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Towards Decentralized Optimization over Digraphs: Effective metrics, lower bound, and optimal algorithms

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Joint work with

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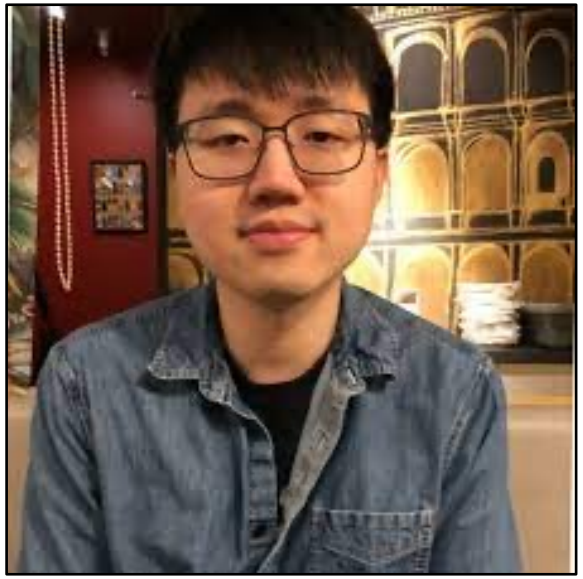

Ran Xin (ByteDance)

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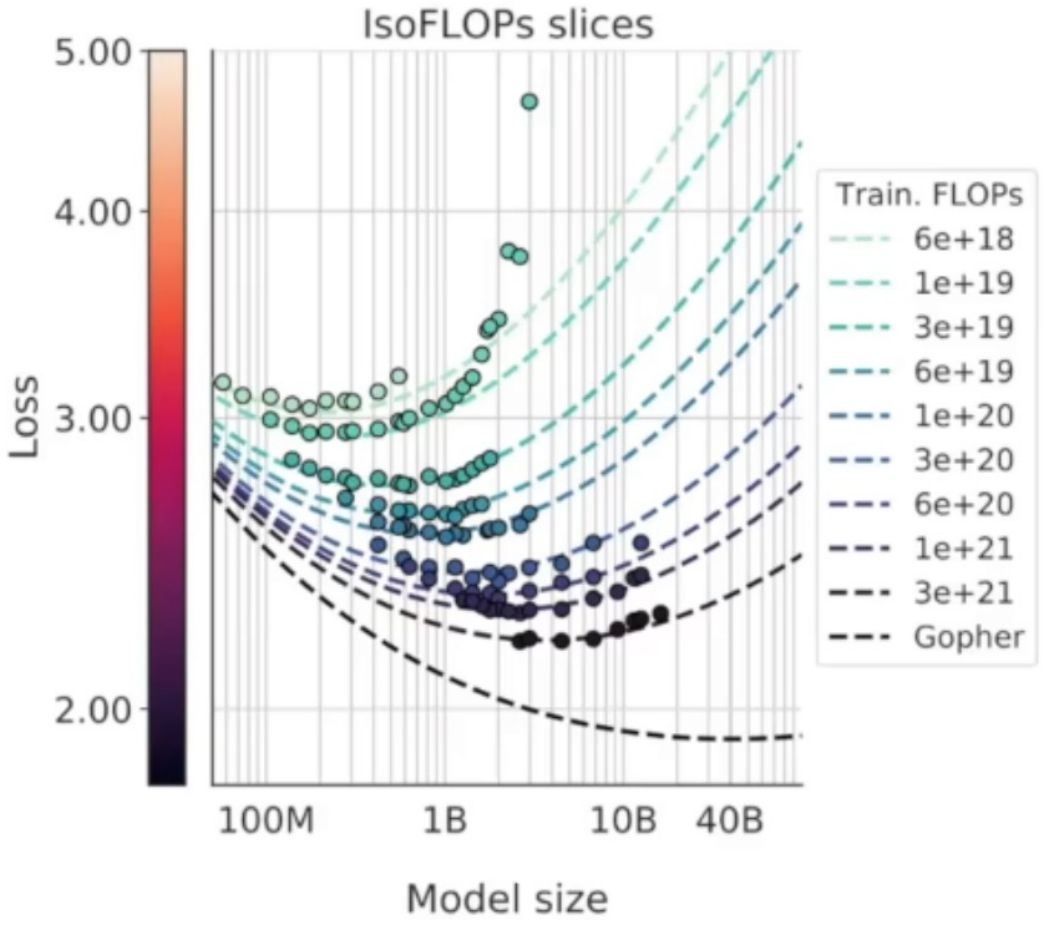
Part 01

Decentralized Optimization

Chinchilla law

[Training Compute-Optimal Large Language Models, 2022]



The “RGB” elements in LLM:

Larger dataset

+

Bigger model

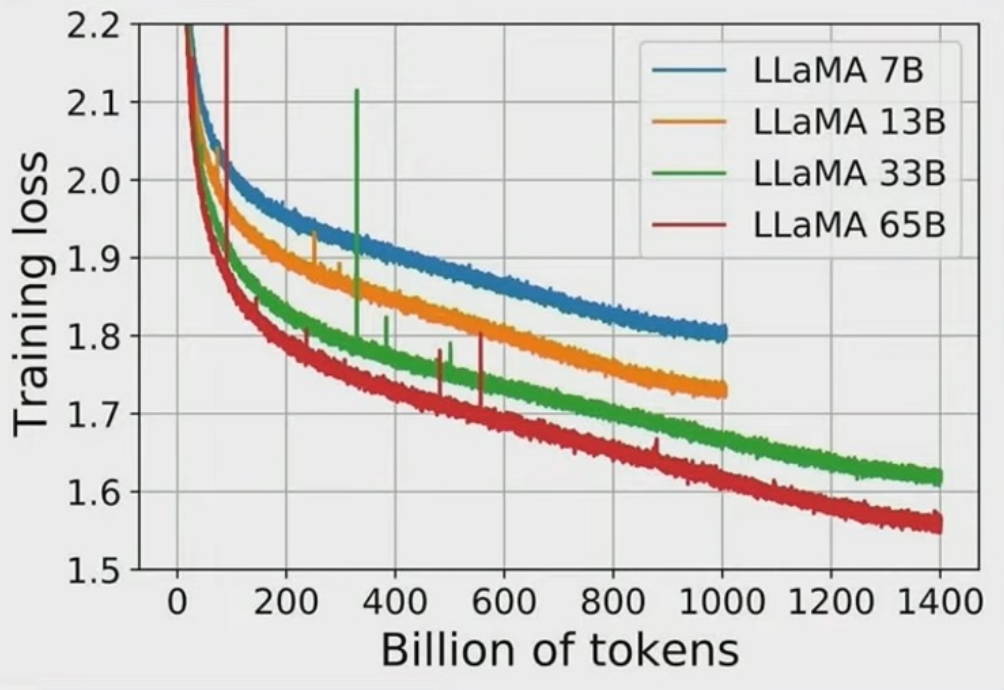
+

Longer training

= Better LLM

LLaMA follows Chinchilla law

[LLaMA: Open and Efficient Foundation Language Models,2023]



According to Chinchilla law:

- LLM model gets increasingly bigger
- Dataset gets increasingly larger
- Distributed training gets increasingly important

Thousands of GPUs are needed to train LLM

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[State of GPT, 2023]

2 example models

GPT-3 (2020)

50,257 vocabulary size
2048 context length
175B parameters
Trained on 300B tokens

Model Name	n_{params}	n_{layers}	d_{model}	n_{heads}	d_{head}	Batch Size	Learning Rate
GPT-3 Small	125M	12	768	12	64	0.5M	6.0×10^{-4}
GPT-3 Medium	350M	24	1024	16	64	0.5M	3.0×10^{-4}
GPT-3 Large	760M	24	1536	16	96	0.5M	2.5×10^{-4}
GPT-3 XL	1.3B	24	2048	24	128	1M	2.0×10^{-4}
GPT-3 2.7B	2.7B	32	2560	32	80	1M	1.6×10^{-4}
GPT-3 6.7B	6.7B	32	4096	32	128	2M	1.2×10^{-4}
GPT-3 13B	13.0B	40	5140	40	128	2M	1.0×10^{-4}
GPT-3 175B or "GPT-3"	175.0B	96	12288	96	128	3.2M	0.6×10^{-4}

Table 2.1: Sizes, architectures, and learning hyper-parameters (batch size in tokens and learning rate) of the models which we trained. All models were trained for a total of 300 billion tokens.

Training: (rough order of magnitude to have in mind)

- O(1,000 - 10,000) V100 GPUs
- O(1) month of training
- O(1-10) $M

LLaMA (2023)

32,000 vocabulary size
2048 context length
65B parameters
Trained on 1-1.4T tokens

params	dimension	n heads	n layers	learning rate	batch size	n tokens
6.7B	4096	32	32	$3.0e^{-4}$	4M	1.0T
13.0B	5120	40	40	$3.0e^{-4}$	4M	1.0T
32.5B	6656	52	60	$1.5e^{-4}$	4M	1.4T
65.2B	8192	64	80	$1.5e^{-4}$	4M	1.4T

Table 2: **Model sizes, architectures, and optimization hyper-parameters.**

Training for 65B model:

- 2,048 A100 GPUs
- 21 days of training
- $5M

Recent GPU clusters scale with more than 10000 GPUs

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Feb. 23, 2024

MegaScale: Scaling Large Language Model Training to More Than 10,000 GPUs

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Abstract

We present the design, implementation and engineering experience in building and deploying MegaScale, a production system for training large language models (LLMs) at the scale of more than 10,000 GPUs. Training LLMs at this scale brings unprecedented challenges to training efficiency and stability. We take a full-stack approach that co-designs the algorithmic and system components across model block and optimizer design, computation and communication overlapping, operserving billions of users, we have been aggressively integrating AI into our products, and we are putting LLMs as a high priority to shape the future of our products.

Training LLMs is a daunting task that requires enormous computation resources. The scaling law [3] dictates that the model size and the training data size are critical factors that determine the model capability. To achieve state-of-the-art model capability, many efforts have been devoted to train large models with hundreds of billions or even trillions of parameters on hundreds of billions or even trillions of tokens. For example, GPT-3 [4] has 175 billion parameters and

Llama 3 uses 24000 GPUs in LLM training

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Apr. 18, 2024

Build the future of AI with Meta Llama 3

To train our largest Llama 3 models, we combined three types of parallelization: data parallelization, model parallelization, and pipeline parallelization. Our most efficient implementation achieves a compute utilization of over 400 TFLOPS per GPU when trained on 16K GPUs simultaneously. We performed training runs on two custom-built 24K GPU clusters. To maximize GPU uptime, we developed an advanced new training stack that automates error detection, handling, and maintenance. We also greatly improved our

[Introducing Meta Llama 3: The most capable openly available LLM to date]

Distributed training over massive GPUs is extremely challenging

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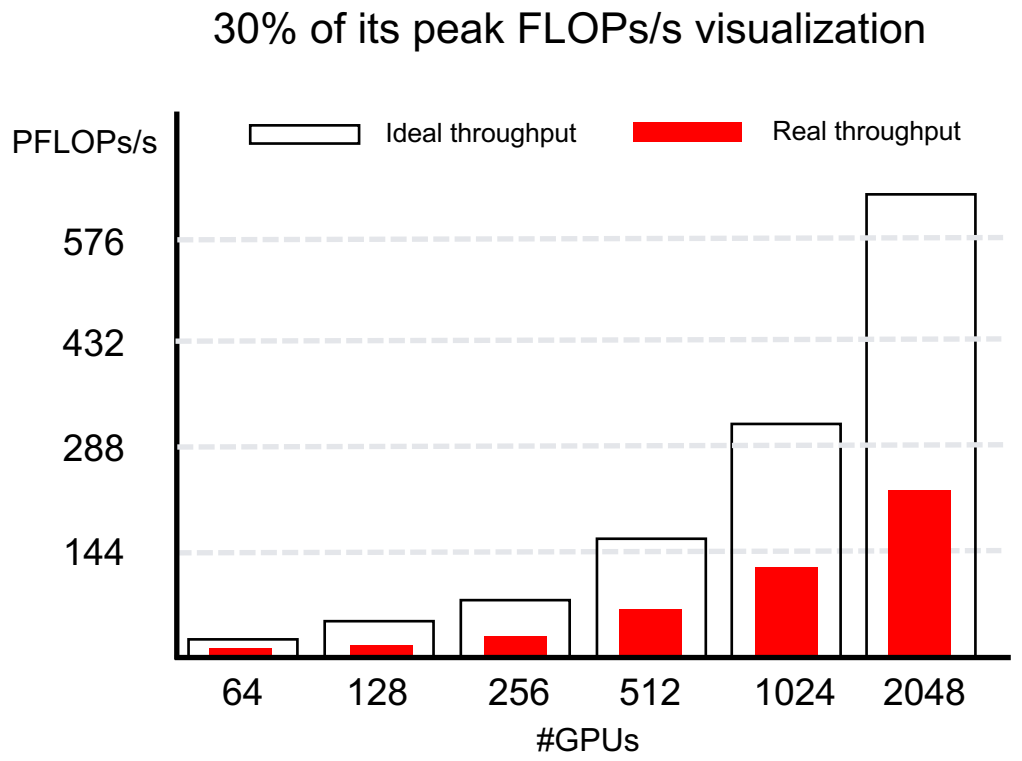
- Distributed training over thousands of GPUs is extremely challenging
- Two major challenges: **stability** and **scalability**
- Stability: very common that some GPU crashes during training LLMs
  - Meta OPT-175B: 175+ job restarts caused by hardware failures in 2 months
  - 书生大模型: Waste 41700 GPU hours due to training crashes (>80% are infrastructure failures)
- Stability in LLM training is very important, but this talk will not discuss it

Distributed training over massive GPUs is extremely challenging

- The **communication overhead** and **GPU idle time** severely hamper the scalability
- Each GPU can only achieve **30%~55%** of its peak FLOPs/s during LLM training
- When GPU achieves 30% of peak FLOP/s, we say the system achieves 30% scalability



