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Optimization for Deep Learning

Lecture 13-3: Decentralized Deterministic Optimization

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Part 1

Decentralized gradient descent (DGD)

Distributed optimization

- A network of n nodes collaborate to solve the following deterministic optimization problem:

$$\min_{x\in\mathbb{R}^d} \quad \sum_{i=1}^{n} f_i(x)$$

We assume **data heterogeneity** exists, i.e., $f_i(x) \neq f_j(x)$

- Given a connected network, decentralized gradient descent iterates as follows [NO09]

$$x_i^{(k+1)} = \sum_{j\in\mathcal{N}_i} w_{ij} x_j^{(k)} - \gamma \nabla f_i(x_i^{(k)}) \qquad \forall i \in [n] \qquad \text{(adapt-with-combine)}$$

A different adapt-then-combine version was also proposed in [LS07]

$$x_i^{(k+1)} = \sum_{j\in\mathcal{N}_i} w_{ij} \left(x_j^{(k)} - \gamma \nabla f_j(x_j^{(k)})\right) \qquad \forall i \in [n] \qquad \text{(adapt-then-combine)}$$

Decentralized gradient descent (DGD)

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- Both adapt-with-combine and adapt-then-combine versions are used a lot
- Adapt-with-combine version is easier to analyze while adapt-then-combine performs better in real implementations
- For simplicity, we mainly discuss DGD in the adapt-with-combine version
- We introduce the following notations for convenience

$$\mathbf{x} = [x_1^T; x_2^T; \cdots; x_n^T] \in \mathbb{R}^{n \times d}$$
$$\nabla \mathbf{F}(\mathbf{x}) = [\nabla f_1(x_1)^T; \nabla f_2(x_2)^T; \cdots; \nabla f_n(x_n)^T] \in \mathbb{R}^{n \times d}$$

Decentralized gradient descent (DGD)

- With the notations, DGD can be rewritten in a neat manner

$$\mathbf{x}^{(k+1)} = W\mathbf{x}^k - \gamma\nabla\mathbf{F}(\mathbf{x}^{(k)})$$

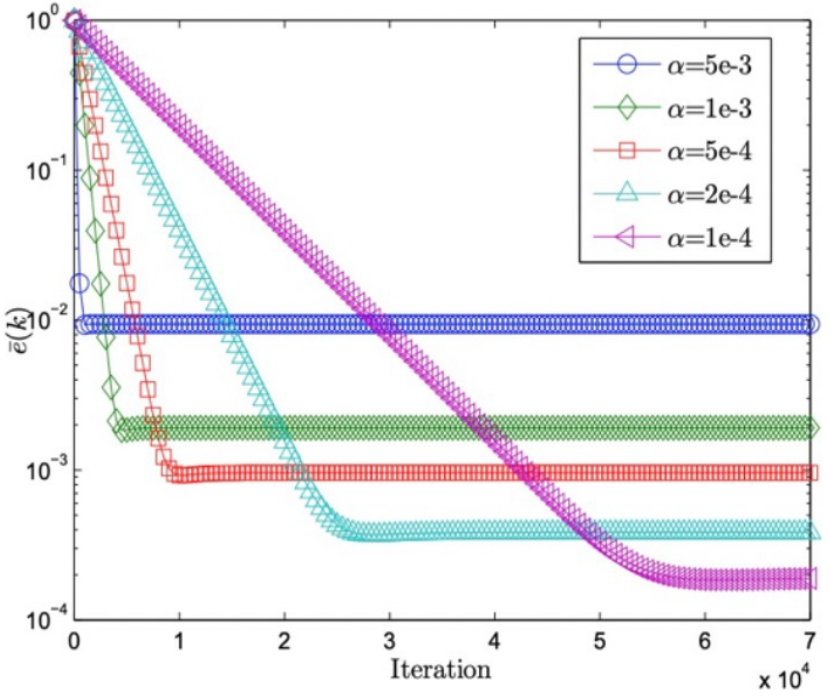
- Very intuitive, but **does not exactly converge** to the desired solution with constant learning rate

Theorem 2 [DGD Convergence] [YLY16]

Assume each $f_i(x)$ is L-smooth and μ-strongly convex. If the learning rate is set as $\gamma = O(1/L)$, DGD converges as follows

$$\frac{1}{n}\sum_{i=1}^{n}\|x_i^{(k)} - x^\star\| = O(\rho^k + \frac{\gamma b}{1-\rho})$$

where $\rho \in (0,1)$ and $b^2 = \frac{1}{n}\sum_{i=1}^{n}\|\nabla f_i(x^\star)\|^2$.



