Python OR-tools Notes

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Preface

1 Introduction

This book covers the usage of Google OR-Tools to solve optimization problems in Python. There are several major chapters in this book:

In Chapter 2, we explain the steps needed to setup OR-Tools in a Python environment.

In Chapter 3, we use an example to illustrate the modeling capability of OR-Tools to solve linear programming problems.

In Chapter 4, we go through the modeling techniques made available in OR-Tools.

2 Environment Setup

In this chapter, we explain the steps needed to set up Python and Google OR-Tools. All the steps below are based on MacBook Air with M1 chip and macOS Ventura 13.1.

2.1 Install Homebrew

The first tool we need is Homebrew, 'the Missing Package Manager for macOS (or Linux)', and it can be accessed at https://brew.sh/. To install Homebrew, just copy the command below and run it in the Terminal.

```
/bin/bash -c "$(curl -fsSL https://raw.githubusercontent.com/Homebrew/install/HEAD/install
```

We can then use the `brew --version` command to check the installed version. On my system, it shows the info below.

```
  ~/ brew --version
Homebrew 3.6.20
Homebrew/homebrew-core (git revision 5f1582e4d55; last commit 2023-02-05)
Homebrew/homebrew-cask (git revision fa3b8a669d; last commit 2023-02-05)
```

2.2 Install Anaconda

Since there are several Python versions available for our use and we may end up having multiple Python versions installed on our machine, it is important to use a consistent environment to work on our project in. Anaconda is a package and environment manager for Python and it provides easy-to-use tools to facilitate our data science needs. To install Anaconda, run the below command in the Terminal.

```
  ~/ brew install anaconda
```

After the installation is done, we can use `conda --version` to verify whether it is available on our machine or not.

```
  ~/ conda --version
conda 23.1.0
```

2.3 Create a Conda Environment

Now we will create a Conda environment named 'ortools'. Execute the below command in the Terminal, which effectively creates the required environment with Python version 3.10.

```
  ~/ conda create -n ortools python=3.10
Retrieving notices: ...working... done
Collecting package metadata (current_repodata.json): done
Solving environment: done

## Package Plan ##

  environment location: /opt/homebrew/anaconda3/envs/test

  added / updated specs:
    - python=3.10


The following packages will be downloaded:

    package                    |            build
    ---------------------------|-----------------
    setuptools-67.4.0          |     pyhd8ed1ab_0         567 KB  conda-forge
    ------------------------------------------------------------
                                           Total:         567 KB

The following NEW packages will be INSTALLED:

  bzip2              conda-forge/osx-arm64::bzip2-1.0.8-h3422bc3_4
  ca-certificates    conda-forge/osx-arm64::ca-certificates-2022.12.7-h4653dfc_0
  libffi             conda-forge/osx-arm64::libffi-3.4.2-h3422bc3_5
  libsqlite          conda-forge/osx-arm64::libsqlite-3.40.0-h76d750c_0
  libzlib            conda-forge/osx-arm64::libzlib-1.2.13-h03a7124_4
  ncurses            conda-forge/osx-arm64::ncurses-6.3-h07bb92c_1
  openssl            conda-forge/osx-arm64::openssl-3.0.8-h03a7124_0
  pip                conda-forge/noarch::pip-23.0.1-pyhd8ed1ab_0
  python             conda-forge/osx-arm64::python-3.10.9-h3ba56d0_0_cpython
```

```
  readline           conda-forge/osx-arm64::readline-8.1.2-h46ed386_0
  setuptools         conda-forge/noarch::setuptools-67.4.0-pyhd8ed1ab_0
  tk                 conda-forge/osx-arm64::tk-8.6.12-he1e0b03_0
  tzdata             conda-forge/noarch::tzdata-2022g-h191b570_0
  wheel              conda-forge/noarch::wheel-0.38.4-pyhd8ed1ab_0
  xz                 conda-forge/osx-arm64::xz-5.2.6-h57fd34a_0


Proceed ([y]/n)?
```

Type 'y' to proceed and Conda will create the environment for us. We can use `cnoda env list` to show all the created environments on our machine:

```
 ~/ conda env list
# conda environments:
#
base                     /opt/homebrew/anaconda3
ortools                   /opt/homebrew/anaconda3/envs/ortools
```

Note that we need to manually activate an environemnt in order to use it: `conda activate ortools`. On my machine, the activated environment `ortools` will appear in the beginning of my prompt.

```
 ~/ conda activate ortools
(ortools)  ~/
```

2.4 Install Google OR-Tools

As of this writing, the latest version of Google OR-Tools is 9.5.2237, and we can install it in our newly created environment using the command `pip install ortools==9.5.2237`. We can use `conda list` to verify whether it is available in our environment.

```
(ortools)  ~/ conda list
# packages in environment at /opt/homebrew/anaconda3/envs/ortools:
#
# Name                    Version                   Build  Channel
absl-py                   1.4.0                     pypi_0    pypi
bzip2                     1.0.8                h3422bc3_4    conda-forge
ca-certificates           2022.12.7            h4653dfc_0    conda-forge
libffi                    3.4.2                h3422bc3_5    conda-forge
```

```
libsqlite                 3.40.0               h76d750c_0    conda-forge
libzlib                   1.2.13               h03a7124_4    conda-forge
ncurses                   6.3                  h07bb92c_1    conda-forge
numpy                     1.24.2                   pypi_0    pypi
openssl                   3.0.8                h03a7124_0    conda-forge
ortools                   9.5.2237                 pypi_0    pypi
pip                       23.0.1             pyhd8ed1ab_0    conda-forge
protobuf                  4.22.0                   pypi_0    pypi
python                    3.10.9        h3ba56d0_0_cpython    conda-forge
readline                  8.1.2                h46ed386_0    conda-forge
setuptools                67.4.0             pyhd8ed1ab_0    conda-forge
tk                        8.6.12               he1e0b03_0    conda-forge
tzdata                    2022g                h191b570_0    conda-forge
wheel                     0.38.4             pyhd8ed1ab_0    conda-forge
xz                        5.2.6                h57fd34a_0    conda-forge
```

Now we have Python and Google OR-Tools ready, we can start our next journey.

3 Linear Programming

In this chapter, we first go through the modeling capabilities provided by Google OR-Tools to solve linear programming problems. Then we get our hands dirty by solving some linear programming problems.

3.1 Modeling Capabilities

There are three components in a mathematical model, namely, decision varaibles, constraints and objective, for which we will go over in the following sections.

3.1.1 Solver

In Google OR-Tools, a `Solver` instance must be created first so that variables, constraints and objective can be added to it. The `Solver` class is defined in the `ortools.linear_solver.pywraplp` module and it requires a solver id to instantiate an object. In the code snippet below, the required module is imported first and a `solver` object is created with `GLOP`, Google's own optimization engine for solving linear programming problems. It is good practice to verify whether the desired solver is indeed created successfully or not.

```
from ortools.linear_solver import pywraplp

solver = pywraplp.Solver.CreateSolver("GLOP")

if solver:
    print("solver creation success!")
else:
    print("solver creation failure!")
```

```
solver creation success!
```

3.1.2 Decision Variables

The `Solver` class defines a number of ways to create decision variables:

1. `Var(lb, ub, integer, name)`
2. `NumVar(lb, ub, name)`
3. `IntVar(lb, ub, name)`
4. `BoolVar(name)`

- Function `Var()`

The `Var()` method is the most flexible way to define variables, as it can be used to create numerical, integral and boolean variables. In the following code, a numerical variable named 'var1' is created with bound (0.0, 1.0). Note that the parameter `integer` is set to `False` in the call to function `Var()`.

```
var1 = solver.Var(lb=0, ub=1.0, integer=False, name="var1")
```

