Finite Block Length Analysis on Quantum Coherence Distillation and Incoherent Randomness Extraction

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***Abstract*—We give the *first* systematic study on the second order asymptotics of the operational task of coherence distillation with and without assistance. In the unassisted setting, we introduce a variant of randomness extraction framework where free incoherent operations are allowed before the incoherent measurement and the randomness extractors. We then show that the maximum number of random bits extractable from a given quantum state is precisely equal to the maximum number of coherent bits that can be distilled from the same state. This relation enables us to derive tight second order expansions of both tasks in the independent and identically distributed setting. Remarkably, the incoherent operation classes that can empower coherence distillation for generic states all admit the same second order expansions, indicating their operational equivalence for coherence distillation in both asymptotic and large block length regime. We then generalize the above line of research to the assisted setting, arising naturally in bipartite quantum systems where Bob distills coherence from the state at hand, aided by the benevolent Alice possessing the other system. More precisely, we introduce a new assisted incoherent randomness extraction task and establish an exact relation between this task and the assisted coherence distillation. This strengthens the one-shot relation in the unassisted setting and confirms that this cryptographic framework indeed offers a new perspective to the study of quantum coherence distillation. Likewise, this relation yields second order characterizations to the assisted tasks. As by-products, we show the strong converse property of the aforementioned tasks from their second order expansions.**

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I. Introduction

A central problem in quantum resource theories [1] is the *resource distillation*: the process of extracting canonical units of resources from a given quantum state using free operations. The usual asymptotic approach in quantum information theory is to assume that there is an unbounded number of independent and identically distributed (i.i.d.) copies available and that the error measure asymptotically goes to zero. However, for many practical applications there are natural restrictions on the code length, for example, by limitations on how much quantum information can be processed coherently. It is thus crucial to go beyond the asymptotic regime and understand the intricate tradeoff between different operational parameters of concern.

This work systematically studies the second order asymptotics of coherence distillation under various settings within the resource theory of quantum coherence [2]–[13], a physical resource framework that is essential for various information processing tasks [14]–[19]. Our results are derived from an intrinsic relation between two seemingly different tasks: coherence distillation and randomness extraction. Below, we present an outline of the main results in both unassisted and assisted settings while deferring details to the technical manuscript [20].

II. Unassisted coherence distillation and randomness extraction

In this section we first review the resource theory of quantum coherence and the task of quantum coherence distillation and then introduce a variant of randomness extraction framework in the context of quantum coherence theory.

A. Resource theory of quantum coherence

The resource theory of quantum coherence consists of the following ingredients [12]: the set of *free states* and the set of *free operations*, that is, a set of quantum operations that do not generate coherence. The free states, so-called incoherent states, are the quantum states which are diagonal in a given reference orthonormal basis $\{|b\rangle\}_{b\in\mathcal{B}}$, where $\mathcal{B}$ is the alphabet. We will use $\Delta_B(\cdot) := \sum_{b\in\mathcal{B}} |b\rangle\langle b| \cdot |b\rangle\langle b|$ to denote the diagonal map (completely dephasing channel) in this basis. Then the set of incoherent states is denoted as $\mathcal{I}(B) := \{\rho \in \mathcal{S}(B) : \rho = \Delta_B(\rho)\}$, where $\mathcal{S}(B)$ is the set of quantum states in $\mathcal{H}_B$. For convenience, we will also use the cone of diagonal positive semidefinite operators, denoted as $\mathcal{I}^{**}(B) := \{X \in \mathcal{P}(B) : X = \Delta_B(X)\}$, where $\mathcal{P}(B)$ is the set of positive semidefinite operators in $\mathcal{H}_B$. The maximal resource state on $\mathcal{H}_B$ is the *maximally coherent state (MCS)* $|\Psi_B\rangle := 1/\sqrt{|B|}\sum_{b=1}^{|B|} |b\rangle$ with dimension $|B|$. Denote its density operator as $\Psi_B := |\Psi\rangle\langle\Psi|_B$. The resource theory of coherence is known not to admit a unique physically-motivated choice of allowed free operations [2], [13], [21]–[23]. The relevant choices of free operations that we will focus on are: *maximally incoherent operations (MIO)* [9], defined to be all operations Λ such that

