Sketches of Solutions

Assignment 4 Linear Algebra 2

1. Find the solutions for the following system of linear equations

$$\begin{bmatrix} 5 & 2 & 0 & 0 \\ 2 & 5 & 2 & 0 \\ 0 & 2 & 5 & 2 \\ 0 & 0 & 2 & 5 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{Bmatrix} = \begin{Bmatrix} 12 \\ 17 \\ 14 \\ 7 \end{Bmatrix}$$

1.1 Using Jacobi Iteration Method with initial values of $x_1=x_2=x_3=x_4=0$. Show your work for 4 iterations. How many iterations will the Jacobi Iteration Method take to solve this system if ε = 0.001%? Write a program to support your answer.

We first rearrange the given sub-equations to iterative form as follow

$$x_1^{k+1} = \frac{b_1 - a_{12}x_2^k - a_{13}x_3^k - a_{14}x_4^k}{a_{11}} \quad (1)$$

$$x_2^{k+1} = \frac{b_2 - a_{21}x_1^k - a_{23}x_3^k - a_{24}x_4^k}{a_{22}} \quad (2)$$

$$x_3^{k+1} = \frac{b_3 - a_{31}x_1^k - a_{32}x_2^k - a_{34}x_4^k}{a_{33}} \quad (3)$$

$$x_4^{k+1} = \frac{b_4 - a_{41}x_1^k - a_{42}x_2^k - a_{43}x_3^k}{a_{44}} \quad (4)$$

After that, we substitute the initial guesses and other values to iteratively determine x_1 to x_4

First approximation :

$$x_1^2 = \frac{12-2(0)-0-0}{5} = 2.4, \varepsilon_1^1 = \left|\frac{2.4-0}{2.4}\right| \times 100 = 100\%$$

$$x_2^2 = \frac{17-2(0)-2(0)-0}{5} = 3.4, \varepsilon_2^1 = \left|\frac{3.4-0}{3.4}\right| \times 100 = 100\%$$

$$x_3^2 = \frac{14-0-2(0)-2(0)}{5} = 2.8, \varepsilon_3^1 = \left|\frac{2.8-0}{2.8}\right| \times 100 = 100\%$$

$$x_4^2 = \frac{7-0-0-2(0)}{5} = 1.4, \varepsilon_4^1 = \left|\frac{1.4-0}{1.4}\right| \times 100 = 100\%$$

Then, we substitute the findings to determine values of the variables in next iteration and calculate the errors and so on.

It can be summarized into the table below

Sketches of Solutions

Iter. No.	x_1^k	x_2^k	x_3^k	x_4^k	x_1^{k+1}	x_2^{k+1}	x_3^{k+1}	x_4^{k+1}	ε_1^k (%)	ε_2^k (%)	ε_3^k (%)	ε_4^k (%)	Max. ε^k (%)
1	0.000	0.000	0.000	0.000	2.400	3.400	2.800	1.400	100.000	100.000	100.000	100.000	100.000
2	2.400	3.400	2.800	1.400	1.040	1.320	0.880	0.280	130.769	157.576	218.182	400.000	400.000
3	1.040	1.320	0.880	0.280	1.872	2.632	2.160	1.048	44.444	49.848	59.259	73.282	73.282
4	1.872	2.632	2.160	1.048	1.347	1.787	1.328	0.536	38.955	47.269	62.651	95.522	95.522
5	1.347	1.787	1.328	0.536	1.685	2.330	1.871	0.869	20.053	23.294	29.011	38.306	38.306
...	...	...	...	...	...	...	...	...	...	...	...	...	...
29	1.554	2.114	1.660	0.736	1.554	2.114	1.660	0.736	0.001	0.001	0.001	0.001	0.001
30	1.554	2.114	1.660	0.736	1.554	2.114	1.660	0.736	0.000	0.000	0.001	0.001	0.001

The calculation will be stopped at the 30th iteration since the ε_a is lesser or equal to ε_s = 0.001% and we get x_1 = 1.554, x_2 = 2.114, x_3 = 1.660 and x_4 = 0.736.

