Visualizing the equilibrium manifold*

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Abstract

The characterisation of the equilibrium manifold using a differentiable approach is a major contribution in economic theory and mathematical economics. While there is a detailed characterisation of the equilibrium manifold, there are no worked example to illustrate the various properties that have been derived. This paper presents several examples, one with a globally unique equilibrium, and three where there can be multiplicity of equilibria. These examples help visualize a concept which is well understood but has so far been a chimera.

Keywords: Equilibrium manifold, Regular Economies, No-Trade Equilibira, Multiplicity, Transfer Paradox.

JEL Classification: *D50, D51, C62, C63.*

*This paper has been inspired by learning the differentiable approach from Yves Balasko, and conversations with Dave Cass, Christian Ghilglino, Atshushi Kajii, Andrea Loi, Stefano Matta, Christophe Prechac, Karl Shell, Steve Spear, and Mich Tvede over many years. The usual disclaimer applies.

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I. INTRODUCTION

Establishing the existence of competitive equilibrium and the two theorems of welfare were one of the peaks of economic theory. However, the promise of using the model to study applied problems was moderated by a series of examples: Scarf's example of a unique equilibrium that is not locally stable in tatonnement dynamics (Scarf, 1960), uniqueness of equilibrium requireing very strong restrictions on demand (see Arrow (1971)), the equilibrium set of prices can be an arbitrary compact set (Mas-Colell, 1977), redistributions of endowments can lead to utility reversals (Gale, 1974; Aumann and Peleg, 1974). The question that whether these examples are pathological or not, was addressed in the study of the equilibrium manifold, largely developed in this journal.

The elegant theory shows that the set of equilibrium prices and endowments such that the markets clear is a smooth manifold diffeomorphic to the Euclidean space when preferences can be represented by smooth and well-behaved utility functions. The set of endowments can also be classified into regular, $\mathcal{R}$, and singular economies, Σ, and the structure of these economies is well understood.[1] Issues of number of equilibria, continuity, stability, transfer paradox, etc. are related to the classification of the economy and the index at an equilibrium for a regular economy. New insights about paths on the equilibrium manifold have emerged using using differential geometry.[2]

While the theory is very well developed and we have a clear understanding of the equilibrium manifold and the set of economies, parametrized by endowments,[3] its use by economists for comparative statics and policy design has been limited.[4] As far as we are aware, there are no computed examples that illustrate the equilibrium manifold. All we have are heuristic diagrams and without clear visualisation of it can be confusing.[5]

This paper presents the equilibrium manifold for several well-known examples such as the log-linear utility where the equilibrium is always unique, and those that generate multiple equilibria: the example in Mas-Colell, Whinston and Green (1995) for quasi-linear utility, the example due to Shapley and Shubik (1977) for exponential utility, the example of Bergstrom, Shimomura and Yamato (2009) of quadratic utility. We compute the equilibrium manifold for these four economies. This provides a visualization of the properties that are known to hold.

[1] The results are summarized in four monographs Balasko (2009, 2011, 2016); Mas-Colell (1985).
[2] For example, Loi and Matta (2008, 2021)
[3] The monograph by Mas-Colell (1985) also studies perturbations of preferences.
[4] See Garratt and Goenka (1995) for an application to tax policy
[5] For example, suggesting that the set of singular economies intersects the set of no-trade endowments which cannot be the case.

II. NOTATION

As we want to map the equilibrium manifold and set of economies we restrict attention to 2×2 economies. The subscript i indicates a consumer, $i = 1, 2$ and the superscript, $j \in \{1, 2\}$ denotes a commodity. Thus, x_i^j is consumption of commodity j by consumer i and $x_i = (x_i^1, x_i^2),\ i = 1, 2$. Similarly, endowments are denoted by $\omega_i = (\omega_i^1, \omega_i^2),\ i = 1, 2$. As we hold the preferences of consumers fixed an economy is determined by the endowments $\Omega = (\omega_1, \omega_2)$. The price of a good is denoted by $p^j > 0$. We normalize the price of a good (the second good) $= 1$, so that the set of prices $S = \{p^1 : p^1 > 0\}$. The set of endowments, or the set of economies, $\Omega = \{\omega_1, \omega_2 : \omega_i >> 0\}$.[6] Given the price vector and endowments consumers maximize a utility function $u_i : \mathbb{R}^2 \to \mathbb{R}$ which is $\mathcal{C}^\infty$, satisfy $Du_i > 0$, smooth quasi-concavity, and the indifference surfaces are closed.. In this case, the demand function: $f_i(p, \omega_i)$ is a smooth diffeomorphism. All the information is summarized in the excess demand functions $z_i(p, \omega_i) = f_i(p, \omega_i) - \omega_i$ of consumers that can be further summarized by the aggregate excess demand: $z(p, \omega) = \sum_i f_i(p, p \cdot \omega_i) - \sum_i \omega_i$. An *Equilibrium Price Vector* for the economy $\omega \in \Omega$ is a vector $p \in S$ such that $z_i(p, \omega) = 0$. The set of equilibrium price vectors associated with an economy ω is denoted as $W(\omega)$. This is non-empty for all ω from the existence theorem. It need not be single-valued. Thus, the mapping $\omega \to S$ given by $W(\omega)$ is called the Walrasian correspondence or the equilibrium set correspondence. This correspondence is upper hemicontinuous but need not be lower hemicontinuous. Debreu (1970) introducing the differentiable approach to general equilibrium theory showing that the correspondence $W(\omega)$ is lower hemicontinuous for an open set of full measure, $\mathcal{R}$ which is termed a regular economy. The complement, the set of singular economies, Σ has Lebesgue measure zero. The Equilibrium set, $\mathcal{E}$ is a closed subset of $S \times \Omega$ (Balasko, 2016) and is defined as:

$$\mathcal{E} = \{(p, \omega) \in S \times \Omega : \sum f_i(p, p \cdot \omega_i) - \sum_i \omega_i = 0.\}. \tag{1}$$

An important subset of $\mathcal{E}$ is the set of no-trade equilibria, $\mathcal{T}$ which is defined by:

$$\mathcal{T} = \{(p, \omega) \in \mathcal{E} : f_i(p, p \cdot \omega_i) = \omega_i\}. \tag{2}$$

The projection $\pi : S \times \Omega \to \Omega$ is the natural projection. An equilibrium, $(p, \omega) \in \mathcal{E}$ is critical if it is a critical point of π. The set of critical equilibria is denoted as $\mathcal{E}_c$. If the equilibrium is not critical, it is called a regular equilibrium. Going from $\mathcal{E}$ to the set of economies (endowments) we say an economy is regular (resp. singular)

[6] The theory does not place sign restrictions on the endowments, but these are required for the parametric examples in this paper.

if it is a regular value (resp. singular value) of π. The set of regular economies is $\mathcal{R}$ and set of singular economies is Σ. Note that we are following the definitions in Balasko (2016).

III. THE EQUILIBRIUM MANIFOLD

A. Log Utility

The utility function and endowments for $i = 1, 2$ are:

$$u_i = \alpha_i \log(x_i^1) + \beta_i \log(x_i^2), \quad \omega_i = (\omega_i^1, \omega_i^2) \gg 0. \tag{3}$$

The demand function obtained by maximizing Equation 3 subject to the budget constraint for $i = 1, 2$ are:

$$x_1 = \begin{cases} x_1^1 = \alpha_1(\omega_1^1 + \dfrac{1}{p}\omega_1^2) \\ x_1^2 = \beta_1(p\omega_1^1 + \omega_1^1) \end{cases}, x_2 = \begin{cases} x_2^1 = \alpha_2(\omega_2^1 + \dfrac{1}{p}\omega_2^2) \\ x_2^2 = \beta_2(p\omega_2^1 + \omega_2^2) \end{cases}. \tag{4}$$

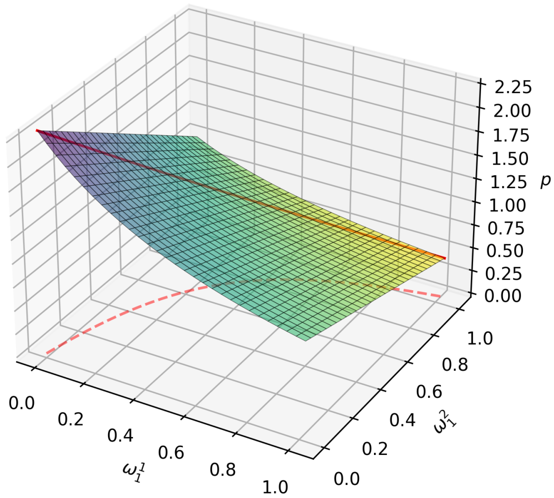
The aggregate excess demand is:

$$z_1 = \alpha_1\omega_1^1 + \alpha_2\omega_2^1 + \frac{1}{p}(\alpha_1\omega_1^2 + \alpha_2\omega_2^2) - (\omega_1^1 + \omega_2^1), \tag{5}$$

$$z_2 = \beta_1\omega_1^2 + \beta_2\omega_2^2 + p(\beta_1\omega_1^1 + \beta_2\omega_2^1) - (\omega_1^2 + \omega_2^2). \tag{6}$$