$\Lambda(\rho) \in \mathcal{I}$ for every $\rho \in \mathcal{I}$; *dephasing-covariant incoherent operations (DIO)* [21], [22], which are maps Λ such that $\Delta \circ \Lambda = \Lambda \circ \Delta$; *incoherent operations (IO)* [12], which admit a set of incoherent Kraus operators $\{K_l\}$ such that $K_l \rho K_l^\dagger \in \mathcal{I}^{**}$ for all l and $\rho \in \mathcal{I}$; the intersection of IO and DIO is denoted as DIIO := DIO$\cap$IO [24]. The inclusion relations between free operation classes can be summarized as DIIO $\subsetneq$ IO $\subsetneq$ MIO, DIIO $\subsetneq$ DIO $\subsetneq$ MIO, while IO and DIO are not contained by each other.

B. Framework of quantum coherence distillation

The task of *coherence distillation* aims to transform a given quantum state ρ_B to a maximally coherent state Ψ_C such that the obtained maximally coherent state has dimension as large as possible and that the transformation error is within a given threshold. More formally, for any free operation class $\mathcal{O} \in \{\mathrm{MIO}, \mathrm{DIO}, \mathrm{IO}, \mathrm{DIIO}\}$, any given state $\rho_B \in \mathcal{S}(B)$ and error threshold $\varepsilon \in [0,1]$, the one-shot distillable coherence is defined as

$$C^{\varepsilon}_{d,\mathcal{O}}(\rho_B) := \max_{\Lambda \in \mathcal{O}} \{\log |C| : P(\Lambda_{B\to C}(\rho_B), \Psi_C) \leq \varepsilon\}, \quad (1)$$

where all logarithms in this work are taken base two, and the purified distance P is defined in terms of the generalized quantum fidelity F as $P(\rho,\sigma) := \sqrt{1 - F(\rho,\sigma)^2}$ with $F(\rho,\sigma) := \|\sqrt{\rho}\sqrt{\sigma}\|_1 + \sqrt{(1-\operatorname{Tr}\rho)(1-\operatorname{Tr}\sigma)}$ [25]. Note that some previous works (e.g. [6], [7], [24]) estimate the performance of distillation by the error criterion $P(\Lambda_{B\to C}(\rho_B), \Psi_C) \leq \sqrt{\varepsilon}$. Here we use the definition in (1) for convenience.

C. Framework of incoherent randomness extraction

To study $C^{\varepsilon}_{d,\mathcal{O}}$, we introduce a variant of randomness extraction framework that closely follows the one in [26]. This task aims to obtain as many random bits as possible in Bob's laboratory that is secure from the possible adversary Eve. A general incoherent randomness extraction protocol is characterized by a triplet (Λ, Δ, f), where Λ is an incoherent operation in a certain class, Δ is a completely dephasing channel and f is a hash function. A detailed procedure of randomness extraction by (Λ, Δ, f) is depicted in Fig. 1. Here, we assume that Eve has unlimited power in her system and all the information of Eve about Bob's systems is encoded in a purification. That is, for any given quantum state ρ_B held by Bob, we denote its purification as

$$|\psi\rangle_{BR} := \sum_{b \in \mathcal{B}} \sqrt{p_b} |b\rangle_B |\psi_b\rangle_R, \quad (2)$$

where R is the reference system held by Eve and $\operatorname{Tr}_R \psi_{BR} = \rho_B$. The extraction protocol consists of three steps:

1) Bob first performs a free operation $\Lambda_{B\to C} \in \mathcal{O}$ on his part of the system. We assume that the environment system $\mathcal{H}_E$ of the free operation Λ is also controlled by Eve. Hence, Eve has control over the composite system $\mathcal{H}_{RE}$. To cover this scenario, we consider the Stinespring representation U^Λ of Λ, where U^Λ is the isometry from $\mathcal{H}_B$ to $\mathcal{H}_C \otimes \mathcal{H}_E$. After the action of Λ, the total output state is a pure state

$$\rho[\Lambda]_{CER} := U^\Lambda |\psi\rangle\langle\psi| (U^\Lambda)^\dagger. \quad (3)$$

