Sketches of Solutions

Assignment 4 Linear Algebra 1

1. Solve this linear system

$$\begin{array}{cccccccc} -2x_1 & + & 3x_2 & + & x_3 & & 9 \\ 3x_1 & + & 4x_2 & - & 5x_3 & = & 0 \\ x_1 & - & 2x_x & + & x_3 & & -4 \end{array}$$

1.1 Using Cramer's Rule

From the given linear equation system, we can rewrite it in a form of $[A]\{x\} = \{B\}$ as

$$\begin{bmatrix} -2 & 3 & 1 \\ 3 & 4 & -5 \\ 1 & -2 & 1 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{Bmatrix} 9 \\ 0 \\ -4 \end{Bmatrix}$$

To find the solution using Cramer's rule, we have to find $\det(A)$

$$\det(A) = \begin{vmatrix} -2 & 3 & 1 \\ 3 & 4 & -5 \\ 1 & -2 & 1 \end{vmatrix} = (-8-15-6)-(4-20+9) = -22$$

Let A_i be a matrix A with replacing column i by {B}, we can determine $\det(A_i)$ where i=1,2,3 as follow

$$\det(A_1) = \begin{vmatrix} 9 & 3 & 1 \\ 0 & 4 & -5 \\ -4 & -2 & 1 \end{vmatrix} = (36+60+0)-(-16+90+0) = 22$$

$$\det(A_2) = \begin{vmatrix} -2 & 9 & 1 \\ 3 & 0 & -5 \\ 1 & -4 & 1 \end{vmatrix} = (0-45-12)-(0-40+27) = -44$$

$$\det(A_3) = \begin{vmatrix} -2 & 3 & 9 \\ 3 & 4 & 0 \\ 1 & -2 & -4 \end{vmatrix} = (32+0-54)-(36+0-36) = -22$$

Then we can find x_1, x_2 and x_3 as follow

$$x_1 = \frac{\det(A_1)}{\det(A)} = \frac{22}{-22} = -1$$

$$x_2 = \frac{\det(A_2)}{\det(A)} = \frac{-44}{-22} = 2$$

$$x_3 = \frac{\det(A_3)}{\det(A)} = \frac{-22}{-22} = 1$$

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1.2 Using Gauss Elimination Method

From $[A]\{x\}=\{B\}$ = $\begin{bmatrix} -2 & 3 & 1 \\ 3 & 4 & -5 \\ 1 & -2 & 1 \end{bmatrix}\begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix}=\begin{Bmatrix} 9 \\ 0 \\ -4 \end{Bmatrix}$, we can rewrite it in the form of $[A|B]$

$$\left[\begin{array}{ccc|c} -2 & 3 & 1 & 9 \\ 3 & 4 & -5 & 0 \\ 1 & -2 & 1 & -4 \end{array}\right]$$

First step of Gaussian Elimination is Forward Elimination. We begin this step by multiplying the first row by -1 and add the third row to it, we get

$$\left[\begin{array}{ccc|c} 1 & -1 & -2 & -5 \\ 3 & 4 & -5 & 0 \\ 1 & -2 & 1 & -4 \end{array}\right]$$

Then we multiply the first row by -3 and add it to the second row to eliminate x_1 from the second row. We obtain

$$\left[\begin{array}{ccc|c} 1 & -1 & -2 & -5 \\ 0 & 7 & 1 & 15 \\ 1 & -2 & 1 & -4 \end{array}\right]$$

To eliminate x_1 from the third row, we multiply the first row by -1 and add it to the third row to obtain

$$\left[\begin{array}{ccc|c} 1 & -1 & -2 & -5 \\ 0 & 7 & 1 & 15 \\ 0 & -1 & 3 & 1 \end{array}\right]$$

To make the coefficient of x_2 to 1, we multiply the third row by 6 and add it to the second row

$$\left[\begin{array}{ccc|c} 1 & -1 & -2 & -5 \\ 0 & 1 & 19 & 21 \\ 0 & -1 & 3 & 1 \end{array}\right]$$

Then we have to get rid of a_{32} by adding the second row to the third row and we get

$$\left[\begin{array}{ccc|c} 1 & -1 & -2 & -5 \\ 0 & 1 & 19 & 21 \\ 0 & 0 & 22 & 22 \end{array}\right]$$

Then we make the coefficient of x_3 to be 1 by dividing the third row by 22 to obtain

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$$\left[\begin{array}{ccc|c} 1 & -1 & -2 & -5 \\ 0 & 1 & 19 & 21 \\ 0 & 0 & 1 & 1 \end{array}\right]$$

After Forward Elimination step, we get the upper triangular matrix. Then we can move on to Back Substitution step which we will obtain $x_3 = 1, x_2 = 2, x_1 = -1$

1.3 Using Gauss-Jordan Method

From Gaussian Elimination, we continue to eliminate the members in the upper triangular of the matrix A.

$$\left[\begin{array}{ccc|c} 1 & -1 & -2 & -5 \\ 0 & 1 & 19 & 21 \\ 0 & 0 & 1 & 1 \end{array}\right]$$

We add the second row to the first row to get

$$\left[\begin{array}{ccc|c} 1 & 0 & 17 & 16 \\ 0 & 1 & 19 & 21 \\ 0 & 0 & 1 & 1 \end{array}\right]$$

To eliminate a_{13} and a_{23}, we multiply the third row by 17 and 19 and subtract it from the first and second row respectively.