Existing systems suffer from severe scalability issue

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[Efficient Large-Scale Language Model Training on GPU Clusters Using Megatron-LM, 2021]

Number of parameters (billion)	Attention heads	Hidden size	Number of layers	Tensor model-parallel size	Pipeline model-parallel size	Number of GPUs	Batch size	Achieved teraFlOP/s per GPU	Percentage of theoretical peak FLOP/s	Achieved aggregate petaFLOP/s
1.7	24	2304	24	1	1	32	512	137	44%	4.4
3.6	32	3072	30	2	1	64	512	138	44%	8.8
7.5	32	4096	36	4	1	128	512	142	46%	18.2
18.4	48	6144	40	8	1	256	1024	135	43%	34.6
39.1	64	8192	48	8	2	512	1536	138	44%	70.8
76.1	80	10240	60	8	4	1024	1792	140	45%	143.8
145.6	96	12288	80	8	8	1536	2304	148	47%	227.1
310.1	128	16384	96	8	16	1920	2160	155	50%	297.4
529.6	128	20480	105	8	35	2520	2520	163	52%	410.2
1008.0	160	25600	128	8	64	3072	3072	163	52%	502.0

Nvidia Megatron: the most popular distributed training framework

Existing systems suffer from severe scalability issue

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- Open AI and Meta does not disclose its training scalability, but we can infer them

- End-to-end training time can be estimated by

$$\text{Training time} = \frac{\text{Total FLOPS to train the model}}{\text{FLOPs/s of the GPU cluster}} \approx \frac{8\,T\,P}{N\,F\,U}$$

T: the number of tokens

P: the number of parameters

N: the number of GPUs

F: Peak FLOP/s per GPU

U: Utilization

[Efficient Large-Scale Language Model Training on GPU Clusters Using Megatron-LM, 2021]

Existing systems suffer from severe scalability issue

- **Open AI GPT3-175B**: 1024 A100 GPUs, 300B tokens, training time is 35 days

$$U \approx \frac{8 \times (300 \times 10^9) \times (175 \times 10^9)}{1024 \times (312 \times 10^{12}) \times (35 \times 24 \times 3600)} = \mathbf{0.44}$$

  The scalability is around 0.44

- **Meta LLaMA-65B**: 2048 A100 GPUs, 1.4TB tokens, training time is 21 days

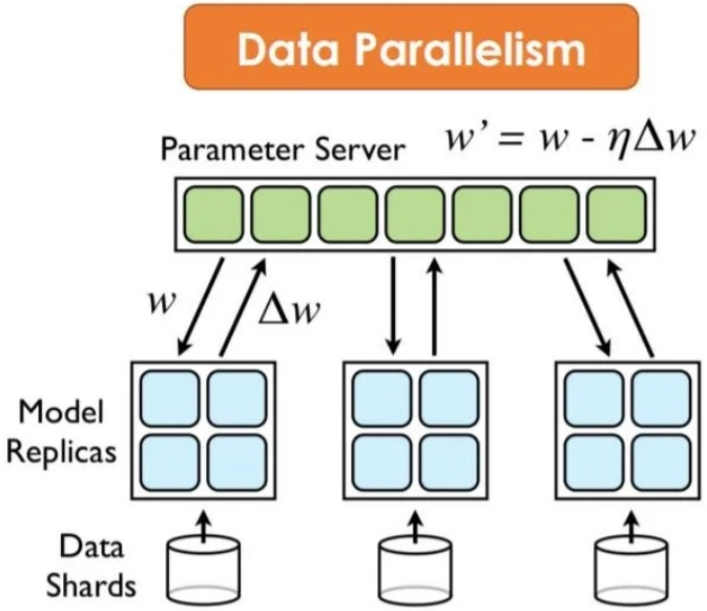
$$U \approx \mathbf{0.3}$$

  The scalability is around 0.3

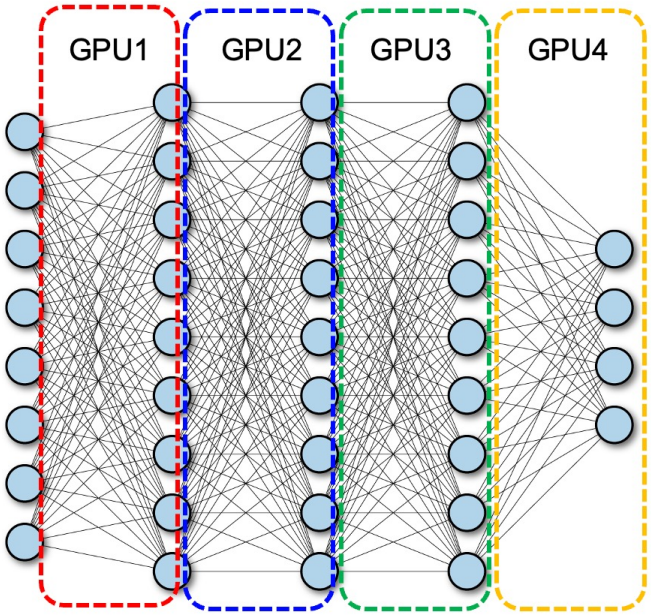
- We find even Nvidia, Open AI, and Meta **cannot** achieve strong scalability

Communication overhead is the top factor hampering the scalability

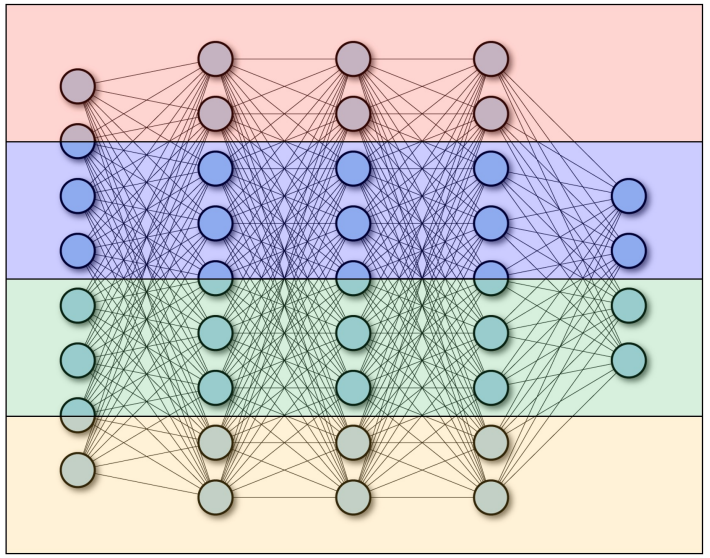
- 3D parallelism indicates 3 orthogonal parallel techniques used to train LLM



Data parallelism



Pipeline parallelism



Tensor parallelism

Communication overhead is the top factor hampering the scalability

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- Mata Llama 3 mainly uses 3D parallelism

> To train our largest Llama 3 models, we combined three types of parallelization: data parallelization, model parallelization, and pipeline parallelization. Our most efficient implementation achieves a compute utilization of over 400 TFLOPS per GPU when trained on 16K GPUs simultaneously. We performed training runs on two custom-built 24K GPU clusters. To maximize GPU uptime, we developed an advanced new training stack that

- This talk mainly focuses on **data parallelism**, and discuss how to save communication for it.

Data-parallel distributed learning

- Training deep neural networks typically requires **massive** datasets; efficient and scalable distributed optimization algorithms are in urgent need

- A network of n nodes (devices such as GPUs) collaborate to solve the problem:

$$\min_{x\in\mathbb{R}^d} \quad f(x)=\frac{1}{n}\sum_{i=1}^{n} f_i(x), \quad \text{where} \quad f_i(x)=\mathbb{E}_{\xi_i\sim D_i} F(x;\xi_i)$$

  - Each component $f_i : \mathbb{R}^d \to \mathbb{R}$ is local and private to node i
  - Random variable ξ_i denotes the local data that follows distribution D_i
  - Each local distribution D_i is different; data heterogeneity exists

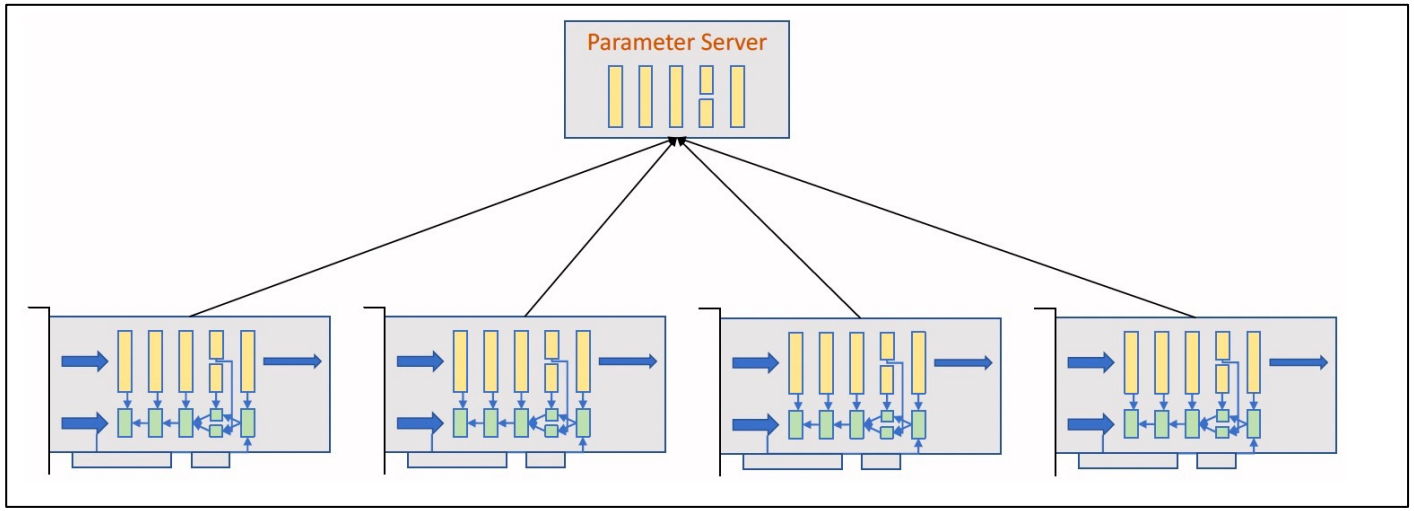
Vanilla parallel stochastic gradient descent (PSGD)

$$g_i^{(k)} = \nabla F(x^{(k)}; \xi_i^{(k)}) \qquad \text{(Local compt.)}$$

$$x^{(k+1)} = x^{(k)} - \frac{\gamma}{n} \sum_{i=1}^{n} g_i^{(k)} \qquad \text{(Global comm.)}$$

- Each node i samples data $\xi_i^{(k)}$ and computes gradient $\nabla F(x^{(k)}; \xi_i^{(k)})$
- All nodes synchronize (i.e. globally average) to update model x per iteration

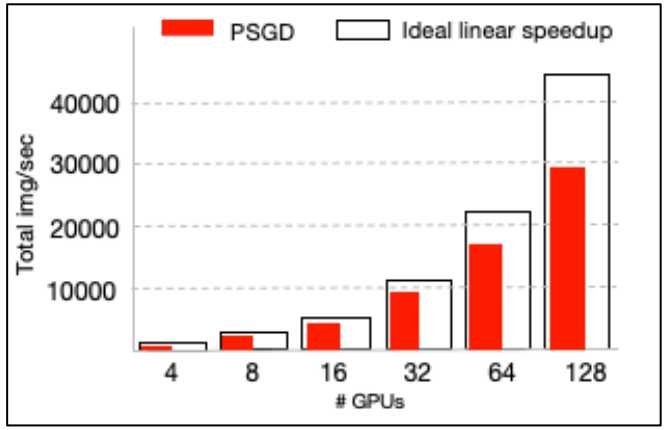
Vanilla parallel stochastic gradient descent (PSGD)



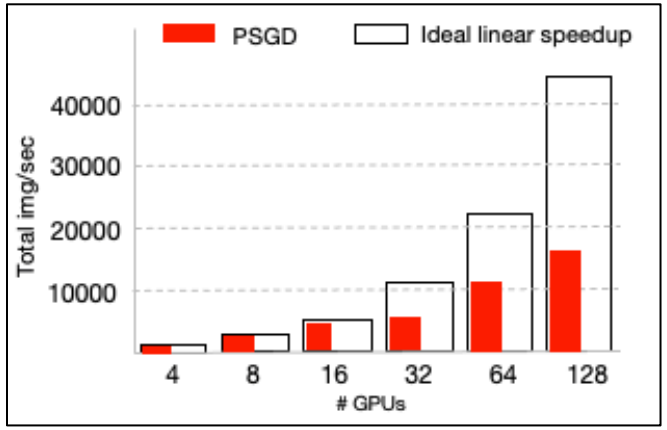
- Global average incurs $O(n)$ comm. overhead; **proportional to network size n**
- When network size n is large, PSGD suffers severe communication overhead

PSGD cannot achieve linear speedup due to comm. overhead

- PSGD cannot achieve ideal linear speedup in throughput due to comm. overhead
- Larger comm-to-compt ratio leads to worse performance in PSGD



Small comm.-to-compt. ratio



Large comm.-to-compt. ratio

- How can we accelerate PSGD? **Decentralized SGD is a promising paradigm**

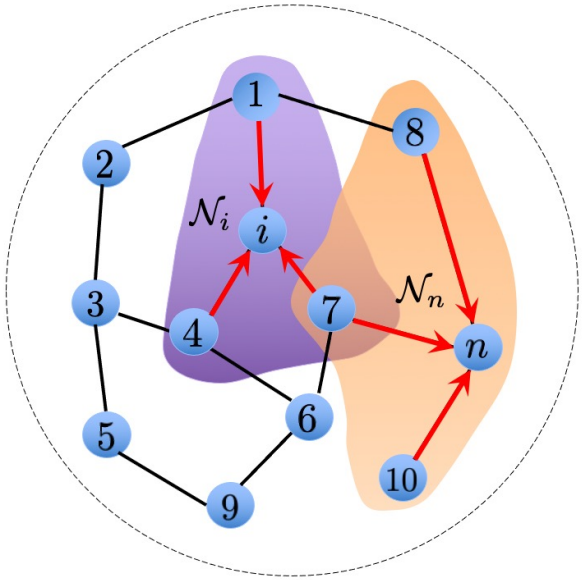
B. Ying, K. Yuan, H. Hu, Y. Chen and W. Yin, “BlueFog: Make decentralized algorithms practical for optimization and deep learning”, arXiv: 2111. 04287, 2021