As the excess demand satisfies the gross substitute property, the equilibrium is always unique. It is, however, instructive to visualize the equilibrium manifold, $\mathcal{E}$ is shown in Figure 1 where total endowments have been restricted to be 1 for each good. The surface is the equilibrium manifold, $\mathcal{E}$ and the red line on equilibrium manifold is the set of no-trade equilibria, $\mathcal{T}$ and its projection on the Edgeworth box by the natural projection, π, is set of Pareto efficient economies. We have restricted total endowments to be equal to 1 for both goods. We can see that $\mathcal{E}$ is a smooth manifold.

Figure 1. Manifold for Log Utility



Note: This illustrates for the log-linear economy, the Equilibrium manifold, the set of no-trade equilibria (solid red line), and the projection of the no-trade equilbria to the endowment space which is the set of Pareto efficient economies.

B. *Quasi-linear utility (Mas-Colell, Whinston and Green, 1995)*

The following example is in Mas-Colell, Whinston and Green (1995) (p.521). The utility functions are given by:

$$\begin{aligned} u_1 &= x_1^1 - \frac{1}{k}(x_1^2)^{-k}, \\ u_2 &= -\frac{1}{k}(x_2^1)^{-k} + x_2^2. \end{aligned} \tag{7}$$

The utility functions are quasi-linear but for each consumer in one good and in the other good for the other consumer. These utility functions satisfy the required smoothness, monotonicity and quasi-concavity properties. We can solve for the demand functions:

$$x_1 = \begin{cases} x_1^1 = \omega_1^1 + \frac{1}{p}\omega_1^2 - \left(\frac{1}{p}\right)^{\frac{k}{k+1}} \\ x_1^2 = \left(\frac{1}{p}\right)^{-\frac{1}{k+1}} \end{cases}, \quad x_2 = \begin{cases} x_2^1 = (p)^{-\frac{1}{k+1}} \\ x_2^2 = p\omega_2^1 + \omega_2^2 - (p)^{\frac{k}{k+1}} \end{cases}. \tag{8}$$

Thus, the aggregate excess demand functions are:

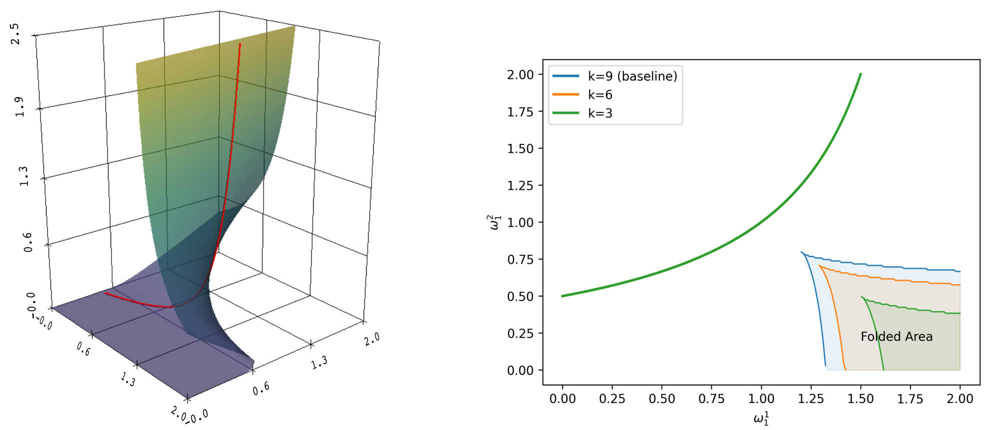
$$z_1 = \omega_1^1 + \left(\frac{1}{p}\right)\omega_1^2 - \left(\frac{1}{p}\right)^{\frac{k}{k+1}} + (p)^{-\frac{1}{k+1}} - (\omega_1^1 + \omega_2^1), \tag{9}$$

$$z_2 = \left(\frac{1}{p}\right)^{-\frac{1}{k+1}} + p\omega_2^1 + \omega_2^2 - p^{\frac{k}{k+1}} - (\omega_1^2 + \omega_2^2). \tag{10}$$

The equilibrium manifold is $\mathcal{E} = \{(p, \omega) : z_2(p, \omega) = 0\}$.

Panel (a) of Figure 2 shows the manifold for the baseline case in Mas-Colell, Whinston and Green (1995), where $k = 9$ and total endowments are equal to 2 for each good. The prices are always strictly positive as the preferences are strongly monotone. As in Mas-Colell, Whinston and Green (1995), when there are 3 equilibria, one of the equilibrium prices is the reciprocal of another one. When we vary endowments, one price becomes very large and the other one very small. In this Figure, we have truncated the high price to better represent the equilibrium manifold.