We could create an integer variable using the same function:

```
var2 = solver.Var(lb=0, ub=1.0, integer=True, name="var2")
```

- Function `NumVar()`

`var1` could be creatd alternatively using the specialized function `NumVar()`:

```
var1 = solver.NumVar(lb=0, ub=1.0, name="var1")
```

- Function `IntVar()`

Similarly, `var2` could be created alternatively using the specialized function `IntVar()`:

```
var2 = solver.IntVar(lb=0, ub=1.0, name="var2")
```

- Function `BoolVar()`

A boolean variable could be created using the `BoolVar()` function:

```
var3 = solver.BoolVar(name="var3")
```

3.1.3 Constraints

Constraints limit the solution space of an optimization problem, and there are two ways to define constraints in Google OR-Tools. In the first approach, we could use the `Add()` function to create a constraint and automatically add it to the model at the same time, as the below code snippet illustrates.

```
cons1 = solver.Add(constraint=var1 + var2 <= 1, name="cons1")

type(cons1)
```

```
ortools.linear_solver.pywraplp.Constraint
```

Note that the `Add()` function returns an object of the `Constraint` class defined in the `pywraplp` module, as shown in the code output. It is a good practice to retain the reference of the newly created constraint, as we might want to query its information later on.

The second approach works in a slightly different way. It starts with an empty constraint, with potential lower bound and upper bounds provided, and add components of the constraint gradually. The code snippet below shows an example of adding a second constraint to the model. In this approach, we must retain the reference to the constraint, as it is needed to add decision variables to the constraint in following steps.

```
cons2 = solver.Constraint(-solver.infinity(), 10.0, "cons2")
cons2.SetCoefficient(var1, 2)
cons2.SetCoefficient(var2, 3)
cons2.SetCoefficient(var3, 4)
type(cons2)
```

```
ortools.linear_solver.pywraplp.Constraint
```

3.1.4 Objective

Similar to constraints, there are two ways to define the objective in Google OR-Tools. In the first approach, we directly add an objective to the model by using the `Maximize()` or `Minimize()` function. Below is an example:

```
solver.Minimize(var1 + var2 + var3)
```

Note that the function itself does not return a reference to the newly created objective function, but we could use a dedicated function to retrive it:

```
obj = solver.Objective()
print(obj)
```

```
<ortools.linear_solver.pywraplp.Objective; proxy of <Swig Object of type 'operations_researcl
```

In the second approach, we build the objective incrementally, just as in the second approach of creating constraints. Specifically, we start with an empty objective function, and gradually add components to it. In the end, we specify the optimization sense - whether we want to maximize or minimnize the objective.

```
obj = solver.Objective()
obj.SetCoefficient(var1, 1.0)
obj.SetCoefficient(var2, 1.0)
obj.SetCoefficient(var3, 1.0)
obj.SetMinimization()
print(obj)
```

```
<ortools.linear_solver.pywraplp.Objective; proxy of <Swig Object of type 'operations_researcl
```

3.1.5 Objective and Constraint Expressions

When we build constraints or objective functions, sometimes they comprise of complex expressions that we would like to build incrementally, possibly within loops. For example, we might have a mathematical expression of the form $expr = 2x_1 + 3x_2 + 4x_3 + x_4$, which could be part of the objective function or any constraints. In this case, we can either use the aforementioned `SetCoefficient()` function to add each element of the expression to the constraint or objective, or we could build an expression first and add it once in the end. The code snippet below shows an example.

```
infinity = solver.Infinity()
x1 = solver.NumVar(0, infinity, name="x1")
x2 = solver.NumVar(0, infinity, name="x2")
x3 = solver.NumVar(0, infinity, name="x3")
x4 = solver.NumVar(0, infinity, name="x4")

expr = []
expr.append(2 * x1)
expr.append(3 * x2)
expr.append(4 * x3)
```

```
expr.append(x4)

constr = solver.Add(solver.Sum(expr) <= 10)
print(constr)

solver.Minimize(solver.Sum(expr))
```

```
<ortools.linear_solver.pywraplp.Constraint; proxy of <Swig Object of type 'operations_resear
```

Of course, it is not obvious here that the retitive calls to the `append()` method are any more convenient than the `SetCoefficient()` method. Let's say that we have a slightly more complex expression of the form $\sum_{0 \leq i < 4} w_i \cdot x_i$, now we could build the expression using a loop:

```
w = [2, 3, 4, 1]
x = [x1, x2, x3, x4]
expr = []
for i in range(4):
    expr.append(w[i] * x[i])

constr = solver.Add(solver.Sum(expr) <= 10)
```

3.1.6 Query the Model

After we build the model, we can query it using some helper functions. For example, to get the total number of constraints, we use the `NumVariables()` function. In a similar fashion, we can retrieve the total number of constraints with the `NumConstraints()` function.

```
num_vars = solver.NumVariables()
print(f"there are a total of {num_vars} variables in the model")

num_constr = solver.NumConstraints()
print(f"there are a total of {num_constr} constraints in the model")
```

```
there are a total of 9 variables in the model
there are a total of 4 constraints in the model
```

3.2 Applications

In this section, we use some examples to showcase the modeling capability of Google OR-Tools.

3.2.1 Trivial Problem

We now consider an simple linear programming problem with two decision variables x and y. The formal mathematical model is defined as below:

$$\begin{aligned} \text{max.} \quad & x + 2y && (3.1)\\ \text{s.t.} \quad & x + y \leq 10 && (3.2)\\ & x >= 1 && (3.3)\\ & y >= 1 && (3.4) \end{aligned}$$

Figure 3.1 shows the three defining constraints represented in blue lines and the feasible space depicted by the orange shaded area. The objective function is indicated by the red dashed lines. It can be seen from the figure that the point in green circle gives the maximal objective value of 19.

Let's now use Google OR-Tools to model and solve this problem. The code snippet below shows the complete program.

```
# import Google OR-Tools library
from ortools.linear_solver import pywraplp

# create a solver
solver = pywraplp.Solver.CreateSolver("GLOP")

# create decision variables
x = solver.NumVar(1.0, solver.Infinity(), "x")
y = solver.NumVar(1.0, solver.Infinity(), "y")

# create constraints
constr = solver.Add(x + y <= 10)

# create objective
solver.Maximize(x + 2 * y)

# solve the problem
```

