

Hands-on Large Scale Optimization in Python

From Beginning to Giving Up

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Preface

1 Environment Setup

In this chapter, we explain the steps needed to set up Python and Google OR-Tools. All the steps below are based on MacBook Air with M1 chip and macOS Ventura 13.1.

1.1 Install Homebrew

The first tool we need is Homebrew, ‘the Missing Package Manager for macOS (or Linux)’, and it can be accessed at <https://brew.sh/>. To install Homebrew, just copy the command below and run it in the Terminal.

```
/bin/bash -c "$(curl -fsSL https://raw.githubusercontent.com/Homebrew/install/HEAD/install.sh)"
```

We can then use the `brew --version` command to check the installed version. On my system, it shows the info below.

```
~/ brew --version
Homebrew 3.6.20
Homebrew/homebrew-core (git revision 5f1582e4d55; last commit 2023-02-05)
Homebrew/homebrew-cask (git revision fa3b8a669d; last commit 2023-02-05)
```

1.2 Install Anaconda

Since there are several Python versions available for our use and we may end up having multiple Python versions installed on our machine, it is important to use a consistent environment to work on our project in. Anaconda is a package and environment manager for Python and it provides easy-to-use tools to facilitate our data science needs. To install Anaconda, run the below command in the Terminal.

```
~/ brew install anaconda
```

After the installation is done, we can use `conda --version` to verify whether it is available on our machine or not.

```
~/ conda --version
conda 23.1.0
```

1.3 Create a Conda Environment

Now we will create a Conda environment named 'ortools'. Execute the below command in the Terminal, which effectively creates the required environment with Python version 3.10.

```
~/ conda create -n ortools python=3.10
Retrieving notices: ...working... done
Collecting package metadata (current_repodata.json): done
Solving environment: done
```

```
## Package Plan ##
```

```
environment location: /opt/homebrew/anaconda3/envs/test
```

```
added / updated specs:
- python=3.10
```

The following packages will be downloaded:

package	build		
-----	-----		
setuptools-67.4.0	pyhd8ed1ab_0	567 KB	conda-forge
-----	-----		
Total:		567 KB	

The following NEW packages will be INSTALLED:

bzip2	conda-forge/osx-arm64::bzip2-1.0.8-h3422bc3_4
ca-certificates	conda-forge/osx-arm64::ca-certificates-2022.12.7-h4653dfc_0
libffi	conda-forge/osx-arm64::libffi-3.4.2-h3422bc3_5
libsqlite	conda-forge/osx-arm64::libsqlite-3.40.0-h76d750c_0
libzlib	conda-forge/osx-arm64::libzlib-1.2.13-h03a7124_4
ncurses	conda-forge/osx-arm64::ncurses-6.3-h07bb92c_1
openssl	conda-forge/osx-arm64::openssl-3.0.8-h03a7124_0
pip	conda-forge/noarch::pip-23.0.1-pyhd8ed1ab_0
python	conda-forge/osx-arm64::python-3.10.9-h3ba56d0_0_cpython
readline	conda-forge/osx-arm64::readline-8.1.2-h46ed386_0

```

setuptools      conda-forge/noarch::setuptools-67.4.0-pyhd8ed1ab_0
tk              conda-forge/osx-arm64::tk-8.6.12-he1e0b03_0
tzdata          conda-forge/noarch::tzdata-2022g-h191b570_0
wheel           conda-forge/noarch::wheel-0.38.4-pyhd8ed1ab_0
xz              conda-forge/osx-arm64::xz-5.2.6-h57fd34a_0

```

Proceed ([y]/n)?

Type 'y' to proceed and Conda will create the environment for us. We can use `conda env list` to show all the created environments on our machine:

```

~/ conda env list
# conda environments:
#
base                /opt/homebrew/anaconda3
ortools             /opt/homebrew/anaconda3/envs/ortools

```

Note that we need to manually activate an environment in order to use it: `conda activate ortools`. On my machine, the activated environment `ortools` will appear in the beginning of my prompt.

```

~/ conda activate ortools
(ortools) ~/

```

1.4 Install Google OR-Tools

As of this writing, the latest version of Google OR-Tools is 9.5.2237, and we can install it in our newly created environment using the command `pip install ortools==9.5.2237`. We can use `conda list` to verify whether it is available in our environment.

```

(ortools) ~/ conda list
# packages in environment at /opt/homebrew/anaconda3/envs/ortools:
#
# Name                Version                Build    Channel
absl-py               1.4.0                  pypi_0   pypi
bzip2                 1.0.8                  h3422bc3_4  conda-forge
ca-certificates       2022.12.7              h4653dfc_0  conda-forge
libffi                 3.4.2                  h3422bc3_5  conda-forge
libsqlite              3.40.0                 h76d750c_0  conda-forge

```

libzlib	1.2.13	h03a7124_4	conda-forge
ncurses	6.3	h07bb92c_1	conda-forge
numpy	1.24.2	pypi_0	pypi
openssl	3.0.8	h03a7124_0	conda-forge
ortools	9.5.2237	pypi_0	pypi
pip	23.0.1	pyhd8ed1ab_0	conda-forge
protobuf	4.22.0	pypi_0	pypi
python	3.10.9	h3ba56d0_0_cpython	conda-forge
readline	8.1.2	h46ed386_0	conda-forge
setuptools	67.4.0	pyhd8ed1ab_0	conda-forge
tk	8.6.12	he1e0b03_0	conda-forge
tzdata	2022g	h191b570_0	conda-forge
wheel	0.38.4	pyhd8ed1ab_0	conda-forge
xz	5.2.6	h57fd34a_0	conda-forge

Now we have Python and Google OR-Tools ready, we can start our next journey.

2 Introduction

This is a book created from markdown and executable code.

See Knuth (1984) for additional discussion of literate programming.

3 Environment Setup

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```
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```

3.3 Create a Conda Environment

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libsqlite	conda-forge/osx-arm64::libsqlite-3.40.0-h76d750c_0
libzlib	conda-forge/osx-arm64::libzlib-1.2.13-h03a7124_4
ncurses	conda-forge/osx-arm64::ncurses-6.3-h07bb92c_1
openssl	conda-forge/osx-arm64::openssl-3.0.8-h03a7124_0
pip	conda-forge/noarch::pip-23.0.1-pyhd8ed1ab_0
python	conda-forge/osx-arm64::python-3.10.9-h3ba56d0_0_cpython
readline	conda-forge/osx-arm64::readline-8.1.2-h46ed386_0

```

setuptools      conda-forge/noarch::setuptools-67.4.0-pyhd8ed1ab_0
tk              conda-forge/osx-arm64::tk-8.6.12-he1e0b03_0
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wheel           conda-forge/noarch::wheel-0.38.4-pyhd8ed1ab_0
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```

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```

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# Name                Version                Build    Channel
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bzip2                 1.0.8                  h3422bc3_4  conda-forge
ca-certificates       2022.12.7              h4653dfc_0  conda-forge
libffi                 3.4.2                  h3422bc3_5  conda-forge
libsqlite              3.40.0                 h76d750c_0  conda-forge

```

libzlib	1.2.13	h03a7124_4	conda-forge
ncurses	6.3	h07bb92c_1	conda-forge
numpy	1.24.2	pypi_0	pypi
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ortools	9.5.2237	pypi_0	pypi
pip	23.0.1	pyhd8ed1ab_0	conda-forge
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python	3.10.9	h3ba56d0_0_cpython	conda-forge
readline	8.1.2	h46ed386_0	conda-forge
setuptools	67.4.0	pyhd8ed1ab_0	conda-forge
tk	8.6.12	he1e0b03_0	conda-forge
tzdata	2022g	h191b570_0	conda-forge
wheel	0.38.4	pyhd8ed1ab_0	conda-forge
xz	5.2.6	h57fd34a_0	conda-forge

Now we have Python and Google OR-Tools ready, we can start our next journey.

Part I

Benders Decomposition

4 Benders Decomposition

In this chapter, we will explain the theories behind Benders decomposition and demonstrate its usage on a trial linear programming problem. Keep in mind that Benders decomposition is not limited to solving linear programming problems. In fact, it is one of the most powerful techniques to solve some large-scale mixed-integer linear programming problems.

In the following sections, we will go through the critical steps during the decomposition process when applying the algorithm on optimization problems represented in standard forms. This is important as it helps build up the intuition of when we should consider applying Benders decomposition to a problem at hand. Often times, recognizing the applicability of Benders decomposition is the most important and challenging step when solving an optimization problem. Once we know that the problem structure is suitable to solve via Benders decomposition, it is straightforward to follow the decomposition steps and put it into work.

Generally speaking, Benders decomposition is a good solution candidate when the resulting problem is much easier to solve if some of the variables in the original problem are fixed. We will illustrate this point using an example in the following sections. In the optimization world, the first candidate that should come to mind when we say a problem is easy to solve is a linear programming formulation, which is indeed the case in Benders decomposition applications.

4.1 The Decomposition Logic

To explain the workings of Benders decomposition, let us look at the standard form of linear programming problems that involve two vector variables, \mathbf{x} and \mathbf{y} . Let p and q indicate the dimensions of \mathbf{x} and \mathbf{y} , respectively. Below is the original problem (\mathbf{P}) we intend to solve.

$$\min. \quad \mathbf{c}^T \mathbf{x} + \mathbf{f}^T \mathbf{y} \tag{4.1}$$

$$\text{s.t.} \quad \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{y} = \mathbf{b} \tag{4.2}$$

$$\mathbf{x} \geq 0, \mathbf{y} \geq 0 \tag{4.3}$$

In this formulation, \mathbf{c} and \mathbf{f} in the objective function represent the cost coefficients associated with decision variables \mathbf{x} and \mathbf{y} , respectively. Both of them are column vectors of corresponding dimensions. In the constraints, matrix \mathbf{A} is of dimension $m \times p$, and matrix \mathbf{B} is of dimension $m \times q$. \mathbf{b} is a column vector of dimension m .

Suppose the variable \mathbf{y} is a complicating variable in the sense that the resulting problem is substantially easier to solve if the value of \mathbf{y} is fixed. In this case, we could rewrite problem **P** as the following form:

$$\min. \quad \mathbf{f}^T \mathbf{y} + g(\mathbf{y}) \quad (4.4)$$

$$\text{s.t.} \quad \mathbf{y} \geq 0 \quad (4.5)$$

where $g(\mathbf{y})$ is a function of \mathbf{y} and is defined as the subproblem **SP** of the form below:

$$\min. \quad \mathbf{c}^T \mathbf{x} \quad (4.6)$$

$$\text{s.t.} \quad \mathbf{A}\mathbf{x} = \mathbf{b} - \mathbf{B}\mathbf{y} \quad (4.7)$$

$$\mathbf{x} \geq 0 \quad (4.8)$$

Note that the \mathbf{y} in constraint (4.7) takes on some known values when the problem is solved and the only decision variable in the above formulation is \mathbf{x} . The dual problem of **SP**, **DSP**, is given below.

$$\max. \quad (\mathbf{b} - \mathbf{B}\mathbf{y})^T \mathbf{u} \quad (4.9)$$

$$\text{s.t.} \quad \mathbf{A}^T \mathbf{u} \leq \mathbf{c} \quad (4.10)$$

$$\mathbf{u} \text{ unrestricted} \quad (4.11)$$

A key characteristic of the above **DSP** is that its solution space does not depend on the value of \mathbf{y} , which only affects the objective function. According to the Minkowski's representation theorem, any $\bar{\mathbf{u}}$ satisfying the constraints (4.10) can be expressed as

$$\bar{\mathbf{u}} = \sum_{j \in \mathbf{J}} \lambda_j \mathbf{u}_j^{point} + \sum_{k \in \mathbf{K}} \mu_k \mathbf{u}_k^{ray} \quad (4.12)$$

where \mathbf{u}_j^{point} and \mathbf{u}_k^{ray} represent an extreme point and extreme ray, respectively. In addition, $\lambda_j \geq 0$ for all $j \in \mathbf{J}$ and $\sum_{j \in \mathbf{J}} \lambda_j = 1$, and $\mu_k \geq 0$ for all $k \in \mathbf{K}$. It follows that the **DSP** is equivalent to

$$\max. \quad (\mathbf{b} - \mathbf{B}\mathbf{y})^T \left(\sum_{j \in \mathbf{J}} \lambda_j \mathbf{u}_j^{point} + \sum_{k \in \mathbf{K}} \mu_k \mathbf{u}_k^{ray} \right) \quad (4.13)$$

$$\text{s.t.} \quad \sum_{j \in \mathbf{J}} \lambda_j = 1 \quad (4.14)$$

$$\lambda_j \geq 0, \quad \forall j \in \mathbf{J} \quad (4.15)$$

$$\mu_k \geq 0, \quad \forall k \in \mathbf{K} \quad (4.16)$$

We can therefore conclude that

- The **DSP** becomes unbounded if any \mathbf{u}_k^{ray} exists such that $(\mathbf{b} - \mathbf{B}\mathbf{y})^T \mathbf{u}_k^{ray} > 0$. Note that an unbounded **DSP** implies an infeasible **SP** and to prevent this from happening, we have to ensure that $(\mathbf{b} - \mathbf{B}\mathbf{y})^T \mathbf{u}_k^{ray} \leq 0$ for all $k \in \mathbf{K}$.
- If an optimal solution to **DSP** exists, it must occur at one of the extreme points. Let g denote the optimal objective value, it follows that $(\mathbf{b} - \mathbf{B}\mathbf{y})^T \mathbf{u}_j^{point} \leq g$ for all $j \in \mathbf{J}$.

Based on this idea, the **DSP** can be reformulated as follows:

$$\min. \quad g \quad (4.17)$$

$$\text{s.t.} \quad (\mathbf{b} - \mathbf{B}\mathbf{y})^T \mathbf{u}_k^{ray} \leq 0, \quad \forall k \in \mathbf{K} \quad (4.18)$$

$$(\mathbf{b} - \mathbf{B}\mathbf{y})^T \mathbf{u}_j^{point} \leq g, \quad \forall j \in \mathbf{J} \quad (4.19)$$

$$j \in \mathbf{J}, k \in \mathbf{K} \quad (4.20)$$

Constraints (4.18) are called **Benders feasibility cuts**, while constraints (4.19) are called **Benders optimality cuts**. Now we are ready to define the Benders Master Problem (**BMP**) as follows:

$$\min. \quad \mathbf{f}^T \mathbf{y} + g \quad (4.21)$$

$$\text{s.t.} \quad (\mathbf{b} - \mathbf{B}\mathbf{y})^T \mathbf{u}_k^{ray} \leq 0, \quad \forall k \in \mathbf{K} \quad (4.22)$$

$$(\mathbf{b} - \mathbf{B}\mathbf{y})^T \mathbf{u}_j^{point} \leq g, \quad \forall j \in \mathbf{J} \quad (4.23)$$

$$j \in \mathbf{J}, k \in \mathbf{K}, \mathbf{y} \geq 0 \quad (4.24)$$

Typically J and K are too large to enumerate upfront and we have to work with subsets of them, denoted by J_s and K_s , respectively. Hence we have the following Restricted Benders Master Problem (**RBMP**):

$$\min. \quad \mathbf{f}^T \mathbf{y} + g \quad (4.25)$$

$$\text{s.t.} \quad (\mathbf{b} - \mathbf{B}\mathbf{y})^T \mathbf{u}_k^{ray} \leq 0, \quad \forall j \in \mathbf{J}_s \quad (4.26)$$

$$(\mathbf{b} - \mathbf{B}\mathbf{y})^T \mathbf{u}_j^{point} \leq g, \quad \forall k \in \mathbf{K}_s \quad (4.27)$$

$$j \in \mathbf{J}, k \in \mathbf{K}, \mathbf{y} \geq 0 \quad (4.28)$$

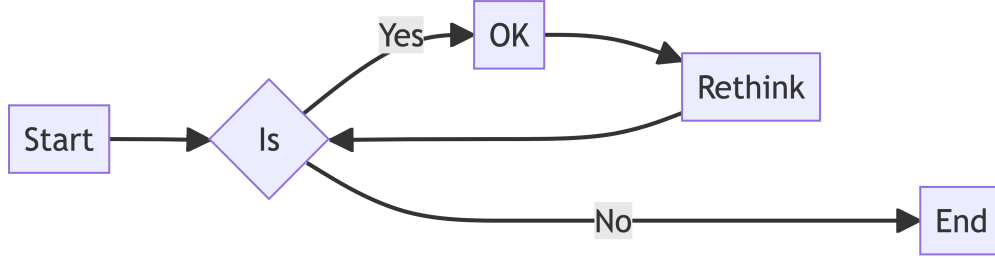


Figure 4.1: Benders decomposition workflow

4.2 Solving Linear Programming Problems with Benders Decomposition

In this section, we use Benders decomposition to solve several linear programming (LP) problems in order to demonstrate the decomposition logic, especially how the restricted Benders master problem interacts with the subproblem in an iterative approach to reach final optimality. Most linear programs could be solved efficiently nowadays by either open source or commercial solvers without resorting to any decomposition approaches. However, by working through the example problems in the following sections, we aim to showcase the implementation details when applying Benders decomposition algorithm on real problems, which helps solidify our understanding of Benders decomposition. Hopefully, by the end of this chapter, we will build up enough intuition as well as hands-on experience such that we are ready to tackle most involved problems in the following chapters.

