INTRODUCTION TO NUMERICAL METHODS

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WHY NEED TO STUDY NUMERICAL METHODS

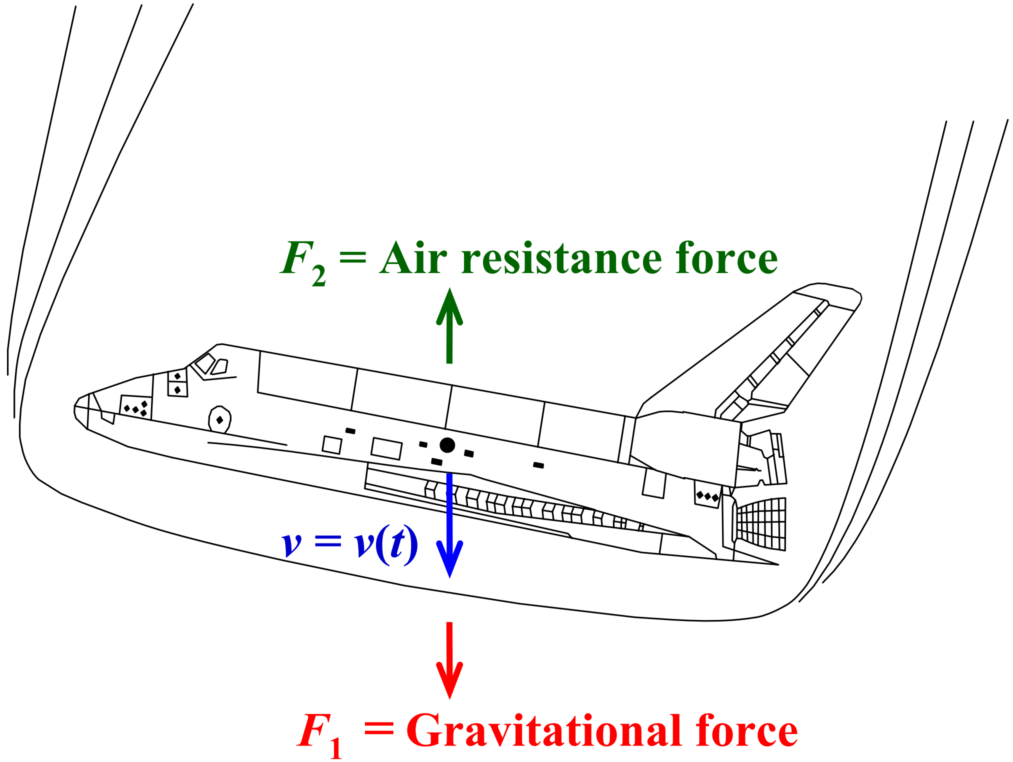
- Large or complex problems can not be solved analytically.
- “Black-box programs” are expensive and incapable to solve problems needed.
- Most of future courses will include Numerical Methods to solve for solutions.
- First step toward higher-level computations, as well as higher education.

WAYS TO SOLVE PROBLEMS

- Through Mathematics **for exact solutions**, or
- Through Numerical Methods, e.g.
  - Finite Difference Method
  - Finite Element Method

for approximate solutions.

MOTIVATING EXAMPLE



Newton's second law: $$F = ma$$

Here F is the net force, $$F = F_1 - F_2$$

MOTIVATING EXAMPLE

Where F_1 is the gravitational force, i.e.,

$$F_1 = mg$$

m is the mass of the Shuttle, g is the gravitational constant. The air resistance force F_2 may be assumed to vary linearly with the velocity v,

$$F_2 = cv$$

where c is the drag coefficient. Then the Newton's second law becomes,

$$mg - cv = ma$$

MOTIVATING EXAMPLE

Writing acceleration in term of velocity,

$$mg - cv \quad = \quad m\frac{dv}{dt}$$

leads to a linear ordinary differential equation,

$$\frac{dv}{dt} + \frac{c}{m}v \quad = \quad g$$

Here, the velocity v which is function of time t can be solved,

(a) analytically for exact solution

(b) numerically for approximate solution

EXACT SOLUTION

ordinary

ODE:
$$\frac{dv}{dt} = g - \frac{c}{m}v$$

Separating the variables then performing integration,

$$\int \frac{dv}{g - \frac{c}{m}v} = \int dt$$

which yields,

$$-\frac{m}{c} ln\left(g - \frac{c}{m}v \right) = t + A$$

where A is the integrating constant that can be determined from an initial condition, e.g. $v\ (t = 0) = 0$, to get