Decentralized SGD (DSGD)

- To break $O(n)$ comm. overhead, we replace global average with partial average

$$x_i^{(k+\frac{1}{2})} = x_i^{(k)} - \gamma \nabla F(x_i^{(k)}; \xi_i^{(k)}) \quad \text{(Local update)}$$

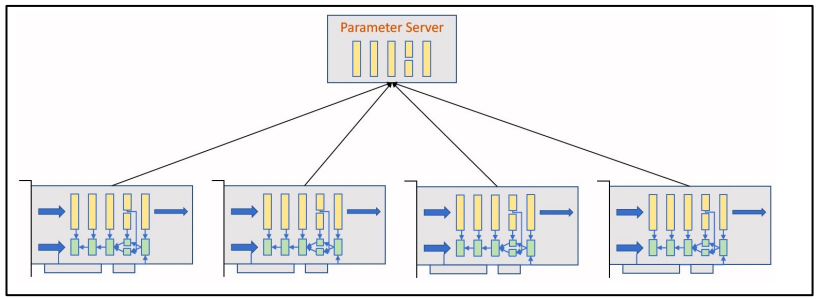
$$x_i^{(k+1)} = \sum_{j \in \mathcal{N}_i} w_{ij} x_j^{(k+\frac{1}{2})} \quad \text{(Partial averaging)}$$



- DSGD = local SGD update + partial averaging [LS08]
- $\mathcal{N}_i$ is the set of neighbors at node i ; w_{ij} scales information from j to i and satisfies $\sum_{j \in \mathcal{N}_i} w_{ij} = 1$
- Incurs $O(d_{\max})$ comm. overhead per iteration where $d_{\max} = \max_i \{|\mathcal{N}_i|\}$ is the graph maximum degree

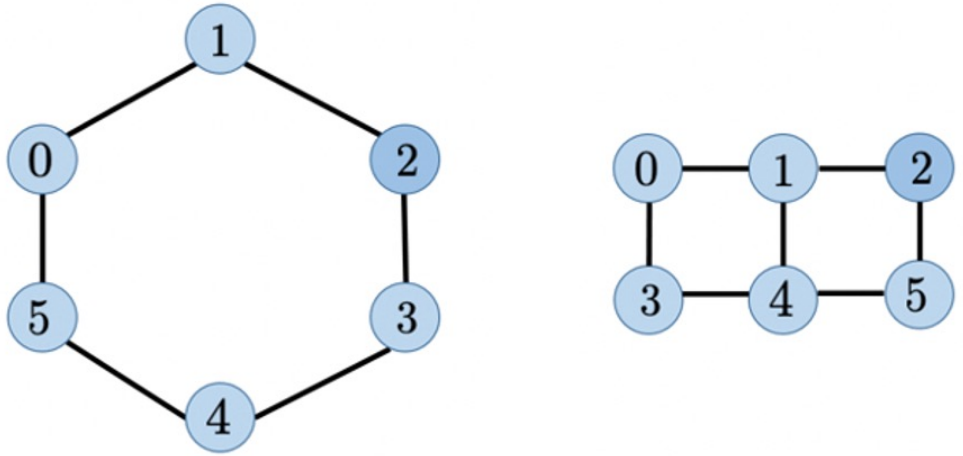
DSGD is more communication-efficient than PSGD

- Incurs $O(1)$ comm. overhead on **sparse** topologies; much less than global average $O(n)$



Global averaging over centralized network

comm. overhead $O(n)$

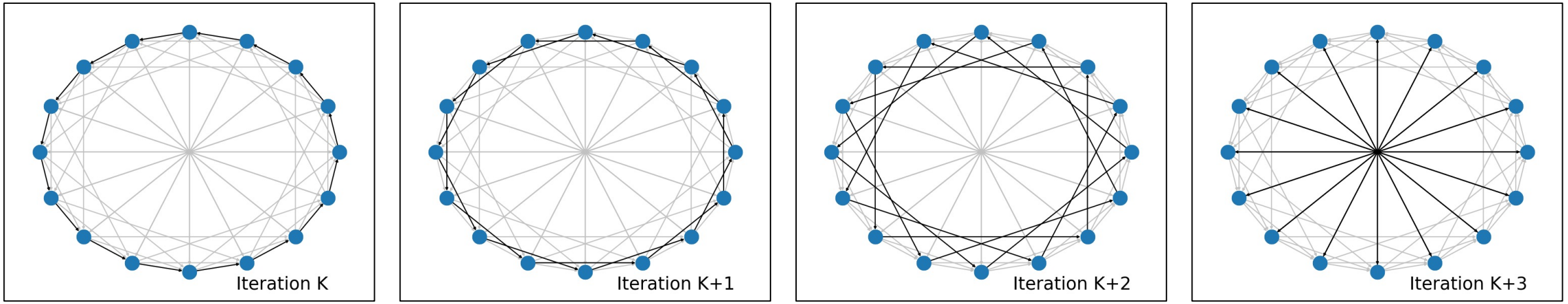


Partial averaging over ring or grid

comm. overhead $O(1)$

DSGD is more communication-efficient than PSGD

- Incurs $O(1)$ comm. overhead on **sparse** topologies; much less than global average $O(n)$



One-peer exponential graph incurs O(1) comm. overhead

B. Ying, K. Yuan, Y. Chen, H. Hu, and W. Yin, “Exponential Graph is Provably Efficient for Decentralized Deep Training”, NeurIPS 2021

DSGD is more communication-efficient than PSGD

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- A real experiment on a 256-GPUs cluster [CYZ+21]

Model	Ring-Allreduce	Partial average
ResNet-50 (25.5M)	278 ms	150 ms

Table. Comparison of per-iter comm. time in terms of runtime with 256 GPUs

- DSGD saves more communications per iteration for larger models

[CYZ+21] Y. Chen*, K. Yuan*, Y. Zhang, P. Pan, Y. Xu, and W. Yin, ``Accelerating Gossip SGD with Periodic Global Averaging", ICML 2021

DSGD is more communication-efficient than PSGD

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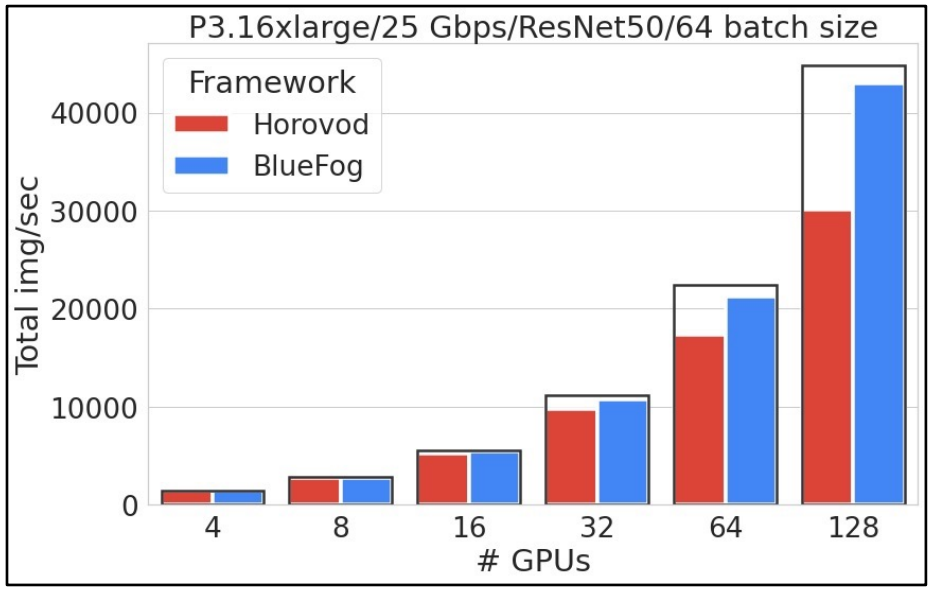
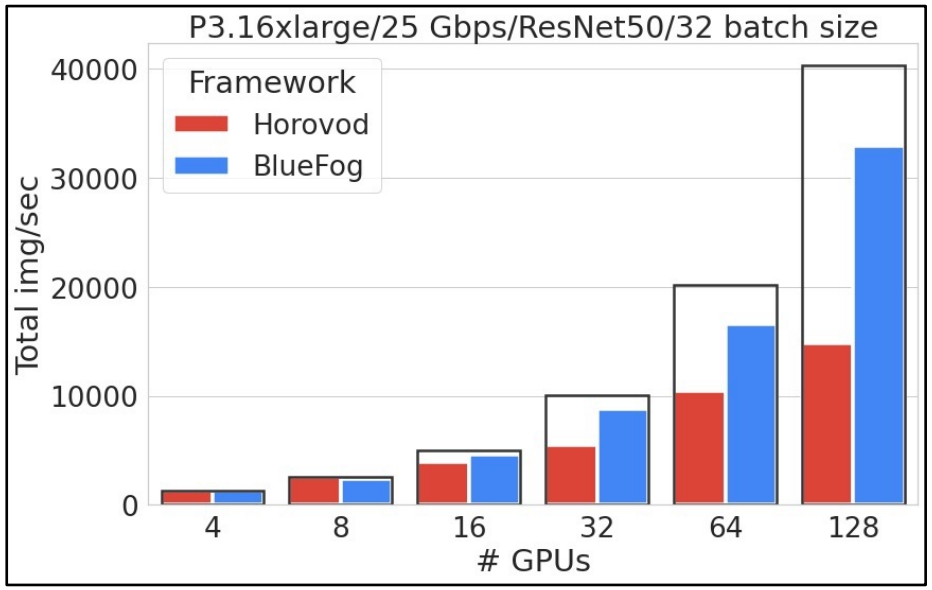
- DSGD (BlueFog) has **better linear speedup** than PSGD (Horovod) due to its small comm. overhead



Small comm.-to-compt. ratio



Large comm.-to-compt. ratio

B. Ying, K. Yuan, H. Hu, Y. Chen and W. Yin, “BlueFog: Make decentralized algorithms practical for optimization and deep learning”, arXiv: 2111. 04287, 2021

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Part 02

Influence of Undirected Network Topology

Does decentralized SGD converge? Yes!

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- Recall the PSGD and DSGD recursions

Parallel SGD

$$x_i^{(k+\frac{1}{2})} = x_i^{(k)} - \gamma \nabla F(x_i^{(k)}; \xi_i^{(k)}) \quad \text{(Local update)}$$

$$x_i^{(k+1)} = \frac{1}{n}\sum_{j=1}^{n} x_j^{(k+\frac{1}{2})} \quad \text{(Global averaging)}$$

Decentralized SGD

$$x_i^{(k+\frac{1}{2})} = x_i^{(k)} - \gamma \nabla F(x_i^{(k)}; \xi_i^{(k)}) \quad \text{(Local update)}$$

$$x_i^{(k+1)} = \sum_{j \in \mathcal{N}_i} w_{ij} x_j^{(k+\frac{1}{2})} \quad \text{(Partial averaging)}$$

- DSGD will converge if the **partial average** asymptotically converge to the **global average**

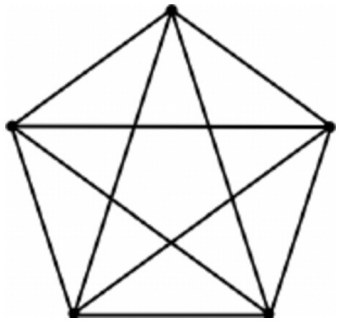
- This argument is true if the weight matrix $W = [w_{ij}]_{i=1,j=1}^{n} \in \mathbb{R}^{n \times n}$ is doubly-stochastic

$$W\mathbf{1}_n = \mathbf{1}_n \quad \text{and} \quad \mathbf{1}_n^T W = \mathbf{1}_n^T$$

Doubly-stochastic weight matrix is easy to construct

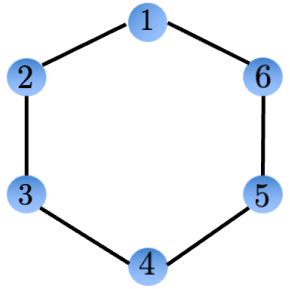
- Doubly-stochastic weight matrix is easy to construct over **undirected** graphs

Fully-connected graph



$$W = \frac{1}{5}\mathbb{1}\mathbb{1}^T = \begin{bmatrix} \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & \frac{1}{5} \\ \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & \frac{1}{5} \\ \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & \frac{1}{5} \\ \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & \frac{1}{5} \\ \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & \frac{1}{5} \end{bmatrix}$$

Ring graph



$$W = \begin{bmatrix} 1/3 & 1/3 & 0 & 0 & 0 & 1/3 \\ 1/3 & 1/3 & 1/3 & 0 & 0 & 0 \\ 0 & 1/3 & 1/3 & 1/3 & 0 & 0 \\ 0 & 0 & 1/3 & 1/3 & 1/3 & 0 \\ 0 & 0 & 0 & 1/3 & 1/3 & 1/3 \\ 1/3 & 0 & 0 & 0 & 1/3 & 1/3 \end{bmatrix}$$

A numerical illustration

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- We use examples to test whether partial average will converge to the global average
- Partial average: $x_i^{(k+1)} = \sum_{j \in \mathcal{N}_i} w_{ij} x_j^{(k)}$ for k = 0, 1, …, T