In the following sections, we employ several steps to illustrate the problem solving process of Benders decomposition.

- We will first create two linear programming solvers based on Gurobi and SCIP that can solve any linear programs defined in the standard form. They are used in later section to validate the correctness of the solutions produced by Benders decomposition.
- Next, we use a specific linear program and give the corresponding RBMP and DSP to prepare for the implementations.

- Then, we will solve the example linear program step by step by examining the outputs of the RBMP and DSP to decide the next set of actions.
- Furthermore, a holistic Benders decomposition implementation is then developed to solve the example linear program.
- Following the previous step, a more generic Benders decomposition implementation is created.
- Then, we will examine an alternative implementation using Gurobi callback functions.
- We will also provide an implementation based on SCIP.
- In the final section, we will do several benchmarking testing.

4.2.1 LP solvers based on Gurobi and SCIP

We aim to use Benders decomposition to solve several linear programming problems in the following sections. To do that, we intentionally decompose the LP problem under consideration into two sets, one set of *complicating* variables and the other set containing the remaining variables. In order to validate the correctness of the results obtained by Benders decomposition, we implement two additional ways of solving the target linear programming problems directly. The first option is based on the Gurobi API in python and the other is based on the open source solve SCIP. The two implementations defined here assume the LP problems under consideration follow the below standard form:

$$\min. \quad \mathbf{c}^T \mathbf{x} \quad (4.29)$$

$$\text{s.t.} \quad \mathbf{Ax} = \mathbf{b} \quad (4.30)$$

$$\mathbf{x} \geq 0 \quad (4.31)$$

Listing 4.1 defines a solver for LP problems using Gurobi. It takes three constructor parameters:

- `obj_coeff`: this corresponds to the objective coefficients \mathbf{c} .
- `constr_mat`: this refers to the constraint matrix \mathbf{A} .
- `rhs`: this is the right-hand side \mathbf{b} .

Inside the constructor `__init__()`, a solver environment `_env` is first created and then used to initialize a model object `_model`. The input parameters are then used to create decision variables `_vars`, constraints `_constrs` and objective function respectively. The `optimize()` function simply solves the problem and shows the solving status. Finally, the `clean_up()` function frees up the computing resources.

Listing 4.2 presents an LP solver implementation in class `LpSolverSCIP` using SCIP. The constructor requires the same of parameters as defined in `LpSolverGurobi`. The model building

process is similar with minor changes when required to create decision variables, constraints and the objective function.

Listing 4.3 generates a LP problem with 20 decision variables and 5 constraints.

Listing 4.4 solves the generated LP using `LpSolverGurobi` and the solver output shows that an optimal solution was found with objective value of 36.90.

```
Optimal solution found!
Optimal objective = 36.90
```

Listing 4.5 solves the same LP problem using SCIP and not surprisingly, the same optimal objective value was found. This is not exciting, as it only indicates that the two solvers agree on the optimal solution on such a small LP problem as expected. However, they will become more useful in the following sections when we use them to validate our Benders decomposition results.

```
Optimal solution found!
Optimal objective = 36.90
```

4.2.2 A serious LP problem that cannot wait to be decomposed!

With the validation tools available for use, we are ready to solve some serious LP problems using Benders decomposition! What we have below is a LP problem with five decision variables, three of which are denoted by $\mathbf{x} = (x_1, x_2, x_3)$ and the remaining two variables are denoted by $\mathbf{y} = (y_1, y_2)$. We assume that \mathbf{y} are the complicating variables.

$$\begin{aligned} \min. \quad & 8x_1 + 12x_2 + 10x_3 + 15y_1 + 18y_2 \\ \text{s.t.} \quad & 2x_1 + 3x_2 + 2x_3 + 4y_1 + 5y_2 = 300 \\ & 4x_1 + 2x_2 + 3x_3 + 2y_1 + 3y_2 = 220 \\ & x_i \geq 0, \quad \forall i = 1, \dots, 3 \\ & y_j \geq 0, \quad \forall j = 1, 2 \end{aligned}$$

According to the standard LP form presented in the previous section, $\mathbf{c}^T = (8, 12, 10)$, $\mathbf{f}^T = (15, 18)$ and $\mathbf{b}^T = (300, 220)$. In addition,

$$\mathbf{A} = \begin{bmatrix} 2 & 3 & 2 \\ 4 & 2 & 3 \end{bmatrix} \quad \mathbf{B} = \begin{bmatrix} 4 & 5 \\ 2 & 3 \end{bmatrix}$$

Listing 4.6 solves the LP problem using the two solvers define above. Both solvers confirm the optimal objective value is 1091.43. With this information in mind, we will apply Benders decomposition to see if the same optimal solution could be identified or not.

```
Optimal solution found!
Optimal objective = 1091.43
Optimal solution found!
Optimal objective = 1091.43
```

4.2.3 Benders decomposition formulations

With \mathbf{y} being the complicating variable, we state the Benders subproblem (**SP**) below for the \mathbf{y} assuming fixed values $\bar{\mathbf{y}} = (\bar{y}_1, \bar{y}_2)$:

$$\begin{aligned} \min. \quad & 8x_1 + 12x_2 + 10x_3 \\ \text{s.t.} \quad & 2x_1 + 3x_2 + 2x_3 = 300 - 4\bar{y}_1 - 5\bar{y}_2 \\ & 4x_1 + 2x_2 + 3x_3 = 220 - 2\bar{y}_1 - 3\bar{y}_2 \\ & x_i \geq 0, \quad \forall i = 1, \dots, 3 \end{aligned}$$

We define the dual variable $\mathbf{u} = (u_1, u_2)$ to associate with the constraints in the (**SP**). The dual subproblem (**DSP**) could then be stated as follows:

$$\begin{aligned} \max. \quad & (300 - 4\bar{y}_1 - 5\bar{y}_2)u_1 + (220 - 2\bar{y}_1 - 3\bar{y}_2)u_2 \\ \text{s.t.} \quad & 2u_1 + 4u_2 \leq 8 \\ & 3u_1 + 2u_2 \leq 12 \\ & 2u_1 + 3u_2 \leq 10 \\ & u_1, u_2 \text{ unrestricted} \end{aligned}$$

The (**RBMP**) can be stated as below. Note that $\mathbf{u} = (0, 0)$ is a feasible solution to (**DSP**) and the corresponding objective value is 0, which is the reason we restrict the variable g to be nonnegative.

$$\begin{aligned} \min. \quad & 15y_1 + 18y_2 + g \\ \text{s.t.} \quad & y_1, y_2 \geq 0 \\ & g \geq 0 \end{aligned}$$

4.2.4 Benders decomposition step by step

Benders decomposition defines a problem solving process in which the restricted Benders master problem and the dual subproblem interact iteratively to identify the optimal solution or conclude infeasibility/unboundedness. To facilitate our understanding of the process, we demonstrate in this section the workings of Benders decomposition by solving the target LP problem step by step.

Listing 4.7 shows the codes that initialize the lower bound `lb`, upper bound `ub` and threshold value `eps`. Furthermore, the restricted Benders master problem `rbmp` is created with three variables and the minimizing objective function. Note that no constraints are yet included in the model at all.

Listing 4.8 initializes the Benders subproblem with two decision variables and three constraints.

In Listing 4.9, we solve the (**RBMP**) for the first time. It has an optimal solution with $(y_1, y_2, g) = (0, 0, 0)$ and optimal objective value of 0. This is expected as all the variables assume their minimal possible values in order to minimize the objective function. This objective value also serves as the new lower bound.

```
Optimal solution found! Objective value = 0.00
(y1, y2, g) = (0.00, 0.00, 0.00)
lb=0.0, ub=1e+100
```

Given that $(\bar{y}_1, \bar{y}_2, \bar{g}) = (0, 0, 0)$, we now feed the values of \bar{y}_1 and \bar{y}_2 into the Benders dual subproblem (**DSP**) by updating its objective function, as shown in Listing 4.10:

```
Optimal objective = 1200.00
(u1, u2) = (4.00, 0.00)
lb=0.0, ub=1200.0
```

We see that the dual subproblem has an optimal solution. The upper bound `ub` is also updated. Since the optimal objective value of the subproblem turns out to be 1200 and is greater than $\bar{g} = 0$, which implies that an optimality cut is needed to make sure that the variable g in the restricted Benders master problem reflects this newly obtained information from the subproblem.

```
Optimal solution found! Objective value = 1080.00
(y1, y2, g) = (0.00, 60.00, 0.00)
lb=1080.0, ub=1200.0
```

Now we solve the subproblem again with the newly obtained solution $(\bar{y}_1, \bar{y}_2, \bar{g}) = (0, 60, 0)$.

```
Optimal objective = 80.00
(u1, u2) = (0.00, 2.00)
lb=1080.0, ub=1160.00
```

Since the optimal objective value of the subproblem, 80, is still bigger than $\bar{g} = 0$, an optimality cut is needed. In the below code snippet, we add the new cut and solve the restricted Benders master problem again.

```
Optimal solution found! Objective value = 1091.43
(y1, y2, g) = (0.00, 54.29, 114.29)
lb=1091.43, ub=1160.00
```

Note that a new lower bound is obtained after solving the master problem. Since there is still a large gap between the lower bound and upper bound, we continue solving the subproblem.

```
Optimal objective = 114.29
(u1, u2) = (0.00, 2.00)
lb=1091.43, ub=1091.43
```

4.2.5 Putting it together

Certainly we don't want to manually control the interaction between the master problem and subproblem to find the optimal solution. Therefore, in this section, we will put every together to come up with a control flow to help us identify the optimal solution automatically.

```
import gurobipy as gp
from gurobipy import GRB
import numpy as np
from enum import Enum

class OptStatus(Enum):
    OPTIMAL = 0
    UNBOUNDED = 1
    INFEASIBLE = 2
    ERROR = 3
```

```

class MasterSolver:

    def __init__(self, env):
        self._model = gp.Model(env=env, name='RBMP')

        # create decision variables
        self._y1 = self._model.addVar(vtype=GRB.CONTINUOUS, lb=0, name='y1')
        self._y2 = self._model.addVar(vtype=GRB.CONTINUOUS, lb=0, name='y2')
        self._g = self._model.addVar(vtype=GRB.CONTINUOUS, lb=0, name='g')

        # create objective
        self._model.setObjective(15*self._y1 + 18*self._y2 + self._g, GRB.MINIMIZE)

        self._opt_obj = None
        self._opt_y1 = None
        self._opt_y2 = None
        self._opt_g = None

    def solve(self) -> OptStatus:
        print('-' * 50)
        print(f'Start solving master problem.')
        self._model.optimize()

        opt_status = None
        if self._model.status == GRB.OPTIMAL:
            opt_status = OptStatus.OPTIMAL
            self._opt_obj = self._model.objVal
            self._opt_y1 = self._y1.X
            self._opt_y2 = self._y2.X
            self._opt_g = self._g.X
            print(f'\tmaster problem is optimal.')
            print(f'\topt_obj={self._opt_obj:.2f}')
            print(f'\topt_y1={self._opt_y1:.2f}, opt_y2={self._opt_y2:.2f}, opt_g={self._opt_g:.2f}')
        elif self._model.status == GRB.INFEASIBLE:
            print(f'\tmaster problem is infeasible.')
            opt_status = OptStatus.INFEASIBLE
        else:
            print(f'\tmaster problem encountered error.')
            opt_status = OptStatus.ERROR

        print(f'Finish solving master problem.')
        print('-' * 50)

```

```

    return opt_status

def add_feasibility_cut(self, opt_u1, opt_u2) -> None:
    self._model.addConstr((300 - 4*self._y1 - 5*self._y2) * opt_u1 +
                          (220 - 2*self._y1 - 3*self._y2) * opt_u2 <= 0)
    print(f'Benders feasibility cut added!')

def add_optimality_cut(self, opt_u1, opt_u2) -> None:
    self._model.addConstr((300 - 4*self._y1 - 5*self._y2) * opt_u1 +
                          (220 - 2*self._y1 - 3*self._y2) * opt_u2 <= self._g)
    print(f'Benders optimality cut added!')

def clean_up(self):
    self._model.dispose()

@property
def opt_obj(self):
    return self._opt_obj

@property
def opt_y1(self):
    return self._opt_y1

@property
def opt_y2(self):
    return self._opt_y2

@property
def opt_g(self):
    return self._g

```

```

class DualSubprobSolver:

```

```

    def __init__(self, env):
        self._model = gp.Model(env=env, name='DSP')

        # create decision variables
        self._u1 = self._model.addVar(vtype=GRB.CONTINUOUS, name='u1')
        self._u2 = self._model.addVar(vtype=GRB.CONTINUOUS, name='u2')

        # create constraints
        self._model.addConstr(2*self._u1 + 4*self._u2 <= 8, name='c1')

```



```

self._model.addConstr(3*self._u1 + 2*self._u2 <= 12, name='c2')
self._model.addConstr(2*self._u1 + 3*self._u2 <= 10, name='c3')

self._model.setObjective(1, GRB.MAXIMIZE)
self._model.update()

self._opt_obj = None
self._opt_u1 = None
self._opt_u2 = None

def solve(self):
    print('-' * 50)
    print(f'Start solving dual subproblem.')
    self._model.optimize()

    status = None
    if self._model.status == GRB.OPTIMAL:
        self._opt_obj = self._model.objVal
        self._opt_u1 = self._u1.X
        self._opt_u2 = self._u2.X
        status = OptStatus.OPTIMAL
        print(f'\tdual subproblem is optimal.')
        print(f'\topt_obj={self._opt_obj:.2f}')
        print(f'\topt_y1={self._opt_u1:.2f}, opt_y2={self._opt_u2:.2f}')
    elif self._model.status == GRB.UNBOUNDED:
        status = OptStatus.UNBOUNDED
    else:
        status = OptStatus.ERROR

    print(f'Finish solving dual subproblem.')
    print('-' * 50)
    return status

def update_objective(self, opt_y1, opt_y2):
    self._model.setObjective((300-4*opt_y1-5*opt_y2)*self._u1 + (220-2*opt_y1-3*opt_y2)*self._u2)
    print(f'dual subproblem objective updated!')

def clean_up(self):
    self._model.dispose()

@property
def opt_obj(self):

```

```

        return self._opt_obj

    @property
    def opt_u1(self):
        return self._opt_u1

    @property
    def opt_u2(self):
        return self._opt_u2

```

```

class BendersDecomposition:

    def __init__(self, master_solver, dual_subprob_solver):
        self._master_solver = master_solver
        self._dual_subprob_solver = dual_subprob_solver

    def optimize(self) -> OptStatus:
        eps = 1.0e-5
        lb = -np.inf
        ub = np.inf

        while True:
            # solve master problem
            master_status = self._master_solver.solve()
            if master_status == OptStatus.INFEASIBLE:
                return OptStatus.INFEASIBLE

            # update lower bound
            lb = np.max([lb, self._master_solver.opt_obj])
            print(f'Bounds: lb={lb:.2f}, ub={ub:.2f}')

            opt_y1 = self._master_solver.opt_y1
            opt_y2 = self._master_solver.opt_y2

            # solve subproblem
            self._dual_subprob_solver.update_objective(opt_y1, opt_y2)
            dsp_status = self._dual_subprob_solver.solve()

            if dsp_status == OptStatus.OPTIMAL:
                # update upper bound
                opt_obj = self._dual_subprob_solver.opt_obj

```

```

        ub = np.min([ub, 15*opt_y1 + 18*opt_y2 + opt_obj])
        print(f'Bounds: lb={lb:.2f}, ub={ub:.2f}')

        if ub - lb <= eps:
            break

        opt_u1 = self._dual_subprob_solver.opt_u1
        opt_u2 = self._dual_subprob_solver.opt_u2
        self._master_solver.add_optimality_cut(opt_u1, opt_u2)
    elif dsp_status == OptStatus.UNBOUNDED:
        opt_u1 = self._dual_subprob_solver.opt_u1
        opt_u2 = self._dual_subprob_solver.opt_u2
        self._master_solver.add_feasibility_cut(opt_u1, opt_u2)

```

```

env = gp.Env('benders')
env.setParam("OutputFlag",0)
master_solver = MasterSolver(env)
dual_subprob_solver = DualSubprobSolver(env)

benders_decomposition = BendersDecomposition(master_solver, dual_subprob_solver)
benders_decomposition.optimize()

```

```

Set parameter Username
Set parameter LogFile to value "benders"

```

```

-----
Start solving master problem.
    master problem is optimal.
    opt_obj=0.00
    opt_y1=0.00, opt_y2=0.00, opt_g=0.00
Finish solving master problem.

