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Optimization for Deep Learning

Lecture 13-5: Network topologies

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PART 01

Trade-off between sparsity and connectivity

DSGD is communication-efficient but has slower convergence

- The efficient comm. comes with a cost: slower convergence
- Partial average $x_i^+ = \sum w_{ij} x_j$ is less effective to aggregate information than global average
- The average effectiveness can be evaluated by **graph spectral gap**:

$$\rho = \|W - \frac{1}{n}\mathbb{1}\mathbb{1}^T\|_2 \in (0,1) \text{ where } W = [w_{ij}] \in \mathbb{R}^{n\times n}$$

  - Well-connected topology has $\rho \to 0$, e.g. fully-connected topology
  - Sparsely-connected topology has $\rho \to 1$, e.g. ring has $\rho = O(1 - \frac{1}{n^2})$
  - ρ or $1-\rho$ essentially gauges the **graph connectivity**

DSGD convergence rate

- Convergence comparison (non-convex and data-homogeneous scenario) [KLB+20]:

$$\text{P-SGD}: \quad \frac{1}{T}\sum_{k=1}^{T}\mathbb{E}\|\nabla f(\bar{x}^{(k)})\|^2 = O\Big(\frac{\sigma}{\sqrt{nT}}\Big)$$

$$\text{D-SGD}: \quad \frac{1}{T}\sum_{k=1}^{T}\mathbb{E}\|\nabla f(\bar{x}^{(k)})\|^2 = O\Big(\frac{\sigma}{\sqrt{nT}} + \underbrace{\frac{\rho^{2/3}\sigma^{2/3}}{T^{2/3}(1-\rho)^{1/3}}}_{\text{extra overhead}}\Big)$$

where σ^2 is the gradient noise, and T is the number of iterations

- D-SGD can asymptotically converge as fast as P-SGD when $T \to \infty$; the first term dominates; reach **linear speedup** asymptotically

- But D-SGD **requires more iteration** to reach that stage due to the overhead caused by partial average

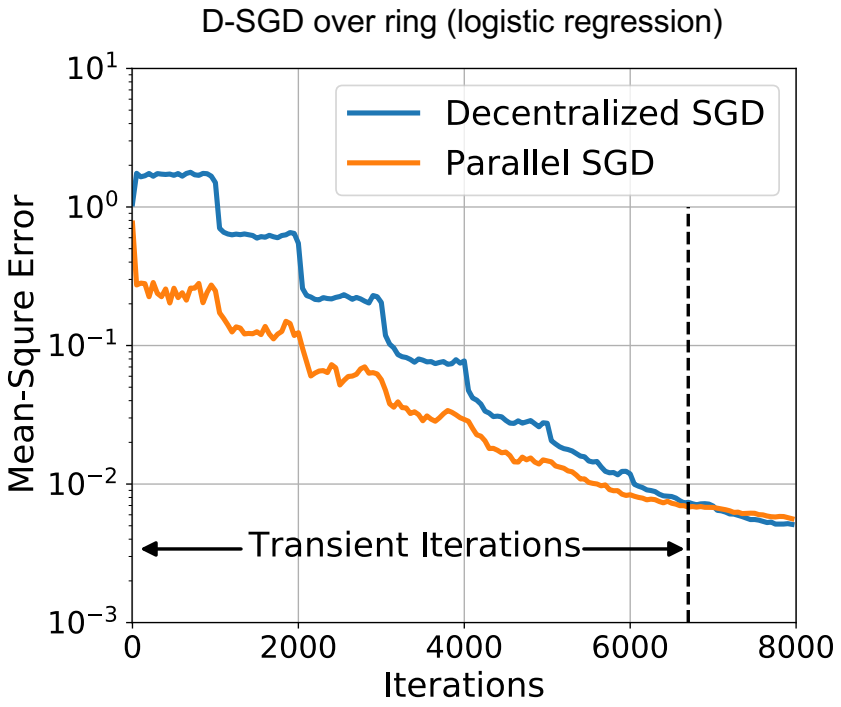
Transient iterations

- Definition [POP21]: number of iterations before D-SGD achieves linear speedup

- D-SGD for non-convex and data-homogeneous scenario has $O(n^3(1-\rho)^{-2})$ transient iterations

$$\frac{\rho^{2/3}\sigma^{2/3}}{T^{2/3}(1-\rho)^{1/3}} \leq \frac{\sigma}{\sqrt{nT}} \quad \Longrightarrow \quad O(\frac{\rho^4 n^3}{(1-\rho)^2})$$

- Sparse topology $\rho \to 1$ incurs longer tran. Iters.



Trade-off between comm. cost and trans. iters.

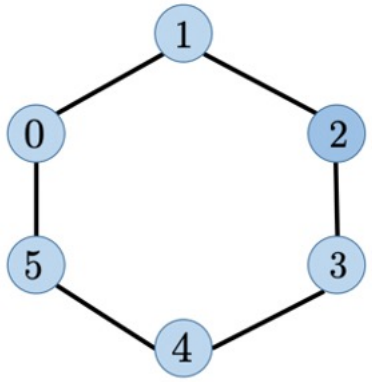
- Recall per-iter comm. $O(d_{\max})$ and trans. iters. $\Omega(n^3(1-\rho)^{-2})$

- Trade-off between per-iteration communication and transient iteration complexity

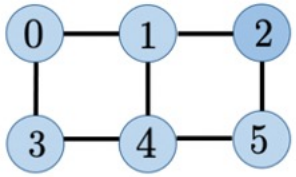
	Sparse topology	Dense topology
per-iter comm.	✓	×
trans. iter. complexity	×	✓

What topology to use?

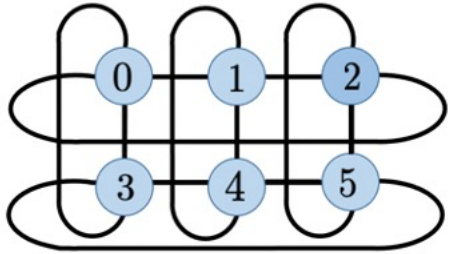
- Shall we use these common topologies to organize all nodes?



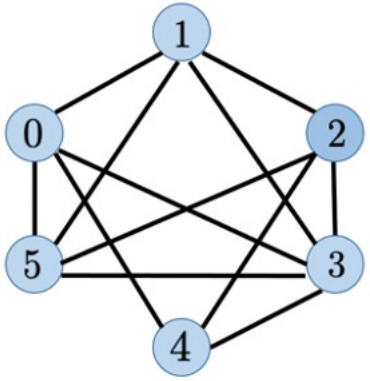
Ring



Grid



Torus



Erdos-Renyi Random

What topology to use?