EXACT SOLUTION

$$A = -\frac{m}{c} \ln g$$

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Substitute back and rearrange terms to get exact solution,

$$v = \frac{mg}{c}\left(1 - e^{-\frac{c}{m}t}\right)$$

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Note that as time t approaches infinity, which means the air resistance and the gravitational forces are equal, then

$$v(t \to \infty) = \frac{mg}{c}$$

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EXACT SOLUTION

If we let

$$m = 90{,}000 \quad \text{kg}$$

$$c = 450 \quad \text{kg/sec}$$

$$g = 9.8 \quad \text{m/sec}^2$$

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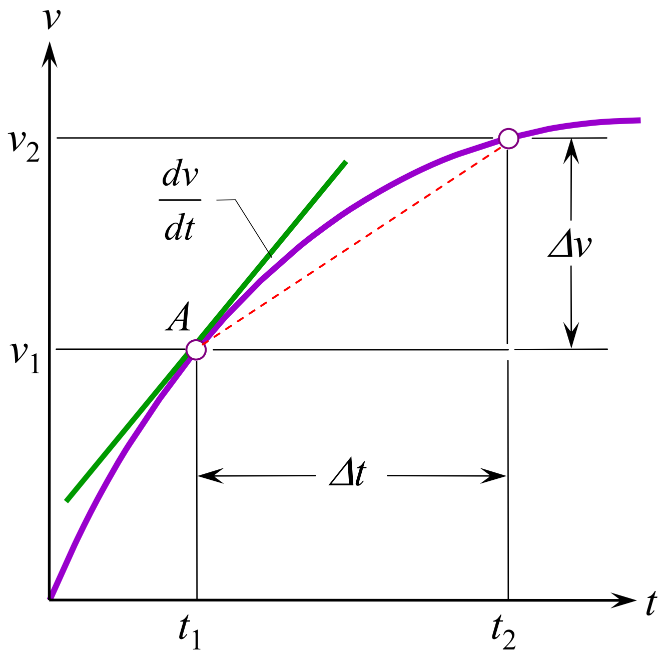
then,

$$v(t) = 1{,}960\left(1 - e^{-0.005t}\right)$$

Time *t*, sec	**Velocity *v*, m/sec**
0	0
30	273
60	508
90	710
120	884
150	1,034
180	1,163
⋮	⋮
∞	1,960

NUMERICAL SOLUTION



We approximate dv/dt which is the slope by

$$\boxed{\frac{\Delta v}{\Delta t} = \frac{v_2 - v_1}{t_2 - t_1}}$$

i.e., as $\Delta t \to 0$ then $\Delta v/\Delta t \to dv/dt$ where Δt is called then time step

Substitute into ODE to get,

$$\frac{v_2 - v_1}{\Delta t} + \frac{c}{m} v_1 = g$$

NUMERICAL SOLUTION

Or, in a more general form,

$$\frac{v_{i+1} - v_i}{\Delta t} + \frac{c}{m} v_i \quad = \quad g$$

where $i = 1, 2, 3, \ldots\ldots\ldots$ Rearrange terms to get,

$$v_{i+1} \quad = \quad v_i + \Delta t \left(g - \frac{c}{m} v_i \right)$$

With the values assigned, then

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$$v_{i+1} \quad = \quad v_i + \Delta t \, (9.8 - 0.005 v_i)$$

i.e., knowing v_i we can compute v_{i+1} directly.

COMPARATIVE SOLUTIONS

			Velocity, m/sec	
i	$i+1$	t, sec	Numerical	Exact
0	1	30	294	273
1	2	60	544	508
2	3	90	756	710
3	4	120	937	884
4	5	150	1090	1034
5	6	180	1221	1163
.	.	.	.	.
.	.	.	.	.
.	.	.	.	.
49	50	1500	1959	1959

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Note: Time step Δt used for numerical solution is 30 sec.

