# The Asset Market Game: A Review

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# Outline

# References


- Alós-Ferrer, C., Ania, A., *The Asset Market Game*, Department of Economics, Universität Wien (2002).

# Section 1: Aditya Iyengar (180100007)

## How can we model a Financial Market?

- Finitely many states and assets in the world
- Assets pay according to an arbitrary matrix of returns (known?)
- The task of an investor is to decide what fraction of their wealth they invest in each of the assets

## Definition (Market Clearing Price)

The *market-clearing price* of a good is the price of a good at which the quantity supplied is equal to the quantity demanded. In other words, this is the highest price at which all goods can be sold.

Throughout this analysis, it is assumed that asset prices result from a market-clearing condition.

# Defining the Static Asset Market Game

- Let the states of the world be $s = 1, ..., S; S \geq 2$; that each occur with probability $q_s > 0$, $\sum_{s=1}^{S} q_s = 1$
- Let the assets in the market be $k = 1, ..., K; K \geq 2$; such that in some state $s$, asset $k$ yields gross return $A_k(s) > 0$
- The most fundamental assumption here is that $\forall k \exists s$, $A_k(s) > 0$, thus there is **at least one** state of the world where each asset gives a positive return
- Let there be $N$ investors ('players'), $i = 1, ..., N$, each with initial wealth $r_0^i > 0$, normalized such that $\sum_{i=1}^{N} r_0^i = 1$
- Player $i$ chooses a strategy $\alpha^i = (\alpha_1^i, ..., \alpha_K^i) \in \Delta^K$ (the $K - 1$ dimensional simplex), where $\alpha_k^i$ denotes the fraction of player $i$'s wealth invested in asset $k$
- Thus, the vector $\overline{\alpha}_k = (\alpha_k^1, ..., \alpha_k^N)$ denotes the proportions of wealth invested in asset $k$ by all the investors

# Defining the Static Asset Market Game (contd.)

- From the fundamental assumption of market-clearing, we backtrack to compute the actual price of an asset $k$ as $p_k(\overline{\alpha}_k) = \sum_{i=1}^{N} \alpha_k^i r_0^i$
- The number of units of asset $k$ bought by investor $i$ is thus given by $x_k^i(\overline{\alpha}_k) = \frac{\alpha_k^i r_0^i}{p_k(\overline{\alpha}_k)}$ if $\alpha_k^i > 0$, and zero otherwise

**Lemma (Negative Price Effect)**

*The number of units of asset $k$ purchased by an investor $i$ is a* **strictly increasing** *and* **strictly concave** *function of the fraction of wealth allocated to $k$, holding the investments of the other investors constant.*

This paves the way for one of the most important results in economics.

# Defining the Static Asset Market Game (contd.)

- If an investor increases the fraction of wealth allocated to an asset at a constant price, he will possess more units of the asset. However, the increase in demand will lead to an increase in the price of the asset (market-clearing!).
- The negative price effect lemma states that the price of an asset cannot increase so much that an investor ends up with lesser units of the asset after increasing their wealth share in the asset.
- Moreover, concavity of $x_k^i$ ensures that more the share of wealth allocated to asset $k$, more dominant is the negative price effect - thus proving the famous **Law of Diminishing Returns to Investment**.

# Defining the Static Asset Market Game (contd.)

- Given a strategy profile $\overline{\alpha} = (\alpha^i, \alpha^{-i})$, the expected payoff for player $i$ can be given by

$$\pi^i(\alpha^i, \alpha^{-i}) = \sum_{s=1}^{S} q_s \left( \sum_{k=1}^{K} A_k(s) x_k^i(\overline{\alpha}_k) \right) = \sum_{k=1}^{K} E_k x_k^i(\overline{\alpha}_k)$$

- Thus, the **Asset Market Game** is completely defined by the tuple $(\Delta^K, r_0^i, \pi^i)_{i=1}^N$
- The next question we pose is of the Nash equilibrium of this game.
- Intuitively, at equilibrium it must be that **all assets are traded**. If this is not the case, a single rogue investor can acquire the entire supply of the asset by investing an infinitesimal amount $\epsilon > 0$.
- Since all assets are defined to have gross positive return, it is possible to make a profit on the asset, thus providing investors with an incentive to deviate from their chosen strategy.
- The next step is to mathematically consolidate this argument.

# Nash Equilibrium

**Lemma**

*If $\overline{\alpha}$ is a Nash equilibrium, then all assets are traded. Thus, $p_k(\overline{\alpha}_k) > 0 \, \forall k$.*

The above can be easily shown by following the argument previously described.