2) Next, Bob performs an incoherent measurement, with respect to the computational basis, on his part of the state. The output state is then given by

$$\rho[\Lambda, \Delta]_{CER} := \Delta_C(\rho[\Lambda]_{CER}). \quad (4)$$

3) Finally, a hash function f is applied on his part of the system to extract the randomness that is secure from Eve. For any deterministic function $f : \mathcal{C} \to \mathcal{L}$, and any classical-quantum (CQ) state $\rho_{CR} = \sum_{c\in\mathcal{C}} t_c |c\rangle\langle c|_C \otimes \rho_{R|c}$, denote $\rho_{f(C)R} := \sum_{c\in\mathcal{C}} t_c |f(c)\rangle\langle f(c)|_L \otimes \rho_{R|c}$. The output state is given by

$$\rho[\Lambda, \Delta, f]_{LER} := \rho[\Lambda, \Delta]_{f(C)ER}. \quad (5)$$

To quantify the security of randomness in a quantum state ρ_{BR} with respect to Eve, we employ the following measure:

$$d_{sec}(\rho_{BR}|R) := \min_{\sigma_R \in \mathcal{S}(R)} P(\rho_{BR}, \pi_B \otimes \sigma_R), \quad (6)$$

where π_B is the maximally mixed state on $\mathcal{H}_B$. Based on this security measure, the one-shot extractable randomness of ρ_B under free operation class $\mathcal{O}$ is defined as:

$$\ell^{\varepsilon}_{\mathcal{O}}(\rho_B) := \max_{\Lambda \in \mathcal{O}} \max_{f} \{\log |L| : d_{sec}(\rho[\Lambda, \Delta, f]_{LER}|ER) \leq \varepsilon\}, \quad (7)$$

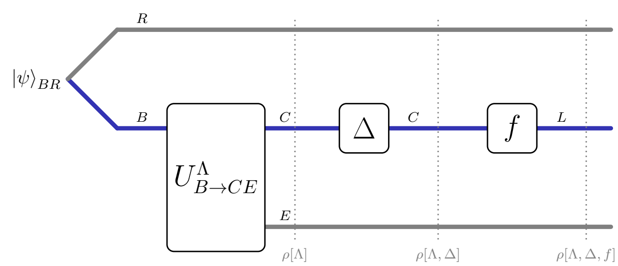

Fig. 1. Schematic diagram of an incoherent randomness extraction protocol given by (Λ, Δ, f). $|\psi\rangle_{BR}$ is a purification of ρ_B. $U^\Lambda_{B\to CE}$ is the Stinespring representation of $\Lambda_{B\to C} \in \mathcal{O}$. Δ is a completely dephasing channel and f is a hash function from alphabet $\mathcal{C}$ to $\mathcal{L}$. $\rho[\Lambda]$, $\rho[\Lambda, \Delta]$ and $\rho[\Lambda, \Delta, f]$ are respectively the output states in each step of the protocol. The systems in blue belong to Bob and the systems in gray belong to Eve.

D. One-shot equivalence

The randomness extraction and quantum coherence distillation are defined in clearly different ways and for different operational purposes. However, we can establish an equivalent relation between them: the maximum number of secure random bits extractable from a given quantum state is exactly equal to the maximum number of coherent bits that can be distilled from the same state.

Theorem 1 *For any quantum state ρ_B and error tolerance $\varepsilon \in [0, 1]$, it holds that*

$$C^{\varepsilon}_{d,\mathcal{O}}(\rho_B) = \ell^{\varepsilon}_{\mathcal{O}}(\rho_B), \tag{8}$$

where the free class $\mathcal{O} \in \{\mathrm{MIO}, \mathrm{DIO}, \mathrm{IO}, \mathrm{DIIO}\}$.