DGD is essentially a GD algorithm

- DGD was proposed by heuristics, and it cannot converge to the desired solution
- Why DGD cannot converge to the desired solution? Can we correct the bias?
- We need to treat decentralized optimization in a more "**scientific**" manner

Lemma 2

Assume matrix W is symmetric, primitive, and doubly stochastic, we have

$$Wx = x \quad \Longleftrightarrow \quad x_1 = x_2 = \cdots = x_n$$

DGD is essentially a GD algorithm

- The distributed optimization problem can be reformulated into a constrained problem

$$\min_{x\in\mathbb{R}^d} \quad \sum_{i=1}^{n} f_i(x_i) \quad \text{s.t.} \quad x_1 = x_2 = \cdots = x_n \tag{1}$$

where the constraint $x_1 = x_2 = \cdots = x_n$ is typically referred to as **consensus constraint**

- With Lemma 2, the above constrained problem is equivalent to [YLY16]

$$\min_{\mathbf{x}\in\mathbb{R}^{n\times d}} \quad \mathbf{F}(\mathbf{x}) \quad \text{s.t.} \quad (I-W)\mathbf{x} = 0 \tag{2}$$

where $\mathbf{F}(\mathbf{x}) = \sum_{i=1}^{n} f_i(x_i)$

DGD is essentially a GD algorithm

- Since W is symmetric, problem (2) is further equivalent to

$$\min_{\mathbf{x}\in\mathbb{R}^{n\times d}} \quad \mathbf{F}(\mathbf{x}) \quad \text{s.t.} \quad (I-W)^{1/2}\mathbf{x}=0$$

- To solve the above constrained problem, we penalize the constraint to the objective function

$$\min_{\mathbf{x}\in\mathbb{R}^{n\times d}} \quad \mathbf{F}(\mathbf{x})+\frac{1}{2\gamma}\|(I-W)^{1/2}\mathbf{x}\|^2 \qquad \text{(penalized problem)}$$

- Solve the above problem with gradient descent

$$\begin{aligned}\mathbf{x}^{(k+1)} &= \mathbf{x}^{(k)}-\gamma\nabla_{\mathbf{x}}\left(\mathbf{F}(\mathbf{x})+\frac{1}{2\gamma}\|(I-W)^{1/2}\mathbf{x}\|^2\right)\\ &= \mathbf{x}^{(k)}-\gamma\nabla F(\mathbf{x}^{(k)})-(I-W)\mathbf{x}^{(k)}\\ &= W\mathbf{x}^{(k)}-\gamma\nabla F(\mathbf{x}^{(k)})\end{aligned}$$

We recovered DGD!

DGD is essentially a GD algorithm

- The DGD algorithm targeting to solve the consensus-contrained problem

$$\min_{x\in\mathbb{R}^d} \quad \sum_{i=1}^{n} f_i(x_i) \quad \text{s.t.} \quad x_1 = x_2 = \cdots = x_n$$

is essentially a gradient descent algorithm that solves an approximate penalized problem

$$\min_{\mathbf{x}\in\mathbb{R}^{n\times d}} \quad \mathbf{F}(\mathbf{x}) + \frac{1}{2\gamma}\|(I-W)^{1/2}\mathbf{x}\|^2$$

- Since DGD does not solve the original problem, it cannot converge to the desired solution

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Part 2

An Exact First-Order algorithm (EXTRA)

EXTRA algorithm

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- Recall the consensus-constrained optimization problem

$$\min_{\mathbf{x}\in\mathbb{R}^{n\times d}} \quad \mathbf{F}(\mathbf{x}) \quad \text{s.t.} \quad (I-W)^{1/2}\mathbf{x}=0$$

- When solve the above problem with penalty method, we reduce to DGD; cannot converge exactly

- Many effective algorithms to handle the constraints, such as Augmented Lagrangian Method

- The Augmented Lagrangian function is

$$L(\mathbf{x},\mathbf{y})=\mathbf{F}(\mathbf{x})+\frac{1}{\gamma}\langle\mathbf{y},(I-W)^{1/2}\mathbf{x}\rangle+\frac{1}{2\gamma}\|(I-W)^{1/2}\mathbf{x}\|^2$$