Figure 2. Equilibrium Manifold for Mas-Colell, Whinston and Green (1995)



(a) Equilibrium Manifold **(b)** Contract Curve

Notes: (i) Panel (a) illustrates the Equilibrium manifold (for total endowments equal to 2 for each good) and the set of no-trade equilibria (solid red line). Note that when there are multiple equilibria, one equilibrium price becomes large and we have truncated it to better represent the equilibrium manifold. (ii) Panel (b) shows the projection from $\mathcal{E}$ to Edgeworth box. The shaded areas are the regions with multiple equilibria and its boundary are the singular economies, Σ. The contract curve is the projection from the set of no-trade equilibria, $\mathcal{T}$. We vary the parameter k in the utility functions.

Panel (b) of Figure 2 shows the contract curve and the set of economies with multiple equilibria that is the folded area under different parameters (The projection from the $\mathcal{E}$ to the set of endowments). The shaded area in this Panel is the set of economies with multiple equilibria. The economies on the boundary of this set are singular economies, Σ, and those in the interior are regular economies. As the parameter k increases, the set of economies with multiple equilibria increases. In this example, this is due to the increasing income effect as k increases. (See Appendix A for details).

Looking at the these two figures, we see that the example of Mas-Colell, Whinston and Green (1995) is robust to perturbations of the endowments as well as the preferences. Thus, the restriction on endowments in their text is only necessary for computational ease. For the regular economies, the number of equilibria is odd (at most 3). The set of singular values, Σ, has measure zero in the Ω. This is the boundary of the shaded area shown in Figure 3 for the three different parametric values of k. The set of regular economies $\mathcal{R}$ is open in Ω, and for every $\omega \in R$, the set of equilibrium prices is odd and finite (at most 3). The 'stack of records' theorem holds (see Propostion 2.6.4. in Balasko (2009)). If we take a regular economy, $\omega \in \mathcal{R}$, then there exists an open neighbourhood $U \subset \mathcal{R}$ of ω such that the number of equilibria is constant in U. The number of equilibria is constant in a connected component of $\mathcal{R}$, and $\mathcal{R}$ is open and with full measure in Ω. The set of singular economies is arc-connected (Balasko 1978a). The example also illustrates that the measure of economies with multiple equilibria is decreasing in the number of equilibria. The projection of the no-trade equilibria to the Ω is the set of Pareto-efficent allocations. We know that there is a transfer paradox for a regular economy if and only if the index is (-1) (Balasko, 1978, 2014). We can see that the area where this can take place is in the interior of the shaded area in Panel (b) of Figure 2. Thus, this area is increasing in k or when the income effects become larger.

C. Exponential utility (Shapley and Shubik, 1977)

The utility functions of the two consumers are:

$$\begin{aligned} u_1 &= x_1^1 + k_1(1 - e^{-\frac{x_1^2}{\theta}}), \\ u_2 &= x_2^1 + k_2(1 - e^{-\frac{x_2^2}{\theta}}). \end{aligned} \tag{11}$$

The demand functions are:

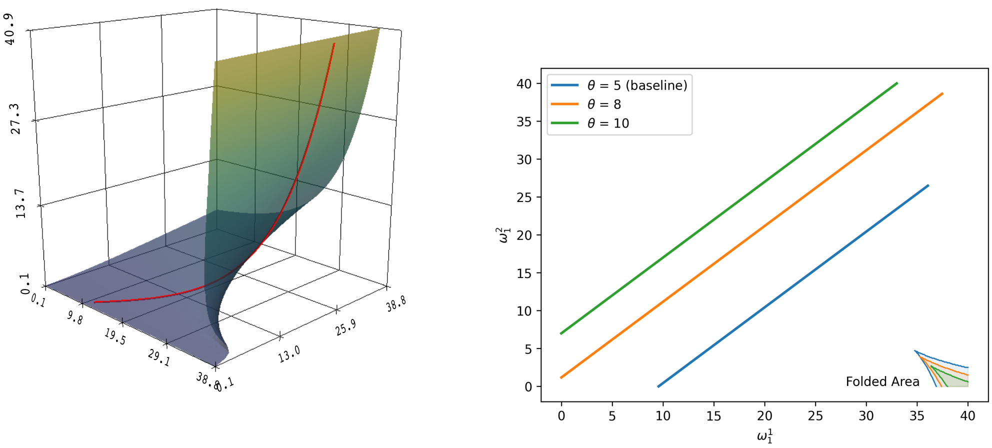
$$\begin{aligned} x_1 &= \begin{cases} x_1^1 = \omega_1^1 + \left(\frac{1}{p}\right)\omega_1^2 - \left(\frac{1}{p}\right) \cdot \theta \left(\log(\frac{k_1}{\theta}) - \log(\frac{1}{p})\right) \\ x_1^2 = \theta \left(\log(\frac{k_1}{\theta}) - \log(\frac{1}{p})\right) \end{cases}, \\ x_2 &= \begin{cases} x_2^1 = \theta \left(\log(\frac{k_2}{\theta}) - \log(p)\right) \\ x_2^2 = p\omega_2^1 + \omega_2^2 - p \cdot \theta \left(\log(\frac{k_2}{\theta}) - \log(p)\right) \end{cases}. \end{aligned}$$

