Lecture 19: Generalized Time Constant

ECE3110J, Electronic Circuits

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Frequency Response

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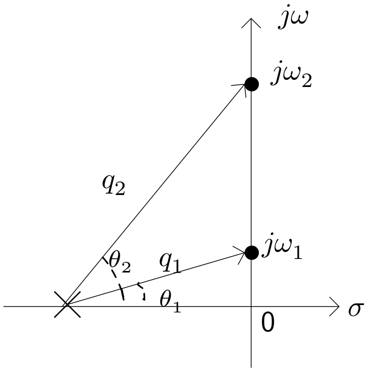
y substituting $j\omega$ for s

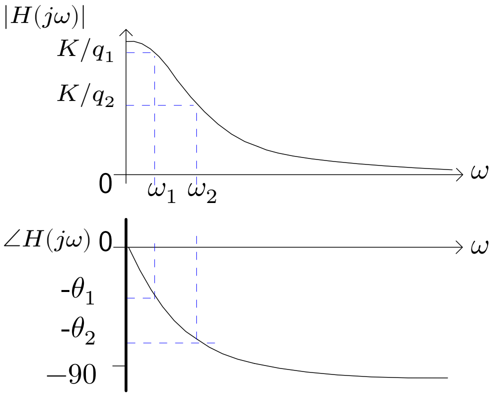
$$H(j\omega) = K\frac{(j\omega - z_1)(j\omega - z_2)\ldots(j\omega - z_{m-1})(j\omega - z_m)}{(j\omega - p_1)(j\omega - p_2)\ldots(j\omega - p_{n-1})(j\omega - p_n)} \tag{1}$$

$$\mid j\omega - p_i \mid = \sqrt{\sigma_i^2 + (\omega - \omega_i)^2} \tag{2}$$

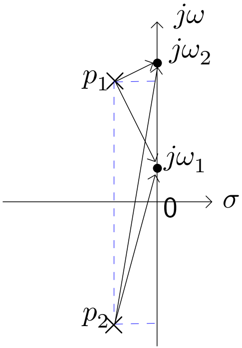
$$\angle(s - p_i) = \tan^{-1}\left(\frac{\omega - \omega_i}{-\sigma_i}\right) \tag{3}$$

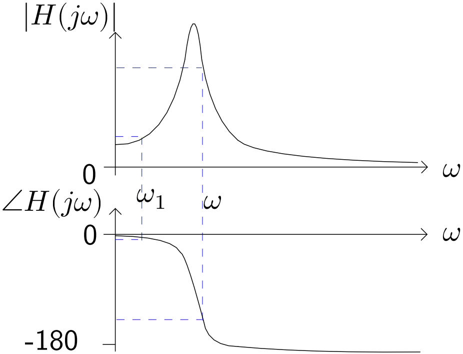
Frequency Response Example



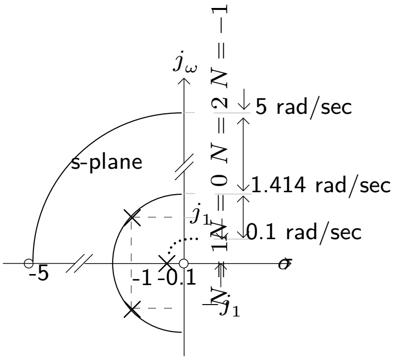


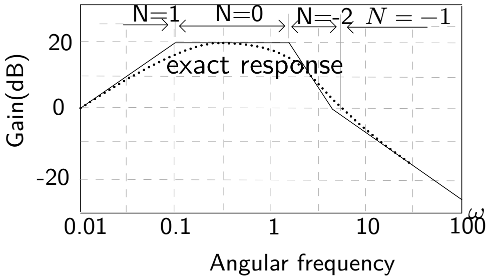
Frequency Response Example





Frequency Response Example





Each pole or zero contributes a change in the slope of $\pm$ 20 dB/decade above its break frequency. A complex conjugate pole or zero pair gives a change of $\pm$ 40 dB/decade.

Constructing Magnitude Bode Plot

- For lefthand plane
  - Slope changes by –20 dB/decade
  - Phase decreases by 90°
- Zero
  - Slope changes by 20 dB/decade
  - Phase increase by 90°

Constructing Magnitude Bode Plot

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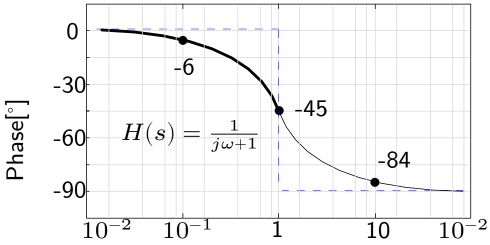
$$H(s) = \frac{1}{s+1} \Rightarrow H(j\omega) = \frac{1}{j\omega + 1} \quad (4)$$

$$\angle H(j\omega) = -\tan^{-1}(\omega) \quad (5)$$

$$\angle H(j\omega) = \begin{cases} 0^\circ & \omega \ll 1 \\ -90^\circ & \omega \gg 1 \end{cases} \quad (6)$$

The behavior for a zero is similar. The phase increases by 90° and passes though the midpoint of 45° at the break point



Constructing Magnitude Bode Plot

We can extend the results for simple repeated poles and zeroes as before using the more general function

$$H(s) = (s+a)^{\pm r} \Rightarrow H(j\omega) = (j\omega + a)^{\pm r} \tag{7}$$

$$\angle H(j\omega) = \begin{cases} 0^\circ & \omega \ll a \\ \pm r 90^\circ & \omega \gg a \end{cases} \tag{8}$$

$$\angle H(ja) = \pm r 45^\circ \tag{9}$$

Unstable (right half plane) poles and zeros have opposite behavior

Zero @ Origin

$$H(s) = s \tag{10}$$

$$|H(j\omega)| = \omega \text{ @ } \omega = 1 \tag{11}$$

$$\Rightarrow \log|H(j\omega)| = 0 \text{ dB} \tag{12}$$

- Slope 20 dB/decade

Pole @ Origin

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$$H(s) = \frac{1}{s} \text{ and } s = j\omega \tag{13}$$

$$\angle H(j\omega) = -90^\circ \tag{14}$$

Phase always at -90°

$$|H(j\omega)| = \frac{1}{\omega} \text{ @ } \omega = 1 \Rightarrow \log(1) = 0 \text{ dB} \tag{15}$$

- Slope -20 dB/ decade

Single Zero @ $-\sigma$

Single Pole @ $-\sigma$

Bode Plot Example

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$$H(s) = \frac{s}{\left(1+\frac{s}{10}\right)\left(1+\frac{s}{10^4}\right)} \tag{16}$$

Zero @ 0
Pole @ -10
Pole @ -10^4

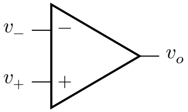
Bode Plot Example

$$H(s) = \frac{1000}{s+100} = \frac{10}{1+\frac{s}{100}} \tag{17}$$

$$G_{DC} = 10 = 20\log 10 = 20 \text{ dB} \tag{18}$$

Pole at -100.