Partial average over **ring** graph

Iter T	x1	x2	x3	x4	x5
0	0	1	2	3	4
1	1.67	1	2	3	2.33
5	1.95	1.93	2.00	2.07	2.05
10	1.998	1.996	2.000	2.000	2.002
20	1.999	1.999	2.000	2.000	2.000
50	**2.000**	**2.000**	**2.000**	**2.000**	**2.000**

Global average

Partial average over **exponential** graph

Iter T	x1	x2	x3	x4	x5
0	0	1	2	3	4
1	1.75	1.5	2.5	2.25	2.0
5	2.002	2.001	2.00	1.999	1.998
10	1.999	1.999	2.000	2.000	2.002
20	**2.000**	**2.000**	**2.000**	**2.000**	**2.000**

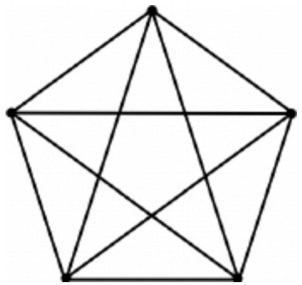
Global average

Network topology determines the consensus rate

- The above numerical example implies that
  - Partial average **asymptotically converges** to global average
  - **Network topology determines how fast** that partial average will converge to global average

- We introduce quantity ρ to gauge the graph connectivity

$$\rho = \|W - \frac{1}{n}\mathbb{1}\mathbb{1}^T\|_2 \in (0,1) \text{ where } W = [w_{ij}] \in \mathbb{R}^{n\times n}$$



$$W = \frac{1}{5}\mathbb{1}\mathbb{1}^T = \begin{bmatrix} \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & \frac{1}{5} \\ \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & \frac{1}{5} \\ \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & \frac{1}{5} \\ \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & \frac{1}{5} \\ \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & \frac{1}{5} \end{bmatrix}$$

  - Well-connected topology has $\rho \to 0$, e.g. fully-connected topology
  - Sparsely-connected topology has $\rho \to 1$, e.g. ring has $\rho = O(1 - \frac{1}{n^2})$

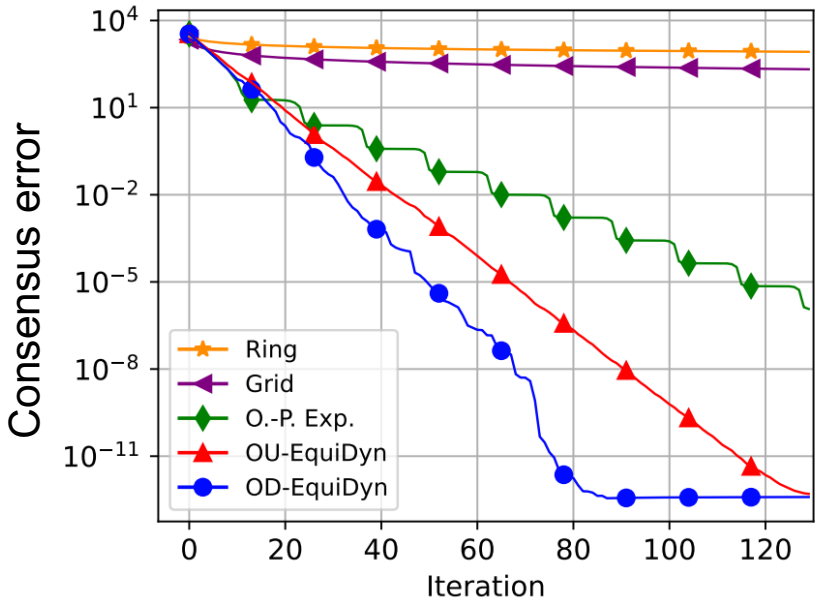
Network topology determines the consensus rate

- With graph connectivity indicator ρ, it can be proved that

$$\|k\text{-th partial average} - \text{global average}\| \leq C\rho^k$$

Based on the above fact, ρ is also referred to as **consensus rate**

Network topology	Consensus rate ρ
Ring	$O(1-\frac{1}{n^2})$
Grid	$O(1-\frac{1}{n\ln(n)})$
Torus	$O(1-\frac{1}{n})$
ExpoGraph	$O(1-\frac{1}{\ln(n)})$
GeoMedian	$O(1-\frac{\ln(n)}{n})$
Erdos-Renyi	$O(1)$
EquiGraph	$O(1)$



Song et. al., “Communication-Efficient Topologies for Decentralized Learning with O(1) Consensus Rate”, NeurIPS 2022

DSGD convergence rate

- Convergence comparison (non-convex and **data-homogeneous** scenario) [KLB+20]:

$$\text{P-SGD}: \quad \frac{1}{T}\sum_{k=1}^{T}\mathbb{E}\|\nabla f(\bar{x}^{(k)})\|^2 = O\left(\frac{\sigma}{\sqrt{nT}}\right)$$

$$\text{D-SGD}: \quad \frac{1}{T}\sum_{k=1}^{T}\mathbb{E}\|\nabla f(\bar{x}^{(k)})\|^2 = O\left(\frac{\sigma}{\sqrt{nT}} + \underbrace{\frac{\rho^{2/3}\sigma^{2/3}}{T^{2/3}(1-\rho)^{1/3}}}_{\text{extra overhead}}\right)$$

where σ^2 is the gradient noise, and T is the number of iterations

- D-SGD can asymptotically converge as fast as P-SGD when $T \to \infty$; the first term dominates; reach **linear speedup** asymptotically

- But D-SGD **requires more iteration** to reach that stage due to the overhead caused by partial average

Transient iterations

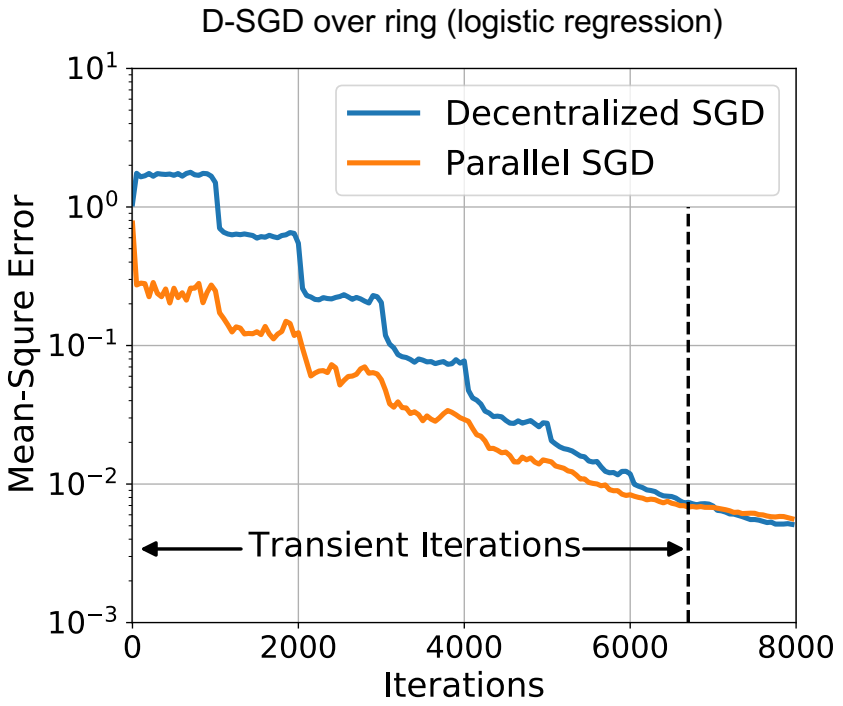
- Definition [POP21]: number of iterations before D-SGD achieves linear speedup

- D-SGD for non-convex and data-homogeneous scenario has $O(n^3(1-\rho)^{-2})$ transient iterations

$$\frac{\rho^{2/3}\sigma^{2/3}}{T^{2/3}(1-\rho)^{1/3}} \leq \frac{\sigma}{\sqrt{nT}} \quad \Longrightarrow \quad O(\frac{\rho^4 n^3}{(1-\rho)^2})$$

- Topology significantly influences the trans. stage.

- Sparse topology $\rho \to 1$ incurs longer tran. Iters.



Techniques to reduce transient iterations

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- Remove the influence of data heterogeneity
  - Exact-Diffusion [YAYS20, YAH23] (also known as D2 [TLY+18])
  - Gradient tracking [KLS22, AY22]
- Develop multi-step gossip strategy to accelerate convergence
  - DeTAG [LS21]; MG-DSGD[YHC+22]; MG-Exact-Diffusion[YAH23]
- Develop sparse and effective network topologies
  - Exponential Graph [YYC+21]; EquiTopo [SLJ+22]
  - CECA-DSGD [DJY+23]

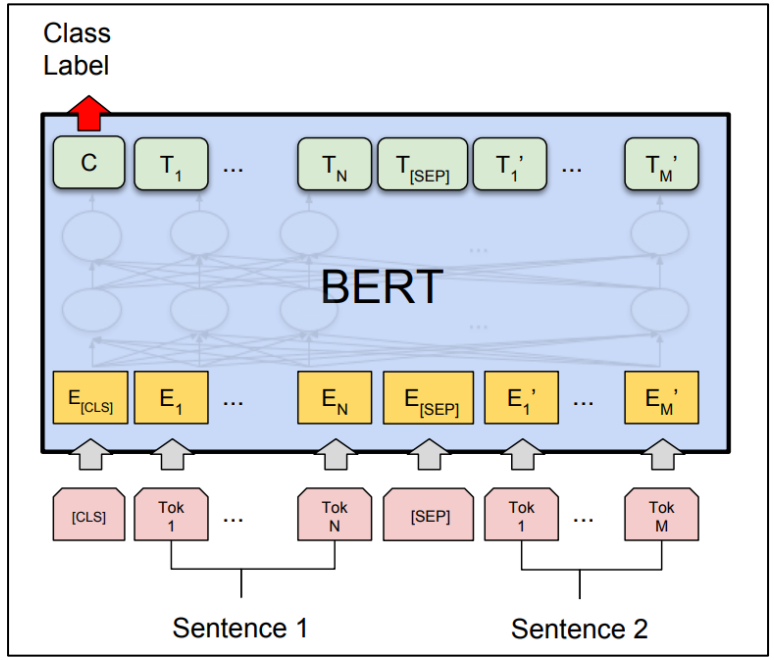
Algorithm	Tran. Iters.
DSGD	$\mathcal{O}(n^3/(1-\rho)^4)$
+ ED/D2	$\mathcal{O}(n^3/(1-\rho)^2)$
+ MG	$\mathcal{O}(n/(1-\rho))$
+ EquiTopo	$\mathcal{O}(n)$

DSGD achieves better linear speedup

nodes topology	4(4x8 GPUs) acc.	 time	8(8x8 GPUs) acc.	 time	16(16x8 GPUs) acc.	 time	32(32x8 GPUs) acc.	 time
P-SGD	76.32	11.6	76.47	6.3	76.46	3.7	76.25	2.2
Ring	76.16	11.6	76.14	6.5	76.16	3.3	75.62	1.8
one-peer exp.	76.34	11.1	76.52	5.7	76.47	2.8	76.27	1.5

DSGD shows very impressive linear speedup performance and saves more time than PSGD!

Experiments in deep learning (language modeling)



Model: BERT-Large (330M parameters)

Dataset: Wikipedia (2500M words) and BookCorpus (800M words)

Hardware: 64 GPUs

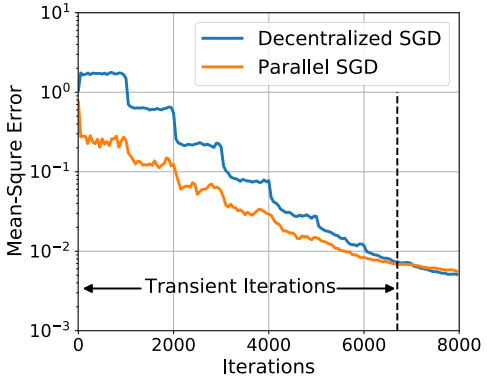
Table. Comparison in loss and training time [CYZ+21]

Method	Final Loss	Wall-clock Time (hrs)
P-SGD	1.75	59.02
D-SGD	1.77	30.4

[CYZ+21] Y. Chen*, K. Yuan*, Y. Zhang, P. Pan, Y. Xu, and W. Yin, ``Accelerating Gossip SGD with Periodic Global Averaging", ICML 2021

Review the research path on decentralized optimization

- We find that DSGD is efficient in communication but slow in convergence



- We identify the spectral gap metric to quantify the influence of network topology on convergence

$$\|W - \mathbf{1}_n \mathbf{1}_n^T / n\| \leq \rho$$

- We establish the transient iteration complexity

$$\mathcal{O}(n^3/(1-\rho)^4)$$

- We develop techniques to reduce transient iterations

Remove data heterogeneity | Multiple gossip | Effective graphs

The key step is to **identify an effective metric** that captures the influence of network topology

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Part 03

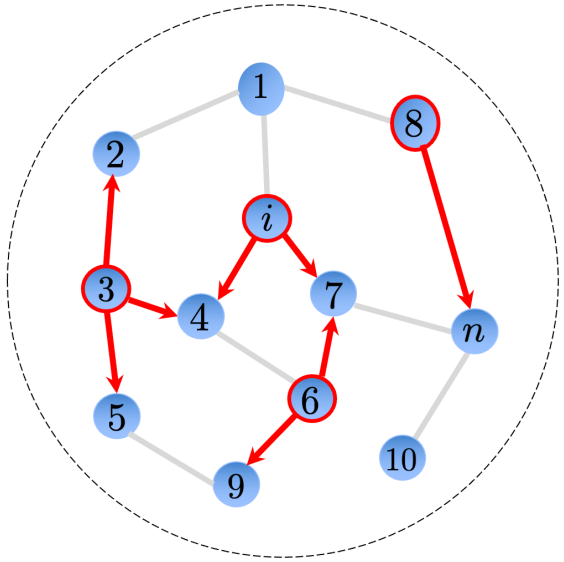
Decentralized Optimization over Digraphs