```

```

-----
Bounds: lb=0.00, ub=inf
dual subproblem objective updated!

```

```

-----
Start solving dual subproblem.
    dual subproblem is optimal.
    opt_obj=1200.00
    opt_y1=4.00, opt_y2=0.00
Finish solving dual subproblem.

```

```

-----
Bounds: lb=0.00, ub=1200.00

```

```

Benders optimality cut added!
-----
Start solving master problem.
  master problem is optimal.
  opt_obj=1080.00
  opt_y1=0.00, opt_y2=60.00, opt_g=0.00
Finish solving master problem.
-----
Bounds: lb=1080.00, ub=1200.00
dual subproblem objective updated!
-----
Start solving dual subproblem.
  dual subproblem is optimal.
  opt_obj=80.00
  opt_y1=0.00, opt_y2=2.00
Finish solving dual subproblem.
-----
Bounds: lb=1080.00, ub=1160.00
Benders optimality cut added!
-----
Start solving master problem.
  master problem is optimal.
  opt_obj=1091.43
  opt_y1=0.00, opt_y2=54.29, opt_g=114.29
Finish solving master problem.
-----
Bounds: lb=1091.43, ub=1160.00
dual subproblem objective updated!
-----
Start solving dual subproblem.
  dual subproblem is optimal.
  opt_obj=114.29
  opt_y1=0.00, opt_y2=2.00
Finish solving dual subproblem.
-----
Bounds: lb=1091.43, ub=1091.43

```

4.2.6 A generic solver

In this section, we will create a more generic Benders decomposition based solver for linear programming problems.

```

class GenericLpMasterSolver:

    def __init__(self, f: np.array, B: np.array, b: np.array):
        # save data
        self._f = f
        self._B = B
        self._b = b

        # env and model
        self._env = gp.Env('MasterEnv')
        # self._env.setParam("OutputFlag",0)
        self._model = gp.Model(env=self._env, name='MasterSolver')

        # create variables
        self._num_y_vars = len(f)
        self._y = self._model.addVars(self._num_y_vars, lb=0, vtype=GRB.CONTINUOUS, name='y')
        self._g = self._model.addVar(vtype=GRB.CONTINUOUS, lb=0, name='g')

        # create objective
        self._model.setObjective(gp.quicksum(self._f[i] * self._y.get(i)
                                             for i in range(self._num_y_vars)) + self._g,
                                GRB.MINIMIZE)

        self._model.update()

        self._opt_obj = None
        self._opt_obj_y = None
        self._opt_y = None
        self._opt_g = None

    def solve(self) -> OptStatus:
        print('-' * 50)
        print(f'Start solving master problem.')
        print(self._model.display())
        self._model.optimize()

        opt_status = None
        if self._model.status == GRB.OPTIMAL:
            opt_status = OptStatus.OPTIMAL
            self._opt_obj = self._model.objVal
            self._opt_y = {
                i: self._y.get(i).X
                for i in range(self._num_y_vars)
            }

```

```

    }
    self._opt_g = self._g.X
    self._opt_obj_y = self._opt_obj - self._opt_g
    print(f'\tmaster problem is optimal.')
    print(f'\topt_obj={self._opt_obj:.2f}')
    print(f'\topt_g={self._opt_g:.2f}')
    # for i in range(self._num_y_vars):
    #     print(f'\topt_y[{i}]= {self._opt_y.get(i)}')
elif self._model.status == GRB.INFEASIBLE:
    print(f'\tmaster problem is infeasible.')
    opt_status = OptStatus.INFEASIBLE
else:
    print(f'\tmaster problem encountered error.')
    opt_status = OptStatus.ERROR

print(f'Finish solving master problem.')
print('-' * 50)
return opt_status

def add_feasibility_cut(self, opt_u: dict) -> None:
    constr_expr = [
        opt_u.get(u_idx) * (self._b[u_idx] - gp.quicksum(self._B[u_idx][j] * self._y.get
                                                             for j in range(self._num_y_vars)))
        for u_idx in opt_u.keys()
    ]
    self._model.addConstr(gp.quicksum(constr_expr) <= 0)
    print(f'Benders feasibility cut added!')

def add_optimality_cut(self, opt_u: dict) -> None:
    constr_expr = [
        opt_u.get(u_idx) * (self._b[u_idx] - gp.quicksum(self._B[u_idx][j] * self._y.get
                                                             for j in range(self._num_y_vars)))
        for u_idx in opt_u.keys()
    ]
    self._model.addConstr(gp.quicksum(constr_expr) <= self._g)
    self._model.update()
    print(self._model.display())
    print(f'Benders optimality cut added!')

def clean_up(self):
    self._model.dispose()
    self._env.dispose()

```

```

@property
def f(self):
    return self._f

@property
def opt_obj(self):
    return self._opt_obj

@property
def opt_obj_y(self):
    return self._opt_obj_y

@property
def opt_y(self):
    return self._opt_y

@property
def opt_g(self):
    return self._g

```

```

class GenericLpSubprobSolver:

```

```

    def __init__(self, A: np.array, c: np.array, B: np.array, b: np.array):
        # save data
        self._A = A
        self._c = c
        self._b = b
        self._B = B

        # env and model
        self._env = gp.Env('SubprobEnv')
        self._env.setParam("OutputFlag",0)
        self._model = gp.Model(env=self._env, name='SubprobSolver')

        # create variables
        self._num_vars = len(b)
        self._u = self._model.addVars(self._num_vars, vtype=GRB.CONTINUOUS, name='u')

        # create constraints
        for c_idx in range(len(c)):
            self._model.addConstr(gp.quicksum(A[:,c_idx][i] * self._u.get(i)

```

```

for i in range(len(b))) <= c[c_idx])

self._opt_obj = None
self._opt_u = None
self._extreme_ray = None

def solve(self):
    print('-' * 50)
    print(f'Start solving dual subproblem.')
    self._model.setParam(GRB.Param.DualReductions, 0)
    self._model.setParam(GRB.Param.InfUnbdInfo, 1)
    self._model.optimize()

    status = None
    if self._model.status == GRB.OPTIMAL:
        self._opt_obj = self._model.objVal
        self._opt_u = {
            i: self._u.get(i).X
            for i in range(self._num_vars)
        }
        status = OptStatus.OPTIMAL
        print(f'\tdual subproblem is optimal.')
        print(f'\topt_obj={self._opt_obj:.2f}')
        # for i in range(self._num_vars):
        #     print(f'\topt_u[{i}]= {self._opt_u.get(i)}')
    elif self._model.status == GRB.UNBOUNDED:
        status = OptStatus.UNBOUNDED
        self._extreme_ray = {
            i: self._u.get(i).UnbdRay
            for i in range(self._num_vars)
        }
        print(f'dual subproblem is unbounded')
        # for i in range(self._num_vars):
        #     print(f'\topt_u[{i}]= {self._extreme_ray.get(i)}')
    else:
        status = OptStatus.ERROR

    print(f'Finish solving dual subproblem.')
    print('-' * 50)
    return status

def update_objective(self, opt_y: dict):

```



```

        obj_expr = [
            self._u.get(u_idx) * (self._b[u_idx] - sum(self._B[u_idx][j] * opt_y.get(j)
                                                         for j in range(len(opt_y))))
            for u_idx in range(self._num_vars)
        ]
        self._model.setObjective(gp.quicksum(obj_expr), GRB.MAXIMIZE)
        print(f'dual subproblem objective updated!')

    def clean_up(self):
        self._model.dispose()
        self._env.dispose()

    @property
    def opt_obj(self):
        return self._opt_obj

    @property
    def opt_u(self):
        return self._opt_u

    @property
    def extreme_ray(self):
        return self._extreme_ray

```

```

class GenericBendersSolver:

    def __init__(self, master_solver, dual_subprob_solver):
        self._master_solver = master_solver
        self._dual_subprob_solver = dual_subprob_solver

    def optimize(self,) -> OptStatus:
        eps = 1.0e-5
        lb = -np.inf
        ub = np.inf

        while True:
            # solve master problem
            master_status = self._master_solver.solve()
            if master_status == OptStatus.INFEASIBLE:
                return OptStatus.INFEASIBLE

```

```

# update lower bound
lb = np.max([lb, self._master_solver.opt_obj])
print(f'Bounds: lb={lb:.2f}, ub={ub:.2f}')

opt_y = self._master_solver.opt_y

# solve subproblem
self._dual_subprob_solver.update_objective(opt_y)
dsp_status = self._dual_subprob_solver.solve()

if dsp_status == OptStatus.OPTIMAL:
    # update upper bound
    opt_obj = self._dual_subprob_solver.opt_obj
    opt_obj_y = self._master_solver.opt_obj_y
    ub = np.min([ub, opt_obj_y + opt_obj])
    print(f'Bounds: lb={lb:.2f}, ub={ub:.2f}')

    if ub - lb <= eps:
        break

    opt_u = self._dual_subprob_solver.opt_u
    self._master_solver.add_optimality_cut(opt_u)
elif dsp_status == OptStatus.UNBOUNDED:
    extreme_ray = self._dual_subprob_solver.extreme_ray
    self._master_solver.add_feasibility_cut(extreme_ray)

```

```

import gurobipy as gp
from gurobipy import GRB
import numpy as np

c = np.array([8, 12, 10])
f = np.array([15, 18])
A = np.array([
    [2, 3, 2],
    [4, 2, 3]
])
B = np.array([
    [4, 5],
    [2, 3],
])
b = np.array([300, 220])

```

```

master_solver = GenericLpMasterSolver(f, B, b)
dual_subprob_solver = GenericLpSubprobSolver(A, c, B, b)

benders_solver = GenericBendersSolver(master_solver, dual_subprob_solver)
benders_solver.optimize()

```

```

Set parameter Username
Set parameter LogFile to value "MasterEnv"
Set parameter Username
Set parameter LogFile to value "SubprobEnv"
-----

Start solving master problem.
Minimize
    15.0 y[0] + 18.0 y[1] + g
Subject To
None
Gurobi Optimizer version 10.0.3 build v10.0.3rc0 (mac64[arm])

CPU model: Apple M1
Thread count: 8 physical cores, 8 logical processors, using up to 8 threads

Optimize a model with 0 rows, 3 columns and 0 nonzeros
Model fingerprint: 0xb00b2c2a
Coefficient statistics:
  Matrix range      [0e+00, 0e+00]
  Objective range   [1e+00, 2e+01]
  Bounds range      [0e+00, 0e+00]
  RHS range         [0e+00, 0e+00]
Presolve removed 0 rows and 3 columns
Presolve time: 0.00s
Presolve: All rows and columns removed
Iteration   Objective      Primal Inf.    Dual Inf.      Time
          0      0.0000000e+00   0.000000e+00   0.000000e+00    0s

Solved in 0 iterations and 0.00 seconds (0.00 work units)
Optimal objective  0.000000000e+00
    master problem is optimal.
    opt_obj=0.00
    opt_g=0.00
Finish solving master problem.
-----

Bounds: lb=0.00, ub=inf

```

```

dual subproblem objective updated!
-----
Start solving dual subproblem.
    dual subproblem is optimal.
    opt_obj=1200.00
Finish solving dual subproblem.
-----

Bounds: lb=0.00, ub=1200.00
Minimize
    15.0 y[0] + 18.0 y[1] + g
Subject To
    R0: -16.0 y[0] + -20.0 y[1] + -1.0 g <= -1200
None
Benders optimality cut added!
-----

Start solving master problem.
Minimize
    15.0 y[0] + 18.0 y[1] + g
Subject To
    R0: -16.0 y[0] + -20.0 y[1] + -1.0 g <= -1200
None
Gurobi Optimizer version 10.0.3 build v10.0.3rc0 (mac64[arm])

CPU model: Apple M1
Thread count: 8 physical cores, 8 logical processors, using up to 8 threads

Optimize a model with 1 rows, 3 columns and 3 nonzeros
Coefficient statistics:
  Matrix range      [1e+00, 2e+01]
  Objective range   [1e+00, 2e+01]
  Bounds range      [0e+00, 0e+00]
  RHS range         [1e+03, 1e+03]
Iteration    Objective          Primal Inf.    Dual Inf.      Time
     0         0.0000000e+00      1.500000e+02   0.000000e+00     0s
     1         1.0800000e+03      0.000000e+00   0.000000e+00     0s

Solved in 1 iterations and 0.00 seconds (0.00 work units)
Optimal objective  1.080000000e+03
    master problem is optimal.
    opt_obj=1080.00
    opt_g=0.00
Finish solving master problem.
-----

```

```

Bounds: lb=1080.00, ub=1200.00
dual subproblem objective updated!
-----
Start solving dual subproblem.
    dual subproblem is optimal.
    opt_obj=80.00
Finish solving dual subproblem.
-----
Bounds: lb=1080.00, ub=1160.00
Minimize
    15.0 y[0] + 18.0 y[1] + g
Subject To
    R0: -16.0 y[0] + -20.0 y[1] + -1.0 g <= -1200
    R1: -4.0 y[0] + -6.0 y[1] + -1.0 g <= -440
None
Benders optimality cut added!
-----
Start solving master problem.
Minimize
    15.0 y[0] + 18.0 y[1] + g
Subject To
    R0: -16.0 y[0] + -20.0 y[1] + -1.0 g <= -1200
    R1: -4.0 y[0] + -6.0 y[1] + -1.0 g <= -440
None
Gurobi Optimizer version 10.0.3 build v10.0.3rc0 (mac64[arm])

CPU model: Apple M1
Thread count: 8 physical cores, 8 logical processors, using up to 8 threads

Optimize a model with 2 rows, 3 columns and 6 nonzeros
Coefficient statistics:
  Matrix range      [1e+00, 2e+01]
  Objective range   [1e+00, 2e+01]
  Bounds range      [0e+00, 0e+00]
  RHS range         [4e+02, 1e+03]
Iteration    Objective          Primal Inf.    Dual Inf.      Time
     0         1.0800000e+03      4.000000e+01   0.000000e+00     0s
     1         1.0914286e+03      0.000000e+00   0.000000e+00     0s

Solved in 1 iterations and 0.00 seconds (0.00 work units)
Optimal objective  1.091428571e+03
    master problem is optimal.
    opt_obj=1091.43

```

```

    opt_g=114.29
Finish solving master problem.
-----
Bounds: lb=1091.43, ub=1160.00
dual subproblem objective updated!
-----
Start solving dual subproblem.
    dual subproblem is optimal.
    opt_obj=114.29
Finish solving dual subproblem.
-----
Bounds: lb=1091.43, ub=1091.43