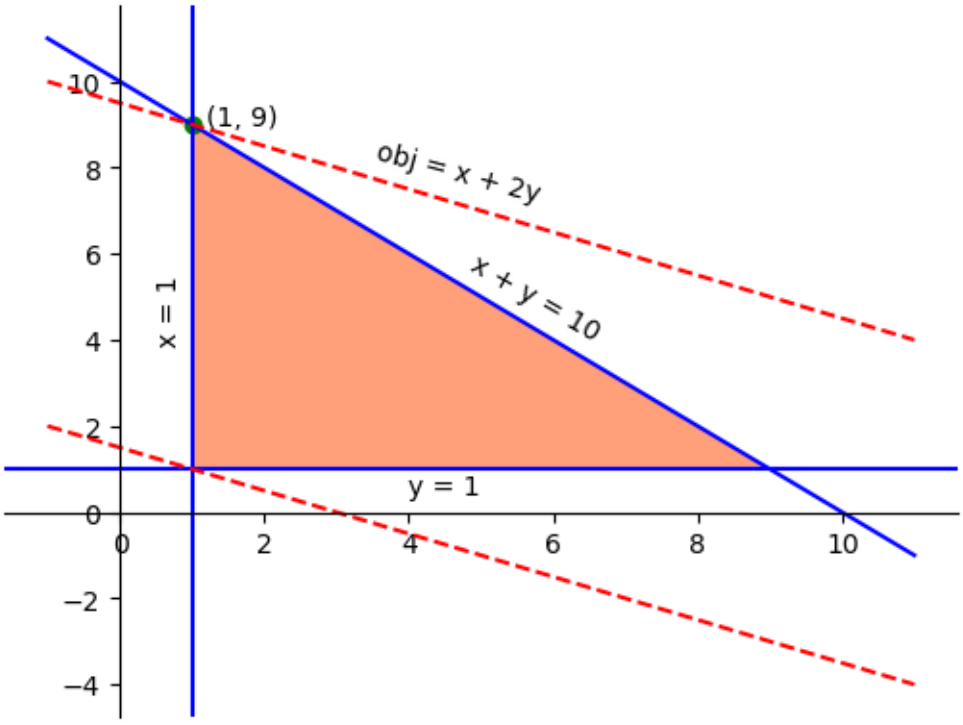


Figure 3.1: A simple LP example

```
status = solver.Solve()

if status == pywraplp.Solver.OPTIMAL or status == pywraplp.Solver.FEASIBLE:
    print(f"obj = {solver.Objective().Value()}")
    print(f"x = {x.solution_value()}, reduced cost = {x.reduced_cost()}")
    print(f"y = {y.solution_value()}, reduced cost = {y.reduced_cost()}")
    print(f"constr dual value = {constr.dual_value()}")
```

```
obj = 19.0
x = 1.0, reduced cost = -1.0
y = 9.0, reduced cost = 0.0
constr dual value = 2.0
```

We can see from the output that the optimal solution is $x = 1.0$ and $y = 9.0$, and the optimal objective is 19.0. This can also be validated from Figure 3.1 that the optimal solution is exactly the green point that sits at the intersection of the three lines $x = 1$, $x + y = 10$ and $x + 2y = 19$.

Figure 3.1 also shows that the point (1, 1) should give us the minimal value of the objective function. To validate this, we can actually change the optimization sense of the objective function from maximization to minimization using the function `SetOptimizationDirection()`, as shown in the code below:

```
solver.Objective().SetOptimizationDirection(maximize=False)

solver.Solve()

print(f"obj = {solver.Objective().Value()}")
print(f"x = {x.solution_value()}, reduced cost = {x.reduced_cost()}")
print(f"y = {y.solution_value()}, reduced cost = {y.reduced_cost()}")
print(f"constr dual value = {constr.dual_value()}")
```

```
obj = 3.0
x = 1.0, reduced cost = 1.0
y = 1.0, reduced cost = 2.0
constr dual value = 0.0
```

3.2.2 Transportation Problem

The transportation problem involves moving goods from its sources $\mathcal{S}$ to destinations $\mathcal{D}$. Each source $s \in \mathcal{S}$ has a total amount of goods p_s it could supply, and each destination $s \in \mathcal{D}$ has a

certain amount of demands m_d. There is a transportation cost, denoted by c_{sd}, to move one unit of goods from a source to a destination. The problem is to find the best set of goods to move from each source to each destination such that all the destination demands are met with the lowest transportation costs.

To model this transportation problem, we define the decision variable x_{sd} to be the amount of goods moving from source s to destinaiton d. Then we could state the problem mathematically as below.

$$\text{min.} \quad \sum_{s \in \mathcal{S}} \sum_{d \in \mathcal{D}} c_{sd} x_{sd} \tag{3.5}$$

$$\text{s.t.} \quad \sum_{d \in \mathcal{D}} x_{sd} = p_s, \ \forall s \in \mathcal{S} \tag{3.6}$$

$$\sum_{s \in \mathcal{S}} x_{sd} = m_d, \ \forall d \in \mathcal{D} \tag{3.7}$$

$$x_{sd} \geq 0, \ \forall s \in \mathcal{S}, d \in \mathcal{D} \tag{3.8}$$

The objective function (3.5) aims to minimize the total transportation costs going from all sources to all destinations. Constraints (3.6) make sure that the sum of goods leaving a source node s must equal to its available supply p_s. Constraints (3.7) require that the sum of goods gonig to a destination node d must equal to its demand m_d. Constraints (3.8) state that the flow variables from sources to destination can only be nonnegative values.

Table 3.1 shows an instance of the transportation problem in which there are four sources and five destinations. Entries in the last row give the corresponding demand from each destination, and the last column list the available supply at each source. The entries in the middle of the table show the transportation cost associated with moving from a specific source to a specific destination. For example, it costs \$18 to move one unit of good from source S2 to D3.

Table 3.1: A transportation problem

	D1	D2	D3	D4	D5	Supply
S1	8	5	13	12	12	58
S2	8	7	18	6	5	55
S3	11	12	5	11	18	64
S4	19	13	5	10	18	71
Demand	44	28	36	52	88	248

We show here two modeling flavors of using OR-Tools to solve this problem. In the first approach, decision variables are created using the `NumVar()` function, cnostraints are defined using the `Add()` function and the objective function is added using the `Minimize()` function.

Note that both constraints and the objective function are generated with the help of of `Sum()` function that creates an expression.

```
from ortools.linear_solver import pywraplp

# gather data
num_sources = 4
num_destinations = 5
supplies = [58, 55, 64, 71]
demands = [44, 28, 36, 52, 88]
costs = [[8, 5, 13, 12, 12],
         [8, 7, 18, 6, 5],
         [11, 12, 5, 11, 18],
         [19, 13, 5, 10, 18]]

# create solver
solver = pywraplp.Solver.CreateSolver("GLOP")

# create decision variables
var_flow = []
for src_idx in range(num_sources):
    vars = [
        solver.NumVar(0, solver.Infinity(),
                      name=f"var_{src_idx}, {dest_idx}")
        for dest_idx in range(num_destinations)
    ]
    var_flow.append(vars)

# create constraints
for src_idx in range(num_sources):
    expr = [var_flow[src_idx][dest_idx]
            for dest_idx in range(num_destinations)]
    solver.Add(solver.Sum(expr) == supplies[src_idx])

for dest_idx in range(num_destinations):
    expr = [var_flow[src_idx][dest_idx]
            for src_idx in range(num_sources)]
    solver.Add(solver.Sum(expr) == demands[dest_idx])

# create objective function
obj_expr = []
for src_idx in range(num_sources):
```

```
    for dest_idx in range(num_destinations):
        obj_expr.append(var_flow[src_idx][dest_idx] * costs[src_idx][dest_idx])
solver.Minimize(solver.Sum(obj_expr))

status = solver.Solve()

opt_flow = []
if status == pywraplp.Solver.OPTIMAL:
    print(f"optimal obj = {solver.Objective().Value()}")
    for src_idx in range(num_sources):
        opt_vals = [var_flow[src_idx][dest_idx].solution_value()
                    for dest_idx in range(num_destinations)]
        opt_flow.append(opt_vals)
```

```
optimal obj = 2013.0
```

The optimal solution is shown in Table 3.2.

