1.

(a)

$$y = \sigma(x) = \frac{1}{1+e^{-x}}$$

$$\frac{\partial \sigma(x)}{\partial x} = \frac{\partial\left(\frac{1}{1+e^{-x}}\right)}{\partial x}$$

$$= \frac{-1}{(1+e^{-x})^2} \cdot (-e^{-x})$$

$$= \frac{e^{-x}}{(1+e^{-x})^2}$$

$$= \frac{A}{(1+e^{-x})} + \frac{B}{(1+e^{-x})^2}$$

$$A + Ae^{-x} + B = e^{-x}$$

$$\begin{cases} A = 1 \\ A + B = 0 \end{cases} \Rightarrow A = 1 \quad B = -1$$

$$\frac{\partial \sigma(x)}{\partial x} = \frac{1}{1+e^{-x}} - \frac{1}{(1+e^{-x})^2}$$

$$= \sigma(x) - \sigma(x)^2$$

$$= \sigma(x)\,(1 - \sigma(x))$$ ，得證 #

(b)

$$\sigma(-x) = \frac{1}{1+e^{x}}$$

$$= \frac{1e^{-x}}{(1+e^{x})\,e^{-x}}$$

$$= \frac{e^{-x}}{e^{-x}+1}$$

$$= \frac{1+e^{-x}}{1+e^{-x}} - \frac{1}{1+e^{-x}}$$

$$= 1 - \frac{1}{1+e^{-x}}$$

$$= 1 - \sigma(x)$$ ，得證 #

(c)

$$y = \sigma(x) = \frac{1}{1+e^{-x}}$$

$$y + ye^{-x} = 1$$

$$1 + e^{-x} = \frac{1}{y}$$

$$e^{-x} = \frac{1}{y} - 1$$

$$e^{-x} = \frac{1-y}{y}$$

$$\ln(e^{-x}) = \ln\left(\frac{1-y}{y}\right)$$

$$-x = \ln\left(\frac{1-y}{y}\right)$$

$$x = \ln\left(\frac{y}{1-y}\right)$$. 得證 #

2.

$$\nabla L(w) = -\sum_{n=1}^{N} \left\{ y_n \frac{\partial}{\partial w} \ln(\sigma(w^T\phi_n)) + (1-y_n)\frac{\partial}{\partial w}\ln(1-\sigma(w^T\phi_n)) \right.$$

令 $x = w^T\phi_n$，使用 chain rule

$$= -\sum_{n=1}^{N} \left\{ y_n \frac{\partial(\ln(\sigma(x)))}{\partial x}\frac{\partial x}{\partial w} + (1-y_n)\frac{\partial}{\partial x}\ln(1-\sigma(x))\frac{\partial x}{\partial w} \right.$$

$$= -\sum_{n=1}^{N} \left\{ y_n \underbrace{\frac{1}{\sigma(x)}\sigma(x)(1-\sigma(x))}_{\text{根據 1.(a) 的證明}}\frac{\partial x}{\partial w} - (1-y_n)\frac{1}{1-\sigma(x)} \right.$$

$$\left. \times (1-\sigma(x))\sigma(x)\frac{\partial x}{\partial w} \right\}$$

$$= -\sum_{n=1}^{N} \left\{ y_n(1-\sigma(x)) - (1-y_n)\sigma(x) \right\} \frac{\partial x}{\partial w}$$

$$= -\sum_{n=1}^{N} \left\{ y_n - \sigma(x)y_n - \sigma(x) + y_n\sigma(x) \right\} \frac{\partial x}{\partial w}$$

$$= -\sum_{n=1}^{N} \left\{ y_n - \sigma(x) \right\} \frac{\partial x}{\partial w}$$

$$= -\sum_{n=1}^{N} \left\{ y_n - \sigma(w^T\phi_n) \right\} \frac{\partial w^T\phi_n}{\partial w}$$

$$= -\sum_{n=1}^{N} \left\{ y_n - \hat{y}_n \right\} \phi_n = \sum_{n=1}^{N} (\hat{y}_n - y_n)\phi_n$$ ，得證

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3.

(a)

使 $x_n = \frac{y_n + 1}{2}$，$\hat{x}_n = \frac{\hat{y}_n + 1}{2}$ 將 range 轉回 $\{0,1\}$

$$L = -\sum_{n=1}^{N} \{ x_n \ln(\hat{x}_n) + (1-x_n)\ln(1-\hat{x}_n) \}$$

$$= -\sum_{n=1}^{N} \left\{ \frac{y_n+1}{2} \ln\left(\frac{\hat{y}_n+1}{2}\right) + \left(1-\frac{y_n+1}{2}\right)\ln\left(1-\frac{\hat{y}_n+1}{2}\right) \right\}$$

$$= -\sum_{n=1}^{N} \left\{ \frac{y_n+1}{2} \ln\left(\frac{\hat{y}_n+1}{2}\right) + \frac{1-y_n}{2}\ln\left(\frac{1-\hat{y}_n}{2}\right) \right\}$$

$$\propto -\sum_{n=1}^{N} \{ (y_n+1)(\ln(\hat{y}_n+1) - \ln 2) + (1-y_n)(\ln(1-\hat{y}_n) - \ln 2) \}$$

∵ $\frac{1}{2}$ 只是一個倍率關係，像 learning rate 一樣

$$= -\sum_{n=1}^{N} \{ (y_n+1)\ln(\hat{y}_n+1) + (1-y_n)\ln(1-\hat{y}_n) - 2\ln 2 \}$$

$$\approx -\sum_{n=1}^{N} \{ (y_n+1)\ln(\hat{y}_n+1) + (1-y_n)\ln(1-\hat{y}_n) \}$$

∵ $2\ln 2$ 是常數項，在求梯度時就會為 0，故可消除 #

(b)

原本 range 為 {0, 1} 的 $\hat{y_n}$

$$\hat{y_n} \longrightarrow \left(\hat{y_n} - \frac{1}{2}\right) \cdot 2$$

修正後 range 為 {-1, 1} 的 $\hat{y_n}$

$$\Rightarrow \frac{2}{1+e^{-x}} - 1$$

$$\Rightarrow \frac{2-(1+e^{-x})}{1+e^{-x}}$$

$$\Rightarrow \frac{1-e^{-x}}{1+e^{-x}}$$

$$\Rightarrow \tanh\left(\frac{x}{2}\right)$$

$$\Rightarrow \tanh\left(\frac{w^T \phi_n}{2}\right) \quad \#$$