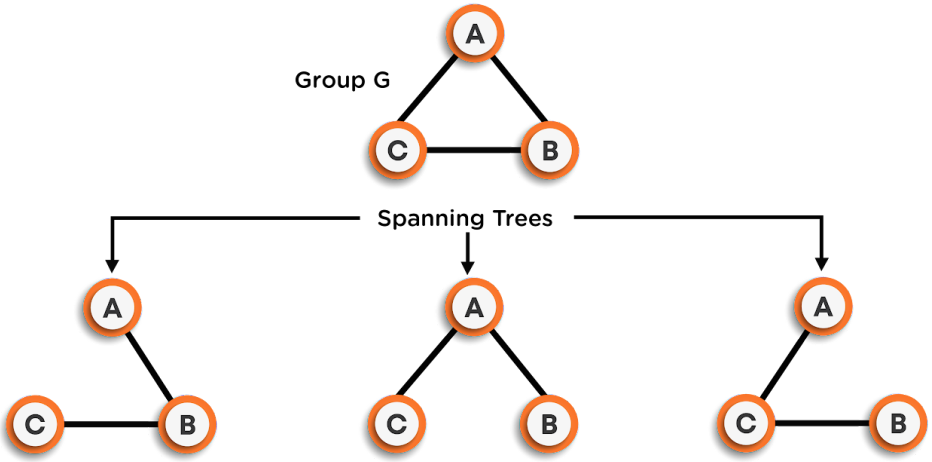
Minimum Spanning Tree(MST)

A tree is a graph, which:

- Is always connected.
- Contains no cycle.

If we are given an undirected and connected graph, a **spanning tree** is a tree that contains all the vertices(V) of the graph and |V|-1 edges. For a given graph, we can have multiple spanning trees.

Refer to the example below for a better understanding of spanning trees.



If there are *n* vertices and *e* edges in the graph, then any spanning tree corresponding to that graph contains *n* vertices and *n-1* edges.

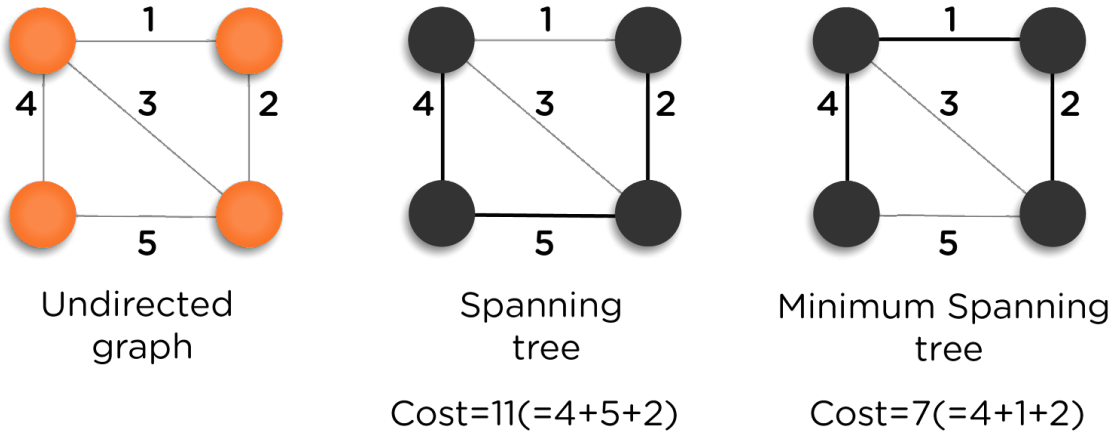
Properties of spanning trees:

- A connected and undirected graph can have more than one spanning tree.
- The spanning tree is free of loops, i.e., it is acyclic.
- Removing any one of the edges will make the graph disconnected.

- Adding an extra edge to the spanning tree will create a loop in the graph.