Finite Op-amp Bandwidth



For a simple RC circuit

$$V_{\text{out}} = \frac{1}{1 + j\omega CR} V_{in} \tag{19}$$

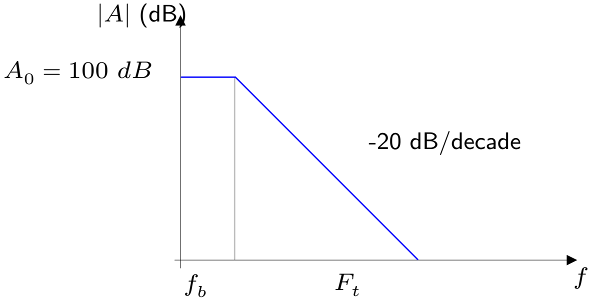
$$\omega_p = -\frac{1}{RC} \tag{20}$$

$$Av(\omega) = \frac{1}{1 + j\omega/\omega_p} \tag{21}$$

- Ideal : Gain is Infinite
- Gain is finite, bandwidth finite
- Assume single pole amplifier

$$A(j\omega) = \frac{A_0}{1 + j\omega/\omega_p} \tag{22}$$

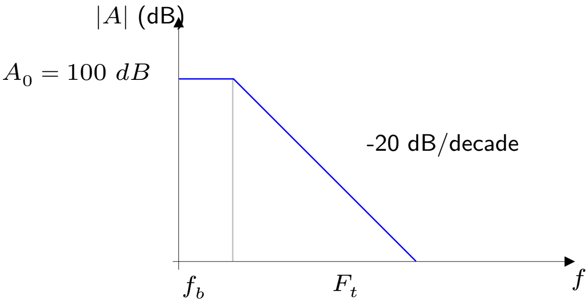
Finite Op-amp Bandwidth



$$A(j\omega) = \frac{A_0}{1 + j\omega/\omega_p} \qquad (23)$$

- Typical op-amps have every low 3dB bandwidth

Finite Op-amp Bandwidth



- $f(t)$ (ω_t) is ω where $|A(j\omega)| = 1$

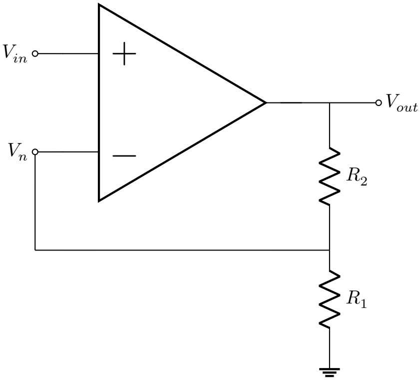
$$A(j\omega) = \frac{A_0}{1 + j\omega/\omega_b} \quad (24)$$

$$1 \approx \frac{A_0}{\omega_{T/\omega_b}} \Rightarrow \omega_T \approx A_0\omega_b \quad (25)$$

$$\omega_T \approx A_0\omega_b \quad (26)$$

- Gain bandwidth product (GBW) important parameter

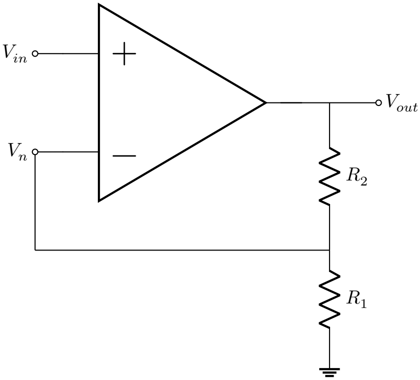
Non-Inverting Amplifier



$$v_n = v_0 \frac{R_1}{R_1 + R_2} = v_0 \beta \tag{27}$$

$$\beta = \frac{R_1}{R_1 + R_2} \tag{28}$$

Non-Inverting Amplifier



$$V_0(\omega) = A(\omega)V_{Id} = A(\omega)\left(V_I - V_n\right) \tag{29}$$

$$= A(\omega)\left(V_I - \beta V_0\right) \tag{30}$$

$$= \frac{A_0}{1 + j\omega/\omega_b}\left(V_I - \beta V_0\right) \tag{31}$$

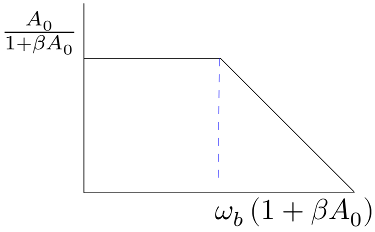
Non-Inverting Amplifier

We pick $\omega_p = -\omega_b$ that frequency looks positive

$$v_0 = \frac{A_0}{1 - \frac{s}{\omega_p}} \left(v_I - \beta v_0\right) \tag{32}$$

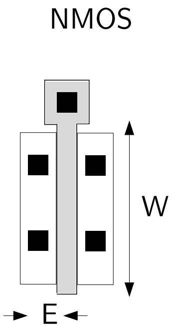
$$v_0 \left(1 - \frac{s}{\omega_p}\right) + A_0 \beta v_0 = A_0 v_I \tag{33}$$

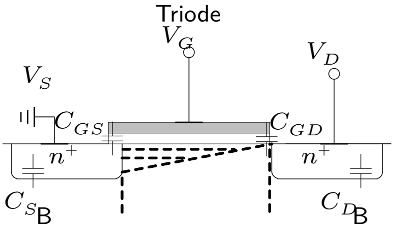
$$\frac{v_0}{V_I} = \frac{A_0}{1 + \beta A_0 - s/\omega_p} = \frac{A_0}{1 + \beta A_0} \frac{1}{1 - \frac{s}{\omega_p (1+\beta A_0)}} \tag{34}$$



- This tells me the bandwidth of a noninverting amplifier is $\omega_p \left(1 + \beta A_0\right)$

Parasitic Capacitance in Triode Region





- Bottom-plate capacitance associated with the bottom of the junction, C_j

Parasitic Capacitance in Triode Region

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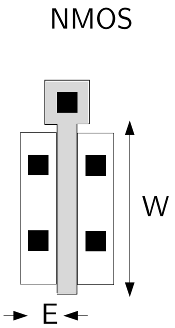
$$C_{GS} = WC_{ov} + 1/2\,(WLC_{ox}) \quad (35)$$

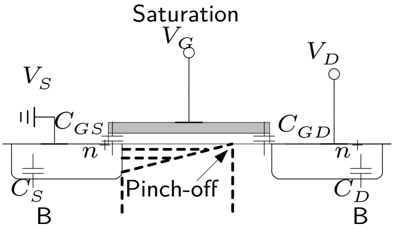
$$C_{GD} = WC_{ov} + 1/2\,(WLC_{ox}) \quad (36)$$

$$C_{SB} = WEC_j + 2(W+E)C_{jsw} \quad (37)$$

$$C_{DB} = WEC_j + 2(W+E)C_{jsw} \quad (38)$$

Parasitic Capacitance in Saturation Mode





- C_{jsw} the sidewall capacitance due to the perimeter of the junction

Parasitic Capacitance in Saturation Mode

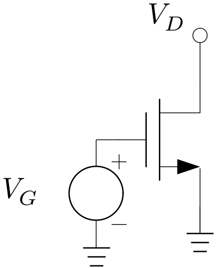
$$C_{GS} = WC_{ov} + 2/3\,(WLC_{ox}) \quad (39)$$

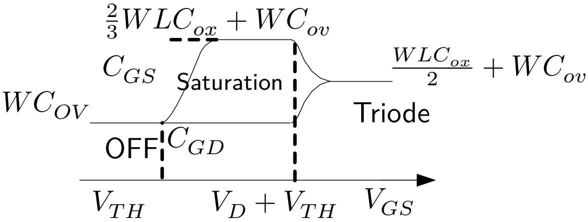
$$C_{GD} = WC_{ov} \quad (40)$$

$$C_{SB} = WEC_j + 2(W+E)C_{jsw} \quad (41)$$

$$C_{DB} = WEC_j + 2(W+E)C_{jsw} \quad (42)$$

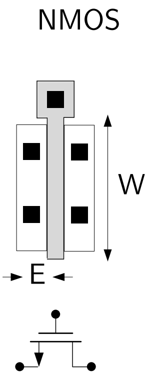
Parasitic Capacitance (Razavi 2.4.2)

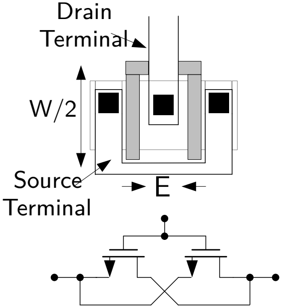




Parasitic Capacitance

Calculate C_{SB} and C_{DB} of the two structures below.