We can evaluate our answers by substituting them back into the original equations

$5x_1 + 2x_2 = 5(1.554) + 2(2.114) = 7.77 + 4.228 = 11.998 \approx 12$

$2x_1 + 5x_2 + 2x_3 = 2(1.554) + 5(2.114) + 2(1.660) = 3.108 + 10.57 + 3.32 = 16.998 \approx 17$

$2x_2 + 5x_3 + 2x_4 = 2(2.114) + 5(1.660) + 2(0.736) = 4.228 + 8.3 + 1.472 = 14$

$2x_3 + 5x_4 = 2(1.660) + 5(0.736) = 3.32 + 3.68 = 7$

So the answers we obtained are converged and valid.

Here is the example of program to solve this problem

1.2 Using Gauss-Seidel Iteration Method with initial values of x1=x2=x3=0. Show your work for 4 iterations. How many iterations will the Gauss-Seidel Iteration Method take to solve this system if ε = 0.001%? Write a program to support your answer.

It is quite similar to question 1.1. However, once we determine x1 we use it to determine the other variables right away.

First Approximation :

$$x_1^2 = \frac{12-2(0)-0-0}{5} = 2.4, \varepsilon_1^1 = \left|\frac{2.4-0}{2.4}\right| \times 100 = 100\%$$

$$x_2^2 = \frac{17-2(2.4)-2(0)-0}{5} = 2.44, \varepsilon_2^1 = \left|\frac{2.44-0}{2.44}\right| \times 100 = 100\%$$

$$x_3^2 = \frac{14-0-2(2.44)-2(0)}{5} = 1.824, \varepsilon_3^1 = \left|\frac{1.824-0}{1.824}\right| \times 100 = 100\%$$

$$x_4^2 = \frac{7-0-0-2(1.824)}{5} = 0.670, \varepsilon_4^1 = \left|\frac{0.670-0}{0.670}\right| \times 100 = 100\%$$

It can be summarized into the table below

Iter. No.	x_1^k	x_2^k	x_3^k	x_4^k	x_1^{k+1}	x_2^{k+1}	x_3^{k+1}	x_4^{k+1}	ε_1^k (%)	ε_2^k (%)	ε_3^k (%)	ε_4^k (%)	Max. ε^k (%)
1	0.000	0.000	0.000	0.000	2.400	2.440	1.824	0.670	100.000	100.000	100.000	100.000	100.000
2	2.400	2.440	1.824	0.670	1.424	2.101	1.692	0.723	68.539	16.146	7.832	7.325	68.539
3	1.424	2.101	1.692	0.723	1.560	2.100	1.671	0.732	8.699	0.061	1.238	1.131	8.699
4	1.560	2.100	1.671	0.732	1.560	2.108	1.664	0.734	0.033	0.383	0.393	0.356	0.393
5	1.560	2.108	1.664	0.734	1.557	2.111	1.662	0.735	0.207	0.185	0.157	0.142	0.207
...	...	...	...	...	...	...	...	...	...	...	...	...	...
11	1.554	2.114	1.660	0.736	1.554	2.114	1.660	0.736	0.001	0.001	0.001	0.001	0.001
12	1.554	2.114	1.660	0.736	1.554	2.114	1.660	0.736	0.001	0.000	0.000	0.000	0.001

After 12 iterations, we get x_1 = 1.554, x_2 = 2.114, x_3 = 1.660 and x_4 = 0.736 and the ε_a is lesser or equal to ε_s = 0.001%.

Here is the example of program to solve this problem

1.3 Using Conjugate Gradient Method with initial values of x1=x2=x3=x4=0. Show your work for 4 iterations. How many iterations will the Conjugate Gradient Method take to solve this system if ε = 0.001%? Write a program to support your answer.