```

```

import gurobipy as gp
from gurobipy import GRB
import numpy as np

c = np.array([1, 1, 1, 1, 1, 1])
f = np.array([1, 1, 1, 1])
A = np.array([
    [1, 1, 1, 1, 1, 1]
])
B = np.array([
    [1, 1, 1, 1]
])
b = np.array([1])

master_solver = GenericLpMasterSolver(f, B, b)
dual_subprob_solver = GenericLpSubprobSolver(A, c, B, b)

benders_solver = GenericBendersSolver(master_solver, dual_subprob_solver)
benders_solver.optimize()

```

```

Set parameter Username
Set parameter LogFile to value "MasterEnv"
Set parameter Username
Set parameter LogFile to value "SubprobEnv"
-----
Start solving master problem.
Minimize
    y[0] + y[1] + y[2] + y[3] + g
Subject To

```

```

None
Gurobi Optimizer version 10.0.3 build v10.0.3rc0 (mac64[arm])

CPU model: Apple M1
Thread count: 8 physical cores, 8 logical processors, using up to 8 threads

Optimize a model with 0 rows, 5 columns and 0 nonzeros
Model fingerprint: 0xbadf3d1d
Coefficient statistics:
  Matrix range      [0e+00, 0e+00]
  Objective range   [1e+00, 1e+00]
  Bounds range      [0e+00, 0e+00]
  RHS range         [0e+00, 0e+00]
Presolve removed 0 rows and 5 columns
Presolve time: 0.00s
Presolve: All rows and columns removed
Iteration   Objective      Primal Inf.    Dual Inf.      Time
           0    0.0000000e+00    0.000000e+00    0.000000e+00      0s

Solved in 0 iterations and 0.00 seconds (0.00 work units)
Optimal objective 0.000000000e+00
  master problem is optimal.
  opt_obj=0.00
  opt_g=0.00
Finish solving master problem.
-----
Bounds: lb=0.00, ub=inf
dual subproblem objective updated!
-----
Start solving dual subproblem.
  dual subproblem is optimal.
  opt_obj=1.00
Finish solving dual subproblem.
-----
Bounds: lb=0.00, ub=1.00
Minimize
  y[0] + y[1] + y[2] + y[3] + g
Subject To
  R0: -1.0 y[0] + -1.0 y[1] + -1.0 y[2] + -1.0 y[3] + -1.0 g <= -1
None
Benders optimality cut added!
-----
Start solving master problem.

```

```

Minimize
    y[0] + y[1] + y[2] + y[3] + g
Subject To
    R0: -1.0 y[0] + -1.0 y[1] + -1.0 y[2] + -1.0 y[3] + -1.0 g <= -1
None
Gurobi Optimizer version 10.0.3 build v10.0.3rc0 (mac64[arm])

CPU model: Apple M1
Thread count: 8 physical cores, 8 logical processors, using up to 8 threads

Optimize a model with 1 rows, 5 columns and 5 nonzeros
Coefficient statistics:
  Matrix range      [1e+00, 1e+00]
  Objective range   [1e+00, 1e+00]
  Bounds range      [0e+00, 0e+00]
  RHS range         [1e+00, 1e+00]
Iteration    Objective          Primal Inf.    Dual Inf.      Time
     0      0.0000000e+00      1.000000e+00   0.000000e+00     0s
     1      1.0000000e+00      0.000000e+00   0.000000e+00     0s

Solved in 1 iterations and 0.00 seconds (0.00 work units)
Optimal objective  1.000000000e+00
    master problem is optimal.
    opt_obj=1.00
    opt_g=0.00
Finish solving master problem.
-----
Bounds: lb=1.00, ub=1.00
dual subproblem objective updated!
-----
Start solving dual subproblem.
    dual subproblem is optimal.
    opt_obj=-0.00
Finish solving dual subproblem.
-----
Bounds: lb=1.00, ub=1.00

```

```

import gurobipy as gp
from gurobipy import GRB
import numpy as np
import scipy.sparse as sp

```



```

class GurobiLpSolver:

    def __init__(self, c, f, A, B, b):
        self._env = gp.Env('GurobiEnv')
        self._model = gp.Model(env=self._env, name='GurobiLpSolver')

        # prepare data
        self._obj_coeff = np.concatenate((c, f))
        print(self._obj_coeff)
        self._constr_mat = np.concatenate((A, B), axis=1)
        print(self._constr_mat)
        self._rhs = b
        self._num_vars = len(self._obj_coeff)
        self._num_constrs = len(b)

        # create decision variables
        self._vars = self._model.addMVar(self._num_vars, vtype=GRB.CONTINUOUS, lb=0)

        # create constraints
        self._constrs = self._model.addConstr(self._constr_mat@self._vars == self._rhs)

        # create objective
        self._model.setObjective(self._obj_coeff @ self._vars, GRB.MINIMIZE)

    def optimize(self):
        self._model.update()
        print(self._model.display())
        self._model.optimize()
        pass

    def clean_up(self):
        pass

import gurobipy as gp
from gurobipy import GRB
import numpy as np

np.random.seed(142)
c = np.random.randint(2, 6, size=20)
f = np.random.randint(1, 15, size=10)
A = np.random.randint(2, 6, size=(20, 20))

```

```

B = np.random.randint(2, 26, size=(20, 10))
b = np.random.randint(20, 50, size=20)

model = GurobiLpSolver(c, f, A, B, b)
model.optimize()

```

Set parameter Username

Set parameter LogFile to value "GurobiEnv"

```

[ 3  3  5  5  2  4  3  2  3  3  4  2  5  5  2  5  4  4  5  5  3  6  8 12
  3  5  4  4  9  3]
[[ 2  4  3  5  5  2  5  4  5  3  3  4  2  2  2  2  3  2  3  3 11 11 25 20
  9  5  7  8  9  2]
 [ 5  2  2  4  3  5  4  2  3  5  4  3  2  5  2  3  5  4  5  4 23 19 23 19
 12 12 13 15 19 19]
 [ 2  3  2  2  5  5  4  5  5  2  5  2  5  4  5  4  3  5  3  3 11 10  8 20
 17  9 25 25 10 12]
 [ 3  5  4  3  4  2  4  2  4  2  4  2  3  3  2  5  3  2  4  2 16 23 18 20
 20 25 14 10  2  6]
 [ 5  4  3  4  3  4  5  4  4  2  5  4  2  3  2  2  5  2  4  3  3  8 10  4
 22 25 11 15 15  9]
 [ 5  5  5  3  2  5  2  2  3  4  2  5  4  4  2  5  4  5  5  4 21 14 22 19
 15 19 16  8 22 23]
 [ 4  2  2  5  2  5  5  4  2  3  2  3  2  5  4  3  3  4  5  3 18  8 11  2
 19 23 23  8 18 25]
 [ 3  2  4  5  3  2  3  5  3  4  5  2  5  4  2  4  2  4  3  5 20 19 13 24
 19  7  4 15 24  3]
 [ 4  4  2  3  2  2  5  5  2  3  5  5  4  5  3  4  2  2  4  2 19  6 20 16
  5 14 20 18 19  6]
 [ 4  2  3  4  4  4  2  5  2  5  5  2  3  4  4  5  4  4  2  3 25 11 17 14
 15 12  6 23 24  6]
 [ 3  5  2  5  3  2  3  3  5  5  3  2  5  3  4  3  5  5  4  4  3 23  9 16
 22  3 14 16 12 16]
 [ 2  3  4  3  5  5  3  5  4  4  2  5  4  3  5  2  3  4  5  4 13 11  7  2
 15 24 13 16  4  6]
 [ 4  4  2  2  4  4  3  2  2  2  5  5  2  5  3  3  5  2  4  3  9 13 10 16
 25 16 24 21  6 16]
 [ 3  5  5  4  3  2  3  5  2  3  5  3  5  3  3  3  5  5  5  5  3 20  5 18
  8 19 25  2 17  7]
 [ 3  2  5  2  2  3  3  3  4  3  5  5  4  4  4  3  3  4  4  4  9 15  6 23
 14 21 21 22 18 22]
 [ 3  2  2  3  4  3  2  4  5  3  2  2  2  3  5  3  2  2  4  4 19 17 17 17
  4 22 12 18  4 18]

```

```

[ 4 2 2 5 2 3 5 4 2 3 2 5 5 5 5 4 4 4 2 5 2 13 23 8
 22 17 10 24 20 23]
[ 2 4 2 3 2 2 2 4 4 5 2 4 4 4 2 4 3 4 5 3 10 6 20 21
 16 15 13 3 24 16]
[ 2 5 4 3 3 3 4 2 5 4 2 4 5 2 2 2 3 5 4 5 12 18 21 8
 19 8 11 20 21 21]
[ 4 2 2 2 2 2 4 3 4 3 2 2 4 2 4 3 5 2 4 3 11 4 2 9
 13 20 23 16 16 14]]

```

Minimize

$3.0 C_0 + 3.0 C_1 + 5.0 C_2 + 5.0 C_3 + 2.0 C_4 + 4.0 C_5 + 3.0 C_6 + 2.0 C_7 + 3.0 C_8$
 $+ 3.0 C_9 + 4.0 C_{10} + 2.0 C_{11} + 5.0 C_{12} + 5.0 C_{13} + 2.0 C_{14} + 5.0 C_{15} + 4.0 C_{16} + 4.0 C_{17}$
 $+ 5.0 C_{18} + 5.0 C_{19} + 3.0 C_{20} + 6.0 C_{21} + 8.0 C_{22} + 12.0 C_{23} + 3.0 C_{24} + 5.0 C_{25}$
 $+ 4.0 C_{26} + 4.0 C_{27} + 9.0 C_{28} + 3.0 C_{29}$

Subject To

$R_0: 2.0 C_0 + 4.0 C_1 + 3.0 C_2 + 5.0 C_3 + 5.0 C_4 + 2.0 C_5 + 5.0 C_6 + 4.0 C_7 + 5.0 C_8 +$
 $3.0 C_9 + 3.0 C_{10} + 4.0 C_{11} + 2.0 C_{12} + 2.0 C_{13} + 2.0 C_{14} + 2.0 C_{15} + 3.0 C_{16} + 2.0 C_{17} +$
 $3.0 C_{18} + 3.0 C_{19} + 11.0 C_{20} + 11.0 C_{21} + 25.0 C_{22} + 20.0 C_{23} + 9.0 C_{24} + 5.0 C_{25} + 7.0$
 $C_{26} + 8.0 C_{27} + 9.0 C_{28} + 2.0 C_{29} = 21$

$R_1: 5.0 C_0 + 2.0 C_1 + 2.0 C_2 + 4.0 C_3 + 3.0 C_4 + 5.0 C_5 + 4.0 C_6 + 2.0 C_7 + 3.0 C_8 +$
 $5.0 C_9 + 4.0 C_{10} + 3.0 C_{11} + 2.0 C_{12} + 5.0 C_{13} + 2.0 C_{14} + 3.0 C_{15} + 5.0 C_{16} + 4.0 C_{17} +$
 $5.0 C_{18} + 4.0 C_{19} + 23.0 C_{20} + 19.0 C_{21} + 23.0 C_{22} + 19.0 C_{23} + 12.0 C_{24} + 12.0 C_{25} +$
 $13.0 C_{26} + 15.0 C_{27} + 19.0 C_{28} + 19.0 C_{29} = 26$

$R_2: 2.0 C_0 + 3.0 C_1 + 2.0 C_2 + 2.0 C_3 + 5.0 C_4 + 5.0 C_5 + 4.0 C_6 + 5.0 C_7 + 5.0 C_8 +$
 $2.0 C_9 + 5.0 C_{10} + 2.0 C_{11} + 5.0 C_{12} + 4.0 C_{13} + 5.0 C_{14} + 4.0 C_{15} + 3.0 C_{16} + 5.0 C_{17} +$
 $3.0 C_{18} + 3.0 C_{19} + 11.0 C_{20} + 10.0 C_{21} + 8.0 C_{22} + 20.0 C_{23} + 17.0 C_{24} + 9.0 C_{25} + 25.0$
 $C_{26} + 25.0 C_{27} + 10.0 C_{28} + 12.0 C_{29} = 38$

$R_3: 3.0 C_0 + 5.0 C_1 + 4.0 C_2 + 3.0 C_3 + 4.0 C_4 + 2.0 C_5 + 4.0 C_6 + 2.0 C_7 + 4.0 C_8 +$
 $2.0 C_9 + 4.0 C_{10} + 2.0 C_{11} + 3.0 C_{12} + 3.0 C_{13} + 2.0 C_{14} + 5.0 C_{15} + 3.0 C_{16} + 2.0 C_{17} +$
 $4.0 C_{18} + 2.0 C_{19} + 16.0 C_{20} + 23.0 C_{21} + 18.0 C_{22} + 20.0 C_{23} + 20.0 C_{24} + 25.0 C_{25} +$
 $14.0 C_{26} + 10.0 C_{27} + 2.0 C_{28} + 6.0 C_{29} = 42$

$R_4: 5.0 C_0 + 4.0 C_1 + 3.0 C_2 + 4.0 C_3 + 3.0 C_4 + 4.0 C_5 + 5.0 C_6 + 4.0 C_7 + 4.0 C_8 +$
 $2.0 C_9 + 5.0 C_{10} + 4.0 C_{11} + 2.0 C_{12} + 3.0 C_{13} + 2.0 C_{14} + 2.0 C_{15} + 5.0 C_{16} + 2.0 C_{17} +$
 $4.0 C_{18} + 3.0 C_{19} + 3.0 C_{20} + 8.0 C_{21} + 10.0 C_{22} + 4.0 C_{23} + 22.0 C_{24} + 25.0 C_{25} + 11.0$
 $C_{26} + 15.0 C_{27} + 15.0 C_{28} + 9.0 C_{29} = 35$

$R_5: 5.0 C_0 + 5.0 C_1 + 5.0 C_2 + 3.0 C_3 + 2.0 C_4 + 5.0 C_5 + 2.0 C_6 + 2.0 C_7 + 3.0 C_8 +$
 $4.0 C_9 + 2.0 C_{10} + 5.0 C_{11} + 4.0 C_{12} + 4.0 C_{13} + 2.0 C_{14} + 5.0 C_{15} + 4.0 C_{16} + 5.0 C_{17} +$
 $5.0 C_{18} + 4.0 C_{19} + 21.0 C_{20} + 14.0 C_{21} + 22.0 C_{22} + 19.0 C_{23} + 15.0 C_{24} + 19.0 C_{25} +$
 $16.0 C_{26} + 8.0 C_{27} + 22.0 C_{28} + 23.0 C_{29} = 37$

$R_6: 4.0 C_0 + 2.0 C_1 + 2.0 C_2 + 5.0 C_3 + 2.0 C_4 + 5.0 C_5 + 5.0 C_6 + 4.0 C_7 + 2.0 C_8 +$
 $3.0 C_9 + 2.0 C_{10} + 3.0 C_{11} + 2.0 C_{12} + 5.0 C_{13} + 4.0 C_{14} + 3.0 C_{15} + 3.0 C_{16} + 4.0 C_{17} +$
 $5.0 C_{18} + 3.0 C_{19} + 18.0 C_{20} + 8.0 C_{21} + 11.0 C_{22} + 2.0 C_{23} + 19.0 C_{24} + 23.0 C_{25} + 23.0$
 $C_{26} + 8.0 C_{27} + 18.0 C_{28} + 25.0 C_{29} = 28$

$R_7: 3.0 C_0 + 2.0 C_1 + 4.0 C_2 + 5.0 C_3 + 3.0 C_4 + 2.0 C_5 + 3.0 C_6 + 5.0 C_7 + 3.0 C_8 +$

$4.0 C_9 + 5.0 C_{10} + 2.0 C_{11} + 5.0 C_{12} + 4.0 C_{13} + 2.0 C_{14} + 4.0 C_{15} + 2.0 C_{16} + 4.0 C_{17} + 3.0 C_{18} + 5.0 C_{19} + 20.0 C_{20} + 19.0 C_{21} + 13.0 C_{22} + 24.0 C_{23} + 19.0 C_{24} + 7.0 C_{25} + 4.0 C_{26} + 15.0 C_{27} + 24.0 C_{28} + 3.0 C_{29} = 22$