Minimum Spanning Tree(MST)

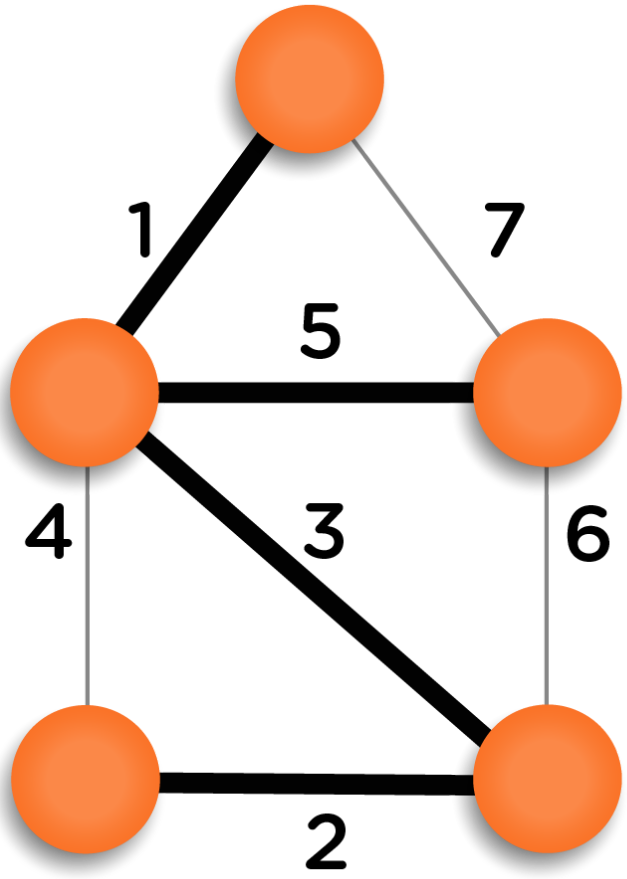
In a weighted graph, the MST is a spanning tree with minimum weight than all other spanning trees of that graph. Refer to the image below for better understanding, where the edges marked in bold represent the edges of the spanning tree

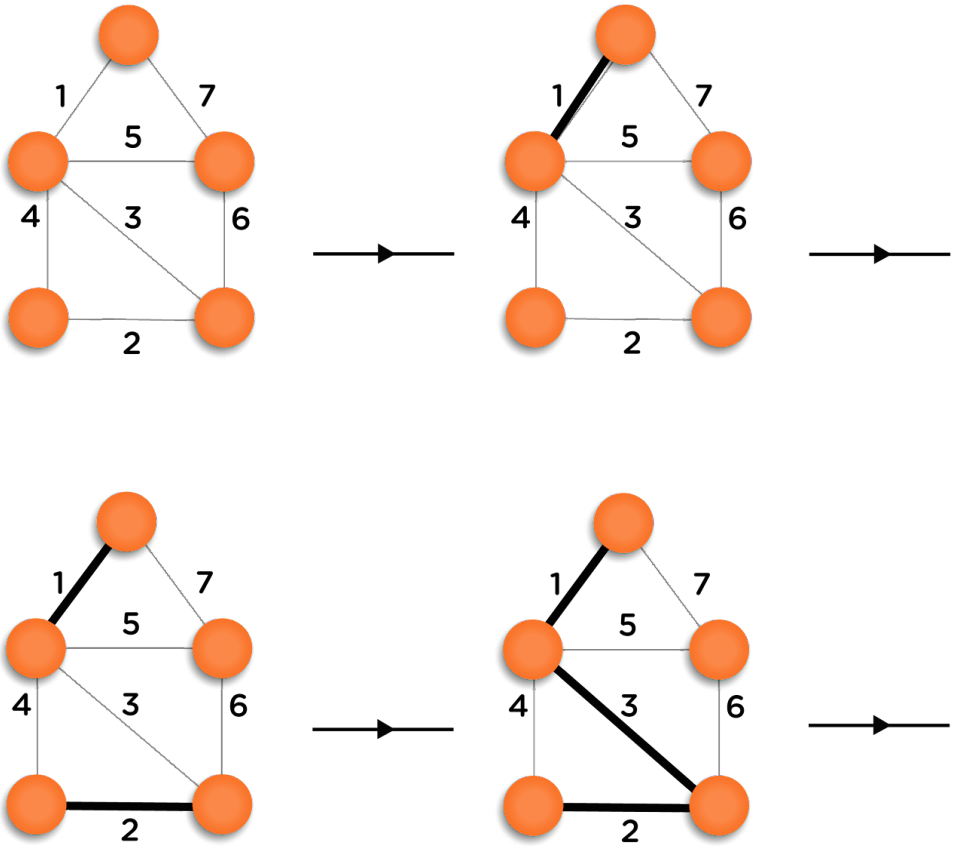


Kruskal's Algorithm:

This algorithm is used to find MST for the given graph. It builds the spanning tree by adding edges one at a time. We start by picking the edge with minimum weight, adding that edge into the MST, and increasing the count of edges in the spanning tree by one. Now, we will be picking the minimum weighted edge by excluding the already chosen ones and correspondingly increasing the count. While choosing the edge, we will also make sure that the graph remains acyclic after including the same. This process will continue until the count of edges in the MST reaches *n-1*. Ultimately, the graph obtained will be MST.

Refer to the example below for a better understanding of the same.


1
7
5
4
3
6
2



This is the final MST obtained using Kruskal's algorithm. It can be checked manually that the final graph is the MST for the given graph.

Cycle Detection

While inserting a new edge in the MST, we have to check if introducing that edge makes the MST cyclic or not. If not, then we can include that edge, otherwise not.

Now, let's figure out a way to detect the cycle in a graph. The following are the possible cases:

- By including an edge between nodes A and B, if both nodes A and B are not present in the graph, then it is safe to include that edge as including it, will not bring a cycle to the graph.
- Out of two vertices, if any one of them has not been visited (or not present in the MST), then that vertex can also be included in the MST.
- If both the vertices are already present in the graph, they can introduce a cycle in the MST. It means we can't use this method to detect the presence of the cycle.

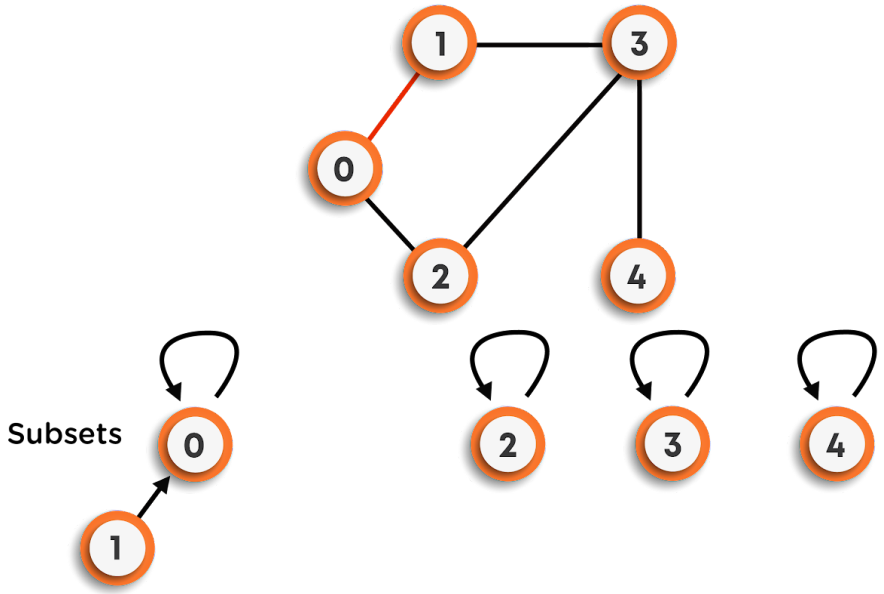
Union-Find Algorithm:

Before adding any edge to the graph, we will check if the two vertices of the edge lie in the same component of MST or not, in other words, we are checking whether the addition of an edge will lead to a cycle or not. If not, then it is safe to add that edge to the MST.