**Lemma (Competitive Equilibrium)**

*If $\overline{\alpha}$ is a Nash equilibrium, then $p_k(\overline{\alpha}_k) = R_k = \frac{E_k}{E}$ where $E = \sum_{k=1}^{K} E_k$.*

Here is an intuitive explanation. If an asset is 'undervalued' i.e. its expected returns exceed the price of the asset, there must also exist an asset that is 'overvalued' i.e. where the price of the asset exceeds the expected returns.

In such a situation, it is possible for investors to deviate and realize a direct positive gain, bringing the market back to competitive equilibrium.

# Nash Equilibrium (contd.)

**Theorem (Existence and Uniqueness of the Nash Equilibrium)**

*The only pure strategy Nash equilibrium of the asset market game is the profile* $\alpha^* = (\alpha_1^*, ..., \alpha_K^*)$ *with* $\alpha_k^* = R_k \, \forall k$. *Moreover, it is a strict equilibrium.*

**Corollary (Constant Number of Units)**

*At the Nash equilibrium, for each investor, the number of units held from each asset is constant across assets and is numerically equal to the initial wealth.*

The significance of this result is that investors now hold an equally strong position in terms of wealth invested relative to prices.
Thus, when assets are priced according to their relative returns, an investor who holds a stronger position on an asset can profitably deviate by moving it to a more balanced allocation.

# Nash Equilibrium (contd.)

What should investors take away from this result?

- The only Nash equilibrium occurs when all investors split their wealth proportional to the expected returns of the assets. This is often labelled 'investing according to the fundamentals'.
- Uniqueness ensures that at any other market position, there exist arbitrage opportunities.

However, there is a catch here.

- We assumed that all investors know the distributions of returns for all assets and always act strategically.
- Investors are not necessarily well informed about the characteristics that define the asset market and do not necessarily behave strategically. Instead they tend to follow strategies that exhibit good performance.

We relax some of these assumptions in the following sections, to permit boundedly rational investors.

Section 2: Chaitanya Johari (18D100008)

# Nash Equilibrium (Example)

- Consider a static game as follows:
- There are two equally likely states of nature $s = 1, 2$, and two assets $k = 1, 2$.
- Asset 1 pays 2 units in state 1 and 0 otherwise, while asset 2 pays 3 units in state 2 and 0 otherwise. Hence $E_1 = 1$ and $E_2 = 3/2$.
- Consider two investors, Adam and Eve, endowed with the same initial wealth $r_0^A = r_0^E = 1/2$.
- In a Nash equilibrium both should invest a fraction $(1/(5/2)) = 0.4$ of their wealth in asset 1 and the remaining 0.6 in asset 2, which yields prices $p_1 = 0.4$ and $p_2 = 0.6$. And both invest the same amounts, so both own 0.5 of each asset.
- This gives both of them an expected payoff of 1.25.

Suppose instead that Adam invests 0.7 of his wealth in asset 1, while Eve only invests 0.1 of her wealth in the same asset. The asset prices remain the same, as do their respective expected payoffs.

# Nash Equilibrium (contd.)

- Currently, Eve has invested more in the more expensive asset (k=2).
- Since asset prices are increasing with demand, but strictly convex, by increasing her stake in asset 1 to 0.2, she drives up the price to $p_1' = 0.45$, and drives down $p_2$ to $p_2' = 0.55$.
- By increasing her stake in $(k = 1)$, she now has 0.222 of asset-1 and 0.727 of asset-2
- This leads to an expected payoff of $(0.22 * 1 + 0.727 * 1.5 =)1.313$
- This leads to Eve being more profitable than before, so she has incentive to deviate.
- Adam can similarly reallocate his wealth to improve his expected payoff.
- This makes the status quo unstable. Both Adam and Eve can keep reallocating their wealth, until they reach the Nash Equilibrium, $\alpha^*$, at which point they no longer have incentive to deviate.

# Evolutionary Stability

## What if investors only have limited information?

- Investors are not informed about the assets distributions of returns
- Investors do not act strategically
- The investor still needs to decide what fraction of their wealth they invest in each of the assets

## Definition (Evolutionary Stable Strategy)

An *evolutionary stable strategy* (ESS) is an investment strategy such that, once adopted by all investors, it cannot be outperformed by any different investment strategy. In other words, no investor would want to deviate from such a strategy.

It is important to note, however, that this concept of evolutionary stability is not related to Nash equilibrium in general.