Parasitic Capacitance

Right:

$$C_{DB} = \frac{W}{2} E C_j + 2\left(\frac{W}{2} + E\right) C_{jsw} \tag{43}$$

Left:

$$C_{DB} = C_{SB} = W E C_j + 2(W + E) C_{jsw} \tag{44}$$

$$C_{SB} = 2\left[\frac{W}{2} E C_j + 2\left(\frac{W}{2} + E\right) C_{jsw}\right] \tag{45}$$

$$= W E C_j + 2(W + 2E) C_{jsw} \tag{46}$$

- Drain junction capacitance is greatly reduced

Complete Small-Signal Model

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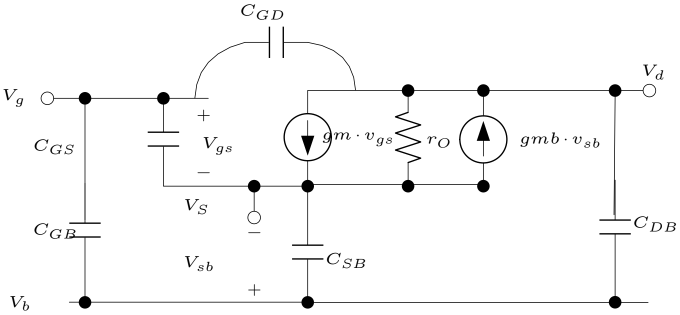

Only when MOSFET is off should we need to consider C_{GB}, because the inversion layer shields gate from body. so change of V_b does not affect V_g

Spice Model

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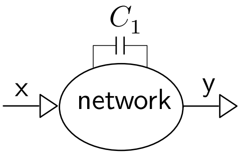
NMOS Model

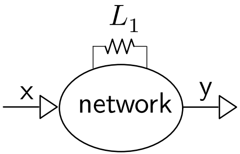
LEVEL $= 1$	VTO $=0.7$	GAMMA $= 0.45$	PHI $= 0.9$
NSUB $= 9e + 14$	LD $= 0.08e - 6$	UO $= 350$	LAMBDA $= 0.1$
TOX $= 9e - 9$	PB $= 0.9$	CJ $= 0.56e - 3$	CJSW $= 0.35e - 11$
MJ $= 0.45$	MJSW $= 0.2$	CGDO $= 0.4e - 9$	JS $= 1.0e - 8$

PMOS Model

LEVEL $= 1$	VTO $= -0.8$	GAMMA $= 0.4$	PHI $= 0.8$
NSUB $= 5e + 14$	LD $= 0.09e - 6$	UO $= 100$	LAMBDA $= 0.2$
TOX $= 9e - 9$	PB $= 0.9$	CJ $= 0.94e - 3$	CJSW $= 0.32e - 11$
MJ $= 0.5$	MJSW $= 0.3$	CGDO $= 0.3e - 9$	JS $= 0.5e - 8$

First Order Systems





$$H(S) = \frac{a_0 + a_1 S}{1 + bS} \tag{47}$$

- No energy is store in the circle. In general, a one pole one zero system

First Order Systems

- The low frequency is represented by a_0

$$a_0 = H(s)\mid_{c_1=0} = H^{\circ} \tag{48}$$

- The time constant determines b_1 (This is the pole of the system)

$$b_1 = \tau = RC_1 \tag{49}$$

- The ratio of a_0 and a_1 determines the location of the zero

First Order Systems

$$H(s) = \frac{a_0 + a_1 s + a_2 s^2 + \cdots + a_m s^m}{1 + b_1 s + b_2 s^2 + \cdots + b_n s^n} = a_0 \cdot \frac{\left(1 - \frac{s}{z_1}\right)\left(1 - \frac{s}{z_2}\right)\cdots\left(1 - \frac{s}{z_m}\right)}{\left(1 - \frac{s}{p_1}\right)\left(1 - \frac{s}{p_2}\right)\cdots\left(1 - \frac{s}{p_n}\right)} \tag{50}$$

$$b_1 = -\sum_i \frac{1}{p_i} \tag{51}$$

$$\frac{a_1}{a_0} = -\sum_i \frac{1}{z_i} \tag{52}$$

$$b_2 = \sum_i \sum_j^{i<j} \frac{1}{p_i p_j} \tag{53}$$

$$\frac{a_2}{a_0} = \sum_i \sum_j^{i<j} \frac{1}{z_i z_j} \tag{54}$$

First Order Systems

$$H(S) = \frac{a_0 + a_1 S}{1 + bs} \tag{55}$$

$$H_i(s) = \frac{a_0 + \alpha_1^i C_i s}{1 + \beta_1^i C_i s} \tag{56}$$

- (It is easy to convince ourselves that a_1 is also related to the capacitor.)
- The transfer function shall be valid for all cap values including zero and infinity.
- the b_1 is simply given by the sum of these zero-value time constants (ZVT)

$$b_1 = \sum_{i=1}^{N} \tau_i^0 \tag{57}$$

First Order Systems

$$H_i(s) = \frac{a_0 + \alpha_1^i C_i s}{1 + \beta_1^i C_i s} \tag{58}$$

C_1 goes to ∞

$$H(s) = \frac{\alpha_1}{\beta_1} \tag{59}$$

$$H(s) = \frac{H^0 + H^1 \tau s}{1 + \tau s} \tag{60}$$

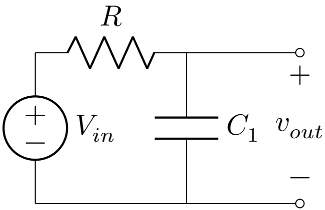
First Order Systems

- If it is an inductor

$$\tau = \frac{L_1}{R_0} \tag{61}$$

- To find the time constant, remove the cap/ind nulling all the sources, find the resistance.
- To find transfer constant H^0, it is just the low frequency gain.
- To find the transfer constant H^1, we look into high frequency response, so the cap shall be shorted. For inductor it is the opposite.

First Order Systems



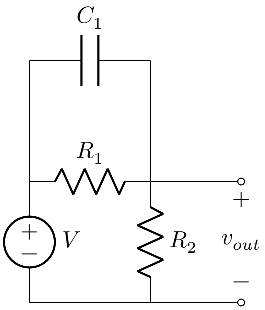
$$H^0 = 1 \tag{62}$$

$$\tau = RC_1 \tag{63}$$

$$H^1 = 0 \tag{64}$$

$$H(s) = \frac{1}{1 + RCS} \tag{65}$$

First Order Systems



- We look at zero frequency response to find out H^0

$$H^0 = \frac{R_1}{R_1 + R_2} \tag{66}$$

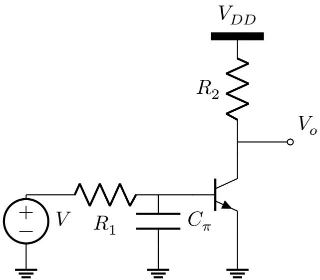
- We null sources to find τ_1

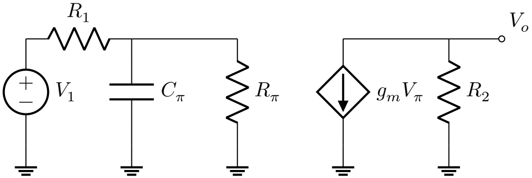
$$\tau_1 = (R_1 \parallel R_2)\, C_1 \tag{67}$$

- We short circuit to find $H^1 = 1$

$$H(S) = \frac{R_1}{R_1 + R_2} \frac{1 + R_2 C_1\ S}{1 + (R_1 \parallel R_2)\, C_1\ S} \tag{68}$$