It is given that $\{x\}^0 = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix}$. From R = Ax – b, we get

$$\{R\}^0 = [A]\{x\}^0 - \{b\} = \begin{bmatrix} 5 & 2 & 0 & 0 \\ 2 & 5 & 2 & 0 \\ 0 & 2 & 5 & 2 \\ 0 & 0 & 2 & 5 \end{bmatrix} \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix} - \begin{Bmatrix} 12 \\ 17 \\ 14 \\ 7 \end{Bmatrix} = \begin{Bmatrix} -12 \\ -17 \\ -14 \\ -7 \end{Bmatrix} \text{ and } \{D\}^0 = -\{R\}^0 = \begin{Bmatrix} 12 \\ 17 \\ 14 \\ 7 \end{Bmatrix}$$

Then we begin to iteratively determine {x} as follow

Iteration k = 0:

$$\lambda_0 = -\frac{\lfloor D \rfloor^0 \{R\}^0}{\lfloor D \rfloor^0 [A] \{D\}^0} = \frac{-\{12 \quad 17 \quad 14 \quad 7\}\begin{Bmatrix} -12 \\ -17 \\ -14 \\ -7 \end{Bmatrix}}{\{12 \quad 17 \quad 14 \quad 7\}\begin{bmatrix} 5 & 2 & 0 & 0 \\ 2 & 5 & 2 & 0 \\ 0 & 2 & 5 & 2 \\ 0 & 0 & 2 & 5 \end{bmatrix}\begin{Bmatrix} 12 \\ 17 \\ 14 \\ 7 \end{Bmatrix}} = \frac{(144+289+196+49)}{\{94 \quad 137 \quad 118 \quad 63\}\begin{Bmatrix} 12 \\ 17 \\ 14 \\ 7 \end{Bmatrix}}$$

$$= \frac{678}{(1128+2329+1652+441)} = \frac{678}{5550} = 0.122162$$

$$\{x\}^1 = \{x\}^0 + \lambda_0 \{D\}^0 = 0.122162\begin{Bmatrix} 12 \\ 17 \\ 14 \\ 7 \end{Bmatrix} = \begin{Bmatrix} 1.465944 \\ 2.076754 \\ 1.710268 \\ 0.855134 \end{Bmatrix}$$

$$\{R\}^1 = [A]\{x\}^1 - \{b\} = \begin{bmatrix} 5 & 2 & 0 & 0 \\ 2 & 5 & 2 & 0 \\ 0 & 2 & 5 & 2 \\ 0 & 0 & 2 & 5 \end{bmatrix}\begin{Bmatrix} 1.465944 \\ 2.076754 \\ 1.710268 \\ 0.855134 \end{Bmatrix} - \begin{Bmatrix} 12 \\ 17 \\ 14 \\ 7 \end{Bmatrix} = \begin{Bmatrix} -0.516757 \\ -0.263784 \\ 0.415135 \\ 0.696216 \end{Bmatrix}$$

$$\text{Error}^1 = \sqrt{\lfloor R \rfloor^1 \{R\}^1} = \sqrt{\{-0.516757 \quad -0.263784 \quad 0.415135 \quad -0.696216\}\begin{Bmatrix} -0.516757 \\ -0.263784 \\ 0.415135 \\ 0.696216 \end{Bmatrix}}$$

$$= 0.996832$$

$$\alpha_0 = \frac{\lfloor R \rfloor^1 [A]\{D\}^0}{\lfloor D \rfloor^0 [A]\{D\}^0}$$