The aggregate excess demand functions are:

$$\begin{aligned} z_1 =& \omega_1^1 + \left(\frac{1}{p}\right)\omega_1^2 - \left(\frac{1}{p}\right)\cdot\theta\left(\log(\frac{k_1}{\theta}) - \log(\frac{1}{p})\right) + \theta\left(\log(\frac{k_2}{\theta}) - \log(p)\right) \\ & - (\omega_1^1 + \omega_2^1), \\ z_2 =& \theta\left(\log(\frac{k_1}{\theta}) - \log(\frac{1}{p})\right) + p\omega_2^1 + \omega_2^2 - p\cdot\theta\left(\log(\frac{k_2}{\theta}) - \log(p)\right) \\ & - (\omega_1^2 + \omega_2^2). \end{aligned} \tag{12}$$

Following the same strategy as before, we can plot the equilibrium manifold under $\theta = 5$ in Panel (a) of Figure 3. The projection from the $\mathcal{E}$ to the set of endowments is shown in the Panel (b) of Figure 3 where we vary the parameter θ in $\{5, 8, 10\}$.

Figure 3. Equilibrium Manifold for Shapley and Shubik (1977)



(a) Equilibrium Manifold **(b)** Contract Curve

Notes: (i) Panel (a) illustrates the Equilibrium manifold and the set of no-trade equilibria (solid red line). (ii) Panel (b) shows the projection from $\mathcal{E}$ to Edgeworth box. The shaded areas are the regions with multiple equilibria and its boundary are the singular economies, Σ. The contract curve is the projection from the set of no-trade equilibria, $\mathcal{T}$. Note, that when there are multiple equilibria, one equilibrium price becomes large and we have truncated it to better represent the equilibrium manifold. We vary the parameter θ in the utility function.

D. Quadratic Utility (Bergstrom, Shimomura and Yamato, 2009)

The utility functions of the two consumers are:

$$\begin{aligned} u_1 &= x_1^1 + \alpha x_1^2 - \frac{1}{2}(x_1^2)^2, \\ u_2 &= x_2^2 + \beta x_2^1 - \frac{1}{2\theta}(x_2^1)^2. \end{aligned} \tag{13}$$

The demand functions are:

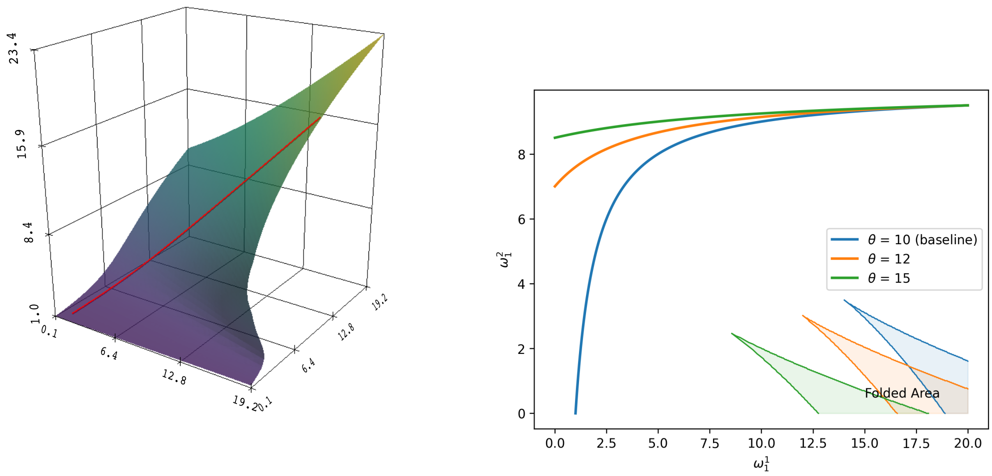
$$\begin{aligned} x_1 &= \begin{cases} x_1^1 = \omega_1^1 + \left(\frac{1}{p}\right)\omega_1^2 - \left(\frac{1}{p}\right)\left(\alpha - \frac{1}{p}\right) \\ x_1^2 = \alpha - \frac{1}{p} \end{cases}, \\ x_2 &= \begin{cases} x_2^1 = \theta(\beta - p) \\ x_2^2 = p\omega_2^1 + \omega_2^2 - \theta p\,(\beta - p) \end{cases}. \end{aligned}$$