R8: $4.0 C_0 + 4.0 C_1 + 2.0 C_2 + 3.0 C_3 + 2.0 C_4 + 2.0 C_5 + 5.0 C_6 + 5.0 C_7 + 2.0 C_8 + 3.0 C_9 + 5.0 C_{10} + 5.0 C_{11} + 4.0 C_{12} + 5.0 C_{13} + 3.0 C_{14} + 4.0 C_{15} + 2.0 C_{16} + 2.0 C_{17} + 4.0 C_{18} + 2.0 C_{19} + 19.0 C_{20} + 6.0 C_{21} + 20.0 C_{22} + 16.0 C_{23} + 5.0 C_{24} + 14.0 C_{25} + 20.0 C_{26} + 18.0 C_{27} + 19.0 C_{28} + 6.0 C_{29} = 39$

R9: $4.0 C_0 + 2.0 C_1 + 3.0 C_2 + 4.0 C_3 + 4.0 C_4 + 4.0 C_5 + 2.0 C_6 + 5.0 C_7 + 2.0 C_8 + 5.0 C_9 + 5.0 C_{10} + 2.0 C_{11} + 3.0 C_{12} + 4.0 C_{13} + 4.0 C_{14} + 5.0 C_{15} + 4.0 C_{16} + 4.0 C_{17} + 2.0 C_{18} + 3.0 C_{19} + 25.0 C_{20} + 11.0 C_{21} + 17.0 C_{22} + 14.0 C_{23} + 15.0 C_{24} + 12.0 C_{25} + 6.0 C_{26} + 23.0 C_{27} + 24.0 C_{28} + 6.0 C_{29} = 33$

R10: $3.0 C_0 + 5.0 C_1 + 2.0 C_2 + 5.0 C_3 + 3.0 C_4 + 2.0 C_5 + 3.0 C_6 + 3.0 C_7 + 5.0 C_8 + 5.0 C_9 + 3.0 C_{10} + 2.0 C_{11} + 5.0 C_{12} + 3.0 C_{13} + 4.0 C_{14} + 3.0 C_{15} + 5.0 C_{16} + 5.0 C_{17} + 4.0 C_{18} + 4.0 C_{19} + 3.0 C_{20} + 23.0 C_{21} + 9.0 C_{22} + 16.0 C_{23} + 22.0 C_{24} + 3.0 C_{25} + 14.0 C_{26} + 16.0 C_{27} + 12.0 C_{28} + 16.0 C_{29} = 28$

R11: $2.0 C_0 + 3.0 C_1 + 4.0 C_2 + 3.0 C_3 + 5.0 C_4 + 5.0 C_5 + 3.0 C_6 + 5.0 C_7 + 4.0 C_8 + 4.0 C_9 + 2.0 C_{10} + 5.0 C_{11} + 4.0 C_{12} + 3.0 C_{13} + 5.0 C_{14} + 2.0 C_{15} + 3.0 C_{16} + 4.0 C_{17} + 5.0 C_{18} + 4.0 C_{19} + 13.0 C_{20} + 11.0 C_{21} + 7.0 C_{22} + 2.0 C_{23} + 15.0 C_{24} + 24.0 C_{25} + 13.0 C_{26} + 16.0 C_{27} + 4.0 C_{28} + 6.0 C_{29} = 38$

R12: $4.0 C_0 + 4.0 C_1 + 2.0 C_2 + 2.0 C_3 + 4.0 C_4 + 4.0 C_5 + 3.0 C_6 + 2.0 C_7 + 2.0 C_8 + 2.0 C_9 + 5.0 C_{10} + 5.0 C_{11} + 2.0 C_{12} + 5.0 C_{13} + 3.0 C_{14} + 3.0 C_{15} + 5.0 C_{16} + 2.0 C_{17} + 4.0 C_{18} + 3.0 C_{19} + 9.0 C_{20} + 13.0 C_{21} + 10.0 C_{22} + 16.0 C_{23} + 25.0 C_{24} + 16.0 C_{25} + 24.0 C_{26} + 21.0 C_{27} + 6.0 C_{28} + 16.0 C_{29} = 47$

R13: $3.0 C_0 + 5.0 C_1 + 5.0 C_2 + 4.0 C_3 + 3.0 C_4 + 2.0 C_5 + 3.0 C_6 + 5.0 C_7 + 2.0 C_8 + 3.0 C_9 + 5.0 C_{10} + 3.0 C_{11} + 5.0 C_{12} + 3.0 C_{13} + 3.0 C_{14} + 3.0 C_{15} + 5.0 C_{16} + 5.0 C_{17} + 5.0 C_{18} + 5.0 C_{19} + 3.0 C_{20} + 20.0 C_{21} + 5.0 C_{22} + 18.0 C_{23} + 8.0 C_{24} + 19.0 C_{25} + 25.0 C_{26} + 2.0 C_{27} + 17.0 C_{28} + 7.0 C_{29} = 28$

R14: $3.0 C_0 + 2.0 C_1 + 5.0 C_2 + 2.0 C_3 + 2.0 C_4 + 3.0 C_5 + 3.0 C_6 + 3.0 C_7 + 4.0 C_8 + 3.0 C_9 + 5.0 C_{10} + 5.0 C_{11} + 4.0 C_{12} + 4.0 C_{13} + 4.0 C_{14} + 3.0 C_{15} + 3.0 C_{16} + 4.0 C_{17} + 4.0 C_{18} + 4.0 C_{19} + 9.0 C_{20} + 15.0 C_{21} + 6.0 C_{22} + 23.0 C_{23} + 14.0 C_{24} + 21.0 C_{25} + 21.0 C_{26} + 22.0 C_{27} + 18.0 C_{28} + 22.0 C_{29} = 45$

R15: $3.0 C_0 + 2.0 C_1 + 2.0 C_2 + 3.0 C_3 + 4.0 C_4 + 3.0 C_5 + 2.0 C_6 + 4.0 C_7 + 5.0 C_8 + 3.0 C_9 + 2.0 C_{10} + 2.0 C_{11} + 2.0 C_{12} + 3.0 C_{13} + 5.0 C_{14} + 3.0 C_{15} + 2.0 C_{16} + 2.0 C_{17} + 4.0 C_{18} + 4.0 C_{19} + 19.0 C_{20} + 17.0 C_{21} + 17.0 C_{22} + 17.0 C_{23} + 4.0 C_{24} + 22.0 C_{25} + 12.0 C_{26} + 18.0 C_{27} + 4.0 C_{28} + 18.0 C_{29} = 34$

R16: $4.0 C_0 + 2.0 C_1 + 2.0 C_2 + 5.0 C_3 + 2.0 C_4 + 3.0 C_5 + 5.0 C_6 + 4.0 C_7 + 2.0 C_8 + 3.0 C_9 + 2.0 C_{10} + 5.0 C_{11} + 5.0 C_{12} + 5.0 C_{13} + 5.0 C_{14} + 4.0 C_{15} + 4.0 C_{16} + 4.0 C_{17} + 2.0 C_{18} + 5.0 C_{19} + 2.0 C_{20} + 13.0 C_{21} + 23.0 C_{22} + 8.0 C_{23} + 22.0 C_{24} + 17.0 C_{25} + 10.0 C_{26} + 24.0 C_{27} + 20.0 C_{28} + 23.0 C_{29} = 46$

R17: $2.0 C_0 + 4.0 C_1 + 2.0 C_2 + 3.0 C_3 + 2.0 C_4 + 2.0 C_5 + 2.0 C_6 + 4.0 C_7 + 4.0 C_8 + 5.0 C_9 + 2.0 C_{10} + 4.0 C_{11} + 4.0 C_{12} + 4.0 C_{13} + 2.0 C_{14} + 4.0 C_{15} + 3.0 C_{16} + 4.0 C_{17} + 5.0 C_{18} + 3.0 C_{19} + 10.0 C_{20} + 6.0 C_{21} + 20.0 C_{22} + 21.0 C_{23} + 16.0 C_{24} + 15.0 C_{25} + 13.0 C_{26} + 3.0 C_{27} + 24.0 C_{28} + 16.0 C_{29} = 35$

R18: 2.0 C0 + 5.0 C1 + 4.0 C2 + 3.0 C3 + 3.0 C4 + 3.0 C5 + 4.0 C6 + 2.0 C7 + 5.0 C8 + 4.0 C9 + 2.0 C10 + 4.0 C11 + 5.0 C12 + 2.0 C13 + 2.0 C14 + 2.0 C15 + 3.0 C16 + 5.0 C17 + 4.0 C18 + 5.0 C19 + 12.0 C20 + 18.0 C21 + 21.0 C22 + 8.0 C23 + 19.0 C24 + 8.0 C25 + 11.0 C26 + 20.0 C27 + 21.0 C28 + 21.0 C29 = 46

R19: 4.0 C0 + 2.0 C1 + 2.0 C2 + 2.0 C3 + 2.0 C4 + 2.0 C5 + 4.0 C6 + 3.0 C7 + 4.0 C8 + 3.0 C9 + 2.0 C10 + 2.0 C11 + 4.0 C12 + 2.0 C13 + 4.0 C14 + 3.0 C15 + 5.0 C16 + 2.0 C17 + 4.0 C18 + 3.0 C19 + 11.0 C20 + 4.0 C21 + 2.0 C22 + 9.0 C23 + 13.0 C24 + 20.0 C25 + 23.0 C26 + 16.0 C27 + 16.0 C28 + 14.0 C29 = 32

None

Gurobi Optimizer version 10.0.3 build v10.0.3rc0 (mac64[arm])

CPU model: Apple M1

Thread count: 8 physical cores, 8 logical processors, using up to 8 threads

Optimize a model with 20 rows, 30 columns and 600 nonzeros

Model fingerprint: 0x05cb70d4

Coefficient statistics:

Matrix range [2e+00, 2e+01]

Objective range [2e+00, 1e+01]

Bounds range [0e+00, 0e+00]

RHS range [2e+01, 5e+01]

Presolve time: 0.00s

Presolved: 20 rows, 30 columns, 600 nonzeros

Iteration	Objective	Primal Inf.	Dual Inf.	Time
0	3.3913043e+00	5.119565e+01	0.000000e+00	0s

Solved in 13 iterations and 0.00 seconds (0.00 work units)

Infeasible model

```
import gurobipy as gp
from gurobipy import GRB
import numpy as np

np.random.seed(142)
c = np.random.randint(2, 6, size=20)
f = np.random.randint(1, 15, size=10)
A = np.random.randint(2, 6, size=(20, 20))
B = np.random.randint(2, 26, size=(20, 10))
b = np.random.randint(20, 50, size=20)

master_solver = GenericLpMasterSolver(f, B, b)
```

```
dual_subprob_solver = GenericLpSubprobSolver(A, c, B, b)
```

```
benders_solver = GenericBendersSolver(master_solver, dual_subprob_solver)
benders_solver.optimize()
```

Set parameter Username

Set parameter LogFile to value "MasterEnv"

Set parameter Username

Set parameter LogFile to value "SubprobEnv"

Start solving master problem.

Minimize

3.0 y[0] + 6.0 y[1] + 8.0 y[2] + 12.0 y[3] + 3.0 y[4] + 5.0 y[5] + 4.0 y[6] + 4.0 y[7]
+ 9.0 y[8] + 3.0 y[9] + g

Subject To

None

Gurobi Optimizer version 10.0.3 build v10.0.3rc0 (mac64[arm])

CPU model: Apple M1

Thread count: 8 physical cores, 8 logical processors, using up to 8 threads

Optimize a model with 0 rows, 11 columns and 0 nonzeros

Model fingerprint: 0xb01aa8a2

Coefficient statistics:

Matrix range [0e+00, 0e+00]

Objective range [1e+00, 1e+01]

Bounds range [0e+00, 0e+00]

RHS range [0e+00, 0e+00]

Presolve removed 0 rows and 11 columns

Presolve time: 0.00s

Presolve: All rows and columns removed

Iteration	Objective	Primal Inf.	Dual Inf.	Time
0	0.0000000e+00	0.000000e+00	0.000000e+00	0s

Solved in 0 iterations and 0.00 seconds (0.00 work units)

Optimal objective 0.000000000e+00

master problem is optimal.

opt_obj=0.00

opt_g=0.00

Finish solving master problem.

Bounds: lb=0.00, ub=inf

dual subproblem objective updated!

Start solving dual subproblem.

dual subproblem is optimal.

opt_obj=28.16

Finish solving dual subproblem.

Bounds: lb=0.00, ub=28.16

Minimize

3.0 y[0] + 6.0 y[1] + 8.0 y[2] + 12.0 y[3] + 3.0 y[4] + 5.0 y[5] + 4.0 y[6] + 4.0 y[7]
+ 9.0 y[8] + 3.0 y[9] + g

Subject To

R0: -8.5625 y[0] + -9.859375 y[1] + -9.5625 y[2] + -7.9375 y[3] + -12.125 y[4] +
-11.828125 y[5] + -11.21875 y[6] + -11.46875 y[7] + -10.21875 y[8] + -10.5 y[9] + -1.0 g
 <= -28.1562

None

Benders optimality cut added!

Start solving master problem.

Minimize

3.0 y[0] + 6.0 y[1] + 8.0 y[2] + 12.0 y[3] + 3.0 y[4] + 5.0 y[5] + 4.0 y[6] + 4.0 y[7]
+ 9.0 y[8] + 3.0 y[9] + g

Subject To

R0: -8.5625 y[0] + -9.859375 y[1] + -9.5625 y[2] + -7.9375 y[3] + -12.125 y[4] +
-11.828125 y[5] + -11.21875 y[6] + -11.46875 y[7] + -10.21875 y[8] + -10.5 y[9] + -1.0 g
 <= -28.1562

None

Gurobi Optimizer version 10.0.3 build v10.0.3rc0 (mac64[arm])

CPU model: Apple M1

Thread count: 8 physical cores, 8 logical processors, using up to 8 threads

Optimize a model with 1 rows, 11 columns and 11 nonzeros

Coefficient statistics:

Matrix range [1e+00, 1e+01]

Objective range [1e+00, 1e+01]

Bounds range [0e+00, 0e+00]

RHS range [3e+01, 3e+01]

Iteration	Objective	Primal Inf.	Dual Inf.	Time
0	0.0000000e+00	1.759766e+00	0.000000e+00	0s
1	6.9664948e+00	0.000000e+00	0.000000e+00	0s

Solved in 1 iterations and 0.00 seconds (0.00 work units)

Optimal objective 6.966494845e+00

master problem is optimal.

opt_obj=6.97

opt_g=0.00

Finish solving master problem.

Bounds: lb=6.97, ub=28.16

dual subproblem objective updated!

Start solving dual subproblem.

dual subproblem is optimal.

opt_obj=11.82

Finish solving dual subproblem.

Bounds: lb=6.97, ub=18.78

Minimize

$3.0 y[0] + 6.0 y[1] + 8.0 y[2] + 12.0 y[3] + 3.0 y[4] + 5.0 y[5] + 4.0 y[6] + 4.0 y[7] + 9.0 y[8] + 3.0 y[9] + g$

Subject To

R0: $-8.5625 y[0] + -9.859375 y[1] + -9.5625 y[2] + -7.9375 y[3] + -12.125 y[4] + -11.828125 y[5] + -11.21875 y[6] + -11.46875 y[7] + -10.21875 y[8] + -10.5 y[9] + -1.0 g$
 ≤ -28.1562

R1: $-8.76923076923077 y[0] + -6.153846153846153 y[1] + -8.307692307692307 y[2] + -7.692307692307692 y[3] + -2.0 y[4] + -8.923076923076923 y[5] + -6.769230769230769 y[6] + -8.307692307692307 y[7] + -4.153846153846153 y[8] + -6.461538461538462 y[9] + -1.0 g$
 ≤ -16.4615

None

Benders optimality cut added!

Start solving master problem.

