Introduction to Large Language Models

Momentum SGD

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Main contents in this lecture

- Momentum SGD
- Convergence analysis
- Lower bound

Gradient descent can be slow

- Gradient descent can be very slow for ill-conditioned problems
- For example, GD converges very slow when μ/L is sufficiently small[1]

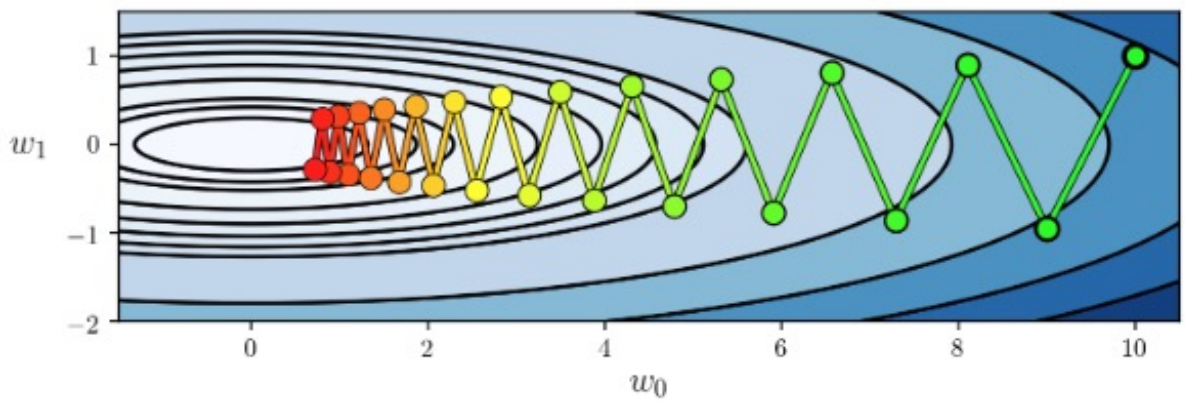

Figure: GD converges slow for ill-conditioned problem

[1] Image is from `https://github.com/jermwatt/machine_learning_refined`

Gradient descent with Polyak's momentum

- We have to alleviate the "Zig-Zag" to accelerate the algorithm

- **Polyak's momentum** method, a.k.a, **heavy-ball** gradient method

$$x_k = x_{k-1} - \gamma \nabla f(x_{k-1}) + \beta(x_{k-1} - x_{k-2})$$

where $\beta \in (0, 1)$ is the momentum parameter

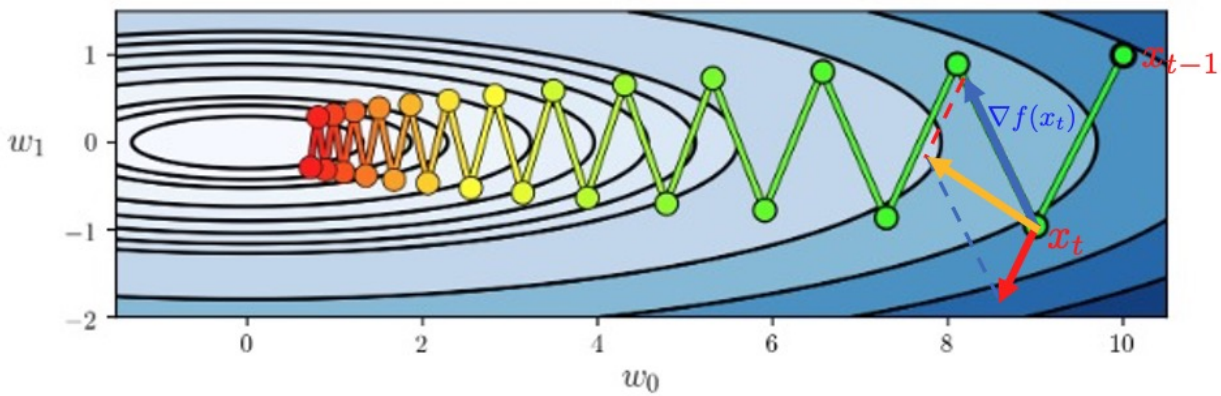

Figure: Momentum can alleviate the "Zig-Zag"

