$$\left[\begin{array}{cccc|c} \boxed{0} & 4 & -8 & 0 & 4 \\ 2 & \boxed{6} & -6 & -2 & -4 \\ 2 & 7 & -8 & \boxed{0} & -1 \end{array}\right] \xrightarrow{\text{row reduction}} \left[\begin{array}{cccc|c} \boxed{1} & 0 & 3 & 0 & -4 \\ 0 & \boxed{1} & -2 & 0 & 1 \\ 0 & 0 & 0 & \boxed{1} & 1 \end{array}\right]$$

Definition

A *pivot position* in a matrix is a position that after row reduction contains a leading one.

A *pivot column* of a matrix is a column that contains a pivot position.

Theorem

1) A system of linear equations is inconsistent if and only if the last column of its augmented matrix is a pivot column.

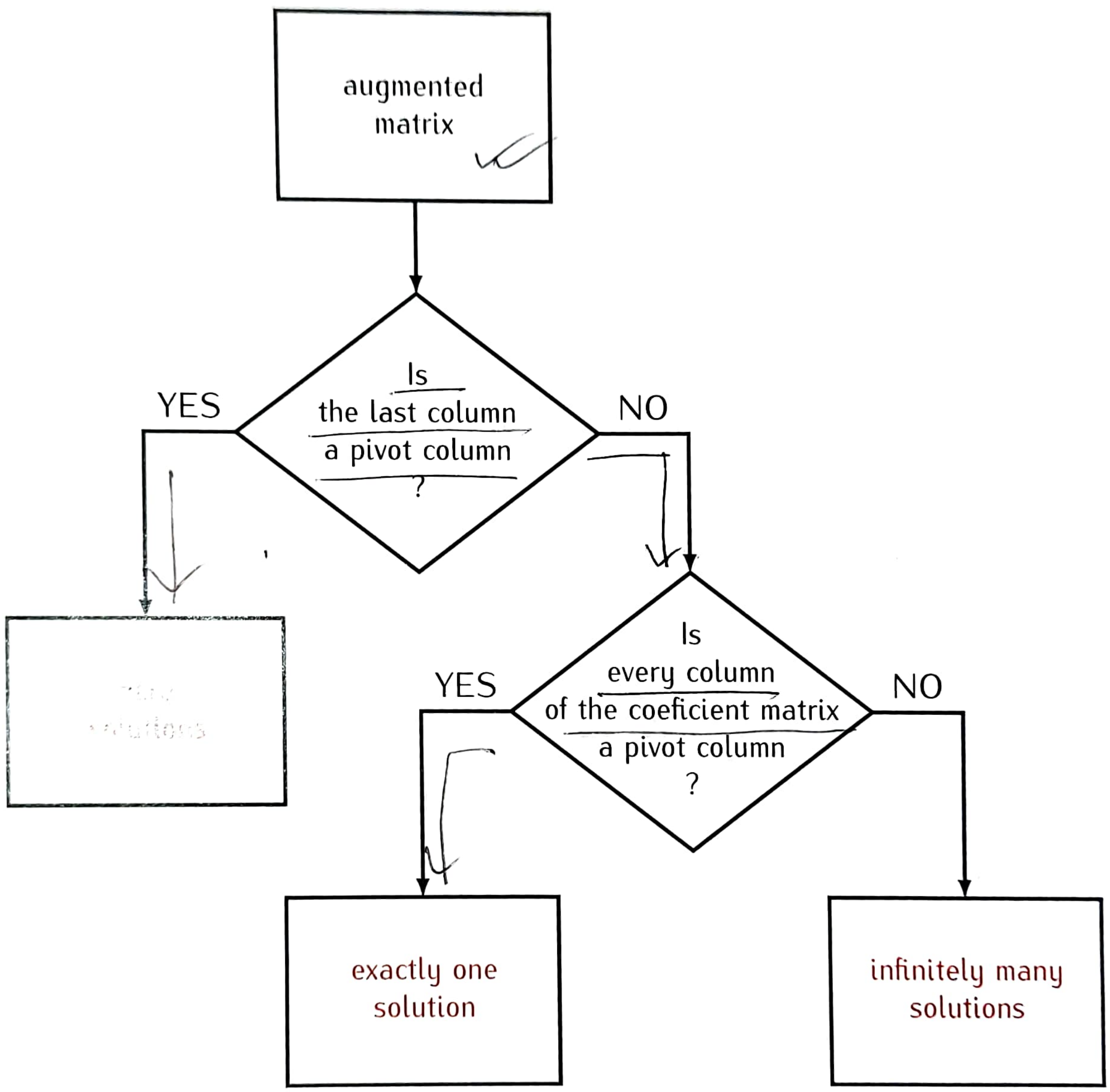
2) Free variables of the system correspond to non-pivot columns of the coefficient matrix.

3) The system has only one solution if and only if every column of its augmented matrix is a pivot column, except for the last column.

Theorem

A system of linear equations can have either 0, 1, or infinitely many solutions.

Proof.



$$x_1 - 2x_2 + x_3 = 0$$
$$2x_2 - 8x_3 = 8$$
$$5x_1 - 5x_3 = 10$$

Augmented matrix:

$$\begin{bmatrix} \boxed{1} & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ \boxed{5} & 0 & -5 & 10 \end{bmatrix}$$

$$\downarrow R_3 - 5R_1$$

$$\begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ 0 & 10 & -10 & 10 \end{bmatrix}$$

$$\downarrow \tfrac{1}{2} R_2.$$

$$\begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 1 & -4 & 4 \\ 0 & 10 & -10 & 10 \end{bmatrix}$$

$$\downarrow \begin{matrix} R_3 - 10R_2. \\ R_1 + 2R_2. \end{matrix}$$

$$\begin{bmatrix} 1 & 0 & -7 & 8 \\ 0 & 1 & -4 & 4 \\ 0 & 0 & 30 & -30 \end{bmatrix}$$

$$\xrightarrow{R_3 \times \frac{1}{30}} \begin{bmatrix} 1 & 0 & -7 & 8 \\ 0 & 1 & -4 & 4 \\ 0 & 0 & 1 & -1 \end{bmatrix}$$

$$\downarrow \begin{matrix} R_1 + 7R_3 \\ R_2 + 4R_3 \end{matrix}$$

$$\begin{bmatrix} \boxed{1} & 0 & 0 & 1 \\ 0 & \boxed{1} & 0 & 0 \\ 0 & 0 & \boxed{1} & -1 \end{bmatrix}$$

Unique soln.

$$\left.\begin{aligned} x_1 &= 1 \\ x_2 &= 0 \\ x_3 &= -1 \end{aligned}\right\}$$