Assignment 7 NUMERICAL INTEGRATION

1. Evaluate the following integral

$$I = \int_2^8 (4x^5 - 3x^4 + x^3 - 6x + 2)dx \qquad (1)$$

1.1 Using Single Trapezoidal Rule and calculate the true error.

1.2 Using Composite Trapezoidal Rule with n= 2, 4, 6 and calculate the true and approximation errors.

1.3 Write a program to do as in 1.1

1.4 Write a program to do as in 1.2

1.1 Trapezoidal Rule : $I \approx (b-a)\left(\frac{f(a)+f(b)}{2}\right)$

≈ (8-2)*(f(a)+(f(b))*0.5 ≈ 3*(f(a)+(f(b))

$$f(a) = 4(2^5) - 3(2^4) + 2^3 - 6(2) + 2 = 78$$

$$f(b) = 4(8^5) - 3(8^4) + 8^3 - 6(8) + 2 = 119250$$

$$\therefore I \approx 3(119250 - 78) \approx 357516$$

Analytic :

$$I = \frac{2x^6}{3} - \frac{3x^5}{5} + \frac{x^4}{4} - 3x^2 + 2x \Big|_2^8$$

$$= \left(\frac{2(8^6)}{3} - \frac{3(8^5)}{5} + \frac{8^4}{4} - 3(8^2) + 2(8)\right) - \left(\frac{2(2^6)}{3} - \frac{3(2^5)}{5} + \frac{2^4}{4} - 3(2^2) + 2(2)\right)$$

$$= \left(\frac{2*262144}{3} - \frac{3*32768}{5} + \frac{4096}{4} - 192 + 16\right) - \left(\frac{2*64}{3} - \frac{3*32}{5} + \frac{16}{4} - 12 + 4\right)$$

$$= (174762.6666667 - 19660.8 + 1024 - 192 + 16) - (42.6666667 - 19.2 + 4 - 12 + 4)$$

$$= 155949.8666667 - 19.4666667$$

$$= 155930.4$$

error = $100*\left(\frac{155930.4 - 357516}{155930.4}\right) = -129.2792168\%$

1.2 Composite Trapezoidal Rule

<u>Case n=2</u>; h = (8-2)/2 = 3

From Trapezoidal Rue

$$f(x_0 = 2) = 4(2^5) - 3(2^4) + 2^3 - 6(2) + 2 = 78$$

$$f(x_1 = 5) = 4(5^5) - 3(5^4) + 5^3 - 6(5) + 2 = 10722$$

$$f(x_2 = 8) = 4(8^5) - 3(8^4) + 8^3 - 6(8) + 2 = 119250$$

$$I \approx \frac{h}{2}\left(f(x_0) + f(x_2) + 2 * \sum_{i=1}^{2-1} f(x_i) \right) \approx \frac{3}{2}(78 + 119250 + 2 * 10722) \approx 211158$$

$$error = 100 * \left(\frac{155930.4 - 211158}{155930.4} \right) = -35.418109618\%$$

Case n=4; h = (8-2)/4 = 1.5

From Trapezoidal Rue

$$f(x_0 = 2) = 4(2^5) - 3(2^4) + 2^3 - 6(2) + 2 = 78$$

$$f(x_1 = 3.5) = 4(3.5^5) - 3(3.5^4) + 3.5^3 - 6(3.5) + 2 = 1674.5625$$

$$f(x_2 = 5) = 4(5^5) - 3(5^4) + 5^3 - 6(5) + 2 = 10722$$

$$f(x_3 = 6.5) = 4(6.5^5) - 3(6.5^4) + 6.5^3 - 6(6.5) + 2 = 41294.0625$$

$$f(x_4 = 8) = 4(8^5) - 3(8^4) + 8^3 - 6(8) + 2 = 119250$$

$$I \approx \frac{h}{2}\left(f(x_0) + f(x_4) + 2 * \sum_{i=1}^{4-1} f(x_i) \right)$$

$$\approx \frac{1.5}{2}(78 + 119250 + 2(1674.5625 + 10722 + 41294.0625)) \approx 170031.9375$$

$$error = 100 * \left(\frac{155930.4 - 170031.9375}{155930.4} \right) = -9.0434818\%$$