First Order Systems

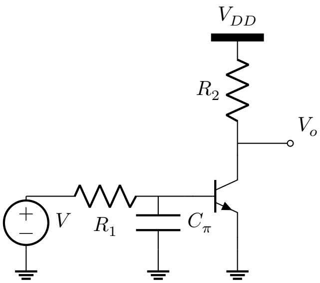


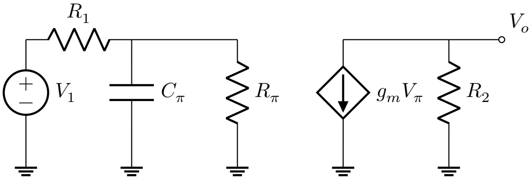


$$H^0 = -\frac{r_\pi}{R_1 + r_\pi} g_m R_2 \tag{69}$$

$$H^\pi = 0 \tag{70}$$

First Order Systems



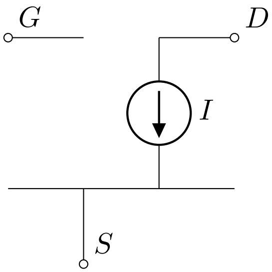


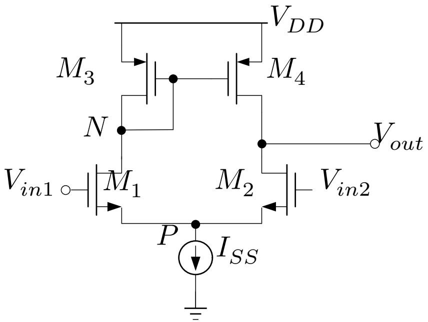
$$H(S) = \frac{H^0}{1 + \tau S} \tag{72}$$

$$\tau = (R_1 \parallel r_\pi) C_\pi \tag{71}$$

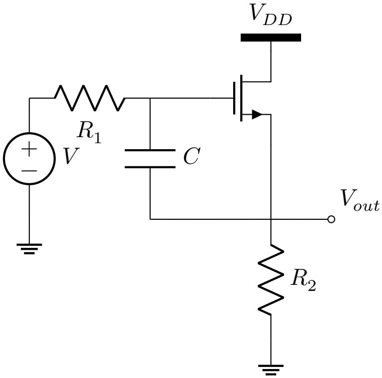
If there is a zero in the system, then we can test it with a shorted cap/ open ind and see if the output still have some value.

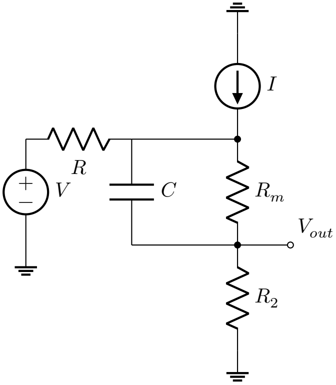
MOSFET T Model





First Order Systems



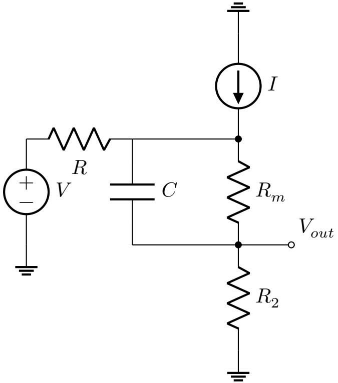


First Order Source Follower

- Low frequency, $V_x = V_{in}$
- V_{out} is voltage divider

$$H^0 = \frac{R_2}{R_2 + R_m} \quad (73)$$

$$H^1 = \frac{R_2}{R_2 + r_1} \quad (74)$$

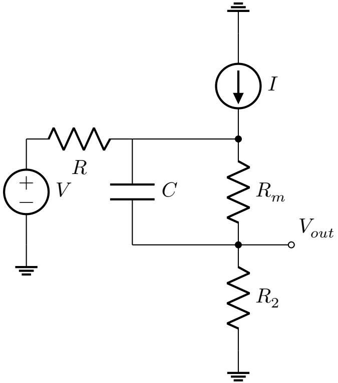


First Order Source Follower

$$R_2 (g_m v_x - i_x) + v_x = R_1 i_x \tag{75}$$

$$R = \frac{R_1 + R_2}{1 + g_m R_2} \tag{76}$$

- You can imagine one zero and one pole.



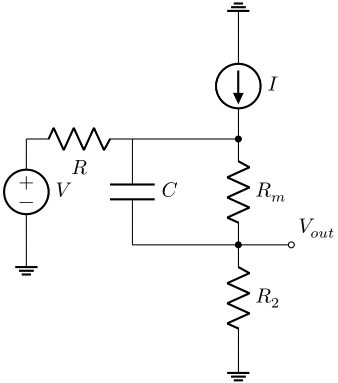
First Order Source Follower

$$H(s) = \frac{R_2}{R_2 + R_m} \cdot \frac{1 + \frac{R_m + R_2}{R_1 + R_2} \cdot \frac{R_1 + R_2}{1 + g_m R_2} C_\pi S}{1 + R_\pi^0 C_\pi S} \quad (77)$$

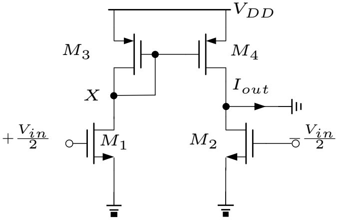
$$= \frac{R_2}{R_2 + R_m} \cdot \frac{1 + r_m C_\pi S}{1 + R_\pi^0 C_\pi S} \quad (78)$$

- $Z = -\frac{g_m}{C_\pi}$ @ high frequency
- $P = -\frac{1}{R_\pi^0 C_\pi}$ @ high frequency

Therefore, source follower is quite wide band



Hybrid-pi model

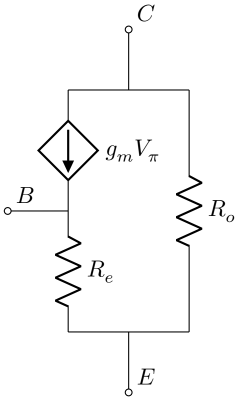


$$g_m = \left.\frac{i_c}{v_{be}}\right|_{v_{ce}=0} = \frac{I_C}{V_T} \quad (79)$$

$$r_\pi = \left.\frac{v_{be}}{i_b}\right|_{v_{ce}=0} = \frac{V_T}{I_B} = \frac{\beta_0}{g_m} \quad (80)$$

$$r_o = \left.\frac{v_{ce}}{i_C}\right|_{v_{be}=0} = \frac{1}{I_C}\left(V_A + V_{CE}\right) \approx \frac{V_A}{I_C} \quad (81)$$

BJT T-model



where

$$g_m = \frac{I_C}{V_T} \tag{82}$$

$$I_C = I_S \cdot e^{\frac{V_{BE}}{V_T}} \tag{83}$$

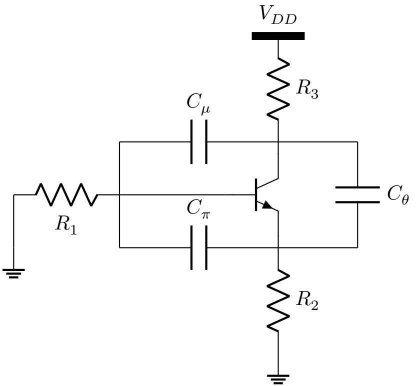
$$r_e = \frac{V_T}{I_E} = \frac{\alpha}{g_m} \tag{84}$$