- Communication cost v.s. transient iteration complexity in DSGD

Topology	Per-iter. Comm.	Trans. Iters. (iid scenario)
Ring	$O(1)$	$O(n^7)$
2D-Grid	$O(1)$	$\tilde{O}(n^5)$
2D-Torus	$O(1)$	$O(n^5)$
$\frac{1}{2}$-RandGraph	$O(n)$	$O(n^3)$

The smaller both comm. cost and tran. Iters. are, the better

- These topologies either have expensive communication cost or longer transient stage

- **Is there any topology that enables both cheap communication and fast convergence?**

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PART 02

Exponential graphs

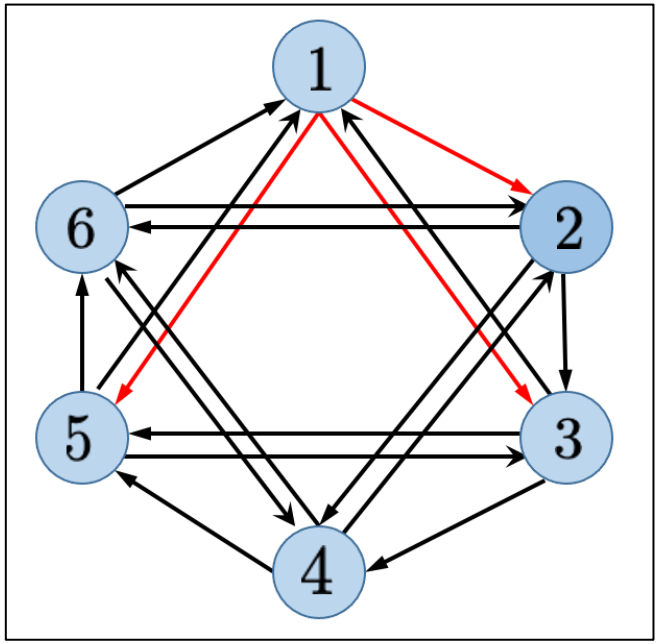
Static exponential graph: topology and per-iteration comm.

- Each node links to neighbors that are $2^0, 2^1, \cdots, 2^{\lfloor \log_2(n-1) \rfloor}$ away [ALB+19]

- In the figure, node 1 connects to node 2, 3 and 5.

- Each node has $\lceil \log_2(n) \rceil$ neighbors; per-iter comm. cost is $O(\log_2(n))$

- Empirically successful in deep training but less theoretically understood

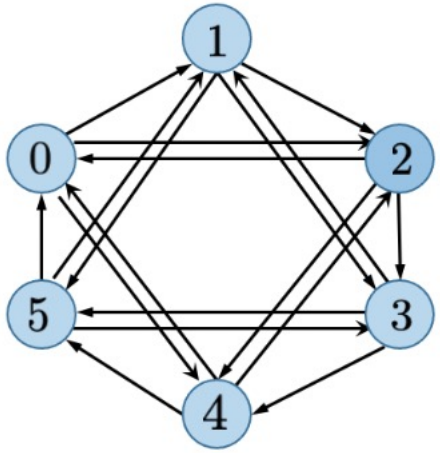


Static exponential graph: weight matrix

- The weight matrix associated with exponential graph is defined as

$$w_{ij}^{\exp} = \begin{cases} \frac{1}{\lceil \log_2(n) \rceil + 1} & \text{if } \log_2(\mathrm{mod}(j-i, n)) \text{ is an integer or } i = j \\ 0 & \text{otherwise.} \end{cases}$$

- An illustrating example:



$$W = \begin{bmatrix} \frac{1}{4} & 0 & \frac{1}{4} & 0 & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} & 0 & \frac{1}{4} & 0 & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & 0 & \frac{1}{4} & 0 \\ 0 & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & 0 & \frac{1}{4} \\ \frac{1}{4} & 0 & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & 0 \\ 0 & \frac{1}{4} & 0 & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \end{bmatrix}$$

Static exponential graph: connectivity

- Is the static exponential graph well connected?

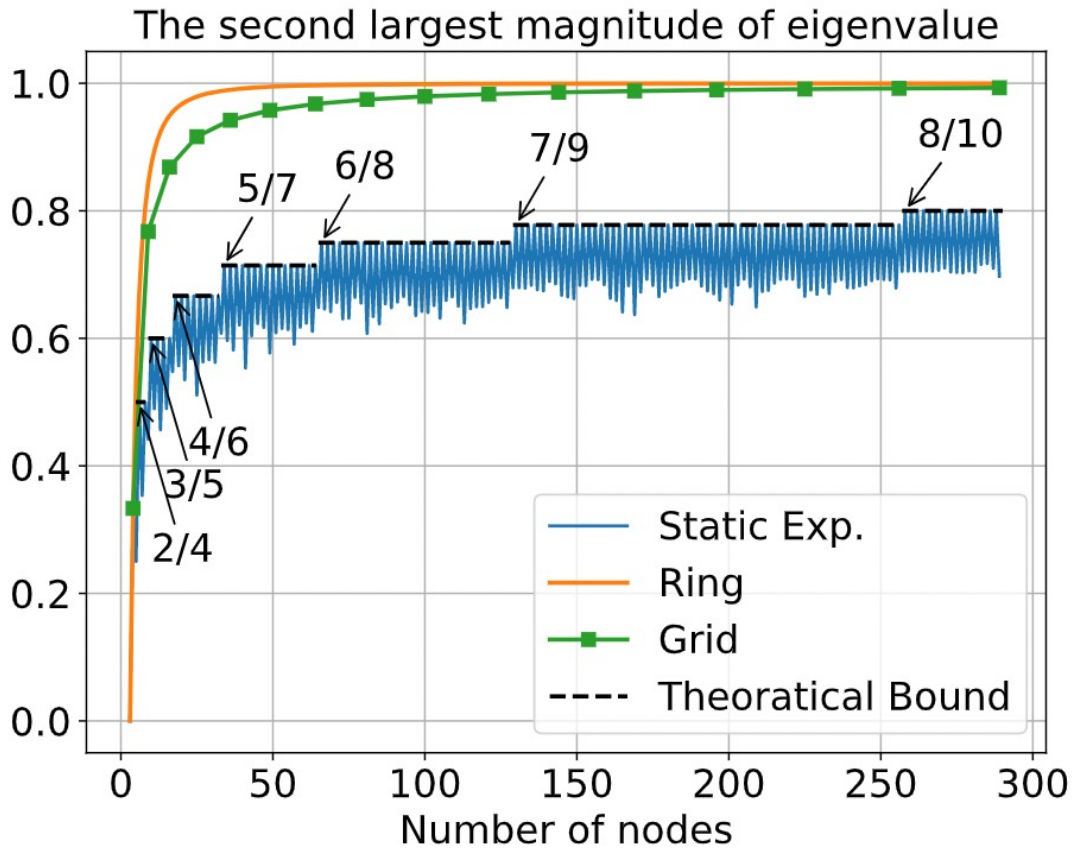
Theorem. Let $\tau = \lceil \log_2(n) \rceil$, and $\rho = \|W - \frac{1}{n}\mathbb{1}\mathbb{1}^T\|_2$ be the spectral gap. It holds that

$$\begin{cases} \rho = 1 - \dfrac{2}{\tau + 1}, & \text{when } n \text{ is even} \\ \rho < 1 - \dfrac{2}{\tau + 1}, & \text{when } n \text{ is odd} \end{cases}$$

- This theorem implies that exponential graph has $\rho(W) = O(1 - 1/\log_2(n))$

- Highly non-trivial proofs; requires smart utilization of Fourier transform

Static exponential graph: illustration of the spectral gap



- Our theoretical bound is very tight
- Spectral gap increases slowly when n grows