$$= \frac{\{-0.516757 \quad -0.263784 \quad 0.415135 \quad -0.696216\}\begin{bmatrix} 5 & 2 & 0 & 0 \\ 2 & 5 & 2 & 0 \\ 0 & 2 & 5 & 2 \\ 0 & 0 & 2 & 5 \end{bmatrix}\begin{Bmatrix} 12 \\ 17 \\ 14 \\ 7 \end{Bmatrix}}{\{12 \quad 17 \quad 14 \quad 7\}\begin{bmatrix} 5 & 2 & 0 & 0 \\ 2 & 5 & 2 & 0 \\ 0 & 2 & 5 & 2 \\ 0 & 0 & 2 & 5 \end{bmatrix}\begin{Bmatrix} 12 \\ 17 \\ 14 \\ 7 \end{Bmatrix}}$$

$$= \frac{\{-3.111351 \quad -1.522162 \quad 2.940541 \quad 4.311351\}\begin{Bmatrix} 12 \\ 17 \\ 14 \\ 7 \end{Bmatrix}}{5550} = \frac{8.134054}{5550}$$

$$= 0.001466$$

$$\{D\}^1 = -\{R\}^1 + \alpha_0 \{D\}^0 = -\begin{Bmatrix} -0.516757 \\ -0.263784 \\ 0.415135 \\ 0.696216 \end{Bmatrix} + 0.001466\begin{Bmatrix} 12 \\ 17 \\ 14 \\ 7 \end{Bmatrix} = \begin{Bmatrix} 0.534344 \\ 0.288699 \\ -0.394617 \\ -0.685957 \end{Bmatrix}$$

Iteration k = 1:

$$\lambda_1 = -\frac{\lfloor D \rfloor^1 \{R\}^1}{\lfloor D \rfloor^1 [A]\{D\}^1}$$

$$= -\frac{\{0.534344 \quad 0.288699 \quad -0.394617 \quad -0.685957\}\begin{Bmatrix} -0.516757 \\ -0.263784 \\ 0.415135 \\ 0.696216 \end{Bmatrix}}{\{0.534344 \quad 0.288699 \quad -0.394617 \quad -0.685957\}\begin{bmatrix} 5 & 2 & 0 & 0 \\ 2 & 5 & 2 & 0 \\ 0 & 2 & 5 & 2 \\ 0 & 0 & 2 & 5 \end{bmatrix}\begin{Bmatrix} 0.534344 \\ 0.288699 \\ -0.394617 \\ -0.685957 \end{Bmatrix}}$$

$$= 0.159761$$

$$\{x\}^2 = \{x\}^1 + \lambda_1\{D\}^1 = \begin{Bmatrix} 1.465944 \\ 2.076754 \\ 1.710268 \\ 0.855134 \end{Bmatrix} + 0.159761 \begin{Bmatrix} 0.534344 \\ 0.288699 \\ -0.394617 \\ -0.685957 \end{Bmatrix} = \begin{Bmatrix} 1.551313 \\ 2.122879 \\ 1.647226 \\ 0.745546 \end{Bmatrix}$$

$$\{R\}^2 = [A]\{x\}^2 - \{b\} = \begin{bmatrix} 5 & 2 & 0 & 0 \\ 2 & 5 & 2 & 0 \\ 0 & 2 & 5 & 2 \\ 0 & 0 & 2 & 5 \end{bmatrix} \begin{Bmatrix} 1.551313 \\ 2.122879 \\ 1.647226 \\ 0.745546 \end{Bmatrix} - \begin{Bmatrix} 12 \\ 17 \\ 14 \\ 7 \end{Bmatrix} = \begin{Bmatrix} 0.002324 \\ 0.011476 \\ -0.027018 \\ 0.022183 \end{Bmatrix}$$

$$\text{Error}^2 = \sqrt{\lfloor R \rfloor^2 \{R\}^2} = \sqrt{\{0.002324 \quad 0.011476 \quad -0.027018 \quad 0.022183\} \begin{Bmatrix} 0.002324 \\ 0.011476 \\ -0.027018 \\ 0.022183 \end{Bmatrix}}$$

$$= 0.036867$$

$$\alpha_1 = \frac{\lfloor R \rfloor^2 [A]\{D\}^1}{\lfloor D \rfloor^1 [A]\{D\}^1}$$