First Order Systems

$$R_\pi^0 = r_\pi \parallel \frac{R_1 + R_2}{1 + g_m R_2} \tag{85}$$

$$R_\mu^0 = R_{\text{left}} + R_{\text{right}} + G_m R_{\text{left}} R_{\text{right}} \tag{86}$$

$$R_{\text{left}} = R_B \parallel [r_\pi + (1+\beta) R_E] \tag{87}$$



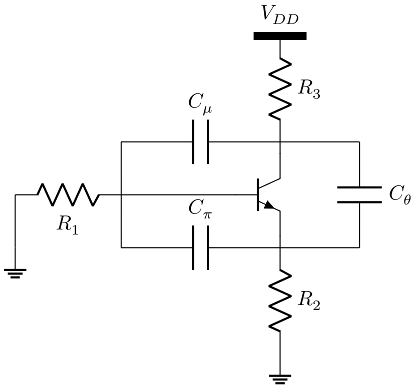
First Order Systems

$$R_{\text{right}} = R_3 \tag{88}$$

$$G_m = \frac{g_m}{1 + g_m R_2} \tag{89}$$

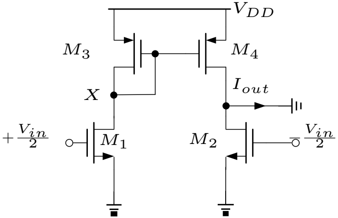
$$R_\theta^0 \approx \frac{R_2 + R_3}{1 + g_m R_2} \tag{90}$$

$$b_1 = \sum_{i=1}^{N} \tau_i^0 \tag{91}$$



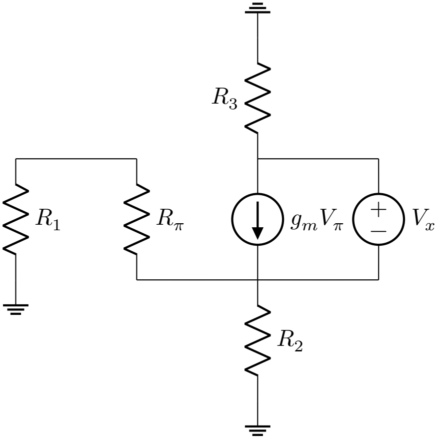
First Order Systems

Proof:



$$R_\mu^0 = R_{\text{left}} + R_{\text{right}} + G_m R_{\text{left}} R_{\text{right}} \tag{92}$$

First Order Systems

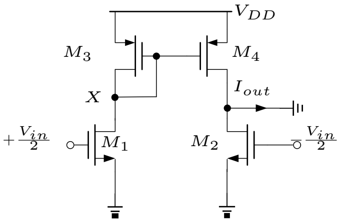


Proof:

$$R_\theta^0 \approx \frac{R_2 + R_3}{1 + g_m R_2} \quad (93)$$

$$\frac{V_a + V_x}{R_3} = i_x + g_m v_a = \frac{-V_a}{R_2} \quad (94)$$

First Order Systems

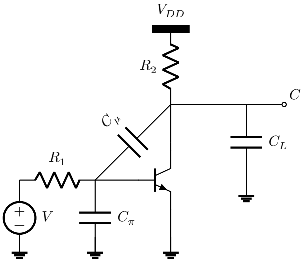


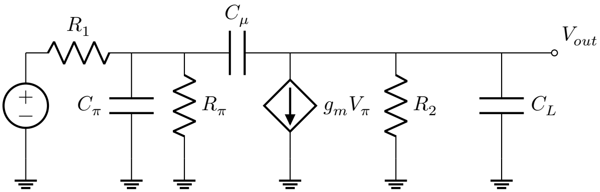
Proof:

$$R_{\text{left}} = R_B \parallel [r_\pi + (1+\beta)R_E] \quad (95)$$

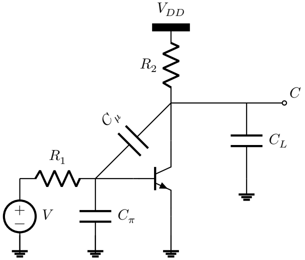
First Order Systems

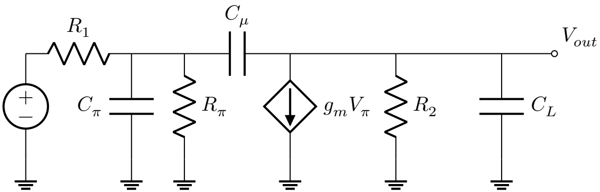
.model NPN ($C_{je} = 20$ fF, $C_{jc} = 20$ fF, $\beta_0 = 100$, $C_{js} = 50$ fF, $\tau_F = 2$ ps, $C_L = 150$ fF) c_{je} Zero bias B-E depletion capacitance





First Order Systems





- A collector current of 1 mA gives us a $G_m = \frac{I_c}{V_T} = 40\ \mathrm{mS}$
- $R_1 = 1K\Omega, R_2 = 1K\Omega$

First Order Systems

$$C_b = g_m \tau_F = 80 \text{ fF} \tag{96}$$

$$C_\pi = C_{je} + C_b = 100 \text{ fF} \tag{97}$$

$$C_L = C_{out} + C_{js} = 200 \text{ fF} \tag{98}$$

$$C_\mu = C_{jc} = 20 \text{ fF} \tag{99}$$

$$C_\theta = C_{js} = 50 \text{ fF} \tag{100}$$

$$b_1 = \sum_i \tau_i^0 = \tau_\pi^0 + \tau_\mu^0 + \tau_L^0 \tag{101}$$

First Order Systems

$$\frac{\beta_o}{g_m} = 2.5K\ \Omega \tag{102}$$

$$H^0 = -\frac{2.5K}{2.5K + 1K}\ g_m R_2 \tag{103}$$

$$H^0 = -\frac{2.5K}{3.5K} \cdot 40e^{-3} \cdot 2K = -57 \tag{104}$$

$$H^0 = -57 \tag{105}$$

First Order Systems

$$\tau_\pi^0 = C_\pi \left(R_1 \parallel r_\pi\right) = 100 \text{ fF} \times \left(1 \ K\Omega \parallel 2.5 \ K\Omega\right) = 100 \text{ fF} \times 710 \ \Omega \approx 70\text{ps} \tag{106}$$

$$\tau_\mu^0 = C_\mu \left[R_{\text{left}} + R_{\text{right}} + g_m R_{\text{left}} R_{\text{right}}\right] \tag{107}$$

$$C_\mu \left[\left(R_1 \parallel r_\pi\right) + R_2 + g_m R_2 \left(R_1 \parallel r_\pi\right)\right] = C_\mu \left[\left(1 + g_m R_2\right)\left(R_1 \parallel r_\pi\right) + R_2\right] \tag{108}$$

$$20 fF \times \left(700\Omega + 2k\Omega + 56k\Omega\right) \approx 1200 \text{ ps} \tag{109}$$

$$\tau_L^0 = C_L R_2 = 200 \text{ fF} \times 2 \text{ K}\Omega = 400 \text{ ps} \tag{110}$$

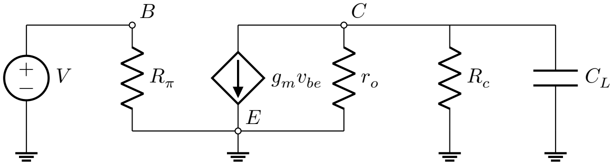
First Order Systems

$$\tau_\pi^0 \approx 70\text{ps} \quad (111) \qquad \tau_\mu^0 \approx 1200\text{ps} \quad (112) \qquad \tau_L^0 = 400\text{ps} \quad (113)$$

$$\omega_h \approx 1/b_1 \approx 2\pi \cdot 95\text{MHz} \quad (114)$$

- This allows you to determine the lowest operating frequency, and also the contribution of each nodes in the circuit.