Static exponential graph: transient iterations in DSGD

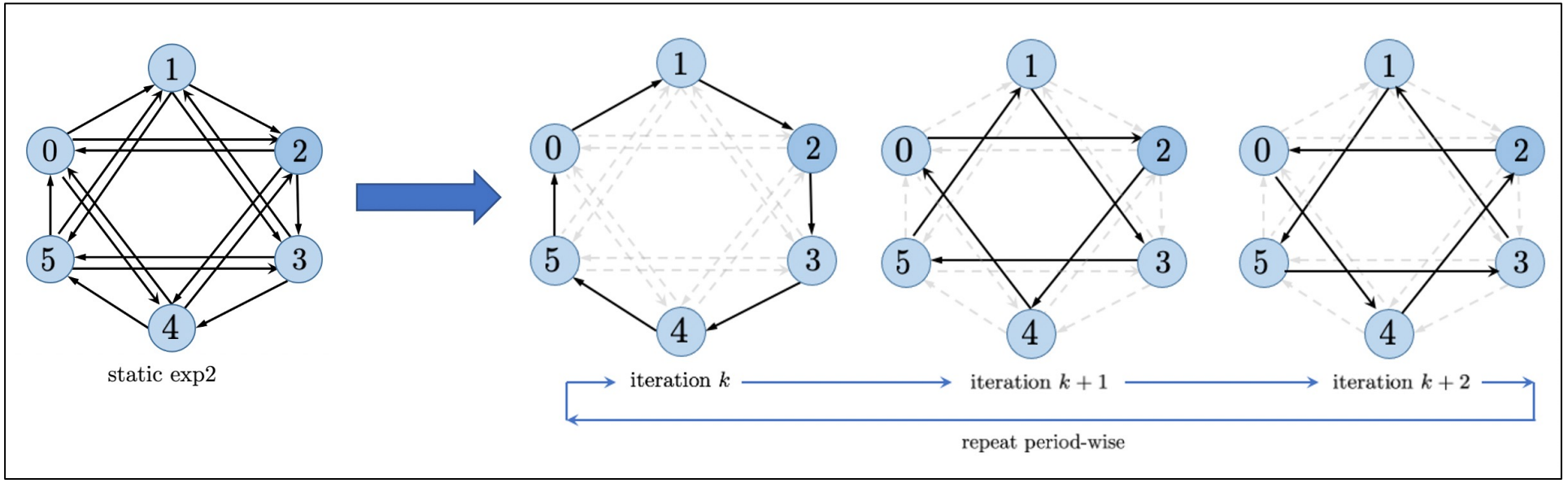
- Recall DSGD has transient iteration complexity $O(n^3/(1-\rho)^2)$ in iid scenarios
- With $\rho(W) = O(1 - 1/\log_2(n))$, exponential graphs have tran. iters. as $O(n^3 \log_2^2(n))$
- Per-iteration communication and transient iteration complexity are **nearly the best** (up to $\log_2(n)$)

Topology	Per-iter. Comm.	Trans. Iters. (iid scenario)
Ring	$O(1)$	$O(n^7)$
2D-Grid	$O(1)$	$\tilde{O}(n^5)$
2D-Torus	$O(1)$	$O(n^5)$
$\frac{1}{2}$-RandGraph	$O(n)$	$O(n^3)$
Static Exp	$\tilde{O}(1)$	$\tilde{O}(n^3)$

- **Can we achieve even better topology?**

One-peer exponential graph: topology

- **Split** exponential graph into a sequence of one-peer realizations;
- Each node has **exactly one** neighbor per iteration
- **O(1) per-iteration communication**; cheaper than ring (2 neighbors) or grid (3 or 4 neighbors)

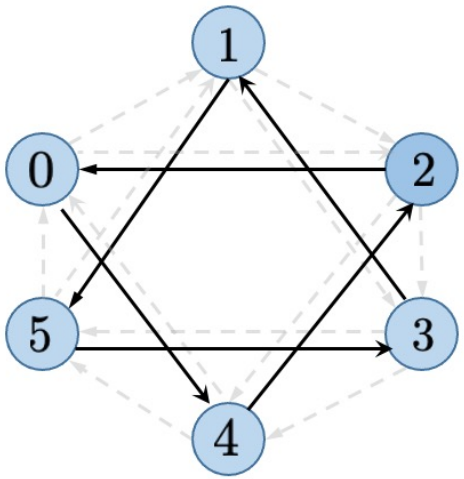


One-peer exponential graph: weight matrix

- We let $\tau = \lceil \log_2(n) \rceil$. The weight matrix $W^{(k)}$ is defined as

$$w_{ij}^{(k)} = \begin{cases} \frac{1}{2} & \text{if } \log_2(\text{mod}(j-i,n)) = \text{mod}(k,\tau) \\ \frac{1}{2} & \text{if } i = j \\ 0 & \text{otherwise.} \end{cases}$$

- An illustrating example



$$W = \begin{bmatrix} \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & 0 & 0 & 0 & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & 0 & 0 & 0 & \frac{1}{2} \end{bmatrix}$$

DSGD over one-peer exponential graph

Sample $W^{(k)}$ over one-peer exponential graph

$$x_i^{(k+\frac{1}{2})} = x_i^{(k)} - \gamma \nabla F(x_i^{(k)}; \xi_i^{(k)}) \quad \text{(Local update)}$$

$$x_i^{(k+1)} = \sum_{j \in \mathcal{N}_i} w_{ij}^{(k)} x_j^{(k+\frac{1}{2})} \quad \text{(Partial averaging)}$$

- DSGD with **time-varying** weight matrix;
- Per-iteration communication cost is **O(1)**; very efficient

One-peer exponential graph: Periodic exact average

- While one-peer exponential graph is **sparse**, it is **effective** to aggregate information

Theorem. Suppose $\tau = \log_2(n)$ is a positive integer. It holds that

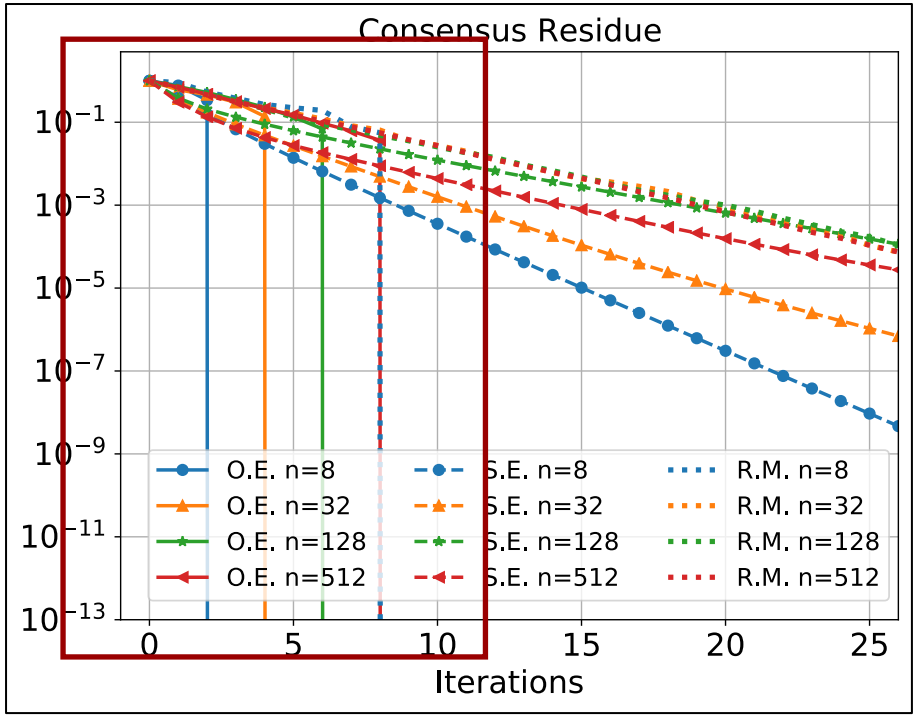
$$W^{(k+\ell)}W^{(k+\ell-1)}\cdots W^{(k+1)}W^{(k)} = \frac{1}{n}\mathbb{1}\mathbb{1}^T$$