EXTRA algorithm

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- The ALM algorithm to solve the constrained problem is

$$\mathbf{x}^{(k+1)} = \mathbf{x}^{(k)} - \gamma \nabla_{\mathbf{x}} L(\mathbf{x}^{(k)}, \mathbf{y}^{(k)})$$
$$\mathbf{y}^{(k+1)} = \mathbf{y}^{(k)} + \gamma \nabla_{\mathbf{x}} L(\mathbf{x}^{(k)}, \mathbf{y}^{(k)})$$

  which can be expanded as

$$\mathbf{x}^{(k+1)} = W\mathbf{x}^{(k)} - \gamma \nabla \mathbf{F}(\mathbf{x}^k) - (I - W)^{1/2}\mathbf{y}^k$$
$$\mathbf{y}^{(k+1)} = \mathbf{y}^{(k)} + (I - W)^{1/2}\mathbf{x}^k$$

- If we remove the dual variable y from the above recursion, we achieve EXTRA [SLWY15]

$$\mathbf{x}^{(k+1)} = \left(\frac{W + I}{2}\right)(2\mathbf{x}^{(k)} - \mathbf{x}^{(k-1)}) - \gamma\left(\nabla \mathbf{F}(\mathbf{x}^{(k)}) - \nabla \mathbf{F}(\mathbf{x}^{(k-1)})\right)$$

EXTRA algorithm

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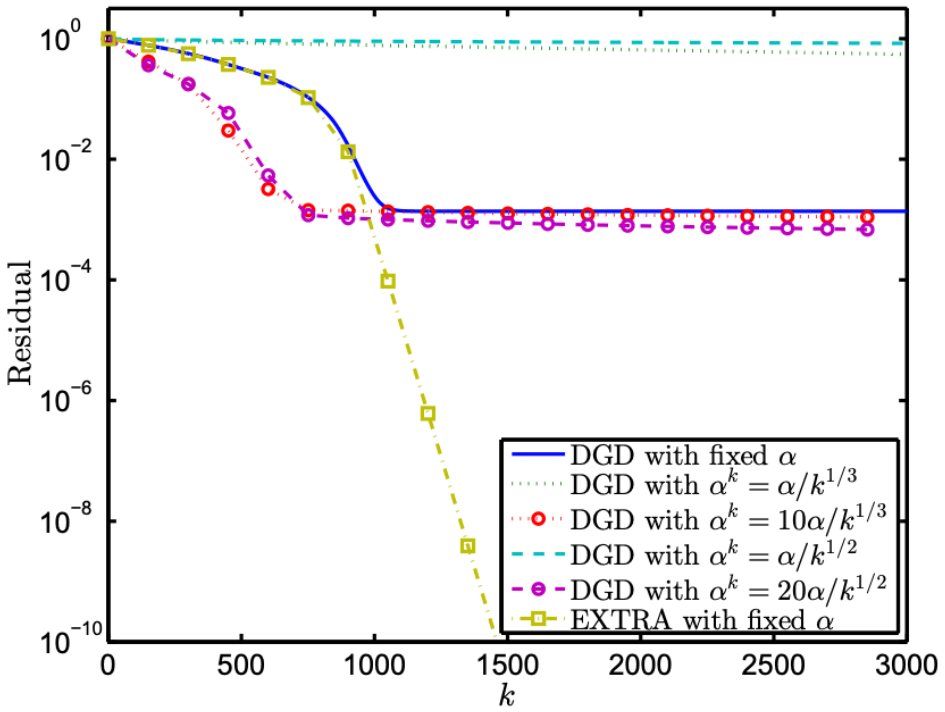
- EXTRA can converge to the exact solution

Theorem 3 [EXTRA Convergence] [SKWY15]

Assume each $f_i(x)$ is L-smooth and μ-strongly convex. If the learning rate is set as $\gamma = O(1/L)$, EXTRA converges as follows

$$\frac{1}{n}\sum_{i=1}^{n}\|x_i^{(k)} - x^\star\| = O(\rho^k)$$



Exact-Diffusion

- Exact-Diffusion is essentially the adapt-then-combine version of EXTRA

$$\mathbf{x}^{(k+1)} = \left(\frac{W+I}{2}\right)\left[2\mathbf{x}^{(k)} - \mathbf{x}^{(k-1)} - \gamma\left(\nabla\mathbf{F}(\mathbf{x}^{(k)}) - \nabla\mathbf{F}(\mathbf{x}^{(k-1)})\right)\right]$$