Minimize

$3.0 y[0] + 6.0 y[1] + 8.0 y[2] + 12.0 y[3] + 3.0 y[4] + 5.0 y[5] + 4.0 y[6] + 4.0 y[7] + 9.0 y[8] + 3.0 y[9] + g$

Subject To

R0: $-8.5625 y[0] + -9.859375 y[1] + -9.5625 y[2] + -7.9375 y[3] + -12.125 y[4] + -11.828125 y[5] + -11.21875 y[6] + -11.46875 y[7] + -10.21875 y[8] + -10.5 y[9] + -1.0 g$
 ≤ -28.1562

R1: $-8.76923076923077 y[0] + -6.153846153846153 y[1] + -8.307692307692307 y[2] + -7.692307692307692 y[3] + -2.0 y[4] + -8.923076923076923 y[5] + -6.769230769230769 y[6] + -8.307692307692307 y[7] + -4.153846153846153 y[8] + -6.461538461538462 y[9] + -1.0 g$
 ≤ -16.4615

None

Gurobi Optimizer version 10.0.3 build v10.0.3rc0 (mac64[arm])

CPU model: Apple M1

Thread count: 8 physical cores, 8 logical processors, using up to 8 threads

Optimize a model with 2 rows, 11 columns and 22 nonzeros

Coefficient statistics:

Matrix range [1e+00, 1e+01]

Objective range [1e+00, 1e+01]

Bounds range [0e+00, 0e+00]

RHS range [2e+01, 3e+01]

Iteration	Objective	Primal Inf.	Dual Inf.	Time
0	6.9664948e+00	7.385755e-01	0.000000e+00	0s
1	7.9710765e+00	0.000000e+00	0.000000e+00	0s

Solved in 1 iterations and 0.00 seconds (0.00 work units)

Optimal objective 7.971076459e+00

master problem is optimal.

opt_obj=7.97

opt_g=0.00

Finish solving master problem.

Bounds: lb=7.97, ub=18.78

dual subproblem objective updated!

Start solving dual subproblem.

dual subproblem is optimal.

opt_obj=14.74

Finish solving dual subproblem.

Bounds: lb=7.97, ub=18.78

Minimize

3.0 y[0] + 6.0 y[1] + 8.0 y[2] + 12.0 y[3] + 3.0 y[4] + 5.0 y[5] + 4.0 y[6] + 4.0 y[7]
+ 9.0 y[8] + 3.0 y[9] + g

Subject To

R0: -8.5625 y[0] + -9.859375 y[1] + -9.5625 y[2] + -7.9375 y[3] + -12.125 y[4] +
-11.828125 y[5] + -11.21875 y[6] + -11.46875 y[7] + -10.21875 y[8] + -10.5 y[9] + -1.0 g
 <= -28.1562

R1: -8.76923076923077 y[0] + -6.153846153846153 y[1] + -8.307692307692307 y[2] +
-7.692307692307692 y[3] + -2.0 y[4] + -8.923076923076923 y[5] + -6.769230769230769 y[6]
+ -8.307692307692307 y[7] + -4.153846153846153 y[8] + -6.461538461538462 y[9] + -1.0 g
 <= -16.4615

R2: -10.75 y[0] + -10.124999999999998 y[1] + -11.75 y[2] + -11.5 y[3] + -8.75 y[4] +
-12.874999999999998 y[5] + -10.25 y[6] + -8.25 y[7] + -5.5 y[8] + -3.7499999999999996

```

y[9] + -1.0 g <= -25.5
None
Benders optimality cut added!
-----

Start solving master problem.
Minimize
3.0 y[0] + 6.0 y[1] + 8.0 y[2] + 12.0 y[3] + 3.0 y[4] + 5.0 y[5] + 4.0 y[6] + 4.0 y[7]
+ 9.0 y[8] + 3.0 y[9] + g
Subject To
R0: -8.5625 y[0] + -9.859375 y[1] + -9.5625 y[2] + -7.9375 y[3] + -12.125 y[4] +
-11.828125 y[5] + -11.21875 y[6] + -11.46875 y[7] + -10.21875 y[8] + -10.5 y[9] + -1.0 g
<= -28.1562
R1: -8.76923076923077 y[0] + -6.153846153846153 y[1] + -8.307692307692307 y[2] +
-7.692307692307692 y[3] + -2.0 y[4] + -8.923076923076923 y[5] + -6.769230769230769 y[6]
+ -8.307692307692307 y[7] + -4.153846153846153 y[8] + -6.461538461538462 y[9] + -1.0 g
<= -16.4615
R2: -10.75 y[0] + -10.124999999999998 y[1] + -11.75 y[2] + -11.5 y[3] + -8.75 y[4] +
-12.874999999999998 y[5] + -10.25 y[6] + -8.25 y[7] + -5.5 y[8] + -3.749999999999996
y[9] + -1.0 g <= -25.5
None
Gurobi Optimizer version 10.0.3 build v10.0.3rc0 (mac64[arm])

CPU model: Apple M1
Thread count: 8 physical cores, 8 logical processors, using up to 8 threads

Optimize a model with 3 rows, 11 columns and 33 nonzeros
Coefficient statistics:
  Matrix range      [1e+00, 1e+01]
  Objective range   [1e+00, 1e+01]
  Bounds range      [0e+00, 0e+00]
  RHS range         [2e+01, 3e+01]
Iteration    Objective          Primal Inf.    Dual Inf.      Time
     0      7.9710765e+00      9.214938e-01   0.000000e+00     0s
     1      8.3297685e+00      0.000000e+00   0.000000e+00     0s

Solved in 1 iterations and 0.00 seconds (0.00 work units)
Optimal objective 8.329768548e+00
  master problem is optimal.
  opt_obj=8.33
  opt_g=0.00
Finish solving master problem.
-----

Bounds: lb=8.33, ub=18.78

```

dual subproblem objective updated!

Start solving dual subproblem.
dual subproblem is optimal.
opt_obj=5.92
Finish solving dual subproblem.

Bounds: lb=8.33, ub=14.25

Minimize

$3.0 y[0] + 6.0 y[1] + 8.0 y[2] + 12.0 y[3] + 3.0 y[4] + 5.0 y[5] + 4.0 y[6] + 4.0 y[7] + 9.0 y[8] + 3.0 y[9] + g$

Subject To

R0: $-8.5625 y[0] + -9.859375 y[1] + -9.5625 y[2] + -7.9375 y[3] + -12.125 y[4] + -11.828125 y[5] + -11.21875 y[6] + -11.46875 y[7] + -10.21875 y[8] + -10.5 y[9] + -1.0 g \leq -28.1562$

R1: $-8.76923076923077 y[0] + -6.153846153846153 y[1] + -8.307692307692307 y[2] + -7.692307692307692 y[3] + -2.0 y[4] + -8.923076923076923 y[5] + -6.769230769230769 y[6] + -8.307692307692307 y[7] + -4.153846153846153 y[8] + -6.461538461538462 y[9] + -1.0 g \leq -16.4615$

R2: $-10.75 y[0] + -10.124999999999998 y[1] + -11.75 y[2] + -11.5 y[3] + -8.75 y[4] + -12.874999999999998 y[5] + -10.25 y[6] + -8.25 y[7] + -5.5 y[8] + -3.7499999999999996 y[9] + -1.0 g \leq -25.5$

R3: $-3.0 y[0] + -8.75 y[1] + -2.75 y[2] + -10.25 y[3] + -5.5 y[4] + -10.0 y[5] + -11.5 y[6] + -6.0 y[7] + -8.75 y[8] + -7.25 y[9] + -1.0 g \leq -18.25$

None

Benders optimality cut added!

Start solving master problem.

Minimize

$3.0 y[0] + 6.0 y[1] + 8.0 y[2] + 12.0 y[3] + 3.0 y[4] + 5.0 y[5] + 4.0 y[6] + 4.0 y[7] + 9.0 y[8] + 3.0 y[9] + g$

Subject To

R0: $-8.5625 y[0] + -9.859375 y[1] + -9.5625 y[2] + -7.9375 y[3] + -12.125 y[4] + -11.828125 y[5] + -11.21875 y[6] + -11.46875 y[7] + -10.21875 y[8] + -10.5 y[9] + -1.0 g \leq -28.1562$

R1: $-8.76923076923077 y[0] + -6.153846153846153 y[1] + -8.307692307692307 y[2] + -7.692307692307692 y[3] + -2.0 y[4] + -8.923076923076923 y[5] + -6.769230769230769 y[6] + -8.307692307692307 y[7] + -4.153846153846153 y[8] + -6.461538461538462 y[9] + -1.0 g \leq -16.4615$

R2: $-10.75 y[0] + -10.124999999999998 y[1] + -11.75 y[2] + -11.5 y[3] + -8.75 y[4] + -12.874999999999998 y[5] + -10.25 y[6] + -8.25 y[7] + -5.5 y[8] + -3.7499999999999996 y[9] + -1.0 g \leq -25.5$

R3: $-3.0 y[0] + -8.75 y[1] + -2.75 y[2] + -10.25 y[3] + -5.5 y[4] + -10.0 y[5] + -11.5$

$y[6] + -6.0 y[7] + -8.75 y[8] + -7.25 y[9] + -1.0 g \leq -18.25$
 None
 Gurobi Optimizer version 10.0.3 build v10.0.3rc0 (mac64[arm])

CPU model: Apple M1
 Thread count: 8 physical cores, 8 logical processors, using up to 8 threads

Optimize a model with 4 rows, 11 columns and 44 nonzeros
 Coefficient statistics:

Matrix range	[1e+00, 1e+01]			
Objective range	[1e+00, 1e+01]			
Bounds range	[0e+00, 0e+00]			
RHS range	[2e+01, 3e+01]			

Iteration	Objective	Primal Inf.	Dual Inf.	Time
0	8.3297685e+00	3.701343e-01	0.000000e+00	0s
1	8.9196232e+00	0.000000e+00	0.000000e+00	0s

Solved in 1 iterations and 0.00 seconds (0.00 work units)
 Optimal objective 8.919623169e+00
 master problem is optimal.
 opt_obj=8.92
 opt_g=0.00
 Finish solving master problem.

 Bounds: lb=8.92, ub=14.25
 dual subproblem objective updated!

 Start solving dual subproblem.
 dual subproblem is optimal.
 opt_obj=6.12
 Finish solving dual subproblem.

 Bounds: lb=8.92, ub=14.25

Minimize

$3.0 y[0] + 6.0 y[1] + 8.0 y[2] + 12.0 y[3] + 3.0 y[4] + 5.0 y[5] + 4.0 y[6] + 4.0 y[7]$
 $+ 9.0 y[8] + 3.0 y[9] + g$

Subject To

R0: $-8.5625 y[0] + -9.859375 y[1] + -9.5625 y[2] + -7.9375 y[3] + -12.125 y[4] +$
 $-11.828125 y[5] + -11.21875 y[6] + -11.46875 y[7] + -10.21875 y[8] + -10.5 y[9] + -1.0 g$
 ≤ -28.1562

R1: $-8.76923076923077 y[0] + -6.153846153846153 y[1] + -8.307692307692307 y[2] +$
 $-7.692307692307692 y[3] + -2.0 y[4] + -8.923076923076923 y[5] + -6.769230769230769 y[6]$
 $+ -8.307692307692307 y[7] + -4.153846153846153 y[8] + -6.461538461538462 y[9] + -1.0 g$

```

<= -16.4615
R2: -10.75 y[0] + -10.124999999999998 y[1] + -11.75 y[2] + -11.5 y[3] + -8.75 y[4] +
-12.874999999999998 y[5] + -10.25 y[6] + -8.25 y[7] + -5.5 y[8] + -3.749999999999996
y[9] + -1.0 g <= -25.5
R3: -3.0 y[0] + -8.75 y[1] + -2.75 y[2] + -10.25 y[3] + -5.5 y[4] + -10.0 y[5] + -11.5
y[6] + -6.0 y[7] + -8.75 y[8] + -7.25 y[9] + -1.0 g <= -18.25
R4: -0.8 y[0] + -5.2 y[1] + -9.200000000000001 y[2] + -3.2 y[3] + -8.8 y[4] +
-6.800000000000001 y[5] + -4.0 y[6] + -9.600000000000001 y[7] + -8.0 y[8] +
-9.200000000000001 y[9] + -1.0 g <= -18.4
None
Benders optimality cut added!
-----
Start solving master problem.
Minimize
3.0 y[0] + 6.0 y[1] + 8.0 y[2] + 12.0 y[3] + 3.0 y[4] + 5.0 y[5] + 4.0 y[6] + 4.0 y[7]
+ 9.0 y[8] + 3.0 y[9] + g
Subject To
R0: -8.5625 y[0] + -9.859375 y[1] + -9.5625 y[2] + -7.9375 y[3] + -12.125 y[4] +
-11.828125 y[5] + -11.21875 y[6] + -11.46875 y[7] + -10.21875 y[8] + -10.5 y[9] + -1.0 g
<= -28.1562
R1: -8.76923076923077 y[0] + -6.153846153846153 y[1] + -8.307692307692307 y[2] +
-7.692307692307692 y[3] + -2.0 y[4] + -8.923076923076923 y[5] + -6.769230769230769 y[6]
+ -8.307692307692307 y[7] + -4.153846153846153 y[8] + -6.461538461538462 y[9] + -1.0 g
<= -16.4615
R2: -10.75 y[0] + -10.124999999999998 y[1] + -11.75 y[2] + -11.5 y[3] + -8.75 y[4] +
-12.874999999999998 y[5] + -10.25 y[6] + -8.25 y[7] + -5.5 y[8] + -3.749999999999996
y[9] + -1.0 g <= -25.5
R3: -3.0 y[0] + -8.75 y[1] + -2.75 y[2] + -10.25 y[3] + -5.5 y[4] + -10.0 y[5] + -11.5
y[6] + -6.0 y[7] + -8.75 y[8] + -7.25 y[9] + -1.0 g <= -18.25
R4: -0.8 y[0] + -5.2 y[1] + -9.200000000000001 y[2] + -3.2 y[3] + -8.8 y[4] +
-6.800000000000001 y[5] + -4.0 y[6] + -9.600000000000001 y[7] + -8.0 y[8] +
-9.200000000000001 y[9] + -1.0 g <= -18.4
None
Gurobi Optimizer version 10.0.3 build v10.0.3rc0 (mac64[arm])

CPU model: Apple M1
Thread count: 8 physical cores, 8 logical processors, using up to 8 threads

Optimize a model with 5 rows, 11 columns and 55 nonzeros
Coefficient statistics:
  Matrix range      [8e-01, 1e+01]
  Objective range   [1e+00, 1e+01]
  Bounds range      [0e+00, 0e+00]

```

RHS range		[2e+01, 3e+01]		
Iteration	Objective	Primal Inf.	Dual Inf.	Time
0	8.9196232e+00	3.825374e-01	0.000000e+00	0s
1	9.5016530e+00	0.000000e+00	0.000000e+00	0s

Solved in 1 iterations and 0.00 seconds (0.00 work units)

Optimal objective 9.501652960e+00

master problem is optimal.

opt_obj=9.50

opt_g=0.00

Finish solving master problem.

Bounds: lb=9.50, ub=14.25

dual subproblem objective updated!

Start solving dual subproblem.

dual subproblem is optimal.

opt_obj=2.23

Finish solving dual subproblem.