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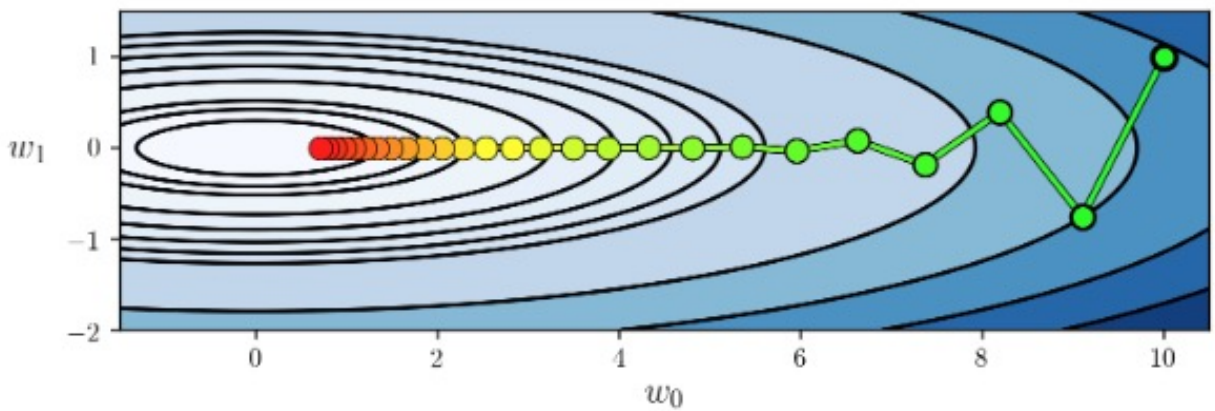

Figure: Momentum can alleviate the "Zig-Zag"

Gradient descent with Nesterov's momentum

- Gradient descent with **Nesterov's momentum**, a.k.a, **Nesterov accelerated gradient (NAG)** method

$$y_{k-1} = x_{k-1} + \beta(x_{k-1} - x_{k-2})$$
$$x_k = y_{k-1} - \gamma \nabla f(y_{k-1})$$

where $\beta \in (0, 1)$ is the momentum parameter

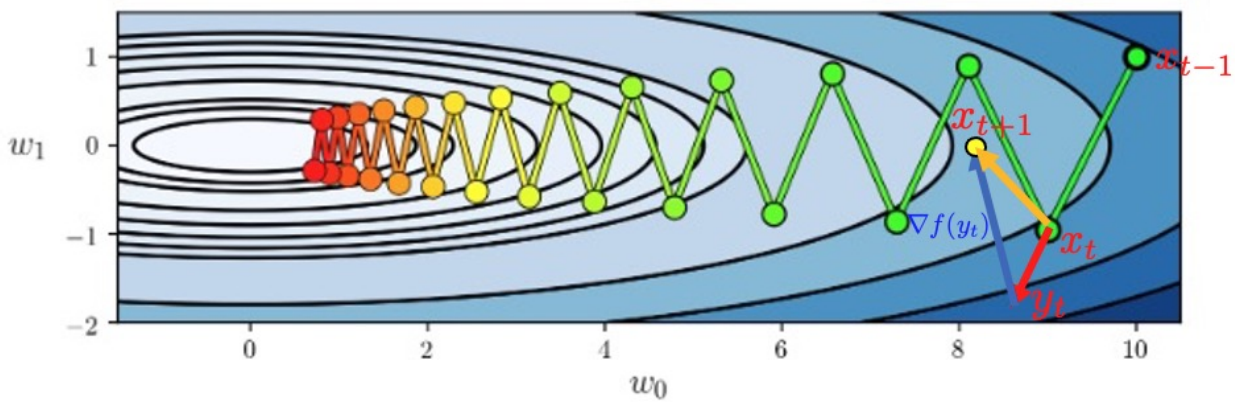

Figure: Nesterov method can alleviate the "Zig-Zag"

The convergence rate of accelerated GD

Method	Convexity	Rate	Complexity
GD	Non-convex	$O(L/k)$	$O(L/\epsilon)$
	Convex	$O(L/k)$	$O(L/\epsilon)$
	Strongly convex	$O((1-\frac{\mu}{L})^k)$	$O(\frac{L}{\mu}\log(1/\epsilon))$
NAG	Non-convex	$O(L/k)$	$O(L/\epsilon)$
	Convex	$O(L/k^2)$	$O(L/\sqrt{\epsilon})$
	Strongly convex	$O((1-\sqrt{\frac{\mu}{L}})^k)$	$O(\sqrt{\frac{L}{\mu}}\log(1/\epsilon))$
Lower bound	Non-convex	$\Omega(L/k)$	$\Omega(L/\epsilon)$
	Convex	$\Omega(L/k^2)$	$\Omega(L/\sqrt{\epsilon})$
	Strongly convex	$\Omega((1-\sqrt{\frac{\mu}{L}})^k)$	$\Omega(\sqrt{\frac{L}{\mu}}\log(1/\epsilon))$

NAG and GD has the **same** rate and complexity in non-convex scenarios (**?**);

GD is optimal and cannot be improved!