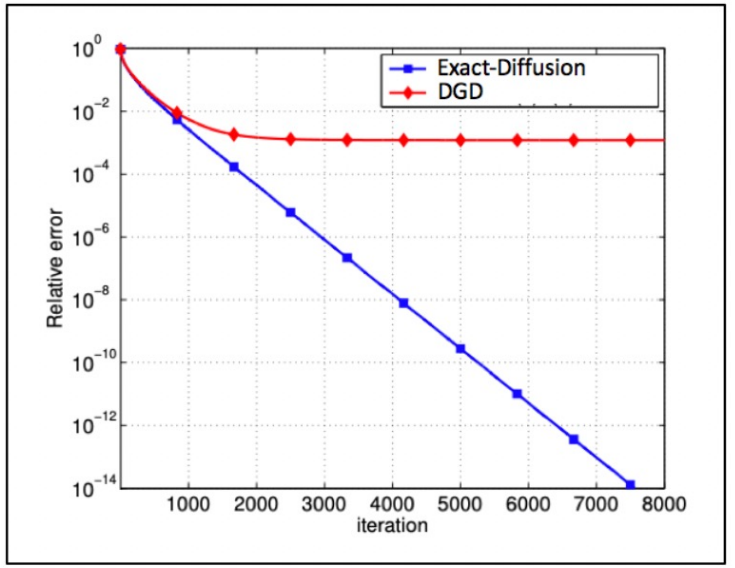
- Exact-Diffusion can be rewritten in a very elegant recursion

DGD

$$\begin{aligned} \psi_i^{(k)} &= x_i^{(k)} - \gamma\nabla f_i(x_i^{(k)}) && \text{(compt.)} \\ x_i^{(k+1)} &= \sum_{j\in\mathcal{N}_i} w_{ij}\psi_j^{(k)} && \text{(comm.)} \end{aligned}$$

Exact Diffusion

$$\begin{aligned} \psi_i^{(k)} &= x_i^{(k)} - \gamma\nabla f_i(x_i^{(k)}) && \text{(compt.)} \\ \phi_i^{(k)} &= \psi_i^{(k)} + x_i^{(k)} - \psi_i^{(k-1)} && \text{(corret.)} \\ x_i^{(k+1)} &= \sum_{j\in\mathcal{N}_i} w_{ij}\phi_j^{(k)} && \text{(comm.)} \end{aligned}$$



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Part 2

Gradient tracking

Gradient tracking

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- EXTRA and Exact-Diffusion can correct the bias in DGD, but have to use symmetric weight matrix
- In real implementations, the graph may not always be symmetric and it can even be time-varying
- **Gradient tracking** is a new state-of-the-art algorithm in decentralized optimization

Revisit DGD

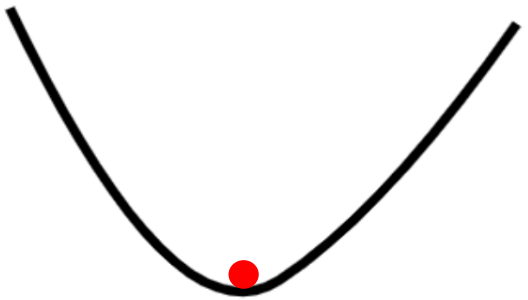
- We revisit the vanilla DGD to figure out the intuition why it does not exactly converge

$$x_i^{(k+1)} = \sum_{j \in \mathcal{N}_i} w_{ij} x_j^{(k)} - \nabla f_i(x_i^{(k)})$$