Practice on BJT Case



$$g_m v_i + \frac{v_0}{r_0 \parallel R_c} + v_0 S C_L = 0 \quad (115)$$

$$g_m v_i + v_0 \frac{1 + r_0 \parallel R_c S C_L}{r_0 \parallel R_c} = 0 \quad (116)$$

- Single Pole at
- Often $r_0 >> RC$

Further discussion on P and Z

Suggested Readings

- Lee, Thomas H. The design of CMOS radio-frequency integrated circuits. Cambridge university press, 2003.
- Hajimiri, Ali. "Generalized time-and transfer-constant circuit analysis." IEEE Transactions on Circuits and Systems I: Regular Papers 57.6 (2009): 1105-1121.

Further discussion on P and Z

$$H(s) = \frac{H^0 + H^1 \tau s}{1 + \tau s} = H^0 \frac{1 + \frac{H^1}{H^0}\tau s}{1 + \tau s} \tag{118}$$

$$P = -\frac{1}{\tau} \tag{119}$$

Further discussion on P and Z

$$Z = -\frac{1}{\frac{H^1}{H^0}\tau} = \frac{H^0}{H^1}\left(-\frac{1}{\tau}\right) \tag{120}$$

$$Z = \frac{H^0}{H^1}P \tag{121}$$

- This tells us that we can look at the sign of H^0 and H^1 to infer the location of zeros and poles
- The ratio also tells whether poles or zeros come earlier.

Further discussion on P and Z

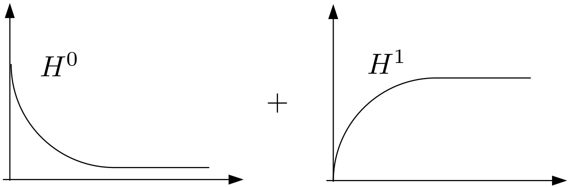
	$\lvert\frac{H_0}{H_1} < 1\rvert$	$\lvert\frac{H_0}{H_1} > 1\rvert$
$\frac{H_0}{H_1} > 0$	P and Z in same half plane $\lvert Z\rvert < \lvert P\rvert$	P & Z in same half plane $\lvert Z\rvert > \lvert P\rvert$
$\frac{H_0}{H_1} < 0$	P & Z opposite HP $\lvert Z\rvert < \lvert P\rvert$	P & Z opposite HP $\lvert Z\rvert > \lvert P\rvert$

Further discussion on P and Z

$$H(s) = \frac{H^0 + H^1 \tau S}{1+\tau S} = H^0 \frac{1}{1+\tau S} + H^1 \frac{\tau S}{1+\tau S} \tag{122}$$

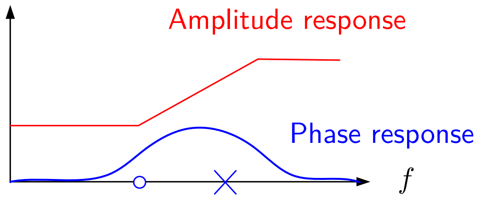
Low path filter   High path filter

$$s(t) = H^0\left(1 - e^{-t/\tau}\right)u(t) + H^1 e^{-t/\tau}u(t) \tag{123}$$



- The final waveform also tells you the existence of poles and zeros.

Further discussion on P and Z

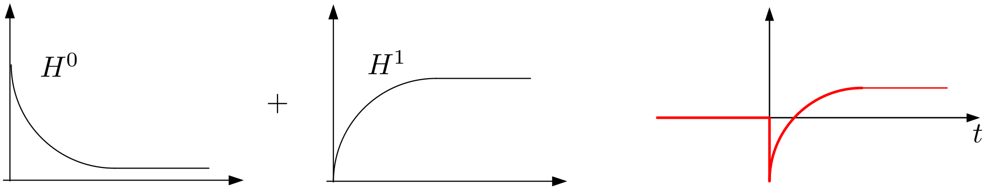


P and Z in same left half plane $|Z| < |P|$

- Phase response of a zero gives you increasing phase.
- Phase response of a pole gives you decreasing phase.
- This tells me, if I look at the frequency response (AC simulation), we can infer the existence of a zero/ pole.

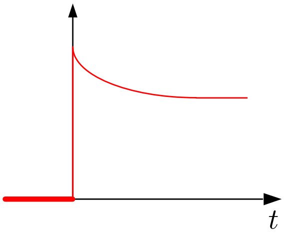
Further discussion on P and Z

	$\lvert \frac{H_0}{H_1} < 1 \rvert$	$\lvert \frac{H_0}{H_1} > 1 \rvert$
$\frac{H_0}{H_1} < 0$	P & Z opposite HP $\lvert Z \rvert < \lvert P \rvert$	P & Z opposite HP $\lvert Z \rvert > \lvert P \rvert$



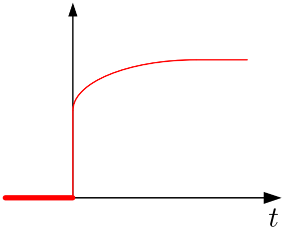
- Opposite sign resulting in undershoot, LHP pole and RHP zero.

Further discussion on P and Z



$$0 < \frac{H_0}{H_1} < 1 \qquad (124)$$

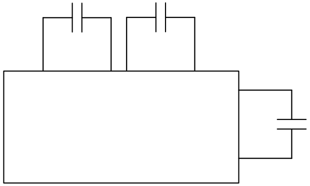
LHP low-freq. zero happens before the pole in a first order system.



$$\frac{H_0}{H_1} > 1 \qquad (125)$$

Zero happens after the pole freq.

Nth Order System



$$H(s) = \frac{a_0 + a_1 S + a_2 S^2 + \dots}{1 + b_1 S + b_2 S^2 + \dots} \tag{126}$$

- Only the caps and inductors produces S.
- To get a_1 we have to have a cap (or inductor)

Nth Order System

We can also infer that the s^2 term comes from two capacitors.

$$H(s) = \frac{a_0 + \left(\sum_{i=1}^{N} \alpha_1^i c_i\right) s + \left(\sum_i^{1\le i\le j\le N} \alpha_j^{ij} c_i c_j\right) s^2 + \cdots}{1 + \left(\sum_{i=1}^{N} \beta_1^i c_i\right) s + \left(\sum_i^{1\le i\le j\le N} \sum_j \beta_2^{ij} c_i c_j\right) s^2 + \cdots} \tag{127}$$

Nth Order System

If we set all c's except c_i as zeros By comparing with a first-order system.

$$H_i(s) = \frac{a_0 + \alpha_1^i c_i s}{1 + \beta_1^i c_i s} \tag{128}$$

$$\beta_1^i = R_i^0 \tag{129}$$

$$\tau_i^0 = R_i^0 C_i \tag{130}$$

$$b_1 = \sum_{i=1}^{N} \tau_i^0 \tag{131}$$

Nth Order System

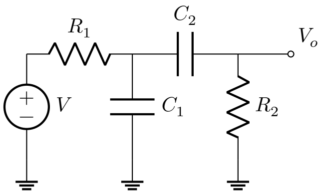
$$b_1 = \sum_{i=1}^{N} \tau_i^0 \tag{132}$$

coefficient b_1 is the sum of all zero valued time constant.

$$a_1 = \sum_{i=1}^{N} \tau_i^0 H^i \tag{133}$$

This allows us to find out the dominant time constant.

N^{th} Order System Example 1



- It is a two-cap system, therefore the number of poles are two.
- It is a system with an infinite response of zero if we short both capacitors, so there is no zeros.