The aggregate excess demand functions are:

$$\begin{aligned} z_1 =& \omega_1^1 + \left(\frac{1}{p}\right)\omega_1^2 - \left(\frac{1}{p}\right)\cdot\left(\alpha - \frac{1}{p}\right) + \theta\left(\beta - \frac{1}{p}\right) - (\omega_1^1 + \omega_2^1). \\ z_2 =& \alpha - p + \frac{1}{p}\omega_2^1 + \omega_2^1 - \theta p\left(\beta - \frac{1}{p}\right) - (\omega_1^2 + \omega_2^2) \end{aligned} \tag{14}$$

Following the same strategy as before, we can plot the equilibrium manifold and the projection from the equilibrium manifold in Panel (a) and (b) of Figure 4

Figure 4. Equilibrium Manifold for Bergstrom, Shimomura and Yamato (2009)



(a) Equilibrium Manifold **(b)** Contract Curve

Notes: (i) For Quadratic utilities, Panel (a) illustrates the Equilibrium manifold and the set of no-trade equilibria (solid red line). We have set $\alpha = 10, \beta = 2$, $\theta = 10$. (ii) Panel (b) shows the projection from $\mathcal{E}$ to Edgeworth box. The shaded areas are the regions with multiple equilibria and its boundary are the singular economies, Σ. The contract curve is the projection from the set of no-trade equilibria, $\mathcal{T}$. We vary the parameter θ in the utility function.

IV. CONCLUSION

This paper visualizes equilibrium manifolds for some well-known economies. With modern computing methods the equilibrium manifold can easily be computed and visualized. It is hoped that the there will be a resurgence in using a global approach to doing comparative static analysis especially for policy analysis.

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Appendices

A Derivation

The Slutsky equation for Mas-Colell, Whinston and Green (1995) (1995, p. 99) (17.E.3) is:

$$Dz(p) = \sum_i [\mathcal{S}_i - D_{y_i} x_i(p, p \cdot \omega_i) z_i(p)^T] \tag{15}$$

The $\mathcal{S}_i$ in Equation 15 is the Slutsky matrix that captures the substitutional effect which is N.S.D. The second term is the wealth effect. Without the wealth effect, excess function $Dz(p)$ inherits the N.S.D. property from $\mathcal{S}_i$ and the general equilibrium is unique. However, if wealth effect is large enough, $Dz(p)$ may not be N.S.D. any more and generates multiple equilibria.

We now show this equation in the example.

$$Dz(p) = \sum_i \{\mathcal{S}_i - \mathcal{W}_i\} \tag{16}$$

The first matrix $\mathcal{S}_i$ is exactly Slutsky matrix [7] that captures the substitution effect.

The second matrix $\mathcal{W}_i$ is exactly $D_{y_i} x_i z_i^T$ in Equation 15. It captures the wealth effect.

The Slutsky matrix for 1 is

$$\mathcal{S}_1 = \; = \frac{1}{k+1} \begin{bmatrix} -p_1^{\frac{-2k-1}{k+1}} p_2^{\frac{k}{k+1}} & \left(\frac{p_2}{p_1}\right)^{-\frac{1}{k+1}} \frac{1}{p_1} \\ \left(\frac{p_1}{p_2}\right)^{-\frac{k}{k+1}} \frac{1}{p_2} & -\left(\frac{p_2}{p_1}\right)^{\frac{-2-k}{k+1}} \frac{1}{p_1} \end{bmatrix} \tag{19}$$

Similarly, for consumer 2, we have a similar matrix

$$\mathcal{S}_2 = \frac{1}{k+1} \begin{bmatrix} -\left(\frac{p_1}{p_2}\right)^{\frac{-2-k}{k+1}} \frac{1}{p_2} & \left(\frac{p_2}{p_1}\right)^{-\frac{k}{k+1}} \frac{1}{p_1} \\ \left(\frac{p_1}{p_2}\right)^{-\frac{1}{k+1}} \frac{1}{p_2} & -p_2^{\frac{-2k-1}{k+1}} p_1^{\frac{k}{k+1}} \end{bmatrix} \tag{20}$$

[7] In the 2 by 2 example, the Slutsky matrix reads

$$\mathcal{S}_i = \begin{bmatrix} s_{11} & s_{12} \\ s_{21} & s_{22} \end{bmatrix} \quad i \in \{a, b\} \tag{17}$$

where

$$s_{jk} = \frac{\partial x_{ij}}{\partial p_k} + \frac{\partial x_{ij}}{\partial y_i} x_{ik} \quad i = \{1, 2\} \tag{18}$$

and $y_i = p_1\omega_i^1 + p_2\omega_i^2$ is the income for consumer i.