Directed network topology

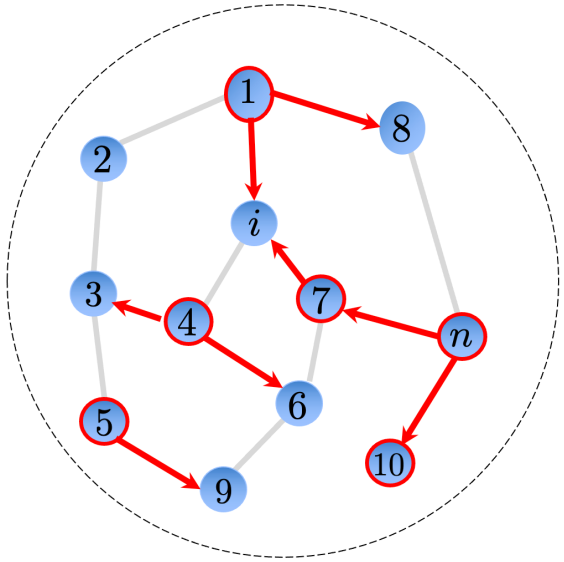
Social networks are directed

Asynchronous networks are directed

(nodes are activated at different iteration)



iteration k



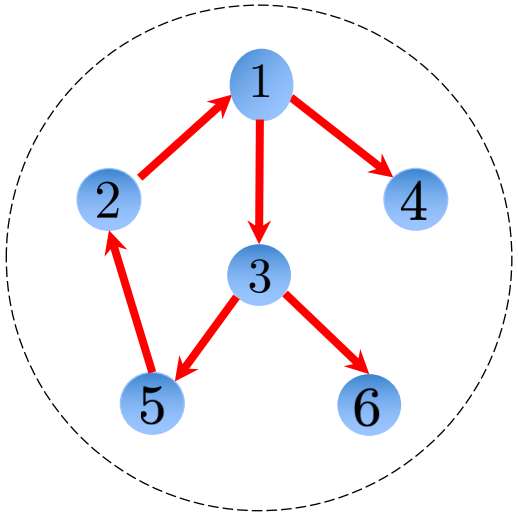
iteration k+1

Column-stochastic weight matrix

- This talk focuses on the **column-stochastic** weight matrix associated with a directed network

- A common way to construct the column-stochastic weight matrix

$$w_{ij} = \begin{cases} 1/(1+d_i^{\text{out}}) & \text{if directed edge } (j,i) \in \mathcal{E} \text{ or } j = i \\ 0 & \text{otherwise} \end{cases}$$



$$\begin{bmatrix} \frac{1}{3} & \frac{1}{2} & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 & \frac{1}{2} & 0 \\ \frac{1}{3} & 0 & \frac{1}{3} & 0 & 0 & 0 \\ \frac{1}{3} & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{1}{3} & 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{3} & 0 & 0 & 1 \end{bmatrix}$$

Column-stochastic weight matrix

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Assumption 2.

The mixing matrix W is entry-wisely non-negative, primitive (i.e., all entries of W^{k_0} are positive for sufficiently large $k_0 \in \mathbb{N}_+$), and satisfies $\mathbf{1}_n^\top W = \mathbf{1}_n^\top$.

Lemma 2.

Under Assumption 2, there exists a unique equilibrium vector $\pi \in \mathbb{R}^n$ with positive entries such that $W\pi = \pi$ and $\mathbf{1}_n^\top \pi = 1$. Moreover, it holds that

$$\lim_{k\to\infty} W^k = \pi \mathbf{1}_n^\top$$

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Part 04

Push-sum decentralized averaging

Column-stochastic matrix cannot enable global average

- Assume each node i maintains a local vector $z_i \in \mathbb{R}^n$
- We introduce $\mathbf{z} = [z_1^\top; z_2^\top; \cdots; z_n^\top] \in \mathbb{R}^{n \times d}$ and let $\mathbf{z}^{(0)} = \mathbf{z}$
- Column-stochastic matrix cannot enable global average

$$\mathbf{z}^{(k)} = W\mathbf{z}^{(k-1)} = W^k\mathbf{z}^{(0)} \longrightarrow \pi\mathbf{1}^\top\mathbf{z}^{(0)} \quad (\text{as } k \to \infty)$$

  which implies that $z_i^{(k)} \to \pi_i \sum_{j=1}^n z_j$ rather than $z_i^{(k)} \to (1/n)\sum_{j=1}^n z_j$

- We can correct the bias to enable global average

$$\mathbf{w}^{(k)} = \operatorname{diag}(n\pi)^{-1}\mathbf{z}^{(k)} \longrightarrow \operatorname{diag}(n\pi)^{-1}\pi\mathbf{1}_n^\top\mathbf{z} = (1/n)\mathbf{1}_n\mathbf{1}_n^\top\mathbf{z}$$

Push-sum decentralized averaging

(Bias correction) $\mathbf{w}^{(k)} = \mathrm{diag}(n\pi)^{-1}\mathbf{z}^{(k)} \longrightarrow \mathrm{diag}(n\pi)^{-1}\pi\mathbf{1}_n^\top\mathbf{z} = (1/n)\mathbf{1}_n\mathbf{1}_n^\top\mathbf{z}$

- However, the equilibrium vector π is not known in advance
- Push-sum decentralized averaging [BBT+10, TLR12, NO13]

$$\begin{aligned}\mathbf{z}^{(k+1)} &= W\mathbf{z}^{(k)} \\ v^{(k+1)} &= Wv^{(k)} \qquad \text{(starting with } v^{(0)} \text{ satisfying } \mathbf{1}_n^\top v^{(0)} = n\text{)} \\ V^{(k+1)} &= \mathrm{diag}(v^{(k+1)}) \\ \mathbf{w}^{(k+1)} &= {V^{(k+1)}}^{-1}\mathbf{z}^{(k+1)}.\end{aligned}$$

- It is guaranteed that $w_i^{(k)} \to (1/n)\sum_{j=1}^n z_j$

Push-sum decentralized optimization

- Many optimization algorithms over digraphs have been proposed based on push-sum averaging
  - Push-sum subgradient method [NO13]
  - Push-sum dual averaging [TLR12]
  - Push-sum EXTRA [ZY17; XK17]
  - Push-sum Gradient-tracking [NOS17]
  - Push-sum SGD [ALBR19]

Push-sum decentralized optimization

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- Existing results show that algorithms over digraphs asymptotically converge as fast as centralized algorithms

Algorithm	Rate (A.)	Rate (F.T.)	Transient Stage
Gradient-Push [4]	$\frac{\sigma\sqrt{L\Delta}}{\sqrt{nK}}$	N.A.	N.A.
Push-DIGing [15]	$\frac{\sigma\sqrt{L\Delta}}{\sqrt{nK}}$	N.A.	N.A.

- How much slower decentralized optimization over digraphs are compared to centralized optimization?
  We do not know

- **It is not known how the digraphs will affect the convergence rate**

- **The transient iteration complexity is not known neither**

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Part 05

Effective metrics to evaluate digraphs

Spectral gap

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- In undirected graphs, the connectivity is gauged by spectral gap $1-\beta$ where

$$\beta = \|W - (1/n)\mathbf{1}_n\mathbf{1}_n^\top\|_2 \in [0,1)$$

- Inspired by this, can we use the same metric to capture the connectivity of digraphs? **No we cannot!**

$$\beta = \|W - (1/n)\mathbf{1}_n\mathbf{1}_n^\top\|_2 > 1 \text{ for digraphs}$$

- No problem. Recall that $W^k \to \pi\mathbf{1}_n^\top$, can we use the following metric? **No we cannot!**

$$\beta = \|W - \pi\mathbf{1}_n^\top\|_2 > 1 \text{ for digraphs}$$

Spectral gap

- According to [XSKK19], digraph connectivity can be gauged by **Generalized Spectral Gap** $1-\beta_\pi$

$$\beta_\pi = \|W - \pi \mathbf{1}_n^\top\|_\pi \in [0,1)$$

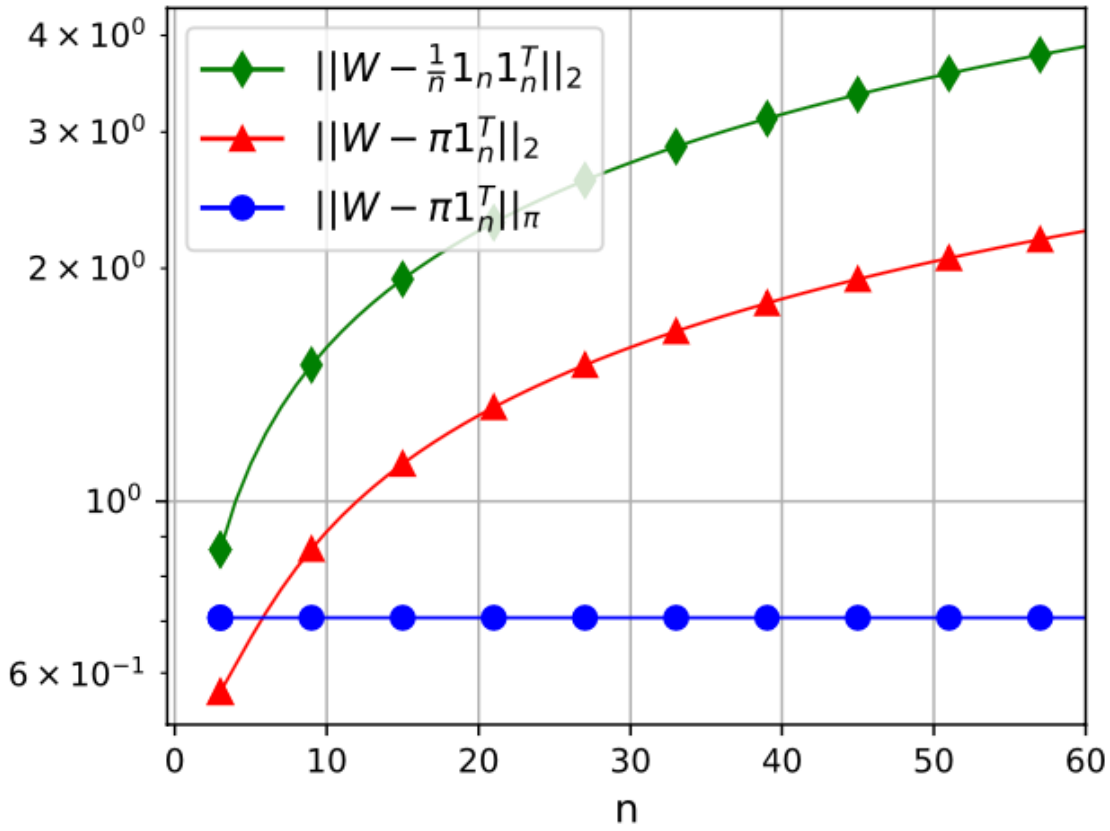
- For a vector v, its π-norm is defined as

$$\|v\|_\pi := \|\mathrm{diag}(\sqrt{\pi})^{-1} v\|_2$$

- For a vector A, its π-norm is defined as

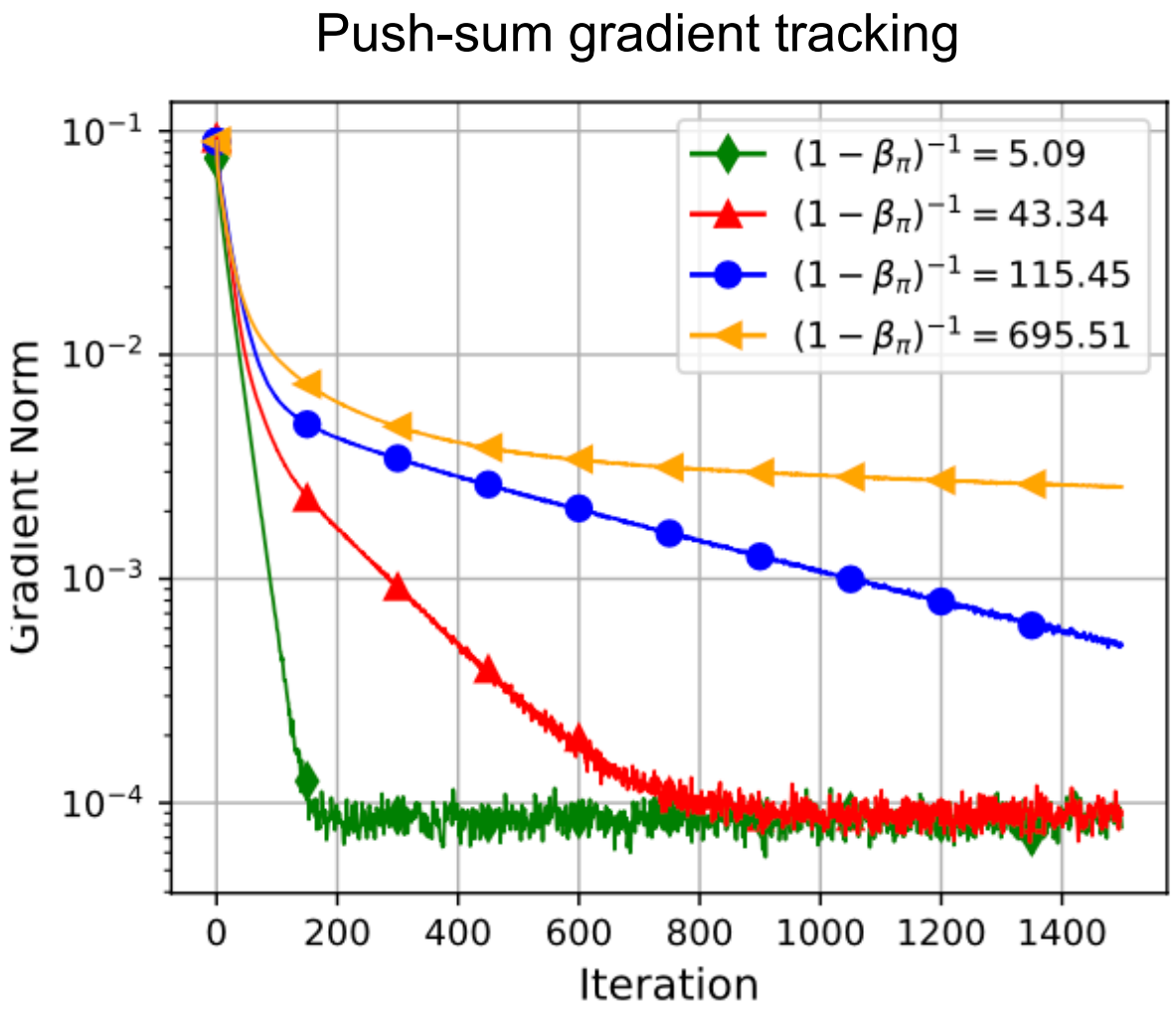
$$\|A\|_\pi := \|\mathrm{diag}(\sqrt{\pi})^{-1} A \ \mathrm{diag}(\sqrt{\pi})\|_2$$