$$= \frac{\{0.002324 \quad 0.011476 \quad -0.027018 \quad 0.022183\} \begin{bmatrix} 5 & 2 & 0 & 0 \\ 2 & 5 & 2 & 0 \\ 0 & 2 & 5 & 2 \\ 0 & 0 & 2 & 5 \end{bmatrix} \begin{Bmatrix} 0.534344 \\ 0.288699 \\ -0.394617 \\ -0.685957 \end{Bmatrix}}{\{0.534344 \quad 0.288699 \quad -0.394617 \quad -0.685957\} \begin{bmatrix} 5 & 2 & 0 & 0 \\ 2 & 5 & 2 & 0 \\ 0 & 2 & 5 & 2 \\ 0 & 0 & 2 & 5 \end{bmatrix} \begin{Bmatrix} 0.534344 \\ 0.288699 \\ -0.394617 \\ -0.685957 \end{Bmatrix}}$$

$$= \frac{0.008508}{6.219767} = 0.001368$$

$$\{D\}^2 = -\{R\}^2 + \alpha_1\{D\}^1 = -\begin{Bmatrix} 0.002324 \\ 0.011476 \\ -0.027018 \\ 0.022183 \end{Bmatrix} + 0.001368 \begin{Bmatrix} 0.534344 \\ 0.288699 \\ -0.394617 \\ -0.685957 \end{Bmatrix} = \begin{Bmatrix} -0.001593 \\ -0.011081 \\ 0.026479 \\ -0.023122 \end{Bmatrix}$$

Iteration k = 2:

$$\lambda_2 = -\frac{\lfloor D \rfloor^2 \{R\}^2}{\lfloor D \rfloor^2 [A]\{D\}^2} = 0.417782$$

$$\{x\}^3 = \{x\}^2 + \lambda_2\{D\}^2 = \begin{Bmatrix} 1.550647 \\ 2.118250 \\ 1.658288 \\ 0.735886 \end{Bmatrix}$$

$$\{R\}^3 = [A]\{x\}^3 - \{b\} = \begin{Bmatrix} -0.010263 \\ 0.009122 \\ -0.000285 \\ -0.003991 \end{Bmatrix}$$

$$\text{Error}^3 = \sqrt{\lfloor R \rfloor^3 \{R\}^3} = 0.014302$$

$$\alpha_2 = \frac{\lfloor R \rfloor^3 [A]\{D\}^2}{\lfloor D \rfloor^2 [A]\{D\}^2} = 0.150494$$

$$\{D\}^3 = -\{R\}^3 + \alpha_2\{D\}^2 = \begin{Bmatrix} 0.010023 \\ -0.010790 \\ 0.004270 \\ 0.000511 \end{Bmatrix}$$

Iteration k = 3:

$$\lambda_3 = -\frac{\lfloor D \rfloor^3 \{R\}^3}{\lfloor D \rfloor^3 [A]\{D\}^3} = 0.359698$$

$$\{x\}^4 = \{x\}^3 + \lambda_3\{D\}^3 = \begin{Bmatrix} 1.554252 \\ 2.114370 \\ 1.659824 \\ 0.736070 \end{Bmatrix}$$

$$\{R\}^4 = [A]\{x\}^4 - \{b\} = \begin{Bmatrix} 0.000000 \\ 0.000000 \\ 0.000000 \\ 0.000000 \end{Bmatrix}$$

$$\text{Error}^4 = \sqrt{\lfloor R \rfloor^4 \{R\}^4} = 0.000000$$

We can stop since the error is lesser or equal to ε_s = 0.001%. We do not have to determine α_3 and $\{D\}^4$.