The proof of this result follows from a key observation that we can construct a coherence distillation protocol from a given randomness extraction protocol, and vice versa.

We remark that a one-shot relation between distillable coherence and extractable randomness has appeared in [24, Eq. (80)]. Unlike the precise equation in (8), the relation in [24] is given in the form of one-shot lower and upper bounds with unmatched error dependence and additional correction terms. However, the clean form in (8) plays a pivotal role in deriving the second order expansions where the error dependence matters.

E. Second order analysis

The above equivalence relation unveils a deep connection between two different kinds of resources embedded in the same quantum state. It not only sharpens our understanding on the relation between coherence and randomness, but also leads to an interesting application in their second order estimations as follows.

Theorem 2 *For any quantum state ρ_B and error tolerance $\varepsilon \in [0, 1]$, it holds that*

$$\begin{aligned} & C^{\varepsilon}_{d,\mathcal{O}}(\rho^{\otimes n}) \\ &= \ell^{\varepsilon}_{\mathcal{O}}(\rho^{\otimes n}) \\ &= nD(\rho\|\Delta(\rho)) + \sqrt{nV(\rho\|\Delta(\rho))}\,\Phi^{-1}(\varepsilon^2) + O(\log n), \end{aligned} \tag{9}$$

where the free class $\mathcal{O} \in \{\mathrm{MIO}, \mathrm{DIO}, \mathrm{IO}, \mathrm{DIIO}\}$, $D(\cdot\|\cdot)$ is the quantum relative entropy, $V(\cdot\|\cdot)$ is the quantum information variance, and Φ^{-1} is the inverse of the cumulative distribution function of a standard normal random variable.

This result follows from a combination of the one-shot estimation of the distillable coherence and the second-order expansion of hypothesis testing relative entropy.

Our second order characterization reveals that the distillable coherence under MIO/DIO/IO/DIIO not only have the same first order asymptotics as observed in [2], [6], [24] but also have the same second order asymptotics, indicating that they are equivalently powerful for coherence distillation in the large block length regime. The same argument goes to the incoherent randomness extraction.

III. Assisted Coherence Distillation and Randomness Extraction

We generalize the above line of research to the assisted setting [3], [27]–[32] that arises naturally in bipartite quantum systems where Bob distills coherence from the state at hand, aided by the benevolent Alice possessing the other system [33]–[46].

A. Resource theory of quantum coherence in distributed scenarios

In this distributed resource theory, there are two separated parties, Alice and Bob, that are connected via a classical channel and restricted to performing local incoherent operations. We think of Alice as assistant who helps Bob to manipulate coherence. Here we briefly summarize several sets of free bipartite quantum operations widely studied in this resource theory:

- LICC: the set of *local incoherent operations and classical communication* [47]. That is, Alice and Bob perform local incoherent operations and share their outcomes via a classical channel. Throughout this work, we assume without loss of generality that the free local operations are chosen to be MIO;
- LQICC: the set of *local quantum-incoherent operations and classical communication* [47]. That is, Alice can adopt arbitrary quantum operations while Bob is restricted to quantum incoherent operations, and they share the outcomes via a classical channel;
- SI: the set of *separable incoherent operations* [27]:

$$\Lambda_{AB\to A'B'}(\cdot) := \sum_i (A_i \otimes B_i)(\cdot)(A_i \otimes B_i)^{\dagger}, \tag{10}$$

  where both A_i and B_i are incoherent Kraus operators satisfying $\sum_i A_i^{\dagger} A_i \otimes B_i^{\dagger} B_i = I_{AB}$;
- SQI: the set of *separable quantum-incoherent operations* [27] of the form (10), where B_i are incoherent Kraus operators satisfying $\sum_i A_i^{\dagger} A_i \otimes B_i^{\dagger} B_i = I_{AB}$.