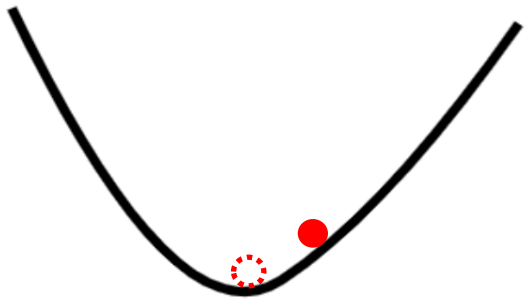
- Assume all nodes initialize at $x_i^{(k)} = x^\star$, we examine whether $x_i^{(k+1)}$ remains at $x^\star$

$$x_i^{(k+1)} = \sum_{j \in \mathcal{N}_i} w_{ij} x^\star - \nabla f_i(x^\star) = x^\star - \nabla f_i(x^\star) \neq x^\star$$

where the last equality does not hold due to $f_i(x) \neq f_j(x)$ and hence $\nabla f_i(x^\star) \neq \frac{1}{n} \sum_{i=1}^{n} \nabla f_i(x^\star) = 0$



$x_i^{(k)} = x^\star$



$x_i^{(k+1)} \neq x^\star$

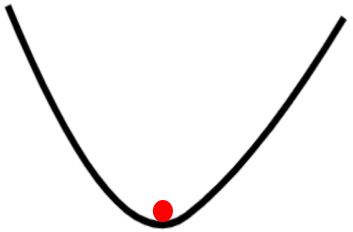
A crude idea

- To remove the bias, an impractical but intuitive algorithm is

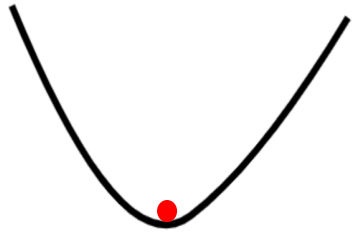
$$x_i^{(k+1)} = \sum_{j \in \mathcal{N}_i} w_{ij} x_j^{(k)} - \frac{1}{n} \sum_{j=1}^{n} \nabla f_j(x_j^{(k)})$$

- Assume all nodes initialize at $x_i^{(k)} = x^\star$, we find $x_i^{(k+1)}$ still remains at $x^\star$

$$x_i^{(k+1)} = \sum_{j \in \mathcal{N}_i} w_{ij} x^\star - \frac{1}{n} \sum_{j=1}^{n} \nabla f_j(x^\star) = x^\star - \frac{1}{n} \sum_{j=1}^{n} \nabla f_j(x^\star) = x^\star$$



$x_i^{(k)} = x^\star$



$x_i^{(k+1)} = x^\star$

Dynamic average consensus

- How to get the dynamic global average $\frac{1}{n}\sum_{j=1}^{n}\nabla f_j(x_j^{(k)})$ in a decentralized manner ?

- Recall the dynamic average consensus algorithm

$$x^{(k+1)} = Wx^{(k)} + z^{(k+1)} - z^{(k)} \quad \text{where} \quad x^{(0)} = z^{(0)}$$

- When the dynamic $z^{(k)}$ converges to stationary points or oscillates slowly, we have

$$x_i^{(k)} \to \frac{1}{n}\sum_{j=1}^{n} z_j^{(k)} \quad \text{as} \quad k \to \infty, \quad \forall i \in [n]$$

Gradient tracking

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- Inspired by dynamic average consensus, we can track the globally averaged gradient as follows

$$\mathbf{y}^{(k+1)} = W\mathbf{y}^{(k)} + \nabla\mathbf{F}(\mathbf{x}^{(k+1)}) - \nabla\mathbf{F}(\mathbf{x}^{(k)}) \quad \text{with} \quad \mathbf{y}^{(0)} = \nabla\mathbf{F}(\mathbf{x}^{(0)})$$

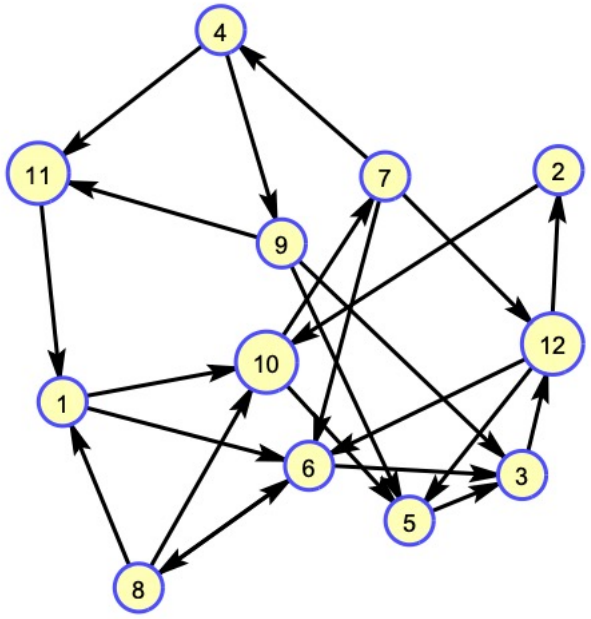
- When $\nabla\mathbf{F}(\mathbf{x}^{(k+1)})$ converges to stationary points, we have $y_i^{(k)} \to \frac{1}{n}\sum_{i=1}^{n} \nabla f_j(x_j^{(k)})$