Table 3.2: The optimal solution

	D1	D2	D3	D4	D5	Supply
S1	0	28	0	0	30	58
S2	0	0	0	0	55	55
S3	44	0	20	0	0	64
S4	0	0	16	52	3	71
Demand	44	28	36	52	88	248

In the second approach shown in the code snippet below, decision variables are created with the `Var(integer=False)` method instead of the `NumVar()` method. In addition, both constraints and the objective function are created using the `SetCoefficient()` method. In the case of constraints, a lower bound and upper bound are used to generated an empty constraint, and variables are then added to the constraint one by one with their corresponding coefficient. In the case of the objective function, an empty objective is first initialized and variables are then added to it sequentially. Note that the optimization sense is set using the `SetMinimization()` function.

```
from ortools.linear_solver import pywraplp

# gather data
```

```
num_sources = 4
num_destinations = 5
supplies = [58, 55, 64, 71]
demands = [44, 28, 36, 52, 88]
costs = [[8, 5, 13, 12, 12],
        [8, 7, 18, 6, 5],
        [11, 12, 5, 11, 18],
        [19, 13, 5, 10, 18]]

# create solver
solver = pywraplp.Solver.CreateSolver("GLOP")

# create decision variables
var_flow = []
for src_idx in range(num_sources):
    vars = [
        solver.Var(
            0, solver.Infinity(), integer=False,
            name=f"var_{src_idx}, {dest_idx}"
        )
        for dest_idx in range(num_destinations)
    ]
    var_flow.append(vars)

# create constraints
for src_idx in range(num_sources):
    constr = solver.Constraint(supplies[src_idx], supplies[src_idx])
    for dest_idx in range(num_destinations):
        constr.SetCoefficient(var_flow[src_idx][dest_idx], 1.0)

for dest_idx in range(num_destinations):
    constr = solver.Constraint(demands[dest_idx], demands[dest_idx])
    for src_idx in range(num_sources):
        constr.SetCoefficient(var_flow[src_idx][dest_idx], 1.0)

# create objective function
obj = solver.Objective()
for src_idx in range(num_sources):
    for dest_idx in range(num_destinations):
        obj.SetCoefficient(var_flow[src_idx][dest_idx], costs[src_idx][dest_idx])
obj.SetMinimization()
```

```
status = solver.Solve()

opt_flow = []
if status == pywraplp.Solver.OPTIMAL:
    print(f"optimal obj = {solver.Objective().Value()}")
    for src_idx in range(num_sources):
        opt_vals = [var_flow[src_idx][dest_idx].solution_value()
                    for dest_idx in range(num_destinations)]
        opt_flow.append(opt_vals)
```

```
optimal obj = 2013.0
```

To validate the results, Table 3.3 shows the optimal solution produced by the second modeling approach, which is the same as in the previous approach.

Table 3.3: The optimal solution

	D1	D2	D3	D4	D5	Supply
S1	0	28	0	0	30	58
S2	0	0	0	0	55	55
S3	44	0	20	0	0	64
S4	0	0	16	52	3	71
Demand	44	28	36	52	88	248

3.2.3 Resource Allocation Problem

The resource allocation problems involves distributing scarce resources among alternative activities. The resources could be machines in a manufacturing facility, money available to spend, or CPU runtime. The activities could be anything that brings profit at the cost of consuming resources. The objective of this problem is therefore to allocate the available resources to activities such that the total profit is maximized.

Here, we give a general resource allocation model devoid of any practical meanings. To this end, we define a few input parameters to this problem:

- $\mathcal{A}$: the set of candidate activities
- $\mathcal{R}$: the set of avaiable resources
- p_a: the profit of performing one unit of activity $a \in \mathcal{A}$
- c_{ar}: the amount of resource $r \in \mathcal{R}$ required by one unit of activity $a \in \mathcal{A}$
- b_r: the total amount of available quantities for resource $r \in \mathcal{R}$

The decision variable x_a represents the amount of activity $a \in \mathcal{A}$ we select to perform, and the mathematical mode is defined below:

$$\text{max.} \quad \sum_{a \in \mathcal{A}} p_a x_a \tag{3.9}$$

$$\text{s.t.} \quad \sum_{a \in \mathcal{A}} c_{ar} \leq b_r, \ \forall r \in \mathcal{R} \tag{3.10}$$

$$x_a \geq 0, \ a \in \mathcal{A} \tag{3.11}$$

Table 3.4 shows an instance of the resource allocation problem, in which there are three type of resources and five candidate activities. The last row gives the profit of performing each unit of an activity, while the last column shows the available amount of resources. The remaining entries in the table refer to the resource consumption for each activity. For example, selecting one unit of activity 1 (A1) requires 90, 64 and 55 units of resources R1, R2 and R3, respectively.

Table 3.4: A resource allocation problem

	A1	A2	A3	A4	A5	Available
R1	90	57	51	97	67	2001
R2	64	58	97	56	93	2616
R3	55	87	77	52	51	1691
Profit	1223	1238	1517	1616	1027	

In the code snippet below, we use Google OR-Tools to solve this problem instance. Again, we start with initializing a solver object, follwed by creation of five decision variables, one for each activity. Both the constraints and objective function are created using the first modeling approach demonstrated previously. The optimal solution is outputed in the end.

```
from ortools.linear_solver import pywraplp

# gather instance data
num_resources = 3
num_activities = 5
profits = [1223, 1238, 1517, 1616, 1027]
available_resources = [2001, 2616, 1691]
costs = [[90, 57, 51, 97, 67],
         [64, 58, 97, 56, 93],
         [55, 87, 77, 52, 51]]
```

```
# initialize a solver object
solver = pywraplp.Solver.CreateSolver("GLOP")

infinity = solver.Infinity()
# create decision variables
var_x = [solver.NumVar(0, infinity, name=f"x_P{a}")
        for a in range(num_activities)]

# create objective function
solver.Maximize(solver.Sum([profits[a] * var_x[a]
                            for a in range(num_activities)]))

# create constraints
for r_idx in range(num_resources):
    cons = solver.Add(
            solver.Sum([costs[r_idx][a_idx] * var_x[a_idx]
                        for a_idx in range(num_activities)])
                    <= available_resources[r_idx])

status = solver.Solve()
if status != pywraplp.Solver.OPTIMAL:
    print("solver failure!")

print("solve complete!")
opt_obj = solver.Objective().Value()
print(f"optimal obj = {opt_obj:.2f}")

opt_sol = [var_x[a_idx].solution_value()
           for a_idx in range(num_activities)]
for a_idx in range(num_activities):
    print(f"opt_x[{a_idx + 1}] = {opt_sol[a_idx]:.2f}")
```

```
solve complete!
optimal obj = 41645.23
opt_x[1] = 0.00
opt_x[2] = 0.00
opt_x[3] = 12.45
opt_x[4] = 14.08
opt_x[5] = 0.00
```

3.2.4 Workforce Planning Problem

In the workforce planning problem, there are a number of time periods and each period has a workforce requirement that must be satisfied. In addition, there are a set of available work patterns to assign workers to and each pattern cover one or more time periods. Note that assignment of workers to a particular pattern incurs a certain cost. The problem is then to identify the number of workers assigned to each pattern such that the total cost is minimized.

Table 3.5 shows a contrived workforce planning problem instance. In this problem, there are a total of 10 time periods and there are four patterns available to assign workers to. The last row gives the work requirement in each time period and the last column shows the cost of assigning a worker to a pattern.

Table 3.5: A workforce planning problem instance

Coverage	1	2	3	4	5	6	7	8	9	10	Cost
Pattern 1	x	x	x	x							10
Pattern 2			x	x	x						30
Pattern 3				x	x	x	x				20
Pattern 4							x	x	x	x	40
Requirement	3	4	3	1	5	7	2	4	5	1	

To model this problem, we use $\mathcal{T}$ and $\mathcal{P}$ to denote the set of time periods and patterns, respectively. The parameter m_{pt} indicates whether a pattern $p \in \mathcal{P}$ covers a certain time period $t \in \mathcal{T}$. The work requirement of each time period and the cost of assigning a pattern is represented as r_t and c_p, respectively.