Following the steps of the algorithm:

- We will assume that initially, the total number of disjoint sets is equal to the number of vertices in the graph starting from 0 to n-1.
- We will maintain a parent array specifying the parent vertex of each of the vertex of the graph. Initially, as each vertex belongs to a different disjoint set (connected component), hence each vertex will be its parent.
- Now, before inserting any edge into the MST, we will check the parent of the vertices. If their parent vertices are equal, they belong to the same connected component; hence it is unsafe to add that edge.
- Otherwise, we can add that edge into the MST, and simultaneously update the parent array so that they belong to the same component(Refer to the code on how to do so).

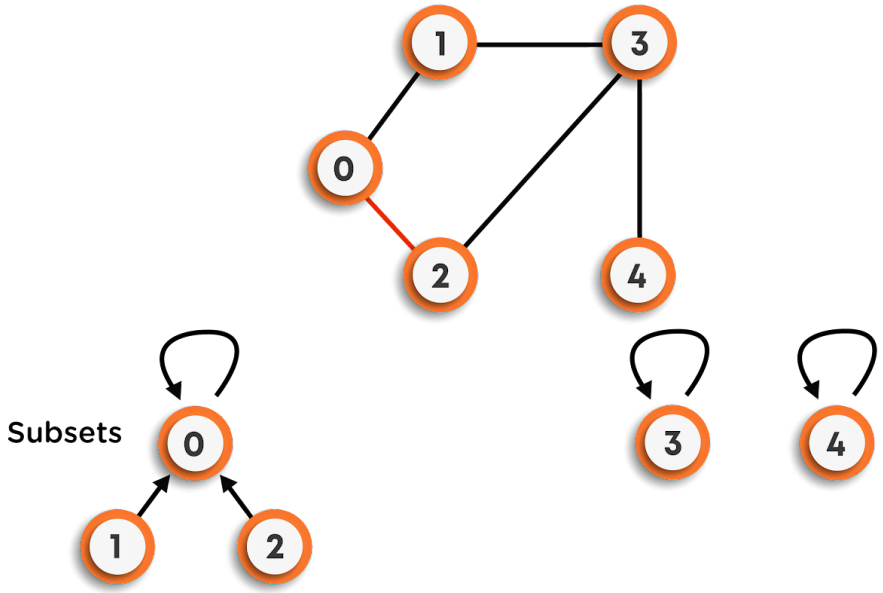
Look at the following example, for better understanding:



Edge 0-1

Find: 0 belongs to subset 0 and 1 belongs to subset 1 so they are in different subsets.

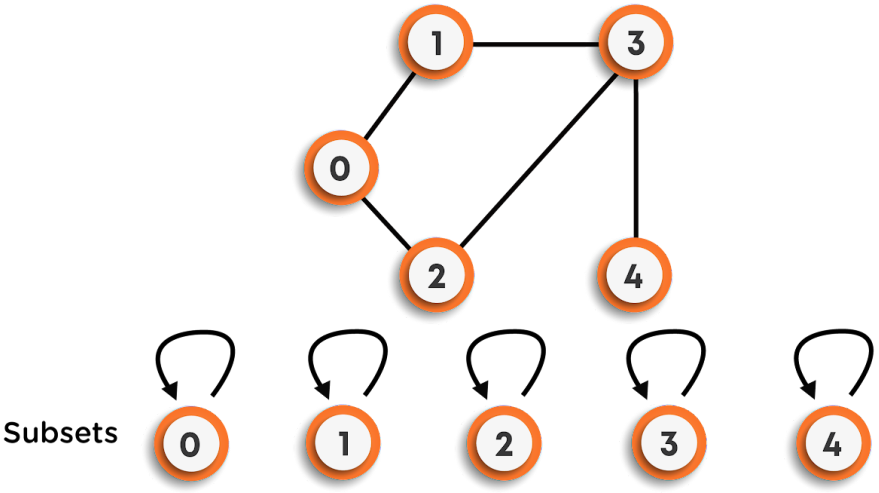
Union: Make 0 as the parent of 1, Updated set is {0,1}. 0 is the set representative since 0 is parent for itself.