for any integer $k \geq 0$ and $\ell \geq \tau - 1$.

- While each realization is sparser, a sequence (with length τ) of one-peer graphs will enable effective global averaging

One-peer exponential graph: Periodic exact average

- We examine $\|\frac{1}{n}\mathbb{1}\mathbb{1}^T x - \prod_{k=0}^{T} W^{(k)} x\|$ for a vector x
- $\frac{1}{n}\mathbb{1}\mathbb{1}^T x$ is the global average
- $\prod_{k=0}^{T} W^{(k)} x$ is the partial average after T iterations
- One-peer exp. achieves global average after $\log_2(n)$ iters.



Apply one-peer exponential graph to DSGD

Assumption *(1) Each $f_i(x)$ is L-smooth; (2) Each gradient noise is unbiased and has bounded variance σ^2; (3) Each local distribution D_i is identical (iid)*

Theorem *Under the above assumptions and with $\gamma = O(1/\sqrt{T})$, let $\tau = \log_2(n)$ be an integer, DSGD with one-peer exponential graph will converge at*

$$\frac{1}{T}\sum_{k=1}^{T}\mathbb{E}\|\nabla f(\bar{x}^{(k)})\|^2 = O\Big(\frac{\sigma}{\sqrt{nT}} + \underbrace{\frac{\sigma^{2/3}\log_2^{1/3}(n)}{T^{2/3}}}_{\text{extra overhead}}\Big)$$

Novel analysis; require new tricks to utilize periodic exact average to establish tight convergence

Static v.s. one-peer exponential graph

- Convergence rate of DSGD over static and one-peer exponential graphs are

$$\text{Static exp.} \quad O\Big(\frac{\sigma}{\sqrt{nT}} + \frac{\sigma^{2/3}}{T^{2/3}(1-\rho)^{1/3}}\Big) \quad (\text{where } 1-\rho = O(1/\log_2(n)))$$

$$\text{One-peer exp.} \quad O\Big(\frac{\sigma}{\sqrt{nT}} + \frac{\sigma^{2/3}\log_2^{1/3}(n)}{T^{2/3}}\Big)$$

- DSGD with one-peer exp. converges **as fast as** static exp.; **a surprising result**.

- DSGD with both graphs are with the **same** transient iteration complexity $O(n^3\log_2^2(n))$

- The communication cost saving in one-peer exponential graph is a **free lunch**

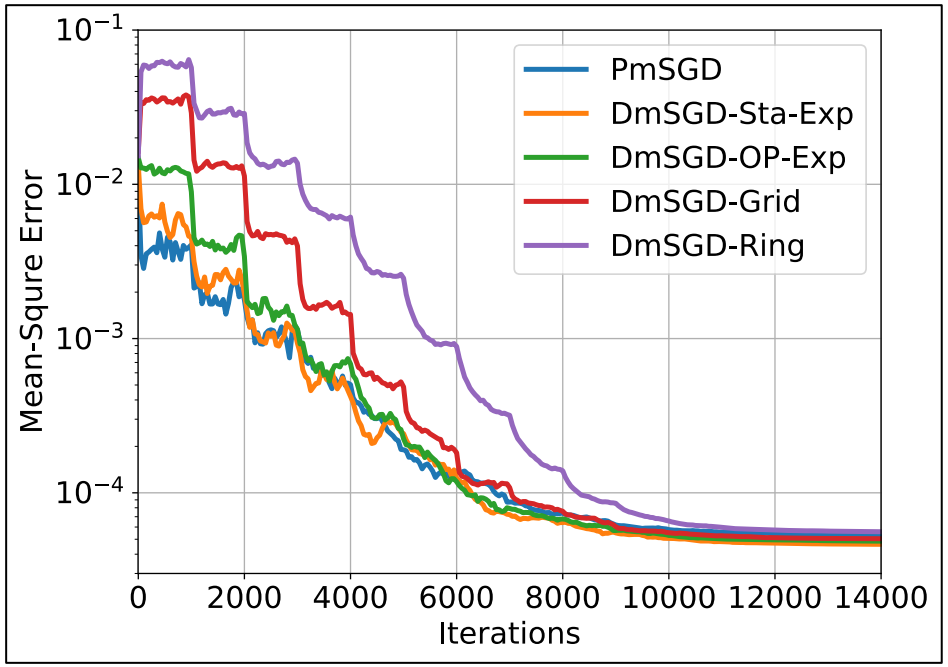
Compare one-peer exp. with other topologies

Topology	Per-iter. Comm.	Trans. Iters. (iid scenario)
Ring	$O(1)$	$O(n^7)$
2D-Grid	$O(1)$	$\tilde{O}(n^5)$
2D-Torus	$O(1)$	$O(n^5)$
$\frac{1}{2}$-RandGraph	$O(n)$	$O(n^3)$
Static Exp	$\tilde{O}(1)$	$\tilde{O}(n^3)$
One-Peer Expo	$O(1)$	$\tilde{O}(n^3)$

We recommend using one-peer exponential graph in deep training.

Exponential graphs have shorter tran. iters.