The two free classes LQICC and SQI lead to the same set of free states, which is called the *quantum-incoherent states* [27], [47] (system A is quantum and system B is incoherent) and bears the form

$$\mathcal{QI}(A{:}B) := \left\{ \sigma_{AB} = \sum_{b\in\mathcal{B}} p_b \sigma_A^b \otimes |b\rangle\langle b|_B : \right. \\ \left. p_b \geq 0, \sum_{b\in\mathcal{B}} p_b = 1,\ \sigma_A^b \in \mathcal{S}(A) \right\}. \tag{11}$$

This motivates us to define the maximal set of free operations that preserves $\mathcal{QI}$: the set of *quantum-incoherent state preserving operations* QIP [48].

We assume $\mathcal{F} \in \{\mathrm{LICC}, \mathrm{LQICC}, \mathrm{SI}, \mathrm{SQI}, \mathrm{QIP}\}$ be some chosen free *bipartite* operation class, which is different from the free class $\mathcal{O} \in \{\mathrm{MIO}, \mathrm{DIO}, \mathrm{IO}, \mathrm{DIIO}\}$ in the previous section.

B. Framework of assisted coherence distillation

In the task of assisted coherence distillation, Alice and Bob work together to transform a given quantum state ρ_{AB} (not necessarily pure) to a MCS in system B such that the error is within certain threshold and the obtained MCS has dimension as large as possible, under the constraint that the available quantum operations are chosen from $\mathcal{F}$. We call this task the assisted coherence distillation since we can think of Alice as a helpful environment who holds an *extension* ρ_{AB} of ρ_B possessing certain amount of quantum coherence. See Fig. 2 for illustration. More formally, for any free operation class $\mathcal{F}$, any given state $\rho_{AB} \in \mathcal{S}(AB)$ and error tolerance $\varepsilon \in [0,1]$, the one-shot assisted distillable coherence of ρ_{AB} is defined as

$$C^{\varepsilon}_{d,\mathcal{F}}(\rho_{AB}) := \max_{\Lambda\in\mathcal{F}} \{\log|B'| : P\left(\operatorname{Tr}_{A'} \Lambda_{AB\to A'B'}(\rho_{AB}), \Psi_{B'}\right) \leq \varepsilon\}, \tag{12}$$

where system A' is at Alice's hand, system B' is at Bob's hand, and $\Psi_{B'}$ is a MCS in B'.

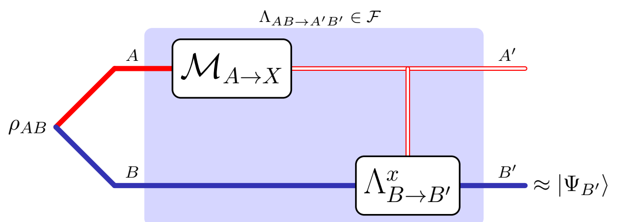

Fig. 2. Schematic diagram of the assisted coherence distillation. Alice and Bob together perform a free bipartite quantum operation $\Lambda_{AB\to A'B'} \in \mathcal{F}$ to distill a MCS $|\Psi_{B'}\rangle$. The system in red belongs to Alice and the system in blue belongs to Bob. The shaded box depicts an one-way LQICC strategy in which Alice performs a measurement $\mathcal{M}_{A\to X}$ and sends the outcome x to Bob. Conditioned on x, Bob performs an incoherent operation $\Lambda^x_{B\to B'}$ to distill $|\Psi_{B'}\rangle$.

C. Framework of assisted incoherent randomness extraction

The task of assisted incoherent randomness extraction aims to obtain as many random bits as possible at Bob's laboratory that is secure from possible attackers such as Eve, under the assistance of a helpful friend Alice. At the beginning, Alice and Bob preshare a bipartite quantum state ρ_{AB} with purification $|\psi\rangle_{RAB}$ such that the reference system R held by Eve. A general assisted incoherent randomness extraction protocol is characterized by a triplet (Λ, Δ, f), where $\Lambda \in \mathcal{F}$, and is composed of three steps:

1) Alice and Bob perform a free operation $\Lambda_{AB\to A'B'} \in \mathcal{F}$ on their joint system. Let $U_{AB\to A'B'E}$ be a Stinespring isometry representation of Λ. We assume the environment system E of Λ is also controlled by Eve. Since Alice is a friend of Bob, Eve has no access to system A'. Hence Eve has control over two systems ER. After the action of Λ, the whole system is in a pure state

$$\rho[\Lambda]_{A'B'ER} := U(|\psi\rangle\langle\psi|_{RAB})U^{\dagger}. \tag{13}$$

2) Bob dephases system B' via the dephasing channel $\Delta_{B'}$. This yields the classical-quantum state

$$\rho[\Lambda,\Delta]_{A'B'ER} := \Delta_{B'}(\rho[\Lambda]_{A'B'ER}) \tag{14}$$

$$= \sum_{b\in\mathcal{B}} p_b |b\rangle\langle b|_{B'} \otimes \sigma^b_{A'ER}, \tag{15}$$

where $p_b := \operatorname{Tr}\langle b|\rho[\Lambda]_{A'B'ER}|b\rangle$ is the probability of outcome b and $\sigma^b_{A'ER} := \langle b|\rho[\Lambda]_{A'B'ER}|b\rangle/p_b$.

3) A hash function f is applied on B' to extract the randomness that is secure from Eve, yielding

$$\rho[\Lambda,\Delta,f]_{A'LER} := \rho[\Lambda,\Delta]_{A'f(B')ER} \tag{16}$$

$$= \sum_{b\in\mathcal{B}} p_b |f(b)\rangle\langle f(b)|_L \otimes \sigma^b_{A'ER}. \tag{17}$$

Using the security measure (6), the one-shot assisted extractable randomness of ρ_{AB} under free operation class $\mathcal{F}$ is defined as

$$\ell^{\varepsilon}_{\mathcal{F}}(\rho_{AB}) := \max_{\Lambda\in\mathcal{F}} \max_{f} \{\log|L| : d_{sec}\left(\rho[\Lambda,\Delta,f]_{LER}|ER\right) \leq \varepsilon\}. \tag{18}$$

D. One-shot equivalence

We present an equivalence relation between the assisted coherence distillation and the assisted incoherent randomness extraction in the one-shot setting, enhancing Theorem 1 to the assisted scenario. The essential ingredient is a systematic method of constructing an assisted coherence distillation protocol from a given assisted randomness extraction protocol with the same performance.

Theorem 3 *For any quantum state ρ_{AB} and error tolerance $\varepsilon \in [0,1]$, it holds that*

$$C^{\varepsilon}_{d,\mathrm{QIP}}(\rho_{AB}) = \ell^{\varepsilon}_{\mathrm{QIP}}(\rho_{AB}), \tag{19}$$

where the definition of the free operation class QIP *can be found in the technical manuscript.*

E. Second order analysis

Building on the one-shot equivalence relation, we are able to derive second order characterizations to these two assisted tasks. We do so by first establishing one-shot achievable and converse bounds in terms of the hypothesis testing relative entropy with matching dependence on the error parameter ε, and then attaining the second order characterization by invoking the second order expansion of the hypothesis testing relative entropy. The result is summarized as follows.

Theorem 4 *For any bipartite quantum state ρ_{AB} and error tolerance $\varepsilon \in [0,1]$, it holds that*

$$C^{\varepsilon}_{d,\mathrm{QIP}}\left(\rho^{\otimes n}_{AB}\right) = nD\left(\rho_{AB}\|\Delta_B(\rho_{AB})\right) + \sqrt{nV\left(\rho_{AB}\|\Delta_B(\rho_{AB})\right)}\Phi^{-1}(\varepsilon^2) + O(\log n), \tag{20a}$$

$$\ell^{\varepsilon}_{\mathcal{F}}\left(\rho^{\otimes n}_{AB}\right) = nD\left(\rho_{AB}\|\Delta_B(\rho_{AB})\right) + \sqrt{nV\left(\rho_{AB}\|\Delta_B(\rho_{AB})\right)}\Phi^{-1}(\varepsilon^2) + O(\log n), \tag{20b}$$

where the free class $\mathcal{F} \in \{\mathrm{LICC}, \mathrm{LQICC}, \mathrm{SI}, \mathrm{SQI}, \mathrm{QIP}\}$.