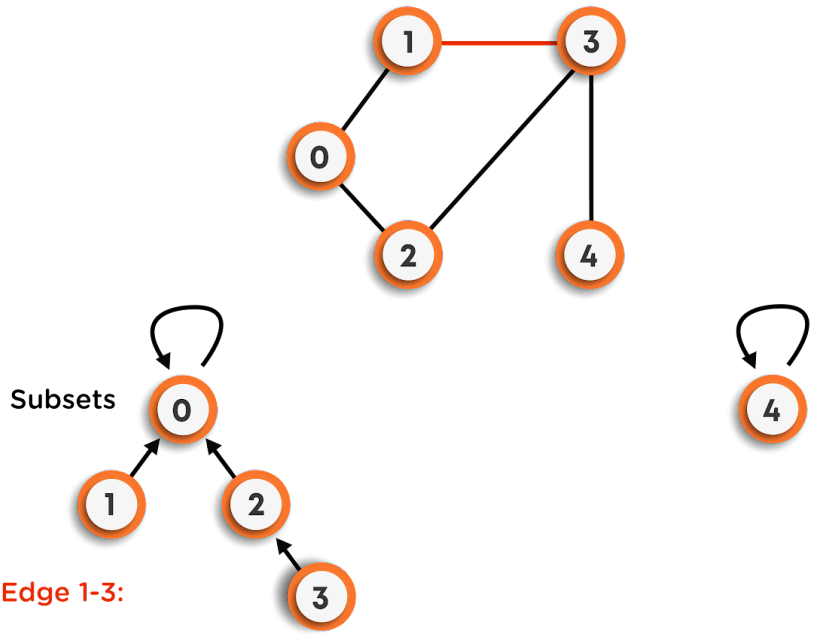
Edge 0-2:

Find: 0 belongs to subset 0 and 2 belongs to subset 2 so they are in different subsets.

Union: Make 0 as the parent of 2, Updated set is {0,1,2}. 0 is the set representative since 0 is parent for itself.



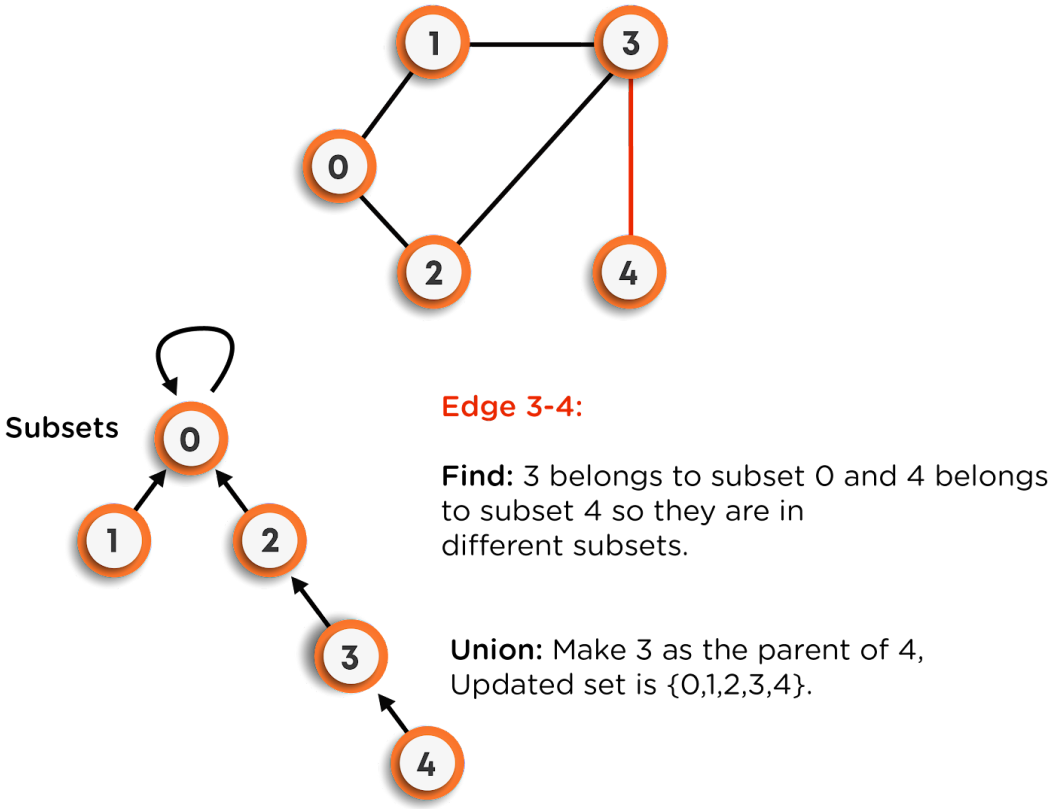
Initially all parent pointers are pointing to self means only one element in each subset