- For undirected graphs, it holds that $\beta_\pi = \beta$



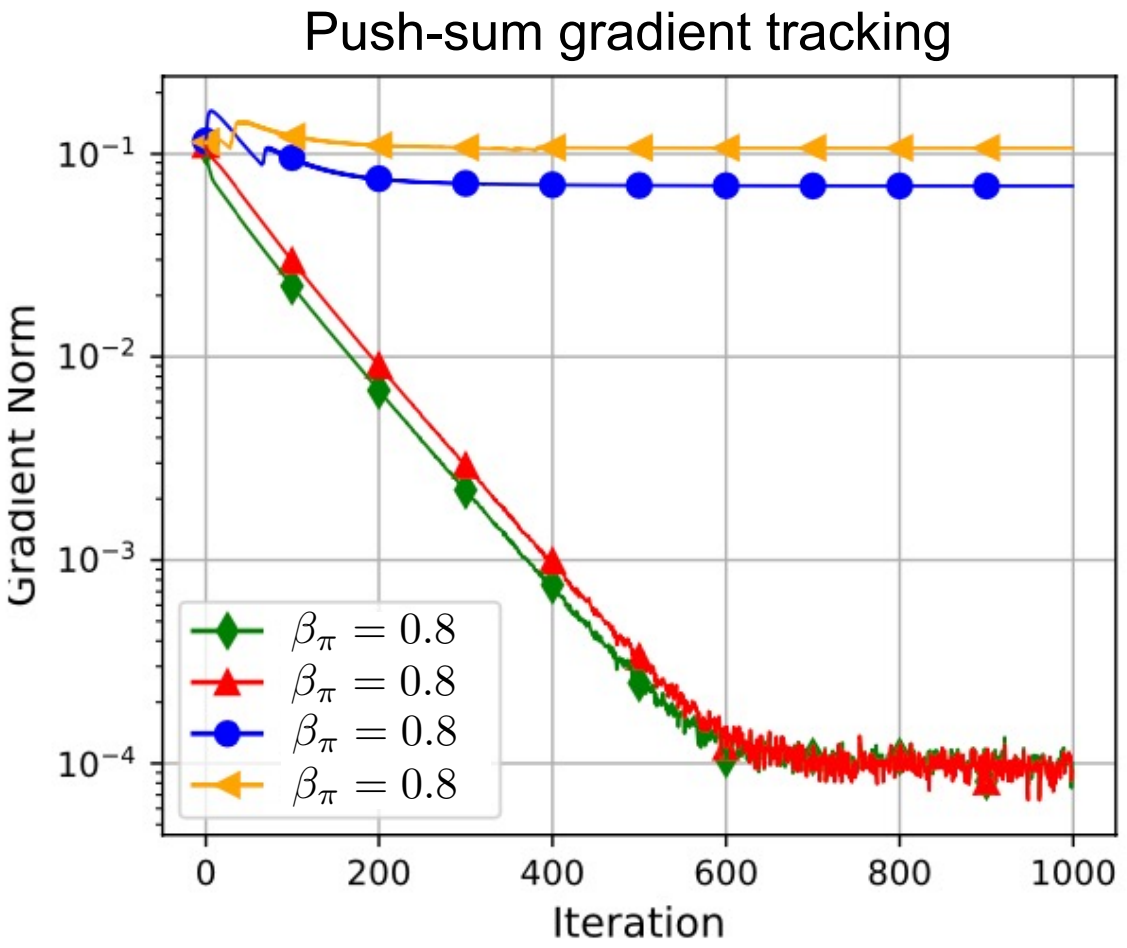
Spectral gap alone cannot precisely reflect the digraph influence

- We are done. We can use β_π to capture the influence of digraphs.



Spectral gap alone cannot precisely reflect the digraph influence

- Wait! Something strange is happening!
- Different curves vary drastically even with the same spectral gap!
- The single spectral gap alone is insufficient to capture the digraph influence



Equilibrium skewness

- Let's revisit the power iteration with column-stochastic W. Initialize $x^{(0)} = x$, we have

$$x^{(k)} = Wx^{(k-1)} = W^k x^{(0)} = W^k x \longrightarrow \pi \mathbf{1}_n^\top x \quad \text{as} \quad k \to \infty.$$

- To evaluate how fast $x^{(k)}$ converges to the global average $(1/n)\mathbf{1}_n\mathbf{1}_n^\top x$:
  - Generalized spectral gap to gauge how fast that $x^{(k)}$ approaches to $\pi \mathbf{1}_n^\top x$
  - A new metric to gauge the disagreement between $\pi \mathbf{1}_n^\top x$ and $(1/n)\mathbf{1}_n\mathbf{1}_n^\top x$

Equilibrium skewness

- Let's revisit the power iteration with column-stochastic W. Initialize $x^{(0)} = x$, we have

$$x^{(k)} = Wx^{(k-1)} = W^k x^{(0)} = W^k x \longrightarrow \pi \mathbf{1}_n^\top x \quad \text{as} \quad k \to \infty.$$

- To evaluate how fast $x^{(k)}$ converges to the global average $(1/n)\mathbf{1}_n\mathbf{1}_n^\top x$:
  - Generalized spectral gap to gauge how fast that $x^{(k)}$ approaches to $\pi \mathbf{1}_n^\top x$
  - **Equilibrium skewness** to gauge the disagreement between $\pi \mathbf{1}_n^\top x$ and $(1/n)\mathbf{1}_n\mathbf{1}_n^\top x$

$$\kappa_\pi := \max_i \pi_i / \min_i \pi_i \in [1, +\infty)$$

Revisit push-sum decentralized averaging

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Push-sum averaging

$$\mathbf{z}^{(k+1)} = W\mathbf{z}^{(k)}$$
$$v^{(k+1)} = Wv^{(k)}$$
$$V^{(k+1)} = \mathrm{diag}(v^{(k+1)})$$
$$\mathbf{w}^{(k+1)} = {V^{(k+1)}}^{-1}\mathbf{z}^{(k+1)}.$$

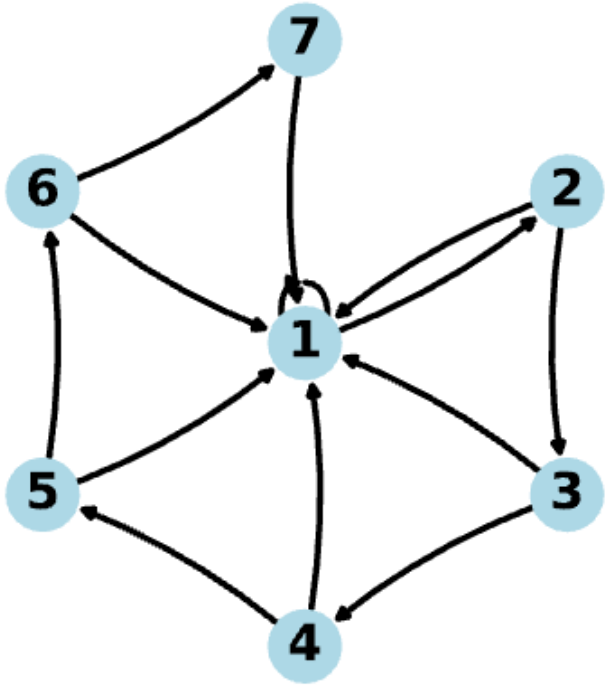
Theorem 1.

Assume $v^{(0)}$ is initialized such that $\mathbf{1}_n^\top v^{(0)} = n$, Push -sum decentralized averaging converges as follows

$$\|\mathbf{w}^{(k)} - \bar{\mathbf{z}}\|_F \leq \kappa_\pi^{3/2}\beta_\pi^k\|\mathbf{z}^{(0)}\|_F$$

The **first** rate reflecting the influence of both κ_π and β_π

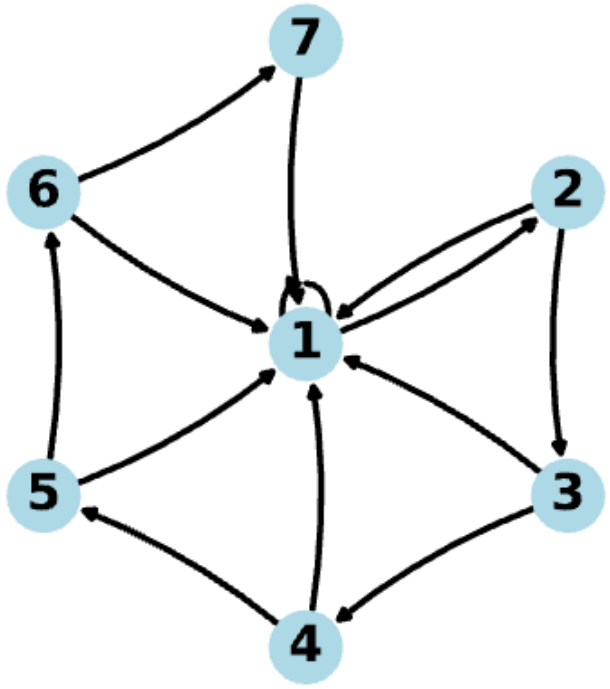
The influence of equilibrium skewness is substantial



$$W = \begin{bmatrix} 1/2 & 1/2 & \cdots & 1/2 & 1 \\ 1/2 & 0 & & & \\ & \ddots & \ddots & & \\ & & 1/2 & 0 & \\ & & & 1/2 & 0 \end{bmatrix} \in \mathbb{R}^{n \times n}.$$

The probability that a message flows from node 1 to node n decays **exponentially** fast

The influence of equilibrium skewness is substantial

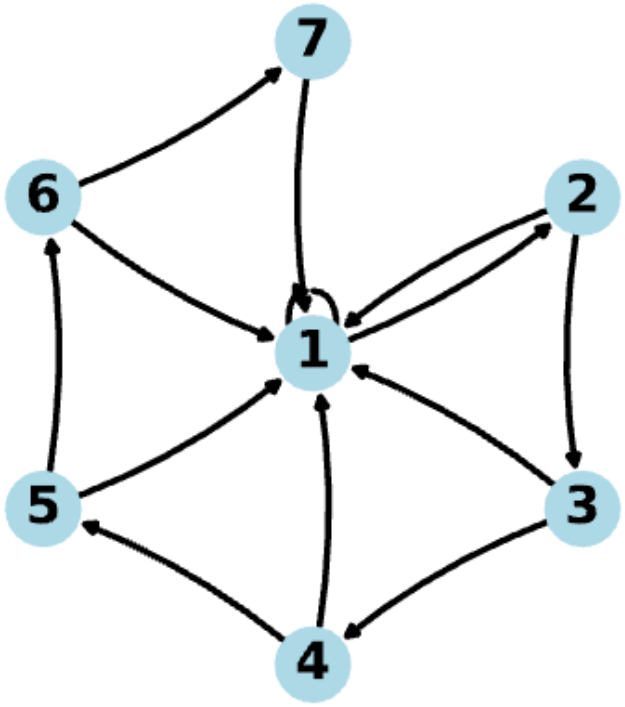


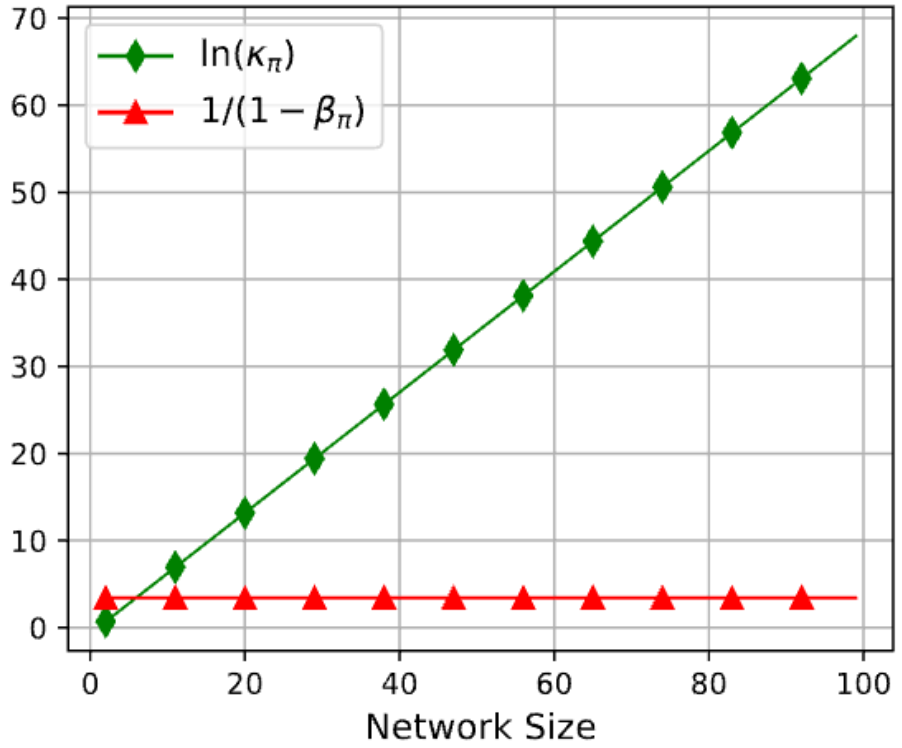
Theorem 2.