Bounds: lb=9.50, ub=11.73

Minimize

3.0 y[0] + 6.0 y[1] + 8.0 y[2] + 12.0 y[3] + 3.0 y[4] + 5.0 y[5] + 4.0 y[6] + 4.0 y[7]
+ 9.0 y[8] + 3.0 y[9] + g

Subject To

R0: -8.5625 y[0] + -9.859375 y[1] + -9.5625 y[2] + -7.9375 y[3] + -12.125 y[4] +
-11.828125 y[5] + -11.21875 y[6] + -11.46875 y[7] + -10.21875 y[8] + -10.5 y[9] + -1.0 g
<= -28.1562

R1: -8.76923076923077 y[0] + -6.153846153846153 y[1] + -8.307692307692307 y[2] +
-7.692307692307692 y[3] + -2.0 y[4] + -8.923076923076923 y[5] + -6.769230769230769 y[6]
+ -8.307692307692307 y[7] + -4.153846153846153 y[8] + -6.461538461538462 y[9] + -1.0 g
<= -16.4615

R2: -10.75 y[0] + -10.124999999999998 y[1] + -11.75 y[2] + -11.5 y[3] + -8.75 y[4] +
-12.874999999999998 y[5] + -10.25 y[6] + -8.25 y[7] + -5.5 y[8] + -3.7499999999999996
y[9] + -1.0 g <= -25.5

R3: -3.0 y[0] + -8.75 y[1] + -2.75 y[2] + -10.25 y[3] + -5.5 y[4] + -10.0 y[5] + -11.5
y[6] + -6.0 y[7] + -8.75 y[8] + -7.25 y[9] + -1.0 g <= -18.25

R4: -0.8 y[0] + -5.2 y[1] + -9.200000000000001 y[2] + -3.2 y[3] + -8.8 y[4] +
-6.800000000000001 y[5] + -4.0 y[6] + -9.600000000000001 y[7] + -8.0 y[8] +
-9.200000000000001 y[9] + -1.0 g <= -18.4

R5: -7.6000000000000005 y[0] + -2.4000000000000004 y[1] + -8.0 y[2] + -6.4 y[3] + -2.0
y[4] + -5.6000000000000005 y[5] + -8.0 y[6] + -7.2 y[7] + -7.6000000000000005 y[8] +
-2.4000000000000004 y[9] + -1.0 g <= -15.6

None

Benders optimality cut added!

Start solving master problem.

Minimize

$3.0 y[0] + 6.0 y[1] + 8.0 y[2] + 12.0 y[3] + 3.0 y[4] + 5.0 y[5] + 4.0 y[6] + 4.0 y[7]$
 $+ 9.0 y[8] + 3.0 y[9] + g$

Subject To

R0: $-8.5625 y[0] + -9.859375 y[1] + -9.5625 y[2] + -7.9375 y[3] + -12.125 y[4] +$
 $-11.828125 y[5] + -11.21875 y[6] + -11.46875 y[7] + -10.21875 y[8] + -10.5 y[9] + -1.0 g$
 ≤ -28.1562

R1: $-8.76923076923077 y[0] + -6.153846153846153 y[1] + -8.307692307692307 y[2] +$
 $-7.692307692307692 y[3] + -2.0 y[4] + -8.923076923076923 y[5] + -6.769230769230769 y[6]$
 $+ -8.307692307692307 y[7] + -4.153846153846153 y[8] + -6.461538461538462 y[9] + -1.0 g$
 ≤ -16.4615

R2: $-10.75 y[0] + -10.124999999999998 y[1] + -11.75 y[2] + -11.5 y[3] + -8.75 y[4] +$
 $-12.874999999999998 y[5] + -10.25 y[6] + -8.25 y[7] + -5.5 y[8] + -3.7499999999999996$
 $y[9] + -1.0 g \leq -25.5$

R3: $-3.0 y[0] + -8.75 y[1] + -2.75 y[2] + -10.25 y[3] + -5.5 y[4] + -10.0 y[5] + -11.5$
 $y[6] + -6.0 y[7] + -8.75 y[8] + -7.25 y[9] + -1.0 g \leq -18.25$

R4: $-0.8 y[0] + -5.2 y[1] + -9.200000000000001 y[2] + -3.2 y[3] + -8.8 y[4] +$
 $-6.800000000000001 y[5] + -4.0 y[6] + -9.600000000000001 y[7] + -8.0 y[8] +$
 $-9.200000000000001 y[9] + -1.0 g \leq -18.4$

R5: $-7.6000000000000005 y[0] + -2.4000000000000004 y[1] + -8.0 y[2] + -6.4 y[3] + -2.0$
 $y[4] + -5.6000000000000005 y[5] + -8.0 y[6] + -7.2 y[7] + -7.6000000000000005 y[8] +$
 $-2.4000000000000004 y[9] + -1.0 g \leq -15.6$

None

Gurobi Optimizer version 10.0.3 build v10.0.3rc0 (mac64[arm])

CPU model: Apple M1

Thread count: 8 physical cores, 8 logical processors, using up to 8 threads

Optimize a model with 6 rows, 11 columns and 66 nonzeros

Coefficient statistics:

Matrix range [8e-01, 1e+01]

Objective range [1e+00, 1e+01]

Bounds range [0e+00, 0e+00]

RHS range [2e+01, 3e+01]

Iteration	Objective	Primal Inf.	Dual Inf.	Time
0	9.5016530e+00	2.782364e-01	0.000000e+00	0s
1	9.6700817e+00	0.000000e+00	0.000000e+00	0s

Solved in 1 iterations and 0.00 seconds (0.00 work units)

Optimal objective 9.670081743e+00

master problem is optimal.

opt_obj=9.67

opt_g=0.00

Finish solving master problem.

Bounds: lb=9.67, ub=11.73

dual subproblem objective updated!

Start solving dual subproblem.

dual subproblem is optimal.

opt_obj=2.23

Finish solving dual subproblem.

Bounds: lb=9.67, ub=11.73

Minimize

3.0 y[0] + 6.0 y[1] + 8.0 y[2] + 12.0 y[3] + 3.0 y[4] + 5.0 y[5] + 4.0 y[6] + 4.0 y[7]
+ 9.0 y[8] + 3.0 y[9] + g

Subject To

R0: -8.5625 y[0] + -9.859375 y[1] + -9.5625 y[2] + -7.9375 y[3] + -12.125 y[4] +
-11.828125 y[5] + -11.21875 y[6] + -11.46875 y[7] + -10.21875 y[8] + -10.5 y[9] + -1.0 g
≤ -28.1562

R1: -8.76923076923077 y[0] + -6.153846153846153 y[1] + -8.307692307692307 y[2] +
-7.692307692307692 y[3] + -2.0 y[4] + -8.923076923076923 y[5] + -6.769230769230769 y[6]
+ -8.307692307692307 y[7] + -4.153846153846153 y[8] + -6.461538461538462 y[9] + -1.0 g
≤ -16.4615

R2: -10.75 y[0] + -10.124999999999998 y[1] + -11.75 y[2] + -11.5 y[3] + -8.75 y[4] +
-12.874999999999998 y[5] + -10.25 y[6] + -8.25 y[7] + -5.5 y[8] + -3.7499999999999996
y[9] + -1.0 g ≤ -25.5

R3: -3.0 y[0] + -8.75 y[1] + -2.75 y[2] + -10.25 y[3] + -5.5 y[4] + -10.0 y[5] + -11.5
y[6] + -6.0 y[7] + -8.75 y[8] + -7.25 y[9] + -1.0 g ≤ -18.25

R4: -0.8 y[0] + -5.2 y[1] + -9.200000000000001 y[2] + -3.2 y[3] + -8.8 y[4] +
-6.800000000000001 y[5] + -4.0 y[6] + -9.600000000000001 y[7] + -8.0 y[8] +
-9.200000000000001 y[9] + -1.0 g ≤ -18.4

R5: -7.6000000000000005 y[0] + -2.4000000000000004 y[1] + -8.0 y[2] + -6.4 y[3] + -2.0
y[4] + -5.6000000000000005 y[5] + -8.0 y[6] + -7.2 y[7] + -7.6000000000000005 y[8] +
-2.4000000000000004 y[9] + -1.0 g ≤ -15.6

R6: -5.0 y[0] + -3.0 y[1] + -10.0 y[2] + -10.5 y[3] + -8.0 y[4] + -7.5 y[5] + -6.5 y[6]
+ -1.5 y[7] + -12.0 y[8] + -8.0 y[9] + -1.0 g ≤ -17.5

None

Benders optimality cut added!

Start solving master problem.


```

Minimize
3.0 y[0] + 6.0 y[1] + 8.0 y[2] + 12.0 y[3] + 3.0 y[4] + 5.0 y[5] + 4.0 y[6] + 4.0 y[7]
+ 9.0 y[8] + 3.0 y[9] + g
Subject To
R0: -8.5625 y[0] + -9.859375 y[1] + -9.5625 y[2] + -7.9375 y[3] + -12.125 y[4] +
-11.828125 y[5] + -11.21875 y[6] + -11.46875 y[7] + -10.21875 y[8] + -10.5 y[9] + -1.0 g
    <= -28.1562
R1: -8.76923076923077 y[0] + -6.153846153846153 y[1] + -8.307692307692307 y[2] +
-7.692307692307692 y[3] + -2.0 y[4] + -8.923076923076923 y[5] + -6.769230769230769 y[6]
+ -8.307692307692307 y[7] + -4.153846153846153 y[8] + -6.461538461538462 y[9] + -1.0 g
    <= -16.4615
R2: -10.75 y[0] + -10.124999999999998 y[1] + -11.75 y[2] + -11.5 y[3] + -8.75 y[4] +
-12.874999999999998 y[5] + -10.25 y[6] + -8.25 y[7] + -5.5 y[8] + -3.7499999999999996
y[9] + -1.0 g <= -25.5
R3: -3.0 y[0] + -8.75 y[1] + -2.75 y[2] + -10.25 y[3] + -5.5 y[4] + -10.0 y[5] + -11.5
y[6] + -6.0 y[7] + -8.75 y[8] + -7.25 y[9] + -1.0 g <= -18.25
R4: -0.8 y[0] + -5.2 y[1] + -9.200000000000001 y[2] + -3.2 y[3] + -8.8 y[4] +
-6.800000000000001 y[5] + -4.0 y[6] + -9.600000000000001 y[7] + -8.0 y[8] +
-9.200000000000001 y[9] + -1.0 g <= -18.4
R5: -7.6000000000000005 y[0] + -2.4000000000000004 y[1] + -8.0 y[2] + -6.4 y[3] + -2.0
y[4] + -5.6000000000000005 y[5] + -8.0 y[6] + -7.2 y[7] + -7.6000000000000005 y[8] +
-2.4000000000000004 y[9] + -1.0 g <= -15.6
R6: -5.0 y[0] + -3.0 y[1] + -10.0 y[2] + -10.5 y[3] + -8.0 y[4] + -7.5 y[5] + -6.5 y[6]
+ -1.5 y[7] + -12.0 y[8] + -8.0 y[9] + -1.0 g <= -17.5
None
Gurobi Optimizer version 10.0.3 build v10.0.3rc0 (mac64[arm])

CPU model: Apple M1
Thread count: 8 physical cores, 8 logical processors, using up to 8 threads

Optimize a model with 7 rows, 11 columns and 77 nonzeros
Coefficient statistics:
  Matrix range      [8e-01, 1e+01]
  Objective range   [1e+00, 1e+01]
  Bounds range      [0e+00, 0e+00]
  RHS range         [2e+01, 3e+01]
Iteration    Objective          Primal Inf.    Dual Inf.      Time
     0         9.6700817e+00      1.396581e-01   0.000000e+00     0s
     1         9.8201416e+00      0.000000e+00   0.000000e+00     0s

Solved in 1 iterations and 0.00 seconds (0.00 work units)
Optimal objective  9.820141589e+00
    master problem is optimal.

```

```

    opt_obj=9.82
    opt_g=0.00
Finish solving master problem.
-----
Bounds: lb=9.82, ub=11.73
dual subproblem objective updated!
-----
Start solving dual subproblem.
    dual subproblem is optimal.
    opt_obj=1.40
Finish solving dual subproblem.
-----
Bounds: lb=9.82, ub=11.22
Minimize
3.0 y[0] + 6.0 y[1] + 8.0 y[2] + 12.0 y[3] + 3.0 y[4] + 5.0 y[5] + 4.0 y[6] + 4.0 y[7]
+ 9.0 y[8] + 3.0 y[9] + g
Subject To
R0: -8.5625 y[0] + -9.859375 y[1] + -9.5625 y[2] + -7.9375 y[3] + -12.125 y[4] +
-11.828125 y[5] + -11.21875 y[6] + -11.46875 y[7] + -10.21875 y[8] + -10.5 y[9] + -1.0 g
<= -28.1562
R1: -8.76923076923077 y[0] + -6.153846153846153 y[1] + -8.307692307692307 y[2] +
-7.692307692307692 y[3] + -2.0 y[4] + -8.923076923076923 y[5] + -6.769230769230769 y[6]
+ -8.307692307692307 y[7] + -4.153846153846153 y[8] + -6.461538461538462 y[9] + -1.0 g
<= -16.4615
R2: -10.75 y[0] + -10.124999999999998 y[1] + -11.75 y[2] + -11.5 y[3] + -8.75 y[4] +
-12.874999999999998 y[5] + -10.25 y[6] + -8.25 y[7] + -5.5 y[8] + -3.749999999999996
y[9] + -1.0 g <= -25.5
R3: -3.0 y[0] + -8.75 y[1] + -2.75 y[2] + -10.25 y[3] + -5.5 y[4] + -10.0 y[5] + -11.5
y[6] + -6.0 y[7] + -8.75 y[8] + -7.25 y[9] + -1.0 g <= -18.25
R4: -0.8 y[0] + -5.2 y[1] + -9.200000000000001 y[2] + -3.2 y[3] + -8.8 y[4] +
-6.800000000000001 y[5] + -4.0 y[6] + -9.600000000000001 y[7] + -8.0 y[8] +
-9.200000000000001 y[9] + -1.0 g <= -18.4
R5: -7.6000000000000005 y[0] + -2.4000000000000004 y[1] + -8.0 y[2] + -6.4 y[3] + -2.0
y[4] + -5.6000000000000005 y[5] + -8.0 y[6] + -7.2 y[7] + -7.6000000000000005 y[8] +
-2.4000000000000004 y[9] + -1.0 g <= -15.6
R6: -5.0 y[0] + -3.0 y[1] + -10.0 y[2] + -10.5 y[3] + -8.0 y[4] + -7.5 y[5] + -6.5 y[6]
+ -1.5 y[7] + -12.0 y[8] + -8.0 y[9] + -1.0 g <= -17.5
R7: -1.2000000000000002 y[0] + -8.0 y[1] + -2.0 y[2] + -7.2 y[3] + -3.2 y[4] +
-7.6000000000000005 y[5] + -10.0 y[6] + -0.8 y[7] + -6.800000000000001 y[8] +
-2.8000000000000003 y[9] + -1.0 g <= -11.2
None
Benders optimality cut added!
-----

```

Start solving master problem.