Case n=6; h = (8-2)/4 = 1

From Trapezoidal Rue

$$f(x_0 = 2) = 4(2^5) - 3(2^4) + 2^3 - 6(2) + 2 = 78$$
$$f(x_1 = 3) = 4(3^5) - 3(3^4) + 3^3 - 6(3) + 2 = 740$$
$$f(x_2 = 4) = 4(4^5) - 3(4^4) + 4^3 - 6(4) + 2 = 3370$$
$$f(x_3 = 5) = 4(5^5) - 3(5^4) + 5^3 - 6(5) + 2 = 10722$$
$$f(x_4 = 6) = 4(6^5) - 3(6^4) + 6^3 - 6(6) + 2 = 27398$$
$$f(x_5 = 7) = 4(7^5) - 3(7^4) + 7^3 - 6(7) + 2 = 60328$$
$$f(x_6 = 8) = 4(8^5) - 3(8^4) + 8^3 - 6(8) + 2 = 119250$$
$$I \approx \frac{h}{2}\left(f(x_0) + f(x_6) + 2 * \sum_{i=1}^{6-1} f(x_i) \right)$$
$$\approx \frac{1}{2}\left(78 + 119250 + 2(740 + 3370 + 10722 + 27398 + 60328)\right) \approx 162222$$
$$error = 100 * \left(\frac{155930.4 - 162222}{155930.4} \right) = -4.0348771\%$$

2. Evaluate the following integral

$$I = \int_{-1}^{2} (x^7 + 2x^3 - 1)dx \qquad (2)$$

2.1 Using SIMPSON'S RULE and calculate the true error.

2.2 Using COMPOSITE SIMPSON'S RULE with n= 2, 4, 6 and calculate the true and approximation errors.

2.3 Write a program to do as in 2.1

2.4 Write a program to do as in 2.2

2.1

Exact solution :

$$I = \int_{-1}^{2} (x^7 + 2x^3 - 1)dx = \frac{x^8}{8} - \frac{x^4}{2} - x \Big|_{-1}^{2} = \left(\frac{2^8}{8} + \frac{2^4}{2} - 2 \right) - \left(\frac{(-1)^8}{8} + \frac{(-1)^4}{2} - (-1) \right)$$
$$= (32 + 8 - 2) - (0.125 + 0.5 + 1) = 38 - 1.625 = 36.375$$

Simpson's rule :

$$h = \frac{(b-a)}{2} = \frac{2-(-1)}{2} = 1.5$$

$$I \approx \frac{(b-a)}{6}(f(x_0) + 4f(x_1) + f(x_2))$$

$$f(x_0 = -1) = (-1)^7 + 2(-1)^3 - 1 = -4$$

$$f(x_1 = 0.5) = (0.5)^7 + 2(0.5)^3 - 1 = -0.7421875$$

$$f(x_2 = 2) = (2)^7 + 2(2)^3 - 1 = 143$$

$$\therefore I \approx \frac{1}{2}\left(-4 + 4(-0.7421875) + 143\right) \approx 68.015265$$

$$error = 100 * \left(\frac{36.375 - 68.015265}{36.375}\right) = -86.9835464\%$$

2.2 Composite Simpson's Rule

Case n=2; it has been done in 2.1

Case n=4;

$$h = \frac{(b-a)}{4} = \frac{2-(-1)}{4} = 0.75$$

$$I \approx \frac{h}{3}\left[f(x_0) + f(x_n) + 4\sum_{i=1,3,5}^{n-1} f(x_i) + 2\sum_{i=2,4,6}^{n-2} f(x_i)\right]$$

$$f(x_0 = -1) = (-1)^7 + 2(-1)^3 - 1 = -4$$

$$f(x_1 = -0.25) = (-0.25)^7 + 2(-0.25)^3 - 1 = -1.03131103515625$$

$$f(x_2 = 0.5) = (0.5)^7 + 2(0.5)^3 - 1 = -0.7421875$$

$$f(x_3 = 1.25) = (1.25)^7 + 2(1.25)^3 - 1 = 7.67462158203125$$

$$f(x_4 = 2) = (2)^7 + 2(2)^3 - 1 = 143$$

$$\therefore I \approx \frac{0.75}{3}\left(-4 + 143 + 4(-1.03131103515625 + 7.67462158203125) + 2(-0.7421875)\right)$$

$$\approx \frac{0.75}{3}(139 + 4(6.643310546875) - 1.484375) \approx \frac{0.75}{3}(139 + 26.5732421875 - 1.484375)$$

$$\approx \frac{0.75}{3}(164.0888671875) \approx 41.022216796875 \approx 41.022$$

$$error = 100 * \left(\frac{36.375 - 41.022}{36.375}\right) = -12.775258\%$$

Case n=6;

$$h = \frac{(b-a)}{6} = \frac{2-(-1)}{6} = 0.5$$

$$I \approx \frac{h}{3}\left[f(x_0) + f(x_n) + 4\sum_{i=1,3,5}^{n-1} f(x_i) + 2\sum_{i=2,4,6}^{n-2} f(x_i) \right]$$