All the iterations can be summarized and shown in the table below

N	$\{x\}^k$	$\{R\}^k$	$\{D\}^k$	λ_k	$\{x\}^{k+1}$	$\{R\}^{k+1}$	$Error^{k+1}$	α_k	$\{D\}^{k+1}$
0	$\begin{Bmatrix}0\\0\\0\\0\end{Bmatrix}$	$\begin{Bmatrix}-12\\-17\\-14\\-7\end{Bmatrix}$	$\begin{Bmatrix}12\\17\\14\\7\end{Bmatrix}$	0.122162	$\begin{Bmatrix}1.465944\\2.076754\\1.710268\\0.855134\end{Bmatrix}$	$\begin{Bmatrix}-0.516757\\-0.263784\\0.415135\\0.696216\end{Bmatrix}$	0.996832	0.001466	$\begin{Bmatrix}0.534344\\0.288699\\-0.394617\\-0.685957\end{Bmatrix}$
1	$\begin{Bmatrix}1.465944\\2.076754\\1.710268\\0.855134\end{Bmatrix}$	$\begin{Bmatrix}-0.516757\\-0.263784\\0.415135\\0.696216\end{Bmatrix}$	$\begin{Bmatrix}0.534344\\0.288699\\-0.394617\\-0.685957\end{Bmatrix}$	0.159761	$\begin{Bmatrix}1.551313\\2.122879\\1.647226\\0.745546\end{Bmatrix}$	$\begin{Bmatrix}0.002324\\0.011476\\-0.027018\\0.022183\end{Bmatrix}$	0.036867	0.001368	$\begin{Bmatrix}-0.001593\\-0.011081\\0.026479\\-0.023122\end{Bmatrix}$
2	$\begin{Bmatrix}1.551313\\2.122879\\1.647226\\0.745546\end{Bmatrix}$	$\begin{Bmatrix}0.002324\\0.011476\\-0.027018\\0.022183\end{Bmatrix}$	$\begin{Bmatrix}-0.001593\\-0.011081\\0.026479\\-0.023122\end{Bmatrix}$	0.417782	$\begin{Bmatrix}1.550647\\2.118250\\1.658288\\0.735886\end{Bmatrix}$	$\begin{Bmatrix}-0.010263\\0.009122\\-0.000285\\-0.003991\end{Bmatrix}$	0.014302	0.150494	$\begin{Bmatrix}0.010023\\-0.010790\\0.004270\\0.000511\end{Bmatrix}$
3	$\begin{Bmatrix}1.550647\\2.118250\\1.658288\\0.735886\end{Bmatrix}$	$\begin{Bmatrix}-0.010263\\0.009122\\-0.000285\\-0.003991\end{Bmatrix}$	$\begin{Bmatrix}0.010023\\-0.010790\\0.004270\\0.000511\end{Bmatrix}$	0.359658	$\begin{Bmatrix}1.554252\\2.114370\\1.659824\\0.736070\end{Bmatrix}$	$\begin{Bmatrix}0.000000\\0.000000\\0.000000\\0.000000\end{Bmatrix}$	0.000000	0.000000	$\begin{Bmatrix}0.000000\\0.000000\\0.000000\\0.000000\end{Bmatrix}$

After 4 iterations, we get x_1 = 1.554252, x_2 = 2.114370, x_3 = 1.659824 and x_4 = 0.736070.

We can evaluate our answers by substituting them back into the original equations

$5x_1 + 2x_2 = 5(1.554252) + 2(2.114370) = 7.7713 + 4.2287 = 12.000000$

$2x_1 + 5x_2 + 2x_3 = 2(1.554252) + 5(2.114370) + 2(1.659824) = 3.1085 + 10.5719 + 3.3196 = 17.000000$

$2x_2 + 5x_3 + 2x_4 = 2(2.114370) + 5(1.659824) + 2(0.736070) = 4.2287 + 8.2991 + 1.472 = 14.000000$

$2x_3 + 5x_4 = 2(1.659824) + 5(0.736070) = 3.3196 + 3.6803 = 7.000000$

So the answers we obtained are converged and valid.

Here is the example of program to solve this problem