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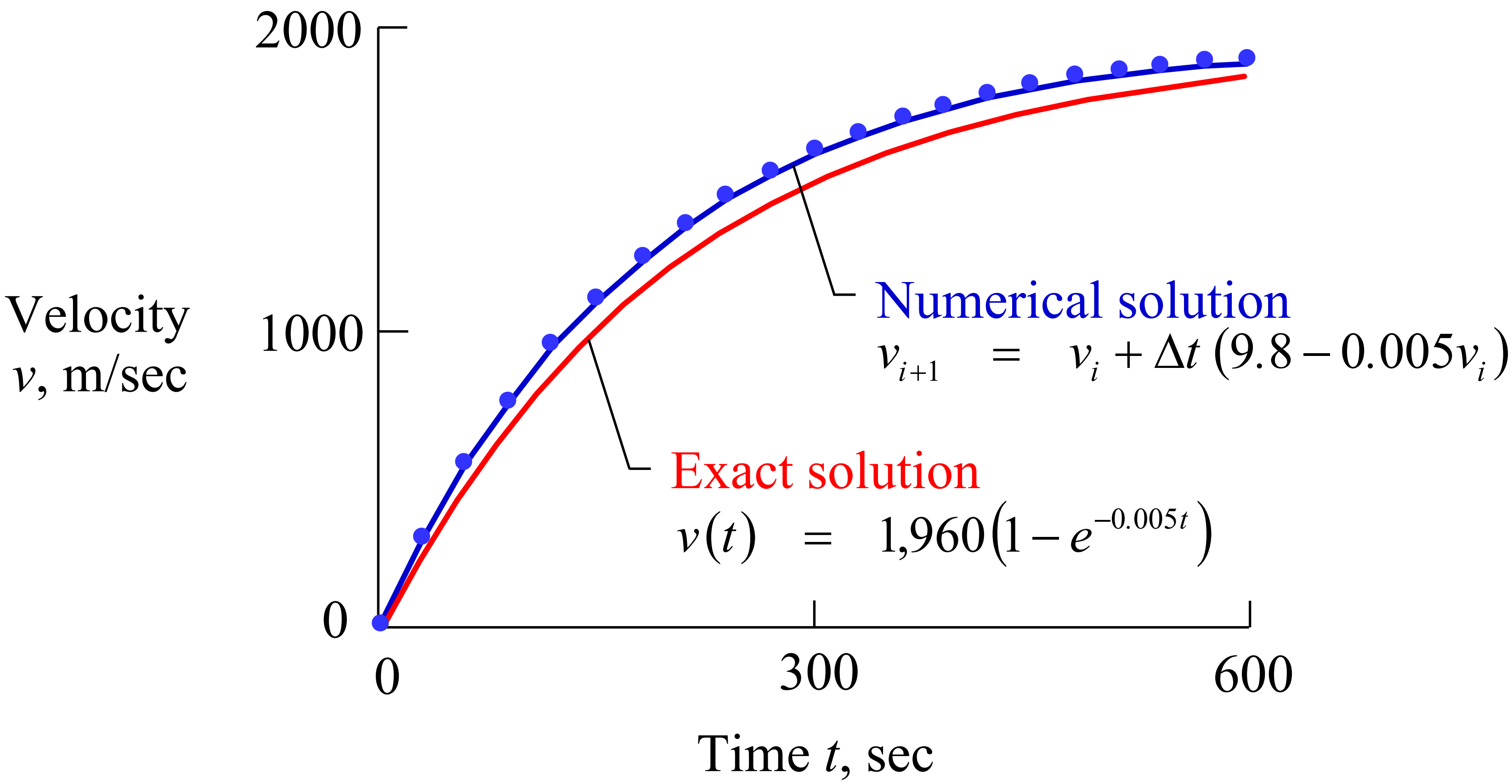
COMPARATIVE SOLUTIONS



<u>Note</u>: Time step Δt used for numerical solution is 30 sec.

ADVANTAGES OF NUMERICAL METHOD

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- Exact solution requires longer time to derive.
- Numerical solution easily obtained by writing a short computer program.
- More accurate solution achieved by reducing time step
- If $F_2 = cv^4$, then ODE is nonlinear,

$$\frac{dv}{dt} + \frac{c}{m} v^4 = g$$

- Exact solution is not easy to derive.
- Numerical solution obtained through same procedure.

COMPUTERS

Types:
1. Microcomputers
2. Work Stations
3. Mainframes
4. Supercomputers

Languages:
1. FORTRAN
2. PASCAL
3. C

Etc.

SAMPLE PROGRAM

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FORTRAN

```
    PROGRAM  SHUTTLE
    T  =  0.
    DT = 30.
    V  =  0.
    DO 10  I=1,50
    V = V + DT*(9.8 - .005*V)
    T = T + DT
    WRITE(6,100)  T, V
100 FORMAT(2F12.0)
 10 CONTINUE
    STOP
    END
```

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PASCAL

```
PROGRAM SHUTTLE;
VAR T,DT,V : real;
    I : integer;
BEGIN
   T  :=  0.;
   DT := 30.;
   V  :=  0.;
   FOR I := 1 TO 50 DO
     BEGIN
        V := V + DT*(9.8 - 0.005*V);
        T := T + DT;
        WRITELN(T : 12 : 0, V : 12 : 0)
     END
END.
```

WRITING COMPUTER PROGRAM

Example Develop a program to compute sine and cosine for angles from zero to 180 degrees with increment every 10 degrees. Given the infinite series of sine and cosine functions,

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$$

where x is in radians.