Edge 1-3:

Find: 1 belongs to subset 0 and 3 belongs to subset 3 so they are in different subsets.

Union: Make 1 as the parent of 3, Updated set is {0,1,2,3}. 0 is the set representative since 0 is parent for itself.



Note: While finding the parent of the vertex, we will be finding the topmost parent (Oldest Ancestor). For example: suppose, the vertex 0 and the vertex 1 were connected, where the parent of 0 is 1, and the parent of 1 is 1. Now, while determining the parent of the vertex 0, we will visit the parent array and check the vertex at index 0. In our case, it is 1. Now we will go to index 1 and check the parent of index 1, which is also 1. Hence, we can't go any further as the index is the parent of itself. This way, we will be determining the parent of any vertex.

The time complexity of the union-find algorithm becomes O(V) for each vertex in the worst case due to skewed-tree formation, where V is the number of vertices in the graph. Here, we can see that time complexity for cycle detection has significantly improved compared to the previous approach.

Time Complexity of Kruskal's Algorithm:

In our code, we have the following three steps: (Here, the total number of vertices is n, and the total number of edges is E)

- Take input in the array of size E.
- Sort the input array based on edge-weight. This step has the time complexity of O(E log(E)).
- Pick (n-1) edges and put them in MST one-by-one. Also, before adding the edge to the MST, we checked for cycle detection for each edge. For cycle detection, in the worst-case time complexity of E edges will be O(E.n), as discussed earlier.

Hence, the total time complexity of Kruskal's algorithm becomes O(E log(E) + n.E). This time complexity is bad and needs to be improved.

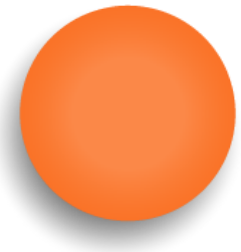
We can't reduce the time taken for sorting, but the time taken for cycle detection can be improved using another algorithm named **Union by Rank and Size, Path Compression**. The basic idea in these algorithms is that we will be avoiding the formation of skewed-tree structure, which reduces the time complexity for each vertex to **O(log(E))**.

Prim's Algorithm

This algorithm is used to find MST for a given undirected-weighted graph (which can also be achieved using Kruskal's Algorithm).

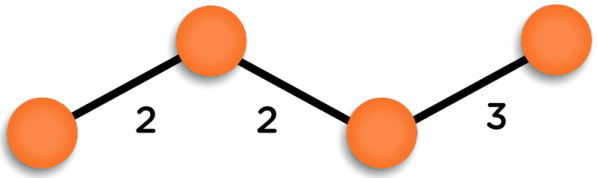
In this algorithm, the MST is built by adding one edge at a time. In the beginning, the spanning tree consists of only one vertex, which is chosen arbitrarily from the set of all vertices of the graph. Then the minimum weighted edge, outgoing from this vertex, is selected and simultaneously inserted into the MST. Now, the tree contains two edges. Further, we will be selecting the edge with the minimum weight such that one end is already present there in the MST and the other one from the unselected set of vertices. This process is repeated until we have inserted a total of (n-1) edges in the MST.

Consider the following example for a better understanding.