Nth Order System Example 1

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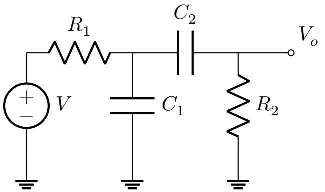

$$H(S) = \frac{1}{1 + 3RCS + (RC)^2S^2} \tag{134}$$

$$b_1 = \sum_{i=1}^{N} \tau_i^0 = \tau_1^0 + \tau_2^0 \tag{135}$$

$$\tau_1^0 = RC \tag{136}$$

$$\tau_2^0 = 2RC \tag{137}$$

$$b_1 = 3RC \tag{138}$$

Nth Order System

$$H(S) = \frac{1}{1 + 3RCS + (RC)^2 S^2} \tag{139}$$

If the impedance seen by one cap does not change as we open or short the other cap, we say that the two-time constant are uncoupled to each other, and the expression can be written as

$$H(S) = \frac{H^0}{(1 + \tau_1 S)(1 + \tau_2 S)} \tag{140}$$

We now have the $3RC$ term. The question is whether it is true in this case to determine $(RC)^2 S^2$

Nth Order System

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$$H(s) = \frac{1}{\left(1+\tau_1^0 s\right)\left(1+\tau_2^0 s\right)} = \frac{1}{1+\left(\tau_1^0+\tau_2^0\right)s+\tau_1^0\tau_2^0 s^2} \tag{141}$$

$$\tau_1^0\tau_2^0 = 2(RC)^2 \tag{142}$$

Nth Order System

We now know how to calculate a_1, b_1, and a_0.

$$H(s) = \frac{a_0 + \left(\sum_{i=1}^{N} \alpha_1^i C_i\right) s + \left(\sum_i^{1\leqslant i<j\leqslant N} \alpha_j^{ij} C_i C_j\right) s^2 + \cdots}{1 + \left(\sum_{i=1}^{N} \beta_1^i C_i\right) s + \left(\sum_i^{1\leqslant i<j\leqslant N} \sum_j \beta_2^{ij} C_i C_j\right) s^2 + \cdots} \tag{143}$$

- A_0 is the zero frequency response.
- B_1 is the summation of time constant
- A_1 can be obtained from infinite time response.

Nth Order System

$$b_2 = \sum_{i}^{i<j<N} \sum_{j}^{0} \tau_j^i \tag{144}$$

- That means you don't repeat $\tau 12$ and $\tau 21$
- τ_j^i means the time constant of element j when element I is infinite frequency

$$a_2 = \sum \sum_{j}^{i<j<N} \tau_i^0 \tau_j^i H^{ij} \tag{145}$$

- We can expect b_n is a multiple summation of the product of many time constants

Nth Order System

$$b_n = \sum_{i}^{1\leqslant i<j<k} \sum_{j} \sum_{k\ldots}^{\ldots} \ldots \tau_i^0 \tau_j^i \tau_k^{ij} \ldots \tag{146}$$

$$a_n = \sum_{i}^{1\leqslant i<j<k} \sum_{j}^{\ldots\ldots N} \ldots \tau_i^0 \tau_j^i \tau_k^{ij} \ldots H^{ijk\ldots} \tag{147}$$

I will skip the derivations and show you how to use it.

Nth Order System

$$H(s) = \frac{a_0 + a_1 s + a_2 s^2}{1 + b_1 s + b_2 s^2} \tag{148}$$

$$H(s) = \frac{H^0 + \left(H^1 R_1^0 C_1 + H^2 R_2^0 C_2\right) s + \left(\alpha_2^{12} C_1 C_2\right) s^2}{1 + \left(R_1^0 C_1 + R_2^0 C_2\right) s + \left(\beta_2^{12} C_1 C_2\right) s^2} \tag{149}$$

- We notice that relabeling C_1 as C_2 and vice versa should not change the derived transfer function. So that $\alpha_2^{12} = \alpha_2^{21}$ and $\beta_2^{12} = \beta_2^{21}$
- R_2^1 is the resistance seen by C_2 when C^1 is infinite valued (shorted).

Nth Order System

$$H(s) = \frac{H^0 + \left(H^1 R_1^0 C_1 + H^2 R_2^0 C_2\right) s + \left(\alpha_2^{12} C_1 C_2\right) s^2}{1 + \left(R_1^0 C_1 + R_2^0 C_2\right) s + \left(\beta_2^{12} C_1 C_2\right) s^2} \tag{150}$$

- I want to determine the value of α_2^{12} and β_2^{12}
- It should be valid for $C_1 \to \infty$
- If I short C_1, $C_1 \to \infty$

$$H(s)|_{C_1 \to \infty} = \frac{C_1 s \cdot \left(H^1 R_1^0 + \alpha_2^{12} C_2 s\right)}{C_1 s \cdot \left(R_1^0 + \beta_2^{12} C_2 s\right)} = H^1 \cdot \frac{1 + \frac{\alpha_2^{12}}{H^1 R_1^0} C_2 s}{1 + \frac{\beta_2^{12}}{R_1^0} C_2 s} \tag{151}$$

Nth Order System

It should be the same as a 1 cap system. Therefore

$$\beta_2^{12} = R_1^0 R_2^1 \tag{152}$$

$$b_2 = R_1^0 C_1 R_2^1 C_2 = \tau_1^0 \tau_2^1 \tag{153}$$

Nth Order System

$$H(s) = \frac{H^0 + \left(H^1 R_1^0 C_1 + H^2 R_2^0 C_2\right) s + \left(\alpha_2^{12} C_1 C_2\right) s^2}{1 + \left(R_1^0 C_1 + R_2^0 C_2\right) s + \left(\beta_2^{12} C_1 C_2\right) s^2} \tag{154}$$

we determine a_2 in by setting $C_1 \to \infty$ and $C_2 \to \infty$

$$H^{12} = \frac{\alpha_2^{12}}{\beta_2^{12}} \tag{155}$$

$$\beta_2^{12} = R_1^0 R_2^1 \tag{156}$$

Nth Order System

$$\alpha_2^{12} = R_1^0 R_2^1 H^{12} \tag{157}$$

$$a_2 = R_1^0 C_1 R_2^1 C_2 H^{12} = \tau_1^0 \tau_2^1 H^{12} \tag{158}$$

Therefore H_{12} is the low-frequency input-output transfer constant with the reactive elements 1 and 2 at their infinite value (C_1 and C_2 shorted).

Nth Order System

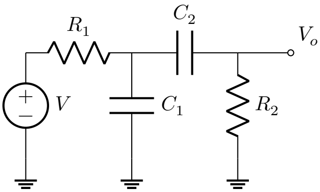
$$b_2 = \tau_1^0 \tau_2^1 = \tau_2^0 \tau_1^2 \tag{159}$$

$$a_2 = \tau_1^0 \tau_2^1 H^{12} = \tau_2^0 \tau_1^2 H^{12} \tag{160}$$

$$b_1 = \tau_1^0 + \tau_2^0 \tag{161}$$

$$a_1 = \tau_1^0 H^1 + \tau_2^0 H^2 \tag{162}$$

Nth Order System

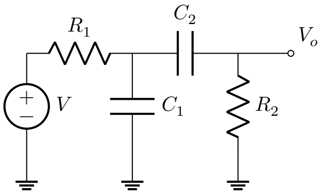


- Two pole and one zero

$$H^0 = 0 \quad (163) \qquad \tau_1^0 = R_1 C_1 \quad (164)$$

$$\tau_2^0 = (R_1 + R_2)\, C_2 \quad (165)$$

Nth Order System



$$\tau_2^1 = R_2 C_L \tag{166}$$

$$\tau_1^2 = (R_1 \parallel R_2)\, C_1 \tag{167}$$

$$\tau_1^0 \tau_2^1 = R_1 C_1 R_2 C_2 = \tau_2^0 \tau_1^2 \tag{168}$$

$$\tau_2^0 \tau_1^2 = (R_1 + R_2)\, C_2 \frac{R_1 R_2}{R_1 + R_2} C_1 \tag{169}$$