Minimize

3.0 y[0] + 6.0 y[1] + 8.0 y[2] + 12.0 y[3] + 3.0 y[4] + 5.0 y[5] + 4.0 y[6] + 4.0 y[7]
+ 9.0 y[8] + 3.0 y[9] + g

Subject To

R0: -8.5625 y[0] + -9.859375 y[1] + -9.5625 y[2] + -7.9375 y[3] + -12.125 y[4] +
-11.828125 y[5] + -11.21875 y[6] + -11.46875 y[7] + -10.21875 y[8] + -10.5 y[9] + -1.0 g
 <= -28.1562

R1: -8.76923076923077 y[0] + -6.153846153846153 y[1] + -8.307692307692307 y[2] +
-7.692307692307692 y[3] + -2.0 y[4] + -8.923076923076923 y[5] + -6.769230769230769 y[6]
+ -8.307692307692307 y[7] + -4.153846153846153 y[8] + -6.461538461538462 y[9] + -1.0 g
 <= -16.4615

R2: -10.75 y[0] + -10.124999999999998 y[1] + -11.75 y[2] + -11.5 y[3] + -8.75 y[4] +
-12.874999999999998 y[5] + -10.25 y[6] + -8.25 y[7] + -5.5 y[8] + -3.749999999999996
y[9] + -1.0 g <= -25.5

R3: -3.0 y[0] + -8.75 y[1] + -2.75 y[2] + -10.25 y[3] + -5.5 y[4] + -10.0 y[5] + -11.5
y[6] + -6.0 y[7] + -8.75 y[8] + -7.25 y[9] + -1.0 g <= -18.25

R4: -0.8 y[0] + -5.2 y[1] + -9.200000000000001 y[2] + -3.2 y[3] + -8.8 y[4] +
-6.800000000000001 y[5] + -4.0 y[6] + -9.600000000000001 y[7] + -8.0 y[8] +
-9.200000000000001 y[9] + -1.0 g <= -18.4

R5: -7.6000000000000005 y[0] + -2.4000000000000004 y[1] + -8.0 y[2] + -6.4 y[3] + -2.0
y[4] + -5.6000000000000005 y[5] + -8.0 y[6] + -7.2 y[7] + -7.6000000000000005 y[8] +
-2.4000000000000004 y[9] + -1.0 g <= -15.6

R6: -5.0 y[0] + -3.0 y[1] + -10.0 y[2] + -10.5 y[3] + -8.0 y[4] + -7.5 y[5] + -6.5 y[6]
+ -1.5 y[7] + -12.0 y[8] + -8.0 y[9] + -1.0 g <= -17.5

R7: -1.2000000000000002 y[0] + -8.0 y[1] + -2.0 y[2] + -7.2 y[3] + -3.2 y[4] +
-7.6000000000000005 y[5] + -10.0 y[6] + -0.8 y[7] + -6.800000000000001 y[8] +
-2.8000000000000003 y[9] + -1.0 g <= -11.2

None

Gurobi Optimizer version 10.0.3 build v10.0.3rc0 (mac64[arm])

CPU model: Apple M1

Thread count: 8 physical cores, 8 logical processors, using up to 8 threads

Optimize a model with 8 rows, 11 columns and 88 nonzeros

Coefficient statistics:

Matrix range [8e-01, 1e+01]

Objective range [1e+00, 1e+01]

Bounds range [0e+00, 0e+00]

RHS range [1e+01, 3e+01]

Iteration	Objective	Primal Inf.	Dual Inf.	Time
0	9.8201416e+00	8.763627e-02	0.000000e+00	0s
1	9.8617265e+00	0.000000e+00	0.000000e+00	0s

```

Solved in 1 iterations and 0.00 seconds (0.00 work units)
Optimal objective  9.861726459e+00
    master problem is optimal.
    opt_obj=9.86
    opt_g=0.00
Finish solving master problem.
-----
Bounds: lb=9.86, ub=11.22
dual subproblem objective updated!
-----
Start solving dual subproblem.
    dual subproblem is optimal.
    opt_obj=0.00
Finish solving dual subproblem.
-----
Bounds: lb=9.86, ub=9.86

```

```
c
```

```
array([3, 3, 5, 5, 2, 4, 3, 2, 3, 3, 4, 2, 5, 5, 2, 5, 4, 4, 5, 5])
```

```
np.random.randint(1, 5, size=(2, 10))
```

```
array([[3, 2, 2, 4, 2, 3, 1, 2, 3, 2],
       [3, 4, 3, 3, 2, 2, 1, 4, 3, 4]])
```

4.2.7 Implementation with callbacks

4.2.8 Implementation with SCIP

```

from pycscipopt import Model
from pycscipopt import quicksum
from pycscipopt import SCIP_PARAMSETTING

# Create a model
model = Model("simple_lp")

# Define variables

```

```

x1 = model.addVar(lb=0, vtype="C", name="x1")
x2 = model.addVar(lb=0, vtype="C", name="x2")

# Set objective function
model.setObjective(x1 + x2, "maximize")

# Add constraints
# model.addCons(2 * x1 + x2 >= 1, "constraint1")
model.addCons(x1 + x2 >= 2, "constraint2")

# Solve the model
model.setPresolve(SCIP_PARAMSETTING.OFF)
model.setHeuristics(SCIP_PARAMSETTING.OFF)
model.disablePropagation()
model.optimize()

# Print results
status = model.getStatus()
print(f'status = {status}')
if model.getStatus() == "optimal":
    print("Optimal solution found.")
    print(f"x1: {model.getVal(x1):.2f}")
    print(f"x2: {model.getVal(x2):.2f}")
    print(f"Objective value: {model.getObjVal():.2f}")
    hasRay = model.hasPrimaryRay()
    print(hasRay)
elif model.getStatus() == 'unbounded':
    hasRay = model.hasPrimalRay()
    print(f'hasRay={hasRay}')
    ray = model.getPrimalRay()
    print(f'ray={ray}')

else:
    print("Model could not be solved.")

```

```
status = unbounded
```

```
hasRay=True
```

```
ray=[0.5, 0.5]
```

```
presolving:
```

```
(0.0s) symmetry computation started: requiring (bin +, int +, cont +), (fixed: bin -, int
```

```
(0.0s) symmetry computation finished: 1 generators found (max: 1500, log10 of symmetry gr
```

```
(0.0s) no symmetry on binary variables present.
```

```

presolving (0 rounds: 0 fast, 0 medium, 0 exhaustive):
  0 deleted vars, 0 deleted constraints, 0 added constraints, 0 tightened bounds, 0 added holes
  0 implications, 0 cliques
presolved problem has 2 variables (0 bin, 0 int, 0 impl, 2 cont) and 2 constraints
  2 constraints of type <linear>
Presolving Time: 0.00

```

```

time | node | left | LP iter|LP it/n|mem/heur|mdpt |vars |cons |rows |cuts |sepa|confs|str
* 0.0s|    1 |    0 |    2 |    - |    LP |    0 |    2 |    2 |    2 |    1 |    0 |    0 |    0

```

```

SCIP Status      : problem is solved [unbounded]
Solving Time (sec) : 0.00
Solving Nodes    : 1
Primal Bound     : +1.0000000000000000e+20 (1 solutions)
Dual Bound       : +1.0000000000000000e+20
Gap              : 0.00 %

```

Testing

Knuth, Donald E. 1984. “Literate Programming.” *Comput. J.* 27 (2): 97–111. <https://doi.org/10.1093/comjnl/27.2.97>.

Listing 4.1 A LP solver based on Gurobi

```
import gurobipy as gp
from gurobipy import GRB
import numpy as np

class LpSolverGurobi:

    def __init__(self, obj_coeff, constr_mat, rhs, verbose=False):
        # initialize environment and model
        self._env = gp.Env('GurobiEnv', empty=True)
        # self._env.setParam('LogToConsole', 1 if verbose else 0)
        self._env.setParam('OutputFlag', 1 if verbose else 0)
        self._env.start()
        self._model = gp.Model(env=self._env, name='GurobiLpSolver')

        # prepare data
        self._obj_coeff = obj_coeff
        # print(self._obj_coeff)
        self._constr_mat = constr_mat
        # print(self._constr_mat)
        self._rhs = rhs
        self._num_vars = len(self._obj_coeff)
        self._num_constrs = len(self._rhs)

        # create decision variables
        self._vars = self._model.addMVar(self._num_vars,
                                          vtype=GRB.CONTINUOUS,
                                          lb=0)

        # create constraints
        self._constrs = self._model.addConstr(
            self._constr_mat@self._vars == self._rhs
        )

        # create objective
        self._model.setObjective(self._obj_coeff@self._vars,
                                GRB.MINIMIZE)

    def optimize(self, verbose=False):
        self._model.optimize()
        if self._model.status == GRB.OPTIMAL:
            print(f'Optimal solution found!')
            print(f'Optimal objective = {self._model.objVal:.2f}')
        elif self._model.status == GRB.UNBOUNDED:
            print(f'Model is unbounded!')
        elif self._model.status == GRB63.INFEASIBLE:
            print(f'Model is infeasible!')
        else:
            print(f'Unknown error occurred!')

    def clean_up(self):
        self._model.dispose()
```

Listing 4.2 A LP solver based on SCIP

```
import pyscipopt as scip
from pyscipopt import SCIP_PARAMSETTING

class LpSolverSCIP:

    def __init__(self, obj_coeff, constr_mat, rhs, verbose=False):
        self._model = scip.Model('LpModel')
        if not verbose:
            self._model.hideOutput()

        # create variables
        self._vars = {
            i: self._model.addVar(lb=0, vtype='C')
            for i in range(len(obj_coeff))
        }

        # create constraints
        for c in range(len(rhs)):
            expr = [
                constr_mat[c][j] * self._vars.get(j)
                for j in range(len(obj_coeff))
            ]
            self._model.addCons(scip.quicksum(expr) == rhs[c])

        # create objective
        obj_expr = [
            obj_coeff[i] * self._vars.get(i)
            for i in range(len(obj_coeff))
        ]
        self._model.setObjective(scip.quicksum(obj_expr), "minimize")

    def optimize(self):
        self._model.optimize()
        status = self._model.getStatus()
        if status == "optimal":
            print(f'Optimal solution found!')
            print(f'Optimal objective = {self._model.getObjVal():.2f}')
        elif self._model.status == "unbounded":
            print(f'Model is unbounded!')
        elif self._model.status == "infeasible":
            print(f'Model is infeasible!')
        else:
            print(f'Unknown error occurred!')
```

Listing 4.3 A randomly generated LP problem

```
import numpy as np

np.random.seed(42)
c = np.random.randint(1, 6, size=20)
A = np.random.randint(-10, 12, size=(5, 20))
b = np.random.randint(20, 100, size=5)
```

Listing 4.4 Solving the generated LP with Gurobi

```
lpsolver_gurobi = LpSolverGurobi(obj_coeff=c, constr_mat=A, rhs=b) # <1>
lpsolver_gurobi.optimize()
```

Listing 4.5 Solving the generated LP with SCIP

```
lpsolver_scip = LpSolverSCIP(obj_coeff=c, constr_mat=A, rhs=b)
lpsolver_scip.optimize()
```

Listing 4.6 Solve the LP problem with Gurobi and SCIP

```
c = np.array([8, 12, 10])
f = np.array([15, 18])
obj_coeff = np.concatenate([c, f])

A = np.array([[2, 3, 2],
              [4, 2, 3]])
B = np.array([[4, 5],
              [2, 3]])
constr_mat = np.concatenate([A, B], axis=1)

rhs = np.array([300, 220])

lpsolver_gurobi = LpSolverGurobi(obj_coeff, constr_mat, rhs, verbose=False)
lpsolver_gurobi.optimize()
# Optimal objective = 1091.43

lpsolver_scip = LpSolverSCIP(obj_coeff, constr_mat, rhs, verbose=False)
lpsolver_scip.optimize()
# Optimal objective = 1091.43
```

Listing 4.7 Gurobi solver setup and restricted master problem initialization

```
import numpy as np
import gurobipy as gp
from gurobipy import GRB

# initialize lower/upper bounds and threshold value
lb = -GRB.INFINITY
ub = GRB.INFINITY
eps = 1.0e-5

# create restricted Benders master problem
env = gp.Env('benders', empty=True) # <1>
env.setParam('OutputFlag', 0)
env.start()
rbmp = gp.Model(env=env, name='RBMP')

# create decision variables
y1 = rbmp.addVar(vtype=GRB.CONTINUOUS, lb=0, name='y1')
y2 = rbmp.addVar(vtype=GRB.CONTINUOUS, lb=0, name='y2')
g = rbmp.addVar(vtype=GRB.CONTINUOUS, lb=0, name='g')

# create objective
rbmp.setObjective(15*y1 + 18*y2 + g, GRB.MINIMIZE)
```

Listing 4.8 Dual subproblem initialization

```
# create dual subproblem
dsp = gp.Model(env=env, name='DSP')

# create decision variables
u1 = dsp.addVar(vtype=GRB.CONTINUOUS, name='u1')
u2 = dsp.addVar(vtype=GRB.CONTINUOUS, name='u2')

# create objective function
dsp.setObjective(300*u1 + 220*u2)

# create constraints
dsp.addConstr(2*u1 + 4*u2 <= 8, name='c1')
dsp.addConstr(3*u1 + 2*u2 <= 12, name='c2')
dsp.addConstr(2*u1 + 3*u2 <= 10, name='c3')

dsp.update()
```

Listing 4.9 Iteration 1 - solving the restricted Benders master problem

```
rbmp.optimize()

if rbmp.status == GRB.OPTIMAL:
    print(f'Optimal solution found! Objective value = {rbmp.objVal:.2f}')

    y1_opt, y2_opt, g_opt = y1.X, y2.X, g.X
    lb = np.max([lb, rbmp.objVal])

    print(f'(y1, y2, g) = ({y1_opt:.2f}, {y2_opt:.2f}, {g_opt:.2f})')
    print(f'lb={lb}, ub={ub}')
elif rbmp.status == GRB.INFEASIBLE:
    print(f'original problem is infeasible!')
```

Listing 4.10 Iteration 1 - solving the dual subproblem

```
# update objective function
dsp.setObjective((300-4*y1_opt-5*y2_opt)*u1 +
                (220-2*y1_opt-3*y2_opt)*u2,
                GRB.MAXIMIZE)

dsp.update()
dsp.optimize()

if dsp.status == GRB.OPTIMAL:
    u1_opt, u2_opt = u1.X, u2.X

    print(f'Optimal objective = {dsp.objVal:.2f}')
    print(f'(u1, u2) = ({u1_opt:.2f}, {u2_opt:.2f})')
    ub = np.min([ub, 15*y1_opt + 18*y2_opt + dsp.objVal])
    print(f'lb={lb}, ub={ub}')
```

Listing 4.11 Iteration 2 - solving the restricted Benders master problem

```
# add optimality cut
rbmp.addConstr((300-4*y1-5*y2)*u1_opt
              + (220-2*y1-3*y2)*u2_opt <= g,
              name='c1')

rbmp.update()
rbmp.optimize()

if rbmp.status == GRB.OPTIMAL:
    print(f'Optimal solution found! Objective value = {rbmp.objVal:.2f}')

    y1_opt, y2_opt, g_opt = y1.X, y2.X, g.X
    lb = np.max([lb, rbmp.objVal])

    print(f'(y1, y2, g) = ({y1_opt:.2f}, {y2_opt:.2f}, {g_opt:.2f})')
    print(f'lb={lb}, ub={ub}')
elif rbmp.status == GRB.INFEASIBLE:
    print(f'original problem is infeasible!')
```

Listing 4.12 Iteration 2 - solving the dual subproblem

```
# update objective function
dsp.setObjective((300-4*y1_opt-5*y2_opt)*u1
                + (220-2*y1_opt-3*y2_opt)*u2,
                GRB.MAXIMIZE)

dsp.update()
dsp.optimize()

if dsp.status == GRB.OPTIMAL:
    u1_opt, u2_opt = u1.X, u2.X

    print(f'Optimal objective = {dsp.objVal:.2f}')
    print(f'(u1, u2) = ({u1_opt:.2f}, {u2_opt:.2f})')
    ub = np.min([ub, 15*y1_opt + 18*y2_opt + dsp.objVal])
    print(f'lb={lb}, ub={ub:.2f}')
```

Listing 4.13 Iteration 3 - solving the restricted Benders master problem

```
# add optimality cut
rbmp.addConstr((300-4*y1-5*y2)*u1_opt
              + (220-2*y1-3*y2)*u2_opt <= g,
              name='c2')

rbmp.update()
rbmp.optimize()

if rbmp.status == GRB.OPTIMAL:
    print(f'Optimal solution found! Objective value = {rbmp.objVal:.2f}')

    y1_opt, y2_opt, g_opt = y1.X, y2.X, g.X
    lb = np.max([lb, rbmp.objVal])

    print(f'(y1, y2, g) = ({y1_opt:.2f}, {y2_opt:.2f}, {g_opt:.2f})')
    print(f'lb={lb:.2f}, ub={ub:.2f}')
elif rbmp.status == GRB.INFEASIBLE:
    print(f'original problem is infeasible!')
```

Listing 4.14 Iteration 3 - solving the dual subproblem

```
# update objective function
dsp.setObjective((300-4*y1_opt-5*y2_opt)*u1
                + (220-2*y1_opt-3*y2_opt)*u2,
                GRB.MAXIMIZE)

dsp.update()
dsp.optimize()

if dsp.status == GRB.OPTIMAL:
    u1_opt, u2_opt = u1.X, u2.X

    print(f'Optimal objective = {dsp.objVal:.2f}')
    print(f'(u1, u2) = ({u1_opt:.2f}, {u2_opt:.2f})')
    ub = np.min([ub, 15*y1_opt + 18*y2_opt + dsp.objVal])
    print(f'lb={lb:.2f}, ub={ub:.2f}')
```