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 1 \end{array}\right]$$

Then we obtain the root of tis linear equation system. $x_1 = -1, x_2 = 2, x_3 = 1$

1.4 Using LU Decomposition Method

Consider matrix A $\begin{bmatrix} -2 & 3 & 1 \\ 3 & 4 & -5 \\ 1 & -2 & 1 \end{bmatrix}$, we can decompose it to L and U matrices.

There are 2 arrangements of L and U matrices which are

Crout arrangement : $L = \begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix}$, $U = \begin{bmatrix} 1 & u_{12} & u_{13} \\ 0 & 1 & u_{23} \\ 0 & 0 & 1 \end{bmatrix}$ and

Doolittle arrangement : $L = \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix}$, $U = \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix}$.

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We will apply Doolittle arrangement in this problem.

From the basic idea we studied in Gaussian Elimination method, we know that to eliminate the coefficient of x_1 from other rows we have to divide the first row by a_{11} and multiply it with the a_{i1}. Then we subtract it from each corresponding row i^{th} where i = 2, 3, ..., n.

From matrix A, we try to eliminate a_{21}, a_{31} and a_{32} so we get U

$$\begin{bmatrix} -2 & 3 & 1 \\ 3 & 4 & -5 \\ 1 & -2 & 1 \end{bmatrix} \text{->} \; R2 = R2 + R1 \times \frac{3}{2}, \; R3 = R3 + R1 \times \frac{1}{2} \; \text{->} \begin{bmatrix} -2 & 3 & 1 \\ 0 & \frac{17}{2} & -\frac{7}{2} \\ 0 & -\frac{1}{2} & \frac{3}{2} \end{bmatrix}$$

$$\text{->} \; R3 = R3 + R2 \times \frac{1}{17} \; \text{->} \begin{bmatrix} -2 & 3 & 1 \\ 0 & \frac{17}{2} & -\frac{7}{2} \\ 0 & 0 & \frac{44}{34} \end{bmatrix} = U$$

Notice that we can obtain l_{21}, l_{31} and l_{32} from $\frac{a_{21}}{a_{11}}$, $\frac{a_{31}}{a_{11}}$ and $\frac{a'_{32}}{a'_{22}}$ respectively.

$$L = \begin{bmatrix} 1 & 0 & 0 \\ -\frac{3}{2} & 1 & 0 \\ -\frac{1}{2} & -\frac{1}{17} & 1 \end{bmatrix}$$

From LUx = b, Let Ux = Y then LY = b. From this we can determine Y

$$\begin{bmatrix} 1 & 0 & 0 \\ -\frac{3}{2} & 1 & 0 \\ -\frac{1}{2} & -\frac{1}{17} & 1 \end{bmatrix} \begin{Bmatrix} y_1 \\ y_2 \\ y_3 \end{Bmatrix} = \begin{Bmatrix} 9 \\ 0 \\ -4 \end{Bmatrix}$$

$$y_1 = 9, y_2 = \frac{3}{2} y_1 = \frac{27}{2}, x_3 = -4 + \frac{1}{2} y_1 + \frac{1}{17} y_2 = \frac{44}{34}$$

Then we can determine x from Ux = Y

$$\begin{bmatrix} -2 & 3 & 1 \\ 0 & \frac{17}{2} & -\frac{7}{2} \\ 0 & 0 & \frac{44}{34} \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{Bmatrix} 9 \\ \frac{27}{2} \\ \frac{44}{34} \end{Bmatrix}$$

$$x_1 = 1, x_2 = (\frac{27}{2} + \frac{7}{2}x_1)\frac{2}{17} = (\frac{34}{2})(\frac{2}{17}) = 2, x_1 = -\frac{(9 - 3x_2 - x_3)}{2} = -\frac{9-6-1}{2} = -1$$

1.5 Write programs to solve this system using 4 methods mentioned above

Cramer's Rule

```
determinant = m =>
  m.length == 1 ?
  m[0][0] :
        m.length == 2 ?
        m[0][0]*m[1][1]-m[0][1]*m[1][0] :
        m[0].reduce((r,e,i) => r+(-1)**(i+2)*e*determinant(m.slice(1).map(c => c.filter((_,j) => i
!= j))),0);

A = [[-2, 3, 1],
    [3, 4, -5],
    [1, -2, 1]];

det_A = determinant(A);
//console.log(det_A);
A1 = [[9, 3, 1],
     [0, 4, -5],
     [-4, -2, 1]];
det_A1 = determinant(A1);
A2 = [[-2, 9, 1],
     [3, 0, -5],
     [1, -4, 1]];
det_A2 = determinant(A2);
```

```
A3 = [[-2, 3, 9],
      [3, 4, 0],
      [1, -2, -4]];
det_A3 = determinant(A3);


console.log('x1 = ' + det_A1/det_A + ', x2 = ' + det_A2/det_A + ', x3 = ' + det_A3/det_A);
```

Gaussian Elimination

Gaussian-Jordan

LU Decomposition