N^{th} Order System

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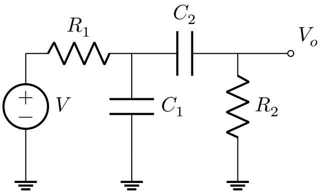

$$H^1 = 0 \qquad (170)$$

$$H^2 = \frac{R_2}{R_2 + R_1} \qquad (171)$$

$$H^{12} = 0 \qquad (172)$$

Nth Order System

$$b_2 = \tau_1^0 \tau_2^1 = \tau_2^0 \tau_1^2 \tag{173}$$

$$a_2 = \tau_1^0 \tau_2^1 H^{12} = \tau_2^0 \tau_1^2 H^{12} \tag{174}$$

$$b_1 = \tau_1^0 + \tau_2^0 \tag{175}$$

$$a_1 = \tau_1^0 H^1 + \tau_2^0 H^2 \tag{176}$$

$$H(s) = \frac{a_0 + a_1 s + a_2 s^2}{1 + b_1 s + b_2 s^2} \tag{177}$$

$$H(s) = \frac{R_2 C_2 S}{1 + (R_1 C_1 + R_1 C_2 + R_2 C_2) S + R_1 C_1 R_2 C_2 S^2} \tag{178}$$

N^{th} Order System

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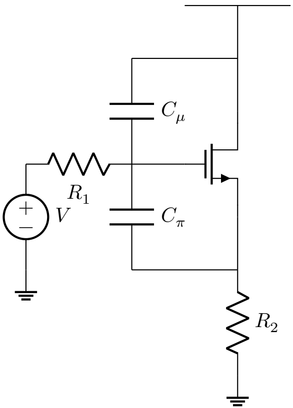
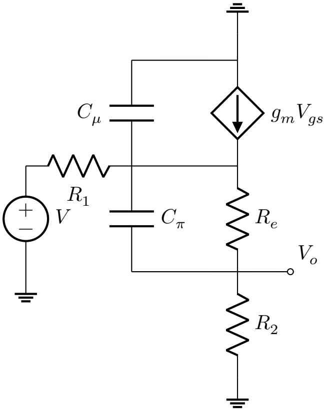




Nth Order System

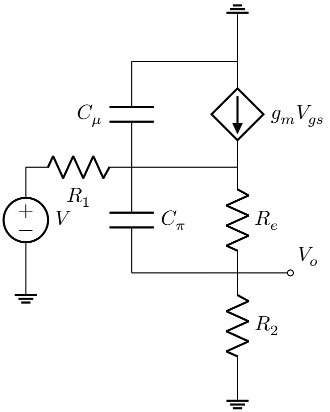
Two poles and one zero

$$H^0 = \frac{R_2}{R_2 + r_e} \tag{179}$$

$$\tau_\pi^0 = C_\pi \frac{R_1 + R_2}{1 + g_m R_2} \tag{180}$$

$$\tau_\mu^0 = C_\mu R_1 \tag{181}$$

$$\tau_\mu^\pi = C_\mu (R_1 \parallel R_2) \tag{182}$$

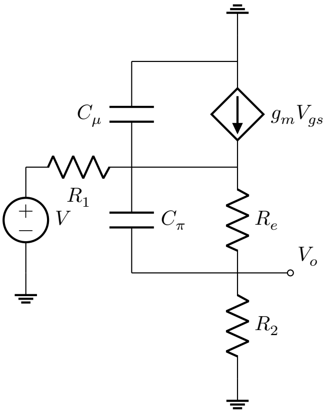


Nth Order System

$$H^{\mu} = 0 \tag{183}$$

$$H^{\pi} = \frac{R_2}{R_1 + R_2} \tag{184}$$

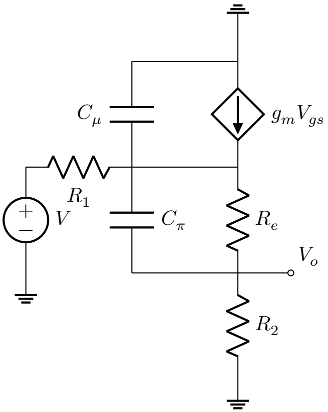
$$H^{\mu\pi} = 0 \tag{185}$$



Nth Order System

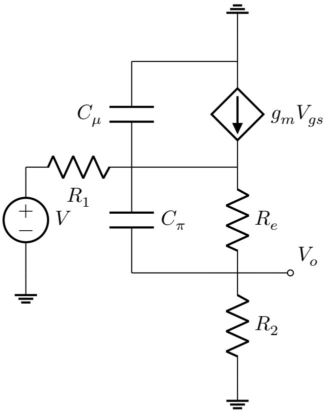
Now we can write out the transfer function. we assume that $R_2 >> R_1$ and $R_2 >> r_e$

$$\tau_\pi^0 = C_\pi \frac{R_1 + R_2}{1 + g_m R_2} = r_e C_\pi \qquad (186)$$



Nth Order System

$$\tau_{\mu}^{\pi} = C_{\mu}\left(R_1 \parallel R_2\right) = C_{\mu}R_1 \quad (187)$$

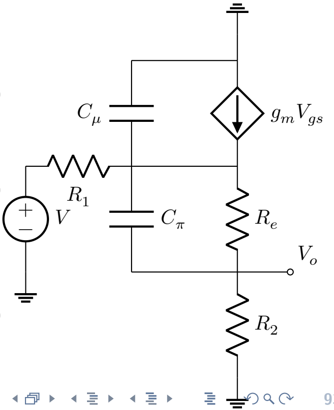


$\mathbf{N}^{th}$ Order System

$$\tau_\pi^0 \tau_\mu^\pi = r_e R_1 C_\pi C_\mu \tag{188}$$

$$H^\pi \tau_\pi^0 = r_e C_\pi \frac{R_2}{r_e + R_2} = H^0 r_e C_\pi \tag{189}$$

$$H(S) = H^0 \frac{1 + r_e C_\pi S}{1 + \left(r_e C_\pi + R_1 C_\mu\right) S + r_e C_\pi R_1 C_\mu S^2} \tag{190}$$



Nth Order System

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$$H(S) = H^0 \frac{1 + r_e C_\pi S}{1 + \left(r_e C_\pi + R_1 C_\mu\right) S + r_e C_\pi R_1 C_\mu S^2} = \frac{H^0}{1 + R_1 C_\mu S} \tag{191}$$

This tells us the dominate pole is the C_μ because C_π shares current between the capacitor and the resistor, so that it tells us to improve the bandwidth of operation, we need to use an inductor or some topology to cancel the effect of C_μ

Bandwidth Estimation

- The whole system can be expressed as the product of a high pass and a low pass transfer function.
- If I'm designing an analog circuit, I can use transfer function to estimate bandwidth, assuming I care about the lowpass one.
- For low frequency system, in many cases we can assume a zeroless system.
- For example, in common-source stage and the source-follower stage, the zero's frequencies are comparable to the cut-off frequency of the transistor itself

Bandwidth Estimation

$$H(s) \approx \frac{a_0}{1 + b_1 s + b_2 s^2 + \cdots + b_n s^n} \tag{192}$$

At DC ($\omega = 0$), the only term in the denominator that matters is the leading 1. As the frequency goes up and starts approaching the ω_h, the first term that becomes non-negligible would be b_1, so in the vicinity of the ω_h The system is pretty much

$$H(s) \approx \frac{a_0}{1 + b_1 s} \tag{193}$$

Bandwidth Estimation

This tells us, in order to find out the cutoff frequency, we calculate b_1

$$\omega_h \approx \frac{1}{b_1} = \frac{1}{\sum_{i=1}^{N} \tau_i^0} \tag{194}$$

$$\frac{a_1}{1 + b_1 S + b_2 S^2} \Rightarrow H(j\omega) = \frac{a_0}{(1 - b_2\omega^2) + j\omega b_1} \tag{195}$$

- It is a conservative estimation of bandwidth