Now we are ready to define the variable x_p as the number of workers that are assigned to pattern p, and the mathematical model can be stated as below.

$$\text{min.} \quad \sum_{p \in \mathcal{P}} c_p x_p \tag{3.12}$$

$$\text{s.t.} \quad \sum_{p \in \mathcal{P}} m_{pt} x_p \geq r_t, \ \forall t \in \mathcal{T} \tag{3.13}$$

$$x_p \geq 0, \ \forall p \in \mathcal{P} \tag{3.14}$$

The code snippet below gives the Python code to solve this problem using Google OR-Tools.

```
from ortools.linear_solver import pywraplp

# import instance data
```

```
num_periods = 10
num_patterns = 4
requirements = [3, 4, 3, 1, 5, 7, 2, 4, 5, 1]
costs = [10, 30, 20, 40]
patterns = [set([1, 2, 3, 4]),
            set([3, 4, 5]),
            set([4, 5, 6, 7]),
            set([7, 8, 9, 10])]

# create solver object
solver = pywraplp.Solver.CreateSolver('GLOP')

infinity = solver.Infinity()
# create decision variables
var_p = [solver.NumVar(0, infinity, name=f"x_{p}")
         for p in range(num_patterns)]

# create objective function
solver.Minimize(
    solver.Sum([costs[p] * var_p[p]
                for p in range(num_patterns)])
)

# create constraints
for t in range(num_periods):
    solver.Add(
        solver.Sum([var_p[p]
                    for p in range(num_patterns)
                    if (t + 1) in patterns[p]])
        >= requirements[t])

# solve the problem and retrieve optimal solution
status = solver.Solve()
if status == pywraplp.Solver.OPTIMAL:
    print(f"obj = {solver.Objective().Value()}")
    for p in range(num_patterns):
        print(f"var_{p + 1} = {var_p[p].solution_value()}")
```

```
obj = 380.0
var_1 = 4.0
var_2 = 0.0
var_3 = 7.0
```

```
var_4 = 5.0
```

3.2.5 Sudoku Problem

In a Sudoku problem, a grid of 9x9 is given and the task is to fill all the cells with numbers 1-9. At the beginning, some of the cells are already gilled with numbers and the requirements are that the remaining cells must be filled so that each row, each column, and each of the 9 3x3 sub-grids contain all the numbers from 1 to 9 without any repitition. The difficulty level of Sudoku problems depends on the number of cells that are already filled in the grid at the beginning of the game. Problems with fewer initial digits filled are considered more challenging. Figure 3.2 illustrate a sample Sudoku problem.

To model this problem, we define set $S = (1, 2, 3, 4, 5, 6, 7, 8, 9)$ and use $i, j \in S$ to index the row and column respectively. In addition, we use $M = \{(i, j, k)|i, j, k \in S\}$ to represent all the known numbers in the grid.

To formulate this problem, we define 9 binary variables for each cell in the 9x9 grid. Each of the 9 variables corresponds to one of the numbers in set S. Formally, x_{ijk} represents whether the value k shows up in cell (i, j) of the grid. Note that $i, j, k \in S$. The mathematical formulation can be stated as below.

$$\text{min.} \quad 0 \tag{3.15}$$

$$\text{s.t.} \quad \sum_{j \in S} x_{ijk} = 1, \ \forall i, k \in S \tag{3.16}$$

$$\sum_{i \in S} x_{ijk} = 1, \ \forall j, k \in S \tag{3.17}$$

$$\sum_{k \in S} x_{ijk} = 1, \ \forall i, j \in S \tag{3.18}$$

$$\sum_{(i-1)\times 3+1}^{(i-1)\times 3+3} \sum_{(j-1)\times 3+1}^{(j-1)\times 3+3} x_{ijk} = 1, \ \forall i, j \in \{1, 2, 3\}, k \in S \tag{3.19}$$

$$x_{ijk} = 1, \ \forall (i, j, k) \in M \tag{3.20}$$

$$x_{ijk} \in \{0, 1\}, \ \forall i, j, k \in S \tag{3.21}$$

Since no feasible soution is more preferable than another, we use a constant value as the objective function, meaning any feasible soution is an optimal solution to this problem. Constraints (3.16) require that the number $k \in S$ shows up once and only once in each row of the grid. Similarly, (3.17) make sure that the number $k \in S$ shows up once and only once in each column of the grid. For each cell in the grid, only one of the numbers in S can appear, which is guaranteed by constraints (3.18). Constraints (3.19) ensure that the numbers in set S show

		6						
		3						
5						3	7	9
2	1	4	9					
					5	4		
3	5	8				9		
4								2
		5						
8	2							

Figure 3.2: A Sudoku problem

up once and only once in each of the sub-grids. Constraints (3.20) make sure that the existing numbers in the grid stay the same in the optimal solution.

We can then solve the problem using Google OR-Tools and the code snippet is given below.

```
import numpy as np
from ortools.linear_solver import pywraplp

# import data
grid_size = 9
subgrid_size = 3
M = [[(1, 3, 6)],
     [(2, 3, 3)],
     [(3, 1, 5), (3, 7, 3), (3, 8, 7), (3, 9, 9)],
     [(4, 1, 2), (4, 2, 1), (4, 3, 4), (4, 4, 0)],
     [(5, 6, 5), (5, 7, 4)],
     [(6, 1, 3), (6, 2, 5), (6, 3, 8), (6, 7, 9)],
     [(7, 1, 4), (7, 9, 2)],
     [(8, 3, 5)],
     [(9, 1, 8), (9, 2, 2)]]

# create solver
solver = pywraplp.Solver.CreateSolver("SCIP")

# # create decision variables
sudoku_vars = np.empty((grid_size, grid_size, grid_size), dtype=object)
for row in range(grid_size):
     for col in range(grid_size):
          for num in range(grid_size):
               sudoku_vars[row][col][num] = solver.Var(0,
                                        1,
                                        integer=True,
                                        name=f"x_{row, col, num}")

# create objective
solver.Minimize(0)

# create constraints
for row in range(grid_size):
     for num in range(grid_size):
          solver.Add(
               solver.Sum([sudoku_vars[row][col][num]
```

```
                              for col in range(grid_size)
                         ]) == 1
                    )

for col in range(grid_size):
     for num in range(grid_size):
          solver.Add(
               solver.Sum([sudoku_vars[row][col][num]
                              for row in range(grid_size)
                         ]) == 1
                    )

for row in range(grid_size):
     for col in range(grid_size):
          solver.Add(
               solver.Sum([sudoku_vars[row][col][num]
                              for num in range(grid_size)
                         ]) == 1
                    )

for row in range(grid_size):
     known_values = M[row]
     for value in known_values:
          row, col, num = value
          solver.Add(
               sudoku_vars[row - 1][col - 1][num - 1] == 1
          )

# solve the problem
status = solver.Solve()

if status == pywraplp.Solver.OPTIMAL or status == pywraplp.Solver.FEASIBLE:
     sudoku_sol = np.zeros((grid_size, grid_size), dtype=int)
     for row in range(grid_size):
          for col in range(grid_size):
               for num in range(grid_size):
                    if sudoku_vars[row][col][num].solution_value() == 1:
                         sudoku_sol[row][col] = num + 1
```

Figure 3.3 shows one solution to the example problem.