- Illustration of the tran. iters. on DSGD (momentum version) for logistic regression
- DSGD over one-peer exponential graph converges faster than other topologies



Comparison over 32 nodes

Experiments in deep training (image classification)

ImageNet-1K dataset

1.3M training images

50K test images

1K classes

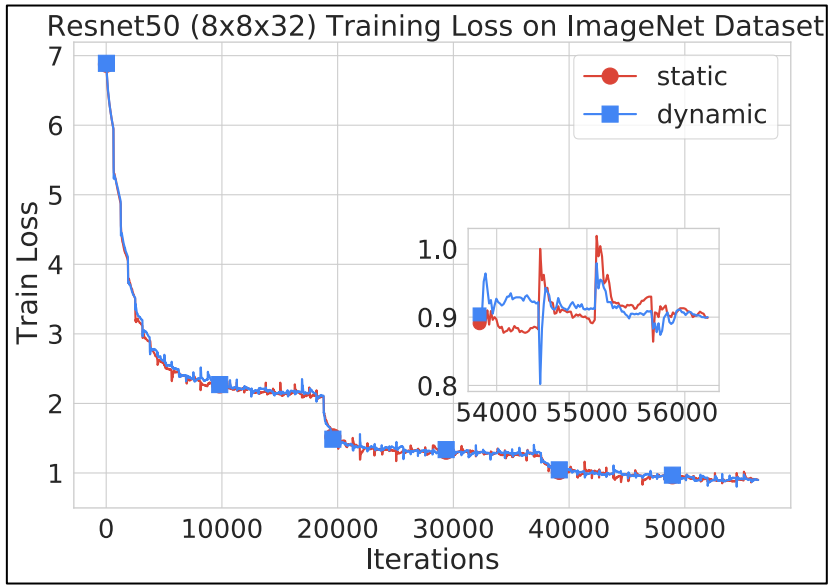
DNN model: ResNet-50 (25.5M parameters)

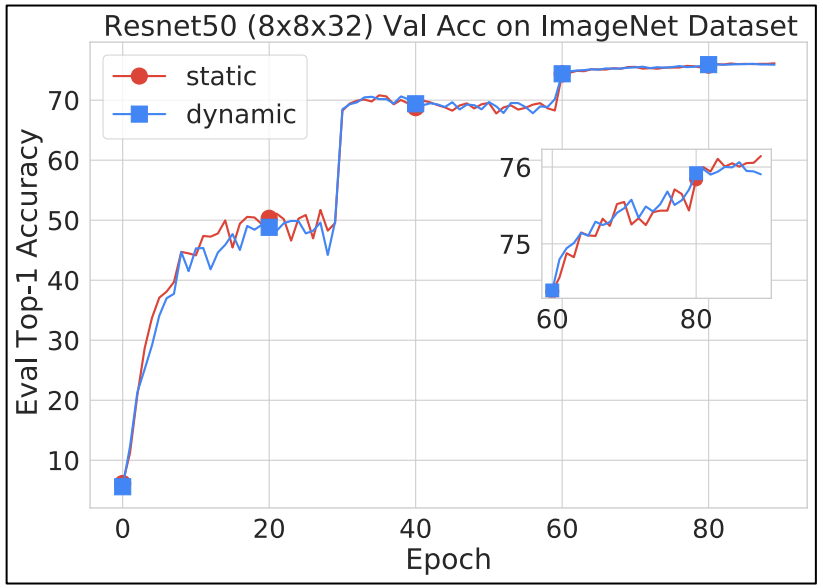
GPU: Up to 256 Tesla V100 GPUs

- **Wall-clock time** to finish 90 epochs of training; measures per-iter communication
- **Validation accuracy** after 90 epochs of training; measures convergence rate

One peer is not slower than static exponential graph

Image classification: ResNet-50 for ImageNet; $8 \times 8 = 64$ GPUs.





One-peer and exponential graphs converge **roughly the same**; but one-peer is more comm. efficient

DSGD over one-peer Exp. achieves better linear speedup

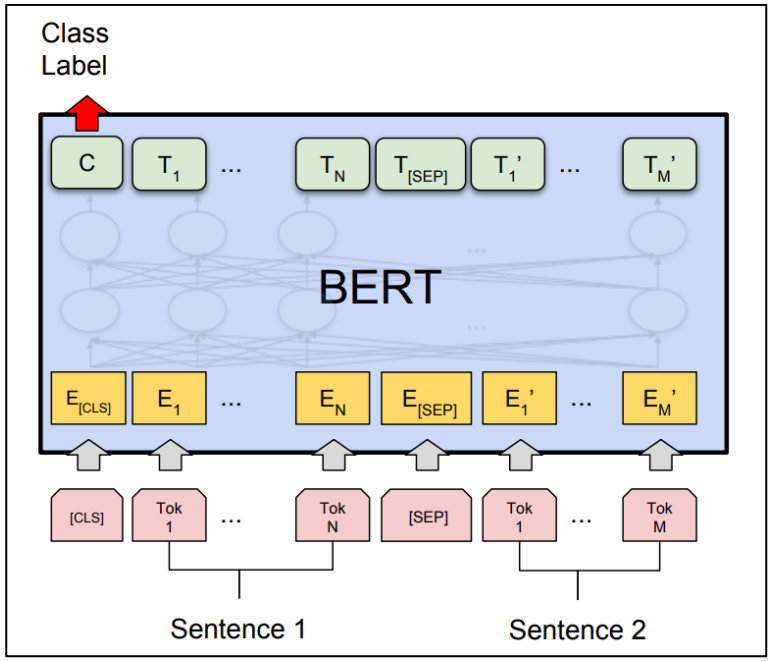
nodes topology	4(4x8 GPUs) acc.	 time	8(8x8 GPUs) acc.	 time	16(16x8 GPUs) acc.	 time	32(32x8 GPUs) acc.	 time
P-SGD	76.32	11.6	76.47	6.3	76.46	3.7	76.25	2.2

DSGD over ring has more efficient comm. than PSGD; **suffers from performance degradation**

DSGD over one-peer exp. graph is more comm.-efficient **without performance degradation**

[YYC+21]B. Ying, K. Yuan, Y. Chen, H. Hu, P. Pan, and W. Yin, ``Exponential Graph is Provably Efficient for Deep Training", NeurIPS 2021

Experiments in deep training (language modeling)



Model: BERT-Large (330M parameters)

Dataset: Wikipedia (2500M words) and BookCorpus (800M words)

Hardware: 64 GPUs

Table. Comparison in loss and training time [CYZ+21]

Method	Final Loss	Wall-clock Time (hrs)
P-SGD	1.75	59.02
D-SGD	1.77	30.4

[CYZ+21] Y. Chen, K. Yuan, Y. Zhang, P. Pan, Y. Xu, and W. Yin, ``Accelerating Gossip SGD with Periodic Global Averaging", ICML 2021

A brief summary

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- Exponential graphs are both sparse and effective. They are nearly best up to logarithm terms
- One-peer exponential graph is even sparser without hurting effectiveness

Topology	Per-iter. Comm.	Trans. Iters. (iid scenario)
Ring	$O(1)$	$O(n^7)$
2D-Grid	$O(1)$	$\tilde{O}(n^5)$
2D-Torus	$O(1)$	$O(n^5)$
$\frac{1}{2}$-RandGraph	$O(n)$	$O(n^3)$
Static Exp	$\tilde{O}(1)$	$\tilde{O}(n^3)$
One-Peer Expo	$O(1)$	$\tilde{O}(n^3)$

However …

- Periodic exact average for one-peer exp. **only holds** when network size **n is a power of 2**
- Not known when one-peer exp. performs well when n is **not** a power of 2
- Not known whether the transient iteration $O(n^3 \log_2^2(n))$ can be further improved to $O(n^3)$