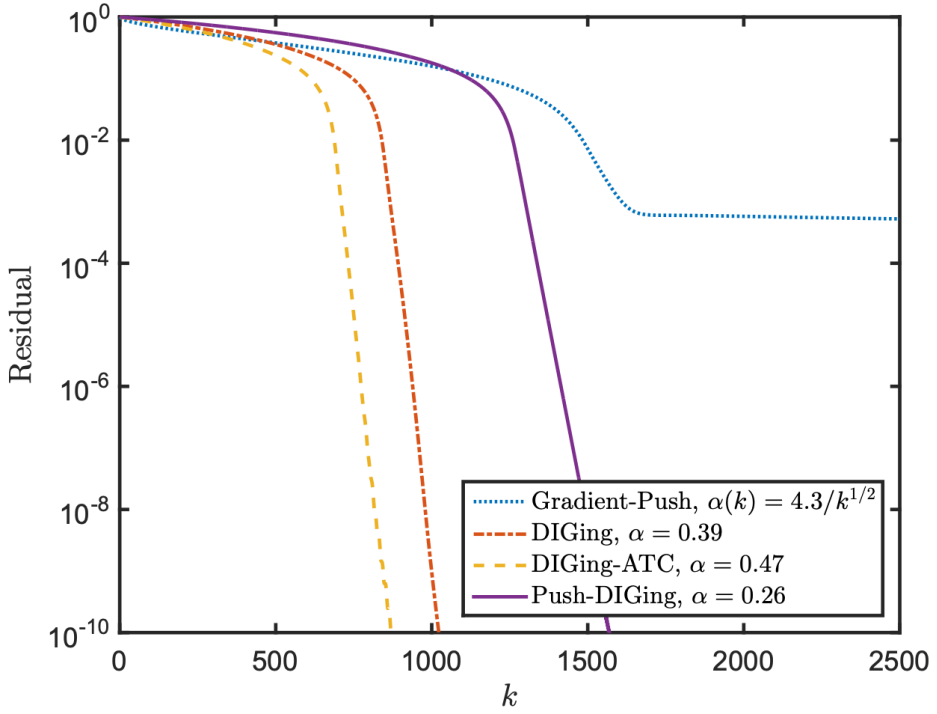
- Gradient tracking [XZSX15 ,LS16, NOS17]

$$\mathbf{x}^{(k+1)} = W\mathbf{x}^{(k)} - \gamma\mathbf{y}^{(k)} \qquad \text{(DSGD)}$$

$$\mathbf{y}^{(k+1)} = W\mathbf{y}^{(k)} + \nabla\mathbf{F}(\mathbf{x}^{(k+1)}) - \nabla\mathbf{F}(\mathbf{x}^{(k)}) \qquad \text{(Gradient tracking)}$$

Gradient tracking: simulations





Gradient tracking: adapt-then-combine

- Adapt-then-combine gradient tracking has slightly better performance

(Semi-ATC)

$$\mathbf{x}^{(k+1)} = W\left(\mathbf{x}^{(k)} - \gamma \mathbf{y}^{(k)}\right)$$
$$\mathbf{y}^{(k+1)} = W\mathbf{y}^{(k)} + \nabla \mathbf{F}(\mathbf{x}^{(k+1)}) - \nabla \mathbf{F}(\mathbf{x}^{(k)})$$

(Fully-ATC)

$$\mathbf{x}^{(k+1)} = W\left(\mathbf{x}^{(k)} - \gamma \mathbf{y}^{(k)}\right)$$
$$\mathbf{y}^{(k+1)} = W\left(\mathbf{y}^{(k)} + \nabla \mathbf{F}(\mathbf{x}^{(k+1)}) - \nabla \mathbf{F}(\mathbf{x}^{(k)})\right)$$