.}$