7	8	6	1	2	3	5	9	4
1	4	3	2	5	9	6	8	7
5	6	2	4	1	8	3	7	9
2	1	4	9	3	6	7	5	8
6	3	7	8	9	5	4	2	1
3	5	8	7	4	2	9	1	6
4	9	1	5	6	7	8	3	2
9	7	5	6	8	1	2	4	3
8	2	9	3	7	4	1	6	5

Figure 3.3: One solution to the Sudoku problem

4 Integer Programming

Integer programming has a wide range of applications across various industries and domains. Some of the classical applications of integer programming include:

- Production Planning and Scheduling: Integer programming is widely used in production planning and scheduling to optimize the allocation of resources, such as machines, workers, and raw materials. It helps to minimize costs and maximize efficiency by determining the optimal production schedule.
- Network Optimization: Integer programming is used in network optimization problems such as routing, scheduling, and allocation of resources in transportation networks, telecommunication networks, and supply chain management.
- Facility Location: Integer programming is used in facility location problems, which involve determining the optimal location for a facility based on various factors such as demand, supply, and transportation costs. It is commonly used in logistics, transportation, and distribution industries.
- Portfolio Optimization: Integer programming is used in finance to optimize investment portfolios, where the goal is to maximize the returns on the investment while minimizing risk.
- Cutting Stock and Bin Packing: Integer programming is used in cutting stock and bin packing problems where items of varying sizes must be packed into containers or cut from a stock. This is commonly used in the packaging and manufacturing industries.
- Crew Scheduling: Integer programming is used in crew scheduling problems, where the goal is to optimize the allocation of crew members to different shifts, duties, or activities. It is commonly used in industries such as airlines, railways, and public transportation.
- Timetabling: Integer programming is used in timetabling problems such as scheduling classes, exams, and events in academic institutions. It helps to minimize scheduling conflicts and maximize resource utilization.

This chapter explores the various methods that Google OR-Tools provides for modeling and solving (mixed) integer linear programming problems. The first step is to review the additional conditions that arise when modeling integer variables and solving integer programs. Following that, we use specific instances to demonstrate how these techniques are applied.

4.1 Modeling Capabilities

When modeling integer programs, there are two main tasks that require attention. The first is declaring integer variables, and the second is selecting a solver that is capable of solving integer programs.

4.1.1 Declaring Integer Variables

As reviewed in Chapter 3, Google OR-Tools provides two options to create integer variables:

- The `Var(lb, ub, integer: bool, name)` function
- The `IntVar(lb, ub, name)` function
- The `Variable.SetInteger(integer: bool)` function

In the code snippet below, we create three integer variables using all the aforementioned approaches:

```
from ortools.linear_solver import pywraplp

solver = pywraplp.Solver.CreateSolver('SCIP')

# option 1
x = solver.Var(lb=0, ub=10, integer=True, name='x')

# option 2
y = solver.IntVar(lb=10, ub=20, name='y')

# option 3
z = solver.NumVar(lb=0, ub=5.5, name='z')
z.SetInteger(integer=True)
```

We can verify the types of variables x, y, z:

```
print(f"x is integer? {x.integer()}")
print(f"y is integer? {y.integer()}")
print(f"z is integer? {z.integer()}")
```

```
x is integer? True
y is integer? True
z is integer? True
```

4.1.2 Selecting an Integer Solver

There are several solvers available for solving integer programs, and some options include:

- CBC_MIXED_INTEGER_PROGRAMMING or CBC
- BOP_INTEGER_PROGRAMMING or BOP
- SAT_INTEGER_PROGRAMMING or SAT or CP_SAT
- SCIP_MIXED_INTEGER_PROGRAMMING or SCIP
- GUROBI_MIXED_INTEGER_PROGRAMMING or GUROBI or GUROBI_MIP
- CPLEX_MIXED_INTEGER_PROGRAMMING or CPLEX or CPLEX_MIP
- XPRESS_MIXED_INTEGER_PROGRAMMING or XPRESS or XPRESS_MIP
- GLPK_MIXED_INTEGER_PROGRAMMING or GLPK or GLPK_MIP

It's important to note that some of these solvers are open-source, while others require a commercial license. The code block above demonstrates how to create an instance of an integer solver. To do so, we simply need to specify the name of the solver in the Solver.CreateSolver() function.

```
solver = pywraplp.Solver.CreateSolver('CBC')
```

4.2 Applications

In this section, we use Google OR-Tools to solve some of the classical integer programming problems.

4.2.1 Job Shop Scheduling Problem

In the job shop scheduling problem (JSSP), there are a set of n jobs $\mathcal{J}$ and a set of m machines $\mathcal{M}$, and each job $j \in \mathcal{J}$ has a list of operations, given by $(o_1^j, \cdots, o_h^j, \cdots, o_m^j)$, that must be carried on the machines. The order of operations in the list also indicates the processing order of the job, and o_h^j represents the h-th operation of the job j. In addition, the processing time of job j on machine i, denoted by p_{ij}, is known in advance and is a non-negative integer. At any moment of time, each machine can only process at most one job and no preemption is allowed which means that a job must completes its processing on a machine once it starts on that machine. The objective is to find a processing schedule of the jobs on the machines such that the makespan, the completion time of the last operation of any job, is minimized.

To test the modeling of JSSP, we use a benchmarking instance from the OR-Library (Beasley (1990)), shown in the box below. The two numbers in the first line represent the number of jobs and the number of machines, respectively. Each remaining line contains the operations,

processing machine and processing time, for each job. Note that the machines are numbered starting from 0.

```
# Instance ft06 from OR-Library
# 6 6
# 2 1   0   3   1   6   3   7   5   3   4   6
# 1 8   2   5   4   10  5   10  0   10  3   4
# 2 5   3   4   5   8   0   9   1   1   4   7
# 1 5   0   5   2   5   3   3   4   8   5   9
# 2 9   1   3   4   5   5   4   0   3   3   1
# 1 3   3   3   5   9   0   10  4   4   2   1
```

Suppose this instance data is saved in a file named *ft06.txt* and the code below defines an utility function to read and parse the instance for later use.

```
def read_jssp_instance(filename: str):
    with open(filename) as f:
        num_jobs, num_machines = [int(x) for x in next(f).split()]
        operations = []
        processing_times = {}
        job_idx = 0
        for line in f:
            info = [int(x) for x in line.split()]
            arr = [info[2 * m] for m in range(num_machines)]
            times = {info[2 * m]: info[2 * m + 1]
                    for m in range(num_machines)}
            operations.append(arr)
            processing_times[job_idx] = times
            job_idx += 1
    return num_jobs, num_machines, operations, processing_times
```

We present here three classical formulations of the JSSP from the literature and implement them using Google OR-Tools.