PROGRAM AND RESULTS

```
      PROGRAM  SINCOS
C.....PROGRAM FOR COMPUTING SIN AND COSINE
C.....FUNCTIONS FOR ANGLES FROM 0 TO 180 DEGREES
C.....WITH INCREMENT AT EVERY 10 DEGREES
      PI = 4.*ATAN(1.)
      DEG = 0.
      DEL = 10.
      WRITE(6,100)
  100 FORMAT(/, 5X, 'DEGREES',
     *         10X, 'SIN', 12X, 'COS', /)
      DO 10  IDEG=1,19
      X = PI*DEG/180.
      SUMS  = X
      SUMC  = 1.
      TERMS = X
      TERMC = 1.
      SIGN  =-1.
      DO 20  N=1,100
      MS = 2*N + 1
      MC = 2*N
      TERMS = TERMS*X*X/(MS*(MS-1))
      TERMC = TERMC*X*X/(MC*(MC-1))
      SUMS  = SUMS + SIGN*TERMS
      SUMC  = SUMC + SIGN*TERMC
      SIGN  = -SIGN
   20 CONTINUE
      WRITE(6,200)  DEG, SUMS, SUMC
  200 FORMAT(F10.0, 2F16.6)
      DEG = DEG + DEL
   10 CONTINUE
      STOP
      END
```

DEGREES	SIN	COS
0.	.000000	1.000000
10.	.173648	.984808
20.	.342020	.939693
30.	.500000	.866025
40.	.642788	.766044
50.	.766044	.642788
60.	.866025	.500000
70.	.939693	.342020
80.	.984808	.173648
90.	1.000000	.000000
100.	.984808	-.173648
110.	.939693	-.342020
120.	.866025	-.500000
130.	.766044	-.642788
140.	.642788	-.766044
150.	.500000	-.866025
160.	.342020	-.939693
170.	.173648	-.984808
180.	.000000	-1.000000

TYPES OF ERROR

error มาจาก

Error introduced from:

1. Mathematical modeling (e.g. discretization)

   ส่งผลจากstepไปยังอีกstepหนึ่ง (เพิ่ม errorขึ้นเรื่อยๆ)
2. Propagation itself (e.g. from step to step)
3. Data (e.g. shuttle mass is not constant)

   ใช้ความประมาท (ผิดพลาดมาก)
4. Blunder (e.g. careless)

   การตัดแบ่ง
5. Truncation (e.g. chopping infinite series)

   ปัดตัวเลขให้ตาม ตน.ที่เราต้องการ
6. Round-off (e.g. value of π)

ROUND-OFF ERROR

For example,

$$\pi \quad = \quad 3.141592653589793238462643\ldots$$

The value of π above consists of 25 significant figures that can be stored in supercomputer, whereas a typical personal computer may store the value of π with only 10 significant figures as,

$$\pi \quad = \quad 3.1415926535$$

This causes round-off error which will also propagate if such value is used <u>repeatedly</u> in computation.

SIGNIFICANT FIGURES

π = 3.14159 has 6 significant figures.

ปัดขึ้น

Easier to write in form of floating point, i.e. $\pi = 0.314159 \times 10^{1}$

This means the values 0.0001278, 0.001278

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0.01278 all have 4 significant figures because they can be written as 0.1278×10^{-3}, 0.1278×10^{-2}, 0.1278×10^{-1}, respectively.

TRUE ERROR

The true error, E_t, is defined by,

$$E_t = v_e - v_a$$

where v_e is the exact solution and v_a is the approximate solution.

Then the true percentage error, ε_t, is,

$$\varepsilon_t = \frac{v_e - v_a}{v_e} \times 100\%$$

For example, the true percentage error of shuttle velocity at 30 sec. is,

$$\varepsilon_t = \frac{273 - 294}{273} \times 100\% = -7.69\%$$

APPROXIMATE ERROR

ใกล้เคียง

For practical problems, exact solution is not available. The approximate percentage error, ε_a, is defined by,

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$$\varepsilon_a = \frac{v_{new} - v_{old}}{v_{new}} \times 100\%$$

For example, velocity at 30 sec using $\Delta t = 30$ sec is 294 m/sec, whereas the velocity at 30 sec using $\Delta t = 10$ sec is 280 m/sec, thus

$$\varepsilon_a = \frac{280 - 294}{280} \times 100\% = -5.00\%$$