Momentum SGD

- Recall the stochastic optimization

$$\min_{x \in \mathbb{R}^d} \quad \mathbb{E}_{\xi \in \mathcal{D}}[F(x; \xi)]$$

- The standard SGD algorithm is

$$x_{k+1} = x_k - \gamma \nabla F(x_k; \xi_k)$$

- Momentum SGD

$$x_{k+1} = x_k - \gamma \nabla F(x_k; \xi_k) + \beta(x_k - x_{k-1}) \tag{1}$$

where $\beta \in [0, 1)$ is the momentum coefficient

Momentum SGD

- Momentum SGD can be rewritten into

$$m_k = \beta m_{k-1} + \nabla F(x_k; \xi_k)$$
$$x_{k+1} = x_k - \gamma m_k$$

  where $m_0 = 0$. This is how PyTorch implement it[2].

- To see it, we have the following recursion from the second line

$$\beta x_k = \beta x_{k-1} - \gamma \beta m_{k-1}.$$

  Subtract it from the second line, we have

$$\begin{aligned} x_{k+1} - \beta x_k &= x_k - \beta x_{k-1} - \gamma(m_k - \beta m_{k-1}) \\ &= x_k - \beta x_{k-1} - \gamma \nabla F(x_k; \xi_k). \end{aligned}$$

  Regrouping the terms, we achieve Momentum SGD recursion (1).

[2] https://pytorch.org/docs/stable/generated/torch.optim.SGD.html

SGD with Nesterov momentum

- Consider SGD with Nesterov momentum

$$\begin{aligned} y_k =& (1-\beta_k)x_{k-1} + \beta_k v_{k-1}, \\ x_k =& y_{k-1} - \gamma \nabla F(y_{k-1}; \xi_k), \\ v_k =& \beta_k^{-1} x_k + (1-\beta_k^{-1}) x_{k-1}, \end{aligned}$$

  where β_k is a time-varying momentum parameter

- As usual, we assume unbiased stochastic gradient and bounded variance

$$\mathbb{E}[\nabla F(y_{k-1}; \xi_k) \mid \mathcal{F}_k] = \nabla f(y_{k-1}), \tag{2}$$

$$\mathbb{E}[\|\nabla F(y_{k-1}; \xi_k) - \nabla f(y_{k-1})\|_2^2 \mid \mathcal{F}_k] \leq \sigma^2. \tag{3}$$

SGD with Nesterov momentum

Theorem

Suppose $f(x)$ is L-smooth and convex, and conditions (2) and (3) hold. If we choose proper γ (see our notes), SGD with Nesterov momentum converges at the following rate:

$$\mathbb{E}[f(x_K) - f^\star] = \mathcal{O}\left(\sqrt{\frac{\sigma^2}{K}} + \frac{1}{K^2}\right)$$

- Reduce to accelerated GD when $\sigma^2 = 0$.
- Recall standard SGD converges as follows

$$\mathbb{E}[f(x_K) - f^\star] = \mathcal{O}\left(\sqrt{\frac{\sigma^2}{K}} + \frac{1}{K}\right)$$

SGD with Nesterov momentum accelerates SGD when σ^2 is small or when $1/K$ dominates; but **cannot** accelerate SGD when K is large

Summary

- Momentum SGD is equivalent to vanilla SGD when learning rate is small and momentum coefficient is not close to 1

- SGD with Nesterov momentum can accelerate SGD when σ^2 is small or when $1/K$ dominates the rate due to

$$\text{(SGD)} \qquad \mathbb{E}[f(x_K) - f^\star] = \mathcal{O}\left(\sqrt{\frac{\sigma^2}{K}} + \frac{1}{K}\right)$$

- When σ^2 or K is large, vanilla SGD has achieved the optimal rate; no algorithm can improve vanilla SGD on the order of convergence rate

- There is a gap between the theoretical understanding and real implementations in momentum