1. Disjunctive model

This model is taken from Ku and Beck (2016) and Manne (1960). The decision variables are defined as follows:

- x_{ij}: the processing starting time of job j on machine i
- z_{ijk}: a binary variable that equals 1 if job j precedes job k on machine i

The disjunctive model can then be stated as below.

$$
\begin{aligned}
\text{min.} \quad & C_{max} && (4.1)\\
\text{s.t.} \quad & x_{ij} \geq 0, \ \forall j \in \mathcal{J}, i \in \mathcal{M} && (4.2)\\
& x_{o^j_h,j} \geq x_{o^j_{h-1},j} + p_{o^j_{h-1},j}, \ \forall j \in \mathcal{J}, h = 2, \cdots, m && (4.3)\\
& x_{ij} \geq x_{ik} + p_{ik} - V \cdot z_{ijk}, \ \forall i \in \mathcal{M}, j, k \in \mathcal{J}, j < k && (4.4)\\
& x_{ik} \geq x_{ij} + p_{ij} - V \cdot (1 - z_{ijk}), \ \forall i \in \mathcal{M}, j, k \in \mathcal{J}, j < k && (4.5)\\
& C_{max} \geq x_{o^j_m,j} + p_{o^j_m,j}, \ \forall j \in \mathcal{J} && (4.6)\\
& z_{ijk} \in \{0, 1\}, \ \forall i \in \mathcal{M}, j, k \in \mathcal{J} && (4.7)
\end{aligned}
$$

The objective (4.1) aims to minimize the maximal completion time of any job $j \in \mathcal{J}$. Constraints (4.2) require that all the job processing starting time must not be negative values. Constraints (4.3) enforce the sequencing order among operations for every job, which state that the h-th operation of job j, o^j_h, cannot start unless its preceeding operation o^j_{h-1} finishes. Constraints (4.4) and (4.5) together make sure that at most one job can be processed on a machine at any time. To be specific, in case of job j preceding job k on machine i, z_{ijk} takes the value of 1 and constraints (4.5) ensure that job k won't start processing on machine i unless job i completes processing; Otherwise, z_{ijk} takes the value of 0 and constraints (4.4) require that job j starts processing after job k. Note that both constraints are needed when we require $j < k$; Otherwise, only one of them is needed if we create a constraint for every pair of j and k on a machine. Constraints (4.6) derive C_{max} across all jobs. The last constraints (4.7) state the variable type of z_{ijk}.

The disjunctive formulation code is presented entirely in the following lines. The data related to the specific case are read between lines 5 to 8, and a solver object is created in line 11. The variable x_{ij} is introduced in lines 16 to 23, followed by the introduction of variable z_{ijk} in lines 25 to 34. The variable C_{max} is defined in lines 36 to 38. The objective of the model is set in line 41, and the constraints are established in lines 44 to 80. The instance is solved, and the optimal solution is obtained from lines 82 to 93.

```
1  from typing import List, Dict
2  from ortools.linear_solver import pywraplp
3
4  # read and parse the data
5  filename = './data/jssp/ft06.txt'
6  num_jobs, num_machines, \
7  operations, processing_times = \
8      read_jssp_instance(filename)
9
10 # create solver
11 solver = pywraplp.Solver.CreateSolver('SCIP')
```

```
12
13  # create variables
14  infinity = solver.Infinity()
15  var_time: List[List] = []
16  for machine in range(num_machines):
17      arr = [
18          solver.NumVar(0,
19                        infinity,
20                        name=f'x_{machine, job}')
21          for job in range(num_jobs)
22          ]
23      var_time.append(arr)
24
25  var_prec: Dict = []
26  for machine in range(num_machines):
27      mac_dict = {}
28      for job_j in range(num_jobs - 1):
29          for job_k in range(job_j + 1, num_jobs):
30              mac_dict[(job_j, job_k)] = \
31              solver.BoolVar(
32                  name=f'z_{machine, job_j, job_k}'
33              )
34      var_prec.append(mac_dict)
35
36  var_makespan = solver.NumVar(0,
37                               infinity,
38                               name='C_max')
39
40  # create objective
41  solver.Minimize(var_makespan)
42
43  # create constraints
44  for job, job_operations in enumerate(operations):
45      for h in range(1, num_machines):
46          curr_machine = job_operations[h]
47          prev_machine = job_operations[h - 1]
48          prev_time = processing_times[job][prev_machine]
49          solver.Add(
50              var_time[curr_machine][job] >=
51                  var_time[prev_machine][job] +
52                  prev_time
53              )
```

```
54
55  V = 0
56  for job in processing_times:
57      V += sum(processing_times[job].values())
58  for machine in range(num_machines):
59      for job_j in range(num_jobs - 1):
60          for job_k in range(job_j + 1, num_jobs):
61              solver.Add(
62                  var_time[machine][job_j] >=
63                  var_time[machine][job_k] +
64                  processing_times[job_k][machine] -
65                  V * var_prec[machine][(job_j, job_k)]
66              )
67              solver.Add(
68                  var_time[machine][job_k] >=
69                  var_time[machine][job_j] +
70                  processing_times[job_j][machine] -
71                  V * (1 - var_prec[machine][(job_j, job_k)])
72              )
73
74  for job in range(num_jobs):
75      last_oper_machine = operations[job][-1]
76      solver.Add(
77          var_makespan >=
78          var_time[last_oper_machine][job] +
79          processing_times[job][last_oper_machine]
80      )
81
82  status = solver.Solve()
83
84  if status == solver.OPTIMAL:
85      print(f"min. makespan = {solver.Objective().Value():.2f}")
86
87      opt_time = []
88      for machine in range(num_machines):
89          arr = [
90              int(var_time[machine][job].solution_value())
91              for job in range(num_jobs)
92          ]
93          opt_time.append(arr)
```

```
min. makespan = 55.00
```

The output of the model indicates that the lowest possible time needed to complete all tasks in this instance is 55. To illustrate the most efficient solution, we have created a function called `show_schedule()` that displays a Gantt chart of the tasks needed to process all jobs. Figure 4.1 displays the optimal solution for this instance.

```
import matplotlib as mpl
import matplotlib.pyplot as plt

def show_schedule(num_jobs, operations, processing_times, opt_time):
    colors = mpl.colormaps["Set1"].colors

    fig, ax = plt.subplots(figsize=[7, 3], dpi=100)

    for idx, job in enumerate(range(num_jobs)):
        machines = operations[job]
        job_start_times = [opt_time[machine][job]
                           for machine in machines]
        job_processing_times = [processing_times[job][machine]
                                for machine in operations[job]]

        if idx >= len(colors):
            idx = idx % len(colors)
        color = colors[idx]

        bars = ax.barh(machines,
                width=job_processing_times,
                left=job_start_times,
                label=f'Job {job + 1}',
                color=color)

        ax.bar_label(bars,
                     fmt=f'{job + 1}',
                     label_type='center')

    ax.set_yticks(machines)
    ax.set_xlabel("Time")
    ax.set_ylabel("Machine")
    fig.tight_layout()
    plt.show()

show_schedule(num_jobs, operations, processing_times, opt_time)
```

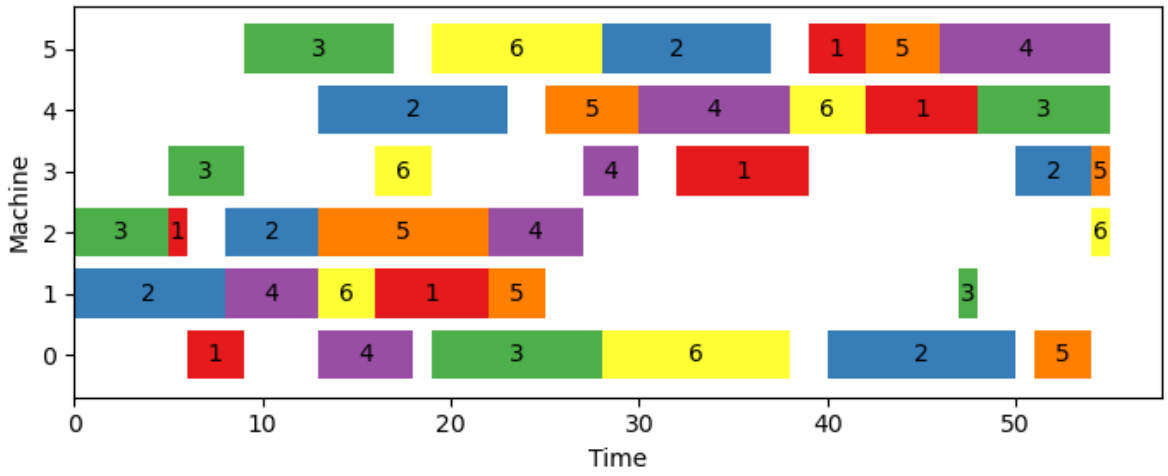


Figure 4.1: Optimal solution of the ft06 instance

2. Time-indexed model

The time-indexed formulation, proposed by Kondili, Pantelides, and Sargent (1988) and Ku and Beck (2016), involves the use of a binary variable x_{ijt} that takes the value of 1 if job j starts at time t on machine i. The model can be expressed as follows.