For any $n \geq 1$, there exists a matrix $W \in \mathbb{R}^{n \times n}$ such that

$$\beta_\pi = \frac{\sqrt{2}}{2} \quad \text{and} \quad \kappa_\pi = 2^{n-1}$$

The influence of equilibrium skewness is substantial



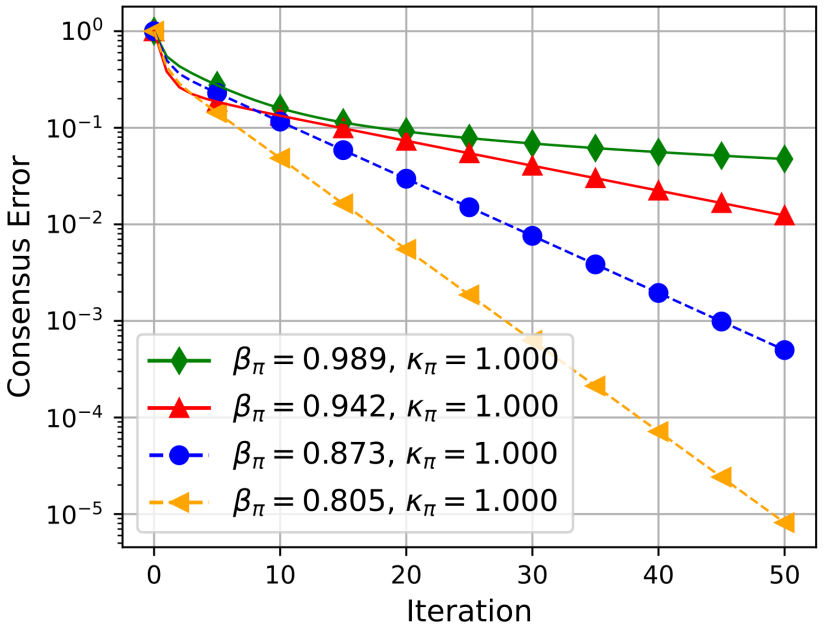


- β_π and κ_π is orthogonal
- The influence of κ_π can be highly non-trivial !

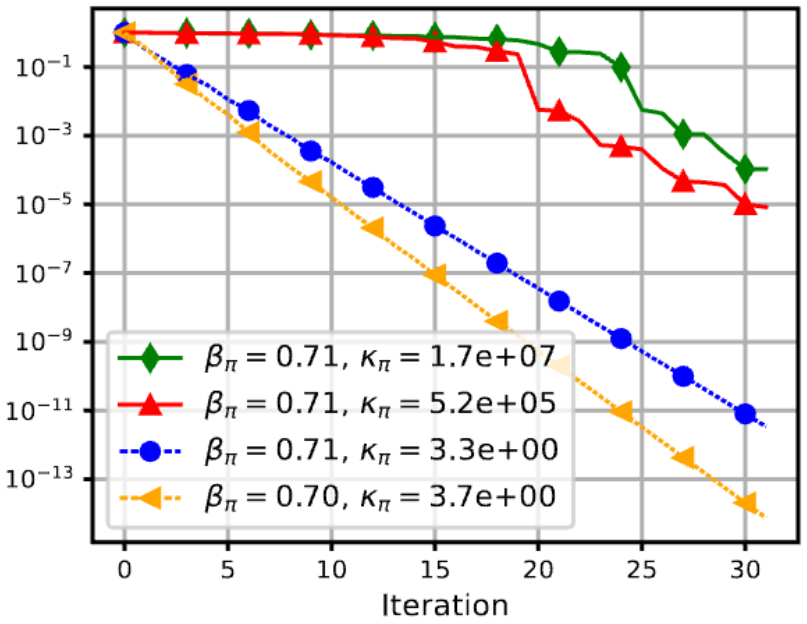
Push-sum decentralized averaging: simulation

- Metrics β_π and κ_π together precisely reflects the influence of network on push-sum averaging

$\kappa_\pi = \text{const. and } \beta_\pi \to 1$



$\beta_\pi = \text{const. and } \kappa_\pi \to \infty$



- Both metrics are indispensable

Push-sum gradient tracking

- Recall the distributed stochastic optimization problem

$$\min_{x\in\mathbb{R}^d} \quad f(x)=\frac{1}{n}\sum_{i=1}^{n} f_i(x), \quad \text{where} \quad f_i(x)=\mathbb{E}_{\xi_i\sim D_i}F(x;\xi_i).$$

- Given a column-stochastic matrix W, the push-sum gradient tracking [NOS17] is

$$\begin{aligned}
\mathbf{x}^{(k+1)} &= W(\mathbf{x}^{(k)}-\gamma\mathbf{y}^{(k)})\\
v^{(k+1)} &= Wv^{(k)}\\
V^{(k+1)} &= \mathrm{diag}(v^{(k+1)})\\
\mathbf{w}^{(k+1)} &= {V^{(k+1)}}^{-1}\mathbf{x}^{(k+1)}\\
\mathbf{y}^{(k+1)} &= W(\mathbf{y}^{(k)}+\nabla F(\mathbf{w}^{(k+1)};\boldsymbol{\xi}^{(k+1)})-\nabla F(\mathbf{w}^{(k)};\boldsymbol{\xi}^{(k)}))
\end{aligned}$$

Push-sum gradient tracking: convergence

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Theorem 3

Suppose each $f_i(x)$ is L-smooth, each local stochastic gradient $\nabla F(x;\xi_i)$ is unbiased and has bounded variance. Given a column-stochastic weight matrix W, the convergence rate of Push-DIGing satisfies

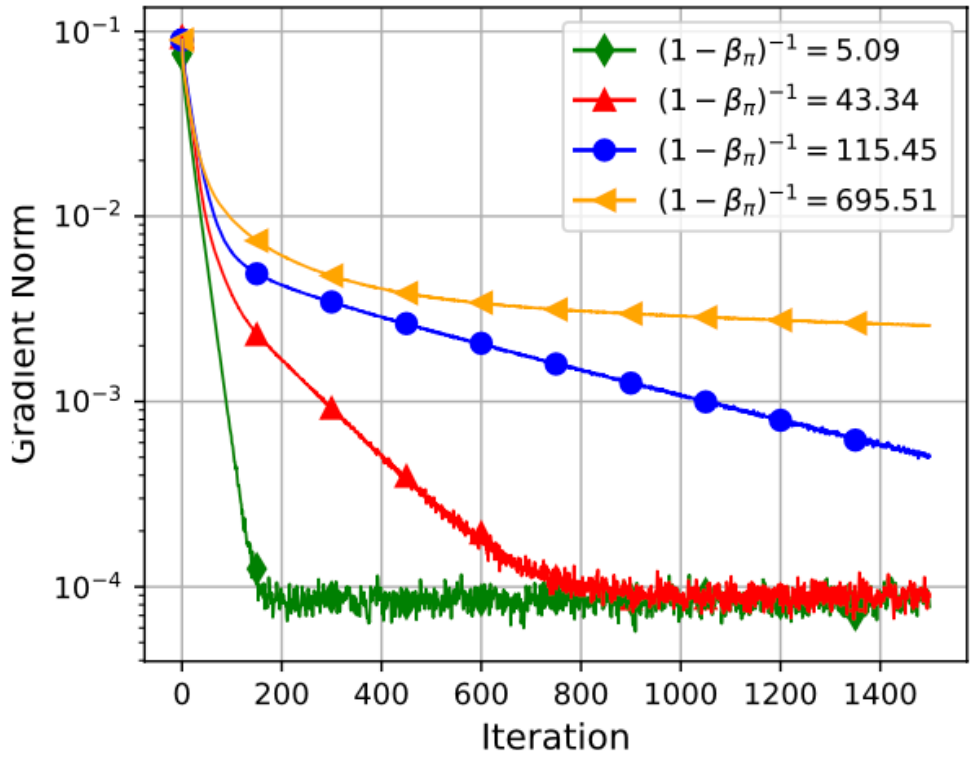
$$\frac{1}{K}\sum_{i=0}^{K-1}\mathbb{E}[\|\nabla f(\bar{x}^{(k)}\|_2^2] \lesssim \frac{\sigma}{\sqrt{nK}} + \frac{\beta_\pi^{\frac{2}{3}}\kappa_\pi^{\frac{5}{3}}\sigma^{\frac{2}{3}}}{(1-\beta_\pi)K^{\frac{2}{3}}} + \frac{\beta_\pi\kappa_\pi^3(1+\kappa_\pi\beta_\pi)}{(1-\beta_\pi)^2K} + \frac{1}{K}.$$

The **first** rate that clarifies the influence of digraphs on decentralized algorithms

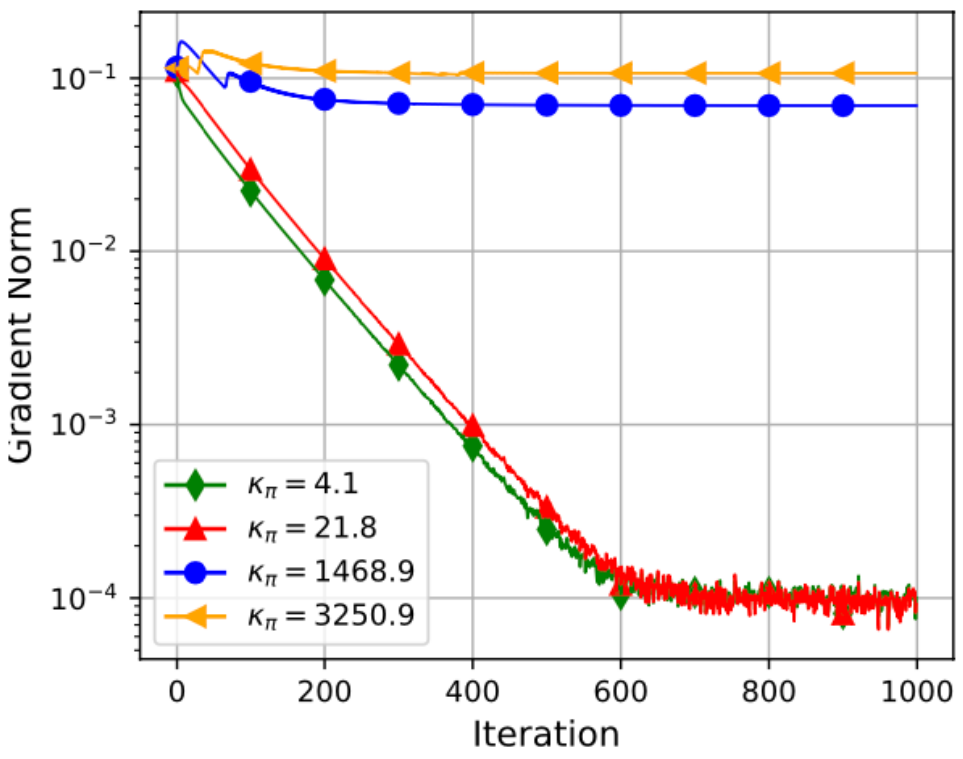
Push-sum gradient tracking: simulation

- Logistic regression with non-convex regularization terms

$\kappa_\pi = \text{const. and } \beta_\pi \to 1$



$\beta_\pi = \text{const. and } \kappa_\pi \to \infty$



Push-sum gradient tracking: comparison with existing works

Algorithm	Rate (A.)	Rate (F.T.)	Transient Stage
Gradient-Push [4]	$\frac{\sigma\sqrt{L\Delta}}{\sqrt{nK}}$	N.A.	N.A.
Push-DIGing [15]	$\frac{\sigma\sqrt{L\Delta}}{\sqrt{nK}}$	N.A.	N.A.
Push-DIGing (Ours)	$\frac{\sigma\sqrt{L\Delta}}{\sqrt{nK}}$	$\frac{\sigma\sqrt{L\Delta}}{\sqrt{nK}} + \frac{\beta_\pi \kappa_\pi^3 (1+\kappa_\pi \beta_\pi)}{(1-\beta_\pi)^2 K}$	$\frac{n\kappa_\pi^8}{(1-\beta_\pi)^4}$

The influence of κ_π is substantial, especially when $\kappa_\pi = O(2^n)$

What is the **best-possible dependence** on κ_π and β_π ?