Observe the expression for $\mathcal{S}1$ and $\mathcal{S}_2$, they are both N.S.D.. When k gets larger, the aggregate substitutional effect $\mathcal{S}_1 + \mathcal{S}_2$ is weaker.

Now lets go to the wealth effect $\mathcal{W}_i$. For consumer 1

$$\begin{aligned}\mathcal{W}_1 = D_{y_1} x_1 z_1^T &= \begin{bmatrix} \frac{\partial x_1^1}{\partial y_1} \\ \frac{\partial x_1^2}{\partial y_1} \end{bmatrix} \begin{bmatrix} z_1^1 & z_1^2 \end{bmatrix} \\ &= \begin{bmatrix} p_1^{-1} \\ 0 \end{bmatrix} \begin{bmatrix} z_1^1 & z_1^2 \end{bmatrix} \\ &= \begin{bmatrix} p_1^{-1} z_1^1 & p_1^{-1} z_1^2 \\ 0 & 0 \end{bmatrix}. \end{aligned} \tag{21}$$

For consumer 2

$$\begin{aligned}\mathcal{W}_2 = D_{y_2} x_2 z_2^T &= \begin{bmatrix} \frac{\partial x_2^1}{\partial y_2} \\ \frac{\partial x_2^2}{\partial y_2} \end{bmatrix} \begin{bmatrix} z_2^1 & z_2^1 \end{bmatrix} \\ &= \begin{bmatrix} 0 \\ p_2^{-1} \end{bmatrix} \begin{bmatrix} z_2^1 & z_2^2 \end{bmatrix} \\ &= \begin{bmatrix} 0 & 0 \\ p_2^{-1} z_2^1 & p_2^{-1} z_2^2 \end{bmatrix} \end{aligned} \tag{22}$$

Thus, the aggregate wealth effect

$$\mathcal{W}_a + \mathcal{W}_b = \begin{bmatrix} p_1^{-1} z_{a1} & p_1^{-1} z_{a2} \\ p_2^{-1} z_{b1} & p_2^{-1} z_{b2} \end{bmatrix} \tag{23}$$

is Full Rank away from the no-trade equilibra.

In summary, we have

$$Dz(p) = (\mathcal{S}_a + \mathcal{S}_b) - (\mathcal{W}_a + \mathcal{W}_b) \tag{24}$$

In this expression, $\mathcal{S}_a + \mathcal{S}_b$ is N.S.D.. At the no-trade equilibria, the wealth effect is 0. Then the $Dz(p)$ inherits the N.S.D. property from the Slutsky matrices. However, as the endowment getting away from the no-trade equilibra, the matrix for the wealth effect is full rank and eventually its effect dominate. Note that in this example, the income effect is increasing in k while the substitution effect is decreasing in it.

B Computation

Here we describe the computation and visualization method. The basic idea and outline for the algorithm are shown below

- Since the excess demand function is continuous and smooth, the Newton method is valid to obtain the solution $z(p, \omega) = 0$ precisely.
- However, different initial guess could lead to different solution as $z(p, \omega)$ is not monotonic. Thus, We obtain the multiple solutions by identifying different proper initial guess.
  - Define the continuous observation for price $\mathcal{P}$.
  - Graph the excess demand function $z(p|\omega)$ for $p \in \mathcal{P}$.
  - Pick up the guess p_k where $z(p_k|\omega)$ is close to 0.
  - Newton Method to solve the precise solution using these guesses.

We demonstrate the full algorithm is shown in Table B1. The corresponding point cloud sets for the three examples with multiple equilibria are given in Figure B1, Figure B2 and Figure B3.