Can we develop topologies that

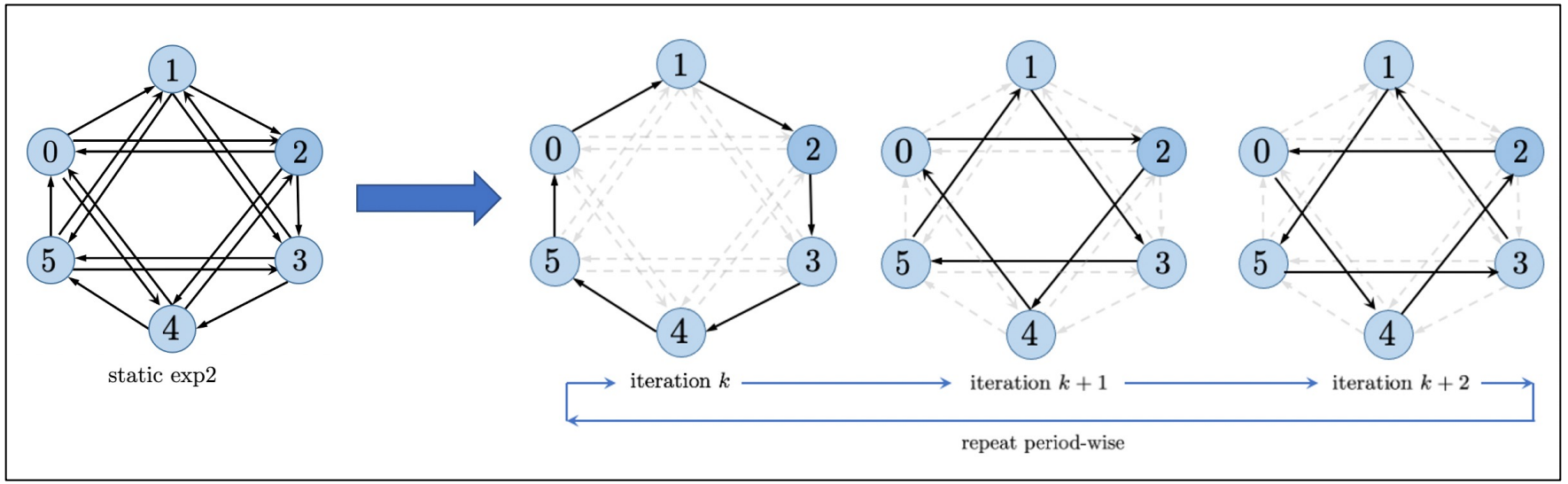
- have $O(1)$ per-iteration communication cost;
- enable DSGD to converge with $O(n^3)$ transient iteration complexity;
- and are valid for any network size n?

PART 03

EquiTopo graphs

Why does exponential graph suffer log(n) deterioration?

- Exponential graphs are still not well-connected



- For example, node 0 never sends messages to nodes 3 and 5

- We need to develop topologies that every pair of nodes is connected in positive probability

Basis weight matrix and graph

Definition Given a graph of size n, we introduce a set of doubly stochastic *basis matrices* $\{A^{(u,n)}\}_{u=1}^{n-1}$, where $A^{(u,n)} = [a_{ij}^{(u,n)}] \in \mathbb{R}^{n\times n}$ with

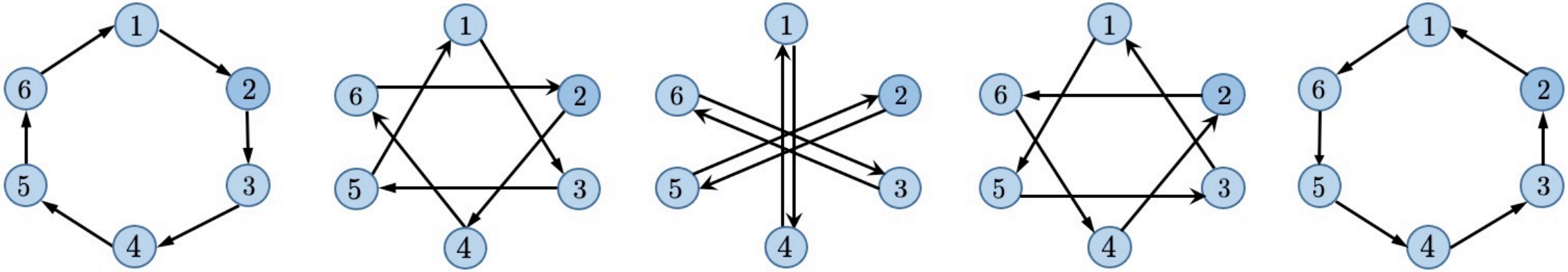
$$a_{ij}^{(u,n)} = \begin{cases} \frac{n-1}{n}, & \text{if } i = (j+u) \mod n, \\ \frac{1}{n}, & \text{if } i = j, \\ 0, & \text{otherwise.} \end{cases}$$

Their associated graphs $\{\mathcal{G}(A^{(u,n)})\}_{u=1}^{n-1}$ are called *basis graphs*.

Z. Song*, W. Li*, K. Jin*, L. Shi, M. Yan, W. Yin, and K. Yuan “Communication-efficient topologies for decentralized learning with O(1) consensus rate”, NeurIPS 2022

Basis weight matrix and graph: illustration

The set of basis graphs $\{\mathcal{G}(A^{(u)})\}_{u=1}^{5}$ for n = 6



- Each basis graph $\mathcal{G}(A^{(u)})$ is an **one-peer** graph; O(1) per-iteration communication overhead
- Each edge in a basis graph has the same **label difference**

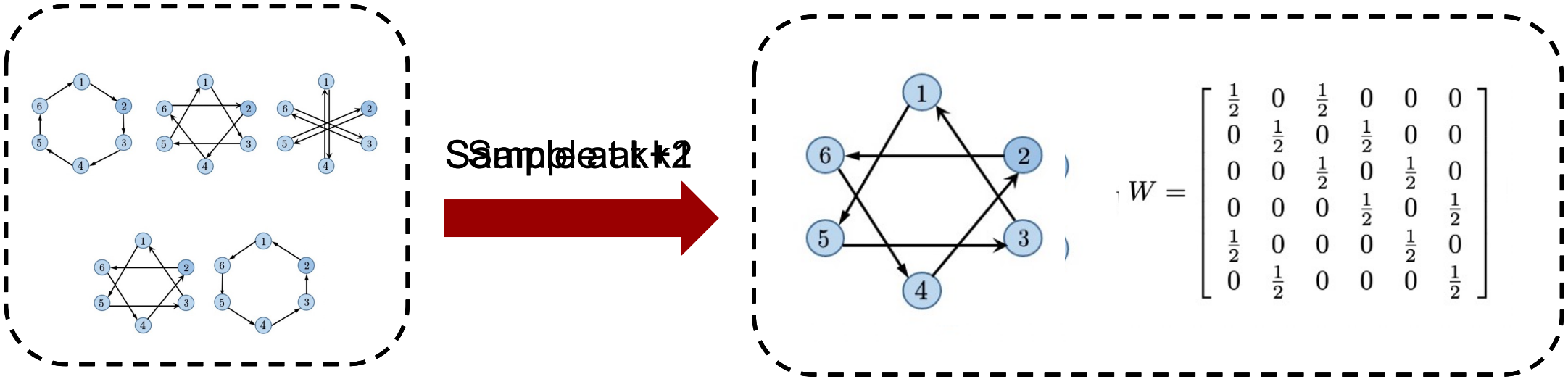
OP-EquiDyn: One-peer dynamic graph with equivalent label differences