$$f(x_0 = -1.0) = (-1)^7 + 2(-1)^3 - 1 = -4$$

$$f(x_1 = -0.5) = (-0.5)^7 + 2(-0.5)^3 - 1 = -1.2578125$$

$$f(x_2 = 0.0) = (0.0)^7 + 2(0.0)^3 - 1 = -1$$

$$f(x_3 = 0.5) = (0.5)^7 + 2(0.5)^3 - 1 = -0.7421875$$

$$f(x_4 = 1.0) = (1.0)^7 + 2(1.0)^3 - 1 = 2$$

$$f(x_5 = 1.5) = (1.5)^7 + 2(1.5)^3 - 1 = 22.8359375$$

$$f(x_6 = 2.0) = (2)^7 + 2(2)^3 - 1 = 143$$

$$\therefore I \approx \frac{0.5}{3}\left(-4 + 143 + 4(-1.2578125 - 0.7421875 + 22.8359375) + 2(-1+2)\right)$$

$$\approx \frac{0.5}{3}(139 + 4(20.8359375) + 2) \approx \frac{1}{6}(139 + 83.34375 + 2) \approx \frac{1}{6}(224.34375)$$

$$\approx 37.390625$$

$$error = 100 * \left(\frac{36.375 - 37.390625}{36.375}\right) = -2.792096\%$$

3. The force on a sailboat mast can be represented by the following function

$$F = \int_0^H 200\left(\frac{z}{5+z}\right)e^{-\frac{2z}{H}}\,dz \qquad (3)$$

where z is the elevation above the deck and H is the height of the mast. This function has no close form solution. Compute F for a case where H equals to 30 using Romberg integration to a tolerance of ε_s = 0.5%.

Let us use Trapezoidal Rule and Composite Trapezoidal Rule to approximate the integral in each step

Given that H = 30, the integral F will be expressed as

$$F = \int_0^{30} 200\left(\frac{z}{5+z}\right)e^{-\frac{2z}{30}}dz$$

Let us try n = 1, 2, 4, 8 to see if we can get to ε_s or not

$$h_1 = \frac{(b-a)}{1} = \frac{30-0}{1} = 30$$

$$I(h_1) \approx \frac{h}{2}\left(f(x_0)+f(x_1)\right) \approx \frac{30}{2}*200\left[\left(\frac{30}{5+30}e^{\frac{-2(30)}{30}}\right)-\left(\frac{0}{5+0}e^{\frac{-2(0)}{0}}\right)\right] \approx 3000(0.1160016713-0)$$

$$\approx 348.0050140370$$

$$h_2 = \frac{(b-a)}{2} = \frac{30-0}{2} = 15$$

$$f(x_0 = 0) = 0$$

$$f(x_1 = 15) = \frac{15}{20}e^{\frac{-2(15)}{30}} = 0.2759098088$$

$$f(x_2 = 30) = 0.1160016713$$

$$I(h_2) \approx \frac{h}{2}*200*\left(f(x_0)+f(x_2)+2*\sum_{i=1}^{2-1}f(x_i)\right) \approx \frac{15}{2}*200(0.1160016713+2*0.2759098088)$$

$$\approx 1001.73193335$$

$$h_4 = \frac{(b-a)}{4} = \frac{30-0}{4} = 7.5$$

$$f(x_0 = 0) = 0$$

$$f(x_1 = 7.5) = \frac{7.5}{12.5}e^{\frac{-2(7.5)}{30}} = 0.3639183598$$

$$f(x_2 = 15) = \frac{15}{20}e^{\frac{-2(15)}{30}} = 0.2759098088$$

$$f(x_3 = 22.5) = \frac{22.5}{27.5}e^{\frac{-2(22.5)}{30}} = 0.1825610401$$

$$f(x_4 = 30) = 0.1160016713$$

$$I(h_4) \approx \frac{h}{2}*200*\left(f(x_0)+f(x_4)+2*\sum_{i=1}^{2-1}f(x_i)\right)$$

$$\approx \frac{7.5}{2}*200(0.1160016713+2(0.3639183598+0.2759098088+0.1825610401))$$