# Evolutionary Stability (contd.)

- An ESS, as we defined it, aims at maximizing the difference between own and opponents' payoffs, a feature that is known as **spiteful behavior**.
- The payoff we want to maximize is returns per-monetary-unit.
- It follows directly that an ESS $\alpha$ solves the following problem:

$$\max_{\alpha^i \in \Delta^K} \left( \frac{1}{r_0^i} \pi^i(\alpha^i, \alpha^{-i}) - \frac{1}{r_0^j} \pi^j(\alpha^j, \alpha^{-j}) \right), \text{ for } i \neq j$$

- In general, the payoff of an experimental strategy could increase with the number of players that adopt it. Thus, an ESS is resistant to the appearance of a single experimenter, but it need not be resistant to the appearance of a larger fraction of experimenters

# Global Stability (contd.)

$$\frac{1}{r_0^i}\pi^i(\alpha^i, \alpha^{-i}) \leq \frac{1}{r_0^j}\pi^j(\alpha^j, \alpha^{-j}), \text{ for } i \neq j$$

**Definition (Global Stability)**

We say that an ESS $\alpha \in \Delta^K$ is globally stable if the above inequality holds for any $m \leq N$ and any strategy profile $\overline{\alpha} = (\alpha', \overset{m}{\ldots}, \alpha', \alpha, \overset{N-m}{\ldots}, \alpha)$ or permutation thereof, where $\alpha \neq \alpha', \alpha^i = \alpha'$, and $\alpha^j = \alpha$

- The strategy $\alpha$ is strictly globally stable if the inequality holds strictly.
- A globally stable ESS is resistant against any fraction of experimenters in the population.
- The intuition for this result is simply that all investors deviating to the same $\alpha' \neq \alpha$ will change prices to their disadvantage and the advantage of their opponents.

# Global Stability (contd.)

**Theorem (Globally Stable Nash Equilibrium)**

*The investment strategy $\alpha^*$ is a strictly globally stable ESS in the asset market game.*

As discussed earlier, $\alpha^*$ is a Nash equilibrium for the Asset Market Game.

- Recall that $x_k^i$ is increasing in $\alpha_k^i$. Therefore, intuitively, when m investors deviate from $\alpha$, they increase the prices of those assets where they hold a stronger position after deviation and decrease the price of those where they hold a weaker position.
- Hence, relative to the traders still investing according to $\alpha^*$, experimenters are more affected by the negative effects of overvalued assets, and less affected by the positive effects of undervalued ones, which leaves them in a worse relative position.

# Evolutionary Stability (contd.)

- So far, we have been analyzing static games
- But the actual Asset Market is not that simple.
- In the next section, we analyze a dynamic form of the game

## Section 3: Pranjal Singh (180100085)

# Dynamic Evolutionary Asset Market Game

**Objective:**

- To compare the dynamic stability of $\alpha^*$ with any other portfolio $\alpha$

The dynamic stability is a measure of **'robustness'** of the strategy

**Assumptions:**

- Each trader has a fixed fraction of his total wealth invested in the market

Throughout this analysis, population dynamics is considered for investment strategies rather that focusing on behaviour of individuals.

# Defining a Dynamic Evolutionary Asset Market Game (contd.)

- Let n be number of investors investing by $\alpha^*$. The remaining $N - n$ will invest according to $\alpha$.
- Hence price of asset $p_k = \frac{n}{N} a_k^* + \frac{N-n}{N} a_k$
- The difference between realized payoffs from $\alpha^*$ and $\alpha$ is given by $\Delta(s, n) = \frac{(a_k^* - a_k) A_k(s)}{N}$ for a state s
- The expected difference in the total payoffs is given by $E\Delta(n) = \sum_{k=1}^{K} q_s \sum_{k=1}^{K} \frac{(a_k^* - a_k) A_k(s)}{N}$,
- For any $\alpha^* \neq \alpha, E\Delta(n) > 0$
- Logically, the more is the difference between realized payoffs from $\alpha^*$ and $\alpha$, the more likely are investors to switch from one to the other.