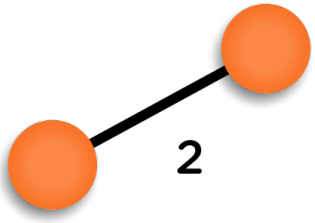
Step: 2

Choose a vertex



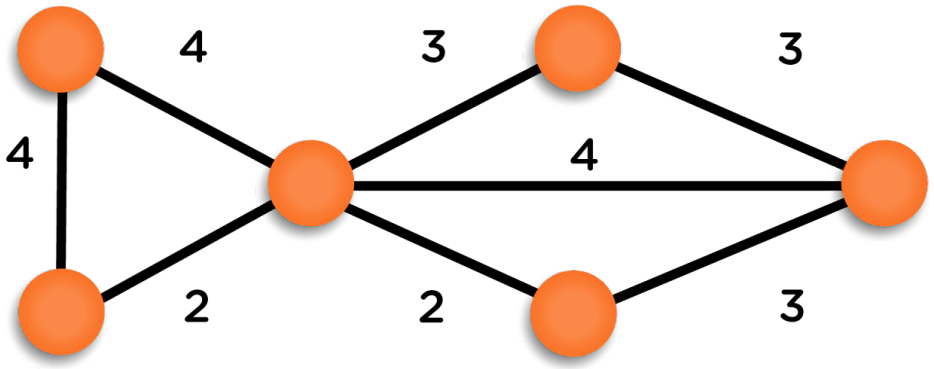
Step: 5

Choose the nearest edge not yet in the solution,
if there are multiple choices, choose one at random



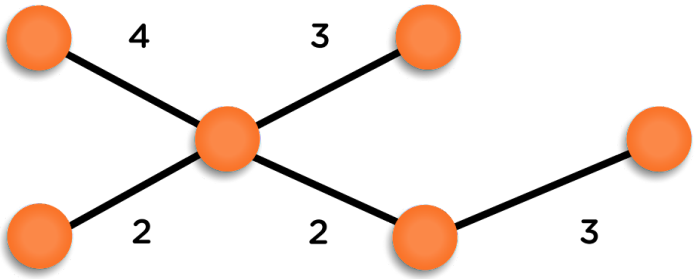
Step: 3

Choose the shortest edge from this vertex add it



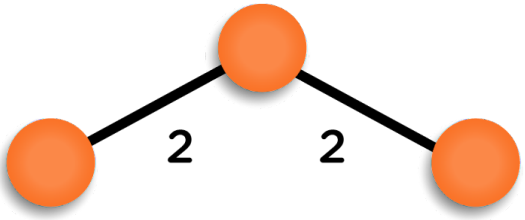
Step: 1

Start with a weighted graph



Step: 6

Repeat until you have a spanning tree



Step: 4

Choose the nearest vertex not yet in the solution

Implementation:

- We are considering the starting vertex to be 0 with a parent equal to -1, and weight is equal to 0 (The weight of the edge from vertex 0 to vertex 0 itself).
- The parent of all other vertices is assumed to be NIL, and the weight will be equal to infinity, which means that the vertex has not been visited yet.
- We will mark the vertex 0 as visited and the rest as unvisited. If we add any vertex to the MST, then that vertex will be shifted from the unvisited section to the visited section.
- Now, we will update the weights of direct neighbors of vertex 0 with the edge weights as these are smaller than infinity. We will also update the parent of these vertices and assign them 0 as we reached these vertices from vertex 0.
- This way, we will keep updating the weights and parents, according to the edge, which has the minimum weight connected to the respective vertex.

Time Complexity of Prim's Algorithm:

Here, n is the number of vertices, and E is the number of edges.

- The time complexity for finding the minimum weighted vertex is O(n) for each iteration. So for (n-1) edges, it becomes O(n2).
- Similarly, for exploring the neighbor vertices, the time taken is O(n2).

It means the time complexity of Prim's algorithm is O(n2). We can improve this in the following ways:

- For exploring neighbors, we are required to visit every vertex because of the adjacency matrix. We can improve this by using an adjacency list instead of a matrix.
- Now, the second important thing is the time taken to find the minimum weight vertex, which is also taking a time of O(n2). Here, out of the available list, we are trying to figure out the one with minimum weight. This can be optimally achieved using a **priority queue** where the priority will be taken as weights of the vertices. This will take O(log(n)) time complexity to remove a vertex from the priority queue.

These optimizations can lead us to the time complexity of **O((n+E)log(n))**, which is much better than the earlier one. Try to write the optimized code by yourself.