$$\approx 750*(0.1160016713+1.6447784174) \approx 1320.5850066525$$

$h_8 = \frac{(b-a)}{8} = \frac{30-0}{8} = 3.75$

$f(x_0 = 0) = 0$

$f(x_1 = 3.75) = \frac{3.75}{8.75} e^{\frac{-2(3.75)}{30}} = 0.3337717642$

$f(x_2 = 7.5) = \frac{7.5}{12.5} e^{\frac{-2(7.5)}{30}} = 0.3639183598$

$f(x_3 = 11.25) = \frac{11.25}{16.25} e^{\frac{-2(11.25)}{30}} = 0.3270229981$

$f(x_4 = 15) = \frac{15}{20} e^{\frac{-2(15)}{30}} = 0.2759098088$

$f(x_5 = 18.75) = \frac{18.75}{23.75} e^{\frac{-2(18.75)}{30}} = 0.2261879975$

$f(x_6 = 22.5) = \frac{22.5}{27.5} e^{\frac{-2(22.5)}{30}} = 0.1825610401$

$f(x_7 = 26.25) = \frac{26.25}{31.25} e^{\frac{-2(26.25)}{30}} = 0.1459701125$

$f(x_8 = 30) = 0.1160016713$

$$I(h_4) \approx \frac{h}{2} * 200 * \left(f(x_0) + f(x_8) + 2 * \sum_{i=1}^{8-1} f(x_i) \right)$$

$$\approx 375 \begin{pmatrix} 0.1160016713 + 2(0.3337717642 + 0.3639183598 + 0.3270229981 + 0.2759098088 + \\ 0.2261879975 + 0.1825610401 + 0.1459701125) \end{pmatrix}$$

≈ 1435.0071874125

Then we calculate Romberg Integration for k= 1, 2, 3

At k= 1; we can combine $I(h_1)$ and $I(h_2)$ as follow

$$I = \frac{4^k}{4^k - 1} I(h_2) - \frac{1}{4^k - 1} I(h_1) = \frac{4}{3}(1001.73193335) - \frac{1}{3}(348.0050140370) =$$

1219.6409064543

$\varepsilon_a = \left| \frac{1219.6409064543 - 1001.73193335}{1219.6409064543} \right| = 17.866513610\%$ which is a maximum error in this level

As for the other level of k, we can summarize and show the results in the following table

n		k=1	k=2	k=3
ε_a		17.8665136110%	0.9589354841%	0.0382084249%
1	348.0050140370	1219.6409064543	1440.6846801030	1476.7973854036
2	1001.73193335	1426.86944425	1476.2331243833	
4	1320.585066525	1473.147894375		
8	1435.0071874125			

Thus, the approximated integral is 1476.7973854036

4. Write a program to evaluate the function in equation (3) as in question 3.

$$a_0 = \frac{\left(\sum_{i=1}^{n} y_i\right)\left(\sum_{i=1}^{n} x_i^2\right) - \left(\sum_{i=1}^{n} x_i y_i\right)\left(\sum_{i=1}^{n} x_i\right)}{n\left(\sum_{i=1}^{n} x_i^2\right) - \left(\sum_{i=1}^{n} x_i\right)^2}$$

$$a_1 = \frac{n\left(\sum_{i=1}^{n} x_i y_i\right) - \left(\sum_{i=1}^{n} x_i\right)\left(\sum_{i=1}^{n} y_i\right)}{n\left(\sum_{i=1}^{n} x_i^2\right) - \left(\sum_{i=1}^{n} x_i\right)^2}$$

$k+1$

$$\begin{bmatrix} n & \sum_{i=1}^{n} x_{1i} & \sum_{i=1}^{n} x_{2i} & \sum_{i=1}^{n} x_{3i} \\ \sum_{i=1}^{n} x_{1i} & \sum_{i=1}^{n} x_{1i}x_{1i} & \sum_{i=1}^{n} x_{1i}x_{2i} & \sum_{i=1}^{n} x_{1i}x_{3i} \\ \sum_{i=1}^{n} x_{2i} & \sum_{i=1}^{n} x_{1i}x_{2i} & \sum_{i=1}^{n} x_{2i}x_{2i} & \sum_{i=1}^{n} x_{2i}x_{3i} \\ \sum_{i=1}^{n} x_{3i} & \sum_{i=1}^{n} x_{1i}x_{3i} & \sum_{i=1}^{n} x_{2i}x_{3i} & \sum_{i=1}^{n} x_{3i}x_{3i} \end{bmatrix} \begin{Bmatrix} a_c \\ a_1 \\ a_2 \\ a_3 \end{Bmatrix} = \begin{Bmatrix} \sum_{i=1}^{n} y_i \\ \sum_{i=1}^{n} x_{1i}y_i \\ \sum_{i=1}^{n} x_{2i}y_i \\ \sum_{i=1}^{n} x_{3i}y_i \end{Bmatrix}$$

ar interpolation Newton's divided-differences

$$f(x) = f(x_0)\, f(x - x_0) \frac{f(x_1) - f(x_0)}{x_1 - x_0}$$