Table B1. Full Algorithm

Algorithm
Request
1. Endowment grid $\omega_1^1 \in \mathcal{W}^1, \omega_1^2 \in \mathcal{W}^2$, $\omega_i = (\omega_i^1, \omega_i^2) \in \mathcal{W}^1 \times \mathcal{W}^2$
2. price grid $\mathcal{P} = \{p_1, p_2, ...p_N\}$ with $p_k > 0, p_k < p_{k+1}$.
3. excess demand function $z^j(\omega, p)$
4. empty set for Point Cloud E
1. **for** $\omega_1 \in \mathcal{W}^1 \times \mathcal{W}^2$, (parallel) **do**:
1. calculate $z_k^1 = z^1(p_k \vert \omega_1)$ with $p_k \in \mathcal{P}$.
2. identify set $\mathcal{K} = \{k : z_k^1 \times z_{k+1}^1 \leq 0\}$
3. if $\#\mathcal{K} > 1$, endowment ω_1 could present multiple equilibria
4. **for** $k^* \in \mathcal{K}$, (parallel) **do**:
1. Newton method to solve $z^1(p \vert \omega_1) = 0$ using k as the initial guess
2. **store** the combinition of endowment and solution (ω_1, p_{ω_1}) into E
2. **return** Point Cloud Set $E = \{(\omega_1, p_{\omega_1}) : \omega_1 \in \mathcal{W}^1 \times \mathcal{W}^2, z^1(\omega_1, p_{\omega_1}) = 0\}$
3. visualize the manifold
1. Delaunay triangulation to construct triangle mesh from point cloud E
2. Pyvista to plot the triangle mesh

Note: Pyvista: https://pyvista.org/index.html

Figure B1. Point cloud set for the example in Mas-Colell, Whinston and Green (1995)

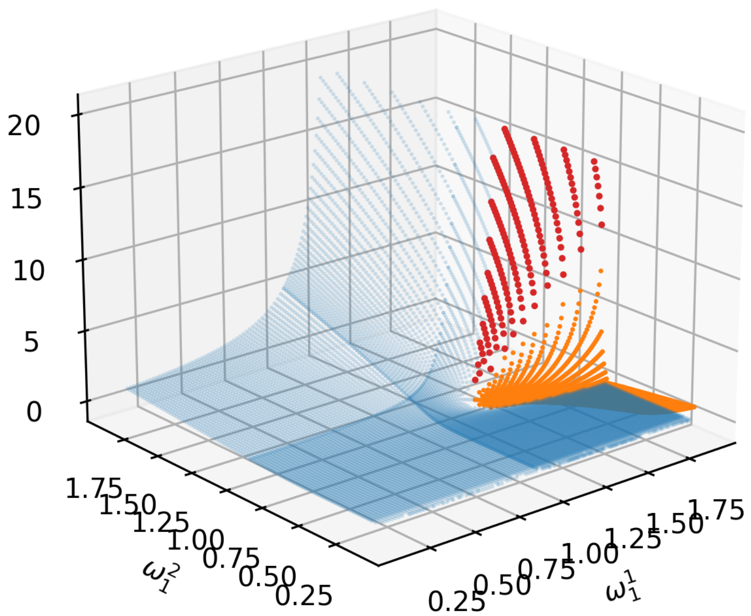

Figure B2. Point cloud set for the example in Shapley and Shubik (1977)

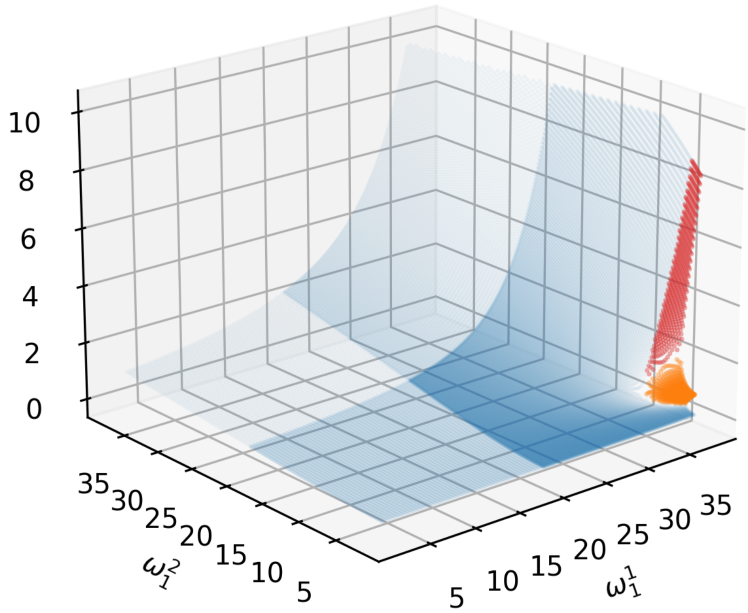

Figure B3. Point cloud set for the quadratic example in Bergstrom, Shimomura and Yamato (2009)