# Darwin Dynamic Process

- Based on the observed payoffs of various portfolios for a market in a given state $s$, some of the investors may change their investment profiles.
- For our asset game, the trader may change his strategy from $\alpha^*$ to $\alpha$ or vice versa.
- Let $\mathrm{Q}_{i,j}$ denote the probability of there being $j$ $\alpha^*$-traders in the next period given that there are $i$ in the current period.
- Then, $Q^s_{n,n+1} = 0.5 + \sigma\Delta(n, s)$ and symmetrically, $Q^s_{n,n-1} = 0.5 - \sigma\Delta(n, s)$.
- When $\Delta(n, s)$ is positive, logically traders are more inclined to switch to $\alpha^*$ from $\alpha$ i.e. $Q^s_{n,n+1} > 0.5$. These results can easily be deduced from the formulae as well.
- It is important to note that the difference in probabilities of realized payoffs is **directly proportional** to the difference in transition probabilities i.e. $Q^s_{n,n+1} - Q^s_{n,n-1} = 2\sigma\Delta(n, s)$.

# Darwin Dynamic Process (contd.)

- In an ideal world, all traders are perfectly rational but in the real world, they may make mistakes.
- In order to consider this factor, we introduce the concept **noise** in our Dynamic Evolutionary Model by introducing a parameter $\epsilon < \frac{1}{2}$ which represents the probability of making mistake in transition.
- The actual probability of making a transition from $n$ to $n+1$ is now $P^s_{n,n+1} = \epsilon Q^s_{n,n-1} + (1-\epsilon) Q^s_{n,n+1}$
- In a single-strategy world with $n = 0$ or $n = N$, only one payoff is observed and hence, the transitions are not based on observations but rather on noises i.e. error of judgement.
  $P^s_{0,0} = 1 - \epsilon \qquad P^s_{N,N} = 1 - \epsilon \qquad P^s_{0,1} = \epsilon \qquad P^s_{N,N-1} = \epsilon$
- The unconditional transition probabilities are given by:
  $P_{n,n+1} = \sum_{s=1}^{S} q_s P^s_{n,n+1} = \frac{1}{2} + (1-2\epsilon)\sigma E \Delta(n)$
  $P_{n,n-1} = \sum_{s=1}^{S} q_s P^s_{n,n-1} = \frac{1}{2} - (1-2\epsilon)\sigma E \Delta(n)$

# Invariant distribution of the portfolio dynamics

The stochastic process described above is a classical **Discrete-Time Birth-Death Markov Process** and is referred to as **Portfolio Dynamics**

The invariant distribution $\mu$ for this Markov Chain is defined in the following manner.

1. $\mu(n)$ represents the probability that exactly $n$ traders follow the $\alpha^*$ investment strategy in the long run
2. $\mu(\mathrm{n})$ can also be calculated as a time based probability. $\mu(\mathrm{n})$ represents the amount of time for which $\alpha^*$ transactions take place along a given path out of the **total trading time period**

The invariant distribution $\mu$ satisfies the following balance condition.

$$\mu(\mathrm{n})\mathrm{P}_{n,n+1} = \mu(\mathrm{n}+1)\mathrm{P}_{n+1,n}$$

# Invariant distribution of the portfolio dynamics (contd.)

One can expect that along any given path, a trader is more likely (or more probable) to invest using the $\alpha^*$ strategy than any other $\alpha$. This intuitive statement can actually be mathematically concluded from our next theorem.

The next theorem is a mathematical representation of the following properties of portfolio dynamics which are quite intuitive:

1. When both $\alpha^*$ & $\alpha$ strategies exist, the population profile with more $\alpha^*$ investors is more frequently observed than the case with more $\alpha$ investors
2. In cases where all investors can invest using only one strategy, the case of all $\alpha^*$ investors is more frequently encountered

# Invariant distribution of the portfolio dynamics (contd.)

**Theorem (Invariant Distribution Theorem)**

*For any $\alpha^* \neq \alpha, \mu$ of the portfolio dynamics verifies*

1. $\forall n = 1, ..., N-1, \mu(n) < \mu(n+1)$
2. $\mu(0) < \mu(N)$

*Hence,* $\mu(n) < \mu(N)\,, \forall n \in N$

# Conclusion

The key takeaways from the paper are as follows:

- For a market with strategic, well-informed institutional investors, the **only Nash equilibrium** occurs when all investors split their wealth proportional to the expected returns of the assets, which are well-known.
- However, such an assumption of investors is utopian. In 'real' markets, investors must switch to an **evolutionary stable strategy**. This is a strategy that, once adopted by the population, cannot be manipulated by an external investor (principle of natural selection).
- Finally, the stochastic and imperfect nature of the system may cause prices to occasionally deviate from the fundamental values - this is not just due to noise but also because investors may be incorrect in their beliefs about the values of assets.
- However, as we have shown $\alpha^*$ to be a **strictly globally stable ESS**, noise trading cannot outperform trading as per the fundamentals in the long run.

Thank you!