$$\text{min.} \quad C_{max} \tag{4.8}$$

$$\text{s.t.} \quad \sum_{t \in H} x_{ijt} = 1, \ \forall j \in \mathcal{J}, i \in \mathcal{M} \tag{4.9}$$

$$\sum_{t \in H} (t + p_{ij}) \cdot x_{ijt} \leq C_{max}, \ \forall j \in \mathcal{J}, i \in \mathcal{M} \tag{4.10}$$

$$\sum_{j \in \mathcal{J}} \sum_{t' \in T_{ijt}} x_{ijt'} \leq 1, \ \forall i \in \mathcal{M}, t \in H, T_{ijt} = \{t - p_{ij} + 1, \cdots, t\} \tag{4.11}$$

$$\sum_{t \in H} (t + p_{o^j_{h-1}, j}) \cdot x_{o^j_{h-1}, jt} \leq \sum_{t \in H} t \cdot x_{o^j_h, jt}, \ \forall j \in \mathcal{J}, h = 2, \cdots, m \tag{4.12}$$

$$x_{ijt} \in \{0, 1\}, \ \forall j \in \mathcal{J}, i \in \mathcal{M}, t \in H \tag{4.13}$$

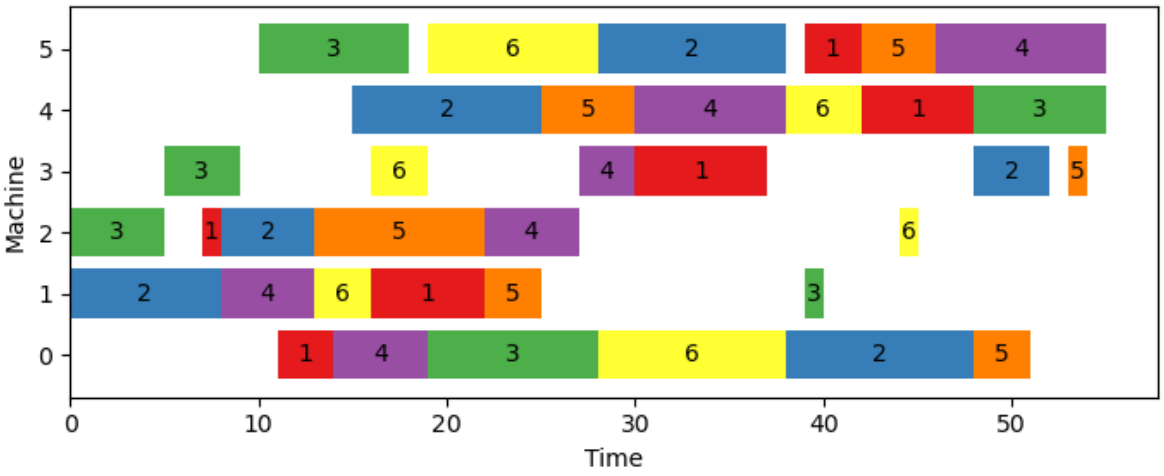
In this formulation, the first set of constraints, referred to as (4.9), state that each job j must start at one specific time within the scheduling horizon H, which is determined as the sum of processing times for all jobs - $H = \sum_{i \in \mathcal{I}, j \in \mathcal{J}} p_{ij}$. Constraints (4.10) are used to calculate the value of C_{max}, while constraints (4.11) ensure that only one job can be processed by a machine at any given time. It's important to note that a job will remain on a machine for its full processing time and cannot be interrupted. Constraints (4.12) make sure that the

order of processing jobs is followed, and constraints (4.13) define the variable types used in the formulation.

```
from typing import List, Dict
from ortools.linear_solver import pywraplp
import numpy as np

# read and parse the data
filename = './data/jssp/ft06.txt'
num_jobs, num_machines, \
operations, processing_times = \
    read_jssp_instance(filename)

# create solver
solver = pywraplp.Solver.CreateSolver('SCIP')

# create variables
H = 1
for job in processing_times:
    H += sum(processing_times[job].values())
var_x = np.empty((num_machines, num_jobs, H), dtype=object)
for machine in range(num_machines):
    for job in range(num_jobs):
        for t in range(H):
            var_x[machine][job][t] = solver.BoolVar(name=f'x_{machine, job, t}')

infinity = solver.Infinity()
var_makespan = solver.NumVar(0,
                             infinity,
                             name='C_max')

# create objective
solver.Minimize(var_makespan)

# create constraints
for machine in range(num_machines):
    for job in range(num_jobs):
        solver.Add(solver.Sum([var_x[machine][job][t] for t in range(H)]) == 1)

for machine in range(num_machines):
    for job in range(num_jobs):
        arr = [var_x[machine][job][t] * (t + processing_times[job][machine]) for t in rang
```

```
        solver.Add(solver.Sum(arr) <= var_makespan)

for machine in range(num_machines):
    for t in range(H):
        arr = [var_x[machine][job][tt]
                    for job in range(num_jobs)
                    for tt in range(t - processing_times[job][machine] + 1, t + 1)]
        solver.Add(solver.Sum(arr) <= 1)

for job in range(num_jobs):
    for oper in range(1, num_machines):
        prev_machine = operations[job][oper - 1]
        curr_machine = operations[job][oper]
        expr_prev = [(t + processing_times[job][prev_machine]) * var_x[prev_machine][job][
                    for t in range(H)]
        expr_curr = [t * var_x[curr_machine][job][t]
                    for t in range(H)]
        solver.Add(solver.Sum(expr_prev) <= solver.Sum(expr_curr))

status = solver.Solve()

if status == solver.OPTIMAL:
    print(f"min. makespan = {solver.Objective().Value():.2f}")

    opt_time = []
    for machine in range(num_machines):
        arr = []
        for job in range(num_jobs):
            for t in range(t):
                if int(var_x[machine][job][t].solution_value()) == 1:
                    arr.append(t)
        opt_time.append(arr)

show_schedule(num_jobs, operations, processing_times, opt_time)
```

```
min. makespan = 55.00
```



3. Rank-based model

5 Column Generation

Column generation is a technique used in linear programming to solve problems with a large number of variables. Some classical/typical applications of column generation include:

Transportation and distribution problems: Column generation can be used to optimize transportation and distribution networks by determining the most efficient routes for goods and services.

Crew scheduling: Column generation is useful in determining optimal crew scheduling for airlines, railways, and other transportation companies.

Cutting stock problems: In the manufacturing industry, column generation can be used to optimize cutting stock problems by finding the best way to cut raw materials into smaller pieces to minimize waste.

Network design: Column generation can be applied to network design problems, such as determining the optimal location of facilities in a supply chain network.

Vehicle routing: Column generation can be used to optimize vehicle routing problems, such as determining the best routes for delivery trucks or garbage trucks.

Resource allocation: Column generation can also be applied to resource allocation problems, such as scheduling employees or assigning tasks to machines in a production facility.

Overall, column generation is a powerful technique that can be applied to a wide range of optimization problems.

6 Summary

In summary, this book has no content whatsoever.

References

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