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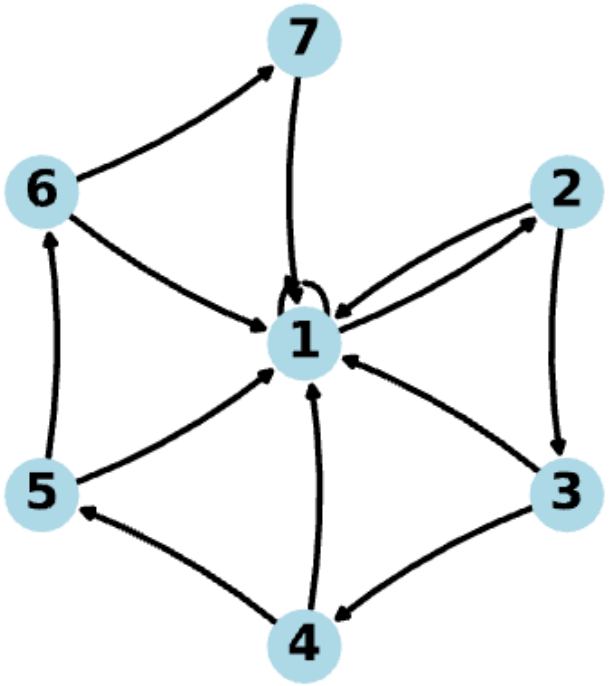
Part 05

Lower bound over digraphs

A challenging digraph



For this graph, we prove that

$$\pi \propto \left(2^{n-1}, 2^{n-2}, \ldots, 1\right)^\top \text{ and } \kappa_\pi = 2^{n-1}$$

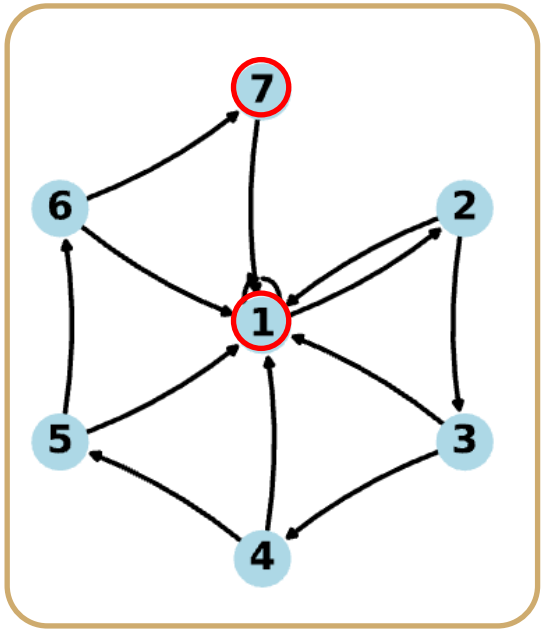
Also, the spectral gap is constant

$$\beta_\pi = 1/\sqrt{2} \quad \forall n$$

Lower bound of communication rounds: core idea

- For non-convex centralized deterministic optimization, the lower bound rate is $\Omega(L\Delta/K)$

- For the digraph below, each effective message passing requires n hops.



require n rounds per iteration

Communication lower bound

$$\Omega(\frac{nL\Delta}{K})$$

$$n = \frac{1+\ln(\kappa_\pi)}{1-\beta_\pi}$$

Communication lower bound

$$\Omega(\frac{(1+\ln(\kappa_\pi))L\Delta}{1-\beta_\pi})$$

Convergence lower bound over digraphs

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Theorem 4

For any given $L \geq 0$, $n \geq 2$, $\sigma \geq 0$, and $\tilde{\beta} \in [1/\sqrt{2}, 1-1/n]$, there exists a set of loss functions $\{f_i\}_{i=1}^n \in \mathcal{F}_{\Delta,L}$, a set of stochastic gradient oracles in $\mathcal{O}_{\sigma^2}$, and a column-stochastic matrix $W \in \mathbb{R}^{n\times n}$ with $\beta_\pi = \tilde{\beta}$ and $\ln(\kappa_\pi) = \Omega(n(1-\beta_\pi))$, such that the convergence of any $A \in \mathcal{A}_W$ starting form $x_i^{(0)} = x^{(0)}, \forall\, 1 \leq i \leq n$ with K iterations is lower bounded by

$$\mathbb{E}[\|\nabla f(x^{(K)})\|_2^2] = \Omega\left(\frac{\sigma\sqrt{L\Delta}}{\sqrt{nK}} + \frac{(1+\ln(\kappa_\pi))L\Delta}{(1-\beta_\pi)K}\right).$$

Big gap between lower and upper bound

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Push-DIGing (Ours)	$\frac{\sigma\sqrt{L\Delta}}{\sqrt{nK}}$	$\frac{\sigma\sqrt{L\Delta}}{\sqrt{nK}} + \frac{\beta_\pi \kappa_\pi^3 (1+\kappa_\pi \beta_\pi)}{(1-\beta_\pi)^2 K}$	$\frac{n\kappa_\pi^8}{(1-\beta_\pi)^4}$
Lower Bound (Ours)	$\frac{\sigma\sqrt{L\Delta}}{\sqrt{nK}}$	$\frac{\sigma\sqrt{L\Delta}}{\sqrt{nK}} + \frac{(1+\ln(\kappa_\pi))L\Delta}{(1-\beta_\pi)K}$	$\frac{n(1+\ln(\kappa_\pi))^2}{(1-\beta_\pi)^2}$

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Part 06

Optimal decentralized algorithm over digraphs

Recall push-sum gradient tracking

$$
\begin{aligned}
\mathbf{x}^{(k+1)} &= W(\mathbf{x}^{(k)} - \gamma \mathbf{y}^{(k)}) \\
v^{(k+1)} &= W v^{(k)} \\
V^{(k+1)} &= \mathrm{diag}(v^{(k+1)}) \\
\mathbf{w}^{(k+1)} &= {V^{(k+1)}}^{-1} \mathbf{x}^{(k+1)} \\
\mathbf{y}^{(k+1)} &= W(\mathbf{y}^{(k)} + \nabla F(\mathbf{w}^{(k+1)}; \boldsymbol{\xi}^{(k+1)}) - \nabla F(\mathbf{w}^{(k)}; \boldsymbol{\xi}^{(k)}))
\end{aligned}
$$

Multiple-Gossip Push-sum Gradient tracking

Improve it with **multiple-gossip** and **mini-batch gradient**

$$
\begin{aligned}
\mathbf{x}^{(k+1)} &= W^R(\mathbf{x}^{(k)} - \gamma \mathbf{y}^{(k)}) \\
v^{(k+1)} &= W^R v^{(k)} \\
V^{(k+1)} &= \mathrm{diag}(v^{(k+1)}) \\
\mathbf{w}^{(k+1)} &= {V^{(k+1)}}^{-1} \mathbf{x}^{(k+1)} \\
\mathbf{y}^{(k+1)} &= W^R(\mathbf{y}^{(k)} + g_R^{(k+1)} - g_R^{(k)})
\end{aligned}
$$

where $g_R^{(k)} = \frac{1}{R}\sum_{r=1}^{R} \nabla F(\mathbf{w}^{(k)}; \boldsymbol{\xi}^{(k,r)})$ is the mini-batch stochastic gradient

Multiple-Gossip Push-sum Gradient tracking

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Theorem 5

Suppose each $f_i(x)$ is L-smooth, each local stochastic gradient $\nabla F(x;\xi_i)$ is unbiased and has bounded variance, and the weight matrix W is column-stochastic, by setting $R = \frac{(1+\sqrt{\ln(\kappa_\pi)})^2}{1-\beta_\pi}$, $T = KR$, the convergence of MG-Push-DIGing satisfies

$$\frac{1}{K}\sum_{k=1}^{K}\mathbb{E}[\|\nabla f(\bar{x}^{(k)})\|_2^2] = \widetilde{\mathcal{O}}\left(\frac{\sigma\sqrt{L\Delta}}{\sqrt{nT}} + \frac{(1+\ln(\kappa_\pi))L\Delta}{(1-\beta_\pi)T}\right),$$

where $\widetilde{\mathcal{O}}(\cdot)$ absorbs logarithmic factors independent of κ_π and β_π.

Lower bound and upper bound are nearly-matched

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Algorithm	Rate (A.)	Rate (F.T.)	Transient Stage
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MG-Push-DIGing (Ours)	$\frac{\sigma\sqrt{L\Delta}}{\sqrt{nK}}$	$\frac{\sigma\sqrt{L\Delta}}{\sqrt{nK}} + \frac{(1+\ln(\kappa_\pi))L\Delta}{(1-\beta_\pi)K}$	$\frac{n(1+\ln(\kappa_\pi))^2}{(1-\beta_\pi)^2}$
Lower Bound (Ours)	$\frac{\sigma\sqrt{L\Delta}}{\sqrt{nK}}$	$\frac{\sigma\sqrt{L\Delta}}{\sqrt{nK}} + \frac{(1+\ln(\kappa_\pi))L\Delta}{(1-\beta_\pi)K}$	$\frac{n(1+\ln(\kappa_\pi))^2}{(1-\beta_\pi)^2}$

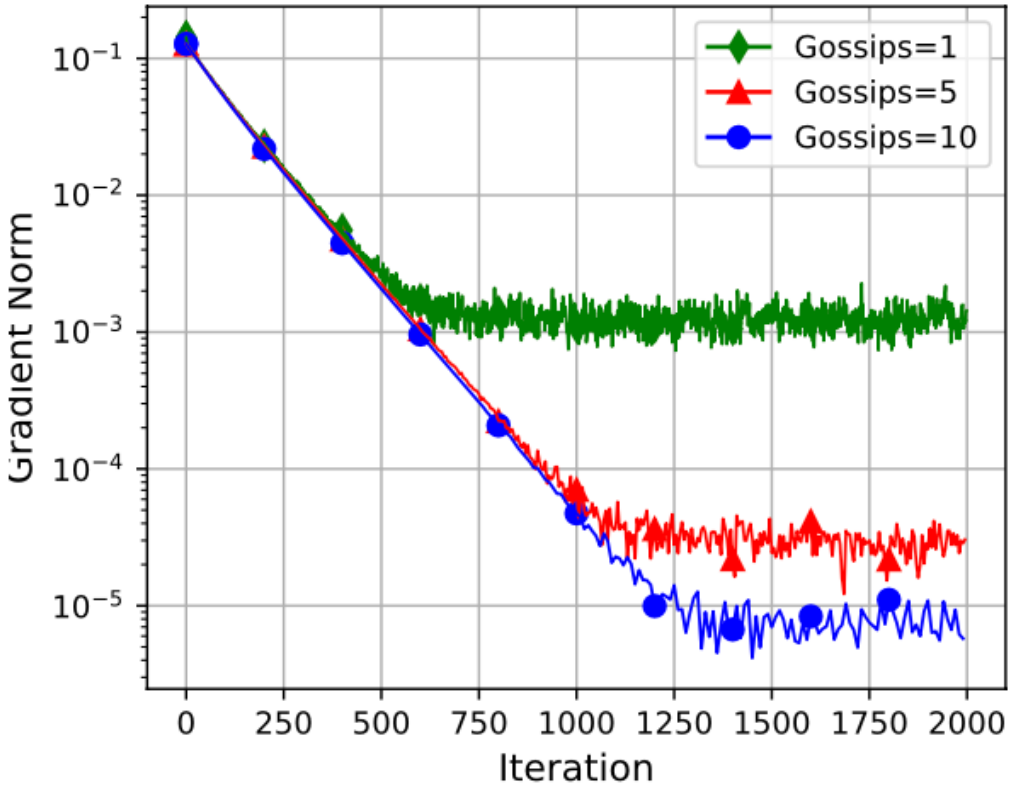
Our lower bound is nearly-tight

Our developed MG-Push-DIGing algorithm is nearly optimal

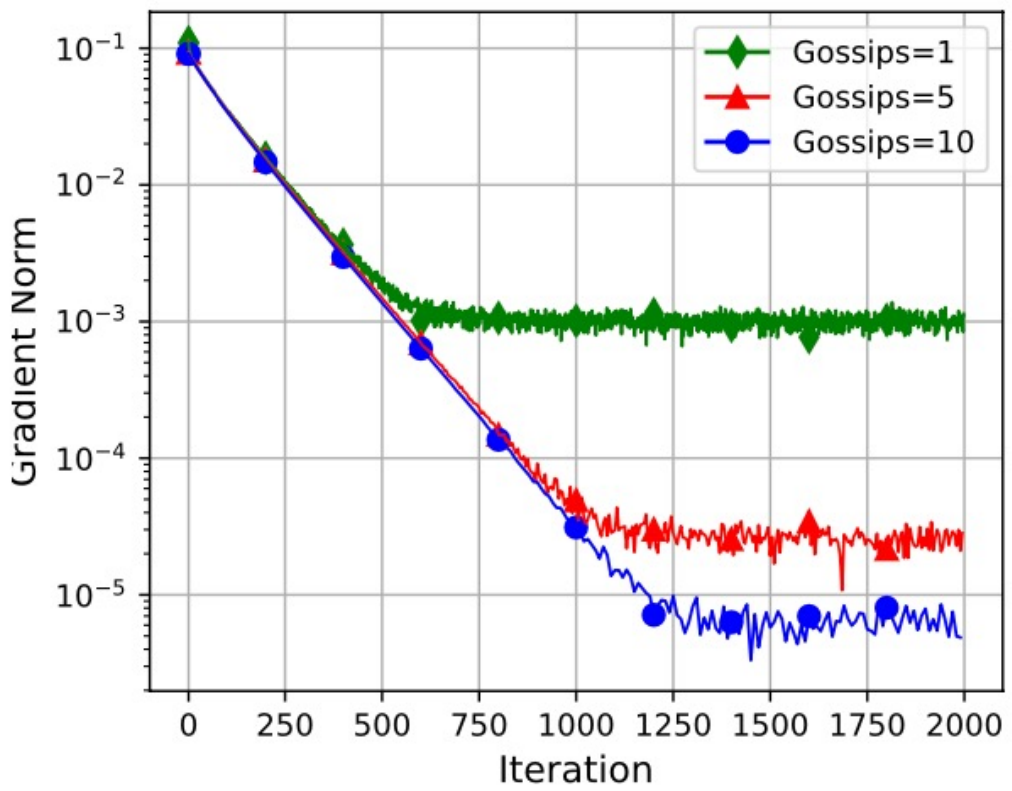
Simulations

- Logistic regression with non-convex regularizer. Multiple Gossip improves both κ_π and $(1-\beta_\pi)^{-1}$

$\kappa_\pi = 4804.49,\ (1-\beta_\pi)^{-1} = 2.34.$



$\kappa_\pi = 6.33,\ (1-\beta_\pi)^{-1} = 51.24.$



Summary

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- The influence of diagraphs on decentralized algorithms is known in previous work
- We identify two effective metrics (spectral gap and equilibrium skewness) that can jointly capture the influence of digraphs
- The two metrics are orthogonal to each other; both of them are indispensable
- We establish the lower bound and develop a nearly-optimal algorithm to attain the lower bound

Future work

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- Clarify the influence of row-stochastic digraphs on decentralized algorithms

  [Ongoing]

- Clarify the influence of digraphs over push-pull algorithms

  [Almost done]

- Lower bound and optimal algorithms for pull-sum and push-sum family

  [Very challenging]

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Thank you!

Liyuan Liang, Xinmeng Huang, Ran Xin, and Kun Yuan, "*Towards Better Understanding the Influence of Directed Networks on Decentralized Stochastic Optimization*", arXiv:2312.04928, 2023