Generate EquiDyn realization $W^{(k)}$

Pick v_k from uniform distribution over the basis index set $[n-1]$

Produce basis matrix $A^{(v_k)}$ according to the definition

Generate weight matrix $W^{(k)} = (1-\eta)I + \eta A^{(v_k)}$



OP-EquiDyn v.s. OP-Exponential

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OP-Exp	Sampled in a **cyclic** manner Nodes with **exponential** label differences can be connected
OP-EquiDyn	Sampled in a **random** manner Nodes with **any** label differences can be connected

OP-EquiDyn has a network-size-independent spectral gap

Theorem. Let the one-peer directed weight matrix $W^{(k)}$ be generated by the above EquiDyn algorithm. If we let $\eta = 1/2$, it then holds that

$$\rho = \mathbb{E}\|W^{(k)} - \frac{1}{n}\mathbb{1}_n\mathbb{1}_n^T\|_2 \leq \frac{\sqrt{2}}{2}$$

- Such spectral gap is **independent of network size**, and holds for **any size n**
- Recall that DSGD has transient iteration complexity $O(n^3(1-\rho)^{-2})$
- Substituting $\rho = \sqrt{2}/2$, DSGD over OP-EquiDyn has tran. iters. $O(n^3)$

OP-EquiDyn has a network-size-independent spectral gap

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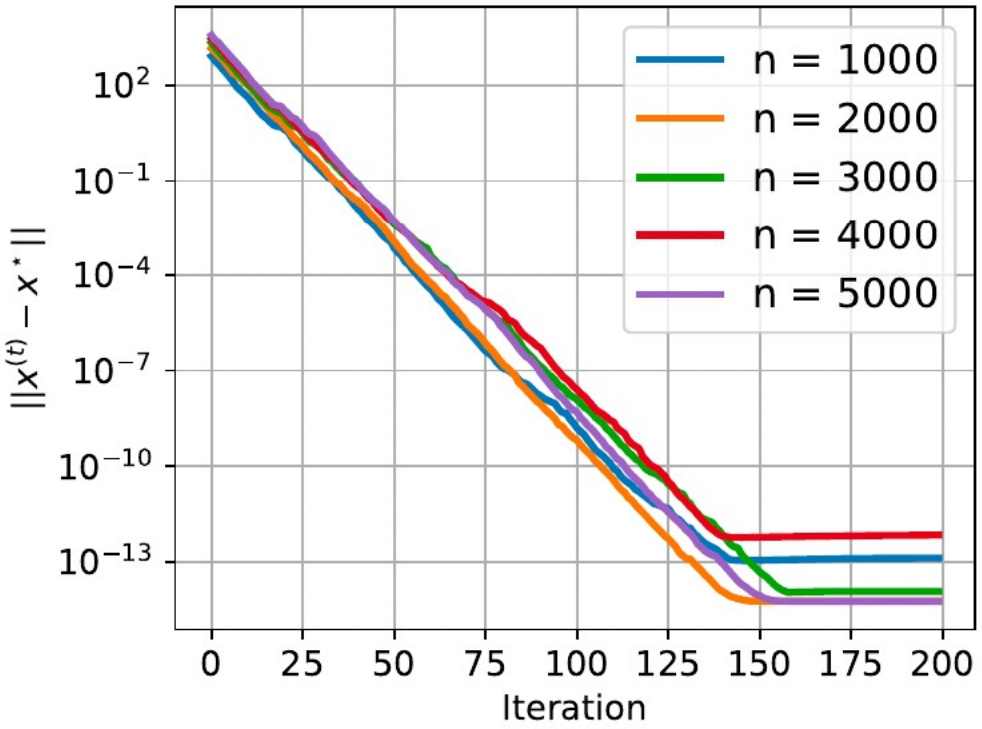
- We examine $\|\frac{1}{n}\mathbb{1}\mathbb{1}^T x - \prod_{k=0}^{T} W^{(k)} x\|$ for a vector x
- OP-EquiDyn has a network-size-independent rate
- While network size increases, consensus rate remains almost unchanged



OP-EquiDyn achieves new SOTA results

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DSGD with different network topology

Topology	Per-iter. Comm.	Trans. Iters. (iid scenario)
Ring	$O(1)$	$O(n^7)$
2D-Grid	$O(1)$	$\tilde{O}(n^5)$
2D-Torus	$O(1)$	$O(n^5)$
$\frac{1}{2}$-RandGraph	$O(n)$	$O(n^3)$
Static Exp	$\tilde{O}(1)$	$\tilde{O}(n^3)$
One-Peer Expo	$O(1)$	$\tilde{O}(n^3)$
O.-P. EquiDyn	$O(1)$	$O(n^3)$

- OP-EquiDyn achieves O(1) comm., $O(n^3)$ transient iteration complexity, and holds for any size n
- Since DSGD has a transient complexity as $O(n^3(1-\rho)^{-2})$, the order $O(n^3)$ cannot be improved

OP-EquiDyn can also accelerate other decentralized methods

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Gradient tracking

$$\begin{aligned}\boldsymbol{x}_i^{(t+1)} &= \sum_{j=1}^n w_{ij}^{(t)}\left(\boldsymbol{x}_j^{(t)} - \gamma \boldsymbol{y}_j^{(t)}\right);\\ \boldsymbol{y}_i^{(t+1)} &= \sum_{j=1}^n w_{ij}^{(t)} \boldsymbol{y}_j^{(t)} + \boldsymbol{g}_i^{(t+1)} - \boldsymbol{g}_i^{(t)}, \quad \boldsymbol{y}_i^{(0)} = \boldsymbol{g}_i^{(0)}.\end{aligned}$$