Theorem 4 constitutes the first second order analysis on the assisted coherence distillation task. It covers many known results as special cases:

1) When $\rho_{AB} = \rho_A \otimes \rho_B$ is a product state, our results reduce to the single partite coherence distillation [2]. This is so because QIP reduces to MIO in single party setting. Theorem 4 matches the second order results for unassisted coherence distillation in Theorem 2.
2) When ρ_{AB} is pure or maximally correlated, Theorem 4 enhances the results of Theorem 5 and Proposition 6 in [27] by stating that the quantum-incoherent relative entropy of coherence $D\left(\rho_{AB} \| \Delta_B(\rho_{AB})\right)$ is the ultimate rate that can be achieved in the assisted coherence distillation even if we make use of the largest free operation class QIP. The same conclusion was previously obtained in [48, Proposition 19] for the pure state case via a different approach.

IV. Strong converse property

In this section we showcase a standard argument how a second order characterization automatically implies the strong converse property. As a concrete example, we consider the task of unassisted coherence distillation whose strong converse property has been pointed out by [24, Theorem 16]. Here we present an alternative proof via the second order characterization. For simplicity, we denote $C_r(\rho) := D(\rho \| \Delta(\rho))$ and $V_r(\rho) := V(\rho \| \Delta(\rho))$. For any achievable rate R_n, we have $R_n \leq \frac{1}{n} C^{\varepsilon}_{d,\mathcal{O}}(\rho^{\otimes n})$. By Theorem 2, we have

$$R_n \leq C_r(\rho) + \sqrt{\frac{V_r(\rho)}{n}}\, \Phi^{-1}(\varepsilon^2) + f(n), \tag{21}$$

where $f(n) = O\left(\log n / n\right)$. Rearranging (21) and using monotonicity of Φ yields

$$\varepsilon^2 \geq \Phi\left(\sqrt{\frac{n}{V_r(\rho)}}(R_n - C_r(\rho)) + g(n)\right), \tag{22}$$

where $g(n) = -\sqrt{n} f(n) / \sqrt{V_r(\rho)}$. Thus $\lim_{n\to+\infty} g(n) = 0$. Note that $\lim_{x\to+\infty} \Phi(x) = 1$. For any achievable rate $R_n > C_r(\rho)$, the argument in (22) diverges to $+\infty$ and thus we have $\varepsilon \to 1$ as $n \to \infty$. This implies the strong converse property of coherence distillation under $\mathcal{O} \in \{\mathrm{MIO}, \mathrm{DIO}, \mathrm{IO}, \mathrm{DIIO}\}$. Similar argument works for the incoherent randomness extraction,

Moreover, we can conclude from Theorem 4 that both the assisted coherence distillation via QIP and the assisted incoherent randomness extraction via arbitrary free operation class $\mathcal{F} \in \{\mathrm{LICC}, \mathrm{LQICC}, \mathrm{SI}, \mathrm{SQI}, \mathrm{QIP}\}$ satisfy the strong converse property. This promises the unique role of $D\left(\rho_{AB} \| \Delta_B(\rho_{AB})\right)$ in the two assisted tasks.

V. Conclusions

This work contributed to the *first* systematic second order analysis on four information-theoretic tasks—the coherence distillation and the randomness extraction, with and without assistance—filling an important gap in the literature. The significance of this work is multi-fold. First, second order expansions of the central quantities in these four tasks provide useful approximations of the corresponding rates in finite block length, refining optimal rates that typically correspond to the first order coefficient in asymptotic expansions. Second, they determine the rate of convergence of these four quantities to the first order coefficient. Finally, second order expansions of these quantities can be used to establish the strong converse property, which rules out a possible tradeoff between the error threshold and the optimal rate.

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