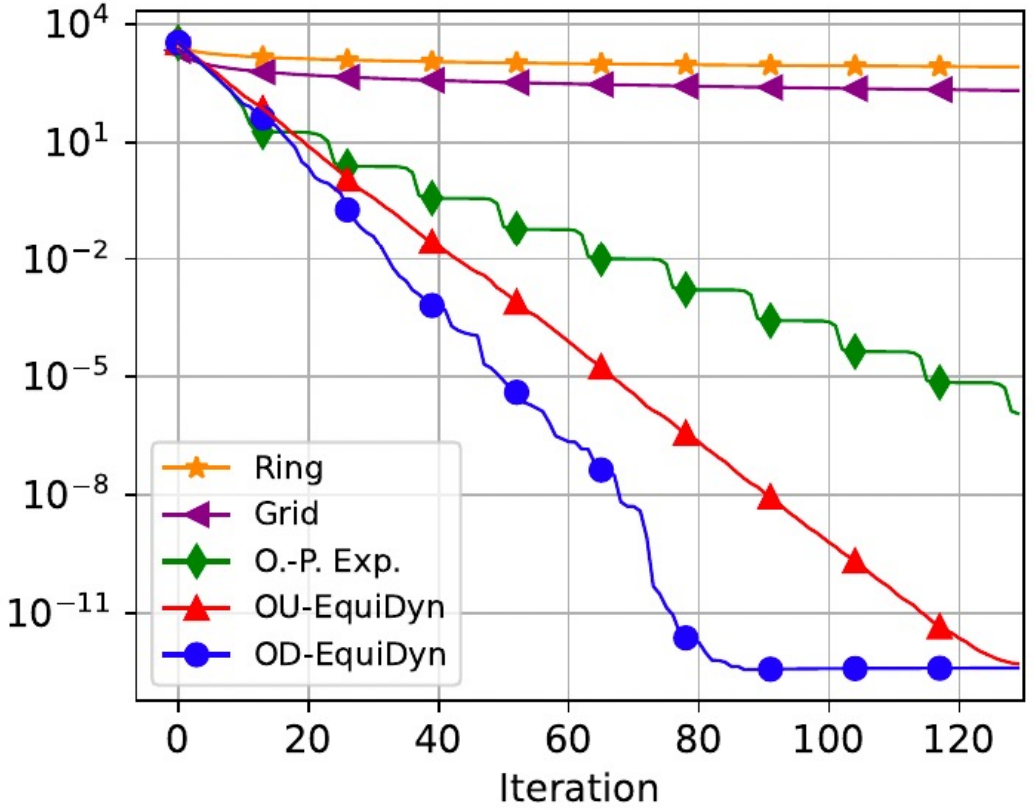
Topology	Per-iter Comm.	Convergence Rate	Trans. Iters.
Ring	$\Theta(1)$	$\mathcal{O}\left(\frac{\sigma}{\sqrt{nT}} + \frac{n^2\sigma^{\frac{2}{3}}}{T^{\frac{2}{3}}} + \frac{n^4}{T}\right)$	$\mathcal{O}(n^{15})$
Torus	$\Theta(1)$	$\mathcal{O}\left(\frac{\sigma}{\sqrt{nT}} + \frac{n\sigma^{\frac{2}{3}}}{T^{\frac{2}{3}}} + \frac{n^2}{T}\right)$	$\mathcal{O}(n^9)$
Static Exp.	$\Theta(\ln(n))$	$\mathcal{O}\left(\frac{\sigma}{\sqrt{nT}} + \frac{\ln(n)\sigma^{\frac{2}{3}}}{T^{\frac{2}{3}}} + \frac{\ln^2(n)}{T}\right)$	$\mathcal{O}(n^3\ln^6(n))$
O.-P. Exp.	1	$\mathcal{O}\left(\frac{\sigma}{\sqrt{nT}} + \frac{\ln(n)\sigma^{\frac{2}{3}}}{T^{\frac{2}{3}}} + \frac{\ln^2(n)}{T}\right)$	$\mathcal{O}(n^3\ln^6(n))$
OD (OU)-EquiDyn	1	$\mathcal{O}\left(\frac{\sigma}{\sqrt{nT}} + \left(\frac{\sigma}{T}\right)^{\frac{2}{3}} + \frac{1}{T}\right)$	$\mathcal{O}(n^3)$

Experiments: compare with other topologies

- We examine $\|\frac{1}{n}\mathbb{1}\mathbb{1}^T x - \prod_{k=0}^{T} W^{(k)} x\|$ for a vector x

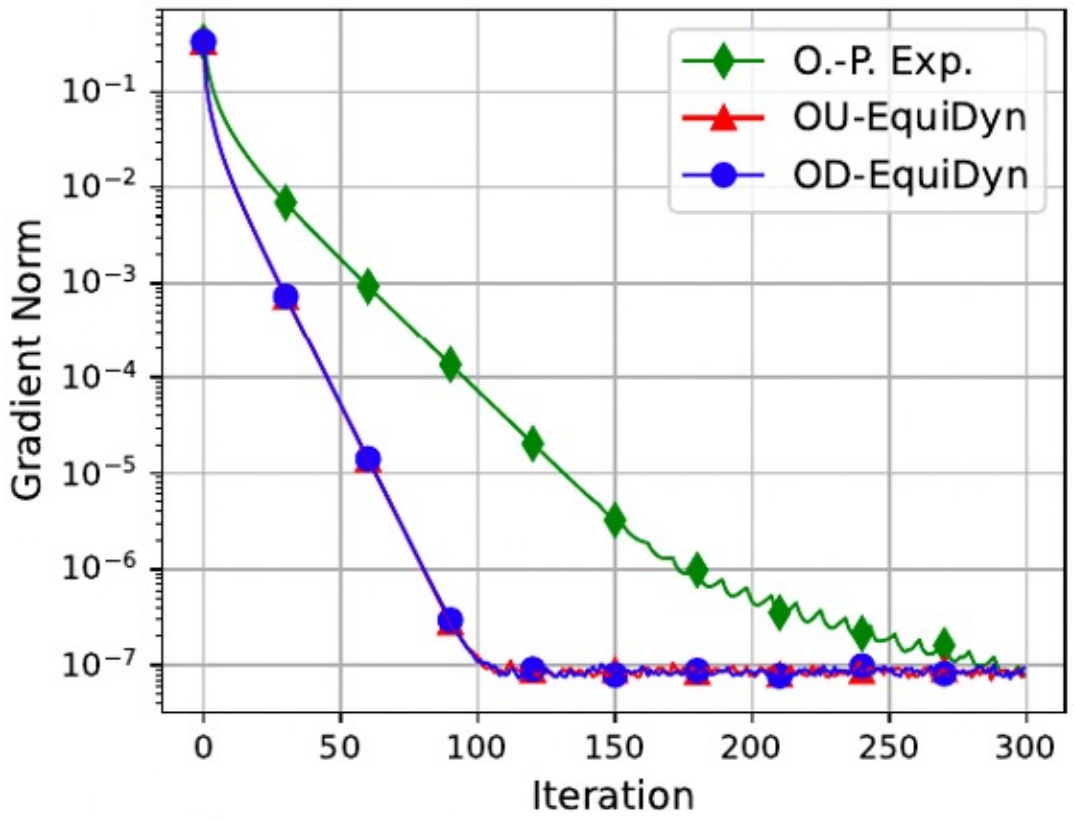
- Network size is 4900

- EquiDyn converges the fastest



Experiments: gradient tracking with different topologies

- We use GT to solve logistic regression with non-convex regularizes
- Network size is 300
- GT with EquiDyn converges faster than OP-Exp



Experiments: deep learning experiments

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- EquiTopo graph has many variants, i.e., OU-EquiDyn supports undirected graphs
- EquiTopo graph outperforms other common topologies with 17 GPUs

Topology	MNIST Acc.	CIFAR-10 Acc.
Centralized SGD	98.34	91.76
Ring	98.32	91.25
Static Exp.	98.31	91.48
O.-P. Exp.	98.17	90.86
D-EquiStatic	98.29	**92.01**
U-EquiStatic	98.26	91.74
OD-EquiDyn	**98.39**	91.44
OU-EquiDyn	98.12	91.56

Summary

Can we develop topologies that

- have $O(1)$ per-iteration communication cost;
- enable DSGD to converge with $O(n^3)$ transient iteration complexity;
- and are valid for any network size n?

One-peer EquiDyn is the answer!